W. D. Wallis

A Beginner's Guide to Finite Mathematics

For Business, Management, and the Social Sciences

Second Edition

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For EAWKW

Preface

When elementary courses in discrete and combinatorial mathematics first became popular, they usually covered a broad spectrum of pure and applied topics. Most of the students taking such courses were from mathematics and computer science, with a handful of brave souls from other disciplines. Those other students usually found the courses quite difficult. However, the applications were useful in a number of areas.

The teaching of discrete topics has evolved into two streams. The more mathematical parts are studied in courses called *Discrete Mathematics*, and more advanced, more rigorous courses called *Combinatorics*, or named for specific areas (*Graph Theory*, *Combinatorial Designs*, *Cryptography*, and so on). Introductions to those areas of applicable discrete mathematics used by students in business, management and the social sciences are usually called *Finite Mathematics*, and elementary courses on this material are now standard at many colleges and universities. These courses are typically offered at the freshman level although many students take them later in their careers.

This book is designed for a one-semester course in finite mathematics and applications. Often this course will be the student's first mathematical experience at the college level, so I have tried to avoid too much sophistication. Those seeking a more mathematical course should look elsewhere, at the many books on the market that are named *Introduction to Discrete Mathematics* or *Introduction to Combinatorics*.

Outline of Topics

The first chapter contains a brief survey of numbers, equations and elementary set theory, including Venn diagrams. I have also defined averages (means, medians); these concepts are omitted by some authors while others define them after discussing probability, but I find that they fit in naturally as properties of sets of numbers.

Counting is covered in Chapter 2. We discuss selections (combinations) and arrangements (permutations), and the binomial theorem for positive integer index; the proof of the binomial theorem could be omitted on a first reading. Only the most elementary parts of this topic are covered; readers wanting more details should look in books on discrete mathematics (such as my *A Beginner's Guide to Discrete Mathematics*, Second Edition, also published by Birkhäuser) or combinatorics.

Counting leads naturally to probability theory. In Chapter 3, I have included the main ideas of discrete probability, up to Bayes' theorem. A brief coverage of random variables and expected values (using the discussion of means in Chapter 1) follows, but this may be omitted.

Chapter 4 starts with a section on relations and functions. Most students can skim this material, reading only the subsection *First definitions* and the first paragraph on functions and working the first couple of exercises in each set, but a few instructors prefer to include the harder material. Then we study graph theory, including Euler and Hamilton cycles and coloring problems. This area is omitted from some finite mathematics courses, and the ideas in this chapter are not covered elsewhere in the book, so the whole chapter could be skipped if desired.

Matrices and vectors are defined and discussed briefly in Chapter 5. This course is not the place for algebraic studies, but matrices are useful for studying systems of linear equations, and are also used for input-output models.

In Chapter 6, we discuss linear programming, including the simplex method. Matrices are again useful in this chapter, providing a concise way of representing the calculations. We include a discussion of the two-phase simplex method ("big M method") for solving linear programming problems in which a basic feasible starting point is not obvious.

This is where the first edition ended. However, a number of users asked for some further material. Those who were teaching stronger students wanted a discussion of two-player, zero-sum games. This is an important application of linear programming. So Chapter 7 is now an introduction to the theory of games.

Some other colleagues suggested that beginning business majors would benefit from a discussion of elementary financial mathematics, compound interest and such. So I added a chapter on this area. I have broadened this slightly to include two other areas—population growth and radioactive decay—so that students can see that the mathematics of compounding has applications outside finance.

Problems and Exercises

A number of worked examples, called Sample Problems, are included in the body of each section. Most of these are accompanied by a practice exercise labeled "Your Turn", designed primarily to test the reader's comprehension of the ideas being discussed. It is recommended that students work all of these exercises; complete solutions are provided at the end of the book.

The book contains a large selection of exercises, collected at the end of sections. There should be enough for students to practice the concepts involved; most of the problems are quite easy. In a departure from the usual method of providing answers to odd-numbered exercises, I have divided the exercises for each section into two sets of roughly the same difficulty, A and B, and provided answers to set A only.

Acknowledgments

This treatment of finite mathematics owes a great deal to many colleagues and mathematicians in other institutions with whom I have taught or discussed this material—too many to mention individually. I would like to thank John George for carefully reading the manuscript and making many useful suggestions, and I am grateful for the constant support and encouragement of the staff at Birkhäuser.

Carbondale, IL, USA W.D. Wallis

Contents

1

Numbers and Sets

1.1 Numbers

Sets and Number Systems

All of discrete mathematics—and, in fact, all of mathematics—rests on the foundations of set theory and numbers. We start this chapter by reminding you of some basic definitions and notations and some further properties of numbers and sets.

A *set* is any collection of objects. The objects in the collection are called the members or elements of the set. If x is a member of a set S, we write $x \in S$, and $x \notin S$ means that x is *not* a member of S. One way of defining a set is to list all the elements, usually between braces; thus if S is the set consisting of the numbers 0, 1, and 3, we could write $S = \{0, 1, 3\}$.

Another method is to use the membership law of the set: for example, since the numbers 0, 1, and 3 are precisely the numbers that satisfy the equation $x^3 - 4x^2 + 3x = 0$, we could write the set S as

$$S = \left\{x : x^3 - 4x^2 + 3x = 0\right\}$$

(which we read as "the set of all x such that $x^3 - 4x^2 + 3x = 0$"). Often we use a vertical line instead of the colon in this expression, as in

$$S = \left\{x \mid x^3 - 4x^2 + 3x = 0\right\}.$$

This form is sometimes called *set-builder notation*.

Sample Problem 1.1. *Write three different expressions for the set with elements* 1 *and* -1.

Solution. Three possibilities are $\{1, -1\}$, $\{x : x^2 = 1\}$, and "the set of square roots of 1". There are others.

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Your Turn. Write three different expressions for the set with elements 1, 2, and 3.

We define several sets of numbers, denoted $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, that we shall use later. The set $\mathbb{N}$ of all positive integers or natural numbers is the first number system we encounter; the natural numbers are used to count things. If one includes zero, to account for the possibility of there being nothing to count, and negatives for subtraction, the result is the set $\mathbb{Z}$ of integers. It is useful to have a notation for the set of all *non-negative* integers—that is, the members of $\mathbb{N}$, together with 0. We shall write $\mathbb{Z}^*$ for that set, but the reader should note that a minority of authors, particularly in Computer Science, include 0 as a member of $\mathbb{N}$.

We sometimes write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The use of a string of dots (an *ellipsis*) is not precise, but is understood to mean that the set continues without end. Such sets are called *infinite* (as opposed to *finite* sets like $\{0, 1, 3\}$). We also write $\{1, 2, \ldots, 20\}$ to mean the set of all positive integers from 1 to 20. A more precise notation for this set, borrowed from the computer language *Pascal*, is (1..20).

The set $\mathbb{Q}$ of rational numbers consists of all fractions, the ratios $\frac{p}{q}$, where p and q are integers and q is not 0. In other words,

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}.$$

In this notation, p is called the *numerator* of the fraction, and q is the *denominator*. An alternative definition is that $\mathbb{Q}$ is the set of all numbers with a repeating or terminating decimal expansion. Examples are

$$\begin{aligned}\frac{1}{2} &= 0.5,\\ \frac{-12}{5} &= -2.4,\\ \frac{3}{7} &= 0.428571428571\ldots.\end{aligned}$$

In the last example, the digit string 428571 repeats forever, and we usually indicate this by writing

$$\frac{3}{7} = 0.\overline{428571}.$$

The denominator q cannot be zero. In fact, division by zero is never possible. This is not a made-up rule, but rather it follows from the definition of division. When we write $x = \frac{p}{q}$, we mean "x is the number that, when multiplied by q, gives p". What would $x = 2/0$ mean? There is no number that, when multiplied by 0, gives 2. Similarly, $x = 0/0$ would be meaningless. In this case, there *are* suitable numbers x, in fact every number will give 0 when multiplied by 0, but we wanted a uniquely defined answer.

Different decimal expansions do not always mean different numbers. The exception is an infinite string of 9s. These can be rounded up: $0.\overline{9} = 1$, $0.7\overline{9} = 0.8$, and so on. Proving this requires some understanding of the theory of limits, but the following discussion illustrates the ideas of the proof.

Suppose x is the number represented by $0.\overline{9}$. Then $x = 0.\overline{9}$, and multiplying both sides by 10, $10x = 9.\overline{9}$. If we subtract x from both sides, we get $9x = 10x - x = 10x - 0.\overline{9}$. So $9x = 9.\overline{9} - 0.\overline{9} = 9$ and $x = 1$.

Integers are rational numbers, and in fact they are the rational numbers with numerator 1. For example, $5 = 5/1$.

Each rational number has infinitely many representations as a ratio:

$$1/2 = 2/4 = 3/6 = \cdots.$$

The final number system we shall use is the set $\mathbb{R}$ of real numbers, consisting of all numbers that are decimal expansions, all numbers that represent lengths. If n is any natural number other than a perfect square (one of $1, 4, 9, 16, \ldots$), then $\sqrt{n}$ is not rational. Another important number that is not rational is the ratio π of the circumference of a circle to its diameter.

Remember that every natural number is an integer; every integer is a rational number; every rational number is a real number. Do not fall into the common error of thinking that "rational number" excludes the integers, and so on. There are special words that exclude: for example, real numbers that are not rational are called *irrational numbers*.

This is not the end of number systems. For example, the set $\mathbb{C}$ of *complex numbers* is derived from the real numbers by including square roots of negative numbers, plus all the sums of these square roots with real numbers. However, we do not encounter them in this book.

Intervals

The set of all real numbers between two bounds is called an *interval*. For example, the set of all real numbers greater than -2 but smaller than 2, which we would write as

$$S = \{x : x \in \mathbb{R}, -2 < x < 2\},$$

is the interval from -2 to 2, exclusive. These intervals occur so often that there is a special notation for them. The above example is written $(-2, 2)$, where the parenthesis (round bracket) indicates that the bound is not included. If the bound is included, a square bracket is used. So

$$\begin{aligned}
(-2, 2) &= \{x : x \in \mathbb{R}, -2 < x < 2\},\\
[-2, 2) &= \{x : x \in \mathbb{R}, -2 \le x < 2\},\\
(-2, 2] &= \{x : x \in \mathbb{R}, -2 < x \le 2\},\\
[-2, 2] &= \{x : x \in \mathbb{R}, -2 \le x \le 2\}.
\end{aligned}$$

The set of all real numbers greater than 1 is denoted $(1, \infty)$, and the set of all real numbers less than or equal to 3 is $(-\infty, 3]$. Read the symbol ∞ as *infinity*.

Sample Problem 1.2. *Write expressions for the following intervals:*

$$\{x : x \in \mathbb{R}, 3 < x \leq 4\}, \qquad \{x : x \in \mathbb{R}, 0 \leq x\},$$
$$\{x : x \in \mathbb{R}, x < 2\}, \qquad \{x : x \in \mathbb{R}, x \geq 2\}.$$

Solution. $(3, 4]$, $[0, \infty)$, $(-\infty, 2)$, $[2, \infty)$.

Your Turn. Write expressions for the following intervals:

$$\{x : x \in \mathbb{R}, 0 < x\}, \qquad \{x : x \in \mathbb{R}, 0 < x < 1\},$$
$$\{x : x \in \mathbb{R}, 1 < x \leq 4\}, \qquad \{x : x \in \mathbb{R}, x < 3\}.$$

Factors

When x and y are integers, we use the phrase "x divides y", and write $x \mid y$, to mean "$\frac{y}{x}$ is an integer", or alternatively "there is an integer z such that $y = x \cdot z$". We say x is a *divisor* of y. If $x \mid y$ is false, one can write $x \nmid y$. Thus 2 divides 6 (because $\frac{6}{2}$ equals the integer 3), -2 divides 6 (because $\frac{6}{-2} = -3$), 2 divides -6 (because $\frac{-6}{2} = -3$). Putting it the other way, the reasons are respectively, $6 = 2 \cdot 3$, $6 = (-2) \cdot (-3)$, and $-6 = 2 \cdot (-3)$. In symbols, we would write $2 \mid 6$ and $-2 \mid 6$. As 4 does not divide 6, we write $4 \nmid 6$.

Some students are confused by the case $y = 0$, but according to our definition x divides 0 for any non-zero integer x. We define the *factors* of a positive integer x to be the positive divisors of x (in our terminology, negative divisors are not called factors so -2 is not a factor of 6).

If x divides both y and z, then we call x a *common divisor* of y and z. Among the common divisors of y and z there is naturally a greatest one, called (not surprisingly) *the greatest common divisor* of y and z, and denoted (y, z), or sometimes $gcd(y, z)$. If $(y, z) = 1$, y and z are called *coprime* or *relatively prime*. For example, $(4, 10) = 2$, so 4 and 10 are not coprime; $(4, 9) = 1$, so 4 and 9 are coprime.

A *prime number* is a positive integer x other than 1 whose only factors are 1 and x. The first few primes are 2, 3, 5, 7, 11, and 13. The number 1 is excluded, and is called a *unit*.

Sample Problem 1.3. *What are the factors of* 12, 22, *and* 17*? What is* $(12, 22)$*?*

Solution.

12 has factors 1, 2, 3, 4, 6, and 12.

22 has factors 1, 2, 11, and 22.

17 has factors 1 and 17.

The common factors of 12 and 22 are 1, 2, so $(12, 22) = 2$.

Your Turn. What are the factors of 24, 15, and 13? What is (24,15)?

Suppose x is any positive integer. If x is not a prime we can find positive integers y and z, neither equal to 1, such that $x = yz$. We could then break down y and z, if possible, until finally we obtain

$$x = x_1, x_2, \ldots, x_k$$

where the x_i are all primes. So every positive integer is the product of prime factors. In fact, it can be shown that such a decomposition is unique (up to the order of the factors): for example, if

$$2^a 3^b 5^c = 2^x 3^y 7^z,$$

it must be true that $a = x$, $b = y$, and c and z are both zero.

Exponents

If x is a positive integer, we know that b^x is the product of x copies of b; in this expression b is ellipsis called the *base* and x the *exponent*. It is easy to deduce such properties as

$$\begin{aligned} b^x b^y &= b^{x+y}, \\ \left(b^x\right)^y &= b^{xy}, \\ (ab)^x &= a^x b^x. \end{aligned}$$

Negative exponents are handled by defining $b^{-x} = \frac{1}{b^x}$, and also $b^0 = 1$ whenever b is non-zero. The rule $b^x b^y = b^{x+y}$ leads us to define $b^{\frac{1}{x}}$ to be the xth root of b. (When x is even, we take the positive root for positive b and say $b^{\frac{1}{x}}$ is not defined for negative b).

Sample Problem 1.4. *Express the following in the simplest form, as decimal numbers if possible:*

$$(612)^0, \quad \left(x^3\right)^5, \quad \left(x^4\right)^0, \quad (-10)^{-4}, \quad \frac{1}{10^{-2}}.$$

Solution. $(612)^0 = 1$ ($b^0 = 1$ for non-zero b); $(x^3)^5 = x^{3\cdot 5} = x^{15}$; $(x^4)^0 = 1$ (again, $b^0 = 1$ for any b, or you could also argue that $(x^4)^0 = x^{4\cdot 0} = x^0 = 1$); $(-10)^{-4} = \frac{1}{(-10)^4} = \frac{1}{(-1)^4 \times 10^4} = \frac{1}{10^4} = 0.0001$; $\frac{1}{10^{-2}} = 10^2 = 100$.

Your Turn. Do the same to

$$1^{15}, \quad 0^5, \quad \left(x^{-1}\right)^0, \quad \frac{x^6}{x^3}, \quad (-2)^{-1}, \quad \frac{1}{5^{-2}}.$$

Sample Problem 1.5. *Express in the simplest possible form, with positive exponents:*

$$\frac{1}{x^{-4}}, \quad \frac{u^{-2}}{v^{-3}}, \quad \left(3a^2\right)\left(5a^{-3}\right).$$

Solution.

$$\frac{1}{x^{-4}} = \left(x^{-4}\right)^{-1} = x^{(-4)\cdot(-1)} = x^4;$$
$$\frac{u^{-2}}{v^{-3}} = \left(u^{-2}\right)\left(v^{-3}\right)^{-1} = u^{-2}v^3 = \frac{v^3}{u^{-2}};$$
$$\left(3a^2\right)\left(5a^{-3}\right) = 3 \cdot 5 \cdot a^2 \cdot a^{-3} = 15a^{-1} = \frac{15}{a}.$$

Your Turn. Express in the simplest possible form, with positive exponents:

$$\frac{t^{-2}}{t^{-3}}, \quad y^{5-2}, \quad \left(4x^{-2}\right)\left(3x^4\right).$$

Absolute Value, Floor, and Ceiling

The *absolute value* or *modulus* of the number x, which is written $|x|$, is the positive number equal to either x or $-x$. For example, $|5.3| = 5.3$, $|-7.2| = 7.2$.

The *floor* $\lfloor x \rfloor$ of x is the largest integer not greater than x. If x is an integer, $\lfloor x \rfloor = x$. Some other examples are $\lfloor 6.1 \rfloor = 6$, $\lfloor -6.1 \rfloor = -7$. It is easy to deduce the following properties:

(1) $\lfloor x \rfloor = n$ if and only if n is an integer and $n \le x < n+1$.

(2) If x is non-negative, then $\lfloor x \rfloor$ equals the integer part of x. If x is negative and non-integral then $\lfloor x \rfloor$ is 1 less than the integer part of x.

(3) If n and k are integers, then k divides n if and only if $\frac{n}{k} = \lfloor \frac{n}{k} \rfloor$.

The *ceiling* $\lceil x \rceil$ is defined analogously as the smallest integer not less than x.

Sample Problem 1.6. *What are* $\lfloor 6.3 \rfloor$, $\lceil 7.2 \rceil$, $\lceil -2.4 \rceil$, $|3.4|$, $|-3.4|$*?*

Solution. 6, 8, −2, 3.4, 3.4.

Your Turn. What are $\lfloor 2.6 \rfloor$, $\lfloor -1.7 \rfloor$, $\lceil 4.8 \rceil$, $\lceil -4 \rceil$, $|3.1|$, $|-4,4|$?

Exercises 1.1 A

1. Is the given statement true or false?

(i) $3 \in \{2, 3, 4, 6\}$; (ii) $4 \notin \{2, 3, 4, 6\}$;
(iii) $5 \in \{2, 3, 4, 6\}$; (iv) $\{3, 2\} = \{2, 3\}$;
(v) $\{1, 2\} \in \{1, 2, 3\}$; (vi) $\{1, 2\} = \{1, 2, 3\}$;
(vii) $2 \in \mathbb{Z}$; (viii) $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$;
(ix) $\mathbb{R} \notin \mathbb{Z}$; (x) $\{4\} \in \{2, 4, 6\}$.

2. In each case, write the list of all members of the set.

(i) $\{x : x$ is an even positive integer less than $12\}$;

(ii) $\{x : x$ is a color on the American flag$\}$;

(iii) $\{x : x$ is a day of the weekend$\}$.

3. Which of the following are true?

(i) All natural numbers are integers.

(ii) All integers are natural numbers.

4. Give an example of a real number that is not a rational number.

5. For each of the following numbers, to which of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ does it belong?

(i) 1.308; (ii) $1.\overline{3}$; (iii) -7;
(iv) $\sqrt{3}$; (v) 10508; (vi) $1 - \sqrt{7}$.

6. Give three different expressions for the set with members 1, -1, and 0.

7. In each case, list all the factors of the number:

(i) 36; (ii) 48; (iii) 50;
(iv) 81; (v) 72; (vi) 61.

8. Decompose the following numbers into primes:

(i) 1044; (ii) 268;
(iii) 256; (iv) 333.

9. In each case, decompose the two numbers into primes, and then compute their greatest common divisor:

(i) 88 and 132; (ii) 256 and 224;
(iii) 1080 and 855; (iv) 168 and 231.

10. Simplify the following expressions, writing the answers using positive exponents only:

(i) $t^3 t^{-3}$; (ii) $\left(2x^2y^{-3}\right)^2$;
(iii) $\dfrac{(xy)^3}{(xy)^2}$; (iv) $\dfrac{5^6 2^4}{10^3}$;

(v) $\left(4b^2\right)^2(2b)^{-3}$; (vi) $(x+y)^{-3}$;

(vii) $\left(\frac{a^{-4}}{a^{-3}}\right)^{-2}$; (viii) $\frac{4p^2q^2}{8p^2q}$.

11. Evaluate the following:

(i) $\lfloor 117 \rfloor$; (ii) $\lceil 44.3 \rceil$;

(iii) $\lfloor 28.4 \rfloor$; (iv) $\lceil -107.7 \rceil$;

(v) $|11.4|$; (vi) $\lceil \sqrt{3} \rceil$;

(vii) $|-27|$; (viii) $\lfloor -73 \rfloor$;

(ix) $\lfloor \sqrt{2} \rfloor$; (x) $\lfloor -7.3 \rfloor$.

Exercises 1.1 B

1. Is the given statement true or false?

(i) $5 \in \{1, 3, 4, 7\}$; (ii) $6 \notin \{1, 3, 4, 7\}$;

(iii) $4 \in \{1, 3, 4, 7\}$; (iv) $\{1, 3\} = \{1, 2, 3\}$;

(v) $3 \in \{1, 2, 3, 4, 5, 6\}$; (vi) $\mathbb{R} \subseteq \mathbb{Z}$;

(vii) $4 \subseteq \{1, 3, 4, 5, 7\}$; (viii) $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1, 0\}$;

(ix) $\{1, 3, 2\} = \{1, 2, 3\}$; (x) $\{2, 3, 4\} = \{1, 2, 3\}$.

2. In each case, write the list of all members of the set:

(i) $\{x : x$ is a month whose name starts with J$\}$;

(ii) $\{x : x$ is an odd integer between -6 and $6\}$;

(iii) $\{x : x$ is a letter in the word "Mississippi"$\}$.

3. Which of the following are true?

(i) All rational numbers are real.

(ii) All real numbers are rational.

(iii) No irrational numbers are rational.

(iv) No irrational numbers are real.

4. For each of the following numbers, to which of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ does it belong?

(i) -2.13; (ii) $\sqrt{5}$; (iii) $\sqrt{4}$; (iv) 2π;

(v) $1.8\overline{34}$; (vi) $2+\sqrt{2}$; (vii) 0; (viii) $\sqrt{2}+\sqrt{2}$.

5. In each case, list all the factors of the number:

(i) 63; (ii) 24; (iii) 40; (iv) 28;
(v) 29; (vi) 66; (vii) 112; (viii) 70.

6. Give three different expressions for the set with members 0, 1, and 2.

7. Decompose the following numbers into primes:

(i) 96; (ii) 1029; (iii) 38 and 56; (iv) 112 and 54;
(v) 187; (vi) 588; (vii) 175; (viii) 127.

8. In each case, decompose the two numbers into primes, and then compute their greatest common divisor:

(i) 240 and 84; (ii) 162 and 45;
(iii) 231 and 275; (iv) 444 and 629;
(v) 95 and 125; (vi) 462 and 252.

9. Simplify the following expressions, writing the answers using positive exponents only:

(i) x^4x^{-4}; (ii) $\dfrac{2^6 3^2}{6^3}$;
(iii) $(2xy)^{-2}$; (iv) $5y^2z^{-3}$;
(v) $(2x)^2y^3(xy)^{-2}$; (vi) $(x-y)^{-2}$;
(vii) $\left(\dfrac{x-3}{x-2}\right)^4$; (viii) $\dfrac{2x^2y}{4xy^2}$;
(ix) $6x^2(3x)^{-2}$; (x) $\dfrac{(2y)^2x^6}{x^4}$;
(xi) $(3xy)^{-2}z$; (xii) $\dfrac{3x^2}{4x^3}$.

10. Evaluate the following:

(i) $\lfloor 8.1 \rfloor$; (ii) $\lfloor -4.6 \rfloor$;
(iii) $\lceil 308 \rceil$; (iv) $\lceil 77.7 \rceil$;
(v) $\lfloor \sqrt{5} \rfloor$; (vi) $|-7.6|$;
(vii) $\lceil 21.5 \rceil$; (viii) $\lceil -7.3 \rceil$;
(ix) $|1.73|$; (x) $|-5.2|$;
(xi) $\lceil -105 \rceil$; (xii) $\lfloor \sqrt{2} \rfloor$;
(xiii) $\lfloor 2\pi \rfloor$; (xiv) $\lceil |-9.6| \rceil$;
(xv) $\lfloor \lfloor 7.5 \rfloor \rfloor$; (xvi) $\lceil \lfloor 7.5 \rfloor \rceil$.

1.2 Equations and Inequalities

Equations

An *equation* is any expression of equality. Sometimes an equation simply expresses a relation between numbers, such as $5-3=2$. But more often we do not write down the values of all the numbers concerned. For example, we might write $x-3=2$.

In the expression $x-3=2$, the quantity x is called a *variable* and can represent any real number. Then $x-3=2$ is *true* when $x=5$, and we say it is *false* otherwise. We say $x=5$ *satisfies* the equation, and we say 5 is the *solution* of the equation.

Sometimes equations can have more than one solution. For example, both $x=1$ and $x=2$ satisfy the equation $x^2-3x=-2$, as you can easily check. We can also show that there are no other solutions. The set of all solutions is called, naturally enough, the *solution set* of the equation. The solution set of $x^2-3x=-2$ is $\{1,2\}$. Two equations with the same solution set are called *equivalent*.

If a problem asks you to solve an equation, you should find all solutions, or in other words find the solution set.

Equations are not restricted to one variable. For example, the equation $x-y=2$ contains the two variables x and y, and we usually think of it as expressing the relationship between x and y. This equation has an infinitude of solutions. One example is $x=3$, $y=1$, which can also be written $(x,y)=(3,1)$. We might also use this equation to express one variable in terms of the other, as in $y=x-2$.

Some equations are satisfied by every number; one such equation is $(x-1)^2=x^2-2x+1$. Such an equation is called an *identity*. There are also equations with no solutions, and these are called *inconsistent* equations or *contradictions*.

Sample Problem 1.7. *Which of the following are solutions of* $2x+3y=1$*?* (i) $x=2, y=3$*;* (ii) $x=-1, y=1$*;* (iii) $x=5, y=-3$.

Solution. If we replace x by 2 and y by 3, the equation becomes $4+9=1$, which is false. The other two suggested solutions lead to $-2+3=1$ and $10-9=1$, respectively, and both of these are true. So (ii) and (iii) are solutions, but (i) is not.

Your Turn. Which of the following are solutions of $3x-2y=4$?
(i) $x=2, y=2$; (ii) $x=4, y=4$; (iii) $x=3, y=1$.

When an equation is first presented to you, you know nothing about the variables. When you solve an equation, you find out more about the variables. Assuming the equation to be true, you know that the variables belong to the solution set. For this reason, the variables are often called *unknowns*.

Linear Equations with One Variable

The *degree* of an equation is the highest power of a variable that occurs in it. For example, $x - y = 2$ is of degree 1, or *first degree*, while $x^2 - 3x = -2$ is of degree 2. The simplest equations are those of the first degree. These equations are called *linear* because linear equations in two variables can be represented by straight lines, as we shall see in Section 5.1.

We shall now discuss the solution of linear equations containing one variable, such as $3x + 4 = 0$, $2x = 3$, and so on. These equations can be written in the form

$$ax = b,$$

which we shall call *standard form*. The symbols a and b may represent any numbers provided $a \neq 0$; they are *constants*, not variables, and in any particular equation their values will be known.

To solve a linear equation with one variable, we first reduce it to standard form. If any term on the right-hand side is a multiple of x, add its negative to both sides. For example, given $x + 2 = 1 - 2x$, add $2x$, the negative of $-2x$, to both sides. The $-2x$ cancels, leaving $x + 2 + 2x = 1$. If any constant is on the left, add its negative to both sides. In the example, the new equation is $x + 2x = 1 - 2$. So you must move all terms involving x to the left-hand side of the equation, and all other terms to the right-hand side; but when you move a term from one side to the other, you must change its sign, from positive to negative, or from negative to positive. Finally, add all terms on each side of the new equation.

Once the equation is in standard form, its solution is immediate. As a is non-zero, we can divide both sides by it, obtaining

$$x = \frac{b}{a},$$

and this is the only possible solution.

Sample Problem 1.8. *Solve*

$$3x - 4 = x + 2.$$

Solution. First we move all x terms to the left-hand side. The term x becomes $-x$ on the left, thus we have:

$$3x - 4 - x = 2.$$

Then move the -4. It becomes 4:

$$3x - x = 4 + 2.$$

Next, gather terms, obtaining

$$2x = 6.$$

Finally, divide by 2:

$$x = 3.$$

Your Turn. Solve $3x + 5 = x + 7$.

Sample Problem 1.9. *Solve*

$$\frac{x}{2} - 4 = -\frac{5x}{3}.$$

Solution. First move $-\frac{5x}{3}$ to the left:

$$\frac{x}{2} - 4 + \frac{5x}{3} = 0.$$

Then move the -4. It becomes 4:

$$\frac{x}{2} + \frac{5x}{3} = 4.$$

Next, gather terms. To simplify this equation, we multiply both sides by the common denominator 6:

$$3x + 10x = 24.$$

Finally, divide by 13:

$$x = \frac{24}{13}.$$

Your Turn. Solve

$$\frac{3x}{2} - 5 = \frac{2x}{3}.$$

Equations in General

In solving linear equations with one variable, our procedure is to change the given equation into a simpler equation that is equivalent to the original. This principle can be applied to any equation. When we add terms to both sides of an equation, or equivalently move terms to the other side of the equation, we are actually using the following general property of equality:

If a, b and c are any quantities, then:

$$\text{If } a = b, \quad \text{then } a + c = b + c;$$
$$\text{If } a = b, \quad \text{then } a - c = b - c.$$

The process of multiplying by a common denominator, and the process of dividing by the coefficient of x, use:

If a, b and c are any quantities, then:

$$\text{If } a = b, \quad \text{then } c \times a = c \times b;$$
$$\text{If } a = b, \quad \text{then } \frac{a}{c} = \frac{b}{c}, \text{ provided } c \neq 0.$$

These ideas can be used in general.

Sample Problem 1.10. *Use the equation* $4x^2 + 3y = y + 6$ *to express* y *in terms of* x.

Solution. Even though x is a variable, we move it to the right-hand side because we want to find an expression for y. So,

$$3y = y + 6 - 4x^2.$$

All y terms belong on the left:

$$3y - y = 6 - 4x^2.$$

Now we simplify:

$$2y = 6 - 4x^2,$$
$$y = 3 - 2x^2.$$

Your Turn. Use the equation $x^2 - 2y = 2y - 8$ to express y in terms of x.

Inequalities

An *inequality* is similar to an equation, with one of the signs $<$, $>$, $\leq$, or $\geq$ replacing the equality. Solving inequalities is similar to solving equations. Again there will be a solution set, and inequalities with the same solution set are called equivalent. The solution sets of inequalities usually involve intervals. A *linear* inequality is one with only constants and first powers of variables.

The most important difference between solving inequalities and solving equations is the effect of multiplying both sides by a constant. If $a < b$, and c is a *positive* constant, then $c \times a < c \times b$, but if d is a *negative* constant, then $d \times a > d \times b$. If the sign is $\leq$, then multiplication by a positive constant leaves the sign unchanged, while multiplication by a negative constant changes $\leq$ to $\geq$.

Sample Problem 1.11. *Solve the following inequality for* x*:*

$$4x - 3 \leq 2x + 1.$$

Solution. We first rewrite the inequality as:

$$4x - 2x \leq 3 + 1$$

or

$$2x \leq 4.$$

Dividing both sides by 2, we obtain $x \leq 2$, or solution set $(-\infty, 2)$.

Your Turn. Solve for x: $2x + 3 \leq x + 6$.

Sample Problem 1.12. *Solve the following inequality for x:*

$$3x - 2 > 5x + 4.$$

Solution. We first rewrite the inequality as:

$$3x - 5x > 2 + 4$$

or

$$-2x > 6.$$

Dividing both sides by -2, we obtain $x < -3$. Observe the change in the inequality sign.

Your Turn. Solve for x: $2x + 3 \leq 4x + 5$.

Sample Problem 1.13. *Use the following inequality to express y in terms of x:*

$$3x - 4y > 2(x + y) - 3.$$

Solution. We first rewrite the inequality with all y terms on the left and all other terms on the right:

$$-4y - 2y > 2x - 3x - 3$$

or

$$-6y > -x - 3.$$

Dividing both sides by -6, we obtain $y < \frac{1}{6}x + \frac{1}{2}$.

Exercises 1.2 A

1. In each case, are the indicated values a solution to the equation?

(i) Equation $3x + y = 4$; solution $x = 3$, $y = -5$;

(ii) Equation $3x + y = 4$; solution $x = 4$, $y = -7$;

(iii) Equation $2x + 3y = 8$; solution $x = 3$, $y = 1$;
(iv) Equation $2x - 5y = -1$; solution $x = 2$, $y = 1$;
(v) Equation $x^2 + y^2 = 4$; solution $x = 2$, $y = 0$;
(vi) Equation $x^2 + y^2 = 1$; solution $x = 1$, $y = 1$.

2. Solve these equations for x:

(i) $2x - 7 = 5$; (ii) $x + 3 = 3 - x$;
(iii) $3x - 5 = x - 3$; (iv) $3(x - 1) = 2(x + 1)$;
(v) $14 = 4 - 5x$; (vi) $3(x - 1) + 2x = 7$;
(vii) $\frac{1}{2}x - 1 = 2$; (viii) $\frac{3}{2}x + 1 = 5 - \frac{4}{3}x$;
(ix) $2x + 2 = -1 - 4x$; (x) $3 - 3x = 4 - 2x$;
(xi) $4x - 8 = 7x - 14$; (xii) $2(3 + x) = 4x - 7$.

3. Use the following equations to express y in terms of x:

(i) $4x - 6y = 9$; (ii) $2x + 2y = 6$;
(iii) $3x + y = 4y + 6$; (iv) $x^2 - 2y = y - 3$.

4. Solve these inequalities for x:

(i) $2x - 5 < 7x - 2$; (ii) $3x + 1 > 5 - x$;
(iii) $2x + 4 \geq 4x$; (iv) $x - 4 \geq 2 - 4x$;
(v) $3 - x \geq x + 5$; (vi) $3x - 3 < 2 + x$;
(vii) $5 - 3x \leq x + 3$; (viii) $2x + 3 > 3 - x$;
(ix) $2 - 2x > 3 + x$; (x) $4(1 - x) < 2(x + 4)$.

5. Use the following inequalities to express y in terms of x:

(i) $4x + 2y \leq 4$; (ii) $x + y \leq 3 - x$;
(iii) $x + 3y > 2y + 5x$; (iv) $3x - 2y < 2x + 3$;
(v) $x + 3y + 2 \geq 3x - y - 6$; (vi) $3 - 2x > 5 - 2y$.

Exercises 1.2 B

1. In each case, are the indicated values a solution to the equation?

(i) Equation $3x + y = 4$; solution $x = 1$, $y = 1$;
(ii) Equation $3x + y = 4$; solution $x = 2$, $y = 2$;
(iii) Equation $4x - 2y = 3$; solution $x = 2$, $y = 2.5$;
(iv) Equation $4x - 2y = 6$; solution $x = 3$, $y = 3$;
(v) Equation $2x + 3y = 5$; solution $x = 1$, $y = -1$;
(vi) Equation $x^2 + 2x = y$; solution $x = 3$, $y = -5$;
(vii) Equation $x^2 + y^2 = 3$; solution $x = 2$, $y = 1$;
(viii) Equation $x^2 = y^2$; solution $x = 1$, $y = -1$.

2. Solve these equations for x:

(i) $2x - 4 = 2$; (ii) $6x - 3 = 9$;
(iii) $8 - 6x = 2$; (iv) $2x + 7 = 5$;
(v) $6x - 2 = 4x - 4$; (vi) $3 + x = 2x + 1$;
(vii) $7x - 3 = 2 - 4x$; (viii) $2 - 6x = 3x - 7$.

3. Solve these equations for x:

(i) $5x - 4 = -4$; (ii) $2x - 3 = 5$;
(iii) $4 - 2x = 3$; (iv) $3x + 7 = 6$;
(v) $2x - 4 = 3x - 3$; (vi) $3 - x = 3x + 1$;
(vii) $5x - 3 = 7 - 2x$; (viii) $2 - 3x = 3x + 7$;
(ix) $3x + 7 = -1 - 5x$; (x) $3 - 3x = 2 - 4x$;
(xi) $3x - 2 = 7x - 14$; (xii) $2(2 + x) = 3x - 7$;
(xiii) $2(x + 3) = 1 - 3x$; (xiv) $5 - x = 2(1 - 2x)$;
(xv) $2x - 5 = 7x - 17$; (xvi) $3(x - 1) = 4x - 2$.

4. Use the following equations to express y in terms of x:

(i) $3x + 2y = x + 4$; (ii) $3 - 2x - 2y = x + y$;
(iii) $2(x + y) = 5 - (x - y)$; (iv) $3 + 3x = 7 + 2y$;
(v) $2x - \frac{1}{3} = y - \frac{1}{2}$; (vi) $3x - y = x - 3y$.

5. Solve these inequalities for x:

(i) $3 - 4x \leq 2$; (ii) $3x - 5 < 5x - 3$;
(iii) $5x + 2 > 8 - x$; (iv) $4x + 6 \leq 2x$;
(v) $x + 10 \geq 12 - 3x$; (vi) $2 - x \leq 2x + 5$;
(vii) $3x - 3 < 2 + 2x$; (viii) $3(x + 1) \leq 3(1 - 2x)$.

6. Solve these inequalities for x:

(i) $3 - 2x \leq 1$; (ii) $5x - 5 < 7x - 3$;
(iii) $4x + 2 > 7 - x$; (iv) $2x + 6 \leq 4x$;
(v) $x + 8 \geq 12 - 5x$; (vi) $3 - x \leq 3x + 1$;
(vii) $7x - 3 < 2 + 4x$; (viii) $2(x + 1) \leq 3(1 - x)$;
(ix) $4 - 7x > 2 + x$; (x) $3(1 - 2x) < 2(x + 3)$;
(xi) $2(x - 3) \geq 1 + x$; (xii) $3x + 2 \leq 3(3 - x)$;
(xiii) $1 - 5x > 2(3 - 4x)$; (xiv) $3(1 + x) < 5(x - 3)$.

7. Use the following inequalities to express y in terms of x:

(i) $2x + 2y \leq 2$;	(ii) $6x - 3y + 12 < 0$;
(iii) $2x + 2y \leq 1 - 2x$;	(iv) $x - 3y \leq 3 - 2x$;
(v) $4x + 3y > 2y + 5x + 3$;	(vi) $2x - 5y < x + y - 3$;
(vii) $4x + 3y - 2 \geq 2(x + y)$;	(viii) $6 + 3x > x + 2y$;
(ix) $5(x + 1) \geq 3x + 2y + 1$;	(x) $3(1 - x) < 2(x - y)$.

8. Use the following inequalities to express x in terms of y:

(i) $2x + y \leq 3$;	(ii) $4x - 2y + 2 < 0$;
(iii) $2x + 2y \leq 2 - 4x$;	(iv) $x - y \leq 2x - 3$;
(v) $2x + 3y > y + 6x + 4$;	(vi) $6x - 4y < x + y - 3$;
(vii) $3x + y - 2 \geq 4(x - y)$;	(viii) $2 + 3y > 2x + 2y$;
(ix) $5(x + 1) \geq 2x + 3y + 2$;	(x) $4(1 - x) < 2(x + y)$.

1.3 Sums

Sigma Notation

Suppose you want to write the sum of the first 16 positive integers. You could write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16$$

but, instead of this clumsy form, it is more usual to write $1 + 2 + \cdots + 16$, assuming that the reader will take the three dots to mean "continue in this fashion until you reach the last number shown" (and, more importantly, hoping it is clear that each number in the sum is obtained by adding 1 to the preceding number).

There is a standard mathematical notation for long sums, which uses the Greek capital letter sigma, or $\sum$. We write the above sum as

$$\sum_{i=1}^{16} i$$

which means we take the sum of all the values $i = 1, i = 2, \ldots,$ up to $i = 16$. In the same way,

$$\sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2;$$

the notation, called *sigma notation*, means "first evaluate the expression after the $\sum$ (that is, i^2) when $i = 1$, then when $i = 2, \ldots,$ then when $i = 6$, and add the results". More generally, suppose a_1, a_2, a_3, and a_4 are any four numbers. (This use of a *subscript* on a letter, like the 1, 2, ... on a, is common in mathematics—otherwise we would run out of symbols!) Then

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4.$$

The definition of a set does not allow for ordering of its elements, or for repetition of its elements. For example, $\{1, 2, 3\}$, $\{1, 3, 2\}$, and $\{1, 2, 3, 1\}$ all represent the same set (which could be written $\{x \mid x \in \mathbb{N} \text{ and } x \leq 3\}$, or $\{x \in \mathbb{N} \mid x \leq 3\}$). However, ordering is often useful, so we define a *sequence* to be an ordered set. Sequences can be denoted by parentheses; (1, 3, 2) is the sequence with first element 1, second element 3 and third element 2, and is different from $(1, 2, 3)$. Sequences can contain repetitions, and $(1, 2, 1, 3)$ is quite different from $(1, 2, 3)$; the two occurrences of object 1 are distinguished by the fact that they lie in different positions in the ordering. Formally, a *sequence* (a_i) of length n is a collection of n objects $\{a_1, a_2, \ldots, a_n\}$, or $\{a_i : 1 \leq i \leq n\}$ together with an *ordering*: a_1 is first, a_2 is second, and so on. Entry a_i, where i is any one of the positive integers $1, 2, \ldots, n$, is called the ith *member of the sequence*. Two members can have the same value, but are counted as different if they are in different positions. For example, $(1, 2, 1, 3)$ is a sequence of length 4, but if we ignore the ordering then its elements form the set $\{1, 2, 3\}$ of order 3, called the *underlying set* of the sequence.

If (a_i) is a sequence of length n or longer, then $\sum_{i=1}^{n} a_i$ is defined by the rules

$$\sum_{i=1}^{1} a_i = a_1,$$

$$\sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n.$$

The use of a_i to mean a "general" or "typical" member of $\{a_1, a_2, \ldots, a_n\}$ is very common. When a set of numbers $\{a_1, a_2, \ldots, a_n\}$ is being discussed, we say a property is true "for all a_i" when we mean it is true for each member of the set.

Usually, the sigma notation is used with a formula involving i for the term following $\sum$, as in the following examples. Notice that the range need not start at 1; we can write $\sum_{i=j}^{n}$ when j and n are any integers, provided $j < n$.

Sample Problem 1.14. *Write out the following as sums and evaluate them:*

$$\sum_{i=1}^{4} i^2; \quad \sum_{i=3}^{6} i(i+1).$$

Solution.

$$\begin{aligned}\sum_{i=1}^{4} i^2 &= 1^2 + 2^2 + 3^2 + 4^2\\ &= 1 + 4 + 9 + 16\\ &= 30;\\ \sum_{i=3}^{6} i(i+1) &= 3{\cdot}4 + 4{\cdot}5 + 5{\cdot}6 + 6{\cdot}7\\ &= 12 + 20 + 30 + 42\\ &= 104.\end{aligned}$$

Your Turn. Write out the following as sums and evaluate them:

$$\sum_{i=3}^{5} i(i-1); \quad \sum_{i=2}^{6} i.$$

Sample Problem 1.15. *Write the following in sigma notation:*

$$2 + 6 + 10 + 14; \quad 1 + 16 + 81.$$

Solution.

$$\sum_{i=1}^{4} (4i-2); \quad \sum_{i=1}^{3} i^4.$$

Your Turn. Write the following in sigma notation:

$$1 + 3 + 5 + 7 + 9; \quad 8 + 27 + 64 + 125.$$

Some properties of sums

It is easy to see that the following properties of sums are true:

(1) If c is any given number, then

$$\sum_{i=1}^{n} c = nc.$$

(2) If c is any given number and (a_i) is any sequence of length n, then

$$\sum_{i=1}^{n} (ca_i) = c\cdot\left(\sum_{i=1}^{n} a_i\right).$$

(3) If (a_i) and (b_i) are any two sequences, both of length n, then

$$\sum_{i=1}^{n}(a_i+b_i)=\left(\sum_{i=1}^{n}a_i\right)+\left(\sum_{i=1}^{n}b_i\right).$$

(4) If (a_i) is any sequence of length n, $1 \le j < n$, then

$$\sum_{i=1}^{n}a_i=\sum_{i=1}^{j}a_i+\sum_{i=j+1}^{n}a_i.$$

The following is a standard result on sums.

Theorem 1. *The sum of the first n positive integers is* $\frac{1}{2}n(n+1)$*, or*

$$\sum_{i=1}^{n}i=\frac{n(n+1)}{2}.$$

Proof. Let us write s for the answer. Then $s=\sum_{i=1}^{n}i$. We shall define two very simple sequences of length n. Write $a_i=i$ and $b_i=n+1-i$. Then (a_i) is the sequence $(1,2,\ldots,n)$ and (b_i) is $(n,n-1,\ldots,1)$. They both have the same elements, although they are written in different order, so they have the same sum,

$$s=\sum_{i=1}^{n}a_i=\sum_{i=1}^{n}b_i.$$

So

$$\begin{aligned}2s&=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i\\&=\sum_{i=1}^{n}(a_i+b_i)\\&=\sum_{i=1}^{n}\big(i+(n+1-i)\big)\\&=\sum_{i=1}^{n}(n+1)\\&=n\cdot(n+1).\end{aligned}$$

Therefore, dividing by 2, we get $s=n(n+1)/2$. □

Since adding 0 does not change a sum, this result could also be stated as

$$\sum_{i=0}^{n}i=\frac{n(n+1)}{2}.$$

That form is sometimes more useful.

Sample Problem 1.16. *Find the sum of the numbers from* 11 *to* 30 *inclusive.*

Solution. $\sum_{i=11}^{30} i$ is required. Now:

$$\sum_{i=1}^{30} i = \sum_{i=1}^{10} i + \sum_{i=11}^{30} i,$$

and, using the theorem and substituting, we have:

$$\frac{30\cdot 31}{2} = \frac{10\cdot 11}{2} + \sum_{i=11}^{30} i,$$

$$465 = 55 + \sum_{i=11}^{30} i,$$

so $\sum_{i=11}^{30} i = 465 - 55 = 410$.

Sample Problem 1.17. *Find* $2 + 6 + 10 + 14 + 18 + 22 + \cdots + 122$.

Solution. This is the sum of the terms $2 + 4i$, where i goes from 0 to 30.

$$\begin{aligned}\sum_{i=0}^{30}(2+4i) &= \left(\sum_{i=0}^{30} 2\right) + \left(\sum_{i=0}^{30} 4i\right)\\ &= \left(\sum_{i=0}^{30} 2\right) + 4\cdot\left(\sum_{i=0}^{30}(i)\right)\\ &= 31\cdot 2 + 4\cdot\frac{30\cdot 31}{2}\\ &= 62 + 1860\\ &= 1922.\end{aligned}$$

Your Turn. Find $3 + 7 + \cdots + 43$.

Two other standard results, that will be proven in the exercises, are:

Theorem 2.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Theorem 3.

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1} \quad \textit{unless } x = 1.$$

Exercises 1.3 A

1. Write each of the following as sums and evaluate them:

(i) $\sum_{i=1}^{6} i^2 + 1$; (ii) $\sum_{i=1}^{3} \frac{1}{10^i}$;

(iii) $\sum_{i=3}^{9} i(i-3)$; (iv) $\sum_{i=2}^{5} \frac{1}{i}$;

(v) $\sum_{i=2}^{4} (-2)^i$; (vi) $\sum_{i=1}^{4} 1 + (-1)^i$;

(vii) $\sum_{i=2}^{5} \frac{1}{i-1}$; (viii) $\sum_{i=1}^{6} 1 + 3i$.

2. Write each of the following sums in sigma notation:

(i) $1 + 4 + 7 + 10 + 13$;

(ii) $2 + 6 + 10$;

(iii) $1 - 4 + 7 - 10 + 13$;

(iv) $0 - 1 + 4 - 9 + 16$;

(v) $3 + 7 + 3 + 7 + 3 + 7$;

(vi) $0 + 3 + 8 + 15$.

3. Suppose $a_i = i^2$, so that $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, and so on.

(i) Say $b_i = i^3 - (i-1)^3$. Prove that $\sum_{i=1}^{n} b_i = n^3$.

(ii) Prove that $b_i = 3i^2 - 3i + 1$, and therefore

$$\sum_{i=1}^{n} b_i = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + n.$$

(iii) Use this to prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. Suppose $\sum_{i=1}^{n} a_i = A$ and $\sum_{i=1}^{n} b_i = B$. Evaluate $\sum_{i=1}^{n} c_i$ in each of the following cases:

(i) $c_i = 2a_i + 1$; (ii) $c_i = 3a_i - b_i$;

(iii) $c_i = a_i + b_i + 2$; (iv) $c_i = a_i + b_i + (-1)^n$.

5. Use the standard results of Theorems 1, 2, and 3 and the four properties of sums to evaluate the following expressions:

(i) $\sum_{i=7}^{12} (i-3)$; (ii) $\sum_{i=3}^{8} (5i^2-4)$;

(iii) $\sum_{i=2}^{6} 2^i$; (iv) $\sum_{i=1}^{n} (i^2+1)$;

(v) $\sum_{i=1}^{n} 3^i$; (vi) $\sum_{i=1}^{n} (6i^2-1)$;

(vii) $\sum_{i=1}^{n} i(i-2)$; (viii) $\sum_{i=1}^{n} \frac{1}{2^i}$.

6. Prove that

$$\sum_{i=1}^{n} 2n-1 = n^2$$

(in other words, the sum of the first n odd positive whole numbers is n^2).

Exercises 1.3 B

1. Write the following as sums and evaluate them:

(i) $\sum_{i=1}^{4} i^2$; (ii) $\sum_{i=1}^{5} 1+(-1)^i$;

(iii) $\sum_{i=3}^{6} 1-i$; (iv) $\sum_{i=6}^{15} 4$;

(v) $\sum_{i=3}^{6} i(i-1)$; (vi) $\sum_{i=2}^{4} \frac{1}{i^2}$.

2. Write the following as sums and evaluate them:

(i) $\sum_{i=1}^{3} i^3$; (ii) $\sum_{i=1}^{6} (-1)^i$;

(iii) $\sum_{i=2}^{4} \sqrt{i}$; (iv) $\sum_{i=1}^{5} 1+\frac{i}{2}$;

(v) $\sum_{i=3}^{6} i(i-2)$; (vi) $\sum_{i=1}^{5} (-1)^{i+1} i^2$;

(vii) $\sum_{i=12}^{15} i + \frac{1}{i}$; (viii) $\sum_{i=2}^{5} i + (-1)^i$;

(ix) $\sum_{i=2}^{5} i^2 - 3$; (x) $\sum_{i=2}^{5} i^2 - 4i$.

3. Write each of the following sums in sigma notation:

(i) $1 + 4 + 9 + 16 + 25$;

(ii) $2 + 5 + 8 + 11$;

(iii) $-1 + 2 - 3 + 4$;

(iv) $6 + 5 + 4 + 3 + 2 + 1$;

(v) $1 + 3 + 9 + 27 + 81$;

(vi) $4 - 6 + 8 - 10 + 12 - 14$;

(vii) $2 + 4 + 10 + 28 + 82$;

(viii) $4 + 8 + 12 + 16 + 20$;

(ix) $-1 - 2 - 3 - 4 - 5 - 6$;

(x) $11 + 17 + 23$.

4. Suppose $a_1 = 1, a_2 = x, a_3 = x^2$, and, in general, $a_i = x^{i-1}$.

(i) Prove that $\sum_{i=2}^{n+1} a_i = x \sum_{i=1}^{n} a_i$;

(ii) Use this to show that $\sum_{i=1}^{n} a_i = 1 + x \sum_{i=1}^{n} a_i - x^n$;

(iii) Use part (ii) to find the value of $\sum_{i=1}^{n} a_i$ when $x \neq 1$;

(iv) Why did we have to require "$x \neq 1$" in the preceding part?

5. Suppose $\sum_{i=1}^{n} a_i = A$ and $\sum_{i=1}^{n} b_i = B$. Evaluate $\sum_{i=1}^{n} c_i$ in each of the following cases:

(i) $c_i = 5a_i$; (ii) $c_i = 2a_i - b_i$;

(iii) $c_i = a_i + 2b_i$; (iv) $c_i = a_i + b_i - 1$;

(v) $c_i = 2a_i + b_i$; (vi) $c_i = 2 - b_i$.

6. Use that standard results of Theorems 1, 2, and 3 in the text and the four properties of sums to evaluate the following expressions:

(i) $\sum_{i=1}^{6}(2i-5)$; (ii) $\sum_{i=1}^{10}\left(2i^2\right)$;

(iii) $\sum_{i=1}^{20}\left(i^2-3i\right)$; (iv) $\sum_{i=0}^{5}\left(\frac{1}{2}\right)^i$;

(v) $\sum_{i=1}^{12} i^2-5i-1$; (vi) $\sum_{i=1}^{n} 2i+1$.

7. Use that standard results of Theorems 1, 2, and 3 and the four properties of sums to evaluate the following expressions:

(i) $\sum_{i=5}^{20}(2i+1)$; (ii) $\sum_{i=1}^{n}\left(i^2-i\right)$;

(iii) $\sum_{i=4}^{10}\left(i^2+1\right)$; (iv) $\sum_{i=1}^{n}(1+i)^2$;

(v) $\sum_{i=1}^{9} 2^{-i}$; (vi) $\sum_{i=0}^{n} 2^{-i}$.

1.4 Elements of Set Theory

More About Sets

We defined the notation

$$s \in S$$

to mean "s belongs to S" or "s is an element of S". If S and T are two sets, we shall write $T \subseteq S$ to mean that every member of T is also a member of S. In other words, "If s is any element of T then s is a member of S", or

$$s \in T \quad \Rightarrow \quad s \in S,$$

where $\Rightarrow$ is shorthand for *implies*. When $T \subseteq S$ we say T is a *subset* of S. Sets S and T are equal, $S = T$, if and only if $S \subseteq T$ and $T \subseteq S$ are both true. If necessary, we can represent the situation where T is a subset of S but S is not equal to T, that is, there is at least one member of S that is not a member of T, by writing $S \subset T$, and we call T a *proper* subset of S.

Suppose $R \subseteq S$ and $S \subseteq T$ are both true. Any member of R will also be a member of S, which means it is a member of T. So $R \subseteq T$. This sort of rule is called a *transitive law*.

It is important not to confuse the two symbols $\in$ and $\subseteq$, or their meanings:

Sample Problem 1.18. *Suppose* $S = \{0, 1\}$. *Which of the following are true:* $0 \in S$, $\{0\} \in S$, $0 \subset S$, $\{0\} \subset S$, $0 \subseteq S$, $\{0\} \subseteq S$, $S \in S$, $S \subset S$, $S \subseteq S$?

Solution. 0 is a member of S, but $\{0\}$ and S are not, so $0 \in S$ is true but $\{0\} \in S$, and $S \in S$ are false. As 0 is a member of S, $\{0\} \subset S$ and $\{0\} \subseteq S$ are true. But 0 is not a *set* of elements of S, so $0 \subset S$ and $0 \subseteq S$ are false. Finally, $S \subseteq S$ is true, but $S \subset S$ would imply $S \neq S$, so it is false.

Among the standard number sets, many subset relationships exist. Every natural number is an integer, every integer is a rational number, and every rational number is a real number, so $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$. We could write all these relationships down in one expression:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

In fact, we know that no two of these sets are equal, so we could write

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Some special sets of numbers are given special names: we have already used $\mathbb{Z}^*$ to denote the set of all *non-negative* integers, and similarly we could write $\mathbb{R}^*$ for the set of all non-negative real numbers. The notation $^+$ is often used for the set of all *positive* members of a number set; in particular, $\mathbb{Z}^+$ is another name for $\mathbb{N}$.

An important concept is the *empty* set, or *null* set, which has no elements. This set, denoted by $\emptyset$, is unique and is a subset of every other set.

In all the discussions of sets in this book, we shall assume (usually without bothering to mention the fact) that all the sets we are dealing with are subsets of some given universal set, U. The set U may be chosen to be as large as necessary in any problem we deal with; in many of our discussions we can choose $U = \mathbb{Z}$ or $U = \mathbb{R}$. The universal set can often be chosen to be a finite set.

The Basic Operations on Sets

Given sets S and T, we define three operations: the *union* of S and T is the set

$$S \cup T = \big\{x : x \in S \text{ or } x \in T \text{ (or both)}\big\};$$

the *intersection* of S and T is the set

$$S \cap T = \{x : x \in S \text{ and } x \in T\};$$

the *relative complement* of T with respect to S (or alternatively the *set-theoretic difference* or *relative difference* between S and T) is the set

$$S \backslash T = \{x : x \in S \text{ and } x \notin T\}.$$

In particular, the relative complement $U\backslash T$ with respect to the universal set U is denoted by $\overline{T}$ and called the *complement* of T.

We may write $R\backslash S = R \cap \overline{S}$, since each of these sets consists of the elements belonging to R but not to S. Hence we see that $R \subseteq S$ if and only if $R\backslash S = \emptyset$. If S is any set, $S\backslash S$ will equal $\emptyset$.

There is also a special relationship between the other operations and subsets. If S is any subset of T, then $S \cap T = S$ and $S \cup T = T$.

Sample Problem 1.19. *If $\mathbb{E}$ is the set of all even integers and Π is the set of all prime numbers, what are $\mathbb{E} \cup \Pi$, $\mathbb{E} \cap \Pi$, $\mathbb{E}\backslash\Pi$, and $\mathbb{Z} \cup \mathbb{N}$, $\mathbb{Z}^*\backslash\Pi$?*

Solution. $\mathbb{E} \cup \Pi$ contains all primes and all even numbers. (2 is the only number we have described twice.) $\mathbb{E} \cap \Pi$ contains only the common element 2, the only even prime. $\mathbb{E}\backslash\Pi$ contains all the even numbers *except* 2. All natural numbers are integers, so $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{Z} \cup \mathbb{N} = \mathbb{Z}$. $\mathbb{Z}^*\backslash\Pi$ contains all the positive numbers that are not primes, and zero. In symbols,

$$\begin{aligned}
\mathbb{E} \cup \Pi &= \{\dots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, 8, 10, 11, \dots\},\\
\mathbb{E} \cap \Pi &= \{2\},\\
\mathbb{E}\backslash\Pi &= \{\dots, -8, -6, -4, -2, 0, 4, 6, 8, \dots\},\\
\mathbb{Z} \cup \mathbb{N} &= \mathbb{Z},\\
\mathbb{Z}^*\backslash\Pi &= \{0, 1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \dots\}.
\end{aligned}$$

Your Turn. What are $\mathbb{Z}\backslash\mathbb{Z}^*$, $\mathbb{Z} \cap \mathbb{N}$, $(\mathbb{N}\backslash\mathbb{E}) \cup \Pi$?

If two sets, S and T, have no common element, so that $S \cap T = \emptyset$, then we say that S and T are *disjoint*. Observe that $S\backslash T$ and T must be disjoint sets; in particular, T and $\overline{T}$ are disjoint.

Sample Problem 1.20. *In each case, are the sets S and T disjoint? If not, what is their intersection?*

(i) *S is the set of perfect squares, $T = \mathbb{R}\backslash\mathbb{R}^*$.*

(ii) *S is the set of perfect squares, $T = \mathbb{R}\backslash\mathbb{R}^+$.*

(iii) *S is the set of all multiples of 5, T is the set of all multiples of 7.*

Solution.

(i) The sets are disjoint.

(ii) They are not disjoint, because 0 is a perfect square ($0 = 0^2$); $S \cap T = \{0\}$.

(iii) They are not disjoint. $S \cap T$ is the set of all multiples of 35.

Your Turn. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) $S = \Pi$, T is the set of even integers.

(ii) $S = \{1, 3, 5, 7, 9\}$, $T = \{2, 4, 6, 8, 10\}$.

Properties of the Operations

We now investigate some of the easier properties of the operations $\cup$, $\cap$, and $\setminus$; for the more difficult problems, we shall introduce some techniques in the next section.

Union and intersection both satisfy *idempotence laws*: for any set S,

$$S \cup S = S \cap S = S.$$

Both operations satisfy *commutative laws*; in other words,

$$S \cup T = T \cup S$$

and

$$S \cap T = T \cap S,$$

for any sets S and T. Similarly, the *associative laws*

$$R \cup (S \cup T) = (R \cup S) \cup T$$

and

$$R \cap (S \cap T) = (R \cap S) \cap T$$

are always satisfied. The associative law means that we can omit brackets in a string of unions (or a string of intersections); expressions like $(A \cup B) \cup (C \cup D)$, $((A \cup B) \cup C) \cup D$, and $(A \cup (B \cup C)) \cup D$ are all equal, and we usually omit all the parentheses and simply write $A \cup B \cup C \cup D$. But be careful not to mix operations. $(A \cup B) \cap C$ and $A \cup (B \cap C)$ are quite different. Combining these two laws, we see that any string of unions can be rewritten in any order: for example,

$$(D \cup B) \cup (C \cup A) = \big(C \cup \big(B \cup (A \cup D)\big)\big) = (A \cup B \cup C \cup D).$$

Sample Problem 1.21. *Prove that* $(A \cup B) \cap C = A \cup (B \cap C)$ *is not always true.*

Solution. To prove that a general rule is not true, it suffices to find just one case in which it is false. As an example we take the case $A = \mathbb{R}$, $B = \mathbb{Z}$, $C = \{0\}$. Then $(A \cup B) \cap C = \{0\}$, while $A \cup (B \cap C) = \mathbb{R}$.

Cartesian Product

We define the *Cartesian product* (or *cross product*) $S \times T$ of sets S and T to be the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$:

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$

There is no requirement that S and T should be disjoint; in fact, it is often useful to consider $S \times S$.

The number of elements of $S \times T$ is $|S| \cdot |T|$.

Sample Problem 1.22. *Suppose* $S = \{0, 1\}$ *and* $T = \{1, 2\}$. *What is* $S \times T$*?*

Solution. $S \times T = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$, the set of all four of the possible ordered pairs.

Your Turn. What is $S \times T$ if $S = \{1, 2\}$ and $T = \{1, 4, 5\}$?

The sets $(R \times S) \times T$ and $R \times (S \times T)$ are not equal; one consists of an ordered pair whose *first* element is itself an ordered pair, and the other of pairs in which the *second* is an ordered pair. So there is no associative law, and no natural meaning for $R \times S \times T$. On the other hand, it is sometimes natural to talk about ordered *triples* of elements, so we define

$$R \times S \times T = \{(r, s, t) : r \in R, s \in S, t \in T\}.$$

This notation can be extended to ordered collections of any length.

Exercises 1.4 A

1. In each case, list all elements of the set described:
   (i) The set of all odd numbers between 2 and 10;
   (ii) The set of all days in a week;
   (iii) The set of all positive multiples of 3 between 10 and 20;
   (iv) The set of all prime numbers smaller than 10;
   (v) The set of all even prime numbers;
   (vi) The set of all positive factors of 36.

2. Write brief descriptions of the following sets. (There will often be many correct answers.)

(i) $\{3, 6, 9\}$; (ii) $\{3, 6, 9, \ldots\}$;
(iii) $\{1, -10\}$; (iv) $\{1, 4, 9\}$;
(v) $\{1, 3\}$.

3. Suppose

$$A = \{a, b, c, d, e\}, \qquad B = \{a, c, e, g, i\}, \qquad C = \{c, f, i, e, o\}.$$

Write down the elements of the following sets:

(i) $A \cup B$; (ii) $A \cap C$;
(iii) $A \backslash B$; (iv) $A \cup (B \backslash C)$.

4. Suppose

$$A = \{1, 2, 4, 5, 6, 7\}, \qquad B = \{1, 3, 5, 7, 9\}, \qquad C = \{2, 4, 6, 7, 8, 9\}.$$

Write down the elements of the following sets:

(i) $A \cup B \cup C$; (ii) $A \cup (B \cap C)$;
(iii) $A \backslash C$; (iv) $A \cap (B \backslash C)$.

5. Suppose

$$A = \{1, 3, 5, 6, 7\}, \qquad B = \{1, 2, 3, 4, 5\}, \qquad C = \{5, 6, 7, 8\}.$$

Write down the elements of the following sets:

(i) $A \cap B$; (ii) $A \cap C$;
(iii) $A \backslash (B \cap C)$; (iv) $A \cup B \cup C$;
(v) $A \backslash (B \cup C)$; (vi) $(A \cup B) \backslash C$.

6. Consider the sets

$$\begin{aligned} S_1 &= \Pi, \\ S_2 &= \{2, 4, 6, 8\}, \\ S_3 &= \{2, 3, 5, 7\}, \\ S_4 &= \{x : x \text{ is a prime divisor of } 30\}, \\ S_5 &= \{x \in \mathbb{N} : x \text{ is a power of } 2\}. \end{aligned}$$

(i) For which i and j, if any, is $S_i \subseteq S_j$? For which i and j, if any, is $S_i = S_j$?
(ii) Write down the elements of $S_i \cap S_j$ for every case where $i < j$.

7. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) S is the set of all multiples of 5; T is the set of all perfect squares.

(ii) S is the set of all students in your class; T is the set of all students in your college.

8. In each case, list all elements of $S \times T$:

(i) $S = \{a, b, c\}$, $T = \{a, c, f\}$;

(ii) $S = \{x \mid x^2 = 4\}$, $T = \{2, 3, 4, 5\}$;

(iii) $S = \{x \mid x \text{ is a prime less than } 4\}$, $T = \{4, 5\}$.

9. In each case, list all elements of $R \times S \times T$:

(i) $R = \{1, 2\}$, $S = \{3, 4\}$, $T = \{5, 6\}$;

(ii) $R = \{x, y\}$, $S = \{z\}$, $T = \{1, 2\}$.

Exercises 1.4 B

1. In each case, list all elements of the set described:

(i) The set of all integers between two and ten inclusive;

(ii) The set of all months in a year;

(iii) The set of odd positive multiples of 5, less than 50;

(iv) The set of all positive factors of 24;

(v) The set of all factors of 24;

(vi) The set of perfect squares smaller than 20;

(vii) The set of all vowels in English;

(viii) The set of all even integers between two and ten inclusive;

(ix) The set of all seasons in a year;

(x) The set of positive multiples of 5, less than 30.

2. Write brief descriptions of the following sets. (There will often be many correct answers.)

(i) $\{2, 4, 6, 8\}$;

(ii) $\{2, 4, 6, 8, \ldots\}$;

(iii) $\{2, 3, 5\}$;

(iv) $\{-1, -2, -3, \ldots\}$;

(v) $\{0, 1\}$.

3. Write at least two different brief descriptions of the set $\{2, 3, 5, 7\}$.

4. Consider the sets

$$A_1 = \{1, 2, 3, 4\},$$
$$A_2 = \{1, 2, 3, 4, 5, 6, 7, 8\},$$
$$A_3 = \{3, 4, 5, 8, 9\},$$

$$A_4 = \{4, 3, 2, 1\},$$
$$A_5 = \{2, 4, 6, 8\}.$$

(i) For which i and j, if any, is $A_i \subseteq A_j$? For which i and j, if any, is $A_i = A_j$?

(ii) Write down the elements of $A_i \cap A_j$ for every case where $i < j$.

5. Consider the sets

$$A_1 = \{1, 2, 4\},$$
$$A_2 = \{1, 2, 3, 4, 5\},$$
$$A_3 = \{1, 3, 4, 6\},$$
$$A_4 = \{4, 2, 1\}.$$

(i) For which i and j, if any, is $A_i \subseteq A_j$? For which i and j, if any, is $A_i = A_j$?

(ii) Write down the elements of $A_i \cap A_j$ for every case where $i < j$.

6. Suppose $A = \{2, 3, 5, 6, 8, 9\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{5, 6, 7, 8, 9\}$. Write down the elements of

(i) $A \cap B$; (ii) $A \cup C$;

(iii) $A \backslash (B \cap C)$; (iv) $(A \cup B) \backslash C$.

7. Suppose $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 5, 7, 8\}$, $C = \{2, 4, 6, 7, 8, 9\}$. Write down the elements of

(i) $(A \cap B) \cup C$; (ii) $A \cup (B \cap C)$;

(iii) $A \backslash C$; (iv) $A \cap B \cap C$.

8. Suppose $A = \Pi$, $B = \{2, 4, 6, 7, 9\}$, $C = \{1, 2, 3, 4, 5, 6\}$. Write down the elements of

(i) $A \cap B$; (ii) $C \backslash A$;

(iii) $A \cap (B \backslash C)$; (iv) $A \cap (B \cup C)$.

9. Suppose $A = \mathbb{Z}^+$, $B = \{-4, -2, 1, 3, 5, 7\}$, $C = \{x \mid x^2 = 1\}$. Write down the elements of

(i) $(A \cap B) \cup C$; (ii) $A \cap B \cap C$;

(iii) $C \backslash A$; (iv) $A \cap (B \backslash C)$.

10. Suppose $A = \{1, 2, 3, 5, 7\}$, $B = \{1, 2, 3, 4, 5, 6, 8\}$, $C = \{2, 4, 5, 6, 7\}$. Write down

(i) $A \cap B$; (ii) $A \cap C$;

(iii) $A \cup B$; (iv) $A \cup C$;

(v) $A \backslash (B \cap C)$; (vi) $A \backslash (B \cup C)$;

(vii) $(A \cup B) \backslash C$; (viii) $A \cap B \cap C$.

11. Consider the sets

$$\begin{aligned} S_1 &= \{2, 5\}, \\ S_2 &= \{1, 2, 4\}, \\ S_3 &= \{1, 2, 4, 5, 10\}, \\ S_4 &= \{x \in \mathbb{N} : x \text{ is a divisor of } 20\}, \\ S_5 &= \{x \in \mathbb{N} : x \text{ is a power of 2 and a divisor of } 20\}. \end{aligned}$$

(i) For which i and j, if any, is $S_i \subseteq S_j$? For which i and j, if any, is $S_i = S_j$?

(ii) Write down the elements of $S_i \cap S_j$ for every case where $i < j$.

12. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) $S = \{1, 2, 3, 4, 5\}$, $T = \{6, 7, 8, 9, 10\}$;

(ii) $S = \{1, 2, 3, 4, 5\}$, $T = \{5, 6, 7, 8, 9\}$;

(iii) $S = \{1, 3, 5, 7, 9\}$, $T = \{2, 4, 6, 8, 10\}$;

(iv) $S = \{1, 2, 3, 4, 5\}$; $T = \{2, 4, 6, 8, 10\}$.

13. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) S is the set of squares $1, 4, 9, \dots$ and T is the set of cubes $1, 8, 27, \dots$ of positive integers.

(ii) S is the set of perfect squares; $T = \mathbb{R} \backslash \mathbb{R}^+$.

(iii) S is the set of perfect squares $1, 4, 9, \dots$; T is the set Π of primes.

14. In each case, list all elements of $S \times T$:

(i) $S = \{1, 3\}$, $T = \{1, 5\}$;

(ii) $S = \{0, 1, 2\}$, $T = \{2, 3, 4\}$;

(iii) $S = \{1, 3, 5, 7\}$, $T = \{1, 2, 3\}$;

(iv) $S = \{1, 3, 4\}$, $T = \{\text{red}, \text{blue}\}$;

(v) $S = \{x \mid x^2 = 1\}$, $T = \{y \mid y^2 = 4\}$;

(vi) $S = \{x \mid x \in \Pi, x < 10\}$, $T = \{1, 2\}$;

(vii) $S = \{1, 2\}$, $T = \{3, 4\}$;

(viii) $S = \{1, 2\}$, $T = \{2, 3\}$;

(ix) $S = \{1, 7\}$, $T = \{1, 2, 3\}$;

(x) $S = \{x \mid x^2 = 1\}$, $T = \{y \mid y^2 = 1\}$.

15. In each case, list all elements of $R \times S \times T$:

(i) $R = \{1, 2\}$, $S = \{1, 3\}$, $T = \{2, 3\}$;

(ii) $R = \{12, 13, 14\}$, $S = \{1\}$, $T = \{1, 2, 3\}$.

16. In each case, list all elements of $R \times S \times T$:

(i) $R = \{1, 2, 3\}$, $S = \{1, 3\}$, $T = \{2\}$;

(ii) $R = \{2, 3\}$, $S = \{1, 3\}$, $T = \{1, 2\}$.

17. In each case, find the sets $A \cap (B \cup C)$ and $(A \cap B) \cup C$:

(i) $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$, $C = \{5, 6, 7, 8\}$;

(ii) $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 4, 6\}$, $C = \{2, 3, 7, 8\}$.

18. (i) If $S = \emptyset$, $T \neq \emptyset$, what is $S \times T$?

(ii) If $S \times T = T \times S$, what can you say about S and T?

19. (i) If $A \subseteq S$, $B \subseteq T$, show that $A \times B \subseteq S \times T$.

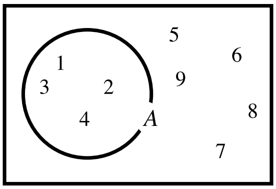
(ii) Find an example of sets A, B, S, T, such that $A \times B \subseteq S \times T$ and $B \subseteq T$, but $A \not\subseteq S$.

1.5 Venn Diagrams

Venn Diagrams

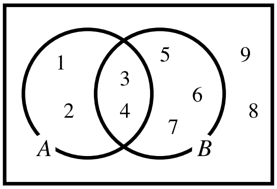
It is common to illustrate sets and operations on sets by diagrams. A set A is represented by a circle, and it is assumed that the elements of A correspond to the points (or some of the points) inside the circle. The universal set is usually shown as a rectangle enclosing all the other sets; if it is not needed, the universal set is sometimes omitted. Such an illustration is called a *Venn diagram*.[1]

For example, suppose the universal set consists of the nine integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then the diagram will be enclosed in a rectangle, and the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 will be represented by points in the rectangle. If A is the set $\{1, 2, 3, 4\}$, then a circle will be drawn inside the rectangle, with 1, 2, 3, and 4 inside it and the other numbers outside:

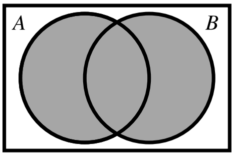


[1] Similar diagram methods were first used by Leibniz and Euler. George Venn used the diagrams extensively. Some textbooks use the name "Euler diagram" for the case where not all possible regions are represented, but we think it more appropriate simply to say "Venn diagram" because Venn formalized and unified diagram methods. For some reason, Leibniz has escaped credit.

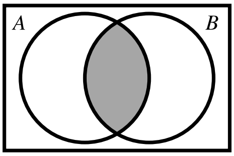
If $B = \{3, 4, 5, 6, 7\}$ then A and B can be shown on the same diagram as follows, with the common elements of A and B in the area common to the two circles:



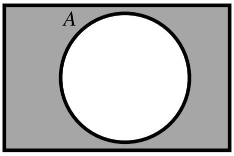
This same method can be used to represent more general sets. The following figures show Venn diagrams representing $A \cup B$, $A \cap B$, $\overline{A}$, and $A \backslash B$; in each case, the set represented is shown by the shaded area. The universal set is shown in each figure. Of course, we do not know the individual members of the sets, but they could be represented by points in the diagram if they were known.



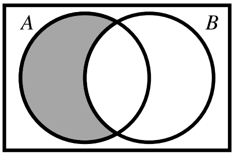
$A \cup B$



$A \cap B$



$\overline{A}$



$A \backslash B$

Two sets are equal if and only if they have the same Venn diagram. In order to illustrate this, consider the set equation

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T),$$

where R, S, and T can be any sets. The Venn diagram for $R \cup (S \cap T)$ is constructed in Figure 1.1, and that for $(R \cup S) \cap (R \cup T)$ is constructed in Figure 1.2. The two are obviously identical.

This rule is called a *distributive law*, and it should remind you of the distributive law

$$a(b + c) = ab + ac$$

for numbers. There are in fact two distributive laws for sets; the second is:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

To prove $A \subseteq B$, it is sufficient to show that all the shaded areas in the diagram for A are also shaded in B. We illustrate this idea with another set law.

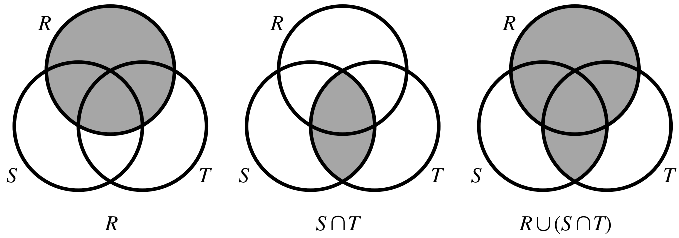


Fig. 1.1. $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$

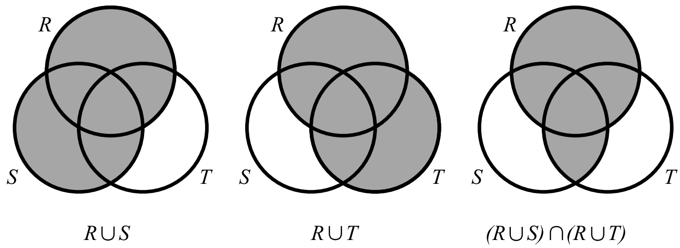
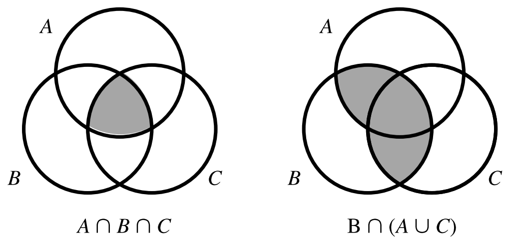


Fig. 1.2. $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$

Sample Problem 1.23. *Use Venn diagrams to prove that*

$$A \cap B \cap C \subseteq B \cap (A \cup C).$$

Solution.



Your Turn. Use Venn diagrams to prove that

$$A \cap C \subseteq (\overline{A} \cap B) \cup C.$$

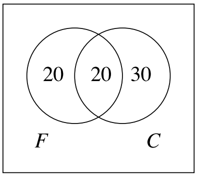
Sometimes we draw a Venn diagram so as to represent some properties of sets. For example, if A and B are disjoint sets, the diagram can be drawn with A and B shown as disjoint circles. If $A \subseteq B$, the circle for A is entirely inside the circle for B.

Counting the Elements

Venn diagrams can also be used to calculate the numbers of elements in sets. We write the numbers of elements in the areas of the diagram, instead of the elements themselves.

Sample Problem 1.24. *There are* 40 *students in Dr. Brown's Finite Mathematics course and* 50 *in his Calculus section. If these are his only classes, and if* 20 *of the students are taking both subjects, how many students does he have altogether? Represent the data in a Venn diagram.*

Solution. We use the notation F for the set of students in the finite class and C for Calculus. There are four areas in the Venn diagram: the set of students in both classes (the center area, $F \cap C$) has 20 members; the set of students in Finite Mathematics only (the left-hand enclosed area) has 20 members—subtract 20 from 40; the area corresponding to Calculus-only students, has $50 - 20 = 30$ members; and the outside area has no members (only Dr. Brown's students are being considered). So Dr. Brown has 70 students; the diagram is

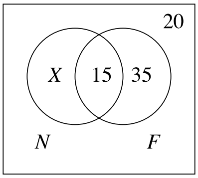


Your Turn. A survey shows that 12 newspaper readers buy the morning edition and seven buy the evening edition; three of these also bought the morning paper. Represent the data in a Venn diagram. How many readers were surveyed?

This method can be used to find the number of elements in the set represented by any of the areas or unions of areas, provided there is enough information.

Sample Problem 1.25. 100 *people were surveyed to find out whether newspaper advertisements or flyers were more efficacious in advertising supermarket specials.* 20 *of them said they pay no attention to either medium.* 50 *said they read the flyers, and* 15 *of those said they also check the newspapers. How many use the newspaper ads, in total?*

Solution. We do not know how many people read the newspaper advertisements but not the flyers. Suppose there are X of them. Then we get the Venn diagram

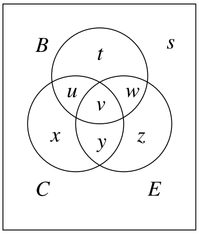


In order for the total to add to 100, $X = 30$, so the total who use the newspaper ads is $X + 15 = 45$.

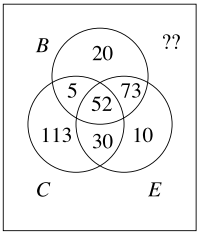
These methods can be applied to three or more sets.

Sample Problem 1.26. 500 *people were asked about their morning vitamin intake. It was found that* 150 *take vitamin B,* 200 *take vitamin C,* 165 *take vitamin E,* 57 *take both B and C,* 125 *take both B and E,* 82 *take both C and E, and* 52 *take all three vitamins. How many take both B and E but do not take C? How many take none of these vitamins?*

Solution. We start with the diagram



From the data, $v = 52$, and $u + v = 57$, so $u = 5$. Similarly, $w = 73$ and $y = 30$. Now $t + u + v + w = 150$, so $t = 20$. The other sizes are calculated similarly, and we get the diagram



The cell corresponding to "B and E but not C" has 73 elements. There are 303 elements in total, so there are 197 people who take none.

Your Turn. In a survey of 150 moviegoers, it was found that 80 like horror movies, 75 like police procedurals, and 60 like romances. In total, 35 like both horror and procedurals, 25 like horror and romance, and 30 like procedurals and romance. 15 like all three types of movie. Represent these data in a Venn diagram. How many like horror and police procedural movies, but do not like romances? How many like romances only? How many like none of these three types?

Exercises 1.5 A

1. Represent the following sets in a Venn diagram:

(i) $R \cup S \cup T$; (ii) $R \cup (S \cap \overline{T})$;

(iii) $(R \cap S) \backslash (S \cap T)$; (iv) $(R \backslash S) \cap T$.

2. Use Venn diagrams to prove the commutative and associative laws for $\cup$:

(i) $(R \cup S) = (S \cup R)$;

(ii) $((R \cup S) \cup T) = (R \cup (S \cup T))$.

3. Prove the following rules using Venn diagrams:

(i) $\overline{S \cap T} = \overline{S} \cup \overline{T}$;

(ii) $(S \cap T) \subseteq S$.

4. 1865 voters were surveyed about a new highway. Of them, 805 were in favor financing it with a new tax, 627 favored tolls on the highway, while 438 said they would vote in favor of either measure.

(i) How many favor taxes but not tolls?

(ii) How many favor tolls but not taxes?

(iii) How many would vote against either form of funding?

5. Some shoppers buy bread, butter, and coffee. Ten buy bread, 15 buy butter, and 14 buy coffee. The number buying both bread and butter, both bread and coffee, and both butter and coffee are four, five, and eight, respectively. Three buy all three products. How many shoppers were there (assuming all bought at least one of the three)?

6. 18 people are interviewed. Of these, seven dislike the Republican party, ten dislike the Democrats, and 11 dislike the Green party. Moreover, five dislike both Democrats and Republicans, five dislike both Republicans and Greens, while six dislike both Democrats and Greens. And four dislike all three types of politicians, on principle. How many *like* all three?

7. In a survey at Dave's Bagel Barn, 75 customers were asked about their preferences. It was found that 35 like poppy seed bagels, 33 like onion bagels, and 26 like chocolate bagels. There were 18 who like both poppy seed and onion, and nine who like both poppy seed and chocolate, while 12 enjoyed only chocolate from the three being discussed.

(i) How many customers enjoy onion and chocolate but do not like poppy seed?

(ii) How many liked onion but neither of the others?

(iii) How many did not like any of those flavors?

(iv) Can you tell how many liked all three?

8. Elizabeth's Chocolate Company sells milk chocolates, dark chocolates, and white chocolates. On one day, 164 customers bought chocolates. 120 of them purchased a selection that included milk chocolates, 114 included dark chocolates in their order, and 84 included white chocolates. There were 80 who bought both dark and milk and 64 who bought both white and milk, while 40 bought all three varieties. Draw a Venn diagram to represent these data.

(i) How many bought both dark and white chocolate but no milk chocolate?

(ii) How many bought a milk and white assortment that included no dark chocolate?

(iii) How many bought white chocolate only?

9. A shopping mall contains K-Mart, Wal-Mart, and Target stores. In a survey of 100 shoppers, it is found that

(a) 47 shopped at Wal-Mart;

(b) 61 shopped at K-Mart;

(c) 52 shopped at Target;

(d) 32 shopped at both Wal-Mart and K-Mart;

(e) 35 shopped at both K-Mart and Target;

(f) 22 shopped at both Wal-Mart and Target;

(g) 12 shopped at all three stores.

(i) How many of those surveyed shopped only at Wal-Mart?

(ii) How many shopped at Wal-Mart and K-Mart, but not at Target?

(iii) How many shopped at exactly one of the stores?

(iv) How many shopped at none of the three stores?

Exercises 1.5 B

1. Represent the following sets in a Venn diagram.

(i) $\overline{R \cup S \cup T}$; (ii) $(R \cap S) \cap T$;

(iii) $R \cap (S \backslash T)$; (iv) $(R \cup T) \backslash (S \cap T)$.

2. Prove: $R = (\overline{\overline{R} \cup S}) \cup (R \cap S)$.

3. Find a simpler expression for $S \cup (\overline{(\overline{R} \cup S) \cap R})$.

4. Use Venn diagrams to prove the commutative and associative laws for $\cap$:

(i) $(R \cap S) = (S \cap R)$;

(ii) $((R \cap S) \cap T) = (R \cap (S \cap T))$.

5. Use Venn diagrams to prove the distributive law

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

6. Prove the following rules using Venn diagrams:

(i) $S \cap \overline{S} = \emptyset$;

(ii) $\overline{S \cup T} = \overline{S} \cap \overline{T}$;

(iii) $S \subseteq (S \cup T)$.

7. For any sets R and S, prove $R \cap (R \cup S) = R$.

8. Among 1000 personal computer users it was found that 375 have a scanner and 450 have a DVD player attached to their computer. Moreover, 150 had both devices.

(i) How many had either a scanner or a DVD player?

(ii) How many had neither device?

9. A survey was carried out. It was found that ten of the people surveyed were drinkers, while five were smokers; only two both drank and smoked, but seven did neither. How many people were interviewed?

10. 50 people were asked about their book purchases. 20 said they had bought at least one fiction book in the last week, 30 had bought at least one non-fiction book, and ten had bought no books. Assuming that any book can be classified as either fiction or non-fiction, how many of the people interviewed had bought both fiction and non-fiction during the week?

11. 60 people were asked about news magazines. It was found that 32 read *Newsweek* regularly, 25 read *Time*, and 20 read *U.S. News and World Report*. Nine read both *Time* and *U.S. News and World Report*, 11 read both *Newsweek* and *Time*, and eight read both *Newsweek* and *U.S. News and World Report*. Eight of the people do not read any of the magazines.

(i) How many read all three magazines?

(ii) Represent the data in a Venn diagram.

(iii) How many people read exactly one of the three magazines?

12. Of 100 high school students, 45 were currently enrolled in Mathematics courses, 38 were enrolled in Science, and 21 were enrolled in Geography. There were 18 in both Math and Science and nine in both Math and Geography, while 12 were taking Geography but neither of the other subjects. Exactly five students were taking all three subjects, and there were 23 not currently enrolled in any of the three subjects. Represent these figures in a Venn diagram. How many students were taking precisely one of the three subjects? How many were taking precisely two?

13. Of 100 personal computer users surveyed, 27 use Dell, 35 use Gateway, and 35 use Hewlett-Packard. Ten of them use both Dell and Gateway, eight use both Dell and Hewlett-Packard, and 12 use both Gateway and Hewlett-Packard. Four use all three.

(i) How many use exactly one of these brands?

(ii) How many only use other brands?

14. Freshmen at New College are required to attend at least one orientation lecture. Three lectures are held. Of the 450 freshmen, 136 attended the morning lecture, 185 the afternoon lecture, and 127 the evening lecture. There were 20 who attended both morning and afternoon, 20 who attended both afternoon and evening, and five who attended both morning and evening. A total of 41 students attended more than one lecture.

(i) How many students attended all three lectures?

(ii) How many students failed to attend any lecture?

15. Researchers at IBM were surveyed about their qualifications in computer science. It was found that 214 have a Bachelor's degree, 123 have a Master's degree, and 99 have a Ph.D. It was further found that 57 have both Bachelor's and Master's degrees, 74 have both Master's and Ph.D., and 22 have both Bachelor's and Ph.D. degrees.

(i) Show that at least eight of the researchers have all three degrees.

(ii) Suppose that all of the researchers with both a Bachelor's degree and a Ph.D. also hold a Master's degree in the field. How many have all three degrees? How many came into computer science from another field: that is, they have either a Master's or a Ph.D., but no Bachelor's degree?

16. A survey of students found that:

(a) 62 were enrolled in Calculus;

(b) 71 were enrolled in Algebra;

(c) 67 were enrolled in Discrete Mathematics;

(d) 37 were enrolled in both Calculus and Algebra;

(e) 32 were enrolled in both Calculus and Discrete Mathematics;

(f) 40 were enrolled in both Algebra and Discrete Mathematics;

(g) 12 were enrolled in all three subjects;

(h) 44 were enrolled in none of the three.

(i) How many were enrolled in both Calculus and Algebra but not in Discrete Mathematics?

(ii) How many were enrolled in exactly one of the three subjects?

(iii) How many students were surveyed?

17. A pet store surveyed 120 of its customers. It was found that 55 of them owned dogs, 50 owned cats, and 40 owned goldfish. Further, 20 own both dogs and cats, 15 own both dogs and goldfish, and 12 own both cats and goldfish. Draw a Venn diagram to represent these data.

(i) How many customers own dogs and cats but do not own goldfish?

(ii) How many own none of these three kinds of pets?

18. Prove, using Venn diagrams, that $(R\backslash S)\backslash T = R\backslash(S\backslash T)$ does not hold for *all* choices of sets R, S and T. In other words, show that there are some choices for R, S and T such that the equation is not true.

1.6 Averages

Central Tendency

Suppose you measure the noon temperature in your home town on ten summer days. The Fahrenheit temperature readings, arranged in ascending order, might be

$$F = \{76, 79, 80, 81, 83, 83, 84, 84, 84, 86\}.$$

If somebody asked you what is a typical noon summer temperature, there are three logical answers.

First, you might add the ten readings and divide by 10, giving the answer 82°. This is called the *mean* or *average* temperature.

Second, you might say 84° because that is the most common answer. It came up three times. This is called the *mode* of F.

Finally, the *median* or half-way point is the reading at the midpoint of the list (after it has been put in ascending or descending order). As F has ten elements, the midpoint lies between the fifth and sixth readings. Each equals 83, so the median is 83°.

All three measures are attempts to answer the question, "what is the center of the collection of temperatures?" For this reason the mean, mode, and median are all called measures of *central tendency*.

Sample Problem 1.27. *What are the mean, mode, and median of the following collections?*

$$S = \{13, 13, 11, 14, 15\},$$
$$T = \{8, 13, 12, 15, 10, 26\}.$$

Solution. S has sum 66, so the mean is $66/5 = 13.2$. The mode is 13, the only reading to appear twice. To find the median, first write the readings in ascending order: 11, 13, 13, 14, 15. The "center" reading is 13, so the median is 13.

T has sum 84, so its mean is 14. Its median is 12.5 (after the readings are reordered as 8, 10, 12, 13, 15, 26, the two "center" values are 12 and 13; when they are unequal, take the average of the two). There is no mode because no reading occurs more frequently than any other.

Your Turn. What are the means, modes, and medians of the following collections?

$$U = \{7, 9, 2, 3, 6, 7, 15\},$$
$$V = \{14, 22, 24, 17, 21, 19\}.$$

Notice the use of the word "collection" rather than "set" because often the data or observations will not form sets; there will be repetitions. Also observe from the examples that neither the mean nor the median need be members of the collection. In fact, all the collections we looked at were integers, but the mean and median were not always integers.

The Mean

The *mean* m_X of an n-element collection $X = \{x_1, x_2, \dots, x_n\}$ is defined by the formula

$$m_X = \frac{\sum_{i=1}^{n} x_i}{n}.$$

If only one set is involved, the subscript X is often omitted.

Suppose a new collection $Y = \{y_1, y_2, \dots, y_n\}$ is defined by the rule

$$y_i = a + x_i.$$

Then

$$m_Y = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{\sum_{i=1}^{n} a + x_i}{n} = \frac{na + \sum_{i=1}^{n} x_i}{n} = a + m_X.$$

So we have the rule:

If the members of Y are obtained by adding a to each member of X, then the mean of Y is obtained by adding a to the mean of X.

The number a can be positive or negative. This rule is useful in practice, as the following example shows.

Sample Problem 1.28. *Find the mean of*

$$X = \{1051, 1053, 1054, 1055, 1059\}.$$

Solution. We define Y by the rule "$y_i = x_i - 1050$". Then $Y = \{1, 3, 4, 5, 9\}$; $m_Y = 22/5 = 4.4$, so $m_X = 4.4 + 1050 = 1054.4$.

Your Turn. Use this method to find the mean of

$$\{833, 834, 836, 837, 841, 841\}.$$

When the collection of data is large, there will sometimes be several copies of the same value. If there are f members all equal to the same value x, then f is called the *frequency* of x. For example, the collection F of temperatures that we discussed at the beginning of this section might be described by saying

> 76, 79, 80, 81, and 86 each have frequency 1; 83 has frequency 2; 84 has frequency 3.

If we wish, we can also say that 77, 78, 82, and 85 (and all values outside the range) have frequency 0.

If the collection X consists of elements $\{x_1, x_2, \ldots, x_k\}$, where x_1 has frequency f_1, x_2 has frequency f_2, and so on, then its mean is

$$m_X = \frac{\sum_{i=1}^{k} x_i f_i}{\sum_{i=1}^{k} f_i}.$$

Sample Problem 1.29. *The scores in a quiz are integers ranging from* 0 *to* 5. *The frequencies of the scores are:*

x	0	1	2	3	4	5
f	2	0	10	14	4	1

What is the mean score?

Solution. The sum of the terms $x_i f_i$ is

$$0 \times 2 + 1 \times 0 + 2 \times 10 + 3 \times 14 + 4 \times 4 + 5 \times 1 = 83$$

while the sum of the frequencies is

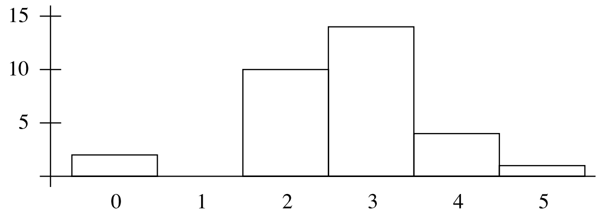
$$2 + 0 + 10 + 14 + 4 + 1 = 31$$

so the mean is $83/31 = 2.6774\ldots$ or about 2.68. In practice we would say the mean is "about 2.7", or even "between 2 and 3", because fractional scores were not given.

Your Turn. Find the mean of the following data:

x	2	3	4	5	6	7	8
f	1	3	7	11	12	4	1

Instead of a table, data like this is often represented graphically in a *histogram*, a diagram with a column for each possible value, where the height of the column is proportional to the frequency of that value. Here is the histogram for the data of Sample Problem 1.29.



Measures of Variation

Even when you have decided which measure of central tendency to use, and calculate it, you do not know all about your collection of readings. For example, consider two towns. Placidville has similar weather on most summer days; the temperatures on 11 consecutive days were

$$P = \{78, 79, 80, 81, 82, 83, 83, 84, 84, 84, 84\}.$$

In Wildtown the weather is far more extreme, with sudden cold and hot changes; over eleven days, the recorded temperatures were

$$W = \{69, 71, 75, 81, 82, 83, 84, 84, 89, 91, 93\}.$$

In both cities, the mean, mode and median were 82°, 84°, and 83°, respectively. However, there is far more temperature variation in Wildtown than in Placidville.

Several measures of *variation*, or *spread*, are used. We shall define the *standard deviation*.

The Standard Deviation

If a set of observations has mean m, the deviation of an observation x from m is $x - m$. At first it seems that the average of the deviations would be a good measure of variation. However, the mean deviation is always zero! The negative and positive contributions cancel each other out.

Better measures can be found by averaging the absolute values of the deviations. Given an n-element collection $X = \{x_1, x_2, \ldots, x_n\}$ with mean m_X, we consider the values

$$|x_1 - m_X|, \quad |x_2 - m_X|, \quad \ldots, \quad |x_m - m_X|.$$

For technical reasons, the most reliable measure is found by averaging the squares of the deviations and taking the square root of this average. So we define the *standard deviation* s_X of the set X to be

$$s_X = \sqrt{\frac{\sum_{i=1}^{n}(x_i - m_X)^2}{n}}.$$

The quantity s_X^2, which must be calculated before the square root is taken, is the *variance* of the set X.

Sample Problem 1.30. *Calculate the standard deviation of the daily temperatures in Wildtown and Placidville.*

Solution. The mean for Wildtown is 82, so the variance is

$$\begin{aligned}&\big[(69-82)^2+(71-82)^2+(75-82)^2+(81-82)^2+(82-82)^2\\&\quad+(83-82)^2+(84-82)^2+(84-82)^2+(89-82)^2+(91-82)^2\\&\quad+(93-82)^2\big]/11\\&=[169+121+49+1+0+1+4+4+49+81+121]/11\\&=599/11=54.4545\ldots\end{aligned}$$

and the standard deviation is about 7.2. The mean for Placidville is again 82, so its variance is

$$\begin{aligned}&\big[(78-82)^2+(79-82)^2+(80-82)^2+(81-82)^2+(82-82)^2\\&\quad+(83-82)^2+(83-82)^2+(84-82)^2+(84-82)^2+(84-82)^2\\&\quad+(84-82)^2\big]/11\\&=[16+9+4+1+0+1+1+4+4+4+4]/11\\&=48/11=4.3636\ldots\end{aligned}$$

and the standard deviation is about 2.1.

The standard deviation of a set of observations will be unchanged if any constant is added to, or subtracted from, each observation. For example, in discussing Placidville, we could just as easily have subtracted 80 from each reading, and found the standard deviation of $\{-2, -1, 0, 1, 2, 3, 3, 4, 4, 4, 4\}$.

If the collection X consists of elements $\{x_1, x_2, \ldots, x_k\}$, where x_1 has frequency f_1, x_2 has frequency f_2, and so on, then its standard deviation is

$$s_X = \frac{\sum_{i=1}^{k} x_i - m_X{}^2 f_i}{\sum_{i=1}^{k} f_i}.$$

A Formula for Standard Deviation

The formula for variance can be rewritten as

$$\begin{aligned} ns_X^2 &= \sum_{i=1}^{n} (x_i - m_X)^2 \\ &= \sum_{i=1}^{n} \left(x_i^2 - 2x_i m_X + m_X^2\right) \\ &= \left(\sum_{i=1}^{n} x_i^2\right) - 2\left(\sum_{i=1}^{n} x_i m_X\right) + \left(\sum_{i=1}^{n} m_X^2\right) \\ &= \left(\sum_{i=1}^{n} x_i^2\right) - 2m_X\left(\sum_{i=1}^{n} x_i\right) + nm_X^2. \end{aligned}$$

Now $\sum_{i=1}^{n} x_i = nm_X$ (this is just the equation for m_X, multiplied on both sides by n), so

$$2m_X\left(\sum_{i=1}^{n} x_i\right) = 2nm_X^2,$$

and

$$ns_X^2 = \left(\sum_{i=1}^{n} x_i^2\right) - 2nm_X^2 + nm_X^2 = \left(\sum_{i=1}^{n} x_i^2\right) - nm_X^2,$$

giving the formula

$$s_X = \sqrt{\frac{(\sum_{i=1}^{n} x_i^2) - nm_X^2}{n}}.$$

This formula is often more convenient for calculations, as the following (very small) example shows.

Sample Problem 1.31. *Find the standard deviation of the data set* $\{1, 1, 1, 2, 3, 3, 5\}$.

Solution. The data add to 16, so the mean is $\frac{16}{7} = 2.\overline{285714}$. Using the original standard deviation formula, each calculation must include several decimal places in order to ensure accuracy. However, using the new formula, we find

$$\begin{aligned}7s_X^2 &= \left[1^2 + 1^2 + 1^2 + 2^2 + 3^2 + 3^2 + 5^2\right] - 7\left(\frac{16}{7}\right)^2\\ &= [1 + 1 + 1 + 4 + 9 + 9 + 25] - \frac{256}{7}\\ &= 50 - 36.5714 = 13.4286\end{aligned}$$

so $s_X^2 = 1.91837$, and $s_X = 1.39$ (correct to two decimal places).

Your Turn. Find the standard deviation of the data $\{1, 1, 2, 2, 3, 4, 4\}$.

Exercises 1.6 A

1. Find the mean and median of the following sets of data:

(i) $\{17, 19, 23, 30\}$;

(ii) $\{8, 13, 20, 12, 29\}$;

(iii) $\{14, 15, 23, 22, 24, 19\}$.

2. Find the mean, mode, and median of the following sets of data:

(i) $\{3, 16, 10, 1, 4, 2, 6, 2\}$;

(ii) $\{-3, 5, 8, 1, -4, 6, 6, 6\}$;

(iii) $\{2, 10, 10, 1, 1, 2, 7, 9, 11, 12\}$.

3. Find the mean of the data $\{2, 3, 5, 2, 7, 9, 7\}$, and list the deviations of the data from the mean. Verify that the mean deviation is zero.

4. For each set of data in Exercise 2, what is the standard deviation?

5. Find the mean and standard deviation of the following data. Under each score is listed its frequency

x	1	2	3	4	5	6
f	5	2	4	4	7	4

6. Draw a histogram representing the data in the preceding exercise.

Exercises 1.6 B

1. Find the mean and median of the following collections of data:

(i) $\{15, 17, 28, 21\}$;

(ii) $\{14, 13, 18, 12, 24\}$;

(iii) $\{17, 25, 13, 22, 14, 29\}$;

(iv) $\{12, 13, 19, 20, 16\}$;

(v) $\{11, 20, 15, 21, 14, 28, 17\}$.

2. Find the mean, mode, and median of the following collections of data:

(i) $\{5, 3, 4, 7, 7\}$;

(ii) $\{8, 11, 5, 4, 9\}$;

(iii) $\{8, 12, 10, 4, 4, 7, 6, 11\}$;

(iv) $\{-3, -5, 6, 4, -4, 6, 6, 8\}$;

(v) $\{9, 10, 7, 9, 1, 2, 7, 9, 11, 7\}$;

(vi) $\{73, 77, 72, 70, 77, 83, 86, 84\}$;

(vii) $\{43, 46, 50, 43, 47, 53, 46, 44\}$;

(viii) $\{1081, 1083, 1088, 1086, 1087\}$;

(ix) $\{944, 942, 940, 947, 949\}$.

3. Find the mean of the data $\{6, 3, 4, 2, 7, 3, 3\}$, and list the deviations of the data from the mean. Verify that the mean deviation is zero.

4. For each set of data in Exercise 2, what is the standard deviation?

5. Find the mean, mode, median, and standard deviation of the following collections of data:

(i) $\{2, 4, 8, 3, 8\}$;

(ii) $\{7, 11, 2, 4, 6, 6\}$;

(iii) $\{-2, -2, 3, 1, 4, 2, 4, -2\}$;

(iv) $\{-3, -2, 1, 1, -2, 3, 5, 5\}$;

(v) $\{2, -2, 5, -3, 3, 7\}$;

(vi) $\{27, 25, 28, 30, 30\}$;

(vii) $\{-5, -6, 4, 14, 17, -7, 6, 4, 3, -5\}$;

(viii) $\{21, 23, 23, 25, 23\}$.

6. In the following data, under each score is listed its frequency. In each case:

(a) Draw a histogram representing the data;

(b) Find the mean and standard deviation:

(i)

x	1	2	3	4	5	7	9
f	2	1	3	2	4	5	3

;

(ii)

x	27	28	30	31	33	35
f	1	2	3	1	2	1

;

(iii)

x	1	2	3	4	5	6	7	8	9
f	3	3	2	3	5	4	3	2	3

;

(iv)

x	17	18	19	20	21	22	23	24
f	2	3	3	2	4	8	4	3

.

2

Counting

2.1 Basic Counting Principles

Inclusion and Exclusion

Suppose S and T are any two sets, and you want to list all members of $S \cup T$. If you list all members of S, then list all the members of T, you will cover all members of $S \cup T$, but those in $S \cap T$ will be listed twice. To count all members of $S \cup T$, you could count all members of both lists, then subtract the number of duplicates. In other words,

$$|S \cup T| = |S| + |T| - |S \cap T|. \tag{2.1}$$

This is the simplest case of a rule called the *principle of inclusion and exclusion*, and you can remember it in the following way. To list all members of the union of two sets, list all members of the first set and all members of the second set. This ensures that all members are included. However, some elements—those in $S \cap T$—will be listed twice, so it is necessary to exclude the duplicates.

Methods for finding the sizes of sets are often used to count the number of members of a universal set that have a particular property. If S is the set of all the objects that have some property A and T is the set of all the objects with property B, then (2.1) expresses the way to count the objects that have *either* property A *or* property B:

(i) count the objects with property A;

(ii) count the objects with property B;

(iii) count the objects with both properties;

(iv) subtract the third answer from the sum of the other two.

As an example, we repeat Sample Problem 1.24, and solve it using (2.1).

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Sample Problem 2.1. *Recall that there are* 40 *students in Dr. Brown's finite mathematics course and* 50 *in his calculus section. If these are his only classes, and if* 20 *of the students are taking both subjects, use* (2.1) *to calculate how many students he has altogether.*

Solution. Using the notation F for the set of students in the finite mathematics class and C for calculus, (2.1) is

$$|F \cup C| = |F| + |C| - |F \cap C| = 40 + 50 - 20 = 70,$$

so he has 70 students.

The Rule of Sum

Another rule, sometimes called the *rule of sum*, can be simply expressed by saying "the number of objects with property A equals the number that have both property A and property B, plus the number that have property A but not property B"; if properties A and B are related to sets S and T as before, this is

$$|S| = |S \cap T| + |S \backslash T|. \tag{2.2}$$

If we rewrite (2.2) as

$$|S \backslash T| = |S| - |S \cap T|$$

and substitute into (2.1), we obtain

$$|S \cup T| = |T| + |S \backslash T|. \tag{2.3}$$

Sample Problem 2.2. *Suppose the set S has* 25 *elements, T has* 17 *elements, and $S \cap T$ has seven elements. Find $|S \cup T|$ and $|S \backslash T|$.*

Solution. From (2.1) we see

$$|S \cup T| = |S| + |T| - |S \cap T| = 25 + 17 - 7 = 35.$$

From (2.2) we get

$$|S \backslash T| = |S| - |S \cap T| = 25 - 7 = 18.$$

Your Turn. Suppose $|S| = 42$, $|T| = 32$, and $|S \cap T| = 22$. Find $|S \cup T|$ and $|S \backslash T|$.

The Multiplication Principle

It is sometimes useful to break an event down into several parts, forming what we shall call a *compound event*. For example, suppose you are planning a trip from Los Angeles to Paris, with a stopover in New York. You have two options for the flight to New York: a direct flight with United or an American flight that stops in Chicago. For the second leg, you consider the direct flight with Air France, a British Airways flight through London, and Lufthansa stopping in Frankfurt. There are two ways to make the first flight and three to make the second, for a total of six combinations.

Suppose S is the set of available flights for the first leg and T is the set of available flights for the second leg. Then the flight combinations correspond to the members of the Cartesian product $S \times T$, and in this example

$$|S| = 2, \qquad |T| = 3, \qquad |S \times T| = |S| \cdot |T| = |2 \cdot 3| = 6.$$

The correspondence between compound events and Cartesian products applies in general. If S is the set of cases where A occurs, and T is the set of cases where B occurs, then the possible combinations correspond to the set $S \times T$, which has $|S| \cdot |T|$ elements. This idea is usually applied without mentioning the sets S and T. Suppose the event A can occur in a ways, and the event B can occur in b ways, then the combination of events A and B can occur in ab ways. This very obvious principle is sometimes called the *multiplication principle* or *rule of product*. It can be extended to three or more sets.

Sample Problem 2.3. *To open a bicycle lock you must know a three-number combination. You must first turn to the left until the first number is reached, then back to the right until the second number, then left to the third number. Any number from* 1 *to* 36 *can be used. How many combinations are possible?*

Solution. There are 36 ways to choose the first number, 36 ways to choose the second, and 36 ways to choose the third. So there are $36 \cdot 36 \cdot 36$ combinations.

Your Turn. Your debit card has a 4-digit PIN. If you can use any digits, how many PINs are possible?

The multiplication principle only works when the events are performed independently—if the result of A is somehow used to affect the performance of B, some combined results may be impossible. In the airline example, if the United flight leaves too late to connect with the Air France flight, then your choices are not independent, and only five combinations would be available.

An Extension of the Multiplication Principle

Suppose a class of 20 students has to elect a main representative and an alternate representative to attend faculty meetings. The alternate will attend only if the main appointee is unavailable, so the two students selected must be different.

There are 20 candidates for the main position. No matter which one is chosen, there will be 19 candidates for alternate. So the total number of possible choices is $20 \cdot 19 = 380$.

The second choice is not independent of the first choice. The set of candidates for the second election will depend on the choice made in the first election. So this is not an application of the multiplication principle. But it uses a similar idea—treating the occurrence, whose possible outcomes are to be counted, as though it were a compound event.

Sample Problem 2.4. *A high school class contains* 10 *boys and* 13 *girls. They have boys' and girls' charity drives, and wish to elect a chair and treasurer for each. The same person cannot be both chair and treasurer. How many combinations are possible?*

Solution. There are 10 ways to select the boys' chair, and when this is done there are nine ways to select their treasurer. So there are 90 ways to select the boys' committee. In the same way, there are $13 \cdot 12 = 156$ ways to select the girls' committee. So the total is $90 \cdot 156 = 14040$.

Your Turn. What is the answer if each committee also has a secretary?

Sometimes the order of the elements in a set is unimportant, but sometimes the order is significant. Suppose S is a set with n elements. How many different ways are there to order the elements of S?

We solve this by treating the ordering as a compound event with n parts. There are n ways to choose the first element of the ordered set. Whichever element is chosen, there remain $n-1$ possible choices for the second element. When two elements have been selected, there are $n-2$ choices for the third element.

In this way, we see that there are $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$ ways to order S. This number is called n factorial, and denoted $n!$. So

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$$

For convenience, we define $0!$ to equal 1.

The different ways of ordering the set S are called *permutations of* S.

Sample Problem 2.5. *Evaluate* $10!$.

Solution. $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362880$.

Your Turn. Evaluate $6!$.

Sample Problem 2.6. *A committee of three people—chair, secretary, and treasurer—is to be elected by a club with* 11 *members. If every member is eligible to stand for each position, how many different committees are possible?*

Solution. We can treat the selection of the committee as a compound event with three parts: choose the chair, choose the secretary, and choose the treasurer. These parts can be performed in 11, 10, and 9 ways, respectively. So there are $11 \cdot 10 \cdot 9$ committees possible.

Your Turn. What is the number of committees if there are 9 members?

This more general method is often combined with the regular multiplication principle.

Sample Problem 2.7. *Three boys and four girls are to sit along a bench. Three boys must sit together, as must the girls. How many ways can this be done.*

Solution. We treat this as a compound event with three parts. First, it is decided whether the boys are to be on the left or on the right. This can be done in two ways. Then the ordering of the boys is chosen. This can be done in $3! = 6$ ways. Finally, the girls are ordered. This can be accomplished in $4! = 24$ ways. So there are $2 \cdot 6 \cdot 24 = 288$ arrangements.

Your Turn. How many ways could five boys and four girls be seated on two benches, if the boys must sit on the back bench and the girls on the front?

Exercises 2.1 A

1. Suppose $|S| = 19$, $|T| = 11$, and $|S \cap T| = 8$. Find $|S \cup T|$ and $|S \backslash T|$.
2. Suppose $|S| = 22$, $|T| = 12$, and $|S \cup T| = 28$. Find $|S \cap T|$ and $|S \backslash T|$.
3. Suppose $|S| = 30$, $|T| = 24$, and $|S \backslash T| = 12$. Find $|S \cap T|$ and $|S \cup T|$.
4. Suppose $|S| = 52$, $|T| = 50$, and $|S \cup T| = 62$. Find $|S \backslash T|$ and $|T \backslash S|$.
5. Suppose $|S| = 20$, $|T| = 18$, and $|S \cap T| = 13$. Find $|S \backslash T|$ and $|T \backslash S|$.
6. A survey of 1100 voters it was found that 275 of them will vote in favor of a $\frac{1}{2}\%$ increase in sales tax for Public Safety funding, 550 will vote in favor of a $\frac{1}{4}\%$ increase in income tax for school funding, and 200 of these will vote for both tax increases.
   (i) How many favor the school tax but not the Public Safety tax?
   (ii) How many will vote for neither option?
7. In a survey of 500 people, 300 said they intended to holiday in another state within the next twelve months, and 100 said they would holiday overseas. Of these, 50 planned to take both types of holiday. Use (2.1) and the rule of sum to calculate: How many people planned to take an overseas holiday but not an interstate holiday? How many did not plan to holiday outside their state?

8. A true–false test consists of eight questions. Assuming you answer all questions, how many ways are there to answer the test?

9. There are three roads from town X to town Y, four roads from town Y to town Z, and two roads from town X to town Z.

(i) How many routes are there from town X to town Z with a stopover in town Y?

(ii) How many routes are there in total from town X to town Z?

(Assume that no road is traveled twice.)

10. One form of Illinois license plate consists of three letters followed by three digits.

(i) How many different licence plates are possible?

(ii) If the letters O and I are never used, how many licence plates are possible?

11. List all different permutations of the set $\{A, B, C\}$.

12. The three boys and four girls in the choir are to sit on two benches. The boys must sit on the back bench and the girls on the front. How many different ways can they be seated?

13. A multiple-choice quiz contains 8 questions. Each has three possible answers.

(i) If you must answer every question, how many different answer sheets are possible?

(ii) If you may either answer a question or leave a blank, how many different answer sheets are possible?

14. In how many different ways can you order the letters of the word "BREAK"?

15. How many different ways can five people stand in line?

16. Ten midsize cars are available for rental. Three customers arrive, and each chooses a midsize. In how many different ways can the choice be made?

17. You have six textbooks for your courses and four notebooks.

(i) In how many ways can you stack your books?

(ii) In how many ways can you stack your books, if all the notebooks are to go on the bottom?

Exercises 2.1 B

1. Suppose $|S| = 20$, $|T| = 14$, and $|S \cap T| = 8$. Find $|S \cup T|$ and $|S \backslash T|$.

2. Suppose $|S| = 22$, $|T| = 23$, and $|S \cup T| = 34$. Find $|S \cap T|$ and $|S \backslash T|$.

3. Suppose $|T| = 37$, $|S \cap T| = 7$, and $|S \backslash T| = 14$. Find $|S|$ and $|S \cup T|$.

4. Suppose $|S| = 44$, $|T| = 18$, and $|S \cap T| = 12$. Find $|S \cup T|$ and $|S \backslash T|$.

5. Suppose $|S| = 17$, $|T| = 19$, and $|S \cup T| = 24$. Find $|S \cap T|$ and $|S \backslash T|$.

6. Suppose $|S| = 34$, $|T| = 34$, and $|S\backslash T| = 18$. Find $|S \cap T|$ and $|S \cup T|$.

7. Suppose $|S| = 24$, $|S \cap T| = 6$, and $|S \cup T| = 28$. Find $|T|$ and $|S\backslash T|$.

8. Suppose $|S \cup T| = 28$, $|S \cap T| = 6$, and $|S\backslash T| = 12$. Find $|S|$ and $|T|$.

9. Suppose $|S \cup T| = 26$, $|S\backslash T| = 2$ and $|T\backslash S| = 3$. Find $|S|$ and $|T|$.

10. Among 440 telephone subscribers, 240 have blocked anonymous callers and 300 have blocked calls from telemarketers. In total, 360 have blocked at least one of these types of call.

(i) How many have blocked both anonymous callers and telemarketers?
(ii) How many have blocked neither?

11. Among 1000 telephone subscribers it was found that 475 have answering machines and 250 call waiting. Moreover 150 had both options.

(i) How many had either an answering machine or call waiting?
(ii) How many had neither option?

12. Out of 400 people surveyed, 100 said they plan to buy a new house within the next three years, and 200 expected to buy a new car in that period. Of these, 50 planned to make both kinds of expenditure. Use (2.1) and the rule of sum to find out how many planned to buy a new house but not buy a new car, and how many planned to buy a new car but not buy a new house.

13. The menu in a restaurant lists three appetizers, five entrees and three desserts. In how many ways can you order a three-course meal of appetizer, entree and dessert?

14. International airport codes consist of three letters. How many codes are possible?

15. There are 13 contestants. No ties are allowed. In how many different ways can the judges award first, second, and third prize?

16. In a multiple-choice test, each question has four different possible answers. If there are five questions, and all questions must be answered, how many different answer sheets are possible?

17. The ACME company uses serial numbers consisting of three letters followed by five digits. How many possible serial numbers are there?

18. Dave's Pizza sells eight different pizzas by the slice and five different brands of soda. If you want to order a pizza slice and a soda for lunch, how many different combinations could you choose?

19. Your PIN number consists of four digits; the first cannot be zero.

(i) How many such PIN numbers are possible?
(ii) How many allowable PIN numbers have no zeros anywhere?
(iii) How many allowable PIN numbers have all digits different?

20. For your soup-and-salad lunch, you must choose between five soups, three salads, and six dressings. Assuming you choose a soup, a salad, and a dressing, how many different possible lunches could you order?

21. List all different permutations of the set $\{1, 2, 3, 4\}$.

22. On your bookshelf you have five novels and five textbooks.

(i) How many different ways can you arrange your books?

(ii) How many different ways can you arrange your books, if all the novels go on the left and all the textbooks go on the right?

23. On your bookshelf you have five Mathematics textbooks and four Chemistry textbooks.

(i) How many different ways can you arrange your books?

(ii) How many different ways can you arrange your books, if all the Chemistry books go on the left?

24. Derive the formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

25. Prove that $(2n)!/n! = 2^n(1 \cdot 3 \cdot 5 \cdots (2n-1))$, and consequently that, for any positive integer n,

$$2 \cdot 6 \cdot 10 \cdots (4n-6) \cdot (4n-2) = (n+1) \cdot (n+2) \cdots (2n-1) \cdot 2n.$$

2.2 Arrangements

Sequences on a Set

Suppose S is a set with s elements. We often need to know how many k-element *ordered* sets or *k-sequences* or *arrangements of size k* can be chosen from S. This number is denoted $P(s, k)$. In particular, $P(s, s)$ denotes the number of s-sequences that can be chosen from an s-set, which is the same as the number of permutations of S. It follows that $P(s, s) = s!$. For this reason arrangements of size k are often called *permutations of size k*.

Given an s-set $S = \{x_1, x_2, \ldots, x_s\}$, there are s different sequences of length 1 on S, namely (x_1), (x_2), ..., and (x_s). So $P(s, 1) = s$. There are $s \cdot (s-1)$ sequences of length 2, because each sequence of length 1 can be extended to length 2 in $s - 1$ different ways, and no two of these $s \cdot (s-1)$ extensions will ever be equal. So $P(s, 2) = s(s-1)$. Similarly, we find

$$
\begin{aligned}
P(s,3) &= s \cdot (s-1) \cdot (s-2), \\
P(s,4) &= s \cdot (s-1) \cdot (s-2) \cdot (s-3), \\
&\vdots \\
P(s,k) &= s \cdot (s-1) \cdot (s-2) \cdots (s-k+1).
\end{aligned}
$$

So $P(s,k)$ is calculated by multiplying, $s, s-1, s-2, \ldots$ until there are k factors.

It follows that

$$P(s,k) = s!/(s-k)!. \tag{2.4}$$

Sample Problem 2.8. *Calculate* $P(10,3)$.

Solution. There are two ways to calculate $P(10,3)$. We could say $P(10,3) = 10 \cdot 9 \cdot 8 = 720$. Or we could use the formula:

$$P(10,3) = 10!/7! = 362880/5040 = 720.$$

The first way is easier.

Your Turn. Calculate $P(6,4)$.

Several of the problems in the preceding section, such as those on selecting a committee, asked for the number of sequences of a certain length, and their solutions can sometimes be stated compactly by using arrangements.

Sample Problem 2.9. *A committee of three people—chair, secretary, and treasurer—is to be elected by a club with* 14 *members. If every member is eligible to stand for each position, how many different committees are possible?*

Solution. We can treat the committee as an ordered set of three elements chosen from the 14-element set of members. So the answer is $P(14,3)$, or 2184.

Your Turn. What is the number of committees if there are 12 members?

Sometimes an added condition makes the solution of a problem easier, not harder. For example, arranging people around a circular table is no more difficult than arranging them in a line, and sometimes easier.

Suppose n people are to sit around a circular table. We start by arbitrarily labeling one seat at the table as "1", the one to its left as "2", and so on. Then there are $n!$ different ways of putting the n people into the n seats. However, we have counted two arrangements as different if one is obtained from the other by shifting every person one place to the left because these two arrangements put different people in "1"; but they are clearly the same arrangement for the purposes of the question. Each arrangement is one of a set of n, all obtained from the others by shifting in a circular fashion. So the number of truly different arrangements is $n!/n$, which equals $(n-1)!$.

Sample Problem 2.10. *How many ways can you make a necklace by threading together seven different beads?*

Solution. Suppose you put the beads on a table before threading them. There would be $(n-1)! = 6! = 720$ ways to arrange them in a circle. However, after the beads are threaded, the necklace could be flipped over, so every necklace has been counted twice (for example, $abcdefg$ and $agfedcb$ are the same necklace). Therefore, the total number is $6!/2 = 360$.

Your Turn. How many ways could the three boys and four girls be arranged around a circular table if the boys must sit together and the girls as well?

Sample Problem 2.11. *Kirsten's Kopying Kompany has eight photocopying machines and seven employees who can operate them. There are four copying jobs to be done simultaneously. How many ways are there to allocate these jobs to operators and machines?*

Solution. Call the jobs A, B, C, D. Choose two arrangements: first, which four operators should do the jobs; second, which machines should be used. The operator choice can be made in $P(7, 4)$ ways, and the machines in $P(8, 4)$ ways. In each case, the first member of the sequence is the one allocated to job A, the second to job B, and so on. There are $P(7, 4) \cdot P(8, 4) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 1411200$ ways.

Sample Problem 2.12. *The club in Sample Problem 2.9 wishes to elect a by-laws committee with three members—Chair, Secretary, and Legal Officer—and requires that no members of the main club committee be members of the by-laws committee. In how many different ways can the two committees be chosen?*

Solution. Suppose the main committee is chosen first. There are $P(14, 3) = 2184$ ways to do this. After the election, there are 11 members eligible for election to the by-laws committee, so it can be chosen in $P(11, 3) = 990$ ways. So there are a grand total of $2184 \cdot 990 = 2162160$.

In the preceding Sample Problem we see that, even in small problems, the numbers get quite large. It might be better to report the answer in its factored form, as $14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$. This form of answer also makes it clear that we could have solved the problem by treating the two committees as one six-member sequence, with $P(14, 6)$ possible solutions.

Arrangements with Repetitions

If repeated elements are allowed, the number of sequences that can be formed from a set is far larger. If there is no restriction, then s^k k-sequences can be formed from an

k	$P(6, k)$	6^k	$P(6, k)/6^k$
1	6	6	1.00000
2	30	36	0.83333
3	120	216	0.55556
4	360	1296	0.27778
5	720	7776	0.09259
6	720	46656	0.01543

Table 2.1. Numbers of sequences without and with repetitions

s-set. To show just how great this difference is, even in small cases, Table 2.1 shows the relative values of $P(6, k)$ and 6^k.

In some cases, repetitions are allowed but limited. Problems of this kind can be modeled by talking about sets with a certain number of copies of each element. For example, if you want to count the number of sequences of length five based on the set $\{A, B\}$ that contain no more than three A's and no more than four B's, you could instead talk about 5-sequences selected from the set $\{A, A, A, B, B, B, B\}$.

To avoid the problems that can arise by having multiple copies of the same element, we often talk of *distinguishable* and *indistinguishable* elements. For example, many problems can be modeled in terms of selecting marbles from an urn; the usual convention is that two sequences are distinguishable only if they differ in color sequence: any two blue marbles are indistinguishable.

One way to tackle these problems is to assume the "indistinguishable" objects can be distinguished, and then take this into account. For example, consider the letters in the word $ASSESS$. In how many distinguishable ways can you order these letters?

Suppose the letters were written on tiles with numbers as subscripts, like scrabble tiles. Label them so that no two copies of the same letter get the same subscript, for example, $A_1S_1S_2E_1S_3S_4$. Then all the 6 letters are different, and there are 6! orderings. Say you have each of these orderings written on slips of paper.

Now collect together into one pile all the slips that differ only in their subscripts. For example, $A_1E_1S_1S_2S_3S_4$ and $A_1E_1S_2S_1S_3S_4$ will be in the same pile, as will $A_1E_1S_2S_3S_1S_4$, $A_1E_1S_4S_2S_3S_1$, and several others. In fact, we can work out how many slips there are in a pile. There are four letters S, and one each of the others. Two slips will be in the same pile when they have the letters in the same order, but the subscripts on the S's are in different order. There are $4! = 24$ ways to order the four subscripts, so there are 4! slips in each pile. Therefore, there are $6!/4! = 30$ piles.

Two orderings can be distinguished if and only if their slips are in different piles, so there are $6!/4! = 30$ distinguishable orderings of $ASSESS$.

The same principle can be applied with several repeated letters. For example, if $SUCCESS$ is written as $S_1U_1C_1C_2E_1S_2S_3$, we see that there are 2! ways of ordering the C's and 3! ways if ordering the S's, so each pile will contain $3! \cdot 2!$ slips, and the number of distinguishable orderings is $7!/(3! \cdot 2!) = 420$.

Sample Problem 2.13. *In how many distinguishable ways can you order the letters of the word $MISSISSIPPI$?*

Solution. There are one M, four I's, four S's and two P's, for a total of 11 letters. So the number of orderings is $11!/(4! \cdot 4! \cdot 2!)$ or 34650.

Your Turn. In how many distinguishable ways can you order the letters of the word $BANANA$?

The problems are significantly harder if not all the available repetitions are used.

Sample Problem 2.14. *An experimenter has an urn containing* 12 *marbles: five red, two blue, and five green. Assuming that marbles of the same color are indistinguishable, how many different sequences of length* 4 *can be chosen from the set?*

Solution. If unlimited numbers of each color were available, this would be the same as the problem of selecting sequences of length 4 from a 3-set with unlimited repetitions allowed, and there would be $3^4 = 81$ solutions. It is necessary to exclude those with three or four blue marbles. In the obvious notation, these are $BBBR$, $BBRB$, $BRBB$, $RBBB$, $BBBG$, $BBGB$, $BGBB$, $GBBB$, and $BBBB$, nine in total. So the answer is $81 - 9 = 72$.

Your Turn. What is the answer if there are four red, three blue, and five green marbles?

Exercises 2.2 A

1. Calculate:

(i) $P(8, 3)$; (ii) $P(4, 4)$;
(iii) $P(5, 4)$; (iv) $P(9, 2)$.

2. A *palindrome* is a "word" (any string of letters) that reads the same forwards or backwards, such as $CIVIC$ or $AABAA$. How many 5-letter palindromes are there (using the ordinary, 26-letter alphabet)?

3. Five people are to sit at a round table.

(i) How many ways can they be seated?
(ii) How many ways can they be seated if two specific people must sit together?

4. What are the answers to the preceding Exercise if the five people sit along a straight bench, rather than a round table?

5. Three men and four women sit in a row. How many different ways can they do it if:
   (i) the men must sit together;
   (ii) the women must sit together?

6. In how many ways can you arrange the letters of the following words?
   (i) *PROFESSOR*; (ii) *ACCESSORY*;
   (iii) *RUBBLE*; (iv) *BOOKKEEPER*.

7. John likes to arrange his books. He has four western, five mystery and six science fiction books, all different.
   (i) In how many ways can he arrange them on a shelf?
   (ii) In how many ways can he arrange them if all the books on the same subject must be grouped together?

8. Suppose your student ID consists of six digits, of which the first cannot be 0 or 9.
   (i) How many different ID numbers are possible?
   (ii) How many different ID numbers are possible if no digit may appear more than once?

9. Your PIN number consists of four digits. No repetitions are allowed, and 0 is not to be used.
   (i) How many PIN numbers are possible?
   (ii) How many PIN numbers are smaller than 4000?
   (iii) How many PIN numbers are even?
   (iv) How many PIN numbers contain no number greater than 7?

Exercises 2.2 B

1. Calculate:
   (i) $P(8,5)$; (ii) $P(7,2)$; (iii) $P(4,1)$;
   (iv) $P(7,5)$; (v) $P(5,2)$; (vi) $P(9,4)$.

2. Calculate:
   (i) $P(8,4)$; (ii) $P(8,2)$; (iii) $P(6,1)$;
   (iv) $P(6,3)$; (v) $P(5,3)$; (vi) $P(9,3)$.

3. (i) How many ways are there of seating six people at a round table?
   (ii) How many ways are there of seating six people at a round table so that two specific people sit together?

4. The deck for a card game consists of 24 different cards. You deal a sequence of three cards from such a deck. How many different sequences are possible?

5. A football league consists of nine teams. Each team must play each other team twice: once as home team, once as visitors.

(i) How many games must be played?

(ii) If each pair plays only once (that is, you don't care which is the home team), how many games must be played?

6. Five stereo systems are to be arranged in a line against the wall of the appliance department.

(i) How many ways can this be done?

(ii) How many ways can they be arranged if the most expensive model must be in the middle?

7. A basketball league consists of eight teams. Each team must play each other team twice: once as home team, once as visitors. How many games must be played?

8. You have a deck of nine cards: three (identical) aces of spades, three (identical) Kings of hearts, and three (identical) Queens of clubs. In how many different ways can you deal a sequence of four cards?

9. A company assigns serial numbers to its computers. Each number consists of four digits followed by three letters.

(i) How many possible serial numbers are there?

(ii) How many possible serial numbers are there, if no digit can be repeated?

(iii) How many, if no repetitions of digits or letters are allowed?

10. Four men and four women are to be seated at a round table, with men and women seated alternately. In how many different ways can they be seated?

11. Five men and four women are to be seated in a row, with men and women seated alternately. In how many different ways can they be seated?

12. In how many ways can you arrange the letters of the following words?

(i) $TODDLER$; (ii) $OFFERED$; (iii) $BORROW$;
(iv) $ARROWROOT$; (v) $MOOSEWOOD$; (vi) $APPLESEED$.

13. In how many ways can you arrange the letters of the following words?

(i) $TOPPLE$; (ii) $PEPPERED$; (iii) $BOBBIN$;
(iv) $LIFELINE$; (v) $HOOSEGOW$; (vi) $BROWBEATEN$.

14. There are 10 speakers in a debate, five on each side. It is agreed that the first speaker must speak in favor of the proposition, followed by a speaker against it, then one in favor, then one against. The remaining six speakers may speak in any order. In how many different ways can the debate be scheduled?

2.3 Selections

Selections

Given a set S, we are often interested in knowing how many different subsets of a given size are contained in S. This depends only on the size of S. We shall write $C(s,k)$ or $\binom{s}{k}$ for the number of k-subsets of an s-set; it is usual to read the symbol as "s choose k". The form $C(s,k)$ is used more frequently in specific practical cases, while we write $\binom{s}{k}$ in general or theoretical discussions. You should be familiar with both notations.

Obviously $\binom{s}{k}$ depends on the values s and k. We often call $\binom{s}{k}$ or $C(s,k)$ the *choice function* (of s and k).

We can use the formula (2.4) to derive expressions for the numbers $C(s,k)$. Suppose S is a set with s elements. It is clear that every k-set that we choose from S gives rise to exactly $k!$ distinct k-sequences on S and that the same k-sequence never arises from different k-sets. So the number of k-sequences on S is $k!$ times the number of k-sets on S, or

$$\binom{s}{k} = \frac{P(s,k)}{k!} = \frac{s!}{(s-k)!k!}. \tag{2.5}$$

When calculating $\binom{s}{k}$ in practice, you would usually calculate $P(s,k)$, then divide by $k!$. So

$$\binom{s}{k} = \frac{s \cdot (s-1) \cdot (s-2) \cdots (s-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

There are k factors in the denominator and in the numerator.

Recall that we agreed to say $0! = 1$. In combination with (2.5) this yields $\binom{s}{0} = 1$. This makes sense: it is possible to choose *no* elements from a set, but one cannot imagine different ways of doing so. We also define $\binom{s}{k} = 0$ if $k > s$. Again this makes sense—there is no way to choose more than s elements from an s-set.

Sample Problem 2.15. *Calculate* $C(8,5)$ *and* $\binom{6}{6}$.

Solution.

$$C(8,5) = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 56,$$
$$\binom{6}{6} = \frac{6!}{0! \times 6!} = 1.$$

There is no need for calculation: the terms 6! in the numerator and denominator cancel.

Your Turn. Calculate $C(9,5)$ and $\binom{6}{0}$.

Sample Problem 2.16. *A student must answer five of the eight questions on a test. How many different ways can she answer, assuming there is no restriction on her choice and the order in which she answers them is unimportant?*

Solution. $\binom{8}{5} = 56$ ways.

Your Turn. How many ways can she answer if she must choose five, one of which is Question 1?

Sample Problem 2.17. *Computers read strings consisting of the digits* 0 *and* 1. *Such a string with k entries is called a k-bit string. How many* 8*-bit strings are there that contain exactly five* 1*'s?*

Solution. To specify a string, it is sufficient to say which positions have 1's. There are $C(8, 5)$ choices, so the answer is $C(8, 5) = 56$.

Your Turn. How many 8-bit strings contain exactly four 1's?

Sample Problem 2.18. *How many ways can a committee of three men and two women be chosen from six men and four women?*

Solution. The three men can be chosen in $\binom{6}{3}$ ways; the two women can be chosen in $\binom{4}{2}$ ways. Using the multiplication principle, the total number of committees possible with no restrictions is

$$\begin{aligned}\binom{6}{3} \cdot \binom{4}{2} &= \frac{6!}{3!3!} \cdot \frac{4!}{2!2!} \\ &= 120.\end{aligned}$$

Your Turn. You wish to borrow two mystery books and three westerns from your friend. He owns five mysteries and seven westerns. How many different selections can you make?

Sample Problem 2.19. *How many different "words" of five letters can you make from the letters of the word*

$$REPUBLICAN,$$

if every word must contain two different vowels and three different consonants?

Solution. The three consonants can be chosen in $\binom{6}{3} = 20$ ways, and the vowels in $\binom{4}{2} = 6$ ways. After the choice is made, the letters can be arranged in $5! = 120$ ways. So there are $20 \cdot 6 \cdot 120 = 14400$ "words."

Your Turn. What is the answer if you use the word

$$DEMOCRAT?$$

An Illustrative Example

The following example illustrates two important facts. First, the numbers that arise in selection problems can be very large—sometimes the intermediate steps are much greater than the answer. And second, there are often two (or more) ways to attack the same problem.

Suppose the sixteen members of the girls' track team are to be divided into four teams of four members each, to train for the relay. Assuming that you don't care who runs in which position, how many ways can they be divided?

Start by arbitrarily ordering the four teams I, II, III, IV. The members of Team I can be chosen in $C(16, 4)$ ways. After they are chosen, there are 12 girls left, so there are $C(12, 4)$ ways to choose Team II. Team III can be chosen in $C(8, 4)$ ways. Then Team IV is decided (there are only four girls left.) We have

$$C(16, 4) \cdot C(12, 4) \cdot C(8, 4)$$

arrangements. But the order of the teams did not really matter, so every arrangement is one of 4! that all give the same solution. (4! is the number of ways of arbitrarily assigning the labels I, II, III, IV to the four teams.) So the answer is

$$\frac{C(16, 4) \cdot C(12, 4) \cdot C(8, 4)}{4!} = \frac{16!}{12! \cdot 4!} \cdot \frac{12!}{8! \cdot 4!} \cdot \frac{8!}{4! \cdot 4!} = \frac{16!}{(4!)^4}$$

(the 12!'s and 8!'s cancel).

Equation (2.5) was useful because of the way the factorials 12! and 8! canceled.

If you try to work out the exact answer to the problem, you will find that 16! is greater than 2×10^{13}. On many calculators this large number will only be represented approximately, and the divisions will give you at best an approximate answer. But if you write out

$$\frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)}$$

and cancel you'll get $15 \cdot 14 \cdot 13 \cdot 11 \cdot 10 \cdot 7 \cdot 6 \cdot 5$, which comes to 63063000. This selection problem involved intermediate numbers nearly a million times as great as the solution. If, instead of canceling the 12! and 8! we had calculated the three C-numbers, the multiplications would have been very large.

As we promised, there is another way to handle this problem. This time, select one girl at random. There are 15 other girls from whom her three teammates can be chosen, so this can be done in $\binom{15}{3}$ ways. For each possible choice here, suppose you select one of the 12 remaining girls. Her team can be completed in $\binom{11}{3}$ ways. Again choose a girl from those remaining; her team can be completed in $\binom{7}{3}$ ways. And the remaining girls must make up the other team. The number of choices is

$$\binom{15}{3} \cdot \binom{11}{3} \cdot \binom{7}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} \cdot \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$$

which comes to the same answer. This time the intermediate numbers were considerably smaller.

Problems Combining Different Methods

Some problems can be solved by combining the methods we have been using with the rule of sum.

Sample Problem 2.20. *In Sample Problem 2.18, how many ways are there to select the committee if one particular couple, say Mr. and Mrs. Smith, do not wish to be on the committee together?*

Solution. The total number of committees possible with no restrictions was 120. If Mr. and Mrs. Smith are both on the committee, then the rest of the committee is found by choosing two of the five remaining men and one of the three remaining women. This can be done in

$$\binom{5}{2} \cdot \binom{3}{1} = \frac{5!}{3!2!} \cdot \frac{3!}{2!1!} = 30$$

ways. So 30 of the 120 possible committees contain both Smiths; when these are excluded there remain 90 ways of choosing the committee.

(We have in fact applied the rule of sum: interpret S as the set of all possible committees, and T as the set of committees not containing both Smiths. Then we found $|S| = 120$ and $|S \cap T| = 30$, we required $|S \backslash T|$, which the rule tells us equals 90.)

Your Turn. A group contains three men and seven women. In how many ways can a committee of three people be chosen if it can contain no more than one man?

Sometimes both arrangement and selection techniques are applied to the same problem.

Sample Problem 2.21. *A copying company has eight photocopying machines and seven employees who can operate them. There are four identical copying jobs to be done.* (*There are* 4000 *booklets to be made; each person makes* 1000 *copies.*) *How many ways are there to allocate these jobs to operators and machines?*

Solution. We shall give two solutions to this problem.

(i) This is like Sample Problem 2.11, but the jobs are now identical. In that problem there were $P(7, 4) \cdot P(8, 4)$ arrangements. As the jobs are indistinguishable, the ordering of A, B, C, D is irrelevant, so we divide the number of arrangements by 4!. The answer is

$$\frac{P(7,4)\cdot P(8,4)}{4!} = \frac{(7\cdot 6\cdot 5\cdot 4)\cdot(8\cdot 7\cdot 6\cdot 5)}{4\cdot 3\cdot 2\cdot 1},$$

which works out to 58800.

(ii) There are $C(7,4)$ ways to choose four operators and $C(8,4)$ ways to choose four machines. When the choice is made, there are 4! ways to assign the four workers to four machines. So the answer is

$$P(7,4)\cdot P(8,4)\cdot 4! = \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}\cdot(4\cdot 3\cdot 2\cdot 1),$$

the same as before. (There is an extra factor 4! in the numerator and in the denominator.)

Exercises 2.3 A

1. List all the selections of size 3 that can be made from the set $\{A, B, C, D, E\}$.

2. Calculate the following quantities:

(i) $C(8,3)$; (ii) $C(9,4)$; (iii) $\binom{6}{3}$;

(iv) $C(7,3)$; (v) $C(7,7)$; (vi) $\binom{8}{6}$.

3. The math department wishes to select four of its 16 members to teach the finite math course. In how many ways can this selection be made?

4. The Student Council consists of six juniors and 12 seniors. A committee of two juniors and three seniors is to be formed. How many ways can this be done?

5. A test has 12 questions, and you must answer nine of them.

(i) How many ways can you choose which questions to answer?

(ii) In how many ways can you make your choice if you must include Question 1 or Question 2 (or maybe both)?

6. How many 12-bit binary strings with five 1's are possible? How many of them start 101?

7. A state lottery requires you to choose five different numbers from $\{1, 2, \ldots, 49\}$. The order in which the numbers are chosen does not matter.

(i) How many possible choices are there?

(ii) The state then draws six different numbers. You win if all five of your numbers are chosen. How many of the possible choices are winners?

8. The directors of a mutual fund select a portfolio of three speculative stocks and three blue-chip stocks. If they must choose from five speculative stocks and seven blue-chip stocks, how many different portfolios could be formed?

9. A businessman wishes to pack three different ties for a business trip. If he has six ties available, how many different selections could be made?

10. There are 18 undergraduates and 14 graduates in the Math club. A committee of five members is to be selected. Calculate how many ways this can be done, if:

(i) There is no restriction.
(ii) There must be exactly two graduates.
(iii) There must be at least two graduates and two undergraduates.

11. A Euchre deck of cards contains 25 cards: 6 spades, 6 hearts, 6 diamonds, 6 clubs and a joker. A five-card hand is dealt.

(i) How many different hands are possible?
(ii) How many of those hands contain only spades?
(iii) How many hands contain three spades and two clubs?
(iv) How many hands contain the joker?
(v) How many hands contain only spades and clubs, at least one of each?

Exercises 2.3 B

1. Calculate the following quantities:

(i) $C(7,7)$; (ii) $\binom{6}{5}$; (iii) $\binom{6}{1}$;

(iv) $C(10,2)$; (v) $\binom{8}{7}$; (vi) $\binom{8}{0}$.

2. Calculate the following quantities:

(i) $C(8,4)$; (ii) $C(10,0)$; (iii) $C(8,3)$;

(iv) $\binom{6}{4}$; (v) $\binom{2}{2}$; (vi) $\binom{7}{2}$.

3. A test has ten questions, five in part A and five in part B.

(i) A student has to choose five questions, two from part A and three from part B. How many ways can she make her choice?

(ii) How many ways can she make her choice if she must choose five questions, at least two from each part?

4. There are eight seniors, six juniors, five sophomores, and five freshmen on the student senate. A bylaws committee with four members is to be selected. Calculate how many ways this can be done, if:

(i) There must be one member from each class.
(ii) There must be exactly two seniors.
(iii) There must be at least two seniors.

5. The homecoming committee wishes to decorate twelve tables in the school colors of red and blue. The tables are in two rows of six and there are to be six red and six blue tables.

(i) How many different arrangements are possible?
(ii) In how many arrangements are there three red tables in each row?

6. A club with 20 members wants to elect a committee consisting of President, Secretary, and two Ordinary Members. How many committees are possible?

7. Ruth wishes to choose five guests for a dinner party from among her nine closest friends.

(i) In how many different ways can she choose her guest list?
(ii) Suppose two of the group are husband and wife and Ruth must include either both of them or neither. How many possible choices are there?
(iii) Suppose instead that two of the group are bitter enemies and Ruth cannot include both. How many possible guest lists are there?

8. Suppose Ruth, in the preceding problem, wishes to choose five guests for a dinner party from among eleven friends, six men, and five women. She wishes the final party to consist of three men and three women (including herself). How many different guest lists are possible?

9. A regular deck of cards contains 52 cards: 13 spades, 13 hearts, 13 diamonds, and 13 clubs. A five-card hand is dealt.

(i) How many hands contain only spades?
(ii) How many hands contain three spades and two clubs?
(iii) How many hands contain only spades and clubs, at least one of each?

10. There are 12 appetizers on the menu, five cold and seven hot. How many ways can you choose four different appetizers for your table? In how many cases does this include two hot and two cold appetizers?

11. A club with 24 members wants to elect a committee consisting of President, Secretary, and three Ordinary Members. How many committees are possible?

12. The motor pool has eight drivers and nine vehicles. Four drivers must be assigned to four vehicles for today's duty. How many ways can it be done?

13. It is required to select, from the set of numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$, a subset of four that will contain either 1 or 2, but not both. How many selections are possible?

14. A baseball squad contains three pitchers, two catchers and 12 players who can play at any other position. How many different teams of nine players, with exactly one pitcher and exactly one catcher, can be formed?

15. Suppose the Senate contains 52 Democrats and 48 Republicans. A committee of six must be chosen.

(i) How many different committees can be chosen?

(ii) How many of these possible committees contain exactly three Democrats and three Republicans?

2.4 More About Selections

Identities Concerning the Choice Function

We observe two important properties of the choice function $\binom{s}{k}$.

Theorem 4. *When* $0 \leq k \leq s$,

$$\binom{s}{k} = \binom{s}{s-k}.$$

Proof. From (2.5),

$$\binom{s}{k} = \frac{s!}{(s-k)!k!} = \frac{s!}{k!(s-k)!} = \binom{s}{s-k}.$$ □

This equality can also be derived as follows. If we wish to choose k things from a set of s, we could just as easily say which $k-s$ will *not* be included. In other words, to specify a subset T of S, we could just as easily specify the complement of T in the set S.

Sample Problem 2.22. *Calculate* $\binom{100}{98}$.

Solution. From Theorem 4,

$$\binom{100}{98} = \binom{100}{2} = \frac{100 \cdot 99}{1 \cdot 2} = 4950.$$

Your Turn. Calculate $\binom{81}{79}$.

Theorem 5. *For all integers k and s such that* $1 \leq k \leq s$,

$$\binom{s}{k} = \binom{s-1}{k} + \binom{s-1}{k-1}.$$

Proof. Suppose S is an s-set, and suppose x is one particular member of S. Each subset of S falls into one of two categories: those that *do* contain x, and those that *do not* contain x. So we calculate the number of k-sets on S that contain x, and then the number that do not. Adding these numbers together, we get the total number of sets.

The k-sets on S that contain x each have $k-1$ other elements, and we get a different k-set whenever we choose a different $(k-1)$-set from those other objects. So their number equals the number of $(k-1)$-sets that can be chosen from $S\backslash\{x\}$, namely $\binom{s-1}{k-1}$.

The k-sets on S that do not contain x are precisely the k-sets of $S\backslash\{x\}$, and there are $\binom{s-1}{k}$ of them. Adding, we get the result. □

Theorem 5 can be used to generate a table of values of the numbers $\binom{s}{k}$. For convenience, we call the top row of the table row 0. In row s we write the values of

$$\binom{s}{0}\binom{s}{1}\binom{s}{2}\cdots\binom{s}{s}$$

in that order. We write the table in a triangular form: $\binom{s}{0}$ occurs a half-space to the left of $\binom{s-1}{0}$. So the table is as shown. Observe that $\binom{s}{k}$ lies in the table with $\binom{s-1}{k}$ and $\binom{s-1}{k}$ above it, a half-step to the left and a half-step to the right, respectively. So Theorem 5 tells us that every term can be constructed by adding together the two numbers above it.

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

This table is called *Pascal's triangle*. When the values are substituted in we get the following array.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

The Binomial Theorem

The choice function also arises when an expression like $x + y$, a *binomial* function, is raised to a positive integer power.

It is easy to verify formulae like

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2,\\(x+y)^3 &= x^3+3x^2y+3xy^2+y^3,\\(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4.\end{aligned}$$

Now suppose n is any positive integer. The following theorem provides an expression for $(x+y)^n$, as a function of x and y. The theorem is called the *binomial theorem* for positive integer index n,

Theorem 6. *If x and y are any numbers and n is any positive integer, then*

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

For example, if $n=4$, the formula is

$$\begin{aligned}(x+y)^4 &= \binom{4}{0}x^{4-0}y^0+\binom{4}{1}x^{4-1}y^1+\binom{4}{2}x^{4-2}y^2+\binom{4}{3}x^{4-3}y^3+\binom{4}{4}x^{4-4}y^4\\&= \binom{4}{0}x^4+\binom{4}{1}x^3y+\binom{4}{2}x^2y^2+\binom{4}{3}xy^3+\binom{4}{4}y^4.\end{aligned}$$

$\binom{4}{0}=1$, so the first term is x^4. As $\binom{4}{1}=4$, the second term is $4x^3y$. Since $\binom{4}{2}=6$, the next term is $6x^2y^2$. In the same way, it is easy to verify the remaining terms in the previously stated expression for $(x+y)^4$.

The proof of the binomial theorem occurs later in this section.

The theorem is useful when one of the two variables is replaced by 1, or when a negative sign or a numerical coefficient is included. Some examples are:

$$\begin{aligned}(1+x)^3 &= 1+3x+3x^2+x^3,\\(x-y)^3 &= x^3-3x^2y+3xy^2-y^3,\\(2x+y)^3 &= 8x^3+12x^2y+6xy^2+y^3.\end{aligned}$$

Sample Problem 2.23. *Write down an expression for* $(1+x)^6$.

Solution. The relevant coefficients are $\binom{6}{0}=\binom{6}{6}=1$, $\binom{6}{1}=\binom{6}{5}=6$, $\binom{6}{2}=\binom{6}{4}=15$, $\binom{6}{3}=20$. So

$$\begin{aligned}(1+x)^6 &= \binom{6}{0}+\binom{6}{1}x+\binom{6}{2}x^2+\binom{6}{3}x^3+\binom{6}{4}x^4+\binom{6}{5}x^5+\binom{6}{6}x^6\\&= 1+6x+15x^2+20x^3+15x^4+6x^5+x^6.\end{aligned}$$

Your Turn. Write down an expression for $(x-y)^6$.

Sample Problem 2.24. *What is the coefficient of* y^3 *in* $(2x - 3y)^5$*?*

Solution. The term involving y^3 is $\binom{5}{3}(2x)^2(-3y)^3$, so the numerical coefficient is

$$\binom{5}{3}2^2(-3)^3 = 10 \cdot 4 \cdot (-27) = -1080$$

and the coefficient is $-1080x^2$.

Your Turn. What is the coefficient of y^2 in $(3x + 2y)^4$?

The binomial theorem can be used to find the approximate values of powers of numbers close to 1.

Sample Problem 2.25. *Find the approximate value* (to within 10^{-3}) *of* 1.02^{10}.

Solution. We write $1.02^{10} = (1 + 2 \cdot 10^{-2})^{10}$. Then it equals

$$1 + 10 \cdot 2 \cdot 10^{-2} + 45 \cdot 4 \cdot 10^{-4} + 120 \cdot 8 \cdot 10^{-6} + \cdots$$

Every subsequent term is at most one-tenth of the one before it. So the approximate value is

$$1 + 0.2 + 0.018 + 0.00096 + \cdots = 1.219 \text{ approx.}$$

Your Turn. Find the approximate value of 0.99^8.

You should observe the relationship between Pascal's triangle and the binomial theorem. The coefficients in row s of the triangle are precisely the coefficients in the expression for $(x + y)^s$. For example, row 6 is

$$1, \quad 6, \quad 15, \quad 20, \quad 15, \quad 6, \quad 1.$$

(Remember, the first row is row 0.)

Proof of the Binomial Theorem

The following proof can be omitted at a first reading.

We consider the following product of n factors:

$$(x_1 + y_1) \cdot (x_2 + y_2) \cdots (x_n + y_n),$$

where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are some $2n$ variables. On expanding the product, we obtain 2^n terms, and each will have n factors, with either x_i or y_i as the ith factor. For example, if $n = 2$ we get

$$x_1x_2 + y_1x_2 + x_1y_2 + y_1y_2,$$

and if $n = 3$ the product is

$$x_1x_2x_3 + y_1x_2x_3 + x_1y_2x_3 + y_1y_2x_3 + x_1x_2y_3 + y_1x_2y_3 + x_1y_2y_3 + y_1y_2y_3.$$

In constructing the terms, we made n decisions:

- from the first factor, take either x_1 or y_1;
- from the second factor, take either x_2 or y_2;

and so on.

In the general case, how many terms are there with exactly k y's? The term with $y_1y_2\cdots y_kx_{k+1}x_{k+2}\cdots x_n$ can arise in one and only one way in the product: you must choose the y term at steps $1, 2, \ldots, k$ and the x term at every later step. Similarly, if $i_1, i_2, \ldots, i_k$ are any k of the numbers $1, 2, \ldots, n$, then the term with $y_{i_1}y_{i_2}\cdots y_{i_k}$ and all other terms x's occurs if and only if you choose y at steps $i_1, i_2, \ldots, i_k$, and x at the other $n - k$ steps. So the number of terms that contain precisely k of the y's will equal the number of ways of selecting k indices $i_1, i_2, \ldots,$ i_k from $\{1, 2, \ldots, n\}$, namely $\binom{n}{k}$.

Now put $x_1 = x_2 = \cdots = x_n = x$ and $y_1 = y_2 = \cdots = y_n = y$. There will be exactly $\binom{n}{k}$ terms equal to $x^{n-k}y^k$. So

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Consequences of the Theorem

An interesting summation formula can be obtained by putting $x = y = 1$ in Theorem 6:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{2.6}$$

Equation (2.6) gives us an interesting way to work out the number of subsets of a set.

Theorem 7. *Suppose S is any set with n elements. The number of subsets of S is 2^s.*

Proof. The number of k-element subsets of S is $\binom{n}{k}$. The total number of subsets equals the number of 0-element subsets (1, for the empty set), plus the number of 1-element subsets, plus the number of 2-element subsets, and so on up to S itself. So the number is $\sum_{k=0}^{n} \binom{n}{k}$, and the result follows from (2.6). □

Sample Problem 2.26. *How many ways are there to choose three or more people from a set of* 11 *people?*

Solution. There are 2^{11} possible ways to choose a subset of the 11 people. However, the subsets with 0, 1, or 2 elements are not allowed. So the number is

$$2^{11} - \binom{11}{0} - \binom{11}{1} - \binom{11}{2} = 2048 - 1 - 11 - 55 = 1981.$$

Your Turn. How many ways are there to choose at most seven books from a selection of ten books?

Exercises 2.4 A

1. Construct the first ten rows of Pascal's triangle.

2. What is the coefficient of:

(i) y^2 in $(1 - 2y)^6$? (ii) x^3 in $(x + 2y)^5$?
(iii) y^5 in $(2x - y)^7$? (iv) y^5 in $(x - y)^6$?

3. Use the binomial theorem to calculate:

(i) $(x - 1)^4$; (ii) $(1 - 2z)^5$;
(iii) $(x + x^{-1})^4$; (iv) $(x + y + z)^3$;
(v) $(x + y - z)^2$.

4. Evaluate 1.01^3.

5. Find an approximation to 1.01^5, using four terms of the binomial theorem.

6. Use the binomial theorem with $x = -1$ and $y = 1$ to prove that

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k} = 0.$$

7. How many subsets are there in a set of seven elements?

Exercises 2.4 B

1. Calculate $\binom{97}{94}$.

2. What is the coefficient of:

(i) x^2 in $(1 + 3x)^6$? (ii) y^3 in $(5x + 2y)^4$?
(iii) y^4 in $(3x - 2y)^6$? (iv) x^3 in $(2x - 2y)^4$?

(v) x^2 in $(1+2x)^5$? (vi) x^2y^2 in $(x+y)^4$?

3. Use the binomial theorem to calculate:

(i) $(x-3y)^3$; (ii) $(1+x^2)^3$;
(iii) $(2x+y)^4$; (iv) $(2x-5y)^3$.

4. Use the binomial theorem to calculate:

(i) $(x+1)^3$; (ii) $(2-4x)^3$;
(iii) $(2x-x^{-1})^3$; (iv) $(x^2-x^{-1})^3$;
(v) $(1+x^2)^2$; (vi) $(1+2x-y)^3$.

5. Use the binomial theorem to evaluate 1.02^2.

6. Find an approximation to 1.01^8, using four terms of the binomial theorem.

7. Find an approximation to 0.99^5, using four terms of the binomial theorem.

8. What is the number of non-empty subsets of a nine-element set?

9. How many ways are there to select a subset of at least two elements from a set with eight elements?

10. Use Theorem 5 to prove that

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} = \binom{n+2}{r}$$

for any integers satisfying $2 \leq r \leq n$.

11. Use the binomial theorem with $x = -1$ and $y = 1$ to prove that

$$\binom{s}{0} + \binom{s}{2} + \binom{s}{4} + \cdots = \binom{s}{1} + \binom{s}{3} + \binom{s}{5} + \cdots$$

and consequently that

$$\binom{s}{0} + \binom{s}{2} + \binom{s}{4} + \cdots = 2^{s-1}.$$

3

Probability

3.1 Events

Experiments and Events

In the next section, we shall try to make sense of two ideas that come up every day in conversation, "randomness" and "probability" (or "chance"). In order to discuss these ideas formally, it is necessary to have a suitable definition of the sort of thing about which it makes sense to say, "is this random?" We shall call such a thing an "event". Since many of the applications of probability lie in the analysis of experiments, we shall use the terminology of the empirical sciences.

An *experiment* is defined to be any activity with well-defined, observable outcomes or results. For example, a coin flip has the outcomes "head" and "tail". The act of looking out the window to check on the weather has the possible results "it is sunny", "it is cloudy", "it is raining", "it is snowing", and so on. Both of these activities fit our definition of an "experiment".

The different possible outcomes of an experiment will be called *sample points* of the experiment. The set of all possible outcomes is the *sample space*. Each subset of the sample space is called an *event*. Those events consisting of one sample point (the singleton sets) are called *simple events*. An event E is called *random* if one cannot predict with certainty, when the relevant experiment is conducted, whether or not the outcome will be a member of E.

Many problems will involve ordinary dice, as used for example in games like Monopoly. These have six faces, with the numbers 1 through 6 on them. The sample space is the set $S = \{1, 2, 3, 4, 5, 6\}$. The event "an odd number is rolled" is $\{1, 3, 5\}$. There are 2^6 events (the number of subsets of the 6-element set S).

Sample Problem 3.1. *Betty and John roll a die. If the result is a* 1 *or* 2*, John wins* \$2 *from Betty; otherwise Betty wins* \$1 *from John. What is the event "John wins"?*

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1_3,

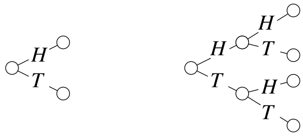


Fig. 3.1. Tree diagrams

Solution. $\{1, 2\}$.

Your Turn. In the above game, what is the event "Betty wins"?

Tree Diagrams

Consider the experiment where a coin is tossed; the possible outcomes are "heads" and "tails" (H and T). The possible outcomes can be shown in a diagram like the one in Figure 3.1(a). A special point (called a *vertex*) is drawn to represent the start of the experiment, and lines are drawn from it to further vertices representing the outcomes, which are called the *first generation*. If the experiment has two or more stages, the second stage is drawn onto the outcome vertex of the first stage, and all these form the *second generation*, and so on. The result is called a *tree diagram*. Figure 3.1(a) shows the tree diagram for the experiment where the coin is tossed twice.

Sample Problem 3.2.

(i) *A coin is flipped three times; each time the result is written down. What is the sample space? Draw a tree diagram for the experiment.*

(ii) *Repeat this example in the case where you stop flipping as soon as a head is obtained.*

Solution. (i) The sample space is $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. The tree diagram is shown in Figure 3.2(a).

(ii) The sample space is $\{H, TH, TTH, TTT\}$. The tree diagram is shown in Figure 3.2(b).

Sample Problem 3.3. *An experiment consists of flipping three identical coins simultaneously and recording the results. What is the sample space?*

Solution. $\{HHH, HHT, HTT, TTT\}$. Notice that the order is not relevant here, so that for example, the events HHT, HTH and THH of Sample Problem 3.2 are all the same event in this case.

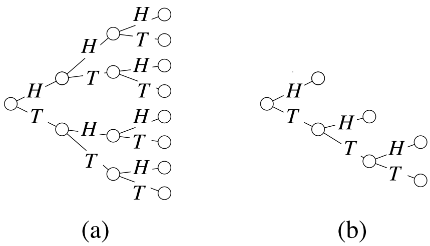


Fig. 3.2. Tree diagrams for Sample Problem 3.2

Your Turn. An experiment consists of flipping a quarter and noting the result, then flipping two pennies and noting the number of heads. What is the sample space? Draw a tree diagram.

One common application of tree diagrams is the *family tree*, which records the descendants of an individual. The start is the person, the first generation represents his or her children, the second the children's children, and so on. (This is why the word "generation" is used.)

Set Language

Since events are sets, we can use the language of set theory in describing them. We define the union and intersection of two events to be the events whose sets are the union and intersection of the sets corresponding to the two events. For example, in Sample Problem 3.1, the event "either John wins or the roll is odd" is the set $\{1, 2\} \cup \{1, 3, 5\} = \{1, 2, 3, 5\}$. (For events, just as for sets, "or" carries the understood meaning, "or both".) The complement of an event is defined to have associated with it the complement of the original set in the sample space; the usual interpretation of the complement $\overline{E}$ of the event E is "E doesn't occur". Venn diagrams can represent events, just as they can represent sets.

The language of events is different from the language of sets in a few cases. If S is the sample space, then the events S and $\emptyset$ are called "certain" and "impossible". If U and V have empty intersection, they are disjoint sets, but we call them *mutually exclusive* events.

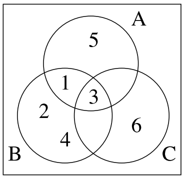
Sample Problem 3.4. *A die is thrown. Represent the following events in a Venn diagram:*

A: An odd number is thrown;

B: A number less than 5 *is thrown;*

C: A number divisible by 3 *is thrown.*

Solution.



Your Turn. Repeat the above example for the events:

A: An even number is thrown;

B: John wins (in the game of Sample Problem 3.1);

C: Betty wins.

Sample Problem 3.5. *A coin is tossed three times. Events A, B, and C are defined as follows:*

A: The number of heads is even;

B: The number of tails is even;

C: The first toss comes down heads.

Write down the outcomes in A, B, C, $A \cup C$, $B \cap C$, and $\overline{C}$. Are any of the sets mutually exclusive? Write a brief description, in words, of $B \cap C$.

Solution. $A = \{HHT, HTH, THH, TTT\}$, $B = \{HHH, HTT, THT, TTH\}$, $C = \{HHH, HHT, HTH, HTT\}$, $A \cup C = \{HHH, HHT, HTH, HTT, THH, TTT\}$, $B \cap C = \{HHH, HTT\}$, and $\overline{C} = \{THH, THT, TTH, TTT\}$. A and B are mutually exclusive, as are A and $B \cap C$. $B \cap C$ consists of all outcomes with a head first and the other two results equal.

Your Turn. For the sets defined above, write down the outcomes in $A \cap C$, $B \cup C$, and $\overline{A}$.

Sample Problem 3.6. *Describe an experiment in which the outcomes correspond to the time one must wait in the checkout line at a supermarket.*

Solution. The experiment might consist of watching the next person to come to the checkout in a supermarket and observing the time, in minutes, until he or she is served. The outcome can be any non-negative real number (although large numbers are extremely unlikely). This example shows that sample spaces need not be finite sets.

Exercises 3.1 A

1. A die is rolled and a coin is tossed, and the results are recorded.

(i) Write down all members of the sample space.

(ii) Write down all members of the event

E: The die roll is even.

2. An experiment consists of studying families with three children. B represents "boy", G represents "girl", and, for example, BGG will represent a family where the oldest child is a boy and the young children are both girls.

(i) What is the sample space for this experiment?

(ii) We define the following events:

E: The oldest child is a boy;

F: There are exactly two boys.

(a) What are the members of E and F?

(b) Describe in words the event $E \cap F$.

(iii) Draw a Venn diagram for this experiment, and show the events E and F on it.

3. Electronic components are being inspected. Initially one is selected from a batch and tested. If it fails, the batch is rejected. Otherwise a second is selected and tested. If the second fails, the batch is rejected; otherwise a third is selected. If all three pass the test, the batch is accepted.

(i) Draw a tree diagram for this experiment.

(ii) How many outcomes are there in the sample space?

(iii) What is the event, "fewer than three components are tested"?

4. An experimenter tosses four coins—two quarters and two nickels—and records the number of heads. For example, two heads on the quarters and one on the nickels is recorded (2, 1).

(i) Write down all members of the sample space.

(ii) Write down all members of the events:

E: There are more heads on the quarters than on the nickels.

F: There are exactly two heads in total.

G: The number of heads is even.

(iii) Write down all members of the events $E \cup F$, $E \cap F$, $E \cap G$, $\overline{F} \cap G$.

(iv) Describe in words the events $E \cap F$ and $\overline{F} \cap G$.

5. A coin is tossed. If the first toss is a head, the experiment stops. Otherwise, the coin is tossed a second time.

(i) Draw a tree diagram for this experiment.

(ii) List the members of the sample space.

(iii) What is the event, “the coin is tossed twice”?

6. A student takes courses in mathematics and computer science. Define the events E and B by:

E: She passes mathematics;
F: She passes computer science.

Write symbolic expressions (using E, F, $\cap$, $\cup$ and $^{-}$) for the following events:
(i) The student fails mathematics;
(ii) The student passes both subjects;
(iii) The student passes exactly one subject;
(iv) The student passes at least one subject;
(v) The student fails both subjects;
(vi) The student passes mathematics but fails computer science.

7. In each of the following cases, are the two events mutually exclusive?

(i) A die shows a 4;
A die shows an odd number.

(ii) A die shows a 4;
A die shows a perfect square.

(iii) A person is male;
The same person is a store clerk.

(iv) A person is a college freshman;
The same person is a college sophomore.

8. Suppose E and F are two events. Must E and $\overline{(E \cup F)}$ be mutually exclusive?

Exercises 3.1 B

1. A die is rolled twice, and the results are recorded as an ordered pair.

(i) How many outcomes are there in the sample space?

(ii) Consider the events:

E: The sum of the throws is 4;

F: Both throws are even;

G: The first throw was 3.

(a) List the members of events E, F, G, $E \cup F$, $E \cap F$, $E \cap G$, $\overline{F} \cap G$.

(b) Write descriptions in words of the events $E \cap F$, $\overline{F} \cap G$.

(c) Are any two of the events E, F, G mutually exclusive?

(iii) Draw a tree diagram for this experiment.

(iv) Represent the outcomes of this experiment in a Venn diagram. Show the events E, F, and G in the diagram.

2. A bag contains two red, two yellow, and three blue balls. In an experiment, one ball is drawn from the bag and its color is noted, and then a second ball is drawn and its color noted.

(i) Draw a tree diagram for this experiment.

(ii) How many outcomes are there in the sample space?

(iii) What is the event, "two balls of the same color are selected"?

3. In an experiment, video tapes are tested until either two defectives have been found or four tapes have been tested in total.

(i) How many outcomes are there in the sample space for this experiment?

(ii) What is the event, "fewer than four tapes are tested"?

4. Two dice, one red and one green, are rolled and the results recorded: for example, 53 means "5 on red, 3 on green".

(i) Write down all members of the sample space.

(ii) Write down all members of the events:

E: The total is 4;

F: The total is 11;

G: There is at least one 6.

(iii) Write down all members of the events $E \cup F$, $E \cap F$, $F \cap G$, $\overline{F} \cap G$.

(iv) Describe in words the events $F \cap G$ and $\overline{F} \cap G$.

5. Repeat the preceding Exercise for the case where the two dice are the same color and indistinguishable (the notation 53 will mean "5 on one die, 3 on the other").

6. Your doctor tests your cholesterol level each month. After the initial reading she makes three tests, and records whether the reading is higher than (H), the same as (S) or lower than (L) the preceding month.

(i) What are the possible outcomes of this experiment? Write down all members of the sample space.

(ii) Consider the events:

E: The cholesterol level never decreases;

F: The cholesterol level decreases at least twice.

(a) Write down the elements of E, F and $\overline{F}$.

(b) Write a description in words of the sets $\overline{E}$, $\overline{F}$.

(c) Are E and F mutually exclusive?

7. Workers in a factory are classified by gender, experience and union membership.

(i) Define events as follows:

E: The worker is male;

F: The worker has one year's experience at least;

G: The worker belongs to the union.

Write descriptions in words of the following events: $\overline{E}$, $\overline{F}$, $\overline{G}$, $E \cap G$, $E \cap \overline{E}$, $E \cup \overline{E}$, $G \cap \overline{F}$, $\overline{E} \cap F$, $E \cap F \cap G$.

(ii) Write expressions in symbols (using E, F, G, $\cap$, $\cup$ and $^-$) for the events:
(a) The worker is a woman who belongs to the union;
(b) The worker has less than one year's experience and is a man;
(c) The worker has at least one year's experience and is a woman union member.

(iii) Represent the set of all workers on a Venn diagram, with the sets E, F and G shown. Shade in the set defined in (ii)(c) above.

8. An experiment consists of drawing three marbles from a jar, one at a time. B represents "blue", G represents "green", R represents "red", and, for example, BGG will represent a drawing where the first marble was blue and the second and third were both green.

(i) What is the sample space for this experiment?

(ii) We define the following events:

E: The first marble is green;

F: There are exactly two red marbles.

(a) What are the members of E and F?

(b) Describe in words the event $E \cap F$.

(c) Are E and F mutually exclusive?

9. Consider the following pairs of events. Which pairs are mutually exclusive?

(i) A coin is tossed and falls heads.
The coin falls tails.

(ii) A student wears a watch.
The student wears sneakers.

(iii) Joe is a freshman.
Joe is a sophomore.

(iv) A die is rolled and gives an odd number.
The roll is greater than 4.

(v) Two dice show a total of 4.
One of the dice shows a 2.

(vi) Two dice show a total of 4.
One of the dice shows a 6.

(vii) A person is male.
The same person is female.

(viii) A student is enrolled in a Physics course.
The same student is enrolled in a Psychology course.

10. The events E and F are mutually exclusive. Which of the following are mutually exclusive pairs? Draw a Venn diagram showing the intersection in each case:

(i) E and $\overline{F}$; (ii) E and $\overline{E}$; (iii) $\overline{E}$ and $\overline{F}$.

11. E and F are two events.

(i) Write symbolic expressions (using the symbols E, F, $\cap$, $\cup$ and $^{-}$) for the events:
  (a) E occurs but F does not;
  (b) E and F both occur;
  (c) E does not occur;
  (d) Exactly one of the two occurs.

(ii) Illustrate the above events in Venn diagrams.

12. E, F, G are three sets.

(i) Write symbolic expressions (using the symbols E, F, G, $\cap$, $\cup$ and $^{-}$) for the events:
  (a) E occurs but neither F nor G does;
  (b) E and F both occur (no information is given about G);
  (c) E does not occur; at least one of F and G does;
  (d) Precisely two of the three occur.

(ii) Illustrate the above events in Venn diagrams.

13. Draw tree diagrams for your family, starting with your grandparents.

3.2 Probability Measures

Chance Events

We use the word "chance" frequently in our everyday conversation. For example, we meet somebody "by chance"; we "chance upon" the solution to a problem; one team has a "better chance" of reaching the Super Bowl than another; the Weather Channel announces a "30% chance" of precipitation.

There are two ideas here. In every case, there is the feature of unpredictability—a chance occurrence is one for which we cannot be certain of the outcome. The other feature is that sometimes chance is *quantitative*—either it can be measured exactly (the weather forecast) or else we can at least say one "chance" is greater than another (the football teams). We shall refer to these two aspects of chance as *randomness* and *probability*, respectively.

For example, say you flip a coin. There is no way to tell whether it will fall heads or tails, so we would say this is a *random* occurrence. If the coin is made uniformly, so that it is equally likely to show a head or a tail, we normally call it a *fair* coin, and we say the probability of a head is $\frac{1}{2}$, and so is the probability of a tail. (Sometimes we say "50%" instead of "$\frac{1}{2}$".)

We shall write $P(E)$ for the probability that the event E occurs. Suppose a fair coin is tossed, if H means "a head is tossed" and T means "a tail is tossed", then what we have just said is

$$P(H) = P(T) = \frac{1}{2}.$$

Suppose the coin in our example was not uniform; for example, say it was made as a sandwich of disks of metal, like a quarter, but the different disks were of different densities. It might be that, if we flipped it enough times, heads would come up 90% of the time and tails only 10%. Then we would say the probability of a head is $\frac{9}{10}$. However, this still qualifies as random, because no one knows beforehand whether a particular flip will be one of the (more common) ones that results in a head or one of those that give a tail. The probabilities do not have to be equal in order for an event to be random. (However, when we say one object is selected "at random" from a collection, this normally means that every object is equally likely to be chosen.)

Probability Distributions

Given any experiment, we could list the probabilities of all the outcomes, like the list "$P(H) = P(T) = \frac{1}{2}$" in the fair coin toss. Such a list is called the *probability distribution* of the experiment. Of course, we don't always know what the probabilities are, and sometimes the whole point of an investigation is to find out the probability distribution.

Sample Problem 3.7. *A fair die is rolled. S_i is the event that the number i is rolled for $i = 1, 2, 3, 4, 5, 6$. E means the roll of an odd number, and F the roll of a number less than* 3. *Find the probabilities of these events.*

Solution. Since the die is fair,

$$P(S_1) = P(S_2) = P(S_3) = P(S_4) = P(S_5) = P(S_6) = 1/6.$$

Clearly, $P(E) = \frac{1}{2}$, and $P(F) = \frac{1}{3}$.

Your Turn. A single die is rolled. What are the probabilities of the following events:

E: A 4 or a 5 is rolled;

F: An even number is rolled;

G: An odd number greater than 2 is rolled?

Observe that in the above example,

$$P(E) = P(S_1) + P(S_3) + P(S_5),$$

so that $P(E)$ equals the sum of the probabilities of the outcomes making up E, and similarly

$$P(F) = P(S_1) + P(S_2).$$

This additive property works in general. Suppose E is the event $\{s_1, s_2, \dots, s_m\}$; as usual, we just write s_i for the simple event $\{s_i\}$. Then

$$P(E) = P(s_1) + P(s_2) + \cdots + P(s_m).$$

We use this to make a general definition of probability.

Definition. Consider an experiment with sample space $S = \{s_1, s_2, \dots, s_m\}$. A *probability distribution* for the experiment is a function P with the following properties:

1. For each s_i, $1 \leq i \leq n$, $P(s_i)$ is a real number and $0 \leq P(s_i) \leq 1$;
2. $P(s_1) + P(s_2) + \cdots + P(s_m) = 1$.

We now define the probability of the event E by the formula

$$P(E) = \sum_{s \in E} P(s).$$

Uniform Experiments

A very important case is a probability distribution in which every sample point has the same probability. In this case, the probability is called *uniform*. We also refer to the experiment as a *uniform experiment*. If a uniform experiment has a sample space with n elements, then each sample point has probability $\frac{1}{n}$. The toss of a fair coin is a uniform experiment with $n = 2$ sample points. The roll of a fair die is a uniform experiment with $n = 6$ sample points. Another way of saying that an experiment is uniform is to call it an experiment with *equally likely outcomes*.

Suppose a uniform experiment has sample space S. then the probability of an event E is

$$P(E) = \frac{|E|}{|S|}.$$

Remember: *this important formula only applies to uniform experiments.*

Sample Problem 3.8. *A black die and a white die are thrown simultaneously. What is the probability that the numbers shown total* 8*, given that the dice are fair?*

Solution. Let us write (x, y) to mean that x shows on the black die and y shows on the white die. Then there are 36 possible outcomes, namely $(1, 1), (1, 2), \ldots, (1, 6), (2, 1), \ldots, (6, 6)$, and they are equally likely. Five of these outcomes—$(6, 2)$, $(5, 3)$, $(4, 4)$, $(3, 5)$, and $(2, 6)$—give a total of 8. So

$$P(E) = \frac{|E|}{|S|} = \frac{5}{36}.$$

Your Turn. A quarter and a nickel are flipped simultaneously. What is the probability that exactly one head shows, assuming that both coins are fair?

Sample Problem 3.9. *A deck of cards is shuffled and one card is dealt. What is the probability that it is a spade?*

Solution. There are 52 cards in a deck, of which 13 are spades. So

$$P(E) = \frac{|E|}{|S|} = \frac{13}{52} = \frac{1}{4}.$$

Your Turn. A deck is shuffled and one card dealt face up. What is the probability that is a picture card (King, Queen, or Jack)?

Non-uniform Experiments

Not all experiments are uniform. But in many cases, given an experiment A, we can find a uniform experiment B such that the outcomes of A are events (not necessarily simple) of B. We shall explain this method by an example.

Consider the following experiment. Two fair dice are rolled, and the total of the points on them is recorded. This experiment has eleven outcomes which we shall denote $s_2, s_3, \ldots, s_{12}$; s_i is the outcome "total i". To calculate the probability of outcome i, we shall look at a slightly different experiment. In it, two fair dice are rolled, and the result is recorded as an ordered pair of digits—$(1, 3)$ means "1 on die 1, 3 on die 2". There are $6^2 = 36$ outcomes, and each has probability $\frac{1}{36}$. In this experiment, we write E_i for the event that the two numbers showing add to i. Then

$E_2 = \{11\}$	$\|E_2\| = 1$ so $P(E_2) = \frac{1}{36}$
$E_3 = \{12, 21\}$	$\|E_3\| = 2$ so $P(E_3) = \frac{2}{36} = \frac{1}{18}$
$E_4 = \{13, 22, 31\}$	$\|E_4\| = 3$ so $P(E_4) = \frac{3}{36} = \frac{1}{12}$
$E_5 = \{14, 23, 32, 41\}$	$\|E_5\| = 4$ so $P(E_5) = \frac{4}{36} = \frac{1}{9}$
$E_6 = \{15, 24, 33, 42, 51\}$	$\|E_6\| = 5$ so $P(E_6) = \frac{5}{36}$
$E_7 = \{16, 25, 34, 43, 52, 61\}$	$\|E_7\| = 6$ so $P(E_7) = \frac{6}{36} = \frac{1}{6}$
$E_8 = \{26, 35, 44, 53, 62\}$	$\|E_8\| = 5$ so $P(E_8) = \frac{5}{36}$
$E_9 = \{36, 45, 54, 63\}$	$\|E_9\| = 4$ so $P(E_9) = \frac{4}{36} = \frac{1}{9}$
$E_{10} = \{46, 55, 64\}$	$\|E_{10}\| = 3$ so $P(E_{10}) = \frac{3}{36} = \frac{1}{12}$;
$E_{11} = \{56, 65\}$	$\|E_{11}\| = 2$ so $P(E_{11}) = \frac{2}{36} = \frac{1}{18}$
$E_{12} = \{66\}$	$\|E_{12}\| = 1$ so $P(E_{12}) = \frac{1}{36}$

Table 3.1. Probabilities when rolling two fair dice

$$P(E) = \frac{|E_i|}{36},$$

and we can calculate all the probabilities; see Table 3.1. But in each case the probability of the outcome s_i in the first experiment is obviously equal to the probability of E_i in the second experiment. So $P(s_2) = \frac{1}{36}$, $P(s_3) = \frac{2}{36}$, and so on.

Sample Problem 3.10. *Three coins are flipped and the number of heads is recorded. What are the possible outcomes and what is the probability distribution?*

Solution. The outcomes are the four numbers 0, 1, 2, and 3. To calculate probabilities, suppose three coins were flipped and all results recorded. E_i is the event "there are i heads showing". Then $|S| = 8$ and

$$E_0 = \{TTT\}, \qquad P(E_0) = \frac{1}{8};$$

$$E_1 = \{HTT, THT, TTH\}, \qquad P(E_1) = \frac{3}{8};$$

$$E_2 = \{HHT, HTH, THH\}, \qquad P(E_2) = \frac{3}{8};$$

$$E_3 = \{HHH\}, \qquad P(E_3) = \frac{1}{8}.$$

Your Turn. A game is played using two dice each of which has the numbers 1, 2, 3 on its faces (so each number appears twice per die). An experiment consists of rolling the dice and adding the numbers showing. What is the sample space? What is the probability distribution?

Sample Problem 3.11. *There are five marbles—two blue, three red—in a box. One is drawn out at random. What is the probability that it is blue?*

Solution. If the marbles were marked v, w, x, y, z, where v and w are blue and the others are red, then the event "blue is chosen" is $\{v, w\}$ and its probability is

$$P(E) = \frac{|E|}{|S|} = \frac{2}{5}.$$

Your Turn. There are two red, three white, and four blue marbles in a box. One is drawn at random. What is the probability that it is *not* blue?

Exercises 3.2 A

1. A sample space contains four outcomes: s_1, s_2, s_3, s_4. In each case, do the probabilities shown form a probability distribution? If not, why not?

(i) $P(s_1) = 0.2$, $P(s_2) = 0.2$, $P(s_3) = 0.2$, $P(s_4) = 0.4$;

(ii) $P(s_1) = 0.2$, $P(s_2) = 0.3$, $P(s_3) = 0.4$, $P(s_4) = 0.5$;

(iii) $P(s_1) = 0.0$, $P(s_2) = 0.2$, $P(s_3) = 0.4$, $P(s_4) = 0.4$;

(iv) $P(s_1) = 0.5$, $P(s_2) = 0.4$, $P(s_3) = 0.3$, $P(s_4) = -0.2$.

2. A fair coin is tossed three times. What are the probabilities that:

(i) At least two heads appear;

(ii) An odd number of heads appear;

(iii) The first two results are tails?

3. In the game of roulette, a wheel is divided into 38 equal parts, labeled with the numbers from 1 to 36, 0, and 00. The spin of the wheel causes the ball to be randomly placed in one of the parts; the chances of the ball landing on any part are equal. Half of the parts numbered from 1 to 36 are red and half are black; the 0 and 00 are green. If the chances of the ball landing in any one part are equal, what are the probabilities of the following events:

(i) The ball lands on a red number;

(ii) The ball lands on a black number;

(iii) The ball lands on a green number;

(iv) The ball lands on 17;

(v) The ball lands on a number from 25 to 36 inclusive.

4. Two fair dice are rolled. What are the probabilities of the following events?

(i) The total is 9;

(ii) The total is odd;

(iii) One odd and one even number are shown.

5. A box contains seven red and five white balls. One ball is drawn at random. What is the probability that it is white?

6. In a lottery there are 90 losing tickets and 10 winning tickets. You drawn one ticket at random. What is the probability that it is a winner?

7. A random number from 0 to 99 is chosen by a computer. What are the probabilities of the following events?

(i) The number is even;

(ii) The number ends in 5;

(iii) The number is divisible by 11.

8. Box A contains tickets numbered 1, 2, 3, 4. Box B contains tickets numbered 2, 3, 4, 5. One ticket is selected from each box.

(i) List all the elements of the sample space of this experiment;

(ii) Find the probability that the tickets have the same number;

(iii) Find the probability that the sum of the two selected numbers is even.

9. A box contains one red, two blue, and two white marbles. One marble is selected at random. What are the probabilities that it is

(i) red? (ii) blue? (iii) white?

10. A bag contains three red marbles and five blue marbles. Three marbles are drawn simultaneously, at random, and the colors are noted.

(i) What are the outcomes of this experiment?

(ii) Calculate the probabilities of the outcomes.

Exercises 3.2 B

1. A sample space contains four outcomes: s_1, s_2, s_3, s_4. In each case, do the probabilities shown form a probability distribution? If not, why not?

(i) $P(s_1) = 0.2$, $P(s_2) = 0.4$, $P(s_3) = 0.3$, $P(s_4) = 0.1$;

(ii) $P(s_1) = 0.2$, $P(s_2) = 0.4$, $P(s_3) = 0.6$, $P(s_4) = 0.2$;

(iii) $P(s_1) = 0.1$, $P(s_2) = 0.2$, $P(s_3) = 0.5$, $P(s_4) = 0.2$;

(iv) $P(s_1) = 0.1$, $P(s_2) = 0.2$, $P(s_3) = 0.3$, $P(s_4) = 1.2$.

2. A fair coin is tossed four times. What are the probabilities that:

(i) At least three heads appear;

(ii) An even number of heads appear;

(iii) The first result is a head?

3. Two fair dice are rolled. What are the probabilities of the following events?

(i) The total is 6;

(ii) The total is even;

(iii) Both scores are even.

4. Two fair dice are rolled. What are the probabilities of the following events?

(i) The total is 7;

(ii) The total is 8;

(iii) The total is a multiple of 3;

(iv) The total is a multiple of 4;

(v) One score is even, the other odd.

5. A box contains 12 cards, one for each month of the year. A card is drawn at random.

(i) What is the probability that the selected card is March?

(ii) What is the probability that the selected card is from a month with r in its name?

6. A box contains six red and four white balls. One ball is drawn at random. What is the probability that it is white?

7. A box contains the 365 pages from a desk calendar, one for each day of the year (not a leap year). A page is drawn at random.

(i) What is the probability that the selected page shows a day in March?

(ii) What is the probability that the selected card shows a day in summer (June 22 to September 21 inclusive)?

8. The weather forecast gives a 25% chance of snow tomorrow and a 35% chance of rain. There is a 10% chance that it will both snow and rain during the day.

(i) Represent the data in a Venn diagram.

(ii) What is the probability that it will snow but not rain?

(iii) What is the probability that it will neither snow nor rain?

9. The moose in a Canadian park are 45% plain brown, 35% mottled, and 20% spotted. One moose is captured at random for the Bronx Zoo. What are the probabilities that:

(i) It is spotted?

(ii) It is not spotted?

(iii) It is not mottled?

10. There are 20 people in a class. There are 12 men (eight physics and four chemistry majors) and eight women (five physics and three chemistry majors). One student's name is selected at random. What is the probability that:

(i) The student is a chemistry major?

(ii) The student is male?

11. A box contains four red, two blue, and three white marbles. One is selected at random. What is the probability that:

(i) The marble is blue?

(ii) The marble is not blue?

12. A box contains six red, three blue, and five white marbles. One is selected at random. What is the probability that:

(i) The marble is blue?

(ii) The marble is not white?

13. An examination has two questions. Of 100 students 75 do Question 1 correctly and 72 do Question 2 correctly. 64 do both questions correctly.

(i) Represent these data in a Venn diagram.

(ii) A student's answer book is chosen at random. What is the probability that:

(a) Question 1 contains an error?

(b) Exactly one question contains an error?

(c) At least one question contains an error?

14. There are 15 members in your club. There are seven men and eight women. The club committee has three members. One member's name is selected at random.

(i) What is the probability that:

(a) The person selected is on the committee?

(b) The person selected is male?

(ii) Can you work out the probability that the person selected is a male committee member?

3.3 Non-uniform Probabilities

More General Probabilities

Even when the outcomes are not equally likely, we calculate the probability of an event from the formula

$$P(E) = \sum_{\{s \in E\}} P(s).$$

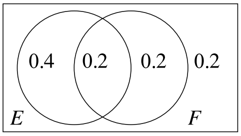


Fig. 3.3. Venn diagram for Sample Problem 3.12

Several important facts can be deduced from this; in particular, if E is any event, $0 \leq P(E) \leq 1$; $P(E) = 0$ if and only if E is impossible, and $P(E) = 1$ if and only if E is certain. If E and F are mutually exclusive events, then

$$P(E \cup F) = P(E) + P(F)$$

and for general E and F

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

The probability of the complement $\overline{E}$ of E is

$$P(\overline{E}) = 1 - P(E).$$

Sample Problem 3.12. *E and F are events in a sample space with $P(E) = 0.6$, $P(F) = 0.4$, and $P(E \cap F) = 0.2$. What is $P(E \cup F)$? What is $P(\overline{E})$?*

Solution.

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\ &= 0.6 + 0.4 - 0.2 \\ &= 0.8; \\ P(\overline{E}) &= 1 - P(E) \\ &= 1 - 0.6 \\ &= 0.4. \end{aligned}$$

Your Turn. E and F are events in a sample space with $P(E) = 0.7$, $P(F) = 0.3$, and $P(E \cap F) = 0.1$. What are $P(E \cup F)$, $P(\overline{E})$, $P(E \cup \overline{F})$?

These probabilities can be represented in a Venn diagram—in each area of the diagram, write the probability of the corresponding event. Then the answer can be obtained by addition. The Venn diagram for the preceding Sample Problem is shown in Figure 3.3.

Sample Problem 3.13. *A discrete mathematics class is restricted to business and math majors, all freshmen or sophomores. The percentage makeup of the class is:*

Freshman business majors	30%;
Freshman math majors	35%;
Sophomore business majors	21%;
Sophomore math majors	14%.

A name is chosen at random from the class list. What are the probabilities of the following events:

(i) *The student is a freshman?*

(ii) *The student is not a freshman?*

(iii) *The student is either a freshman or a math major?*

Solution. We write F for the event that the student is a freshman, S for sophomore, B for business major, and M for math major. Then the data mean

$$P(F \cap B) = 0.3; \qquad P(F \cap M) = 0.35;$$
$$P(S \cap B) = 0.21; \qquad P(S \cap M) = 0.14.$$

(i) Since $M = \overline{B}$ in this problem (all students are math or business majors, and none can be both or else the total would add to more than 100%), the usual equation

$$F = (F \cap B) \cup (F \cap \overline{B})$$

becomes

$$F = (F \cap B) \cup (F \cap M)$$

and, since this is a disjoint union,

$$\begin{aligned} P(F) &= P(F \cap B) + P(F \cap M) \\ &= 0.3 + 0.35 \\ &= 0.65. \end{aligned}$$

(ii)

$$\begin{aligned} P(\overline{F}) &= 1 - P(F) \\ &= 1 - 0.65 \\ &= 0.35. \end{aligned}$$

(iii) $F \cup M = F \cup (S \cap M)$, and this is a disjoint union, so

$$\begin{aligned} P(F \cup M) &= P(F) + P(S \cap M) \\ &= 0.65 + 0.14 \\ &= 0.79. \end{aligned}$$

The data could also be represented as

	B	M
F	0.3	0.35
S	0.21	0.14

Your Turn. An entomologist has found that the butterflies in a certain area can be classified as follows:

Striped males	19%;
Striped females	22%;
Unstriped males	28%;
Unstriped females	31%.

Represent the data in a Venn diagram. Assuming the butterflies appear at random, what is the probability that the next butterfly to be sighted will be: (a) striped; (b) female; (c) either striped or female?

Bernoulli Trials and Binomial Experiments

Consider the experiment, "flip a coin three times and count how many heads occur". This could be viewed as repeating one basic experiment that has exactly two outcomes ("flip a coin") three times and then counting the results.

We shall define a *Bernoulli trial* to be an experiment in which there are exactly two possible outcomes. In many applications, it makes sense to think of one of these events as "success" and the other as "failure" (abbreviated to S and F). We shall often denote the probability of success by p. A *binomial experiment* is one in which a Bernoulli trial is repeated a certain number of times, and the outcome is the number of successes in total. So our initial example was a binomial experiment in which the trial is repeated three times. Since the number of heads is to be counted, we would probably refer to a head as a "success". If the coin were fair, p would equal $\frac{1}{2}$.

In a binomial experiment, if the trial is repeated n times, there are $n+1$ possible outcomes, from "0 successes" to "n successes".

Sample Problem 3.14. *A trial consists of throwing a die; a result of* 1 *or* 2 *is a success, other throws are failures. What is the probability of exactly two successes in three trials?*

Solution. There are three ways in which exactly two successes can occur: the sequences SSF, SFS, and FSS. Now

$$P(SSF) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{27},$$
$$P(SFS) = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{27},$$
$$P(FSS) = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{27}.$$

So the probability of exactly two successes is

$$\frac{2}{27} + \frac{2}{27} + \frac{2}{27} = \frac{6}{27} = \frac{2}{9}.$$

Your Turn. In the experiment described in this Sample Problem, what is the probability of exactly one success in three trials?

More generally, the number of sequences of n trials that contain k successes and $n - k$ failures is $C(n, k)$. The probability of any such sequence is $p^k(1 - p)^{n-k}$. So the probability of k successes is

$$C(n, k)p^k(1 - p)^{n-k}.$$

Sample Problem 3.15. *A sales representative estimates that a sale results from one in four of his calls to companies. If he makes five calls today, what is the probability of at least two sales?*

Solution. Each call is a Bernoulli trial with $p = \frac{1}{4}$, so the five calls in a day can be thought of as a binomial experiment with $p = \frac{1}{4}$, $n = 5$. So

$$P(2 \text{ successes}) = C(5, 2)\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 = 10 \cdot \frac{3^3}{4^5} = \frac{270}{1024},$$
$$P(3 \text{ successes}) = C(5, 3)\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = 10 \cdot \frac{3^2}{4^5} = \frac{90}{1024},$$
$$P(4 \text{ successes}) = C(5, 4)\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 = 5 \cdot \frac{3}{4^5} = \frac{15}{1024},$$
$$P(5 \text{ successes}) = C(5, 5)\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0 = 1 \cdot \frac{1}{4^5} = \frac{1}{1024},$$

and the probability of at least two successes is the sum of these:

$$\frac{270 + 90 + 15 + 1}{1024} = \frac{376}{1024} = \frac{47}{128},$$

or about 37%. Alternatively, we could have noticed that "he makes *at least* two sales per day" is the complement of "he makes 0 or 1 sales per day". The probability of 0 or 1 sales is

$$\begin{aligned}
P(0 \text{ sales}) + P(1 \text{ sale}) &= C(5,0)\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5 + C(5,1)\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^4 \\
&= 1\cdot\frac{3^5}{4^5} + 5\cdot\frac{3^4}{4^5} \\
&= \frac{243 + 5\cdot 81}{1024} = \frac{243+405}{1024} \\
&= \frac{648}{1024} = \frac{81}{128}.
\end{aligned}$$

So the probability of at least two sales is

$$1 - \frac{81}{128} = \frac{47}{128}.$$

Your Turn. If the salesman could improve his record to give him a 50% chance of a sale from each call, what is his probability of at least three successes in the day?

Sample Problem 3.16. *The salesman in the preceding Problem would like to have at least a* 60% *chance of two successes per day. How many calls should he make per day?*

Solution. Suppose he makes n calls. The probability of 0 or 1 success is

$$C(n,0)\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n + C(n,1)\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{n-1} = \frac{(1\cdot 3^n)}{4^n} + \frac{n(1\cdot 3^{n-1})}{4^n}.$$

So what he wants is

$$\frac{3^{n-1}(3+n)}{4^n} < \frac{4}{10}.$$

We do some experimental arithmetic:

$$\begin{aligned}
n = 5: \frac{3^{n-1}(3+n)}{4^n} &= \frac{81\cdot 8}{1024} = 0.63\ldots; \\
n = 6: \frac{3^{n-1}(3+n)}{4^n} &= \frac{243\cdot 9}{4096} = 0.53\ldots; \\
n = 7: \frac{3^{n-1}(3+n)}{4^n} &= \frac{729\cdot 10}{16384} = 0.44\ldots; \\
n = 8: \frac{3^{n-1}(3+n)}{4^n} &= \frac{2187\cdot 11}{65536} = 0.36\ldots.
\end{aligned}$$

So 8 calls are needed.

Your Turn. How many calls are needed to give the salesman at least a 70% chance of two successes per day?

Exercises 3.3 A

1. A sample space contains two events, E and F, and

$$P(E) = 0.70, \qquad P(F) = 0.25, \qquad P(E \cap F) = 0.15.$$

Determine

$$P(\overline{E}), \quad P(E \cup F), \quad P(\overline{E \cup F}).$$

2. A sample space contains two events, E and F, and

$$P(E) = 0.7, \qquad P(F) = 0.5, \qquad P(E \cup F) = 0.9.$$

Determine

$$P(\overline{E}), \quad P(E \cap F), \quad P(\overline{E \cup F}), \quad P(\overline{E \cap F}).$$

3. The events E, F, and G satisfy

$$P(E) = 0.6, \qquad P(F) = 0.6, \qquad P(G) = 0.8,$$
$$P(E \cup F) = 0.8, \qquad P(E \cap G) = 0.5, \qquad P(F \cap G) = 0.5.$$

Determine

$$P(E \cap F), \quad P(E \cup G), \quad P(F \cup G).$$

4. The events E, F, and G satisfy

$$P(E) = 0.5, \qquad P(F) = 0.5, \qquad P(G) = 0.7,$$
$$P(E \cup F) = 0.7, \qquad P(E \cap G) = 0.4, \qquad P(F \cap G) = 0.4.$$

(i) Determine

$$P(E \cap F), \quad P(E \cup G), \quad P(F \cup G).$$

(ii) What is the smallest possible value for $P(E \cap F \cap G)$? What is the largest possible value?

5. Of 1000 researchers at Bell Laboratories, 375 have a degree in mathematics, 450 have a degree in computer science, and 150 of the researchers have degrees in both fields. One researcher's name is selected at random.

(i) What is the probability that the researcher has a degree in mathematics, but not in computer science?

(ii) What is the probability that the researcher has no degree in either mathematics or computer science?

6. Five hundred Illinois farmers were surveyed. 400 of them grow corn and 300 grow soybeans. Every farmer grows at least one of these crops. If one is chosen at random, what is the probability that he grows corn but not soybeans?

7. Two hundred new automobile buyers were surveyed. Their purchases were as follows:

	Sedan	Pickup	Van
Ford	50	10	10
G M	35	10	15
Other	40	10	20

One survey form is chosen at random. What is the probability that the vehicle was:

(i) A sedan?

(ii) A Ford?

(iii) A General Motors van or pickup?

8. Jerry's Gas sells three grades of gasoline at both their full and self-service pumps. One day they kept track of their first 100 customers and their purchases in the following table:

	Regular leaded	Regular unleaded	Premium unleaded
Full service	15	15	20
Self service	20	25	5

If this trend continues, what is the probability that the next customer who comes in

(i) Goes to the self-service pumps?

(ii) Buys regular unleaded at the self-service pumps?

(iii) Buys premium unleaded at the full-service pumps?

9. A coin is weighted so that a head is twice as likely to occur as a tail. It is tossed four times, and the result is noted.

(i) What is the probability that exactly one head occurs?

(ii) What is the probability that at least three heads occur?

10. In a 20-question, true–false test, what is the probability of getting exactly sixteen answers correct by random guessing?

11. A baseball pitcher estimates that he can throw his fast ball for a strike seven times in every eight.

(i) What is his probability of throwing exactly seven strikes in eight pitches?

(ii) What is his probability of throwing at least seven strikes in eight pitches?

12. A children's game uses a die that has 1 on three of its faces, 2 on two of its faces and a 3 on the last face. If it is rolled six times, what is the probability of rolling 2 exactly three times?

13. If a fair coin is tossed five times, what is the probability that exactly three heads occur?

14. A football team has probability $\frac{1}{3}$ of winning any game. Suppose they play three times. What is the probability of:

(i) No wins?

(ii) Three wins?

Exercises 3.3 B

1. A sample space contains two events, E and F, and

$$P(E) = 0.6, \qquad P(F) = 0.4, \qquad P(E \cap F) = 0.2.$$

Determine

$$P(\overline{E}), \quad P(E \cup F), \quad P(\overline{E \cup F}).$$

2. A sample space contains two events, E and F, and

$$P(E) = 0.5, \qquad P(F) = 0.3, \qquad P(E \cap F) = 0.2.$$

Determine

$$P(\overline{F}), \quad P(E \cup F), \quad P(\overline{E} \cap F).$$

3. The events E, F, and G satisfy

$$P(E) = 0.6, \qquad P(F) = 0.5, \qquad P(G) = 0.5,$$
$$P(E \cup F) = 0.8, \qquad P(E \cup G) = 0.8, \qquad P(F \cap G) = 0.2.$$

Determine

$$P(E \cap F), \quad P(E \cap G), \quad P(F \cup G).$$

4. The events E, F, and G satisfy

$$P(E) = 0.5, \qquad P(F) = 0.4, \qquad P(G) = 0.6,$$
$$P(E \cup F) = 0.6, \qquad P(E \cup G) = 0.9, \qquad P(F \cup G) = 0.8.$$

(i) Determine

$$P(E \cap F), \quad P(E \cap G), \quad P(F \cap G), \quad P(G \cap \overline{E}).$$

(ii) What is the smallest possible value for $P(E \cap F \cap G)$? What is the largest possible value?

5. An ice cream store sells three sizes of cones: Large, Giant, and Humongous. Fillings can be ice cream or sorbet. On Monday, the first 100 purchases were as shown in the table:

	Large	Giant	Humongous
Ice cream	25	35	20
Sorbet	5	10	5

If this trend continues, what is the probability that the next customer who comes in

(i) Orders sorbet?

(ii) Buys a Large or Giant ice cream?

6. In a market survey, 50% of those polled said that they usually buy medications at a pharmacy, and the others that they buy them at a supermarket. 80% buy meat at the supermarket, 20% at the butcher's store. 40% buy both meat and medications at the supermarket. One shopper is selected at random from the survey group. What is the probability that he:

(i) Buys meat at the supermarket but buys medications at the pharmacy?

(ii) Does not buy either meat or medications at the supermarket?

7. During the 1992 presidential elections, 200 voters were surveyed in the St. Louis area. Voters were classified according to whether they lived in Missouri (MO), Illinois (IL), or other states (OS), and whether they intended to vote for Bush (REP), Clinton (DEM), or Perot (IND). The results were are follows:

	REP	DEM	IND
MO	44	33	21
IL	20	22	14
OS	14	20	12

One of the voter's responses is selected at random. What are the probabilities of the following events?

(i) The voter is an Illinois democrat?

(ii) The voter is from Missouri?

(iii) The voter intends to vote for Perot?

(iv) The voter is from outside Illinois?

(v) The voter is either a Democrat or is from Missouri?

8. Last year, there were 200 students enrolled in Calculus II. Of these students, 140 also enrolled in Linear Algebra. Their majors were distributed as follows:

Enrollment	Major		
	Math	Science	Business
Calculus only	20	35	5
Calculus and Linear Algebra	90	45	5

Suppose the same trend is followed this year. When the next student comes to enroll in Calculus II, what is the probability that this student

(i) Is a science major?

(ii) Is not enrolling in Linear Algebra this year?

(iii) Is not a mathematics major but will take Linear Algebra?

9. 500 researchers at IBM were surveyed. It was found that 300 have at least one degree in computer science, 180 have a Bachelor's degree in computer science, 123 have a Master's degree in computer science, and 99 have a Ph.D. in computer science. Additionally, it was found that 63 have both a Bachelor's and Master's degree in computer science, 22 have both a Master's and Ph.D. degree in computer science, and 19 have both Bachelor's and Ph.D. degrees in computer science. One employee is selected at random. What are the probabilities of the following events?

(i) She has all three degrees in computer science.

(ii) She has the Bachelor's and Ph.D. degrees but not the Master's.

10. 150 Business students tried to sign up at registration for required Marketing, Accounting, and Finance courses. 70 registered for Marketing, 60 for Accounting, and 50 for Finance. 25 registered for both Marketing and Accounting, 15 for Finance and Marketing, 10 for Finance and Accounting, and 5 signed up for all three courses. One student's registration form is chosen at random. What is the probability that the student:

(i) Was unable to sign up for any of the courses;

(ii) Signed up for exactly one of the courses;

(iii) Signed up for Marketing and Accounting but not for Finance?

11. Of 200 employees at Wal-Mart, 110 have worked at restocking shelves, 90 have worked as cleaners, and 90 have run cash registers. A few have done none of these; they have only worked as managers. 50 of the workers have done both restocking and cleaning; 40 have done cleaning and run registers; 40 have done restocking and run registers; and 30 have done all three jobs. A worker is selected at random.

(i) What is the probability that the worker has a only worked at cleaning?

(ii) What is the probability that the worker has not run a cash register?

(iii) What is the probability that the worker has done only management?

12. You have 60 customers on your newspaper route. Of them, 30 take the Morning News, 40 take the Daily Times, and 34 take the Sunday News-Times. Moreover, 14 take both the Morning News and Daily Times, 24 take both the Daily Times and the Sunday News-Times, and 10 take both the Morning News and the Sunday News-Times. Every customer takes at least one paper. A customer is selected at random.

(i) What is the probability that the customer takes exactly one paper?

(ii) What is the probability that the customer takes exactly two papers?

(iii) What is the probability that the customer takes all three papers?

13. A die is rolled six times. What is the probability of:

(i) Exactly one six?

(ii) At most one six?

14. A fair coin is tossed six times. Which is more probable—that heads and tails occur three times each, or that they divide 4 and 2 (either 4 heads or 4 tails)?

15. A multiple-choice test has eight questions, and each question has three possible answers. A passing grade is five or more correct answers. What is the probability of passing if you guess the answers at random?

16. A pair of fair dice are rolled and the total is recorded. If this is done ten times, what is the probability that 7 is recorded at least three times?

17. On average, one light bulb in 20 is defective. What is the probability that there is more than one defective bulb in a box of ten?

18. A drug is effective in nine cases out of ten. If it is tested on 12 patients, what is the probability that it will be ineffective in more than two cases?

19. The probability that a new tire will last 25000 miles is 0.9. If you replace all four tires now, what is the probability that you will need to replace at least one within the next 25000 miles?

20. A traffic light is green for 50 seconds, yellow for 5 seconds, and red for 45 seconds. If you drive past the light four times in one day, what is the probability that it is red when you arrive at least twice?

21. A candy company puts a prize inside one in every four boxes of candy. If you buy five boxes, what is the chance you get:

(i) No prize?

(ii) Exactly one prize?

(iii) More than one prize?

22. How many times must toss a fair coin in order to have a 90% chance of at least one head?

3.4 Counting and Probability

All of the elementary counting principles which we studied in Chapter 2 are useful in calculating probabilities. We shall show this using several examples.

Choosing Marbles

Many elementary problems involve random drawing of marbles from a jar (or bag or box). These obviously arise in lotteries and other games of chance, but this is not the only reason for studying such problems. The mathematics is the same if we consider many other types of random selection. As an example, suppose the telephone subscribers in your area include 500 Democrat supporters and 400 Republicans. Working out the probabilities of various combinations being contacted in a telephone poll involves exactly the same calculations as the ones made in studying random drawings from a jar of 500 black and 400 white marbles. There are also examples in physical science—for example, molecules escaping from a container of heated gas behave in the same way.

Sample Problem 3.17. *There are* 12 *red marbles and three blue marbles in a jar. Three are selected at random. What is the probability that all are red?*

Solution. There are 15 marbles, so the number of ways of selecting three is $C(15, 3)$. So $|S| = C(15, 3)$. The number of ways of selecting three red marbles is $|E| = C(12, 3)$. So

$$\begin{aligned} P(E) &= \frac{C(12,3)}{C(15,3)} = \frac{12!}{9!3!} \cdot \frac{12!3!}{15!} \\ &= \frac{12\cdot 11\cdot 10}{15\cdot 14\cdot 13} = \frac{88}{91}. \end{aligned}$$

Your Turn. An urn contains five red, four blue and two green marbles. Three marbles are chosen at random. What is the probability that the three are all different colors?

Sample Problem 3.18. *A box contains four red, three white, and two blue balls. Two balls are chosen at random. What is the probability that they are the same color?*

Solution. There are $C(9, 2) = 36$ selections available. There are $C(4, 2) = 6$ ways to choose two red balls, $C(3, 2) = 3$ ways to choose two white balls, and $C(2, 2) = 1$ way to choose two blue balls. So

$$P(\text{two are the same}) = \frac{6+3+1}{C(9,2)} = \frac{10}{36} = \frac{5}{18}.$$

Your Turn. A jar contains six white and four blue balls. Two balls are chosen at random. What is the probability of selecting at least one white ball?

Card Problems

Remember that in a standard deck there are 52 cards, 13 in each of the four suits. So there are four cards in each of the 13 denominations Ace, King, ..., 4, 3, 2. In most card games, the hands of cards are dealt face down to the players, so the order in which cards are received does not matter.

Sample Problem 3.19. *A poker hand of five cards is dealt from a standard deck. What are the probabilities of the following hands:*

(i) *The four Kings and the Ace of spades?*

(ii) *A full house* (*three cards of one denomination, two of another*)*?*

(iii) *A hand with no pair* (*all different denominations*)*?*

Solution. In each case, the hand dealt is one of the $C(52, 5)$ possible selections of five cards from the full deck of 52. So, in each case, $|S| = C(52, 5)$.

(i) The event E: "the hand consists of four Kings and the Ace of spades" contains only one outcome. So

$$\begin{aligned} |E| &= 1; \\ P(E) &= \frac{|E|}{|S|} = \frac{1}{C(52, 5)} \\ &= \frac{1}{2598960}. \end{aligned}$$

(ii) Let's say a full house is of "type (x, y)" if it contains three x's and two y's—one could have "type (7, 5)", "type (King, 4)", and so on. There are 13 ways to choose the denomination of the three, and for each such choice there are 12 ways to select the two. So there are $13 \cdot 12 = 156$ types. Once the type is known, we can calculate the number of hands of that type. For example, if there are three Kings and two fours, then there are $C(4, 3)$ ways of choosing which Kings are to be used (we must choose three out of the four possible Kings), and $C(4, 2)$ ways of choosing which fours. So there are $C(4, 3) \cdot C(4, 2) = 4 \cdot 6 = 24$ hands of each type. So the number of full houses is

$$\begin{aligned} &13 \cdot 12 \cdot C(4, 3) \cdot C(4, 2) \\ &= 13 \cdot 12 \cdot 24, \end{aligned}$$

and the probability is

$$\frac{13 \cdot 12 \cdot 24}{C(52, 5)} = \frac{6}{4165},$$

or about one chance in 700.

(iii) If there is no pair, the hand has one card of each of five denominations. This collection of denominations can be chosen in $C(13, 5)$ ways. The card in each denomination can be selected in 4 ways. So the number of hands is

$$C(13, 5) \cdot 4^5,$$

and the probability is

$$\frac{C(13, 5)\cdot 4^5}{C(52, 5)} = \frac{2112}{4165},$$

which is a little greater than 50%.

Your Turn. A poker hand of five cards is dealt from a standard deck. What is the probability that it contains a flush (all five cards of the same suit)?

The Birthday Coincidence

Sometimes probabilities can be very surprising. One well-known example is the "birthday coincidence" problem. Suppose there are 30 people in a room. What is the probability that two of them share the same birthday (day and month)?

For simplicity, let us ignore leap years. (The answer will still be approximately correct.) What is the probability of the complementary event—that no two have the same birthday? (The probability of a birthday coincidence will found by subtracting this probability from 1.) If we assume the people are ordered in some way (alphabetical order, for example), then there are 365 choices for the first person's birthday, 364 for the second, 363 for the third, and so on. So the total number of possible lists of birthdays (with no repeats) is

$$P(365, 30) = 365 \cdot 364 \cdots 336.$$

If there is no restriction, each person has 365 possible birthdays. So there are 365^{30} possibilities. Therefore, the probability of the event "no two have the same birthday" is

$$P(E) = \frac{|E|}{|S|} = \frac{P(365, 30)}{365^{30}}$$

which is approximately 0.294. So the probability of the "birthday coincidence" is about 0.706.

Similar calculations can, of course, be carried out for any number of people. Table 3.2 shows the probability of a "birthday coincidence" for various numbers of people. It is about a 50% chance when there are 23 people in the room.

Most students find this very surprising, and would guess that many more people would be needed to bring the probability up to 50%.

n	P_n
5	0.027
10	0.117
15	0.253
20	0.411
23	0.507
25	0.569
30	0.706
40	0.891
50	0.970

Table 3.2. P_n is the probability of a "birthday coincidence" among n people

Choosing a Committee

Sample Problem 3.20. *A committee of four people is to be chosen at random from a club with* 12 *members, four men and eight women.*

(i) *What is the probability that at least one man is chosen?*

(ii) *What is the probability that one particular member, Jack Smith, is chosen?*

Solution. The number of possible outcomes—all possible committees—is $C(12, 4)$. This is $|S|$. They are equally likely.

(i) Let E be the event that at least one man is chosen. Then $\overline{E}$ is the event that no man is chosen. $|\overline{E}| = C(8, 4)$. So $|E| = C(12, 4) - C(8, 4)$.

$$P(E) = 1 - \frac{C(8,4)}{C(12,4)} = 1 - \frac{14}{99} = \frac{85}{99}.$$

(ii) The number of committees containing a given individual is $C(11, 3)$. (Jack Smith must be a member; the remaining three are chose from the other 11 members and that can be done in $C(11, 3)$ ways.) Therefore,

$$P(\text{Jack Smith is chosen}) = \frac{C(11,3)}{C(12,4)} = \frac{1}{3}.$$

Your Turn. A class has 18 members—six Math majors, five in Economics, and seven in Computer Science. A class committee of three is chosen at random. What is the probability that all three have the same major?

Exercises 3.4 A

1. There are three red, three blue, and four white marbles in a jar. Two are chosen at random. What are the probabilities that:

(i) Both are blue?
(ii) The two are different colors?

2. A jar contains four red, five blue, and three white marbles. Three are chosen at random. What are the probabilities that:

(i) All three are red?
(ii) None is red?

3. A computer program assigns 4-digit PIN numbers at random. It rejects any PIN that starts with 0. What is the probability that:

(i) Your pin will be smaller than 4000?
(ii) Your PIN will contain a duplicated number (e.g., 3141)?

4. A student must select two courses from a list of two humanities and three science courses. He selects at random. What is the probability that both are science courses?

5. A committee of six people (four men and two women) need to select a chairman and secretary. They do so by drawing names from a hat. What are the probabilities that:

(i) Both are men?
(ii) Exactly one is a man?

6. A poker hand is dealt at random. What is the probability that it contains a straight (A2345, 23456, 34567, ..., 10JKQA)? (In a straight, suits do not matter.)

7. Suppose license plates consist of two letters followed by four numbers. Suppose all possible combinations have been used. If you select a car at random, what is the probability that the plate has:

(i) Both letters the same?
(ii) No repeated symbol?
(iii) The last digit even?

8. A congressional committee contains four democrats and five republicans. To choose a subcommittee they list all groups of four that contain at least one republican and one democrat, and then select one at random. What is the probability that the selected subcommittee consists of two democrats and two republicans?

9. A secretary types three letters and the three corresponding envelopes. She is in a hurry to quit work for the day, and forgets to check that she puts the correct letters in the correct envelopes, so it is as though she puts the letters in the envelopes at random. What are the probabilities that:

(i) Every envelope gets the correct letter?
(ii) No envelope gets the correct letter?

10. A committee of six men and four women select a steering committee of three people. If all combinations are equally likely, what is the probability that all three will be of the same gender?

11. A pinball machine selects a number from 0 to 9 at random; you are awarded a free game if that number equals the last digit in your score.

(i) What is the probability that you win a game in this way on your first attempt?
(ii) What is the probability that you will get no "match" in five games?

12. A jar contains four red, four blue, and two white marbles. Three are chosen at random. What are the probabilities that:

(i) All three are red?
(ii) All three are blue?
(iii) None is red?

13. A supermarket is suspected of overcharging for meat by giving short weight. To test this, a consumer organization purchases five 1-pound packages of meat, selected at random. If there are 25 packages on display, and three are underweight, what is the probability that an underweight package is purchased?

Exercises 3.4 B

1. There are six red and four yellow marbles in a jar.

(i) A pair of marbles is chosen at random. What are the probabilities that:
(a) Both are red?
(b) The two are different colors?
(ii) Suppose the one marble is chosen, its color recorded, the marble is returned to the jar, and another is chosen. What are the probabilities that:
(a) Both are red?
(b) The two are different colors?

2. There are ten red, three white, and four blue marbles in a jar. Two are chosen at random. What are the probabilities that

(i) Neither is red?
(ii) Exactly one is red?
(iii) Both are red?

3. From the jar in the preceding exercise, three marbles are chosen. What are the probabilities that:

(i) All three are red?
(ii) The three are of different colors?

4. A box of 50 matches contains three broken matches. Two matches are drawn at random. What is the probability that neither is broken?

5. Your midterm test contains ten questions, with true–false answers. If you select your answers at random, what are the probabilities of getting:

(i) Exactly four correct answers?
(ii) Exactly two correct answers?
(iii) At most two correct answers?

6. An electrician knows that two switches are faulty out of a batch of five. He tests them one at a time. What is the probability that he finds the two faulty switches in the first two trials?

7. There are five people in a room. What is the probability that two were born in the same month (but not necessarily the same year)?

8. In the game Yahtzee, five fair dice are rolled at once. What are the probabilities of getting:

(i) A Yahtzee, that is, all five dice the same?

(ii) Four of a kind (that is, four of one denomination, with the other different)?

(iii) A full house (that is, three of one denomination, and two of another)?

9. A committee of ten people (four men and six women) need to select an agenda committee with two members. They do so by drawing names at random. What are the probabilities that:

(i) Both are men?

(ii) Exactly one is a man?

10. In one state, 50% of license plates consist of two letters followed by four numbers and the other 50% have three numbers followed by three letters. Each plate is chosen at random from among the possibilities. There is no restriction on the numbers used. What is the probability that the next plate issued contains the number 5?

11. Seven people sit around a circular table, at random. What is the chance that two specific people sit together?

12. Four people apply for two jobs. Among them are a husband and wife. Assume that the four are equally well qualified and choices are made at random.

(i) What is the probability that both husband and wife are appointed?

(ii) What is the probability that at least one is appointed?

13. In a lottery, you must select three different numbers from 1, 2, 3, 4, 5, 6, 7, 8, 9. If your three are the numbers drawn, you win the first prize; if you have two right out of three, you win second prize. (The order of the numbers does not count).

(i) What is the probability of winning first prize?

(ii) What is the probability of winning second prize?

14. A club has 12 members, including Mr. and Mrs. Smith. A committee of three is to be chosen at random.

(i) What is the probability that both are chosen?

(ii) What is the probability that neither are chosen?

15. Your discussion group chooses its chairman at each monthly meeting by drawing names from a hat. If there are 12 members, what is the probability that you will not be chosen at any meeting this year?

16. The scrabble tiles of the letters in the word $TERRORIST$ are placed in a row at random. What is the probability that:
(i) The three Rs occur together?
(ii) The three Rs occur together and the two Ts occur together?

3.5 Stochastic Processes

Sequences of Experiments

Sometimes an experiment can be viewed as a sequence of smaller experiments. The outcome consists of a sequence: "outcome of subexperiment 1" followed by "outcome of subexperiment 2", etc. An experiment of this kind is called a *stochastic process*.

For example, suppose a jar contains three red and two blue marbles. An experiment consists of drawing a marble from the jar, noting its color, drawing another marble, and noting the second marble's color. (The first marble is not replaced.) The possible outcomes are the four ordered pairs of colors: RR, RB, BR, and BB.

To analyze the experiment, we first observe that the first marble drawn is either red (three-fifths of the cases) or blue (two-fifths). If the first marble is red, then the remaining marbles are two red and two blue, so in half of these cases ($\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$ of the original cases) the second marble drawn is red and in the other half ($\frac{3}{10}$ of the original) it is blue. So, in the obvious notation,

$$P(RR) = \frac{3}{10},$$
$$P(RB) = \frac{3}{10}.$$

In the same way, if the first marble is blue, then the second marble drawn is red in $\frac{3}{4}$ and blue in $\frac{1}{4}$ of the cases. So

$$P(BR) = \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10},$$
$$P(BB) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}.$$

So the four outcomes are RR, RB, BR, and BB, and

$$P(RR) = P(RB) = P(BR) = \frac{3}{10}, \qquad P(BB) = \frac{1}{10}.$$

Sample Problem 3.21. *There are two bags A and B. Bag A contains two red and one green ball; bag B contains two red and three green balls. First, a bag is chosen at random, then a ball is chosen at random from it. What are the possible outcomes, and what are their probabilities? What is the probability that a red ball is selected?*

Solution. The outcomes are AR, AG, BR, and BG where A and B are the bags and R and G denote and color of the ball selected. The possible outcomes of the first subexperiment—the selection of bag—are A and B, with probabilities each $\frac{1}{2}$. If A is selected, the probability of R is $\frac{2}{3}$ and the probability of G is $\frac{1}{3}$. If B is selected, R has probability $\frac{2}{5}$ and G has probability $\frac{3}{5}$. So

$$P(AR) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3},$$
$$P(AG) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},$$
$$P(BR) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5},$$
$$P(BG) = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}.$$

So the probability of red is

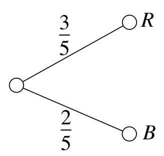
$$P(AR) + P(BR) = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$$

Notice that there are four red and four green balls. If the two bags were emptied and one ball chosen, the probability of red would be $\frac{1}{2}$. The process of dividing the balls into two bags, whose contents are not identical, changes the probabilities.

Your Turn. A die is chosen from two dice, and rolled. One die (die A) is standard; the other has three faces marked 1 and three marked 6. What are the outcomes? Assuming that the die is chosen at random, what is the probability of a 6? Of a 3?

Tree Diagrams

The use of tree diagrams to represent experiments (see Section 3.1) is particularly useful for stochastic processes. Let us go back to the example of three red and two blue marbles. The first stage of the experiment can be represented as a branching into two parts, labeled "red" and "blue", and the branches can be also be labeled with the probabilities $\frac{3}{5}$ (on red) and $\frac{2}{5}$ (on blue):



In each case, the second stage can be represented similarly. The tree diagram for the whole experiment is shown in Figure 3.4.

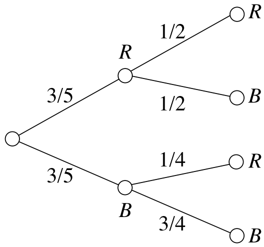


Fig. 3.4. Tree diagram for the whole experiment

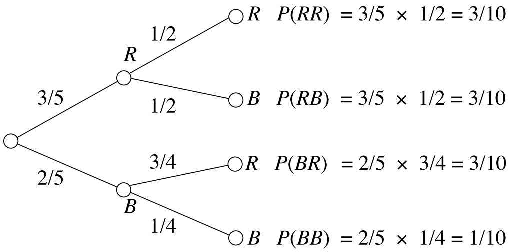


Fig. 3.5. Tree diagram with branch probabilities

Finally, the outcomes are sequences of the results of the subexperiments. Each endpoint on the right of the tree represents an outcome, and the sequence can be read off by tracing back through the branch to the root. The probability of an outcome can be calculated by multiplying the probabilities along the branch. So the experiment under discussion yields the diagram Figure 3.5.

Conditional Probabilities

In the second stage of our experiment, what is the probability of drawing a blue marble? There are two answers. If the first marble was red, then the probability is $\frac{1}{2}$; if it was blue, the probability is $\frac{1}{4}$. We write

$$P(\text{second blue} \mid \text{first red}) = \frac{1}{2},$$
$$P(\text{second blue} \mid \text{first blue}) = \frac{1}{4}.$$

The sign "|" is read as the word "given", so the first probability above is "probability that the second is blue given that the first is red". These probabilities are called *conditional* because they express the probability of a result in the second draw if a certain condition (the result of the first draw) is satisfied.

Conditional probabilities arise all the time. Suppose weather records show that the average temperature on sunny February days is 45°. Then we say "if it is sunny,

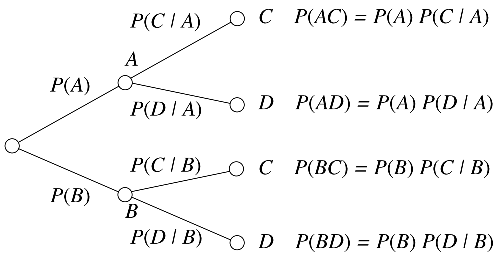


Fig. 3.6. Tree diagram with branch probabilities

the probability of 45° or warmer is 50%". This is a conditional probability—the probability of the temperature being 45° or warmer given the condition that it is sunny in the first place. In general, suppose the fact that event A occurs gives extra information about the probability that B occurs. The *probability that B occurs, given that A occurs*, which is the *conditional probability* $P(B \mid A)$, is the best measure to use in deciding whether or not B will happen.

Sample Problem 3.22. *Two cards are dealt from a standard deck. The first card is not replaced before the second is dealt. What is the probability that the second card is a King, given that the first card is an Ace? What is the probability of a King given that the first card is a King?*

Solution. Given that the first card is an Ace, the second card is a random selection from 51 equally likely possibilities. Four of these outcomes are Kings. So

$$P(K \mid A) = \frac{|E|}{|S|} = \frac{4}{51}.$$

If the first card is a King, there are only three Kings among the cards for the second trial, so

$$P(K \mid K) = \frac{3}{51}.$$

Your Turn. Two cards are dealt, without replacement, from a standard deck. If the first is a spade, what is the probability that the second is (i) a spade? (ii) a heart?

It will be seen that the probabilities written on the later branches of a tree diagram are all conditional probabilities. If an experiment consists of two stages, where the first stage has possible outcomes A and B and the second stage has possible outcomes C and D, then the tree diagram will be the one shown in Figure 3.6. Of course, this all works in a similar way when there are more than two stages, or when there are more than two outcomes at a stage.

Sample Problem 3.23. *Two cards are dealt without replacement from a standard deck, and it is noted whether or not the cards are Kings. What are the outcomes of this experiment, and their probabilities? Represent the information in a tree diagram.*

Solution. We shall write K and N for “a King” and “a card other than a King.” Using Sample Problem 3.22 we get the probabilities

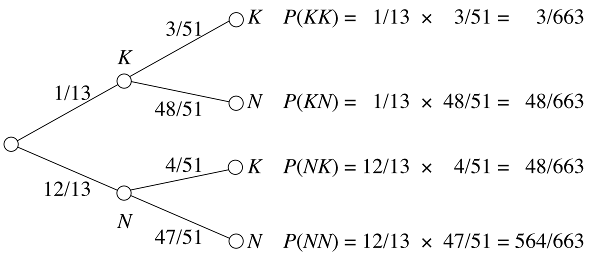
$$P(K \mid K) = \frac{3}{51},$$
$$P(K \mid N) = \frac{4}{51}.$$

(In the Sample Problem we were told that the first card was an Ace, but clearly any non-King denomination would give the same answer.) Given that the first card is a King, the second card is either a King or not, so

$$P(K \mid K) + P(N \mid K) = 1.$$

Therefore, $P(N \mid K) = \frac{48}{51}$, and similarly $P(N \mid N) = \frac{47}{51}$.

So the diagram is



(where, for example, “KN” means “King first, non-King second”).

Your Turn. Repeat this Sample Problem, in the case where the information noted is whether or not the card is a spade.

Exercises 3.5 A

1. A jar contains six marbles, four red and two white. In stage 1 of an experiment, one marble is selected at random and its color noted; it is not replaced. In stage 2, another marble is drawn and its color noted.

(i) Draw a tree diagram for this experiment.

(ii) What are the probabilities of the four outcomes (RR, RW, WR, WW)?

(iii) What is the probability that the two marbles selected are different colors?

2. A cage contains six mice: three white females, one white male, and two gray males. In an experiment, two mice are selected one after the other, without replacement, and sex and color are noted.

(i) Draw a tree diagram for this experiment.

(ii) Find the probabilities of all the outcomes.

(iii) What is the probability that two males are selected?

(iv) What is the probability that both a white and a gray mouse are selected?

(v) What is the probability that the second mouse is gray?

3. Three cards are dealt in order from a standard deck. In each case it is only recorded whether the card is an honor card (Ace, King, Queen, Jack) or a minor card (10 through 2).

(i) Draw a tree diagram for this experiment. Find the probabilities of the different outcomes.

(ii) What is the probability that at least two honor cards are dealt?

4. A jar contains four red, three white, and two blue marbles. One marble is drawn and placed aside; then another marble is drawn. Only the colors are recorded.

(i) Drawn a tree diagram for this experiment.

(ii) What is the probability that the two marbles are of different colors?

5. Box A contains three red pens and four blue pens. Box B contains two red, one green and one blue pen. A pen is selected from Box A at random, and placed in Box B. Then a pen is selected at random from Box B. What is the probability that this pen is:

(i) Red?

(ii) Blue?

(iii) Green?

6. A card is dealt at random from a regular deck. What is the probability that it is a Jack given that it is a face card (King, Queen, or Jack)?

7. Two fair dice are rolled. Consider the events:

A: The sum is 7;

B: The sum is odd;

C: At least one die shows a 4.

What are $P(A \mid B)$, $P(B \mid A)$, $P(A \mid C)$, $P(C \mid A)$?

8. Two cards are selected from a regular deck, without replacement. Consider the events:

A: Both cards are red;

B: The second card is red;

C: At least one card is red;

D: At least one card is a heart.

What are $P(A)$, $P(D)$, $P(A \mid B)$, $P(A \mid C)$, $P(D \mid B)$, $P(D \mid C)$?

9. In Exercise 2, what are the conditional probabilities of the following events:

(i) Two males are selected, given that at least one male is selected?

(ii) Two females are selected, given that at least one white mouse is selected?

10. In Exercise 5, what are the conditional probabilities of the following events:

(i) The pen selected from Box B is blue, given that the pen selected from Box A was blue?

(ii) The pen selected from Box B is blue, given that the pen selected from Box A was red?

Exercises 3.5 B

1. Three cards are dealt in order from a standard deck. Only the color (red or black) of the card is recorded. The cards are not replaced.

(i) Draw a tree diagram for this experiment.

(ii) What is the probability that the three cards are red?

2. Eight evenly-matched horses run a race. Three are bay colts, one is a bay filly, two are brown colts, and two are brown fillies. The color and sex of the first and second finishers are recorded.

(i) Draw a tree diagram for this experiment.

(ii) What is the probability that both the horses recorded are colts?

(iii) What is the probability that the two horses are of different colors?

(iv) What is the probability that the second horse is a filly?

3. A box contains four red and two blue marbles. One marble is selected at random. If it is red, it is put aside; if it is blue, it is put back in the box. In either case, another marble is then drawn.

(i) Draw a tree diagram for this experiment; calculate the probabilities of the outcomes.

(ii) What is the probability that the first marble is blue?

(iii) What is the probability that at least one blue marble is selected?

4. A city worker can either go to work by car or by bus. If she goes by car, she uses the tunnel 40% of the time and the bridge 60% of the time. If she takes the bus, it is equally likely that her bus will use the tunnel or the bridge.

(i) Draw a tree diagram to represent her possible routes to work.

(ii) Suppose she drives on three days out of five and takes the bus twice out of every five times. What are the probabilities of the various outcomes? What is the probability, on a given day, that she crosses the bridge on her way to work?

5. Box 1 contains five red balls and three white balls. Box 2 contains two red and two white balls. An experiment consists of selecting two balls at random from Box 1 and placing them in Box 2, then selecting one ball from Box 2 at random.

(i) Draw a tree diagram for this experiment.

(ii) What is the probability that the ball chosen from Box 2 is red?

6. A card is dealt from a regular deck. What is the probability that it is a 5, given that it is not a face card (King, Queen, or Jack)?

7. Two fair dice are rolled. Consider the events:

A: The sum is 6;

B: Both dice show even numbers;

C: At least one die shows a 4.

What are $P(A \mid B)$, $P(A \mid C)$, $P(B \mid A)$, $P(B \mid C)$, $P(C \mid A)$, $P(C \mid B)$?

8. A jar contains three white and three red marbles. Two are drawn in succession, without replacement, and the colors are noted.

(i) Draw a tree diagram for this experiment.

(ii) Let A, B, and C denote the events:

A: Both marbles are red;

B: The second marble is red;

C: At least one marble is red.

What are $P(A)$, $P(A \mid B)$, $P(A \mid C)$, $P(B \mid C)$?

9. Two cards are dealt in order from a deck of 24 (Ace, King, Queen, Jack, ten and nine of spades, clubs, diamonds and hearts). Only the suit of the card is recorded. The cards are not replaced.

(i) Draw a tree diagram for this experiment.

(ii) What is the probability that both cards are red?

(iii) What is the probability that both cards are hearts?

10. A box contains three red, two white, and two blue marbles. One marble is selected at random. If it is red, it is put aside; otherwise, it is put back in the box. In either case, another marble is then drawn.

(i) Draw a tree diagram for this experiment; calculate the probabilities of the outcomes.

(ii) What is the probability that the first marble is blue?

(iii) What is the probability that the second marble is red?

(iv) What is the probability that at least one white marble is selected?

11. Two teams, A and B, play in the finals of a tournament. The first to win two games wins; once a winner is determined, no further games are played. There are no ties (extra time is always played until there is a result).

(i) What is the smallest number of games they might play? What is the largest?

(ii) Draw a tree diagram to represent the possible outcomes.

(iii) Assume that each team has a 50% chance of winning each game.

(a) Determine the probabilities of the different possible outcomes.

(b) What is the probability that A wins the finals?

(iv) What are the answers to (a) and (b) if team A has a two-thirds chance of winning each game?

12. Box 1 contains four light bulbs, of which one is defective. Box 2 contains eight light bulbs, of which two are defective. Box 3 contains six light bulbs, of which two are defective. A box is chosen at random and a bulb is chosen at random from it. What is the probability that the bulb is defective?

13. Two fair dice are rolled. Consider the events:

A: The sum is 8;

B: Both dice show even numbers;

C: At least one die shows a 6.

What are $P(A \mid B)$, $P(A \mid C)$, $P(B \mid A)$, $P(B \mid C)$, $P(C \mid A)$, $P(C \mid B)$?

14. A jar contains three white, two blue and three red marbles. Two are drawn in succession, without replacement, and the colors are noted. A, B, and C denote the events:

A: Both marbles are red;

B: No marble is blue;

C: At least one marble is red.

What are $P(A)$, $P(B)$, $P(C)$, $P(A \mid B)$, $P(A \mid C)$, $P(B \mid A)$, $P(B \mid C)$?

15. A roulette wheel in Las Vegas contains 38 numbered spaces. Two are green, 18 red or 18 black. You bet on black. Given that you lose, what is the probability that red wins?

16. In Exercise 4, what are the conditional probabilities of the following events?

(i) The worker traveled over the bridge on her way to work, given that she traveled by bus.

(ii) She used the tunnel, given that she came by car.

17. In Exercise 5, what are the probabilities of the following events?

(i) The ball chosen from Box 2 is red, given that both balls chosen from Box 1 were red.

(ii) The ball chosen from Box 2 is red, given that at least one ball chosen from Box 1 was red.

(iii) The ball chosen from Box 2 is red, given that at most one ball chosen from Box 1 was red?

18. Consider Exercises 11, parts (iii) and (iv). In each case, what is the conditional probability that A wins, given that B wins the first game?

19. In Exercise 12, what is the probability that the chosen bulb is defective, given that it was not chosen from box 2?

3.6 More About Conditional Probability

The Conditional Probability Formula

In the preceding section, we used conditional probabilities to calculate probabilities of stochastic processes. The probability of the sequence "A followed by B" is

$$P(AB) = P(A)P(B \mid A). \tag{3.1}$$

This formula can be used more generally, even when A does not happen before B. Whenever we get more information about whether an event B occurs by knowing whether or not event A occurs, we define the conditional probability of B given A by

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}. \tag{3.2}$$

Sample Problem 3.24. *A car pool contains both ten red and ten white Fords and* 15 *red and* 5 *white Buicks. A car is chosen at random, by picking up the keys.*

(i) *What is the probability that a red car is chosen?*

(ii) *You notice that the keys belong to a Buick. What is the probability that a red car is chosen?*

Solution. (i) There are 40 cars, of which 25 are red. So

$$P(R) = \frac{|R|}{|S|} = \frac{25}{40} = \frac{5}{8}.$$

(ii) There are 20 Buicks, of which 15 are red. So

$$P(B) = \frac{20}{40} = \frac{1}{2},$$
$$P(R \cap B) = \frac{15}{40} = \frac{3}{8},$$
$$P(R \mid B) = \frac{P(R \cap B)}{P(B)} = \frac{3}{8} \cdot \frac{2}{1} = \frac{3}{4}.$$

Your Turn. There are five Republicans and three Democrats on a committee. A subcommittee of two is chosen by a random drawing.

(i) What is the probability that both are Democrats?

(ii) You are told that the committee contains at least one Democrat. What is the probability that both are Democrats?

Sample Problem 3.25. *A carpenter uses screws made by two companies, X and Y; 40% of his stock comes from X and the rest from Y. About 2% of the screws from Y are faulty. If he chooses a screw at random, what is the probability that it is a faulty screw from company Y?*

Solution. We have

$$P(Y) = 0.6, \qquad P(F \mid Y) = 0.02.$$

So

$$\begin{aligned} P(Y \cap F) &= P(Y)P(F \mid Y) \\ &= 0.6 \cdot 0.02 = 0.012, \end{aligned}$$

and the probability is 0.012 or 1.2%.

Your Turn. Records show that two March days out of five are dull, and it rains on one out of every three dull days. What is the probability that March 12th next year will be dull and rainy?

Independence

In everyday English, we say two events are independent if they have no effect on each other. In probability theory, events A and B are called *independent* when

$$P(A \mid B) = P(A).$$

Provided neither A nor B is impossible, the relation

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

leads us to the equivalent definition: A and B are independent when

$$P(A \cap B) = P(A) \cdot P(B).$$

This is not quite a translation of the everyday English idea. In the following example, the events are related, but knowing whether one occurs does not give any information about the probability of the other.

Sample Problem 3.26. *A quarter is flipped three times, and the result—heads or tails—is recorded. A is the event "all three are the same" (HHH or TTT) and B is the event "the first result is a head". What are $P(A)$ and $P(B)$? Are A and B independent?*

Solution. There are eight possible outcomes: HHH, HHT, HTH, and so on; they are equally likely. So $|S| = 8$. Since $|A| = 2$ and $|B| = 4$, we have

$$P(A) = \frac{2}{8} = \frac{1}{4},$$
$$P(B) = \frac{4}{8} = \frac{1}{2}.$$

The event $A \cap B$ consists of the single outcome HHH. So

$$P(A \cap B) = \frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2} = P(A) \cdot P(B),$$

so A and B are independent.

Your Turn. Two dice (one red and one green) are rolled and the outcome is recorded. A is the event "5 on the red die" and B is the event "even number on the green die". What are $P(A)$ and $P(B)$? Are A and B independent?

Sample Problem 3.27. *Two dice are rolled and their sum is recorded. Consider the events:*

A: Sum is 2, 8, or 11;

B: Sum is even;

C: Sum is 4, 7, or 10.

Which of these are independent?

Solution. We write E_i for the outcome: "the sum is i". These outcomes are the events shown with their probabilities in Table 3.1. Then

$$A = \{E_2, E_8, E_{11}\},$$
$$B = \{E_2, E_4, E_6, E_8, E_{10}, E_{12}\},$$
$$C = \{E_4, E_7, E_{10}\},$$

$$A \cap B = \{E_2, E_8\},$$
$$A \cap C = \emptyset,$$
$$B \cap C = \{E_4, E_{10}\},$$

and from the table

$$P(A) = \frac{(1+5+2)}{36} = \frac{8}{36},$$
$$P(B) = \frac{(1+3+5+5+3+1)}{36} = \frac{18}{36},$$
$$P(C) = \frac{(3+6+3)}{36} = \frac{12}{36},$$
$$P(A \cap B) = \frac{(1+5)}{36} = \frac{6}{36},$$
$$P(A \cap C) = 0,$$
$$P(B \cap C) = \frac{(3+3)}{36} = \frac{6}{36}.$$

Now

$$P(A) \cdot P(B) = \frac{8}{36} \cdot \frac{18}{36} = \frac{4}{36} \neq P(A \cap B),$$
$$P(A) \cdot P(C) = \frac{8}{36} \cdot \frac{12}{36} = \frac{8}{108} \neq P(A \cap C),$$
$$P(B) \cdot P(C) = \frac{18}{36} \cdot \frac{12}{36} = \frac{6}{36} = P(B \cap C),$$

so B and C are independent, but A and B are dependent, and so are A and C.

Your Turn. An urn contains eight balls numbered 1 through 8. Balls 1, 2, and 3 are red; 4, 5, 6, and 7 are white; 8 is blue. One ball is drawn. Consider the events:

A: It is red;

B: It is blue;

C: The number is odd.

Which of these events are independent?

A Summation Formula

Suppose two events, A and B are considered. Since

$$A = (A \cap B) \cup (A \backslash B) = (A \cap B) \cup (A \cap \overline{B})$$

and the latter events are mutually exclusive, we have

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

so

$$P(A) = P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B}).$$

More generally, if the possible outcomes of an experiment are $B_1, B_2, \ldots, B_k$, then

$$P(A) = \sum_{i=1}^{k} P(A \mid B_i)P(B_i). \tag{3.3}$$

In terms of tree diagrams, this formula is a way of stating in symbols the following rule: "to find the probability of A, find the probabilities associated with all branches that contain A, and sum them".

Sample Problem 3.28. *There are six boxes, two of them round and four square. Each round box contains two green marbles and three blue marbles. Each square box contains one green marble and three blue marbles. A box is chosen at random and a marble is chosen at random from it. What is the probability that the marble is blue? What is the probability that the box was round, given that the marble was blue? Represent the probabilities in a tree diagram.*

Solution. We write R, S, G, B for round, square, green, blue. Then the probabilities of choosing round and square boxes are

$$P(R) = \frac{1}{3}, \qquad P(S) = \frac{2}{3}.$$

So

$$\begin{aligned} P(B) &= P(B \mid R)P(R) + P(B \mid S)P(S) \\ &= \left(\frac{3}{5}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) \\ &= \frac{1}{5} + \frac{1}{2} = \frac{7}{10}. \end{aligned}$$

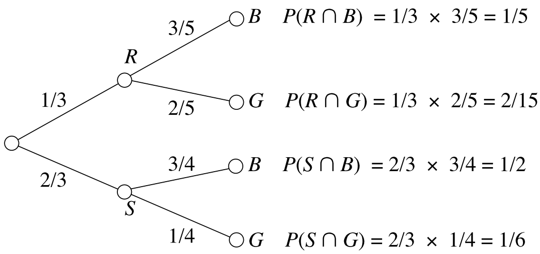
We can calculate $P(R \cap B)$:

$$P(R \cap B) = P(B \mid R)P(R) = \left(\frac{3}{5}\right)\left(\frac{1}{3}\right) = \frac{1}{5}.$$

So

$$P(R \mid B) = \frac{P(R \cap B)}{P(B)} = \frac{\frac{1}{5}}{\frac{7}{10}} = \frac{2}{7}.$$

The diagram is



$$P(B) = P(R \cap B) + P(S \cap B) = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}.$$

(This last equation is the sum of the probabilities of all branches that contain B.)

Your Turn. Consider the carpenter in Sample Problem 3.25. Suppose further that about 4% of the screws from company X are faulty. If he selects a screw at random, what is the probability that it is faulty?

Exercises 3.6 A

1. In each case, illustrate the data with a Venn diagram and calculate $P(A \mid B)$ and $P(B \mid A)$:

(i) $P(A) = 0.7$, $P(B) = 0.4$, $P(A \backslash B) = 0.4$;
(ii) $P(A \cup B) = 0.7$, $P(A \cap B) = 0.3$, $P(B) = 0.3$;
(iii) $P(A) = 0.6$, $P(B) = 0.6$, $P(A \cap B) = 0.4$;
(iv) $P(A) = 0.6$, $P(B) = 0.6$, and $P(A \backslash B) = 0.4$.

2. Suppose A and B are independent events. Find $P(A \cup B)$ and $P(A \cap B)$ if

(i) $P(A) = 0.5$, $P(B) = 0.8$;
(ii) $P(A) = 0.7$, $P(B) = 0.5$;
(iii) $P(A) = 0.9$, $P(B) = 0.7$.

3. Two cards are selected from a regular deck. It is recorded whether or not each card is a spade. Define the events:

A: The first card is a spade;
B: The second card is a spade.

(i) Suppose the first card is not replaced before the second is drawn.
   (a) Draw a tree diagram for this experiment.
   (b) Find $P(A)$, $P(B)$, $P(A \mid B)$, $P(B \mid A)$, and $P(A \cap B)$.
   (c) Are A and B independent?

(ii) Suppose the first card is replaced and the deck is shuffled before the second selection. Repeat (a), (b), and (c).

4. A random sample of 100 people were asked whether their income level is low, medium, or high (L, M, H) and asked which car they prefer of Ford, Chevrolet, or Toyota (F, C, T). The following table gives the results of the survey.

Preference	Income		
	L	M	H
F	10	13	2
C	20	12	8
T	10	15	10

A person in the survey is chosen at random. Writing F to mean that be the person prefers Ford, M to mean that she has medium income, and so on, find

(i) $P(C \mid H)$;
(ii) $P(M \mid T)$;
(iii) $P(\overline{H} \mid T)$;
(iv) $P(M \mid \overline{F})$.

5. A weather forecasting station predicts that on December 18 it will snow with a probability of 0.6, there will be a change in the wind direction with a probability of 0.8, and there will be both snow and a change in the wind direction with a probability of 0.4. If the prediction is correct

(i) What is the probability that there will be neither snow nor a change in the wind direction?
(ii) What is the probability of snow given that the wind direction has changed?

6. The chef at Le Chat Noir has his good days and his bad days. Let A mean "The chef cooked lunch badly" and B mean "the chef cooked dinner badly." If $P(A) = 0.3$, $P(B) = 0.2$, and $P(A \cap B) = 0.1$, what are:

(i) $P(B|A)$?
(ii) $P(\overline{B}|A)$?
(iii) $P(B|\overline{A})$?
(iv) $P(\overline{B}|\overline{A})$?

Suppose the chef cooked a bad dinner. What is the probability that lunch was bad that day?

7. Two dice are rolled and the sum of the top faces is recorded. Let F be the event that the sum is an odd number, let G be the event that the sum is 4, 7, or 10, and let E be the event that the sum is 2, 7, or 11.

(i) Find $P(E \mid F)$.
(ii) Determine whether F and G are independent.

8. Professor Jones has taught the same course for the last 20 years. Each time he teaches it he gives a pretest on the first day of the class. At the end of each semester he compares course grades with whether or not students have passed the pretest. He assigns a course grades C or higher to 60% of the class. He finds that 60% of those students who got C or better in the course have passed the pretest while 40% of the students who got less than a C in the course also passed the pretest.

(i) Draw and label a tree diagram illustrating the process.
(ii) Find the probability that a student who takes the pretest will *not* pass it.
(iii) If a student has not passed the pretest, what is his probability of getting less than a C in the course?

9. Two fair dice, one red and one green, are rolled. Events A, B, and C are defined as follows:

A: The sum of the two dice is 7;
B: The green die shows a 3;
C: The green die shows a 1.

(i) Are A and B independent?
(ii) Are A and C independent?

10. Three jars, labeled A, B, and C, contain red and green jelly beans as follows: A has two red and two green, B has four red and three green, and C has two red and five green. A jar is selected at random, and one jelly bean is selected at random from it. Represent this experiment in a tree diagram, and find the probability that the jelly bean is red.

11. Suppose E, F, and G are any three events. Prove that:

$$P(E \cap F \cap G) = P(E) \cdot P(F \mid G) \cdot P(G \mid E \cap F).$$

Exercises 3.6 B

1. In each case, illustrate the data with a Venn diagram and calculate $P(A \mid B)$ and $P(B \mid A)$:

(i) $P(A) = 0.7$, $P(B) = 0.6$, $P(A \cup B) = 0.9$;
(ii) $P(A) = 0.6$, $P(B) = 0.3$, $P(A \cap B) = 0.2$;
(iii) $P(A \cup B) = 0.7$, $P(A \cap B) = 0.4$, $P(A) = 0.6$;
(iv) $P(A) = 0.4$, $P(B) = 0.6$, $P(A \cup B) = 0.9$;
(v) $P(A) = 0.6$, $P(B) = 0.4$, $P(A \cap B) = 0.1$;
(vi) $P(A) = 0.4$, $P(A \cap B) = 0.1$, $P(A \cup B) = 0.9$;
(vii) $P(A) = 0.6$, $P(B) = 0.6$, $P(A \cap B) = 0.6$;
(viii) $P(A) = 0.7$, $P(B) = 0.6$, $P(A \cup B) = 0.8$;
(ix) $P(A) = 0.8$, $P(B) = 0.4$, $P(A \cap B) = 0.3$.

2. Suppose A and B are independent events. Find $P(A \cup B)$ and $P(A \cap B)$ if

(i) $P(A) = 0.6$, $P(B) = 0.4$;
(ii) $P(A) = 0.4$, $P(B) = 0.4$;
(iii) $P(A) = 0.8$, $P(B) = 0.8$;
(iv) $P(A) = 0.5$, $P(B) = 0.6$;
(v) $P(A) = 0.5$, $P(B) = 0.3$;
(vi) $P(A) = 0.7$, $P(B) = 0.7$.

3. A box contains six red and four blue marbles. Two marbles are drawn from the box and their colors are noted. Define the events:

E: The first marble is red;
F: The second marble is red.

(i) Suppose the first marble is not replaced before the second drawing.

(a) Draw a tree diagram for this experiment.

(b) Find $P(E)$, $P(F)$, $P(E \mid F)$, $P(F \mid E)$, and $P(E \cap F)$.

(c) Are E and F independent?

(ii) Repeat the preceding questions in the case where the first marble is not replaced before the second drawing.

4. A box contains five red and seven blue marbles. Two marbles are drawn from the box and their colors are noted. Define the events:

E: The first marble is red;

F: The second marble is red.

(i) Suppose the first marble is not replaced before the second drawing.

(a) Draw a tree diagram for this experiment.

(b) Find $P(E)$, $P(F)$, $P(E \mid F)$, $P(F \mid E)$, and $P(E \cap F)$.

(c) Are E and F independent?

(ii) Repeat the preceding questions in the case where the first marble is not replaced before the second drawing.

5. A random sample of 100 people were asked about their income level and asked whether they regularly invest in the stock market. The following table gives the results of the survey.

Income level	Regularly invest in stock market	
	Yes	No
High	18	6
Medium	21	16
Low	15	24

A person in the survey is chosen at random. Let H be the event that the person has a high income. Let Y be the event that the person regularly invests in the stock market, and let N be the event that the person does not regularly invest in the stock market. Find

(i) $P(H)$;

(ii) $P(Y)$;

(iii) $P(H \cup Y)$;

(iv) $P(N \mid H)$.

6. Suppose an urn contains two red, four green, and five blue marbles. A marble is selected from the urn and its color noted. If it is red or green it is withdrawn, otherwise it is replaced. Then a second marble is selected from the urn. Determine the probability the first marble was red, given the second was red.

7. 40 percent of those who take drugs also have an alcohol problem and five percent of those who do not take drugs have an alcohol problem. If 32% of the population take drugs, what is the probability that a person who has an alcohol problem also takes drugs?

8. Two fair dice, one red and one green, are rolled. Events A, B, and C are defined as follows:

A: The sum of the two numbers shown is 6;
B: The number showing on the green die is 2;
C: The green die shows a 3.

(i) Are A and B independent?
(ii) Are A and C independent?

9. Two jars, labeled A and B, contain marbles as follows: A has two red, one white and two blue, and B contains six red, two white, and two blue. A jar is selected at random, and a marble is selected at random from it. Represent this experiment in a tree diagram. What is the probability that the marble chosen is blue?

10. A bookstore sells both fiction and non-fiction. Suppose A denotes the event that the next book sold is fiction, and B denotes the event that the next book sold is science fiction (and therefore is fiction). Say $P(A) = 0.5$ and $P(B) = 0.05$. What is $P(B|A)$?

11. At a certain Midwestern University (that shall remain nameless), 30% of the underage students have fake I.D.s, and 20% of the underage students are arrested for drunkenness on a given Friday night. Of those arrested, 60% had fake I.D.s. If an underage student possessed a fake I.D., what is the probability that she was arrested for drunkenness that Friday night?

12. Suppose E and F are events in the sample space S. You may assume that $0 < P(E) < 1$ and $0 < P(F) < 1$. Are the following true, or false, or is it impossible to decide?

(i) $P(E \mid E) = 1$;
(ii) $P(E \mid \overline{E}) = 1$;
(iii) $P(S \mid F) = 1$;
(iv) $P(F \mid S) = 1$;
(v) $P(F \mid S) = P(F)$;
(vi) $P(E \mid E \cap F) = 0$;
(vii) $P(E \mid F) = P(F \mid E)$;
(viii) $P(E \cup F) \geq P(E \cap F)$.

13. E and F are any events such that $P(F) \neq 0$. Show that

$$P(E \mid F) + P(\overline{E} \mid F) = 1.$$

14. D, E, and F are any events such that $P(F) \neq 0$. Show that

$$P(D \mid F) + P(E \mid F) = P(D \cup E \mid F) + P(D \cap E \mid F).$$

3.7 Bayes' Formula and Applications

Bayes' Formula

If A and B are any two events, then $A \cap B$ and $B \cap A$ are, of course, the same event. However, the conditional probability formula gives two different looking expressions for their probabilities:

$$P(A \cap B) = P(B \mid A)P(A),$$
$$P(B \cap A) = P(A \mid B)P(B).$$

It follows that the two expressions must be equal:

$$P(B \mid A)P(A) = P(A \mid B)P(B).$$

This formula is usually written in the form

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}. \tag{3.4}$$

Sample Problem 3.29. *A store gets* 40% *of its stock of light bulbs from factory* X*,* 35% *from* Y*, and* 25% *from* Z*. Of the output from* X*,* 1% *are faulty;* 2% *from* Y *and* 3% *from* Z *are faulty. A light bulb is chosen at random from stock and found to be faulty. What is the chance that it comes from factory* Z*?*

Solution. Write Z for the event "the bulb comes from factory Z" and A for "the bulb is faulty." Then $P(Z \mid A)$ is required. From the data we know $P(Z) = 0.25$ and $P(A \mid Z) = 0.03$. To calculate $P(A)$ we use the formula (3.3):

$$\begin{aligned}P(A) &= P(A \mid X)P(X) + P(A \mid Y)P(Y) + P(A \mid Z)P(Z)\\ &= (0.01)\cdot(0.40) + (0.02)\cdot(0.35) + (0.03)\cdot(0.25)\\ &= 0.004 + 0.007 + 0.0075\\ &= 0.0185.\end{aligned}$$

So

$$\begin{aligned}P(Z \mid A) &= \frac{(0.25)\cdot(0.03)}{0.0185}\\ &= \frac{75}{185} = \frac{15}{37},\end{aligned}$$

or approximately 40.5%.

Your Turn. You are given three coins; one is biased so that it shows heads two-thirds of the time, and the other two are fair (heads and tails are equally likely). You select a coin at random and flip; it shows heads. What is the probability that it is the biased coin?

The formula (3.4) is called Bayes' Formula. It is often combined with (3.3) as follows. Suppose the possible outcomes of an experiment are $B_1, B_2, \dots, B_k$. Then

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A \mid B_1)P(B_1) + \cdots + P(A \mid B_k)P(B_k)} \tag{3.5}$$

(the denominator of (3.5) comes directly from (3.3)).

When Bayes' formula is used, very often the outcomes $B_1, B_2, \dots, B_k$ are the outcomes of an experiment that occurred earlier, and A is an outcome, or set of outcomes, of a later experiment. Bayes' formula answers the question, "given the outcome of the second experiment, what is the probability that the outcome of the first experiment was so-and-so?" This sometimes confuses students because it somehow seems to suggest that the outcome of the later experiment can somehow affect the earlier experiment, and that is impossible—causes must come before effects. What in fact happens is that knowledge of the later experiment can increase our (incomplete) knowledge of the earlier experiment.

Sample Problem 3.30. *The car pool contains ten Fords (five red and five white) and 15 Pontiacs (five red and ten white). You are allocated a car at random. You see from a distance that it is red. What is the probability that you have been given a Ford?*

Solution. We require $P(F \mid R)$. There are 25 cars of which ten are Fords, so $P(F) = \frac{10}{25} = 0.4$, and ten are red, so $P(R) = 0.4$ also. The probability of a red car, given that it is a Ford, is 0.5, since half the Fords are red. So

$$\begin{aligned} P(R)P(F \mid R) &= P(F)P(R \mid F), \\ 0.4 \cdot P(F \mid R) &= 0.4 \cdot 0.5, \\ P(F \mid R) &= 0.5. \end{aligned}$$

Your Turn. Box A contains two blue pens and three red pens. Box B contains two red pens and three blue pens. A box is chosen at random and a pen is chosen at random from it. If the pen is blue, what is the probability that the box was Box A?

Tree Diagrams

It is often convenient to use tree diagrams in Bayes' formula problems. When the diagram is completed, the terms to be added for the denominator of (3.5) are on the right hand side.

Bayes' formula calculations can be carried out as follows:

(i) First, start constructing a tree diagram with the possible outcomes $B_1, B_2, \dots, B_k$ as branches.

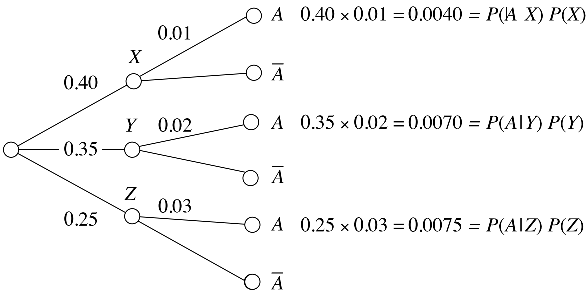
(ii) To each of these branches add the branches A ("the event A occurs") and $\overline{A}$ ("A does not occur"). The paths ending in A together represent all the circumstances in which A can occur.

(iii) Now calculate the probability—the product of the conditional probabilities—for each branch that ends in A. The sum of these probabilities will be $P(A)$, the denominator of (3.5).

(iv) The numerator $P(B_i \mid A)$ will be written next to one of the branches.

Sample Problem 3.31. *Construct a tree diagram for Sample Problem 3.29.*

Solution. In the terminology of experiments, Sample Problem 3.29 could be described as follows. First, a supplier is chosen; then a light bulb is chosen from that supplier. So the outcomes of the first experiment (the $B_1, B_2, \ldots$) are factories X, Y, or Z. The first part of the tree diagram has three branches labeled X, Y, and Z, with probabilities 0.4, 0.35, and 0.25, respectively. Event A is "the bulb is faulty", and the three probabilities are 0.01, 0.02, and 0.03. So we obtain the diagram



From the diagram, the denominator is 0.0185. So

$$P(Z \mid A) = \frac{P(A \mid Z)P(Z)}{0.0185} = \frac{0.0075}{0.0185} = 0.405.$$

Your Turn. Repeat the above for Sample Problem 3.30.

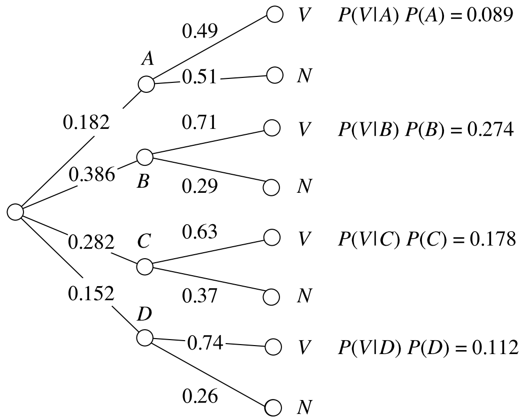
Sample Problem 3.32. *The following table shows the proportion of people over* 18 *who are in various age categories, together with the probability that a person in a given category will vote in a given election. A vote is selected at random. What is the probability that the voter was from the* 18–24 *age group?*

Age group	Proportion	Probability
A: 18–24	18.2%	0.49
B: 25–44	38.6%	0.71
C: 45–64	28.2%	0.63
D: 65+	15.2%	0.74

Solution. We use the following diagram (V and N mean "voter" and "non-voter", respectively).

$$P(A \mid V) = \frac{0.089}{0.653} = 0.136.$$

Your Turn. What is the probability that the voter was 65 or older?



Box Diagrams

In many examples, it is easiest to represent a two-stage experiment using a box diagram. The outcomes of the first experiment are used as labels for rows, and the outcomes of the second experiment are used as labels for columns. Where row A meets column B, we put a box containing $P(A \cap B)$. The sum of entries in row A, which equals $P(A)$, is written at the end of row A, and so on.

As an example, consider Sample Problem 3.29. The first experiment—determination of factory—has outcomes X, Y, and Z; the second experiment—"check: is it faulty"—has outcomes A (faulty) and B (okay). Then

$$P(A \cap X) = P(A \mid X)P(X) = 0.004,$$
$$P(A \cap Y) = P(A \mid Y)P(Y) = 0.007,$$
$$P(A \cap Z) = P(A \mid Z)P(Z) = 0.0075.$$

(See the diagram in Sample Problem 3.31 for these calculations.)

We can calculate $P(B \cap X)$ in various ways. For example, A and B are complements, so

$$\begin{aligned} P(A \cap X) + P(B \cap X) &= P(X), \\ P(B \cap X) &= P(X) - P(A \cap X) \\ &= 0.4 - 0.004 = 0.396; \end{aligned}$$

	A	B	
X	0.004	0.396	0.4
Y	0.007	0.343	0.35
Z	0.0075	0.2425	0.25
	0.0185	0.9815	

Fig. 3.7. Box diagram of data for Sample Problem 3.29

similarly $P(B \cap Y) = 0.343$, $P(B \cap Z) = 0.2425$. So the box diagram is as shown in Figure 3.7.

Sample Problem 3.33. *Use the box diagram in Figure 3.7 to calculate $P(Z \mid A)$ in Sample Problem 3.29.*

Solution. To find $P(Z \mid A)$, look at column A, which has a total of 0.0185. Then look at the (Z, A) entry 0.0075, and take the ratio:

$$P(Z \mid A) = \frac{P(A \cap Z)}{P(Z)} = \frac{0.0075}{0.0185} = 0.406.$$

Your Turn. Produce a box diagram for the problem following Sample Problem 3.29 and *use it* to find the probability that you have tossed the biased coin, given that it shows heads.

Exercises 3.7 A

1. The events E and F satisfy $P(E) = 0.6$, $P(F \mid E) = 0.5$, and $P(F \mid \overline{E}) = 0.75$. Find $P(E \mid F)$ and $P(\overline{E} \mid F)$.
2. One experiment has possible outcomes A, B, C, while a second experiment has possible outcomes E and F. Find $P(A \mid E)$, $P(B \mid E)$, $P(C \mid E)$, $P(A \mid F)$, $P(B \mid F)$, and $P(C \mid F)$ in the following cases:
   (i) $P(A) = 0.4$, $P(B) = 0.4$, $P(E \mid A) = 0.25$, $P(E \mid B) = 0.75$, and $P(E) = 0.6$;
   (ii) $P(A) = 0.25$, $P(B) = 0.25$, $P(E \mid A) = 0.4$, $P(E \mid B) = 0.4$, and $P(E) = 0.5$;
   (iii) $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{10}$, $P(E \mid A) = \frac{1}{3}$, $P(E \mid B) = 0$, and $P(E) = \frac{3}{5}$;
   (iv) $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{10}$, $P(E \mid A) = \frac{1}{3}$, $P(E \mid B) = 0$, and $P(E) = \frac{1}{2}$.
3. An auto rental company has 12 cars available, four small and eight fullsize. There are ten customers; six want small and four want fullsize. If a small car is not available, the company gives a free upgrade to fullsize.
   (i) How many upgrades are necessary?
   (ii) If a customer receives a fullsize, what is the probability that she requested a small car?

4. Jason owns two cars, a Caprice and a Mini Cooper. If he drives the Mini Cooper to his office, he is late to work only 10% of the time (because he can usually find a parking space); but if he drives the Caprice, he is late 30% of the time. He drives the Mini Cooper on four days out of every five, but on the fifth day he needs the larger car.

(i) Draw a box diagram for these data.

(ii) If he arrives on time, what is the probability that he drove the Caprice that morning?

5. An automobile dealership finds that 4% of their customers default on payments, so that the car must be repossessed. On analyzing the records, it is found that among those who did not default, 40% made a large down payment ($2000 or more), while only 10% of those who later defaulted made a large down payment.

(i) Suppose a customer makes a large down-payment. What is the probability that he will default on payments?

(ii) If a customer makes a small down-payment, what is the probability that he will default on payments?

6. In manufacturing metal office equipment, your company uses nuts supplied by three companies, A, B, and C. Company A supplies 30%, company B supplies 45%, and company C supplies 25%. It is known that on average the following percentages of the nuts are defective: 1% of those from A, 1.5% of those from B, and 0.5% of those from C.

(i) If a nut is selected at random, what is the probability that it is defective?

(ii) If a nut is selected at random and is found to be defective, what is the probability that it was made by company B?

7. In a certain city, it is found that equally many people have fair and dark hair. A survey shows that 20% of people with dark hair and 40% of people with fair hair have blue eyes. A person is chosen at random from the population and is found to have blue eyes. What is the probability that this person has fair hair?

8. A class contains 60% women. It is found that 12% of the men students and 7% of the women students are left-handed. A student is chosen at random. If the student is left-handed, what is the probability that the student is male?

9. In a factory, 30% of the workers smoke. It is found that smokers have three times the absentee rate of other workers. If a worker is absent, what is the probability that he is a smoker?

10. An auto insurance company classifies 20% of its drivers as good risks, 60% as medium risks, and 20% as bad risks (called classes A, B, and C, respectively). The probability of at least one accident in a given year is 1% for class C, 0.5% for class B, and 0.1% for class A. If one of their insureds has an accident this year, what is the probability that he is a class B driver?

Exercises 3.7 B

1. The events E and F satisfy $P(E) = 0.85$, $P(F \mid E) = 0.8$, and $P(F \mid \overline{E}) = 0.8$. Find $P(E \mid F)$ and $P(\overline{E} \mid F)$.

2. The events E and F satisfy $P(E) = 0.5$, $P(F \mid E) = 0.4$, and $P(F \mid \overline{E}) = 0.4$. Find $P(E \mid F)$ and $P(\overline{E} \mid F)$.

3. One experiment has the possible outcomes A, B, and C; a second experiment has the possible outcomes E and F. Calculate $P(A \mid E)$, $P(B \mid E)$, $P(C \mid E)$, $P(A \mid F)$, $P(B \mid F)$, and $P(C \mid F)$ when:

(i) $P(A) = \frac{3}{10}$, $P(B) = \frac{1}{2}$, $P(E \mid A) = \frac{2}{3}$, $P(F \mid B) = \frac{2}{5}$, and $P(F \mid C) = \frac{1}{2}$;

(ii) $P(A) = 0.25$, $P(B) = 0.25$, $P(E \mid A) = 0.6$, $P(F \mid B) = 0.6$, and $P(F \mid C) = 0.5$;

(iii) $P(A) = 0.5$, $P(B) = 0.3$, $P(E \mid A) = 0.6$, $P(F \mid B) = 0$, and $P(F \mid C) = 0.5$;

(iv) $P(A) = \frac{1}{5}$, $P(B) = \frac{3}{5}$, $P(E \mid A) = \frac{1}{2}$, $P(F \mid B) = \frac{1}{3}$, and $P(F \mid C) = \frac{1}{2}$.

4. 60 percent of the suitcases that are lost by Incompetence Airlines are eventually recovered. Of those that are recovered, 60% were locked, while only 10% of those not recovered were locked.

(i) If a lost case is locked, what is the probability that it will never be recovered?

(ii) If a lost case is not locked, what is the probability that it will eventually be recovered?

5. A builder buys tiles from two companies, A and B; he gets 80% from A and 20% from B. He finds that 98% of the tiles he gets from A are undamaged, while 96% of those from B are undamaged. If he finds a damaged tile, what is the probability that:

(i) The tile came from A?

(ii) The tile came from B?

6. Suppose the shipping department of a company has three workers who prepare shipping labels. They prepare 60%, 30%, and 10% of the labels, respectively. The respective percentages of errors are 3%, 5%, and 10%. Find the probability that an incorrect label is due to the first person. Also find the probability that an incorrect label is due to each of the other persons.

7. Box 1 contains 2 red pens and 4 blue pens. Box 2 contains 4 red pens, one green pen, and 4 blue pens. A pen is chosen from Box 1 and then it is placed in Box 2. Then a box is chosen at random and a pen is selected.

(i) What is the probability that the pen is blue?

(ii) If the pen is blue, what is the probability it originally came from Box 1?

(iii) Suppose Box 2 is the one from which a pen in finally selected.
   (a) If the pen drawn is blue, what is the probability that the pen drawn earlier from Box 1 was red?
   (b) If the pen drawn is red, what is the probability that the pen drawn earlier from Box 1 was red?

8. In a manufacturing plant three machines, A, B, and C produce 50%, 35%, and 15%, respectively, of the total production. The quality control department of the company has determined that 1% of the items produced by Machine A and 2% of the items produced by each of Machines B and C are defective. If an item is selected at random and found to be defective, what is the probability that it was produced by Machine B?

9. A school tax proposition is submitted to voters. The voters' registered party affiliation as a percentage of all voters, and the percentage of each group who voted in favor of the proposition, are as follows:

Party	Registration	In Favor
Democrat	40	70
Republican	40	20
Independent	10	80
Other	10	50

A voter is selected at random.
(i) What is the probability that she voted in favor of the proposition?
(ii) If she voted in favor of the proposition, what is the probability that she is a Democrat?
(iii) If she voted against the proposition, what is the probability that she is not a Republican?

10. A class contains 20% Math majors, 40% Computer Science majors, and 40% Engineering majors. 50% of the Math Majors, 75% of the Computer Science majors, and 25% of the Engineering majors are women.
(i) Draw a box diagram for these data.
(ii) A student's name is chosen at random from the class list. What is the probability that the student is a woman?
(iii) A student's name is chosen at random from the class list and the student is found to be a woman. What is the probability that the student is an Engineering major?

11. A computer dealer sells 70% PC's, 20% Apple products and 10% Sun platforms. In the first 90 days, an average of 5% of the PCs, 1% of the Apples and 3% of the Suns are returned for repair. What percentage of the repairs are Apples? What percentage are Suns?

3.8 Further Examples of Bayes' Formula

Medical Testing

Probability theory is often misused in the popular press. One particular area where probabilistic ideas are mentioned but not properly analyzed is in the discussion of tests for disease and drugs. This is a very important topic nowadays, when compulsory drug testing is becoming more common in everyday life and testing for diseases such as AIDS is also very important.

You may see in a newspaper that a certain test is "90% accurate". And you would most likely think this means that, if you test positive, there is a 90% probability that you have the disease in question. But this is not so, as the next example shows.

Sample Problem 3.34. *A test for a venereal disease is* 90% *accurate: if you have the disease, the probability is* 0.9 *that you will test positive; and if you don't have the disease, the probability is* 0.9 *that you will test negative. In the whole population, one person in a hundred has the disease. If a given person tests positive, what is the chance that he has the disease?*

Solution. We shall use the abbreviations T (tests positive), N (tests negative), D (has the disease), and H (is healthy). We want to find $P(D \mid T)$. From the data, we know

$$\begin{aligned} P(T \mid D) &= 0.9, & P(N \mid D) &= 0.1, \\ P(T \mid H) &= 0.1, & P(N \mid H) &= 0.9, \\ P(D) &= 0.01, & P(H) &= 0.99. \end{aligned}$$

So

$$\begin{aligned} P(D \mid T) &= \frac{P(T \mid D)P(D)}{P(T \mid H)P(H) + P(T \mid D)P(D)} \\ &= \frac{(0.9) \cdot (0.01)}{(0.1) \cdot (0.99) + (0.9) \cdot (0.01)} \\ &= \frac{0.009}{0.099 + 0.009} = \frac{0.009}{0.108} = 0.083. \end{aligned}$$

So, even if you test positive, it is still most unlikely that you have the disease. This example can be represented by the following box diagram:

	P	N	
D	0.009	0.001	0.01
H	0.099	0.891	0.99
	0.108	0.892	

Your Turn. A disease has infected 5% of the population. The test is 95% effective—95% of those with the disease test positive, and 95% of those without the disease test negative. If your test shows a positive result, what is the probability that you have the disease?

The Three Doors

Marilyn vos Savant's *Ask Marilyn* column in *Parade* magazine for September 9th, 1990 contained a puzzle that generated a lot of interest. It is in fact a version of an older problem called the *Monty Hall Problem*, named for the host of the game show *Let's Make a Deal*.

> The climax of a TV game show is run as follows. The contestant is given a choice of three numbered doors. Behind one closed door is a valuable new car; behind each of the others is a nearly worthless goat. The contestant is to choose a door, and wins the prize behind it. However, after the choice is announced but before the door is opened, the host opens one of the other two doors (not the one she chose) and reveals a goat. (Of course, the host knows where the car is.) He then asks, "Do you want to stay with your original choice? Or would you rather switch to the third door?"

Well, should the contestant stay or switch? Or doesn't it matter?

Before analyzing the problem, we need to agree on three points. First, the car is placed behind the doors at random, so that on any given night the chance that it is behind any particular door is $\frac{1}{3}$. Second, the game always proceeds in the same way: the host *always* opens a door to show a goat, then offers the switch. Third, on those nights when the contestant's first choice is the door with the car, there is an equal chance that the host will open either of the other two doors.

Without loss of generality, let us suppose the contestant chooses door 1 and the host opens door 2. We write $C1$, $C2$, and $C3$ as abbreviations for "the car is behind door 1", "the car is behind door 2", and "the car is behind door 3", and $H2$, $H3$ for "the host opens door 2", "the host opens door 3". (He cannot open door 1.) Then what we want to know is:

$$\text{Is } P(C3 \mid H2) > P(C1 \mid H2)?$$

If so the contestant should switch, otherwise not.

We know the probabilities of the host's actions, given the position of the car. If the car is behind door 1, she is equally likely to open either door, so

$$P(H2 \mid C1) = P(H3 \mid C1) = \tfrac{1}{2}.$$

In the other cases, she must open the remaining "goat" door, so

$$P(H2 \mid C2) = P(H3 \mid C3) = 0,$$
$$P(H2 \mid C3) = P(H3 \mid C2) = 1.$$

Moreover, we know $P(C1) = P(C2) = P(C3) = \frac{1}{3}$.

Now calculation of the probabilities is a simple application of Bayes' formula. First,

$$\begin{aligned} P(H2) &= P(H2 \mid C1)P(C1) + P(H2 \mid C2)P(C2) + P(H2 \mid C3)P(C3) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = \frac{1}{2}, \end{aligned}$$

and similarly $P(H3) = \frac{1}{2}$. So

$$P(C1 \mid H2) = \frac{P(H2 \mid C1)P(C1)}{P(H2)} = \frac{(\frac{1}{2})(\frac{1}{3})}{\frac{1}{2}} = \frac{1}{3},$$
$$P(C3 \mid H2) = \frac{P(H2 \mid C3)P(C3)}{P(H2)} = \frac{(1)(\frac{1}{3})}{(\frac{1}{2})} = \frac{2}{3}.$$

So the odds are 2 to 1 in favor of switching. You may find this very surprising—intuitively, you might argue that "the car was equally likely to be behind any of the doors, so the probabilities are still equal", but the host's choice of doors has actually given you some information.

The most important assumption in this discussion is that, when the contestant has chosen correctly, the host is equally likely to open either door. Suppose this were not true—for example, suppose he always chooses the lowest-numbered available door. Then if the contestant chooses door 1 and the host opens door 3, she should always switch; but if he opens door 2, there is no advantage (or disadvantage) in switching.

This problem is particularly well suited to analysis by box diagrams. Assuming that the contestant chooses door 1, then the fact that the car is equally likely to be behind any door means that the diagram looks like

	$C1$	$C2$	$C3$	
$H2$	$*$	$*$	$*$	$*$
$H3$	$*$	$*$	$*$	$*$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

(where asterisks represent the numbers that have not yet been determined). Since the host never opens the door that hides the car, the $(H2, C2)$ and $(H3, C3)$ entries must be zero, and to make the column sums come out right we have

	$C1$	$C2$	$C3$	
$H2$	$*$	0	$\frac{1}{3}$	$*$
$H3$	$*$	$\frac{1}{3}$	0	$*$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

Assuming the host chooses at random when the car is behind door 1, we can make the two missing numbers in column $C1$ equal, and we have

	$C1$	$C2$	$C3$	
$H2$	$\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{2}$
$H3$	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{2}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

When the host opens door 2, we need only look at the $H2$ row to see that the odds are 2 to 1 ($\frac{1}{3}$ to $\frac{1}{6}$) in favor of switching, and similarly if he opens door 3.

A Card Game

Suppose there are three cards in a hat. One is red on both sides, one black on both sides, and the third has one side red and one side black. One card is withdrawn, and one side is observed to be red. What is the chance that the other side is red?

At first it seems to be an even chance whether the other side is red or black; after all, there are two cards with at least one red side, and exactly one of the two is red on the flip side. However, the flip side is twice as likely to be red as black. Let r and b represent the events "red shows" and "black shows", while RR, RB, and BB represent the three cards: RR means "the RR card was drawn". Then

$$\begin{aligned} P(RR \mid r) &= \frac{P(r \mid RR)P(RR)}{P(r \mid RR)P(RR) + P(r \mid RB)P(RB)} \\ &= \frac{(1)(\frac{1}{3})}{(1)(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3})} = \frac{2}{3}. \end{aligned}$$

Alternatively, one could construct the box diagram

	RB	BB	RR	
r	$\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{2}$
b	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{2}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

and the problem is obviously equivalent to the "three doors" problem.

To understand the apparent paradox, it is easiest to think of the selection process as selecting a *face*, not a *card*. There are three red faces, and two of the three have red backs.

Exercises 3.8 A

1. Suppose the "three doors" game involved four doors, not three. If you decide to switch, you choose at random between the two remaining doors. What is the probability of winning if you switch? What is the probability of winning if you do not switch?

2. Among one strain of cattle in Texas, it is estimated that 30% of the bulls suffer from Ruckel's disease. A blood test has been developed to detect the disease. If a bull has the disease then the test gives a positive response 80% of the time. If a bull does not have the disease, the test is positive in 10% of cases. If a bull is tested for the disease and the test is positive, what is the probability that he really *does not* have the disease?

3. A vaccine produces total immunity against smallpox in 95% of cases. If the vaccination does not produce immunity, then it has no effect—the person's probability of catching smallpox is the same as that of an unvaccinated person. Suppose 30% of the population have been vaccinated. If a person contracts smallpox, what is the probability that she had been vaccinated?

4. A city is divided into three wards: North, South, and West. North ward contains 20% of the registered voters, and the others each have 40% each. The Green Party Committee finds that 10% of the voters in North ward, 5% of the voters in South ward, and 15% of the voters in West ward are Green Party voters.

(i) What is the probability that a registered voter chosen at random in the city is a Green voter?

(ii) A registered voter is chosen at random in the city and is found to be a Green voter. What is the probability that this voter came from South Ward?

5. A test for lupus gives a (true) positive reading in 95% of cases when the disease is present, and gives a (false) positive reading in 10% of cases when the disease is absent. In the population, suppose 2% have lupus.

(i) A subject is chosen at random and tests positive. What is the probability that this person has the disease?

(ii) A subject is chosen at random and tests negative. What is the probability that this person has the disease?

Exercises 3.8 B

1. Suppose the "three doors" game involved five doors, not three. What is the probability of winning if you switch? What is the probability of winning if you do not switch?

2. A test is positive for 94% of subjects who have a certain disease and negative for 96% of those who do not have the disease. If 4% of the population suffers from the disease, then

(i) If the test is positive, what is the probability that the patient actually has the disease?

(ii) If the test is negative, what is the probability that the patient does not have the disease?

3. A laboratory test for a particular disease tests positive 95% of the time when a person has the disease, and tests negative 98% of the time when the person does not have the disease. In general, 0.1% of the population has the disease.

(i) What is the probability that a randomly selected person who tested positive did not have the disease?

(ii) What is the probability that a randomly selected person who tested negative actually had the disease?

4. In order to introduce a new brand of laundry detergent, a marketing company conducted a survey. For the purposes of the survey, the United States was divided into four areas: northeast, southeast, midwest, and west. The company estimates that 35% of the potential customers are in the northeast region, 30% in the southeast, 20% in the midwest, and 15% in the west. The survey indicates that the following percentages of potential customers will try the new detergent: 50% in the northeast, 40% in the southeast, 30% in the midwest, and 40% in the west. A potential customer chosen at random indicates that she would try the detergent. What is the probability that she is from the southeast region?

5. A survey of 100 business professors and 100 science professors finds that 70 of the business professors and 35 of the science professors are overweight. A professor is chosen from the sample at random and found to be overweight. What is the probability that the one chosen was a business professor?

6. A magician puts three cards in a top hat. One is red on both sides, one black on both sides, and the third has one side red and one side black. One card is withdrawn, and one side—the first side you see—is observed to be red. What is the chance that the other side is red?

7. At Western University, 1% of the men and 4% of the women are English majors. Women comprise 40% of the student population.

(i) What proportion of students are English majors?

(ii) A student chosen at random is found to be an English major. What is the probability that this student is a woman?

8. Approximately 5% of men and 1% of women suffer from color-blindness.

(i) Assume the general population is 50% female and 50% male. A person is selected at random and found to be color-blind. What is the probability that the person was a man?

(ii) Your college has 75% women students. A student is selected at random and found to be color-blind. What is the probability that the student was a man? (Assume the percentages for color-blindness are the same as in the general population.)

9. A test for a disease gives a positive reading in $p\%$ of cases when the disease is present, and also gives a (false) positive reading in $p\%$ of cases when the disease is absent. By coincidence, exactly $p\%$ of the population suffer from the disease. Prove that if a person tests positive, the probability that she has the disease is 50%.

3.9 Expected Values

The Theoretical Mean

Suppose the following experiment is carried out 800 times: three coins are tossed, and the number of heads is recorded. If E_0, E_1, E_2, and E_3, respectively, are the events that 0, 1, 2, and 3 heads show on a given toss, then

$$\begin{aligned} P(E_0) &= 1/8, \qquad & P(E_1) &= 3/8, \\ P(E_2) &= 3/8, \qquad & P(E_3) &= 1/8. \end{aligned}$$

Since the chance of E_0 occurring at any turn is $1/8$, we expect it to occur about 100 times out of 800. Similarly, E_1, E_2, and E_3 should occur about 300, 300, and 100 times, respectively. We are not saying that *exactly* these frequencies would occur in 800 tosses; the chance of any specified result is very small. But we expect results roughly like those suggested.

This same reasoning can be applied to any experiment. If the possible outcomes are $E_1, E_2, \ldots, E_k$, then the expected outcomes from n experiments are

$$\begin{aligned} &E_1 \text{ occurs } nP(E_1) \text{ times,} \\ &E_2 \text{ occurs } nP(E_2) \text{ times,} \\ &\vdots \\ &E_k \text{ occurs } nP(E_k) \text{ times.} \end{aligned}$$

In many cases, there is a numerical *value* associated with each outcome. In the coin-tossing example, an obvious value is the number of heads. If we write x_i for the value of E_i in that example, then $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$. If the experiment were carried out 800 times, we would expect the average value of the 800 outcomes to be about

$$\frac{100 \cdot 0 + 300 \cdot 1 + 300 \cdot 2 + 100 \cdot 3}{100 + 300 + 300 + 100}$$
$$= \frac{300 + 600 + 300}{800}$$
$$= \frac{1200}{800} = 1.5.$$

(The procedure is to find the mean, acting as though the expected frequencies were the actual frequencies that occurred.) We shall call this expected average the *expected value of the experiment*. The expected value might be called the *theoretical mean* of the experiment. We shall use μ to denote a theoretical mean; if more than one variable is being discussed, the mean of a variable X will be denoted μ_X.

In general, suppose an experiment has possible outcomes $E_1, E_2, \ldots, E_k$, where E_i has probability $P(E_i) = p_i$ and associated value x_i. If the experiment is repeated n times, the expected frequency of E_i is np_i, and the expected value of the experiment is

$$\mu = \frac{\sum_{i=1}^{k} np_i x_i}{\sum_{i=1}^{k} np_i} = \frac{\sum_{i=1}^{k} p_i x_i}{\sum_{i=1}^{k} p_i}$$
$$= \sum_{i=1}^{k} p_i x_i \quad \left(\text{since} \sum_{i=1}^{k} p_i = 1\right).$$

Sample Problem 3.35. *In an experiment, three balls are drawn at random from a sack containing four red and six blue balls. What is the expected number of red balls drawn?*

Solution. Write E_i for the event that i red balls are drawn and define the value of E_i to be i. What is required is the expected value of this experiment. If S is the sample space, then

$$|S| = C(10, 3) = 120.$$

If a selection contains i red balls, it must contain $3-i$ blue balls. There are $C(4, i)$ ways of selecting i red balls from the four, and $C(6, 3-i)$ ways of selecting $3-i$ blue balls, so

$$|E_i| = C(4, i)C(6, 3 - i).$$

Therefore,

$$|E_0| = C(4, 0) \cdot C(6, 3) = 20,$$
$$|E_1| = C(4, 1) \cdot C(6, 2) = 60,$$
$$|E_2| = C(4, 2) \cdot C(6, 1) = 36,$$
$$|E_3| = C(4, 3) \cdot C(6, 0) = 4,$$

so

$$p_0 = \frac{20}{120}, \qquad p_1 = \frac{60}{120}, \qquad p_2 = \frac{36}{120}, \qquad p_3 = \frac{4}{120}$$

and

$$\begin{aligned} \mu &= \frac{20 \cdot 0 + 60 \cdot 1 + 36 \cdot 2 + 4 \cdot 3}{120} \\ &= \frac{144}{120} = 1.2. \end{aligned}$$

So the expected number is 1.2.

Your Turn. A committee of three is selected from a group of five girls and three boys. If the choice is random, what is the expected number of boys?

Random Variables

When an experiment has a value associated with each outcome, it is convenient to associate a variable x with the experiment. If, for example, the value associated with outcome E_1 is x_1, then we say that $x = x_1$ when E_1 occurs; at any time, x equals the value associated with the current outcome. Then x is called the *random variable* associated with the experiment. The expected value of an experiment with associated random variable x is also called the expected value or mean of x.

In many gambling games, the player pays a fixed amount to play. The play can often be considered as an experiment, with the associated random variable being the payoff. The player's expected net loss or gain is found by subtracting the cost of a play from the expected value of the variable. A *fair* game is one in which the expected net gain equals the price per play.

Sample Problem 3.36. *A gambler flips an unbiased coin. If it shows tails, she loses. If it is a head, she flips again; if it falls tails she stops, but if it falls heads a second time, she flips once more. For three heads she wins* \$20, *two heads wins* \$6, *and one head wins* \$2. *How much does she expect to win? If she has to pay* \$4 *each time she plays this game, should she expect to win or lose in the long run? How much should be charged if this is to be a fair game?*

Solution. Let E_i denote the event that i heads are thrown, and x_i the payout for i heads. Half the time we expect the first flip to fall tails, so E_0 has probability 0.5. In the remaining cases, we expect to see tails at the second toss in 50% of cases, so $P(E_1) = 0.5 \cdot 0.5 = 0.25$. And so on. We have

$$\begin{aligned} &E_0, \text{ probability } 0.5, \text{ value } x_0 = 0, \\ &E_1, \text{ probability } 0.25, \text{ value } x_1 = 2, \\ &E_2, \text{ probability } 0.125, \text{ value } x_2 = 6, \\ &E_3, \text{ probability } 0.125, \text{ value } x_3 = 20. \end{aligned}$$

So the expected value of the payout is

$$0.5 \cdot 0 + 0.25 \cdot 2 + 0.125 \cdot 6 + 0.125 \cdot 20 = 3.75.$$

So she expects to win \$3.75 per play. If she pays \$4 each time, she will have a net loss of 0.25 cents per play, so she expects to lose in the long run. If the game is to be fair, she should be charged \$3.75 per play.

Your Turn. Two gamblers, A and B, each roll a die. If the total is 3 or 9, A pays B \$5; if it is even, B pays A \$2; otherwise there is no payoff. Who expects to win, and by how much?

Binomial Variables

Bernoulli trials and binomial experiments were discussed in Section 3.3. Briefly, a Bernoulli trial is an experiment in which there are exactly two possible outcomes, success and failure, and a binomial experiment is a sequence of Bernoulli trials where the probability of success is the same in each trial. The variable associated with a binomial experiment is the number of successes achieved. We write p for the probability of success in a single trial and $q = 1 - p$ for the probability of failure.

Theorem 8. *Suppose a binomial experiment consists of n trials, and the probability of success at each trial is p. Then the mean of the experiment is np.*

The proof appears later in this section.

Sample Problem 3.37. *A racecar driver wins on average one out of every seven races. If he competes 35 times in the season, how many times does he expect to win?*

Solution. With no information to the contrary, we assume that success or failure in one race does not affect success or failure in the next. So we can model the driver's performance as a binomial experiment, in which each race is a Bernoulli trial and success equates with winning. In this model we have $n = 35$, $p = \frac{1}{7}$, and $\mu = 35 \cdot \frac{1}{7} = 5$.

Your Turn. A bowler averages one strike every five frames. If she bowls 40 frames, what is her expected number of strikes?

The Binomial Mean

This proof can be omitted at a first reading.

We saw in Section 3.3 that the probability of exactly k successes in a binomial experiment is

$$p_k = C(n,k)p^k(1-p)^{n-k}.$$

So the mean is

$$\mu = \sum_{k=1}^{n} kp_k = \sum_{k=1}^{n} kC(n,k)p^k(1-p)^{n-k}.$$

(The sum actually starts from $k = 0$, but the term with $k = 0$ is zero.)

$$\begin{aligned} kC(n,k)p^k(1-p)^{n-k} &= \frac{n!k}{k!(n-k)!}p^k(1-p)^{n-k} \\ &= \frac{n!}{(k-1)!(n-k)!}p^k(1-p)^{n-k} \\ &= n\frac{(n-1)!}{(k-1)!(n-k)!}p^k(1-p)^{n-k} \\ &= npC(n-1,k-1)p^{k-1}(1-p)^{n-k}. \end{aligned}$$

So

$$\mu = \sum_{k=1}^{n} npC(n-1,k-1)p^{k-1}(1-p)^{n-k}.$$

Suppose we define $h = k - 1$. The equation becomes

$$\mu = np\sum_{h=0}^{n-1} C(n-1,h)p^h(1-p)^{(n-1)-h}.$$

From the binomial theorem,

$$\sum_{h=0}^{n-1} C(n-1,h)p^h(1-p)^{(n-1)-h} = \left[(1-p)+p\right]^{n-1},$$

which equals 1 because $(1 - p) + p = 1$. So

$$\mu = np.$$

Exercises 3.9 A

1. A fair coin is tossed three times and the number of heads is recorded. Suppose x is the number of heads.

(i) Calculate $P(x = 0)$, $P(x = 1)$, $P(x = 2)$, and $P(x = 3)$.

(ii) What is the expected value of x?

2. A game has four possible outcomes E_1, E_2, E_3, E_4. If outcome E_i occurs then the player wins \$i. The probabilities of the outcomes are $P(E_1) = \frac{2}{5}$, $P(E_2) = P(E_3) = P(E_4) = \frac{1}{5}$. What does the player expect to win per play?

3. You play a game in which you toss two fair coins. If both land heads, you win \$10; if there is one head, you win \$5; if both are tails, you win nothing. How much should you pay to play, if this is to be a fair game?

4. A player tosses a fair coin 10 times. Suppose x is the number of heads. If x is even, the player wins \$10; if it is odd, she loses \$10. What is the player's expected winnings?

5. An investor wishes to invest \$100000 in a new dotcom company. She estimates that in the first year there is a 25% chance of making 100% profit, a 40% chance of making 50% profit, and a 35% chance of losing 80% of the investment. What is her expected gain (or loss) after one year?

6. A fair die is rolled 30 times. What is the expected number of sixes?

7. A test contains 10 multiple-choice questions, each with four answers. A student answers the test by random guessing. What is the expected number correct?

8. Two dice are rolled and the total is noted. Call the roll a "success" if the total is 9.

(i) What is the probability of a success?

(ii) The experiment is performed twice. Calculate the probability of x successes, for $x = 0, 1, 2$.

(iii) What is the expected value of x?

9. In a raffle with 100000 tickets the first prize is \$20000, there are 10 second prizes of \$1000, and 50 consolation prizes of \$200. What is the expected value of a ticket?

10. A gambler bets \$6 and rolls a fair die. If a six shows, he wins \$9 (in addition to getting his original investment back). If a five or four shows, his money is returned with an additional \$3. Otherwise he loses his bet. How much does he expect to win or lose per game?

Exercises 3.9 B

1. Among one strain of laboratory rats, 25% are born with weak hind legs. Let x denote the number with this defect in a random sample of three rats.

(i) Calculate the probabilities $P(x = 0)$, $P(x = 1)$, $P(x = 2)$, and $P(x = 3)$.

(ii) What is the expected value of x?

2. A game has four possible outcomes E_1, E_2, E_3, E_4. If outcome E_i occurs then the player wins $\$i$. The probabilities of the outcomes are $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{4}$, $P(E_3) = P(E_4) = \frac{1}{8}$. What does the player expect to win per play?

3. In a raffle with 1000 tickets the first prize is \$200, there are five second prizes of \$50, and 10 consolation prizes of \$5. What is the expected value of a ticket?

4. A gambler bets \$2 and tosses three fair coins. If all three land heads, she wins \$6 (in addition to getting her original bet back). If there are two heads, her money is returned. Otherwise she loses her \$2. How much does she expect to win or lose per game?

5. A fair coin is tossed 240 times. What is the expected number of heads?

6. A jar contains three red and seven green marbles. Three marbles are drawn at random, without replacement. E_i is the event that i red marbles are selected.

(i) Calculate $P(E_1)$, $P(E_2)$, and $P(E_3)$.

(ii) What is the expected number of red marbles?

7. 1% of video cassettes are faulty. In a batch of 600, how many are expected to be faulty?

8. A state lottery sells 100000 tickets at \$1 each. The first prize is \$30000. There are four prizes of \$4000, and 100 prizes of \$100. What is the expected value of a ticket?

9. Most American roulette wheels have 38 numbered spaces: 1 through 36, 0, and 00. A number is chosen at random when the ball falls into that slot. Payoffs for bets are determined as though there were only 36 spaces—for example, if a player bets \$10 that a specific number will be selected, say 15, and 15 is the winner, she receives \$360 (including her initial bet); otherwise she loses her investment.

(i) What is the expected value of a \$10 bet on a specific number?

(ii) Wheels at Monte Carlo usually have 37 slots, with no 00. The return is the same as in America. What is the expected value in this case?

10. On an American roulette wheel, as described in the previous question, the 0 and 00 spaces are colored green; half the others are colored red, and the remainder are black. It is possible to bet "red"; if a number in a red space is rolled, you receive your initial bet plus an equal amount. What is the expected value of a \$10 bet?

11. A drug manufacturer finds that 5% of its aspirin bottles contain more than the promised 100 tablets. In a batch of 10000 bottles, what is the expected number of those "overfull" bottles?

12. A box contains eight electrical switches, of which three are faulty. A switch is chosen at random and tested. If it is faulty, another is chosen and tested. The process continues until a nondefective switch is found. find the expected number of switches tested.

13. You play a game in which you toss three fair coins. If all three land heads, you win \$18; if there is one head, you lose \$6; if all three are tails, you lose \$24. There is no payoff for two heads.

(i) What is the expected value of this game?
(ii) Is the game fair?
(iii) If your answer to part (ii) is *no,* how much should you win or lose if two heads occur, in order to make the game fair?

14. Suppose a family has six children. Assuming boys and girls are equally likely, what is the expected number of girls in the family? What is the probability that this number occurs?

15. A coin is weighted so that heads are twice as likely as tails. It is tossed until either a tail occurs or three heads have occurred. What is the expected number of tosses?

4

Graph Theory

4.1 Relations

First Definitions

A (*binary*) *relation* ρ from a set S to a set T is a rule that stipulates, given any element s of S and any element t of T, whether s bears a certain relationship to t (written $s\,\rho\,t$) or not (written $s \not\rho\, t$). For example, if S is the set of living males and T is the set of living females, the relation ρ might be "is the son of"; if s denotes a certain man and t denotes a certain woman, we write $s\,\rho\,t$ if s is the son of t, and $s \not\rho\, t$ otherwise.

Binary relations are very common in mathematics. We have already seen relations such as $\le$, $=$, and $<$, and all of these are relations from $\mathbb{R}$ to $\mathbb{R}$ (or from $\mathbb{Z}$ to $\mathbb{Z}$, or ...). The relation $|$, or "divides," is a relation from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, or from $\mathbb{Z}^+$ to $\mathbb{Z}$. If $\mathcal{P}$ and $\mathcal{Q}$ are any collections of sets, then $\subseteq$ and $\supseteq$ are relations from $\mathcal{P}$ to $\mathcal{Q}$.

Alternatively, we can define a binary relation ρ from the set S to the set T as a set ρ of ordered pairs (s, t), where s belongs to S and t belongs to T, with the notation that $s\,\rho\,t$ when $(s, t) \in \rho$ and $s \not\rho\, t$ otherwise. This means that, formally, a binary relation from S to T can be defined as a subset of $S \times T$, the Cartesian product of S and T defined in Section 1.4.

Sample Problem 4.1. *Suppose* $S = \{1, 2, 3\}$ *and* $T = \{1, 2, 3, 4\}$. *Find the sets corresponding to* $s < t$ *and* $s^2 = t$, *in the usual arithmetical sense.*

Solution. The relation $s < t$ corresponds to the set L, defined by

$$L = \big\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\big\}.$$

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1_4,

$s^2 = t$ has

$$R = \{(1,1), (2,4)\}$$

as its the corresponding set.

Your Turn. What are the sets corresponding to $s \le t$ and $s \ge t$?

Relations on a Set

Suppose α is a binary relation from a set A to itself or, in other words, $\alpha \subseteq A \times A$. Then we say that α is a binary relation *on* A. One special relation on a set is the *identity* relation: ι_A is the identity relation on A if $a\,\iota_A\,b$ is true precisely when $b = a$, that is, $\iota_A = \{(a, a) : a \in A\}$. If A is a set of numbers, ι_A is the same as the ordinary relation of equality, =.

There are several important properties that a relation on a set may (or may not) have. Suppose α is a relation on α.

(i) α is called *reflexive* if and only if $a\,\alpha\,a$ for every $a \in A$. α is *irreflexive* if and only if $a\,\alpha\,a$ is *never* true for any element a of A.

(ii) α is called *symmetric* if and only if $a\,\alpha\,b$ implies $b\,\alpha\,a$ or, in other words, if $(a, b) \in \alpha$ implies $(b, a) \in \alpha$. Since, by the definition of inverse relation, we know that $a\,\alpha\,b$ if and only if $b\,\alpha^{-1}\,a$, we could say that α is symmetric if and only if $\alpha = \alpha^{-1}$. α is called *antisymmetric* if and only if $a\,\alpha\,b$ and $b\,\alpha\,a$ together imply $a = b$ or, in other words, $\alpha \cap \alpha^{-1} \subseteq \iota_A$. If $a\,\alpha\,b$ and $b\,\alpha\,a$ can never both be true, then α is called *asymmetric*, or sometimes *skew*. (If $a\,\alpha\,b$ and $b\,\alpha\,a$ are never both true, then the truth of $(a\,\alpha\,b \wedge b\,\alpha\,a)$ implies that $a\,\alpha\,a$ is vacuously true, so all asymmetric relations are antisymmetric, but this gives us no information.)

(iii) α is called *transitive* if and only if $a\,\alpha\,b$ and $b\,\alpha\,c$ together imply $a\,\alpha\,c$, and *atransitive* if and only if, whenever $a\,\alpha\,b$ and $b\,\alpha\,c$, $a \not\alpha\, c$, or in other words, if and only if $\alpha \cap \alpha\alpha = \emptyset$.

In particular, a relation is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Sample Problem 4.2. *What are the reflexive, symmetric, and transitive properties of $<$ and $\le$ on the set $\mathbb{Z}$? Is either relation an equivalence relation?*

Solution. $<$ is irreflexive ($a < a$ is never true) while $\ge$ is reflexive ($a \le a$ is always true). Both relations are antisymmetric, but $<$ is in fact asymmetric. Both are transitive. Neither is an equivalence relation.

Your Turn. What are the reflexive, symmetric and transitive properties of $=$ and $\geq$ on $\mathbb{Z}$? Is either relation an equivalence relation?

The relation of *adjacency* on the set of integers, defined by $i\,\alpha\,j$ if and only if $|i-j|=1$ for i and j positive integers, is irreflexive (since $|i-i|=0$), symmetric (since $|i-j|=|j-i|$) and not transitive (since $1\,\alpha\,2$ and $2\,\alpha\,3$ but $1\not\alpha\,3$). It is in fact atransitive, for $i\,\alpha\,j$ and $j\,\alpha\,k$ together imply $i=j\pm 1$ and $k=j\pm 1$, so $i=k$ or $i=k\pm 2$, but i cannot equal $c\pm 1$. So $a\not\alpha\,c$.

The properties of relations can be exhibited even in quite small cases. For example, say $A=\{1,2,3\}$ and consider the relation β on A, defined by

$$\beta=\big\{(1,1),(2,2)\big\}.$$

Since $\iota_A\not\subseteq\beta$, it is not reflexive, but since $\iota_A\cap\beta\neq\emptyset$, it is not irreflexive either. Since $\beta=\beta^{-1}$, it is symmetric, and since $\beta\cap\beta^{-1}\subseteq\iota A$, it is also antisymmetric. Since $\beta\beta=\beta$, it is transitive.

Functions

A *function*, or *mapping*, f from a set S to a set T, is a rule that associates with each member of S exactly one member of T. In other words, a function is a special kind of relation from S to T, one in which each element of S occurs as the first element of *precisely one* ordered pair. This uniqueness property is the main point of the idea of a function.

The set S is called the *domain* of the function, and the set T its *codomain*. If we used notation consistent with that of the last section, we would write $f\subseteq S\times T$, but more often when dealing with functions we write $f:S\to T$; the set form is then called the *defining set* or *associated set* of f. To indicate the correspondence between elements, we could again write $(s,t)\in f$, but more often we write $f(s)=t$, as, for example, in familiar algebraic functions like $f(x)=x^3-3x+1$.

The *image* of x under f is the value $f(x)$; if R is any subset of S, then $f(R)=\{f(r):r\in R\}$ is the set of images of elements of R and is called the *image* of R. In particular $f(S)$, the image of S, is called the *range* of f and necessarily $f(S)\subseteq T$. To avoid trivial cases, it is usual to require that the domain of f should not be empty.

The *composition* of two functions is defined just as the composition of relations is. In functions, composition is defined as "first perform one function, then the other" (where *performing* f means replacing the value x by $f(x)$). The composition "first f, then g" is written $g(f(x))$, and this can be interpreted as defining a new function $g(f)$, by the rule

$$g(f)(x)=g\big(f(x)\big).$$

In order for $g(f)(x)$ to be defined, it is necessary that every value of $f(x)$ be in the domain of g. This will not necessarily be true for all x in $\mathcal{D}(f)$; in fact, the domain of $g(f)$ is the set of all elements x in the domain of f such that $f(x)$ is in the domain of g.

This method of combining functions is *not* commutative; quite possibly $g(f)$ and $f(g)$ will be different functions. In fact, one might be defined and the other not. However, composition is associative.

Sample Problem 4.3. *S is the set of all real numbers other than 0 and 1. Consider the following functions from S to S:*

$$f_1(x) = x; \qquad f_2(x) = \frac{1}{x}; \qquad f_3(x) = 1 - x;$$
$$f_4(x) = \frac{1}{1-x}; \qquad f_5(x) = \frac{x}{x-1}; \qquad f_6(x) = \frac{x-1}{x}.$$

Find formulae for $f_2(f_2)$, $f_3(f_5)$, and $f_5(f_3)$, and verify that they are all members of the original set of six functions. Use these to verify that composition is not commutative.

Solution.

$$f_2(f_2)(x) = f_2(1/x) = \frac{1}{1/x} = x = f_1(x);$$
$$f_3(f_5)(x) = f_3\big(x/(x-1)\big) = 1 - \frac{x}{x-1} = \frac{1}{1-x} = f_4(x);$$
$$f_5(f_3)(x) = f_5(1-x) = \frac{1-x}{1-x-1} = \frac{x-1}{x} = f_6(x).$$

So $f_2(f_2) = f_1$, $f_3(f_5) = f_4$, and $f_5(f_3) = f_6$.

Observe that $f_5(f_3) \neq f_3(f_5)$, an example of non-commutativity.

Your Turn. Find formulae for $f_1(f_1)$, $f_2(f_4)$, and $f_5(f_6)$, and verify that they are all members of the original set of six functions.

One-to-One and Onto Functions

Two properties of functions are particularly important: we say that a function f is *one-to-one* if $f(s_1) = f(s_2)$ always implies $s_1 = s_2$ for any $s_1, s_2 \in \mathcal{D}(f)$; we say that f is *onto* if $f(S) = T$ or, in other words, if for every $t \in T$, there exists an $s \in S$ such that $f(s) = t$.

These properties depend on the domain of f. For example, consider the function f defined by $f(x) = x^2$ or $f = \{(x, x^2) : x \in \mathbb{R}\}$, which associates its square with each real number. This satisfies the definition of function, for if we choose x, then x^2 is uniquely determined.

(i) If we define the domain of f to be the set $\mathbb{R}$ of all real numbers, and the codomain to be $\mathbb{R}$ also, then f is neither one-to-one (since $f(x) = f(-x)$) nor onto (since $f(\mathbb{R}) = \{y \in \mathbb{R} : y \geq 0\}$).

(ii) If instead we define the domain of f to be the set of all non-negative real numbers, but let the codomain still be the whole of $\mathbb{R}$, then f is one-to-one but is not onto.

(iii) If we let the domain of f be $\mathbb{R}$ but restrict the codomain to be the set of all non-negative real numbers, then f is onto, but is not one-to-one.

(iv) If both the domain and the codomain are defined to be the set of non-negative reals, then f is both one-to-one and onto.

Inverse Functions

Given any set S, we can define the identity function $\iota_S : S \to S$ by $\iota_S(s) = s$ for every $s \in S$; in other words,

$$\iota_S = \big\{(s, s) : s \in S\big\}.$$

For a function, as for any relation, we can define an inverse relation. If the function is, say $f(x) = x^2$, and if we let the domain and the codomain equal $\mathbb{R}$, then the inverse relation f^{-1} contains the pairs (4, 2) and (4, −2), so it is not a function: $f^{-1}(x)$ would not be uniquely defined. This raises an important question: when is the inverse relation of a function also a function? The following theorem gives the necessary and sufficient conditions. You may omit the proof on a first reading.

Theorem 9. *Let f be a function, $f : S \to T$. Then the inverse relation f^{-1} of f is a function from T to S if and only if f is one-to-one and onto.*

Proof. (i) Suppose f is one-to-one and onto. Since f is one-to-one, each element of T occurs as the second element of at most one ordered pair in f; since f is onto, each element of T occurs as the second element of at least one ordered pair in f. Hence if $t \in T$, exactly one ordered pair $(s, t) \in f$.

Now consider the inverse relation: $(t, s) \in f^{-1}$ exactly when $(s, t) \in f$. Hence for every $t \in T$, exactly one ordered pair $(t, s) \in f^{-1}$, so f^{-1} is a function from T to S.

(ii) Suppose conversely that f^{-1} is a function from T to S. Then for every $t \in T$, exactly one ordered pair $(t, s) \in f^{-1}$ and hence exactly one ordered pair $(s, t) \in f$. Since at least one ordered pair has t as its second element for each $t \in T$, we see that f is onto; since at most one ordered pair has t as its second element for each $t \in T$, we see that f is one-to-one. □

Notice that the function f_1 in Sample Problem 4.3 is the identity function on the set S, and in that problem we showed that f_2 is its own inverse.

Sample Problem 4.4. *What is the inverse of the function f_4 from Sample Problem 4.3?*

Solution. $f_4(x) = 1/(1-x)$. Say $y = 1/(1-x)$. Then $1-x = 1/y$ so $x = 1 - 1/y = (y-1)/y$. So $f_4^{-1}(y) = (y-1)/y$. So $f_4^{-1} = f_6$.

Your Turn. What is the inverse of the function f_3 from Sample Problem 4.3?

Exercises 4.1 A

1. Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In each part, a binary relation from S to itself is defined. For each of these relations, what is the corresponding subset of $S \times S$?

(i) $x \,\alpha\, y$ means $x = y^2$.
(ii) $x \,\beta\, y$ means x divides y.
(iii) $x \,\gamma\, y$ means $x^2 + y = 12$.

2. Relations α and β are binary relations from $S = \{1, 2, 3\}$ to $T = \{1, 2, 3, 4\}$ defined by

$x \,\alpha\, y$ means $x = y^2$;
$x \,\beta\, y$ means $x = y - 1$.

For each of these relations, what is the corresponding subset of $S \times T$?

3. The relations α, β and γ on the set $\{1, 2, 3\}$ are defined by the sets

$$\alpha = \big\{(1,1), (2,2), (3,3)\big\};$$
$$\beta = \big\{(1,3), (2,2), (3,1)\big\};$$
$$\gamma = \big\{(2,3), (1,2), (1,3), (3,2)\big\}.$$

Which of these relations are:

(i) Reflexive; (ii) Irreflexive; (iii) Symmetric;
(iv) Antisymmetric; (v) Asymmetric; (vi) Transitive?

4. The following are relations from the set $\mathbb{Z}^+$ of positive integers to itself. Determine whether each relation has each of the following properties: reflexivity, symmetry, antisymmetry, transitivity:

(i) $m + n \geq 50$; (ii) $m + n$ is odd;
(iii) mn is odd; (iv) $m^2 - n^2$ is even.

5. Which, if any, of the relations in Exercises 3 and 4 are equivalence relations?

6. Suppose $S = \{1, 2, 3, 4\}$. Is the given set a function from S to S?

(i) $\{(1,4), (2,3), (3,2), (2,1), (4,4)\}$;
(ii) $\{(2,1), (3,4), (1,4), (2,1), (4,2)\}$;

(iii) $\{(1, 4), (2, 1), (3, 4), (4, 1), (3, 4)\}$;
(iv) $\{(1, 2), (2, 2), (3, 1), (4, 2)\}$.

7. Find the inverses of the following functions.

(i) $f_6 : S \to S$ from Sample Problem 4.3, where S is the set of all real numbers other than 0 and 1.

(ii) $f : \mathbb{R} \to \mathbb{R}^+$, where $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers and

$$f(x) = \begin{cases} 2 - x & \text{if } x \le 1, \\ 1/x & \text{if } x > 1. \end{cases}$$

8. In each case, is the function one-to-one? Is it onto?

(i) $f : K \to K$ where $K = \{1, 2, \ldots, k\}$ and $f(1) = k$, $f(j) = j - 1$ when $j = 2, 3, \ldots, k$.

(ii) $f : \mathbb{Z}^+ \to \{0, 1, 2, 3\}$, where $\mathbb{Z}^+$ is the set of positive integers, and

$$f(j) = \begin{cases} 0 & \text{if } 7 \nmid j, 3 \mid j, \\ 1 & \text{if } 7 \mid j, 3 \nmid j, \\ 2 & \text{if } 21 \mid j, \\ 3 & \text{otherwise.} \end{cases}$$

(iii) $f : \mathbb{R} \to \mathbb{R}^+$, where $f(y) = y^2$.

Exercises 4.1 B

1. Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In each part, a binary relation from S to itself is defined. For each of these relations, what is the corresponding subset of $S \times S$?

(i) $x \,\alpha\, y$ means $x + y = 9$.
(ii) $x \,\beta\, y$ means $2x < y$.
(iii) $x \,\gamma\, y$ means $x - y = 0$.
(iv) $x \,\delta\, y$ means $x - y > 5$.
(v) $x \,\varepsilon\, y$ means $2x = 3y$.
(vi) $x \,\xi\, y$ means $x^2 < y - 3$?
(vii) $x \,\eta\, y$ means $x^2 > y + 6$.

2. Relations α and β are binary relations from $S = \{1, 2\}$ to $T = \{1, 2, 3\}$ defined by

$x \,\alpha\, y$ means $y = x^2 - 1$;
$x \,\beta\, y$ means $xy > y$.

For each of these relations, what is the corresponding subset of $S \times T$?

3. Relations α and β are binary relations from $S = \{1, 2, 3\}$ to $T = \{1, 2, 3, 4\}$ defined by

$x \,\alpha\, y$ means $y = x^2$;
$x \,\beta\, y$ means $x = y + 1$.

For each of these relations, what is the corresponding subset of $S \times T$?

4. A group of people, denoted by $A, B, \ldots,$ are at a party. Consider the following relations: is each reflexive, symmetric, antisymmetric, asymmetric or transitive?

(i) $A\,\alpha\,B$ means A and B spoke to each other at the party.
(ii) $A\,\beta\,B$ means A spoke to B at the party.
(iii) $A\,\gamma\,B$ means A and B come from the same city.
(iv) $A\,\delta\,B$ means A and B met for the first time at the party.
(v) $A\,\varepsilon\,B$ means A is the father of B.

5. The following are relations from the set $\mathbb{Z}^+$ of positive integers to itself. Determine whether each relation has each of the following properties: reflexivity, symmetry, antisymmetry, transitivity.

(i) m is divisible by n.
(ii) $m + n$ is even.
(iii) mn is even.
(iv) $3 \mid (m + n)$.
(v) $mn > m + n$.

6. Is the given relation α an equivalence relation on the given set S?

(i) $S = \mathcal{R}$, $a\,\alpha\,b$ means $ab \geq 0$.
(ii) $S = \mathcal{R}$, $a\,\alpha\,b$ means $ab > 0$.
(iii) S is the set of all non-zero real numbers, $a\,\alpha\,b$ means $ab > 0$.
(iv) $S = \mathbb{Z}^+$, $a\,\alpha\,b$ means a and b have greatest common divisor 16.

7. Which, if any, of the relations in Exercises 5 and 4 are equivalence relations?

8. The relation α from the set $\mathbb{Z}^+$ of positive integers to itself is defined by

$$r\,\alpha\,s \text{ means } rs \text{ is a perfect square.}$$

Prove that α is an equivalence relation.

9. Suppose $S = \{1, 2, 3, 4\}$. Is the given set a function from S to S?

(i) $\{(1, 1), (3, 1), (4, 3)\}$;
(ii) $\{(1, 4), (2, 4), (3, 3), (4, 2)\}$;
(iii) $\{(1, 1), (3, 1), (1, 2), (4, 2)\}$;
(iv) $\{(2, 2), (1, 2), (2, 1), (3, 4), (4, 2)\}$;
(v) $\{(1, 1), (2, 1), (3, 1), (4, 3)\}$;
(vi) $\{(1, 1), (2, 1), (4, 1)\}$;
(vii) $\{(1, 4), (2, 1), (3, 3), (4, 2)\}$;
(viii) $\{(1, 1), (3, 1), (1, 2), (4, 2), (1, 2)\}$;
(ix) $\{(2, 2), (1, 2), (2, 1), (1, 1)\}$.

10. In each case, is the function one-to-one? Is it onto?

(i) $f : K \to K$ where $K = \{1, 2, \ldots, k\}$ and $f(j) = j + 1$, for $j = 1, 2, \ldots, k-1$, $f(k) = 1$.

(ii) $f : \mathbb{Q} \to \mathbb{Q}$, where $f(q) = q^3$ for $q \in \mathbb{Q}$. ($\mathbb{Q}$ denotes the set of rational numbers.)

(iii) $f : \mathbb{R}^+ \to \mathbb{R}^+$ where $f(x) = \sqrt{x}$.

(iv) $f : \mathbb{R} \to \mathbb{R}^*$, where $f(y) = y^2 + 1$.

11. Find the inverses of the following functions.

(i) $f_5 : S \to S$ from Sample Problem 4.3, where S is the set of all real numbers other than 0 and 1.

(ii) $f_3 : S \to S$ from Sample Problem 4.3, where S is the set of all real numbers other than 0 and 1.

(iii) $g : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and

$$g(x) = \begin{cases} 2x & \text{if } x < 0, \\ 3x & \text{if } x \geq 0. \end{cases}$$

(iv) $g : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and

$$g(x) = \begin{cases} x & \text{if } x < 0, \\ 2\sqrt{x} & \text{if } 0 \leq x < 1, \\ x + 1 & \text{if } x \geq 1. \end{cases}$$

4.2 Graphs

Graphs and Multigraphs

Suppose we want to describe the roads joining various towns in order to plan an itinerary. If our only desire is to list the towns that will be visited, in order, we do not need to know the physical properties of the region, such as hills, whether different roads cross, or whether there are overpasses. The important information is whether or not there is a road joining a given pair of towns. In these cases, we could use a complete road map, with the exact shapes of the roads and various other details shown, but it would be less confusing to make a diagram as shown in Figure 4.1(a), that indicates roads joining B to C, A to each of B and C, and C to D, with no direct roads joining A to D or B to D.

In some cases, there is a choice of routes between two towns. This information may be important: you might prefer the freeway, which is faster, or you might prefer the more scenic coast road. In this case, the diagram does not contain enough information. For example, if there had been two different ways to travel from B to C, this information could be represented as in the diagram of Figure 4.1(b).

Sample Problem 4.5. *Suppose the road system connecting towns A, B, and C consists of two roads from A to B, one road from B to C, and a bypass road directly from A to C. Represent this road system in a diagram.*

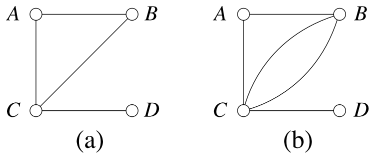


Fig. 4.1. Representing a road system

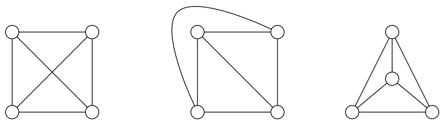
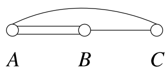


Fig. 4.2. Three representations of the same graph

Solution.



Your Turn. Draw a diagram to represent the road system connecting towns A, B, C, D, with one road from A to B, one from A to C, two from A to D, and one from B to C.

The sort of diagram shown in Figure 4.1(a) is called a *linear graph*, or usually simply called a *graph*. The dots representing towns are called *vertices*, while the lines representing the road links are called *edges*.

We make a formal definition, and set up some notation, as follows: a *graph* G consists of a finite set V or $V(G)$ of objects called *vertices*, together with a set $E(G)$ of pairs of vertices that are called *edges*. The two vertices belonging to an edge are called its *endpoints*; we say they are *adjacent*, and say the edge *joins* them.

In the diagram representing a graph, an edge $\{x, y\}$ is shown as a line from x to y. This is usually drawn as a straight line segment, but not always; the only important thing is that it joins the correct points. Moreover, the position of the points is not fixed. For these two reasons, one graph can give rise to several drawings that look quite dissimilar. For example, the three diagrams in Figure 4.2 all represent the same graph. Although the two diagonal lines cross in the first picture, their point of intersection does not represent a vertex of the graph.

The definition of a graph does not allow for two edges with the same endpoints. Either a pair of vertices is an edge or it is not. However, there are some situations where we wish to represent several different links between two vertices. The road system modeled in Figure 4.1(b) is an example. The lines representing the two roads from B to C in that figure will be called *multiple edges*. We say there is a multiple edge of *multiplicity* 2 joining B to C. The diagram is called a *multigraph*. In order to emphasize the fact that multiple edges are not allowed in graphs, the term *simple graph* is sometimes used.

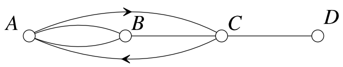
Digraphs

Some road systems contain one-way roads. These can be modeled by assigning a direction to some or all of the edges. Such a model is called a directed graph, or *digraph*.

Formally, we define a digraph to be a finite set v of objects called *vertices* together with a finite set of directed edges, or *arcs*, which are *ordered pairs* of vertices. So a digraph is like a graph except that each edge is allocated a direction—one vertex is designated a *start* and the other is a *finish*. It is important to observe that, unlike a graph, a digraph can have two arcs with the same endpoints, provided they are directed in opposite ways. If *multiple arcs* can occur, the object would be called a *directed multigraph*.

Sample Problem 4.6. *A road system connecting towns A, B, C, and D consists of two roads joining A and B, one road from B to C, one road from C to D, and one-way roads from A to C and from C to A. Represent this road system in a diagram.*

Solution.



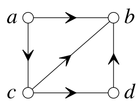
Your Turn. A road system connecting towns A, B, C, and D consists of one road joining A and B, a one-way road from B to C, a one-way road from C to D, and a one-way road from D to B. Represent this road system in a diagram.

The idea of adjacency needs further consideration in digraphs. Associated with a vertex x are the two sets

$$A(x) = \{y : (x, y) \text{ is an arc}\},$$
$$B(x) = \{y : (y, x) \text{ is an arc}\}.$$

($A(x)$ consists of the vertices *after* x, and $B(x)$ means *before* x.) A vertex x is called a *source* in the digraph if $B(x)$ is empty and a *sink* if $A(x)$ is empty.

Sample Problem 4.7. *Describe the adjacencies in the graph*



Solution.

$$\begin{aligned}
A(a) &= \{b, c\}, & B(a) &= \emptyset, \\
A(b) &= \emptyset, & B(b) &= \{a, c, d\}, \\
A(c) &= \{b, d\}, & B(c) &= \{a\}, \\
A(d) &= \{b\}, & B(d) &= \{c\};
\end{aligned}$$

a is a source and b is a sink.

Graphs and Binary Relations

Suppose G is a graph with vertex set $V(G)$. The binary relation $\sim$, where $x \sim y$ means x is adjacent to y, is a symmetric, antireflexive binary relation on $V(G)$.

On the other hand, any binary relation can be represented by a diagram. If the relation is symmetric and antireflexive, the diagram will be a graph. If the binary relation is not symmetric, the lines on the diagram will be directed, and if there are no loops, the diagram is a digraph. So the study of (simple) graphs and digraphs is exactly the same as the study of antireflexive binary relations.

The phrase "looped graph" can be used to describe the diagram of a symmetric relation $\sim$ in which $x \sim x$ is sometimes true, but be careful: a "looped graph" is not really a graph! Similarly, we can discuss looped digraphs. The phrase "infinite graph" means an object defined like a graph, in which the vertex-set can be infinite, and these are useful in discussing binary relations on infinite sets.

Graphical Models

So far we have concentrated on graphs as models of roads or pathways. Graphs and digraphs can be used to model many real-life situations. We shall now describe several examples of graphical models.

In many cases, a *weight* is associated with each edge or with each vertex. This is a real number, usually positive or non-negative, that measures some attribute of the thing being represented. Typical weights include costs, durations, and capacities.

Telephone Networks

Graphs can be used to model telephone systems in several ways. For example, each user might be denoted by a vertex. Two vertices are joined by an edge when the corresponding users can communicate by a local call. In another graph, the vertices could represent local areas. An edge means that a long distance call can be routed directly between two areas. The routing used to make longer connections correspond to sequences of edges.

Electric Circuits

An electric circuit can be viewed as a number of special places, which we shall call *nodes*, joined by wires. At any time a wire either carries a current or does not. A node means any point at which something more can happen than simply a current being carried. For example, a node might be an alarm (which will sound when certain wires connected to it carry a current), a light bulb, a point where current can be input to the system or one where current can flow out to a device or another circuit, or a switch. Nodes are represented by vertices and wires by edges. A computer network is a special example of an electric circuit, with vertices representing the computers and edges representing connections by which computers can communicate.

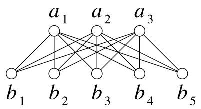
Deliveries

A delivery company might operate as follows. Every day trucks travel on a regular schedule, between some of their depots but not others. Perhaps they travel between St. Louis and Chicago, and between Chicago and Detroit, but there is no service between St. Louis and Detroit. Goods sent from St. Louis to Detroit must be sent to Chicago one day, and on to Detroit the next day. This transportation network can be represented as a graph whose vertices are the depots and whose edges are the routes.

One natural weight on this graph is the capacity. If three trucks travel from Chicago to Detroit each day, and each can carry 10000 pounds of freight, then that edge would be assigned weight 30000. Another weight is the price charged per pound to transport goods. Yet another is the cost of running each truck.

Commodity Flow

Suppose a company manufactures refrigerators in three factories and sells them from five wholesale stores. The movement of computers to retailer can be represented by a directed graph like



where a_1, a_2, a_3 represent the factories and b_1, b_2, b_3, b_4, b_5 represent the wholesale stores.

In this case, there are two obvious weights, the capacity of an edge (the number of truckloads that can be delivered from the particular factory to the particular wholesaler per day) and the cost of sending refrigerators. The usual problems are to ensure that each wholesaler receives as many refrigerators as needed, while no factory exceeds capacity. In other words, can supply meet demand? If this is possible, we are interested to know the cheapest way of satisfying these conditions. Often it will be possible to ship refrigerators between wholesalers. In that case, the model will contain edges joining some of the b_i, but of course using them will entail an extra cost.

Tournament Schedules

Suppose several baseball teams play against each other in a league. The competition can be represented by a graph with the teams as vertices and with an edge xy representing a game between teams x and y. Such a league, where two participants meet in each game, is called a *tournament*. Sometimes multiple edges will be necessary; sometimes two teams do not meet. The particular case where every pair of teams plays exactly once is called a *round robin tournament*.

In a round robin tournament, the result of each game can be shown in a directed graph, where the edge is directed from the winning team to the loser. The digraph that results from this representation is rather special: if x and y are any two vertices, then exactly one of the two possible arcs (x, y) and (y, x) occurs in the digraph. The word "tournament" is also used for directed graphs with this property.

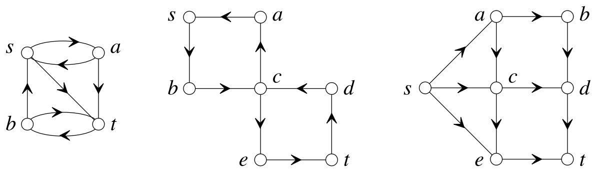
Exercises 4.2 A

1. Draw diagrams to represent the following road systems.

(i) There are three towns, A, B, and C. There are two roads from A to B, two roads from B to C, and a bypass road from A to C.

(ii) There are four towns, A, B, C, and D. There are two roads from A to B, two roads from B to C, two roads from C to D, and one road from A to D.

2. For each of the following digraphs, list all the arcs and list the members of $A(x)$ and $B(x)$ for each vertex x.



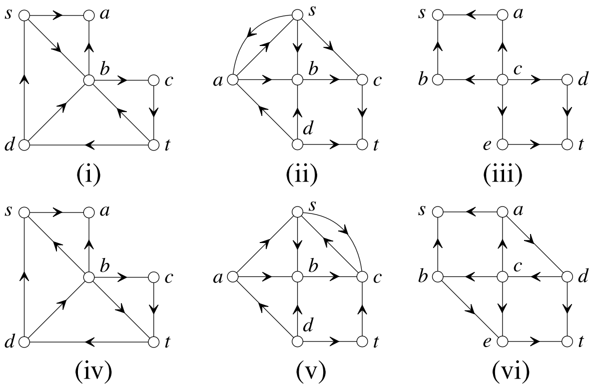
(i) (ii) (iii)

3. Draw a diagram for each binary relation defined in Exercise 4.1A.1. Which if any are graphs? Which if any are digraphs?

4. Draw a diagram for each binary relation defined in Exercise 4.1A.2. Which if any are graphs? Which if any are digraphs?

Exercises 4.2 B

1. For each of the following digraphs, list all the arcs and list the members of $A(x)$ and $B(x)$ for each vertex x.



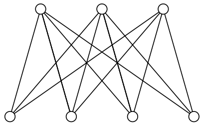
2. In each case, draw a diagram to represent the road system.
   (i) There are four towns, A, B, C, and D. There are two roads from A to B, one road from B to C, two roads from B to D, and one road from C to D.
   (ii) There are four towns, A, B, C, and D. There is one road joining each of the following pairs: A and B, A and C, A and D, B and D, C and D.
   (iii) There are four towns, A, B, C, and D. There is one road from A to B, two roads from B to C, two roads from C to D, and one road from B to D.
3. Draw a diagram for each binary relation defined in Exercise 4.1B.1. Which if any are graphs? Which if any are digraphs?
4. Draw a diagram for each binary relation defined in Exercise 4.1B.3. Which if any are graphs? Which if any are digraphs?
5. Draw a digraph to represent a tournament between teams V, W, X, Y, Z after the following matches. V beat W, V beat X, W beat X, Y beat V, Y beat Z, Z beat X. If it is a round robin tournament, how many matches remain to be played?

4.3 Some Properties of Graphs

Degree

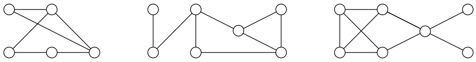
We define the *degree* or *valency* $d(x)$ of the vertex x to be the number of edges that have x as an endpoint.

Sample Problem 4.8. *What are the degrees of the vertices in the following graph?*



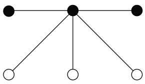
Solution. There are three vertices of degree 4 (the upper vertices in the figure), while the other four have degree 3.

Your Turn. What are the degrees of the vertices on the following graphs?

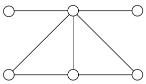


Sample Problem 4.9. *Can you find a graph on six vertices whose vertices have degrees* 5*,* 3*,* 2*,* 2*,* 1*,* 1*?*

Solution. Suppose such a graph exists. The vertex of degree 5 must lie on five edges, and their endpoints must all be different. (The problem asks for a graph, not a multigraph, so there are no multiple edges.) As there are six vertices, the vertex of degree 5 is adjacent to all the other vertices. We know the graph must contain the following configuration:



Moreover, the black vertices lie on no further edges. It is now necessary to add edges, touching the clear vertices only, so that two edges touch one vertex and one edge touches each of the other vertices, in order to make their degrees 3, 2, 2. This is easily done, as follows:



Your Turn. Can you find a graph on six vertices whose vertices have degrees 5, 4, 2, 1, 1, 1?

Theorem 10. *In any graph or multigraph, the sum of the degrees of the vertices equals twice the number of edges* (*and consequently the sum of the degrees is an even integer*).

Proof. Suppose the graph or multigraph has e edges; label the edges, say $y_1, y_2, \ldots, y_e$. Consider a list in which each edge appears twice, once for each endpoint. For example, if y_1 has endpoints x_4 and x_7, you might make entries $y_1 : x_4$ and $y_1 : x_7$. Vertex x will appear in precisely $d(x)$ entries, so the total number of entries equals the sum of the degrees of the vertices. On the other hand, each edge appears twice, so the total number of entries equals twice the number of edges. □

We shall call a vertex *odd* if its degree is odd, and *even* if the degree is even.

Corollary 1. *In any graph or multigraph, the number of odd vertices is even.*

Some Standard Graphs

Given a set S of v vertices, the graph formed by joining each pair of vertices in S is called the *complete* graph on S and denoted K_S. We write K_v to mean any complete graph with v vertices. The three drawings in Figure 4.2 are all representations of K_4. On the other hand, a graph with no edges—consisting of v isolated vertices—will be called *empty*, and denoted $\overline{K}_v$.

Further standard graphs are the *complete bipartite graphs*. If S and T are two disjoint sets of vertices, the complete bipartite graph based on S and T has vertex set $S \cup T$, consisting of all elements of S and all elements of T, and its edges are all the pairs with one vertex in S and the other in T. There are no edges joining members of the same set. We write $K_{m,n}$ for any complete bipartite graph in which one vertex set has m vertices and the other has n. The graph in Sample Problem 4.8 is a copy of $K_{4,3}$.

If G is a graph, it is possible to choose some of the vertices and some of the edges of G in such a way that these vertices and edges again form a graph, say H. H is then called a *subgraph* of G; we write $H \leq G$. A subgraph of a complete bipartite graph is simply called a *bipartite* graph.

Sample Problem 4.10. *Suppose a plant contains m machines, and there are n workers who can operate them. Each worker is qualified to use some of the machines. Model this situation with a graph.*

Solution. We represent the machines by a set of vertices S and the workers by a set of vertices T. The vertex-set is $S \cup T$. There is an edge from (machine) x to (worker) y if y is qualified to use x. The graph will be bipartite, a subgraph of the graph $K_{m,n}$ with vertex sets S and T.

Walks and Connectedness

We define a *walk* in a graph G to be any finite sequence of vertices $x_0, x_1, \ldots, x_n$ and edges $y_1, y_2, \ldots, y_n$ of G:

$$x_0, y_1, x_1, y_2, \ldots, y_n, x_n,$$

where the endpoints of y_i are x_{i-1} and x_i for each i. The *length* of a walk is its number of edges. The walk written above might be called an x_0x_n-*walk* of length n. A *closed* walk, or *circuit*, is one in which the first and last vertices are the same.

Walks are called *simple* if there is no repeated edge. A walk in which no *vertex* is repeated is a *path*.

There is no such thing as a closed path—the last vertex is a repeat of the first—but we shall define a *cycle* to be a closed simple walk with no vertex repeats except for the endpoints.

We say two vertices are *connected* when there is a walk joining them. A graph is called connected if every pair of its vertices are connected. If a graph is not connected, it falls into two or more parts with no edges joining them. The connected parts of a graph are called *components*. So two vertices of G are connected if and only if they lie in the same component of G.

Among connected graphs, some are connected so slightly that removal of a single vertex or edge will disconnect them. Such vertices and edges are quite important. A vertex x is called a *cutpoint* in G if $G-x$ contains more components than does G; in particular if G is connected, then a cutpoint is a vertex x such that $G - x$ is disconnected. Similarly, a *bridge* is an edge whose deletion increases the number of components.

Sample Problem 4.11. *If a graph represents a road network, what is the interpretation of a bridge? Why are such edges important?*

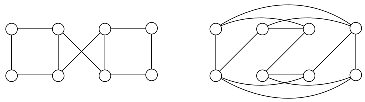
Solution. A bridge represents a road whose deletion would cut people in part of the network off from those in another part. (Often a real bridge, over a river, etc., plays this role.) It is important to recognize such roads so that extra effort can be taken to keep them open to traffic. For example, the highways department might give special priority to keeping them plowed in winter.

Your Turn. In such a graph, what is the interpretation of a cutpoint?

Distance

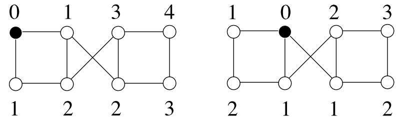
If vertices x and y are adjacent, then there is a path of length 1 from x to y, and we shall say that the *distance* from x to y is 1. If x and y are connected, there may be several paths joining them, and we define the distance between them to be the length of the shortest path joining them; we denote it $D(x, y)$. For completeness, we define $D(x, x)$ to be 0.

Sample Problem 4.12. *Find the distance between every pair of vertices in the graph* (a) *below.*



(a) (b)

Solution. There are two different kinds of vertex in the graph. In each part of the following figure, one vertex is highlighted and the number against a vertex shows the distance from the highlighted one.



Your Turn. Find the distance between every pair of vertices in graph (b).

Distance arises in the example of a communications network. The vertices of a graph might represent computers. If two computers are directly linked, the corresponding vertices are adjacent so that edges represent direct links. Often messages from one computer to another are sent by way of another machine—e-mail messages are often relayed through several intermediate computers. Two computers can communicate if the corresponding vertices are connected.

The physical separation between the computers is irrelevant. The number of times the signal is relayed may well be significant: the more relays, the greater the chance of a delay. (Direct communication between computers, by media such as telephone cables, is practically instantaneous, but delays may occur when an intermediate link is busy or down.) So the distance between vertices is a measure of the possibility of delay.

In this example, we are often interested in the *worst case*—what is the worst delay that can occur? To measure this, we define the *eccentricity* $\varepsilon(x)$ of vertex x in a connected graph to be the largest value of $D(x, y)$, where y ranges through all the vertices. The *diameter* $D = D(G)$ of G is the maximum value of $\varepsilon(x)$ for all vertices x of G, while the *radius* $R = R(G)$ is the smallest value of $\varepsilon(x)$.

Sample Problem 4.13. *Find the diameter and radius of graph* (a) *of Sample Problem* 4.12.

Solution. Four vertices have eccentricity 4, and the others have eccentricity 3. So $D = 4$, $R = 3$.

Your Turn. Find the diameter and radius of the graph (b) of Sample Problem 4.12.

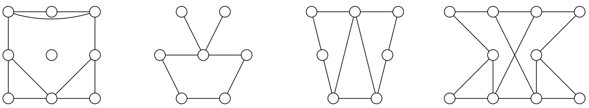
Theorem 11. *In any graph G,*

$$R(G) \le D(G) \le 2R(G).$$

Proof. Clearly, $D(G)$ equals the maximum distance between any two vertices in G. Say x and y attain this maximum distance, so that $D(x, y) = D(G)$, and say z is a vertex for which $\varepsilon(z) = R(G)$. As R is a distance between two vertices, necessarily $R \leq D$. But by definition $D(z, t) \leq \varepsilon(z) = R(G)$ for every vertex t, so $D(G) = D(x, y) \leq D(x, z) + D(z, y) \leq 2R(G)$. □

Exercises 4.3 A

1. What are the degrees of the vertices in the following graphs?



(i) (ii) (iii) (iv)

2. Draw the graphs K_5 and K_6. What are the degrees of the vertices in these graphs?

3. Can you find a connected graph with four vertices whose degrees are 2, 2, 1, 1?

4. Is there a connected graph with four vertices whose degrees are 3, 3, 1, 1?

5. The *n-wheel* W_n has $n + 1$ vertices $\{x_0, x_1, \ldots, x_n\}$; x_0 is joined to every other vertex and the other edges are

$$x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1.$$

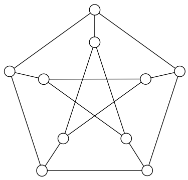
How many edges does W_n have? What are the degrees of its vertices?

6. Identify all cutpoints and bridges in the graphs of Exercise 1.

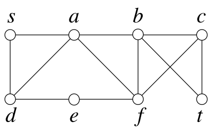
7. How many edges are there in K_7?

8. How many edges are there in K_n?

9. What are the diameter and radius of the following graph, which is called the *Petersen graph*?

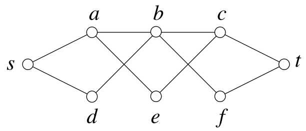


10. Find all paths from s to t in the following graph.



What is the length of each path? What is the distance from s to t?

11. Find all paths from s to t in the following graph.



What is the length of each path? What is the distance from s to t?

Exercises 4.3 B

1. Draw the graphs $K_{2,5}$, $K_{3,3}$, K_7, and $K_{3,4}$. What are the degrees of the vertices in these graphs?

2. Can you find graphs whose vertices have the following sets of degrees?

(i) 3, 2, 2, 1; (ii) 5, 3, 3, 1;

(iii) 5, 3, 3, 2, 2; (iv) 4, 4, 3, 3, 2;

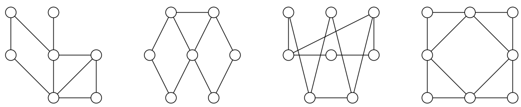
(v) 5, 3, 3, 3, 1, 1; (vi) 3, 3, 3, 3, 2, 2;

(vii) 2, 2, 2, 1; (viii) 3, 3, 3, 1, 1, 1;

(ix) 4, 3, 3, 2, 2; (x) 4, 4, 2, 2, 2, 1, 1;

(xi) 5, 4, 3, 3, 2, 1; (xii) 3, 3, 3, 3, 3, 2, 2.

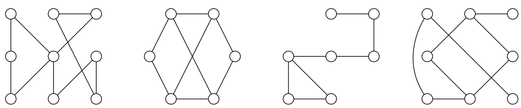
3. What are the degrees of the vertices in the following graphs?



(i) (ii) (iii) (iv)

Identify all cutpoints and bridges in these graphs.

4. What are the degrees of the vertices in the following graphs?



(i) (ii) (iii) (iv)

Identify all cutpoints and bridges in these graphs.

5. The *n-star* $K_{1,n}$ has $n+1$ vertices $\{x_0, x_1, \ldots, x_n\}$; x_0 is joined to every other vertex, but there are no other edges. How many edges does $K_{1,n}$ have? What are the degrees of its vertices?

6. How many edges are there in $K_{3,5}$? How many edges are there in $K_{m,n}$?

7. The graph $K_{p,q,r}$ has three sets of vertices, of sizes p, q, and r, respectively. Two vertices are joined if and only if they belong to different sets.

(i) Sketch $K_{3,3,2}$.

(ii) How many edges are there in $K_{3,3,2}$?

(iii) How many edges are there in $K_{p,q,r}$?

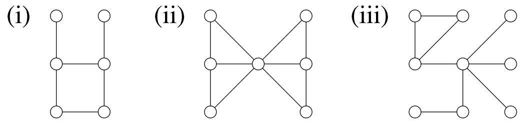
8. The *split graph* $K_{p\backslash q}$ has two sets of vertices, V_1 of size $p-q$ and V_2 of size q. Two vertices are joined unless they both belong to V_2.

(i) Sketch $K_{6\backslash 3}$.

(ii) How many edges are there in $K_{6\backslash 3}$?

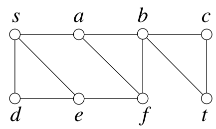
(iii) How many edges are there in $K_{p\backslash q}$?

9. Find all bridges in the following graphs.



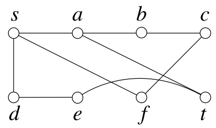
10. What are the diameter and radius of the wheel W_n, which was defined in Exercise 4.3A.5 above?

11. Find all paths from s to t in the following graph.



What is the length of each path? What is the distance from s to t?

12. Find all paths from s to t in the following graph.



What is the length of each path? What is the distance from s to t?

4.4 Euler's Theorem and Eulerizations

The Königsberg Bridges

The first mathematical paper on graph theory was published by the great Swiss mathematician Leonhard Euler in 1736, and had been delivered by him to the St. Petersburg Academy one year earlier.

Euler's paper grew out of a famous old problem. The town of Königsberg in Prussia is built on the river Pregel. The river divides the town into four parts, including an island called The Kneiphof, and in the eighteenth century the town had seven bridges; the layout is shown in Figure 4.3. The question under discussion was whether it is possible from any point on Königsberg to take a walk in such a way as to cross each bridge exactly once.

Euler set for himself the more general problem: given any configuration of river, islands and bridges, find a general rule for deciding whether there is a walk that covers each bridge precisely once.

We first show that it is impossible to walk over the bridges of Königsberg. Suppose there was such a walk. There are three bridges leading to the area C: you can traverse two of these, one leading into C and the other leading out, at one time in your tour. There is only one bridge left: if you cross it going into C, then you cannot leave C again, unless you use one of the bridges twice, so C must be the finish of the walk; if you cross it in the other direction, C must have been the start of the walk. In either event, C is either the place where you started or the place where you finished.

A similar analysis can be applied to A, B, and D, since each has an odd number of bridges—five for B and three for the others. But any walk starts at one place and finishes at one place; either there can be two endpoints, or the walk can start and finish at the same place. Therefore, it is impossible for A, B, C, and D all to be either the start or the finish.

Euler started by finding a multigraph that models the Königsberg bridge problem (considered as a road network, with the islands and river banks as separate "towns"). Vertices A, B, C, and D represent the parts A, B, C, and D of the town, and each bridge is represented by an edge. The multigraph is shown in Figure 4.4. In terms of this model, the original problem becomes: can a simple walk be found that contains every edge of the multigraph?

These ideas can be applied to more general configurations of bridges and islands, and to other problems. A simple walk that contains every edge of a given multigraph will be called an *Euler walk* in the multigraph; if an Euler walk starts and finish at the same vertex, it is called an *Euler circuit*. Following an Euler walk is called *traversing* the multigraph, and finding whether a given multigraph (or a given road network) has an Euler walk is called the *traversability problem*.

You obviously cannot traverse a graph if it can be broken into two parts with no connection between them. For example, the road map of the United States cannot be

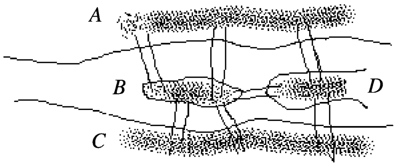


Fig. 4.3. The Königsberg bridges

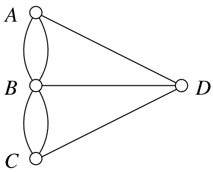


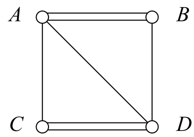
Fig. 4.4. The multigraph representing the Königsberg bridges

traversed, because there is no way to get from Hawaii to the other states. If a graph (or multigraph) is to be traversable, it must be connected.

So far we have seen that if a graph or multigraph has an Euler walk, it can have at most two odd vertices, one for the start of the walk and one for the finish. For an Euler circuit to exist, all vertices of the graph must be even. (Notice that there cannot be exactly one odd vertex, because of Corollary 1.) Euler showed that these conditions are sufficient for the existence of Euler walks in connected multigraphs.

Sample Problem 4.14. *The islands A, B, C, D are joined by seven bridges—two joining A to B, two from C to D, and one each joining the pairs AC, AD, and BD. Represent the system graphically. Could one walk through this system, crossing each bridge exactly once?*

Solution. Islands B and C each have an odd number of bridges, so one must be the start of the walk and the other the finish. With a little experimentation you will find a solution—one example is $BACDABDC$, and there are others.



Your Turn. The islands X, Y, Z, T are joined by seven bridges—three joining X to Y, and one each joining the pairs XZ, XT, and YZ. Represent the system graphically. Could one walk through this system, crossing each bridge exactly once?

Euler's Proof

We begin with a formal statement of Euler's theorem.

Theorem 12. *If a connected multigraph has no odd vertices, then it has an Euler walk starting from any given point and finishing at that point. If a connected multigraph has two odd vertices, then it has an Euler walk whose start and finish are the odd vertices.*

Proof. Consider any simple walk in a multigraph that starts and finishes at the same vertex. If one erases every edge in that walk, one deletes two edges touching any vertex that was crossed once in the walk, four edges touching any vertex that was crossed twice, and so on. (For this purpose, count "start" and "finish" combined as one crossing.) In every case, an *even* number of edges is deleted.

First, consider a multigraph with no odd vertex. Select any vertex x, and select any edge incident with x. Go along this edge to its other endpoint, say y. Then choose any other edge incident with y. In general, on arriving at a vertex, select any edge incident with it that has not yet been used, and go along the edge to its other endpoint. At the moment when this walk has led into the vertex z, where z is not x, an odd number of edges touching z has been used up (the last edge to be followed, and an even number previously). Since z is even, there is at least one edge incident with z that is still available. Therefore, if the walk is continued until a further edge is impossible, the last vertex must be x—that is, the walk is closed. It will necessarily be a simple walk and it must contain every edge incident with vertex x.

Now assume that a connected multigraph with every vertex even is given, and a simple closed walk has been found in it by the method just described. Delete all the edges in the walk, forming a new multigraph. From the first paragraph of the proof, it follows that every vertex of the new multigraph is even. It may be that we have erased every edge in the original multigraph; in that case, we have already found an Euler walk. If there are edges still left, there must be at least one vertex, say c, that was in the original walk and that is still on an edge in the new multigraph—if there were no such vertex, then there could be no connection between the edges of the walk and the edges left in the new multigraph, and the original multigraph must have been disconnected. Select such a vertex c, and find a closed simple walk starting from c. Then unite the two walks as follows: at one place where the original walk contained c, insert the new walk. For example, if the two walks are

$$x, y, \ldots, z, c, u, \ldots, x$$

and

$$c, s, \ldots, t, c,$$

then the resulting walk will be

$$x, y, \ldots, z, c, s, \ldots, t, c, u, \ldots, x.$$

(There may be more than one possible answer if c occurred more than once in the first walk. Any of the possibilities may be chosen.) The new walk is a closed simple walk in the original multigraph. Repeat the process of deletion, this time deleting the

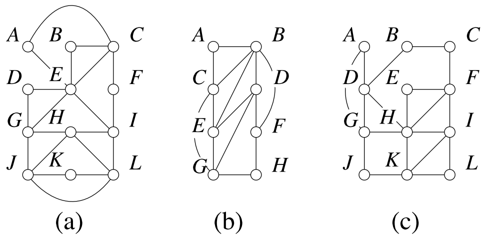


Fig. 4.5. Find Euler walks

newly formed walk. Continue in this way. Each walk contains more edges than the preceding one, so the process cannot go on indefinitely. It must stop: this will only happen when one of the walks contains all edges of the original multigraph, and that walk is an Euler walk.

Finally, consider the case where there are two odd vertices p and q and every other vertex is even. Form a new multigraph by adding an edge pq to the original. This new multigraph has every vertex even. Find a closed Euler walk in it, choosing p as the first vertex and the new edge pq as the first edge. Then delete this first edge; the result is an Euler walk from q to p. □

It is clear that loops make no difference as to whether or not a graph has an Euler walk. If there is a loop at vertex x, it can be added to a walk at some traversing of x.

A good application of Euler walks is planning the route of a highway inspector or mail contractor, who must travel over all the roads in a highway system. Suppose the system is represented as a multigraph G, as was done earlier. Then the most efficient route will correspond to an Euler walk in G.

Sample Problem 4.15. *Find Euler walks in the road networks represented by Figure* 4.5(a,b).

Solution. In the first example, starting from A, we find the walk $ACBEA$. When these edges are deleted (see Figure 4.6(a)) there are no edges remaining through A. We choose C, a vertex from the first walk that still has edges adjacent to it, and trace the walk $CFIHGDEC$, after which there are no edges available at C (see Figure 4.6(b)). E is available, yielding walk $EGJLIE$. As is clear from Figure 4.6(c), the remaining edges form a walk $HJKLH$.

We start with $ACBEA$. We replace C by $CFIHGDEC$, yielding

$$ACFIHGDECBEA,$$

then replace the first E by $EGJLIE$, with result

$$ACFIHGDEGJLIECBEA$$

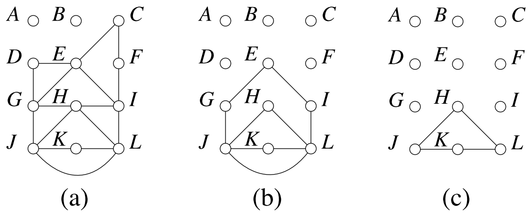


Fig. 4.6. Constructing an Euler walk

(we could equally well replace the second E). Finally, H is replaced by $HJKLH$, and the Euler walk is

$$ACFIHJKLHGDEGJLIECBEA.$$

In the second example, there are two odd vertices, namely B and F, so we add another edge BF and make it the first edge used. The first walk found was

$$BFHGCABFDBCEB,$$

and the second is $DEGD$, exhausting all the vertices and producing the walk

$$BFHGCABFDEGDBCEB.$$

The first edge (the new one we added) is now deleted, giving Euler walk

$$FHGCABFDEGDBCEB$$

in the original graph.

Your Turn. Find an Euler walk in the road network represented by Figure 4.5(c).

Eulerization

If G contains no Euler walk, the highway inspector must repeat some edges of the graph in order to return to his starting point. Let us define an *Eulerization* of G to be a multigraph, with a closed Euler walk, that is formed from G by duplicating some edges. A *good* Eulerization is one that contains the minimum number of new edges, and this minimum number is the *Eulerization number* $eu(G)$ of G. For example, if two adjacent vertices have odd degree, you could add a further edge joining them. This would mean that the inspector must travel the road between them twice. However, if the two vertices were not adjacent, one new edge will not suffice—it would be the same as requiring that a new road be built!

Sample Problem 4.16. *Consider the multigraph G of Figure 4.7. What is $eu(G)$? Find an Eulerization of the road network represented by G that uses the minimum number of edges.*

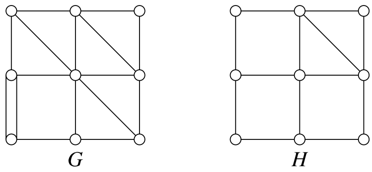


Fig. 4.7. Find Eulerizations

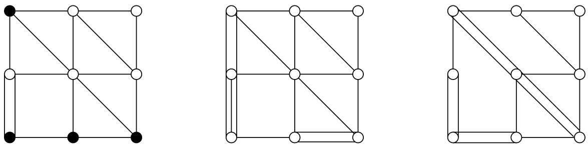


Fig. 4.8. Finding an Eulerization

Solution. Look at the multigraph as shown in Figure 4.8. The black vertices have odd degree, so they need additional edges. As there are four black vertices, at least two new edges are needed; but obviously no two edges will suffice. However, there are solutions with three added edges—two examples are shown—so $eu(G) = 3$.

Your Turn. Consider the graph H of Figure 4.7. What is $eu(H)$? Find an Eulerization of the road network represented by H that uses the minimum number of edges.

An Application of Eulerization

The need for good Eulerization occurs in many practical situations. One good example is the scheduling of snowplows. Plows must drive over all the roads in a city, but the plow does not have to go over a road twice. The ideal route would be an Euler circuit among the roads—a circuit is required because the plow must return to the municipal garage.

If no Euler circuit is available, the plow must traverse a circuit in an Eulerization of the graph. This example shows why Eulerizations must repeat existing edges, and cannot include new connections: a plow can only run on an existing road.

In practical situations, there may be restrictions because of one-way roads. In this case, directed graphs are used. The theory of Euler walks and Eulerization is readily extended to the directed case.

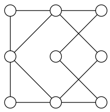
If several plows are available, the schedule is derived by dividing the graph of the city into several circuits that together cover all the edges. Techniques for doing this are quite easily derived from the construction for an Euler circuit.

Other applications include scheduling routes for mailmen and deliveries. There are also applications in the design of electrical circuits.

Exercises 4.4 A

1. In each case, find all odd vertices in the given graph.

(i)



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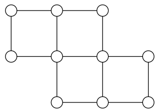
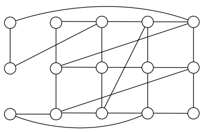


(iii)



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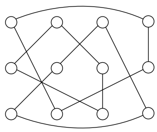
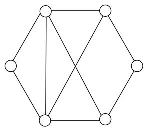


(v)



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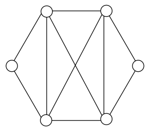

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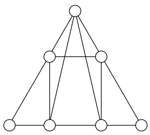

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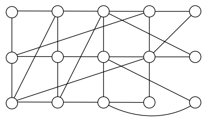
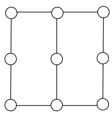


2. Decide whether each graph in Exercise 1 contains an Euler walk. If the graph contains an Euler walk, find one. Is it an Euler circuit?

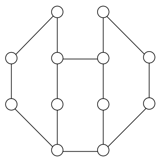
3. For what values of n does the complete graph K_n have an Euler walk?

4. State the Eulerization number of the given graph and find a good Eulerization.

(i)



(ii)



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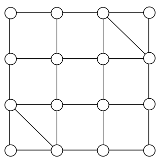

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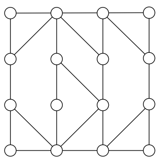
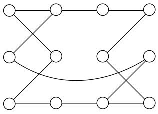


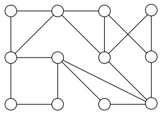
Exercises 4.4 B

1. In each case, how many odd vertices does the given graph contain? Decide whether the graph contains an Euler walk. If it does, find such a walk. Is the walk an Euler circuit?

(i)



(ii)



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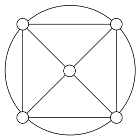

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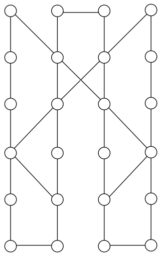

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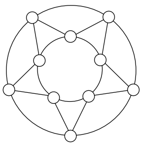

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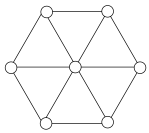

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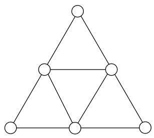

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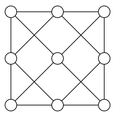
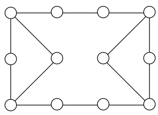


2. In each case, how many odd vertices does the given graph contain? Decide whether the graph contains an Euler walk. If it does, find such a walk. Is the walk an Euler circuit?

(i)



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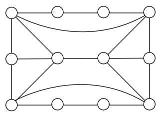

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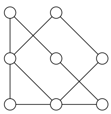

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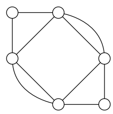

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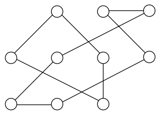

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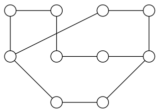

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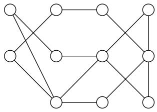

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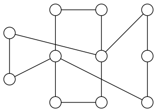


3. Find a graph G such that both G and its complement $\overline{G}$ have Euler circuits.

4. For what values of m and n does $K_{m,n}$ have an Euler walk?

5. Suppose a graph contains an edge whose removal disconnects the graph (when the edge is deleted, the resulting graph is not connected). Show that the original graph contains a vertex of odd degree.

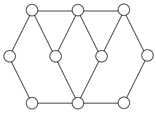
6. If a graph does not contain an Euler circuit, one could *subeulerize* it by deleting edges until the resulting graph has an Euler circuit.

(i) Prove that the number of edges required to subeulerize a graph is at least as many as the number required to Eulerize it.

(ii) Find a graph that can be Eulerized with one edge, but cannot be subeulerized.

7. State the Eulerization number of the given graph and find a good Eulerization.

(i)



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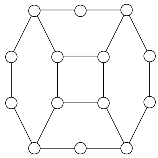

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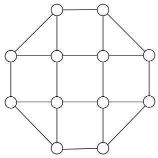

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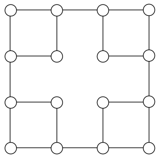

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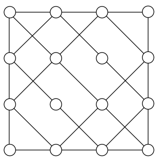

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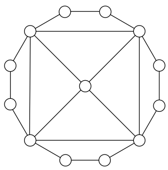

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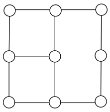

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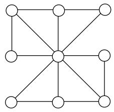

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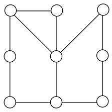
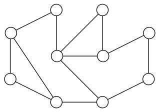
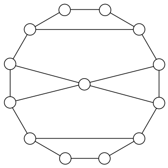


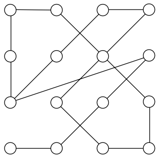
(x)



(xi)



(xii)

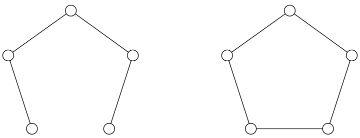


4.5 Types of Graphs

Paths and Cycles

Suppose a graph consists of a single v-vertex path. We then call the graph itself a path, and we write P_v for such a graph. A "cycle" C_v is defined similarly to mean a graph consisting of one v-vertex cycle. If multiple edges are allowed, C_2 would mean a multigraph with two vertices joined by a double edge, and if loops are allowed then C_1 usually means a single vertex with a loop, but for graphs C_v is only defined when $v \geq 3$. The *length* of a path or cycle is its number of edges, so C_v is a cycle of length v, while P_v is a path of length $v - 1$.

Here are examples of P_5 and C_5.



Trees

Sample Problem 4.17. *Suppose an oil field contains five oil wells w_1, w_2, w_3, w_4, w_5, and a depot d. It is required to build pipelines so that oil can be pumped from the wells to the depot. Oil can be relayed from one well to another, at very small cost. The only real expense is building the pipelines. Figure* 4.9(a) *shows*

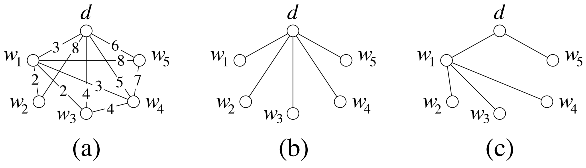


Fig. 4.9. An oil pipeline problem

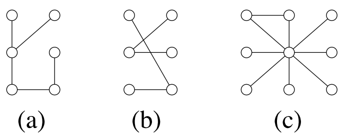
which pipelines are feasible to build, represented as a graph in the obvious way, with the cost (*in hundreds of thousands of dollars*) *shown.* (*A missing edge might mean that the cost would be very high.*) *Which pipelines should be built?*

Solution. Your first impulse might be to connect d to each well directly, as shown in Figure 4.9(b). This would cost \$2600000. However, the cheapest solution is shown in Figure 4.9(c), and costs \$1600000.

Once the pipelines in case (c) have been built, there is no reason to build a direct connection from d to w_3. Any oil being sent from w_3 to the depot can be relayed through w_1. Similarly, there would be no point in building a pipeline joining w_3 to w_4. The conditions of the problem imply that the graph does not need to contain any cycle. However, it must be connected.

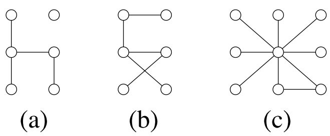
A connected graph that contains no cycle is called a *tree*. So the solution of the problem is to find a subgraph of the underlying graph that is a tree, and among those to find the one that is cheapest to build. We shall return to this topic shortly. But first we look at trees as graphs.

Sample Problem 4.18. *Which of the following graphs are trees?*



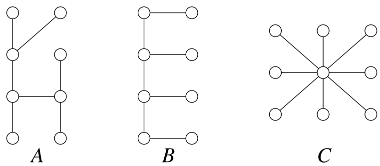
Solution. Graph (a) is a tree. Graph (b) is not a tree because it is not connected, although this might not be obvious immediately. Graph (c) contains a cycle, so it is not a tree.

Your Turn. Which of the following graphs are trees?

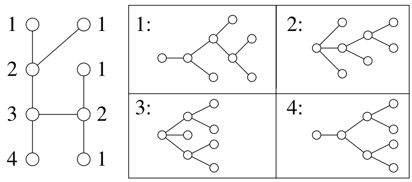


We have already seen trees. The tree diagrams used in Chapter 3 are just trees laid out in a certain way. To form a tree diagram from a tree, select any vertex as a starting point. Draw edges to all of its neighbors. These are the first generation. Then treat each neighbor as if it were the starting point; draw edges to all its neighbors except the start. Continue in this way; to each vertex, attach all of its neighbors except for those that have already appeared in the diagram.

Sample Problem 4.19. *Draw the first of the following three trees (tree A) as a tree diagram. How many different-looking diagrams can be made?*



Solution. There are four different-looking diagrams. In the following illustration, the vertices of the original tree are labeled 1, 2, 3, 4. The diagrams obtained using each type as start are shown.



Your Turn. Repeat this problem for the other trees B and C above.

Every path is a tree, and the star $K_{1,n}$, which was defined in Exercise 4.3B.5, is a tree for every n.

A tree is a minimal connected graph in the following sense: if any vertex of degree at least 2, or any edge, is deleted, then the resulting graph is not connected. In fact, it is easy to prove that a connected graph is a tree if and only if every edge is a bridge.

Trees can also be characterized among connected graphs by their number of edges.

Theorem 13. *A finite connected graph G with v vertices is a tree if and only if it has exactly $v - 1$ edges.*

Proof. (i) Let us call a tree *irregular* if its number of vertices is not greater by 1 than its number of edges. We shall assume that at least one irregular tree exists, and show that a contradiction follows.

There must be a smallest possible value v for which there is an irregular tree on v vertices, and let G be an irregular tree with v vertices. This value v must be at least 2 because the only graph with one vertex is K_1, which has $v - 1 (= 0)$ edges. Choose an edge in G (G must have an edge, or it will be the unconnected graph $\overline{K}_v$) and delete it. The result is a union of two disjoint components, each of which is a tree with fewer than v vertices; say the first component has v_1 vertices and the second has v_2, where $v_1 + v_2 = v$. Neither of these graphs is irregular, so they have $v_1 - 1$ and $v_2 - 1$ edges, respectively. Adding one edge for the one that was deleted, we find that the number of edges in G is

$$(v_1 - 1) + (v_2 - 1) + 1 = v - 1.$$

This contradicts the assumption that G was irregular. So there can be no irregular trees; we could not even choose one smallest example. We have shown that a finite connected graph G with v vertices is a tree *only if* it has exactly $v - 1$ edges.

(ii) On the other hand, suppose G is connected but is not a tree. Select an edge that is part of a cycle, and delete it. If the resulting graph is not a tree, repeat the process. Eventually the graph remaining will be a tree, and must have $v - 1$ edges. So the original graph had more than $v - 1$ edges. □

Corollary 2. *Every tree other than K_1 has at least two vertices of degree* 1.

Proof. Suppose the tree has v vertices. It then has $v - 1$ edges. So, by Theorem 10, the sum of all degrees of the vertices is $2(v - 1)$. There can be no vertex of degree 0, since the tree is connected and it is not K_1; if $v - 1$ of the vertices have degree at least 2, then the sum of the degrees is at least $1 + 2(v - 1)$, which is impossible. So there must be two (or more) vertices with degree 1. □

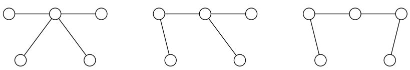
An example of a tree with exactly two vertices of degree 1 is the path P_v, provided $v > 1$. The star $K_{1,n}$ has n vertices of degree 1.

Sample Problem 4.20. *Find all trees with five vertices.*

Solution. If a tree has five vertices, then the largest possible degree is 4. Moreover, there are 4 edges (by Theorem 13), so from Theorem 10 the sum of the five degrees is 8. As there are no vertices of degree 0 and at least two vertices of degree 1, the list of degrees must be one of

$$4, 1, 1, 1, 1 \qquad 3, 2, 1, 1, 1 \qquad 2, 2, 2, 1, 1.$$

In the first case, the only solution is the star $K_{1,4}$. If there is one vertex of degree 3, no two of its neighbors can be adjacent (this would form a cycle), so the fourth edge must join one of those three neighbors to the fifth vertex. The only case with the third degree list is the path P_5. These three cases are



Your Turn. Find all trees with four vertices.

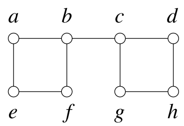
A *spanning* subgraph of a graph G is one that contains every vertex of G. A *spanning tree* in a graph is a spanning subgraph that is a tree when considered as a graph in its own right. For example, it was essential that the tree used to plan the oil pipelines should be a spanning tree.

Spanning Trees

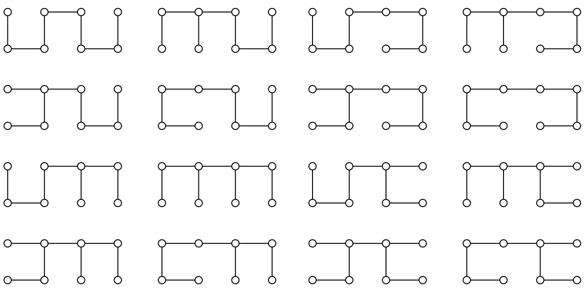
It is clear that any connected graph G has a spanning tree. If G is a tree, then the whole of G is itself the spanning tree. Otherwise G must contain a cycle; delete one edge from the cycle. The resulting graph is still a connected subgraph of G; and, as no vertex has been deleted, it is a spanning subgraph. Find a cycle in this new graph and delete it; repeat the process. Eventually, the remaining graph will contain no cycle, so it is a tree. So when the process stops, we have found a spanning tree.

A given graph might have many different spanning trees. There are algorithms to find all spanning trees in a graph. In small graphs, a complete search for spanning trees can be done quite quickly.

Sample Problem 4.21. *Find all spanning trees in the following graph.*

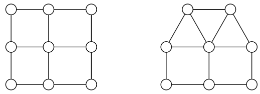


Solution. The graph contains two cycles, $abfe$ and $cdhg$. In order to construct a tree, it is necessary to delete at least one edge from each of these cycles. As the original graph contains eight vertices, any spanning tree will have eight vertices. From Theorem 13, these trees will have seven edges. So exactly one edge must be removed from each cycle, or there will be too few edges. (This argument would need some modification if an edge that was common to both cycles were deleted, but fortunately the graph contains no such edge.) So there are 16 spanning trees, as follows.



Exercises 4.5 A

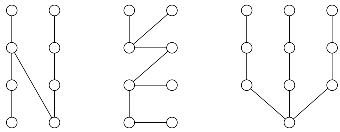
1. Find the diameter and radius of the cycle C_{2n}.
2. The *Petersen graph* was defined in Exercise 4.3A.9. Find cycles of lengths 5, 6, 8, and 9 in this graph.
3. Find a tree with five vertices, three of which have degree 1.
4. Consider the two graphs shown.



(i) (ii)

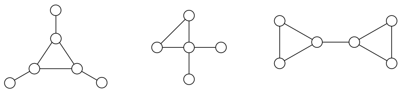
Find cycles of lengths 4, 6, and 8 in the first, and cycles of lengths 3, 4, 5, 6, 7, and 8 in the second.

5. A graph contains at least two vertices. There is exactly one vertex of degree 1, and every other vertex has degree 2 or greater. Prove that the graph contains a cycle. Would this remain true if we allowed graphs to have infinite vertex sets?
6. Represent the following trees as tree diagrams in all possible ways.



(i) (ii) (iii)

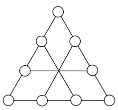
7. Find all spanning trees in the following graphs.



(i) (ii) (iii)

Exercises 4.5 B

1. Find the diameter and radius of the cycle C_{2n+1}.
2. Find all different trees on six vertices.
3. Show that the graph

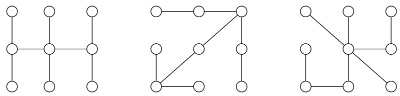


contains cycles of lengths 5, 6, 8, and 9. (Notice that the central point, where three edges cross, is *not* a vertex.)

4. The *wheel* W_n was defined in Exercise 4.3A.5; it consists of a cycle of length n together with one further vertex that is adjacent to all n vertices of the cycle. What are the radius and diameter of W_7?

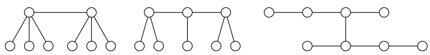
5. For each $n = 2, 3, 4, 5, 6$, find a tree on seven vertices that has n vertices of degree 1.

6. Represent the following trees as tree diagrams in all possible ways.



(i) (ii) (iii)

7. Find all ways to represent the following trees as tree diagrams.



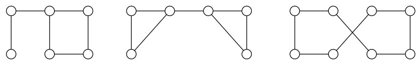
(i) (ii) (iii)

8. Suppose a graph has six vertices.

(i) Suppose two vertices are of degree 3 and four of degree 1. Are the vertices of degree 3 adjacent, or not, or is it impossible to tell?

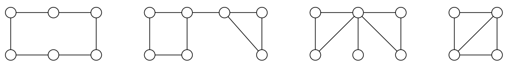
(ii) Two vertices are of degree 4 and four of degree 1. Is such a graph possible?

9. Find all spanning trees in the following graphs.



(i) (ii) (iii)

10. Find all spanning trees in the following graphs.



(i) (ii) (iii) (iv)

4.6 Hamiltonian Cycles

Modeling Road Systems Again

When we discussed modeling road systems in Sections 4.2 and 4.4, the emphasis was on visiting all the roads: crossing the Königsberg bridges, or inspecting a highway.

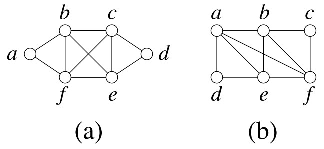


Fig. 4.10. Sample graphs for discussing Hamiltonian cycles

Another viewpoint is that of the traveling salesman or tourist who wants to visit the towns. The salesman travels from one town to another, trying not to pass through any town twice on his trip. In terms of the underlying graph, the salesman plans to follow a cycle.

A cycle that passes through every vertex in a graph is called a *Hamiltonian cycle* and a graph with such a cycle is called *Hamiltonian*. The idea of such a spanning cycle was simultaneously developed by Hamilton in 1859 in a special case, and more generally by Kirkman in 1856.

Sample Problem 4.22. *Consider the graph in Figure* 4.10(a). *Which of the following are Hamiltonian cycles?*

(i) (a, b, e, d, c, f, a)*;*

(ii) (a, b, e, c, d, e, f, a)*;*

(iii) (a, b, c, d, e, f, a)*;*

(iv) (a, b, c, e, f, a).

Solution. (i) and (iii) are Hamiltonian. (ii) is not; it contains a repeat of e. (iv) is not; vertex d is omitted.

Your Turn. Repeat this problem for the graph in Figure 4.10(b) and cycles

(i) (a, b, c, f, e, d, a);

(ii) (a, b, f, c, b, e, d, a);

(iii) (a, b, c, d, e, f, a);

(iv) (a, d, e, b, c, f, a).

Which Graphs Are Hamiltonian?

At first, the problem of deciding whether a graph is Hamiltonian sounds similar to the problem of Euler circuits. However, the two problems are strikingly different in one regard. We found a very easy test for the Eulerian property, but no nice necessary and sufficient conditions are known for the existence of Hamiltonian cycles.

It is easy to see that the complete graphs with 3 or more vertices are Hamiltonian, and any ordering of the vertices gives a Hamiltonian cycle. On the other hand, no tree is Hamiltonian, because trees contain no cycles at all. We can discuss Hamiltonicity in a number of other particular cases, but there are no known theorems that characterize Hamiltonian graphs.

Finding All Hamiltonian Cycles

Suppose you want to find all Hamiltonian cycles in a graph with v vertices. Your first instinct might be to list all possible arrangements of v vertices and then delete those with two consecutive vertices that are not adjacent in the graph. This process can then be made more efficient by observing that the v lists $a_1a_2, \ldots, a_{v-1}a_v$, $a_2a_3, \ldots, a_va_1, \ldots$, and $a_va_1, \ldots, a_{v-2}a_{v-1}$ all represent the same cycle (written with a different starting point), and also $a_va_{v-1}, \ldots, a_2a_1$ is the same Hamiltonian cycle as $a_1a_2, \ldots, a_{v-1}a_v$ (traversed in the opposite direction).

Another shortcut is available. If there is a vertex of degree 2, then any Hamiltonian cycle must contain the two edges touching it. If x has degree 2, and suppose its two neighbors are y and z, then xy and xz are in every Hamiltonian cycle. So it suffices to delete x, add an edge yz, find all Hamiltonian cycles in the new graph, and delete any that do not contain the edge yz. The Hamiltonian cycles in the original graph are then formed by inserting x between y and z.

Sample Problem 4.23. *Find all Hamiltonian cycles in the graph F of Figure 4.11.*

Solution. The graph F has two vertices of degree 2, namely a and f. When we replace these vertices by edges bd and ce, we obtain the complete graph with vertices a, b, c, d (multiple edges can be ignored). This complete graph has three Hamiltonian cycles: there are six arrangements starting with b, namely $bcde$, $bced$, $bdce$, $bedc$, $bdec$, $becd$, and the latter three are just the former three written in reverse. $bcde$ yields a cycle that does not contain edges bd and ce. The other two give the cycles $bcfeda$ and $badcfe$, so these are the two Hamiltonian cycles in F.

Your Turn. Repeat this problem for graphs G and H of Figure 4.11.

The Traveling Salesman Problem

Suppose a traveling salesman wishes to visit several cities. If the cities are represented as vertices and the possible routes between them as edges, then the salesman's preferred itinerary is a Hamiltonian cycle in the graph.

In most cases, there is a cost associated with every edge. Depending on the salesman's priorities, the cost might be a dollar cost such as airfare, or the number of miles

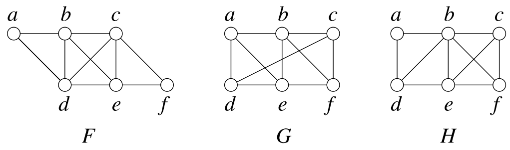


Fig. 4.11. Find all Hamiltonian cycles

to be driven, or the number of hours the trip will take. The most desirable itinerary will be the one for which the sum of costs is a minimum. The problem of finding this cheapest Hamiltonian cycle is called the Traveling Salesman Problem.

We shall continue to speak in terms of a salesman, but these problems have many other applications. They arise in airline and delivery routing and in telephone routing. More recently they have been important in manufacturing integrated circuits and computer chips, and for Internet routing.

In order to solve a Traveling Salesman Problem on v vertices, your first instinct might be to list all Hamiltonian cycles in the graph. But very often we can assume the graph is complete. In that case, listing all the cycles can be a very long task because of the following theorem.

Theorem 14. *The complete graph* K_v *contains* $(v-1)!/2$ *Hamiltonian cycles.*

Proof. There are $v!$ arrangements of the vertices. Each Hamiltonian cycle gives rise to v arrangements in the same cyclic order, and a further v are obtained by reversing them. So there are $v!/(2v) = (v-1)!/2$ Hamiltonian cycles. □

This number grows very quickly. For $n = 3, 4, 5, 6, 7$ the value of $(n-1)!/2$ is $1, 3, 12, 60, 360$; in K_{10}, there are 181440, and in K_{24}, there are about 10^{23} Hamiltonian cycles. 24 vertices is not an unreasonably large network, but performing so many summations and comparing them would be impossible in practice. To give you some idea of the times involved, if you had a computer capable of evaluating and sorting through a million ten-vertex cycles per second, a complete search solution of the Traveling Salesman Problem for K_{10} would take about 0.18 seconds. No problem so far. However, assuming the computer took about twice as much time to process a 24-vertex cycle, the complete search for K_{24} would take about a billion years.

For practical use, fast methods of reaching a "good solution" have been developed. Although they are not guaranteed to give the optimal answer, these approximation algorithms often give a route that is significantly cheaper than the average.

Approximation Algorithms

The *nearest neighbor* method works as follows. Starting at some vertex x, one first chooses the edge incident with x whose cost is least. Say that edge is xy. Then an

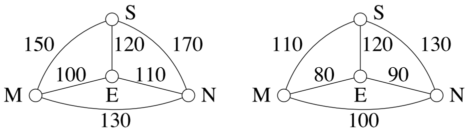


Fig. 4.12. Traveling Salesman Problem example

edge incident with y is chosen in accordance with the following rule: if y is the vertex most recently reached, then eliminate from consideration all edges incident with y that lead to vertices that have already been chosen (including x), and then select an edge of minimum cost from among those remaining. This rule is followed until every vertex has been chosen. The cycle is completed by going from the last vertex chosen back to the starting position x. This algorithm produces a directed cycle in the complete graph, but not necessarily the cheapest one, and different solutions may come from different choices of initial vertex x.

The *sorted edges* method does not depend on the choice of an initial vertex. One first produces a list of all the edges in ascending order of cost. At each stage, the cheapest edge is chosen with the restriction that no vertex can have degree 3 among the chosen edges, and the collection of edges contains no cycle of length less than v, the number of vertices in the graph. This method always produces an undirected cycle, and it can be traversed in either direction.

Sample Problem 4.24. *Suppose the costs of travel between St. Louis, Nashville, Evansville, and Memphis are as shown in dollars on the left in Figure 4.12. You wish to visit all four cities, returning to your starting point. What routes do the two algorithms recommend?*

Solution. The nearest neighbor algorithm, applied starting from Evansville, starts by selecting the edge EM because it has the least cost of the three edges incident with E. The next edge must have M as an endpoint, and ME is not allowed (one cannot return to E, it has already been used), so the cheaper of the remaining edges is chosen, namely MN. The cheapest edge originating at N is NE, with cost \$110, but inclusion of this edge would lead back to E, a vertex that has already been visited, so NE is not allowed, and similarly NM is not available. It follows that NS must be chosen. So the algorithm finds route EMNSE, with cost \$520.

A different result is achieved if one starts at Nashville. Then the first edge selected is NE, with cost \$110. The next choice is EM, then MS, then SN, and the resulting cycle NEMSN costs \$530.

If you start at St. Louis, the first stop will be Evansville (\$120 is the cheapest flight from St. Louis), then Memphis, then Nashville, the same cycle as the Evansville case (with a different starting point), costing \$520. From Memphis, the cheapest leg is to Evansville, then Nashville, and finally St. Louis, for \$530—the same cycle as from Nashville, in the opposite direction.

To apply the sorted edges algorithm, first sort the edges in order of increasing cost: EM($100), EN($110), ES($120), MN($130), MS($150), NS($170). Edge EM is included, and so is EN. The next choice would be ES, but this is not allowed because its inclusion would give degree 3 to E. MN would complete a cycle of length 3 (too short), so the only other choices are MS and NS, forming route EMSNE (or ENSME) at a cost of $530.

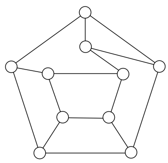
However, in this example, the best route is ENMSE, with cost $510, and it does not arise from the nearest neighbor algorithm, no matter which starting vertex is used, or from the sorted edges algorithm.

Your Turn. A new cut-rate airline offers the fares shown on the right side of the Figure 4.12. What do the algorithms say now?

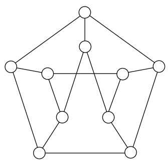
Exercises 4.6 A

1. Find Hamiltonian cycles in the graphs shown.

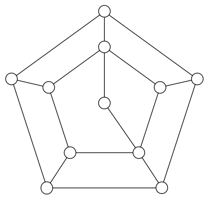
(i)



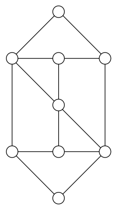
(ii)



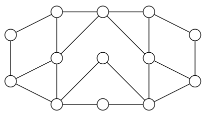
(iii)



(iv)

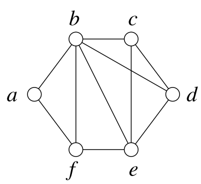


2. Prove that the following graph contains no Hamiltonian cycle.

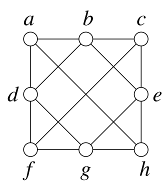


3. List all Hamiltonian cycles in the graphs shown.

(i)



(ii)



4. In the following problems you are given the costs associated with the edges of a complete graph with vertices $abcde$. Find the costs of the routes generated by the nearest neighbor algorithm starting at each of the five vertices in turn and by the sorted edges algorithm.

(i) $ab = 24$, $ac = 23$, $ad = 20$, $ae = 21$, $bc = 27$,
$bd = 22$, $be = 30$, $cd = 26$, $ce = 27$, $de = 28$;

(ii) $ab = 24$, $ac = 22$, $ad = 30$, $ae = 29$, $bc = 17$,
$bd = 19$, $be = 30$, $cd = 18$, $ce = 21$, $de = 25$;

(iii) $ab = 12$, $ac = 18$, $ad = 14$, $ae = 19$, $bc = 22$,
$bd = 16$, $be = 15$, $cd = 18$, $ce = 17$, $de = 11$;

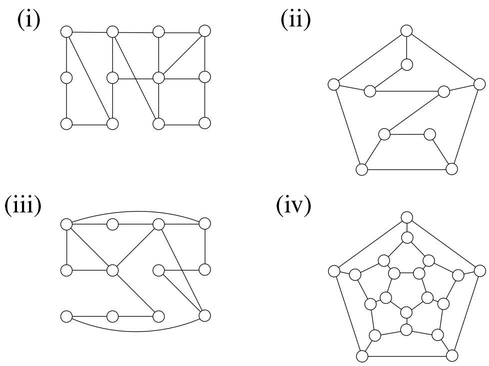
(iv) $ab = 27$, $ac = 22$, $ad = 28$, $ae = 17$, $bc = 33$,
$bd = 14$, $be = 23$, $cd = 24$, $ce = 26$, $de = 19$;

(v) $ab = 24$, $ac = 26$, $ad = 20$, $ae = 21$, $bc = 33$,
$bd = 29$, $be = 23$, $cd = 25$, $ce = 27$, $de = 28$;

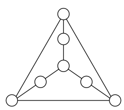
(vi) $ab = 59$, $ac = 45$, $ad = 60$, $ae = 58$, $bc = 56$,
$bd = 46$, $be = 54$, $cd = 58$, $ce = 47$, $de = 48$.

Exercises 4.6 B

1. Find Hamiltonian cycles in the graphs shown.

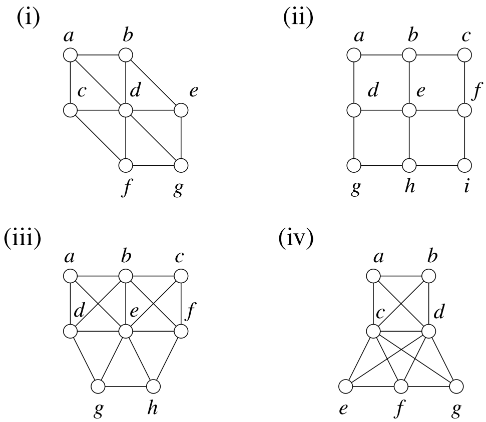


2. Prove that the following graph contains no Hamiltonian cycle.

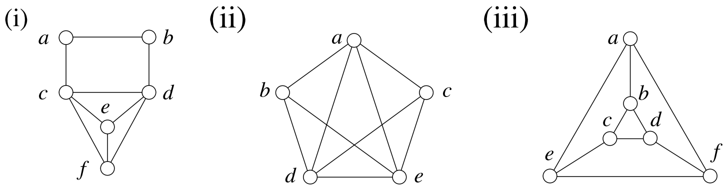


3. Prove that the Petersen graph (see Exercise 4.3A.9) contains no Hamiltonian cycle.

4. List all Hamiltonian cycles in the graphs shown.



5. List all Hamiltonian cycles in the graphs shown.



6. Consider a complete graph K_5 with vertices $abcde$, the costs associated with the edges being

$$ab = 1, \quad ac = 2, \quad ad = 3, \quad ae = 4, \quad bc = 7,$$
$$bd = 8, \quad be = 9, \quad cd = 5, \quad ce = 6, \quad de = 10.$$

Find the cost of the cheapest Hamiltonian cycle in this graph by a complete search. Then find the costs of the routes generated by the nearest neighbor algorithm starting at each of the five vertices in turn and show that the nearest neighbor solution is never the cheapest.

7. In the following problems you are given the costs associated with the edges of a complete graph with vertices $abcde$. Find the costs of the routes generated by the nearest neighbor algorithm starting at each of the five vertices in turn and by the sorted edges algorithm.

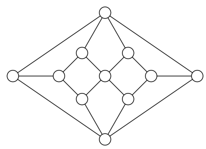
(i) $ab = 24, \quad ac = 26, \quad ad = 20, \quad ae = 21, \quad bc = 33,$
$bd = 29, \quad be = 30, \quad cd = 25, \quad ce = 27, \quad de = 28;$

(ii) $ab = 59, \quad ac = 69, \quad ad = 60, \quad ae = 58, \quad bc = 56,$
$bd = 69, \quad be = 54, \quad cd = 58, \quad ce = 66, \quad de = 61;$

(iii) $ab = 16, \quad ac = 24, \quad ad = 30, \quad ae = 48, \quad bc = 27,$
$bd = 29, \quad be = 44, \quad cd = 16, \quad ce = 46, \quad de = 51;$

(iv)	$ab = 91$,	$ac = 79$,	$ad = 75$,	$ae = 82$,	$bc = 87$,
	$bd = 64$,	$be = 78$,	$cd = 68$,	$ce = 81$,	$de = 88$;
(v)	$ab = 45$,	$ac = 28$,	$ad = 50$,	$ae = 36$,	$bc = 21$,
	$bd = 42$,	$be = 34$,	$cd = 44$,	$ce = 39$,	$de = 25$;
(vi)	$ab = 11$,	$ac = 15$,	$ad = 13$,	$ae = 18$,	$bc = 20$,
	$bd = 12$,	$be = 16$,	$cd = 14$,	$ce = 19$,	$de = 17$;
(vii)	$ab = 14$,	$ac = 12$,	$ad = 15$,	$ae = 24$,	$bc = 27$,
	$bd = 29$,	$be = 44$,	$cd = 16$,	$ce = 46$,	$de = 51$;
(viii)	$ab = 44$,	$ac = 49$,	$ad = 56$,	$ae = 52$,	$bc = 45$,
	$bd = 54$,	$be = 48$,	$cd = 51$,	$ce = 55$,	$de = 50$.

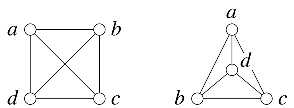
8. Prove that the following graph contains no Hamiltonian cycle.



4.7 Graph Representations and Colorings

Representations and Crossings

Consider these two diagrams.



They represent the same graph, the complete graph K_4 on four vertices. But as diagrams they are quite different: in the left-hand version, two edges cross; in the right-hand diagram there are no crossings. We shall refer to the two diagrams as different *representations* of K_4 in the plane. The *crossing number* of a representation is the number of different pairs of edges that cross; the crossing number $\nu(G)$ of a graph G is the minimum number of crossings in any representation of G. The diagram shows representations of K_4 with crossing numbers 1 and 0, respectively. As one cannot have a representation with fewer than 0 crossings, we have shown that $\nu(K_4) = 0$.

A representation is called *planar* if it contains no crossings, and a *planar graph* is a graph that has a planar representation. In other words, a *planar graph* G is one for which $\nu(G) = 0$. So K_4 is planar.

There are many applications of crossing numbers. An early use was in the design of railway yards, where it is inconvenient to have the different lines crossing, and it is better to have longer track rather than extra intersections. An obvious extension of this idea is freeway design. At a complex intersection, fewer crossings will mean fewer expensive flyover bridges. More recently, small crossing numbers have proven important in the design of VLSI chips; if two parts of a circuit are not to be connected electrically, but they cross, a costly insulation process is necessary.

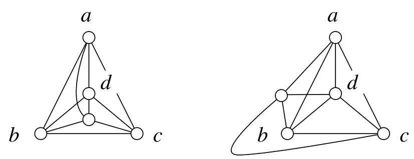
Surprisingly enough, the crossing numbers of complete graphs are not easy to calculate. A very easy (but very bad) upper bound can be found as follows. If the v vertices of K_v are arranged in a circle, and all the edges are drawn as straight lines, then every set of four vertices contributes exactly one crossing. So

$$\nu(K_v) \leq C(v, 4).$$

More sophisticated constructions can be used to find better upper bounds, but the exact answer is not known in general.

Sample Problem 4.25. *Prove that the crossing number of the complete graph on five vertices is* 1.

Solution. We start by considering smaller complete graphs. The representation of K_3 as a triangle is essentially unique: one can introduce a crossing only by a fanciful, twisting representation of one or more edges. Adding one more vertex to obtain a K_4, one always obtains a representation like the left-hand diagram we saw at the start of the section, and any other planar representation is essentially equivalent to this. So any planar representation of K_5 can be obtained by introducing another vertex into this representation of K_4. If a new vertex e is introduced inside the triangle abd, then the representation of ce must cross one of ab, ad, or bd. Similarly, the introduction of e inside any triangle causes a crossing involving the edge joining e to the vertex that is not on the triangle. (The "outer area" is considered to be the triangle abc.) The following diagrams illustrate this for the triangles abc and abc.



Therefore, $\nu(K_5) \geq 1$. A representation with one crossing is easy to find—we have found three of them already—so $\nu(K_5) = 1$.

Your Turn. Prove that the crossing number of the complete bipartite graph $K_{3,3}$ is 1.

It is easy to calculate the crossing numbers of some small graphs. For example, the crossing number of any tree is 0. To see this, simply draw the tree as a tree diagram, as we did in Section 4.5. The diagram has no crossings. Similarly, any cycle has crossing number 0.

Suppose G is planar. If a new graph were constructed by inserting a new vertex of degree 2 into the middle of an edge (*dividing an edge*), or by deleting a vertex of degree 2 and joining the two vertices adjacent to it (*eliding a vertex*), that new graph will also be planar. Graphs that can be obtained from each other in this way are called *homeomorphic*.

As K_5 and $K_{3,3}$ are not planar, it follows that a graph having either as a subgraph could not be planar. Moreover, a graph that is homeomorphic to one with a subgraph homeomorphic to K_5 or $K_{3,3}$ cannot be planar. In fact, it may be shown that this necessary condition for planarity is also sufficient. The proof is too difficult for our purposes.

Theorem 15. *G is planar if and only if G is homeomorphic to a graph containing no subgraph homeomorphic to K_5 or $K_{3,3}$.*

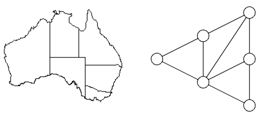
Maps and Planarity

By a *map* we shall mean what is usually meant by a map of a continent (showing countries) or a country (showing states or provinces). However, we shall make one restriction. Sometimes one state can consist of two disconnected parts (in the United States, Michigan consists of two separate land masses, unless we consider man-made constructions such as the Mackinaw Bridge). We shall exclude such cases from consideration.

Given a map, we can construct a graph as follows: the vertices are the countries or states on the map, and the two vertices are joined by an edge precisely when the corresponding countries have a common border.

Sample Problem 4.26. *Represent the map of the mainland of Australia, divided into states and the Northern Territory, as a graph.*

Solution. The map is shown, for reference, alongside its graph.



Vertex T represents the Northern Territory; the states (New South Wales, Queensland, South Australia, Victoria, and Western Australia) are denoted by their initials.

Your Turn. Represent the map of the New England states—Maine, New Hampshire, Massachusetts, Vermont, Connecticut, Rhode Island—as a graph.

Sometimes two states have only one point in common. An example, in the United States, occurs where Utah and New Mexico meet at exactly one point, as do Colorado and Arizona. We shall say that states with only one point in common have no border, and treat them as if they do not touch.

It is always possible to draw the graph corresponding to a map without crossings. To see this, draw the graph on top of the map by putting a vertex inside each state and joining vertices by edges that pass through common state borders so the graph of any map is planar. Conversely, any planar graph is easily represented by a map. Therefore, the theory of maps (with the two stated restrictions) is precisely the theory of planar graphs.

In 1852, William Rowan Hamilton wrote to Augustus de Morgan concerning a problem that had been posed by a student, Frederick Guthrie. Guthrie said: cartographers know that any map (our definition) can be colored using four or less colors; is there a mathematical proof? (Guthrie later pointed out that the question had come from his brother, Francis Guthrie.)

Kempe published a purported proof in 1879. It was thought that the matter was over, but in 1890 Heawood pointed out a fallacy in Kempe's proof. Heawood did, however, repair the proof sufficiently to prove that every planar map can be colored in five colors.

After Heawood's paper appeared, there was renewed interest in the four-color problem. Because it was easy to state and tantalizingly difficult to prove, it became one of the most celebrated unsolved problems in mathematics. In 1976, Appel and Haken finally proved that any map can be colored in at most four colors. Their proof involved computer analysis of a large number of cases, so many that human analysis of all the cases is not feasible. We state their theorem as

Theorem 16 (The Four-Color Theorem). *Every map can be colored using at most four colors.*

Graph Colorings

Instead of coloring a map, we could instead apply the colors to the vertices of the corresponding graph. The result to divide the set of vertices into disjoint subsets, where each subset consists of the vertices that receive a specific color. The defining property is that no two elements of the same subset are adjacent. The terminology of colors and colorings is traditionally used for this sort of problem. Given a set $C = \{c_1, c_2, \ldots\}$ is a set of labels called *colors*, a C-*coloring* (or C-vertex coloring) ξ of a graph G is a function

$$\xi : V(G) \to C,$$

that is, a rule that assigns a uniquely-defined color to each vertex. The sets $V_i = \{x : \xi(x) = c_i\}$ are called *color classes*. A *proper coloring* of G is a coloring in

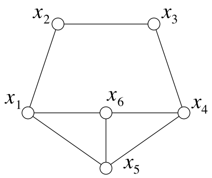


Fig. 4.13. A graph with chromatic number 3

which no two adjacent vertices belong to the same color class. In other words,

$$x \sim y \quad \Rightarrow \quad \xi(x) \neq \xi(y).$$

A proper coloring is called an *n-coloring* if C has n elements. If G has an n-coloring, then G is called *n-colorable*. So the Four-Color Theorem says that every planar graph is 4-colorable.

The *chromatic number* $\chi(G)$ of a graph G is the smallest integer n such that G has an n-coloring. A coloring of G in $\chi(G)$ colors is called *minimal*. We use the phrase "G is *n-chromatic*" to mean that $\chi(G) = n$ (but note that a minority of authors use n-chromatic as a synonym for n-colorable).

Some easy small cases of chromatic number are

$$\chi(K_v) = v,$$
$$\chi(P_v) = 2.$$

A cycle of length v has chromatic number 2 if v is even and 3 if v is odd. The star $K_{1,n}$ has chromatic number 2; this is an example of the next theorem.

Clearly, $\chi(G) = 1$ if G has no edges and $\chi(G) = 2$ if G has at least one edge.

Suppose G is a graph with a subgraph H, and suppose G has been n-colored. If all the vertices and edges that are not in H were deleted, there would remain a copy of H that is colored in at most n colors. So $\chi(H) \leq \chi(G)$ whenever H is a subgraph of G.

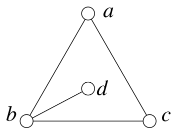
The graph of Figure 4.13 can be colored in three colors:

$$V_0 = \{x_1, x_4\}, \qquad V_1 = \{x_2, x_5\}, \qquad V_2 = \{x_3, x_6\}.$$

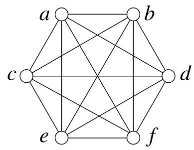
Exercises 4.7 A

1. Which of the following graphs are planar?

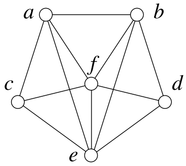
(i)



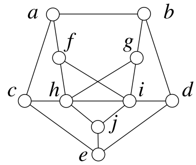
(ii)



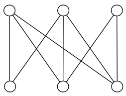
(iii)



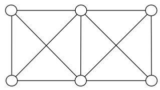
(iv)



(v)

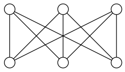


(vi)

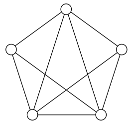


2. For each of the following graphs, find the chromatic number and find a coloring that uses the minimum number of colors.

(i)



(ii)



3. Find the chromatic number and find a coloring that uses the minimum number of colors for the graphs in Exercise 1, parts (v) and (vi).

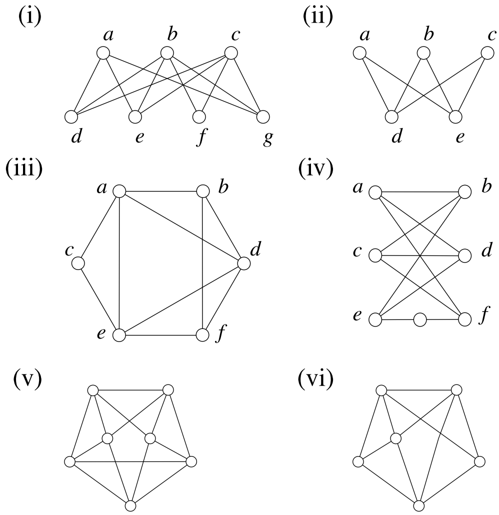
4. The wheel W_n was defined in Exercise 4.3A.5 to consist of an n-cycle together with a further vertex adjacent to all n vertices. What is $\chi(W_n)$?

5. Verify that there are exactly three connected incomplete graphs on four vertices with maximum degree 3, and all can be 3-colored.

6. G is formed from the complete graph K_v by deleting one edge. Prove that $\chi(G) = v - 1$ and describe a way of coloring G in $v - 1$ colors.

Exercises 4.7 B

1. Which of the following graphs are planar?

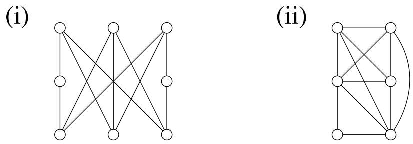


2. Prove that $\nu(K_{1,n}) = \nu(K_{2,n}) = 0$ for any n.

3. Prove that $\nu(K_{3,3}) = 1$. What is $\nu(K_{3,4})$?

4. In Sample Problem 4.26, we drew the graph corresponding to a map of Australia, but omitted the Australian Capital Territory (A.C.T.) and Tasmania. Draw the graph that would result if these two areas were included. (For your information, the A.C.T. lies wholly within New South Wales, and Tasmania is an island to the south.) What is the chromatic number?

5. Find the crossing numbers of the following graphs.

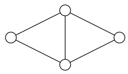


For each graph in the preceding question, find the chromatic number and find a coloring that uses the minimum number of colors.

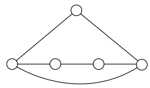
6. G is the union of two graphs G_1 and G_2 that have one common vertex. Show that $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$.

7. For each of the following graphs, find the chromatic number and find a coloring that uses the minimum number of colors.

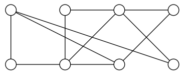
(i)



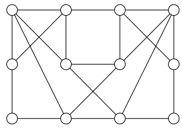
(ii)



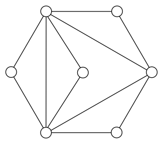
(iii)



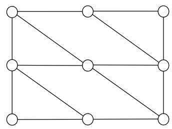
(iv)



(v)



(vi)



8. Write P for the Petersen graph, which was defined in Exercise 4.3A.9

(i) What is $\chi(P)$?

(ii) Prove that P is not planar. What is its crossing number?

5

Linear Equations and Matrices

5.1 Coordinates and Lines

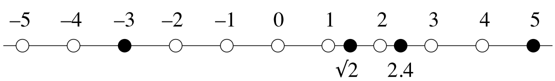
The Number Line

You should be familiar with the method of representing numbers as points on a number line. A horizontal line, or *axis*, is drawn, with two special points marked 0 and 1; 0 is called the *origin*, and is to the left of 1. The positive number x is represented by a point on the axis whose distance to the right of 0 is x times as far as the distance from 0 to 1. We usually say the distance from 0 to 1 is the *unit* distance, and say the point representing x is x *units to the right of the origin*. Negative numbers go to the left of 0, and $-x$ is x units to the left of the origin.

For convenience, other numbers are often shown on the number line.

Sample Problem 5.1. *Show the points corresponding to* 5, -3, $\sqrt{2}$, *and* 2.4 *on a number line.*

Solution. We shall show the integers from -5 to 5 on our line, with -3, $\sqrt{2}$, and 2.4 indicated by heavy dots. The line is:



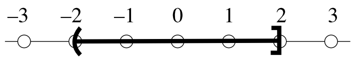
Your Turn. Show the points corresponding to -2, 1, and 2.5 on a number line.

Intervals and Inequalities

One way in which number lines are used is to show sets of numbers. They are particularly useful for showing *intervals*. An interval corresponds to a set of points on the number line. The set is indicated by a heavy line. The diagram is consistent with

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1_5, © Springer Science+Business Media, LLC 2012

the notation for intervals: an open bracket at an endpoint means that the endpoint is *excluded* from the solution, while a closed bracket means it is *included*. In other words, the interval $(-2, 2]$ is shown as

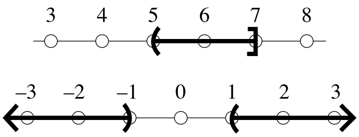


An arrowhead replaces the notation ∞ and means *and so on to infinity.*

This is very useful for illustrating *inequalities*. The solution of an inequality is a collection of intervals.

Sample Problem 5.2. *Show the solution sets of* $5 < x \leq 7$ *and* $x^2 > 1$ *on number lines.*

Solution. The solutions to the two inequalities are $(5, 7]$ and $(-\infty, -1) \cup (1, \infty)$.



Your Turn. Show the solution sets of $-1 \leq x < 3$ and $x^2 \geq 4$ on number lines.

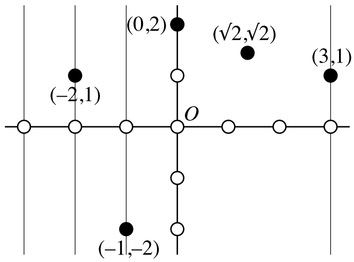
Rectangular Coordinates

If you want to tell somebody how to drive from one point to another in a city, you sometimes say something like “go four blocks East, to Main Street, then go three blocks North”. In other words, you give a distance along one number line, then a distance along another number line. The most efficient way to do this is to set up two standard number lines, or *axes*, one in a west–east direction with positive numbers to the east and the other running south–north with positive numbers to the north.

When two axes are drawn on the page, the west–east axis runs from left to right and is called the x-axis, while the south–north axis runs from bottom to top and is called the y-axis. The point where the two lines meet is called the *origin* and often denoted O. If you start at the origin, the result of going four units to the right is the point 4 units along the x-axis, and is written $(4, 0)$. We also say this point has $x = 4$. If you then go three units up you reach the point $(4, 3)$, which is also called the point $x = 4$, $y = 3$. These values x and y are called the *coordinates* of the point; 4 is the x*-coordinate* and 3 is the y-coordinate. We often write (x, y) to mean a typical point, with x-coordinate value x and y-coordinate value y. This standard ordering—always x-value first—does not have to specify which is the west–east direction and which the south–north, and saves a lot of writing.

Sample Problem 5.3. *Show the points with coordinates* $(3, 1)$, $(\sqrt{2}, \sqrt{2})$, $(0, 2)$, $(-2, 1)$, *and* $(-1, -2)$ *on a set of axes.*

Solution. The points are shown below. Notice that negative values place the points to the left of, or below, the axes, while 0 is on the axis. Coordinates do not need to be integers.



Your Turn. Show the points with coordinates $(-1, -1)$, $(2.3, 1.4)$, and $(0, 1)$ on a set of axes.

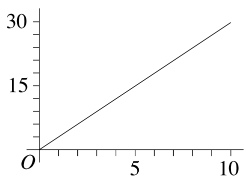
Graphing Equations

If two quantities x and y are related, then the set of all points (x, y) such that x is related to y is called the *graph of the relationship*. For example, consider the temperature at a given time. If the temperature at time t hours after midnight is d degrees, this is represented by the point (t, d). The points trace out a line. Figure 5.1 might show the graph of temperature on a winter's day.

Very often relationships are embedded in equations. Of particular importance is the linear equation, whose graph is the straight line.

Sample Problem 5.4. *A production line moves three feet per minute. The line is* 30 *feet long. Suppose an object is placed on the line at* 3PM. *Draw a graph that shows its position at a given time.*

Solution. We are not interested in any time before 3PM or after 3:10PM. So let x equal the number of minutes after 3PM (for example, 3:05PM is represented by $x = 5$.) The variable y is the distance in feet from the starting point. Then the graph is as shown below. It is the straight line segment joining $(0, 0)$ to $(10, 30)$.



For every point (x, y) on the above graph, the relationship $y = 3x$ is true. The segment is part of the *graph of the equation* $y = 3x$ (the graph of this equation would extend in both directions past the endpoints of the segment).

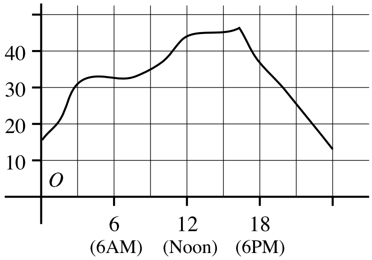


Fig. 5.1. Typical temperature graph

The equation of a straight line is a *linear equation* in the two variables x and y, and every such equation has a straight line graph. (In fact, this is why such equations are called "linear".) Most can be expressed in the form "y equals some non-zero multiple of x, plus possibly a constant". The only exceptions are the two axes, which have equations $x = 0$ (the y axis; all the different y values are represented on it, but no x values) and $y = 0$ (the x axis). We could also say "x equals some non-zero multiple of $y \ldots$", and the most general form is to say a straight line is any one of the sets

$$\{(x, y) : Ax + By + C = 0\},$$

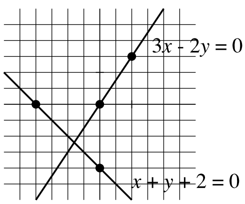
where A, B and C can be any real numbers, provided A and B are not both zero. We also say this is the line "with equation $Ax + By + C = 0$".

Given the equation of a straight line, the simplest way to draw the graph is to find two points on a straight line, join them, and extend this segment in both directions. The easiest method is to find the two points where $x = 0$ and $y = 0$ and join them. These are called the *intercepts* of the line. In those cases where (0, 0) lies on the line, the two intercepts are the same, and the usual method is to find one other point. (For example, put $x = 1$ in the equation, and find the corresponding value of y.)

Sample Problem 5.5. *Draw the straight lines with equations*

$$x + y + 2 = 0, \qquad 3x - 2y = 0.$$

Solution. The line $x + y + 2 = 0$ passes through $(0, -2)$: to see this, put $x = 0$ in the equation, and you get $0 + y + 2 = 0$, or $y = -2$; and $(-2, 0)$. The line $3x - 2y = 0$ passes through (0, 0) and $(1, \frac{3}{2})$. The lines are shown in the diagram:



Your Turn. Draw the straight lines with equations

$$2x - y - 4 = 0, \qquad x + y = 0.$$

You should realize that two different equations can correspond to the same straight line. For example, the equations $Ax + By + C = 0$ and $2Ax + 2By + 2C = 0$ give the same line.

Slope

The graph of $3x - 2y = 0$ passes through the points (0, 0), $(1, \frac{3}{2})$, (2, 3), and (4, 6). Observe that, if we take any two points, the difference in their y coordinates is $\frac{3}{2}$ times the difference in their x-coordinates. We call this common ratio the *slope* of the line; we say $3x - 2y = 0$ has slope $\frac{3}{2}$. Every straight line has a slope.

Say a line has slope m. If (a, b) and (x, y) are any points on the line, then

$$(y - b) = m(x - a).$$

In particular, if $(0, b)$ is an intercept, so that $(0, b)$ is a point on the line,

$$y = mx + b.$$

This is called the *slope-intercept* form of the equation of the line, and corresponds to the standard form

$$mx + (-1)y + b = 0.$$

Every line has a slope-intercept form except for the vertical lines, like $x = 4$. This line has no intercept on the y axis, and we say it "has infinite slope".

Lines with the same slope are *parallel* in the ordinary geometric sense.

Sample Problem 5.6. *What are the slopes of the lines*

$$x + y + 2 = 0, \qquad 3x - 2y = 0$$

of Sample Problem 5.5? Write their equations in slope–intercept form.

Solution. The line $x + y + 2 = 0$ has slope -1. Its slope–intercept form is $y = -x - 2$. The line $3x - 2y = 0$ is $y = \frac{3}{2}x$, with slope $\frac{3}{2}$.

Your Turn. What are the slopes of the lines

$$2x + y - 4 = 0, \qquad x + y = 0?$$

Write their equations in slope–intercept form.

Exercises 5.1 A

1. Represent the following points on a number line:

(i) 0.3; (ii) $\sqrt{5}$; (iii) 1.44;
(iv) -3.1; (v) $\frac{3}{5}$; (vi) 3;
(vii) -2; (viii) $\sqrt{3}-6$; (ix) $6-\sqrt{3}$.

2. Represent the solution sets of the following inequalities on a number line:

(i) $3 \le x \le 5$; (ii) $2 \le x < 4$; (iii) $1 \le x^2 \le 3$;
(iv) $x + 1 \ge 0$; (v) $x^2 \ge 1$; (vi) $x^2 > 1$;
(vii) $x \ge 0$; (vii) $x^2 + 1 > 2$.

3. Represent the following points on a set of axes:

(i) $(1, 0)$; (ii) $(-1, -1)$; (iii) $(\sqrt{2}, 2)$;
(iv) $(2, -1)$; (v) $(-2, -1.5)$; (vi) $(-2, 2)$.

4. Rain is falling at a steady rate. Every three hours, enough rain falls to raise the level in a rain gauge by one inch. Suppose you empty the gauge at 1PM and put it out in the rain again. Draw a graph that shows the level of rain in the gauge at times from 1PM to 10PM.

5. At 2PM you start driving from St. Louis to Chicago at exactly 50 miles per hour. After 30 minutes you change speed to 60 miles per hour.

(i) How far were you from St. Louis when you changed speed?

(ii) Draw a graph that shows your distance from St. Louis at times from 2PM to 6PM.

6. To convert Fahrenheit to Celsius temperature, one subtracts 32 from the Fahrenheit temperature and multiplies by $5/9$.

(i) Suppose the temperature $x°$ Fahrenheit corresponds to $y°$ Celsius. What is the equation linking x and y?

(ii) If the temperature is 68° Fahrenheit, what is it in Celsius?

(iii) If the temperature is 25° Celsius, what is it in Fahrenheit?

(iv) Draw the graph of this relationship.

7. Draw the graphs of straight lines with the following equations:

(i) $2x + y = 0$; (ii) $2x - y - 1 = 0$; (iii) $-x + 2y + 4 = 0$;
(iv) $2x + 3y = 0$; (v) $2x - 2y = 0$; (vi) $2x - y - 3 = 0$.

8. For each line in Exercise 7, what is the slope? Write the equation of the line in slope-intercept form.

9. What is the equation of the line in Exercise 4? What is its slope?

Exercises 5.1 B

1. Represent the following points on a number line:

(i) 1.2; (ii) $1+\sqrt{2}$; (iii) 0.75;
(iv) $\frac{2}{7}$; (v) 3; (vi) -5;
(vii) $3-\sqrt{3}$; (viii) $1-\sqrt{3}$; (ix) -1.3.

2. Represent the solution sets of the following inequalities on a number line:

(i) $1 \le x \le 2$; (ii) $-2 < x \le 1$; (iii) $3 \le x^2 \le 4$;
(iv) $x - 2 < 4$; (v) $x^2 \ge 2$; (vi) $x^2 > 2$;
(vii) $3 \le x < 6$; (viii) $-1 < x \le 3$; (ix) $2 > x^2 > 1$;
(x) $1 < x^2 \le 4$; (xi) $x + 2 \ge 3$; (xii) $x^2 < 3$.

3. Represent the following points on a set of axes:

(i) $(0, 2)$; (ii) $(-2, -1)$; (iii) $(1, -2)$;
(iv) $(2.1, 1.2)$; (v) $(1, \sqrt{3})$; (vi) $(-2, 2.3)$.

4. Represent the following points on a set of axes:

(i) $(0, 0)$; (ii) $(3, 1)$; (iii) $(1.3, 1.7)$;
(iv) $(-2, 1)$; (v) $(-1, 1+\sqrt{2})$; (vi) $(3-\sqrt{5}, -2)$.

5. The state of Missouri decides to change its state income tax laws so that those with annual income less than \$10000 pay no tax and all others pay 10% of their income. Draw a graph that shows the amount of tax paid on incomes up to \$100000.

6. A train runs at a steady rate of 30 miles per hour. It passes a station at 12 noon.

(i) Draw the graph of a line that shows how far the train is from the station at a given time, from noon to 6PM.

(ii) What is the equation of the line in part (i)? What is its slope?

(iii) Suppose the train changed its speed to 35 miles per hour at 2PM.

(a) How far had the train traveled from the station when its speed changed?

(b) Draw a graph that shows how far the train is from the station, at times up to 6PM, in this new case.

7. Your cell phone bill is \$40 per month plus 5 cents per minute for calls over 500 minutes.

(i) Draw a graph showing how much you will pay if you use x minutes.

(ii) What is the equation showing how much you will pay if you use x minutes, where $x \ge 500$?

(iii) What is your bill if you use 800 minutes this month?

8. Draw the graphs of straight lines with the following equations:

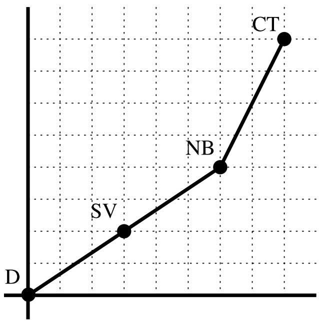
(i) $x + 2y - 1 = 0$; (ii) $3x - 2y + 2 = 0$;
(iii) $-x - 2y + 6 = 0$; (iv) $x + 3y = 0$;
(v) $2x - 4y = 0$; (vi) $x - 3y - 2 = 0$.

9. For each line in Exercise 8, what is the slope? Write the equation of the line in slope-intercept form.

10. Draw the graphs of straight lines with the following equations:

(i) $x + 3y = 0$; (ii) $x + 2y - 1 = 0$;
(iii) $x - 5y = 0$; (iv) $2x + 2y - 2 = 0$;
(v) $2x - 3y + 1 = 0$; (vi) $3x - 2y - 2 = 0$.

11. A freight train leaves its depot (D) at midnight. Traveling at constant speed, it passes South Valley station (SV) at 3AM. It maintains its speed until it crosses the Narrows Bridge (NB), 60 miles further from SB, then changes to a new speed until it reaches the City terminal (CT). The trip is represented by the following graph, with time on the x axis and distance on the y axis.



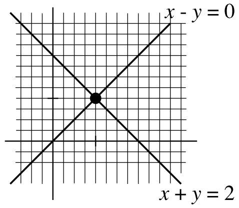
Assuming that the dotted lines are equally spaced,

(i) How far is it from South Valley station to Narrows Bridge?
(ii) How fast was the train traveling between Narrows Bridge and the City terminal?
(iii) When did the train reach City terminal?
(iv) How long did the train take to travel from South Valley station to Narrows Bridge?

5.2 Systems of Linear Equations

Pairs of Linear Equations in Two Variables

Suppose (x, y) is the point common to two straight lines (the *point of intersection* of the lines). The values x and y satisfy the equations of the two lines. For example, say both $x + y = 2$ and $x - y = 0$ are true. The graphs of $x + y = 2$ and $x - y = 0$ are shown in the following diagram, and the common point, marked with a dot, is where both equations are true.



Finding the point of intersection of the two lines is done by simultaneously solving the two equations. The usual method, with two equations in the two unknowns x and y, will be called *solution by substitution*, and is carried out as follows. Choose one of the equations, treat it as though y were a constant, and solve for x. Then substitute that solution into the other equation. For example, if we take $x + y = 2$ and solve it as though y were a constant, we get $x = 2 - y$. Now go back to the equation $x - y = 0$ and replace x by $2 - y$. The resulting equation is $(2 - y) - y = 0$, or $2 - 2y = 0$, which is equivalent to $2y = 2$, or $y = 1$. Finally, since $x = 2 - y$, we must have $x = 2 - 1 = 1$, and the solution is $x = 1$, $y = 1$. The point of intersection, or solution point, is $(1, 1)$.

Sample Problem 5.7. *Simultaneously solve the equations* $3x + 2y = 4$, $x + 3y = -1$.

Solution. From $x + 3y = -1$ we get $x = -3y - 1$. Substituting, $3(-3y - 1) + 2y = 4$, or $7y = -7$, so $y = -1$. Therefore, $x = -3(-1) - 1 = 2$. The solution is $x = 2$, $y = -1$; the point of intersection is $(2, -1)$.

Your Turn. Simultaneously solve the equations $2x + y = 4$, $3x - y = 1$.

A set of two linear equations in two unknowns might have no solutions. For example, the two equations $x - y = 0$ and $x - y = 1$ can have no joint solution: if the values x and y satisfy $x - y = 0$, then $x - y = 1$ must be false. In this case, the graphs of the two equations will be parallel lines. Another possibility is that all solutions of one equation will also be solutions of the second. This means that the two equations are equivalent, and one is a multiple of the other; an example is the set of two equations $x - 2y = 1$ and $2x - 4y = 2$. The two will have the same graph.

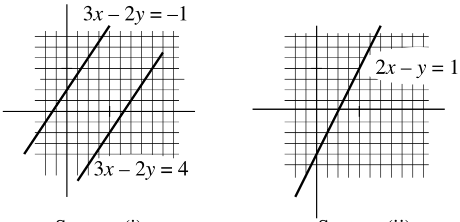
In every other case, the two equations represent two lines that are not parallel. From elementary geometry we know that non-parallel lines meet at exactly one point. So there will be exactly one solution.

The set of all solutions to a system of equations is its *solution set*. We have just seen that the solution set can be empty, can be a one-element set, or can be infinite. If the solution set is empty, the equations are called *inconsistent*, and otherwise they are *consistent*. If the solution set is infinite, the equations are *dependent*; otherwise consistent equations are *independent*.

Sample Problem 5.8. *Find the solution sets of the following systems of equations, and sketch the corresponding graphs.*

$$\text{(i)}\quad \begin{aligned} 3x - 2y &= 4,\\ 3x - 2y &= -1; \end{aligned} \qquad \text{(ii)}\quad \begin{aligned} 2x - y &= 1,\\ -4x + 2y &= -2. \end{aligned}$$

Solution. In case (i), $3x - 2y$ cannot equal both 4 and -1, so there are no solutions, and the solution set is empty. In system (ii), whenever the first equation is true, the second will be true also: each side of the second equation is (-2) times the corresponding side of the first one. The solution set could be written $\{(x, y) \mid 2x - y = 1 \mid x \in \mathbb{R}\}$ or $\{(x, 2x - 1) \mid x \in \mathbb{R}\}$. The graphs are



System (i) System (ii)

Your Turn. Find the solution sets of the following systems of equations, and sketch the corresponding graphs:

$$\begin{aligned} 2x + 2y &= 3,\\ 2x + 2y &= 4; \end{aligned} \qquad \begin{aligned} x - 2y &= 1,\\ 2x - 4y &= 2. \end{aligned}$$

Solution by Elimination

Consider the system of equations

$$\begin{aligned} x + y &= 8,\\ x - y &= 4. \end{aligned} \tag{5.1}$$

If $x + y = 8$ is true, then

$$x + y + c = 8 + c$$

will be true for any real quantity c. In particular,

$$x + y + (x - y) = 8 + (x - y), \tag{5.2}$$

whatever the values of x and y may be. Now suppose x and y form a solution of (5.1). Then $x - y = 4$, so we could write 4 instead of $(x - y)$ on either side of (5.2). Let's make this change on the *right* side only. We have

$$x + y + (x - y) = 8 + 4 = 12,$$

so $2x = 12$ and $x = 6$. So any solution of (5.1) must have $x = 6$. Then (5.2) tells us $6 - y = 4$, so $y = 6 - 4 = 2$. The only possible solution is $x = 6$, $y = 2$. If you check, you'll see that these values make both equations in (5.1) true, so we have solved the system.

The key to this method was the way the two terms y and $-y$ eliminated each other. For this reason we say we solved the equations *by elimination*, and say y was eliminated. The technique is sometimes called the *addition* (or *subtraction*) method because it can be described like this: we *add* the left-hand side of the second equation to the left-hand side of the first, and then add the right-hand side of the second equation to the right-hand side of the first. Usually we simply say we *add the second equation to the first.*

We could also have eliminated x. Starting from the first equation in (5.1), we could write

$$x + y - (x - y) = 8 - (x - y),$$

so

$$x + y - x + y = 8 - 4 = 4,$$

$2y = 4$, $y = 2$, and we then deduce $x = 6$ from the original equations.

Sample Problem 5.9. *Solve the following equations by elimination:*

$$6x + 4y = 10,$$
$$5x - 2y = 3.$$

Solution. We would like to eliminate y from the first equation. The coefficient of y in the second equation is 2, not 4. To get around this problem, we could multiply both sides of the second equation by 2. This will not change the solution set; we could multiply by any constant other than 0. In other words, we proceed to solve the equations

$$6x + 4y = 10,$$
$$10x - 4y = 6.$$

Adding the equations, we find

$$16x = 16,$$

so $x = 1$. Substituting this back into the second equation we get

$$5 \cdot 1 - 2y = 3,$$

so $y = 1$, and the solution is $(x, y) = (1, 1)$. We usually say we added 2·(equation 2) to (equation 1), and do not bother to write down the new set of equations.

Your Turn. Solve the following equations by elimination:

$$\begin{aligned} 4x + 3y &= 11, \\ x - y &= 1. \end{aligned}$$

Sample Problem 5.10. *Solve the following equations by elimination:*

$$\begin{aligned} 2x + 3y &= 12, \\ 3x - 2y &= 5. \end{aligned}$$

Solution. In order to eliminate y from the first equation, we multiply the second equation by $\frac{3}{2}$, obtaining

$$\frac{9}{2}x - 3y = \frac{15}{2}$$

and add this to the first equation, getting

$$\begin{aligned} \frac{13}{2}x &= \frac{39}{2}, \\ x &= 3; \end{aligned}$$

substituting back we get $2y = 4$, or $y = 2$.

Systems of Three or More Equations

These techniques can be applied to any number of equations in any number of variables. However, more variability occurs in dependent systems when there are more than two equations.

Sample Problem 5.11. *Solve the following equations by elimination:*

$$\begin{aligned} 3x + 2y - 2z &= 5, \\ x - 3y + 3z &= -2, \\ 5x - 4y + 4z &= 1. \end{aligned}$$

Solution. We eliminate x from the first and third equations. We add -3 times the second equation to the first equation and -5 times the second equation to the third equation; those two equations become

$$\begin{aligned} 11y - 11z &= 11, \\ 11y - 11z &= 11. \end{aligned}$$

So we get two copies of the same equation. Its solution is $y = 1 + z$, where z can be any real number. When we substitute this into the original second equation, we get

$$\begin{aligned} x - 3(1 + z) + 3z &= -2, \\ x - 3 &= -2. \end{aligned}$$

So $x = 1$, and the solution is

$$(x, y, z) \in \{(1, 1 + z, z) \mid z \in \mathbb{R}\}.$$

The following examples show two further possible forms of dependent solution. Of course, a set of three equations can also be inconsistent (no solutions) or independent (precisely one solution).

Sample Problem 5.12. *Solve the following equations by elimination:*

$$\begin{aligned} x + 2y + z &= 8, \\ 2x + y - z &= 7, \\ 3x - y - 4z &= 3. \end{aligned}$$

Solution. We eliminate z from the second and third equations by adding the first equation to the second equation and 4 times the first equation to the third equation; those two equations become

$$\begin{aligned} 3x + 3y &= 15, \\ 7x + 7y &= 35. \end{aligned}$$

So we get $x = 5 - y$, where y can be any real number. When we substitute this into the original first equation, we get

$$\begin{aligned} (5 - y) + 2y + z &= 8, \\ y + z &= 3. \end{aligned}$$

So $z = 3 - y$, and the solution is

$$(x, y, z) \in \{(5 - y, y, 3 - y) \mid y \in \mathbb{R}\}.$$

Observe that we expressed both x and z in terms of the same variable, y.

Sample Problem 5.13. *Solve the following equations by elimination:*

$$
\begin{aligned}
2x + 4y - 6z &= 6, \\
x + 2y - 3z &= 3, \\
-3x - 6y + 9z &= -9.
\end{aligned}
$$

Solution. We eliminate x from the first and third equations. We add -2 times the second equation to the first equation and 3 times the second equation to the third equation; those two equations both give the form $0 = 0$: both sides are completely eliminated. It follows that any x, y, and z satisfying the original second equation will also satisfy the others. The solution is

$$(x, y, z) \in \{(3 - 2y + 3z, y, z) \mid y \in \mathbb{R}, z \in \mathbb{R}\}.$$

Your Turn. Solve the following equations by elimination:

$$
\begin{aligned}
3x + 2y - z &= 7, \\
x + y &= 3, \\
2x - y - 3z &= 0.
\end{aligned}
$$

The same solution technique can be applied to the case where there are three unknowns but only two equations. In this case, an independent solution is impossible.

Sample Problem 5.14. *Solve the following equations by elimination:*

$$
\begin{aligned}
2x + 3y + 2z &= 4, \\
x + y - z &= 2.
\end{aligned}
$$

Solution. We eliminate z from the second equation by adding the first equation to twice the second equation, obtaining

$$4x + 4y = 8.$$

We have $x = 2 - y$, where y can be any real number. Substituting this into the original first equation yields

$$
\begin{aligned}
2(2 - y) + 3y + 2z &= 4, \\
y + 2z &= 0.
\end{aligned}
$$

So $z = -\frac{1}{2}y$, and the solution is

$$(x, y, z) \in \left\{ \left(2 - y, y, -\frac{1}{2}y\right) \,\middle|\, y \in \mathbb{R} \right\}.$$

Your Turn. Solve the following equations by elimination:

$$
\begin{aligned}
x + 2y - z &= 5, \\
2x - 3y - z &= 3.
\end{aligned}
$$

Exercises 5.2 A

1. In each part, find the complete solution of the system of two linear equations, by substitution.

(i) $2x + y = 12,$
$3x - y = 13;$

(ii) $x + 5y = 11,$
$x + 2y = 2;$

(iii) $5x + 3y = 7,$
$3x - y = 0;$

(iv) $3x - 4y = 4,$
$x + 2y = 3;$

(v) $7x - 2y = 5,$
$12x - 4y = 4;$

(vi) $-2x + 5y = 4,$
$2x - 3y = -2;$

(vii) $3x - y = 5,$
$4x - 2y = 6;$

(viii) $\frac{1}{2}x + 3y = -1,$
$3x - 2y = 2;$

(ix) $5x + 3y = 8,$
$3x + y = 0;$

(x) $2x + y = -1,$
$2x - y = -3.$

2. In each of the parts of Exercise 1, find the complete solution of the system of two linear equations by elimination.

3. In the following problems, say whether the equations are inconsistent, dependent or independent. If they are consistent, write down the solution.

(i) $3x - 2y = 0,$
$6x - 4y = 9;$

(ii) $4x - 5y = 0,$
$2x - 3y = -2;$

(iii) $3x - 2y = 0,$
$2x + 6y = 11;$

(iv) $4x + 2y = 1,$
$-2x + 3y = -\frac{1}{2};$

(v) $4x - 2y = 8,$
$2x - y = 4;$

(vi) $4x - 2y = 4,$
$-2x + y = -2;$

(vii) $6x + 4y = 9,$
$3x + 2y = 4;$

(viii) $3x + y = 30,$
$x + 2y = 12;$

(ix) $x + 2y = 5,$
$10x - y = 6;$

(x) $10x - 16y = 2,$
$-15x + 24y = -3.$

4. In the following problems, say whether the equations are inconsistent, dependent or independent. If they are consistent, write down the solution.

(i) $$\begin{aligned} x + y + z &= 4, \\ 2x - y + z &= 3, \\ x + 2y + 3z &= 4; \end{aligned}$$

(ii) $$\begin{aligned} x + 4y - 3z &= -24, \\ 3x - y + 3z &= 36, \\ x + y + 6z &= 3; \end{aligned}$$

(iii) $$\begin{aligned} x - y - z &= 1, \\ x - 2y + 3z &= 4, \\ 3x - 2y - 7z &= 0; \end{aligned}$$

(iv) $$\begin{aligned} 2x - 2y - 3z &= 6, \\ 4x - 3y - 2z &= 0, \\ 2x - 3y - 7z &= -1; \end{aligned}$$

(v) $$\begin{aligned} 3x - 2y + 2z &= 10, \\ x - 2y + 3z &= 7, \\ 2x + y + z &= 4; \end{aligned}$$

(vi) $$\begin{aligned} x - y - z &= 1, \\ x - 2y + 3z &= 4, \\ 2x - y - 6z &= -1; \end{aligned}$$

(vii) $$\begin{aligned} x + y - z &= -1, \\ 2x - 2y - 3z &= 5, \\ 4x - 3y + 2z &= 16; \end{aligned}$$

(viii) $$\begin{aligned} x - y - z &= 1, \\ 2x + 3y + z &= 2, \\ 3x + 2y &= 0; \end{aligned}$$

(ix) $$\begin{aligned} x - y &= 1, \\ 2x - 3y + z &= 6; \end{aligned}$$

(x) $$\begin{aligned} 3x + y - 6z &= 4, \\ 2x - y + z &= 1; \end{aligned}$$

(xi) $$\begin{aligned} x - y + z &= 20, \\ x + y + z &= 10, \\ 2x + y &= 17; \end{aligned}$$

(xii) $$\begin{aligned} 2x - 2y - z &= 0, \\ 2x - y + 2z &= 4, \\ 2x + 3y + z &= 20; \end{aligned}$$

(xiii) $$\begin{aligned} x + z &= 2, \\ x + y &= 0, \\ y + z &= 2; \end{aligned}$$

(xiv) $$\begin{aligned} x - 2y + 2z &= 2, \\ 2x - 3y + 3z &= 2, \\ 5x - 8y + 8z &= 7. \end{aligned}$$

Exercises 5.2 B

1. In each part, find the complete solution of the system of two linear equations, by substitution.

(i) $$\begin{aligned} 3x - 2y &= 10, \\ 2x - 3y &= 15; \end{aligned}$$

(ii) $$\begin{aligned} \frac{1}{2}x + 3y &= -2, \\ x - 2y &= 8; \end{aligned}$$

(iii) $$\begin{aligned} 7x + 4y &= 2, \\ 3x - 2y &= 0; \end{aligned}$$

(iv) $$\begin{aligned} 2x + y &= -1, \\ 2x - y &= -3; \end{aligned}$$

(v) $5x + 3y = 4,$
$3x - y = 1;$

(vi) $11x + 7y = 1,$
$-2x - 3y = 5;$

(vii) $2x + 3y = 8,$
$-2x - 2y = -4;$

(viii) $3x - 2y = 5,$
$2x + 3y = 12;$

(ix) $5x + 7y = -1,$
$4x + 7y = 2;$

(x) $3x - 5y = 0,$
$2x + 3y = \frac{38}{15};$

(xi) $3x + 2y = 10,$
$2x - 3y = -4;$

(xii) $4x - 3y = 11,$
$2x + 2y = 16;$

(xiii) $6x - 3y = 1,$
$8x + 5y = 7;$

(xiv) $3x + 11y = 5,$
$5x + 15y = 10;$

(xv) $2x + 2y = 12,$
$5x - 3y = 14;$

(xvi) $2x + 3y = 2,$
$2x - y = -6.$

2. In each of the parts of Exercise 1, find the complete solution of the system of two linear equations by elimination.

3. In the following problems, say whether the equations are inconsistent, dependent or independent. If they are consistent, write down the solution.

(i) $4x - 3y = 15,$
$2x + 5y = 1;$

(ii) $2x - 3y = 1,$
$3x + 3y = 9;$

(iii) $2x - 2y = 6,$
$3x - 3y = 7;$

(iv) $3x - 4y = 0,$
$6x - 8y = 7;$

(v) $7x - 4y = 2,$
$4x - 3y = -1;$

(vi) $11x - 7y = 1,$
$3x + 3y = 15;$

(vii) $6x + 4y = 8,$
$3x + 2y = 4;$

(viii) $\frac{1}{2}x - y = 0,$
$x + \frac{1}{2}y = 5;$

(ix) $4x + 3y = 1,$
$-2x + 5y = -7;$

(x) $10x - 14y = 2,$
$-15x + 21y = -3;$

(xi) $2x + 3y = 12,\quad x - 3y = -3;$

(xii) $x - y = 5,\quad x + y = 9;$

(xiii) $x - 2y = 3,\quad 3x - 6y = 9;$

(xiv) $4x - 6y = 0,\quad 6x - 9y = 7;$

(xv) $3x - 2y = 6,\quad x + 2y = 6;$

(xvi) $x + 2y = 5,\quad 2x + 3y = 4.$

4. In the following problems, say whether the equations are inconsistent, dependent or independent. If they are consistent, write down the solution.

(i)
$$\begin{aligned} x - z &= 2,\\ x - 2y + z &= -4,\\ 2x + y - 3z &= 7; \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y - z &= 6,\\ 2x + 4y - 2z &= 12,\\ 2x + 7y + z &= 24; \end{aligned}$$

(iii)
$$\begin{aligned} x + z &= 2,\\ x + y &= 0,\\ y + z &= 0; \end{aligned}$$

(iv)
$$\begin{aligned} 3x - 2y + 2z &= 2,\\ 2x - 4y + 3z &= 2,\\ 5x + 2y &= 4; \end{aligned}$$

(v)
$$\begin{aligned} x + 2y + z &= 2,\\ 3x + 6y + 3z &= 6,\\ 2x + 4y + 2z &= 4; \end{aligned}$$

(vi)
$$\begin{aligned} x - y - z &= 9,\\ x + y + 3z &= -5; \end{aligned}$$

(vii)
$$\begin{aligned} 2x - 2y - z &= 0,\\ 2x - y + 2z &= 2,\\ 2x + 3y + z &= 10; \end{aligned}$$

(viii)
$$\begin{aligned} 2x + y - z &= 1,\\ 4x + 2y + 2z &= 0,\\ 2x - y + z &= 6; \end{aligned}$$

(ix)
$$\begin{aligned} 3x - y + 4z &= 5,\\ x - 2y + 3z &= 7,\\ 2x + y + z &= 4; \end{aligned}$$

(x)
$$\begin{aligned} x - 2y + 2z &= 3,\\ 2x - 4y + z &= 3,\\ 4x - 8y + 3z &= 7; \end{aligned}$$

(xi)
$$\begin{aligned} 2x - y + z &= 9,\\ x + y + z &= 9,\\ x - y + z &= 3; \end{aligned}$$

(xii)
$$\begin{aligned} x + y - z &= 8,\\ x + 4y + z &= 12,\\ x - 2y + z &= -4; \end{aligned}$$

(xiii)
$$\begin{aligned} 4x - 4y + 2z &= 10,\\ 2x + y - z &= -6,\\ 6x - 7y + z &= 12; \end{aligned}$$

(xiv)
$$\begin{aligned} 3x - 4y + 11z &= 6,\\ 2x - 5y + 12z &= 4; \end{aligned}$$

(xv) $\begin{aligned} x + y + 2z &= 20, \\ x - y - 2z &= 6, \\ x + y + z &= 40; \end{aligned}$

(xvi) $\begin{aligned} x + y + 2z &= 4, \\ x + y - 2z &= 0, \\ x + y + 3z &= 5; \end{aligned}$

(xvii) $\begin{aligned} x + 2y - z &= 6, \\ 2x + 4y + 2z &= 8, \\ 2x + 3y + z &= 7; \end{aligned}$

(xviii) $\begin{aligned} x + 2y + z &= 4, \\ x + y + z &= -4, \\ 2x - 2y + 2z &= 8; \end{aligned}$

(xix) $\begin{aligned} x + y + z &= 6, \\ x - y + 2z &= 12, \\ 2x + y + z &= 1; \end{aligned}$

(xx) $\begin{aligned} x - 2y + z &= 3, \\ 2x - 4y + 3z &= 7. \end{aligned}$

5.3 Formal Solution of Systems of Equations

The Augmented Matrix

In this section, we formalize the process of solving a system of linear equations by substitution. In order to talk about a system of m equations in n variables, we use subscripts. We suppose the variables are $x_1, x_2, \ldots, x_n$, and suppose the ith equation to be

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i,$$

where $a_{11}, a_{12}, \ldots, a_{mn}, b_1, \ldots, b_m$ are some constants.

We define the *augmented matrix* of a system to be the following array of numbers:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

The vertical line indicates the division between the two types of element, the coefficients on the left and the constant terms on the right. The horizontal lists of numbers are called *rows* and the vertical lists are called *columns*. The ith row corresponds to the ith equation; the jth column corresponds to the jth variable when $1 \le j \le n$, while column $n + 1$ corresponds to the list of constant terms of the equations. The number lying in the ith row and the jth column is called the (i, j) *element* of the augmented matrix.

Augmented matrices are a special case of more general *matrices*, or rectangular arrays of numbers, and we shall discuss the general case in the next section. Many of the ideas and notations will be repeated there, but you will find them to be consistent with this section.

Sample Problem 5.15. *Write down the augmented matrix of the system*

$$\begin{aligned} 2x + 4y - 4z &= 4, \\ -2y + 4z &= 6, \\ x - y + 4z &= 10. \end{aligned}$$

What is its (2, 3) *element?*

Solution.

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & 4 \\ 0 & -2 & 4 & 6 \\ 1 & -1 & 4 & 10 \end{array}\right].$$

The (2, 3) element equals 4.

Your Turn. Write down the augmented matrix of the system

$$\begin{aligned} 4x + 3y - 2z &= 1, \\ 3x - 2y + 4z &= 6, \\ 2x - 3y + 2z &= 8. \end{aligned}$$

What is its (2, 2) element?

The first step in solving a set of equations is to select a variable to eliminate. This is equivalent to choosing a column in the augmented matrix and selecting a row—an equation—to use for substitution. We say that we are *operating on* the element in that row and column. The only requirement is that the matrix has a non-zero entry in that row and column.

To illustrate this, consider the system in Sample Problem 5.15. Let us choose row 3, column 1, representing variable x in the first equation. It will be convenient to interchange rows 1 and 3, so that we are operating on the (1, 1) entry. This is equivalent to rewriting the equations in a different order. The matrix is now

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & -2 & 4 & 6 \\ 2 & 4 & -3 & 4 \end{array}\right] \quad \begin{array}{l} R1 \leftarrow R3, \\ \\ R3 \leftarrow R1, \end{array}$$

where the annotations mean *the new row* 1 *is the old row* 3 and *the new row* 3 *is the old row* 1.

Now substitute for x in the other equations. No action is required in the second equation, but x must be eliminated from the third. So we subtract twice the first row from the third row. This yields precisely the equation we would get if we used equation 1 to substitute for x in equation 3, but for consistency we have kept all the variables on the left-hand side of the equation. The augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & -2 & 4 & 6 \\ 0 & 6 & -11 & -16 \end{array}\right] \qquad R3 \leftarrow R3 - 2R1,$$

where the legend means *the new row* 3 *is* (*the old row* 3) -2(*the old row* 1). (When we say *old* we are referring to the preceding augmented matrix, not to the original one.)

Now multiply row 2 by $-\frac{1}{2}$. Then eliminate the $6y$ from the third equation. The result is

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 10 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} R2 \leftarrow \frac{1}{2}R2, \\ R3 \leftarrow R3 - 6(\frac{1}{2}R2). \end{array}$$

This could have been broken into two steps.

So far we have done the equivalent of substituting in the later equations. Now we substitute back to find the values. We know from the third equation that $z = 3$. To substitute this in the earlier equations, we add twice row 3 to row 2 and subtract four times row 3 from row 1:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} R1 \leftarrow R1 - 4R3, \\ R2 \leftarrow R2 + 2R3. \\ \end{array}$$

Next add row 2 to row 1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} R1 \leftarrow R1 + R2. \\ \\ \end{array}$$

The resulting array can be translated into the equations

$$x = 3, \qquad y = 1, \qquad z = 2.$$

Elementary Operations

In our example, we used three operations:

E1: *exchange two rows of the matrix;*

E2: *multiply a row by a (non-zero) constant;*

E3: *add a multiple of one row to another row.*

We shall call these the *elementary row operations*. Their importance comes from the following fact:

Theorem 17. *Suppose P is the augmented matrix of a system of linear equations, and Q is obtained from P by a sequence of elementary row operations. Then the system of equations corresponding to Q has the same solutions as the system corresponding to P.*

It is clear that repeated application of the three elementary operations to the augmented matrix will provide a solution. So we can solve systems of linear equations by the following technique.

Stage 1.

1. Find the leftmost column in the matrix of coefficients that contains a non-zero element, say column j. Use E1 to make the row containing this element into the first row, and E2 to convert its leftmost non-zero element to 1. This is called a *leading* 1. Then use E3 to change all entries below the leading 1 to zero. That is, if the (i, j) entry is a_{ij}, then subtract $a_{ij} \cdot$ (row 1) from row j.

At this stage we say column j is *processed*. Processed columns are not disturbed in the first stage.

2. Find the leftmost unprocessed column in the augmented matrix that contains a non-zero element, say column k. Use E1 to make this row the first row under the processed row(s), and E2 to convert its leftmost non-zero element to 1, another leading 1. Use E3 to change all entries below the leading 1 to zero (but do not change the processed row or rows). Now column k is also processed.

3. If you have not either reached the last column of coefficients (the vertical line) or the bottom of the matrix, go back to step 2, make another leading 1 and proceed from there.

Stage 2.

4. Choose the bottom-most leading 1 and eliminate all elements above it in its column by use of E3. Do the same to the next leading 1 up, then the next, until you reach the top.

The process is now finished.

Sample Problem 5.16. *Solve the system*

$$\begin{aligned} 2x + 2y + 4z &= 0, \\ 3x - y + 2z &= 1, \\ 8x + 8z &= 2 \end{aligned}$$

by row operations.

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 3 & -1 & 2 & 1 \\ 8 & 0 & 8 & 2 \end{array}\right].$$

At step 1, we choose the element in the (1, 1) position and divide row 1 by 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & -1 & 2 & 1 \\ 8 & 0 & 8 & 2 \end{array}\right] \quad R1 \leftarrow \tfrac{1}{2}R1 \quad \text{(using E2)}.$$

Then we eliminate the rest of column 1:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -4 & -4 & 1 \\ 0 & -8 & -8 & 2 \end{array}\right] \quad \begin{array}{ll} R2 \leftarrow R2 - 3 \times R1 & \text{(using E3)}, \\ R3 \leftarrow R3 - 8 \times R1 & \text{(using E3)}. \end{array}$$

In step 2, we choose the (2, 2) position and divide by −4, then eliminate the entries below the (2, 2) position, obtaining successively

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & -8 & -8 & 2 \end{array}\right] \quad R2 \leftarrow -\tfrac{1}{4} \times R2 \quad \text{(using E2)},$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R3 \leftarrow R3 + 8 \times R2 \quad \text{(using E3)}.$$

There are no further numbers available for leading 1's, so we move to step 4. We use the (2, 2) element:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{4} \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R1 \leftarrow R1 - R2 \quad \text{(using E3)}.$$

The process is finished. There is no restriction on z. The final augmented matrix converts to the system

$$x + z = \frac{1}{4},$$
$$y + z = -\frac{1}{4}$$

(the third equation can be ignored), and the final solution could be expressed as

$$x = \frac{1}{4} - z, \qquad y = -\frac{1}{4} - z, \qquad \text{any real number } z.$$

Notice that the sequence of calculations is completely determined by the matrix of coefficients, the left-hand part of the augmented matrix.

If the column corresponding to a variable receives a leading 1, we shall call that variable *dependent*; the others are *independent*. One standard way of recording the answer is to give an equation for each dependent variable, with a constant and the independent variables on the right; the independent variables take any real number value. Another way to express the above solution would be to use set notation $\{\frac{1}{4}-z, -\frac{1}{4}-z, z) \mid z \in \mathbb{R}\}$, or perhaps $\{t+\frac{1}{4}, t, -\frac{1}{4}, -t) \mid t \in \mathbb{R}\}$. In this case, t is called a *parameter*.

Sometimes there will be no solution to a system of equations. The equations are then called *inconsistent*.

Sample Problem 5.17. *Solve the system*

$$\begin{aligned} 2x + 2y + 4z &= 0, \\ 3x - y + 2z &= 1, \\ 8x + 8z &= 3. \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 3 & -1 & 2 & 1 \\ 8 & 0 & 8 & 3 \end{array}\right].$$

The left-hand part of this equation is the same as in Sample Problem 5.16, so we go through the same steps, making the appropriate changes to the right-hand column. At the end of Stage 1, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{array}\right].$$

When we convert back to equations, the third row gives the equation

$$0 = 1,$$

which is impossible. No values of x, y and z make this true, so the equations are inconsistent. There is no need to implement Stage 2.

In set-theoretic terms, we could report that the solution set is Ø.

Exercises 5.3 A

1. In each case, the augmented matrix of a system of equations is shown. Assuming the variables are x, y, z, what is the solution of the system?

(i) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$; (ii) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$;

(iii) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$; (iv) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$;

(v) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$; (vi) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$;

(vii) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ -1 & 2 & 1 & -1 \\ 3 & 2 & 4 & 6 \end{array}\right]$; (viii) $\left[\begin{array}{ccc|c} 4 & 3 & 1 & 11 \\ 2 & -2 & 4 & 2 \\ 1 & 3 & -2 & 5 \end{array}\right]$;

(ix) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$; (x) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

2. Solve the following systems of equations:

(i) $\begin{aligned} 2x + 6y &= 6, \\ 4x + 11y &= 10; \end{aligned}$ (ii) $\begin{aligned} 2x + 3y &= 5, \\ 4x + 6y &= 10; \end{aligned}$

(iii) $\begin{aligned} 3x - y &= 4, \\ 6x - 2y &= 2; \end{aligned}$ (iv) $\begin{aligned} x - 2y &= 4, \\ -3x - 4y &= -2, \\ 2x + 3y &= 1. \end{aligned}$

3. Solve the following systems of equations:

(i) $\begin{aligned} x + 2y + z &= 3, \\ x + y - 2z &= 2; \end{aligned}$ (ii) $\begin{aligned} 2x + 2z &= 2, \\ x + 2y + 6z &= 3, \\ 2x - 2y &= 1; \end{aligned}$

(iii) $\begin{aligned} x + 2y + z &= -1, \\ 2x + 3y - 2z &= 7, \\ -2x + 2y - 3z &= -2; \end{aligned}$ (iv) $\begin{aligned} x + y &= 2, \\ x - y + 5z &= 3, \\ -3x - 3y + 2z &= -6. \end{aligned}$

4. Solve the following systems of equations:

(i) $\begin{aligned} 3x - 2y - 8z + 7t &= 1, \\ x + y - z - t &= 3, \\ x - y - 3z + 3t &= -1; \end{aligned}$ (ii) $\begin{aligned} x + 2y + 3z + 4t &= 8, \\ x - 3y + 4z + 4t &= 8, \\ 2x - 2y - z + t &= -3, \\ x - 7y - 7z - 3t &= -11. \end{aligned}$

Exercises 5.3 B

1. In each case, the augmented matrix of a system of equations is shown. Assuming the variables are x, y, z, what is the solution of the system?

(i) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$; (ii) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$;

(iii) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$; (iv) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$;

(v) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$; (vi) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$;

(vii) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 6 \end{array}\right]$; (viii) $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 2 & -1 & 3 & 3 \\ 4 & 1 & 3 & 9 \end{array}\right]$;

(ix) $\left[\begin{array}{ccc|c} 4 & 2 & -3 & 1 \\ 3 & -1 & -1 & 1 \\ 1 & -7 & 3 & 1 \end{array}\right]$; (x) $\left[\begin{array}{ccc|c} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right]$;

(xi) $\left[\begin{array}{ccc|c} 2 & -1 & 1 & -2 \\ 3 & 2 & 3 & 8 \\ 1 & -1 & -1 & 0 \end{array}\right]$; (xii) $\left[\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 2 & -1 & -4 \end{array}\right]$.

2. Solve the following systems of equations:

(i) $\begin{aligned} 3x - 2y &= 4, \\ -6x + 4y &= 2; \end{aligned}$ (ii) $\begin{aligned} x - y &= 3, \\ 2x + y &= 3; \end{aligned}$

(iii) $\begin{aligned} 3x - 2y &= -1, \\ -6x + 4y &= 2; \end{aligned}$ (iv) $\begin{aligned} 3x + 2y &= 4, \\ 2x + 3y &= 1, \\ 5x - 4y &= 14; \end{aligned}$

(v) $\begin{aligned} x + 3y &= 5, \\ 2x + 5y &= 9; \end{aligned}$ (vi) $\begin{aligned} 2x + y &= 4, \\ 4x + 2y &= 8; \end{aligned}$

(vii) $\begin{aligned} 2x - y &= 4, \\ 4x - 2y &= 7; \end{aligned}$ (viii) $\begin{aligned} x + 3y &= 4, \\ 2x - 4y &= -2, \\ 3x + 5y &= 8. \end{aligned}$

3. Solve the following systems of equations:

(i) $\begin{aligned} x + y + 3z &= 2, \\ 4x + 2y + 2z &= 10; \end{aligned}$ (ii) $\begin{aligned} x + y + z &= 3, \\ x + 2y + 2z &= 3, \\ x + y + 2z &= 1; \end{aligned}$

(iii)
$$\begin{aligned} x + z &= 4,\\ x + 4y + z &= 7,\\ x - 2y + z &= 3; \end{aligned}$$

(iv)
$$\begin{aligned} x + y - z &= 4,\\ 3x + 4y - 7z &= 8,\\ -y + 4z &= 4; \end{aligned}$$

(v)
$$\begin{aligned} 2x + 4z &= 6,\\ 2x + y + 5z &= 7,\\ x - y + z &= 2; \end{aligned}$$

(vi)
$$\begin{aligned} x + 2y + 3z &= 4,\\ 4x + 5y + 6z &= 16,\\ 7x + 8y + 9z &= 28. \end{aligned}$$

4. Solve the following systems of equations:

(i)
$$\begin{aligned} x + y + z &= 3,\\ x + 3y + 2z &= 3,\\ 3x + 2y + 4z &= 3; \end{aligned}$$

(ii)
$$\begin{aligned} x - 3z &= 2,\\ x + 2y + z &= 2,\\ x + 4y - z &= 2; \end{aligned}$$

(iii)
$$\begin{aligned} x - 6y - 2z &= 1,\\ 2x + 3y - z &= 3,\\ 3x + 2y - 2z &= 4; \end{aligned}$$

(iv)
$$\begin{aligned} x + 2y + z &= 5,\\ 2x - 3y + 2z &= 3; \end{aligned}$$

(v)
$$\begin{aligned} 2x + y + 3z + t &= 4,\\ x + 3y - 2z + 2t &= 5,\\ 3x - 3y + 13z - t &= 3; \end{aligned}$$

(vi)
$$\begin{aligned} x - 2y + z - t &= 1,\\ 2x - 4y + 3z - 4t &= 2,\\ x - 3y + 3t &= -2. \end{aligned}$$

5. Solve the following systems of equations:

(i)
$$\begin{aligned} 2x - y + z - 3t &= 2,\\ -4x - 3y + t &= 1,\\ 2x - 6y + 3z - 8t &= 4; \end{aligned}$$

(ii)
$$\begin{aligned} 2x + 2y - 2z + 3t &= 2,\\ 4x - 2y - z + t &= -4,\\ 6x - 3z + 4t &= -2,\\ 2x + 8y - 5z + 8t &= 10. \end{aligned}$$

5.4 Pivoting

Dependent and Independent Variables

For convenience, we repeat the list of elementary row operations from the preceding section:

E1: *exchange two rows of the matrix;*

E2: *multiply a row by a* (*non-zero*) *constant;*

E3: *add a multiple of one row to another row.*

In Sample Problem 5.16, we solved the equations

$$\begin{aligned} 2x + 2y + 4z &= 0, \\ 3x - y + 2z &= 1, \\ 8x + 8z &= 2. \end{aligned}$$

The final augmented matrix was

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{4} \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right],$$

so the solution was the set of all (x, y, z) satisfying

$$\begin{aligned} x &= \frac{1}{4} - z, \\ y &= -\frac{1}{4} - z, \\ z &= \text{any real number.} \end{aligned}$$

The variables x and y correspond to columns that contain a leading 1, so those variables are called *dependent*; the other variable z is called *independent*. The reason for this terminology is the way in which the solution is expressed. If a value is chosen for the variable z, then the corresponding values of x and y can be calculated, and in ordinary conversation we might say that the values of x and y depend on the value of z. We shall call this a solution *in terms of* z.

Suppose we wish to express the solutions to Sample Problem 5.16 in terms of variable y. We want to rewrite the solution with x and z the dependent variables and y independent. To do this, we select the row containing the leading 1 corresponding to y, and manipulate the matrix so that the z entry in that column becomes a leading 1. In the example, this is the (2, 3) entry. We subtract row 2 from row 1, obtaining

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R1 \leftarrow R1 - R2 \quad \text{(using E3).}$$

This process is called *pivoting*. The cell that we chose to make into a leading 1, cell (2, 3), is the *pivot position*, and the entry there is the *pivot element*. Row 2 and column 3 are called the *pivot row* and *pivot column*, respectively.

Suppose the variables are $x_1, x_2, \ldots, x_n$; suppose x_k is a dependent variable whose leading 1 is in row i. If the (i, j) entry of the final augmented matrix is non-zero, it is possible to pivot on that entry. The first step is to divide row i by that (i, j) entry (an instance of E2). Then subtract suitable multiples of row i from every other row, so that column j has zeros in all positions except row i (using E3). In the resulting matrix, x_k is independent and x_j is now dependent.

Sample Problem 5.18. *Solve the equations*

$$\begin{aligned} 2x+2y+z+7t &= 13, \\ -x+2y-2z+2t &= 3, \\ x-y+3z-2t &= 3. \end{aligned}$$

Express the solutions in four ways, with t, x, y, z, respectively, as independent variables.

Solution. We reduce the augmented matrix as usual:

$$\left[\begin{array}{rrrr|r} 3 & 2 & 1 & 7 & 13 \\ -1 & 2 & -2 & 2 & 3 \\ 1 & -1 & 3 & -2 & 3 \end{array}\right],$$

$$\left[\begin{array}{rrrr|r} 0 & 5 & -8 & 13 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 1 & -1 & 3 & -2 & 3 \end{array}\right] \quad \begin{array}{ll} R1 \leftarrow R1-3R3 & \text{(using E3)}, \\ R2 \leftarrow R2+3R3 & \text{(using E3)}, \end{array}$$

$$\left[\begin{array}{rrrr|r} 0 & 0 & -13 & 13 & -26 \\ 0 & 1 & 1 & 0 & 6 \\ 1 & 0 & 4 & -2 & 9 \end{array}\right] \quad \begin{array}{ll} R1 \leftarrow R1-5R2 & \text{(using E3)}, \\ & \\ R3 \leftarrow R3+R2 & \text{(using E3)}, \end{array}$$

$$\left[\begin{array}{rrrr|r} 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 4 \\ 1 & 0 & 0 & 2 & 1 \end{array}\right] \quad \begin{array}{ll} R1 \leftarrow -\frac{1}{13}R1 & \text{(using E2)}, \\ R2 \leftarrow R2+\frac{1}{13}R1 & \text{(using E3)}, \\ R3 \leftarrow R3+\frac{4}{13}R1 & \text{(using E3)}. \end{array}$$

So the solution (in terms of t) is

$$\begin{aligned} x &= 1-2t, \\ y &= 4-t, \\ z &= 1+t, \end{aligned}$$

with t any real number.

To solve in terms of x, pivot on position $(1,4)$:

$$\left[\begin{array}{rrrr|r} \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 0 & \frac{9}{2} \\ \frac{1}{2} & 0 & 1 & 0 & \frac{3}{2} \end{array}\right] \quad \begin{aligned} y &= \tfrac{9}{2}+\tfrac{1}{2}x, \\ z &= \tfrac{3}{2}-\tfrac{1}{2}x, \\ t &= \tfrac{1}{2}-\tfrac{1}{2}x. \end{aligned}$$

To solve in terms of y, pivot on position $(2,4)$:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 0 & -7 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 & 5 \end{array}\right] \quad \begin{aligned} x &= -7+2y, \\ z &= 5-y, \\ t &= 4-y. \end{aligned}$$

To solve in terms of z, pivot on position $(3,4)$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right] \qquad \begin{aligned} x &= 3 - 2z, \\ y &= 5 - z, \\ t &= -1 + z. \end{aligned}$$

Your Turn. Solve the equations

$$\begin{aligned} 2x - 3y - z - 2t &= 1, \\ x + 2y - 2z + t &= 1, \\ x - y + z + t &= 4. \end{aligned}$$

Express the solutions in four ways, with t, x, y, z, respectively, as independent variables.

Even in a case where there is one independent variable, it is not always to pivot in every way. To see this, consider the equations

$$\begin{aligned} x + 2y + 2z &= 12, \\ 2x - y + 4z &= 4. \end{aligned}$$

The analysis proceeds

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 12 \\ 2 & -1 & 4 & 4 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 12 \\ 0 & -5 & 0 & -20 \end{array}\right] \qquad R2 \leftarrow R2 - 2R1 \quad \text{(using E3)},$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 4 \end{array}\right] \qquad \begin{array}{l} R1 \leftarrow R1 + \frac{2}{5}R2 \quad \text{(using E3)}, \\ R2 \leftarrow -\frac{1}{5}R2 \quad \text{(using E2)}, \end{array}$$

yielding

$$\begin{aligned} x &= 4 - 2z, \\ y &= 4. \end{aligned}$$

We can pivot on the $(1,3)$ cell, obtaining

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 4 \end{array}\right] \qquad \begin{aligned} y &= 4, \\ z &= 2 - \tfrac{1}{2}x, \end{aligned}$$

but we cannot pivot on the $(2,3)$ entry because it is 0. The solution cannot be expressed with y as the independent variable.

Exercises 5.4 A

1. In each case, the augmented matrix of a system of equations is shown. Assume the variables are x, y, z. Express the solution in three ways:

(a) With x and y dependent variables, z independent;

(b) With x and z dependent variables, y independent;

(c) With y and z dependent variables, x independent.

(i) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$; (ii) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 2 \end{array}\right]$;

(iii) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$; (iv) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right]$.

2. The augmented matrix of a system of equations is shown. Assume the variables are x, y, z, t. Express the solution in six ways, with two variables in terms of the other two.

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

3. In each case solve the equations; express the solutions in three ways:

(a) With x and y dependent variables, z independent;

(b) With x and z dependent variables, y independent;

(c) With y and z dependent variables, x independent.

(i)
$$\begin{aligned} x + z &= 3, \\ x + y + 2z &= 2, \\ 2x - y + z &= 7; \end{aligned}$$

(ii)
$$\begin{aligned} 2x + 2y + 8z &= 16, \\ 3x - 2y + 2z &= 4, \\ x + 3y + 8z &= 16. \end{aligned}$$

Exercises 5.4 B

1. In each case, the augmented matrix of a system of equations is shown. Assume the variables are x, y, z. Express the solution in three ways:

(a) With x and y dependent variables, z independent;

(b) With x and z dependent variables, y independent;

(c) With y and z dependent variables, x independent.

(i) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$; (ii) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 4 & 6 \end{array}\right]$;

(iii) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -2 & 6 \end{array}\right]$; (iv) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right]$.

2. The augmented matrix of a system of equations is shown. Assume the variables are x, y, z, t. Express the solution in six ways, with two variables in terms of the other two.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right].$$

3. In each case solve the equations; express the solutions in three ways:

(a) With x and y dependent variables, z independent;

(b) With x and z dependent variables, y independent;

(c) With y and z dependent variables, x independent.

(i)
$$\begin{aligned} x + 2y - 3z &= 5, \\ 2x - y - z &= 0, \\ 2x + y - 3z &= 4; \end{aligned}$$

(ii)
$$\begin{aligned} 2x + y &= 10, \\ x + 2y - 3z &= 11, \\ 3x - y + 5z &= 5. \end{aligned}$$

4. In each part, solve the equations and express the solutions in four ways, with t, x, y, z, respectively, as independent variables.

(i)
$$\begin{aligned} 2x + y - z + t &= 5, \\ x - 2y + z + 6t &= -4, \\ 2x - y - 2z + t &= 2. \end{aligned}$$

(ii)
$$\begin{aligned} 2x + y + z + 3t &= 9, \\ 3x - 2y + 3z - 4t &= 1, \\ x + 2y + 2z + 3t &= 12. \end{aligned}$$

5. Consider the system of equations

$$\begin{aligned} 2x + y - z + t &= 3, \\ x - 2y - 3z + 3t &= -1, \\ 3x + 2y - z + t &= 5. \end{aligned}$$

Express the solution in five ways, with two variables in terms of the other two, but show that there is no solution with independent variables x and y.

5.5 Matrices and Vectors

Matrices

Suppose a movie theater sells three types of tickets—Adult (A), Student concession (S), and Child (C). The theater charges more after 6PM, so tickets may also be classified as Day (D) or Evening (E). If 43 Adult, 33 Student and 18 Child tickets are sold for the afternoon session, and 78 Adult, 45 Student and 12 Child tickets are sold in the evening, the day's ticket sales could be represented by the following table:

$M =$

	A	S	C
D	43	33	18
E	78	45	12

A rectangular array of data like this is called a *matrix*. We shall usually denote matrices by single upper-case letters. In general, matrices can be used whenever the data is classified in two ways, such as ticket types (A, S, C) and session times (D, E). The horizontal layers are called *rows* and the vertical ones *columns*; for example, the first row in the above matrix M is

43	33	18

and the second column is

33
45

Sample Problem 5.19. *A furniture manufacturer makes tables and chairs. In January, he made* 200 *tables and* 850 *chairs; in February,* 300 *tables and* 1440 *chairs; in March,* 140 *tables and* 880 *chairs. Represent these data in a matrix.*

Solution. Write T for tables, C for chairs.

	T	C
Jan	200	850
Feb	300	1440
Mar	140	880

Your Turn. In April, Joe's Autos sold 32 sedans and 16 pickups. In May, they sold 44 sedans and 12 pickups. Represent the two months' sales in a matrix.

The numbers in a matrix are called its *entries*. Sometimes there are restrictions on the sort of numbers that may be used; for example, one can consider only *integer matrices* or *non-negative matrices*. When this is done, the set of numbers that may be used are called the *scalars* for the problem, and for this reason the word "scalar" is often used to refer to properties involving only numbers.

The augmented matrices we saw in the preceding section were one special example. The rows represent the different equations and the columns represent the variables, except the last column, which represents the constant terms.

Suppose a matrix has m rows and n columns. Then we say it is an $m \times n$ matrix. We refer to $m \times n$ as the *shape* or *size* of the matrix, and the two numbers m and n are its *dimensions*. The above matrix representing ticket sales is a 2×3 matrix.

It is convenient to refer to the entry in the ith row and jth column of a matrix as the (i, j) *element* (or *entry*). The $(1, 2)$ element of M is 33. When there is no confusion possible, we would write m_{ij} to denote the (i, j) element of a matrix M, using the lower-case letter corresponding to the (upper-case) name of the matrix, with the row and column numbers as subscripts. A common shorthand is $M = [m_{ij}]$. (All of this is consistent with the notations for augmented matrices.)

Sample Problem 5.20. *What is the shape of the matrix*

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ -1 & 1 & 4 & 1 \\ 1 & 3 & 0 & 2 \end{bmatrix}?$$

Write down its second row and its third column.

Solution. The matrix has shape 3×4. Its second row is

$$\boxed{\begin{matrix} -1 & 1 & 4 & 1 \end{matrix}},$$

and its third column is

$$\boxed{\begin{matrix} 0 \\ 4 \\ 0 \end{matrix}}.$$

Your Turn. What is the shape of the matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -2 \\ -1 & 4 & 6 \\ 1 & -2 & 2 \end{bmatrix}?$$

Write down its third row and its first column.

Sample Problem 5.21. *Suppose*

$$\begin{bmatrix} x & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & x + y \\ -1 & 2 \end{bmatrix}.$$

What are x and y?

Solution. Two matrices are equal if and only if the corresponding entries are equal. So we have the two equations $x = 4$ and $3 = x + y$. So $x = 4$ and $y = -1$.

Your Turn. Suppose

$$\begin{bmatrix} 2x & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} y & x \\ 1 & 4 \end{bmatrix}.$$

What are x and y?

Adding Matrices

Continuing the movie theater example, let's say the above figures represent sales for Monday. Tuesday's sales are given by

$T =$

	A	S	C
D	38	23	16
E	118	75	14

If the manager wants to know the total number of Adult evening tickets sold over the two days, she simply adds the numbers for Monday and Tuesday. That is, she finds $m_{21} + t_{21}$. In the example, she gets $78 + 118 = 196$. Let us call this sum s_{21}. In the same way she could add the entries in the other positions, and produce a matrix S which we shall call the *sum* of M and T:

$$\begin{aligned} S &= M + T \\ &= \begin{bmatrix} 43 & 33 & 18 \\ 78 & 45 & 12 \end{bmatrix} + \begin{bmatrix} 38 & 23 & 16 \\ 118 & 75 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 81 & 56 & 34 \\ 196 & 120 & 26 \end{bmatrix}. \end{aligned}$$

In general, if M and T are two matrices with the same shape, the *sum* of two matrices $M = [m_{ij}]$ and $T = [t_{ij}]$ is the matrix $S = [s_{ij}]$ defined by

$$s_{ij} = m_{ij} + t_{ij}.$$

We shall not define $M + T$ if M and T are of different shapes.

It is possible to take the sum of a matrix with itself, and we write $2M$ for $M + M$, $3M$ for $M + M + M$, and so on. This can be extended to multipliers other than positive integers: if a is any number, aM will mean the matrix derived from M by multiplying every entry by a. That is,

$$aM \text{ is the matrix with } (i, j) \text{ entry } am_{ij}.$$

We refer to aM as the *scalar product* (or simply *product*) of a with M. It has the same shape as M.

It is easy to see that this addition satisfies the commutative and associative laws: if M, T, and W are any matrices of the same size, then

$$M + T = T + M \quad \text{and} \quad (M + T) + W = M + (T + W).$$

Because of the associative law, we usually omit the brackets and just write $M + T + W$. The scalar product also obeys the laws

$$(a + b)M = aM + bM, \qquad a(bM) = (ab)M, \qquad a(M + T) = aM + aT$$

for any matrices M and T and any numbers a and b. Notice also that $1M = M$ is always true.

We write O_{mn} for a matrix of shape $m \times n$ with every entry zero. Usually, we do not bother to write the subscripts m and n, but simply assume that the matrix is the correct size for our computations. O is called a *zero matrix*, and works like the number zero: if M is any matrix, then

$$M + O = M,$$

provided O has the same shape as M.

It is clear that $0M = O$ for any matrix M (where 0 is the number zero and O is the zero matrix). From the first law for scalar multiplication above, we see that

$$M + (-1)M = 1M + (-1)M = \big(1 + (-1)\big)M = 0M = O,$$

so $(-1)M$ acts like a negation of M. We shall simply write $-M$ instead of $(-1)M$, and call $-M$ the *negative* of M. We can then define *subtraction* by

$$M - T = M + (-T),$$

just as you would expect.

Sample Problem 5.22. *Suppose A, B, and C are the matrices*

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 4 & 3 \\ -1 & -2 & -1 \end{bmatrix}.$$

Find $A + B$, $2A - 3B$, $3A + C$, $-C$.

Solution.
$$A + B = \begin{bmatrix} -1 & 3 \\ 0 & 6 \end{bmatrix}, \qquad 2A - 3B = \begin{bmatrix} -4 & 6 \\ -5 & -8 \end{bmatrix},$$
$$-C = \begin{bmatrix} 1 & -4 & -3 \\ 1 & 2 & 1 \end{bmatrix}.$$

$3A + C$ is not defined, as A and C are of different sizes.

Your Turn. Calculate $-A$, $3A - B$, $B + C$.

Sample Problem 5.23. *Suppose the movie theater manager expects sales to be 10% higher next week, when the new release is shown. What sales does she expect next Monday?*

Solution. The sales for this Monday were shown in the matrix M. She expects 10% higher, or 1.1 times as many sales. So her expected matrix of sales is

$$1.1M = 1.1\begin{bmatrix} 43 & 33 & 18 \\ 78 & 45 & 12 \end{bmatrix} = \begin{bmatrix} 47.3 & 36.3 & 19.8 \\ 85.8 & 46.5 & 13.2 \end{bmatrix}.$$

Of course, she would round to whole numbers, say

$$\begin{bmatrix} 47 & 36 & 20 \\ 86 & 47 & 13 \end{bmatrix}.$$

Transposition

If A is an $m \times n$ matrix, then we can form an $n \times m$ matrix whose (i, j) entry equals the (j, i) entry of A. This new matrix is called the *transpose* of A, and written A^T. A matrix A is called *symmetric* if $A = A^T$.

Sample Problem 5.24. *What is the transpose of the matrix*

$$M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}?$$

Solution.

$$M^T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Your Turn. What is the transpose of the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}?$$

Vectors

A matrix with one of its dimensions equal to 1 is called a *vector*. An $m \times 1$ matrix is a *column vector* of length m, while a $1 \times n$ matrix is a *row vector* of length n. The individual rows and columns of a matrix are vectors, which we call the *row vectors* and *column vectors* of the matrix.

We shall write vectors with boldface lower case letters to distinguish them from matrices and numbers. (We treat vectors separately from matrices because, in many cases, it is not necessary to distinguish between row and column vectors.) In some books, a vector is denoted by a lower case letter with an arrow over it, $\overrightarrow{v}$, rather than a boldface letter $\boldsymbol{v}$.

It is usual to denote the ith entry of a vector by subscript i. The vector $\boldsymbol{v}$ has entries $v_1, v_2, \ldots$, and we usually write $\boldsymbol{v} = (v_1, v_2, \ldots)$.

The two standard operations on vectors follow directly from the matrix operations. One may *multiply by a number*, and one may *add vectors*. If k is any number, and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$, then $k\boldsymbol{v} = (kv_1, kv_2, \ldots, kv_n)$. If $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$, then $\boldsymbol{u} + \boldsymbol{v} = ((u_1 + v_1), (u_2 + v_2), \ldots, (u_n + v_n))$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors of different lengths, then $\boldsymbol{u} + \boldsymbol{v}$ is not defined. We again write $-\boldsymbol{v}$ for $(-1)\boldsymbol{v}$, so $-\boldsymbol{v} = (-v_1, -v_2, \ldots, -v_n)$, and $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$. We define a *zero vector* $\mathbf{0} = (0, 0, \ldots, 0)$ (in fact, a family of zero vectors, one for each possible dimension), and $\boldsymbol{v} + (-\boldsymbol{v}) = 0$.

Sample Problem 5.25. *Calculate* $3(1, -1, 3)$ *and* $(2, 2) + (-1, 3)$.

Solution. $3(1, -1, 3) = (3, -3, 9)$; $(2, 2) + (-1, 3) = (1, 5)$.

Your Turn. Calculate $4(2, 0, -1) + (1, 4, -3)$.

Sometimes there is no important difference between the vector $\boldsymbol{v}$ of length n, the $1 \times n$ matrix (row vector) whose entries are the entries of $\boldsymbol{v}$, and the $n \times 1$ matrix (column vector) whose entries are the entries of $\boldsymbol{v}$. But in the next section, we shall sometimes need to know whether a vector has been written as a row or a column. If this is important, we shall write $row(\boldsymbol{v})$ for the row vector form of $\boldsymbol{v}$, and $col(\boldsymbol{v})$ for the column vector form. If we simply write $\boldsymbol{v}$, you usually can tell from the context whether $row(\boldsymbol{v})$ or $col(\boldsymbol{v})$ is intended.

Sample Problem 5.26. *If* $\boldsymbol{v} = (1, 2, 3)$*, what are* $row(\boldsymbol{v})$ *and* $col(\boldsymbol{v})$*?*

Solution.

$$row(\boldsymbol{v}) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \qquad col(\boldsymbol{v}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Exercises 5.5 A

1. A psychologist has clients who receive individual attention (I) and others who are seen in group sessions (G). She has four private individual clients (P) and six who are sent to her by the court (C). Among her groups are 24 private and 12 court clients. Represent these data in a matrix.

2. A farmer needs to monitor the amounts of vitamins A, B, and C in his chickens' diet. He buys two prepared food mixes. Each bag of food I contains 200 units of vitamin A, 100 units of vitamin B, and 250 units of vitamin C. Each bag of food II contains 250 units of vitamin A, 150 units of vitamin B, and 350 units of vitamin C.

(i) Represent the data in a matrix.

(ii) If he mixes two bags of food I with three bags of food II, how many units of each vitamin will there be in the combination?

3. Carry out the following matrix computations.

(i)
$$\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix};$$

(ii)
$$3\begin{bmatrix} 10 & -1 \\ 2 & 7 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

(iii)
$$2\begin{bmatrix} 6 & -1 & 3 \\ 2 & 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & -5 \end{bmatrix};$$

(iv)
$$3\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}^T.$$

4. Suppose

$$\begin{bmatrix} x & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} y+1 & -1 \\ -1 & x \end{bmatrix}.$$

What are the values of x and y?

5. Find x, y, and z so that

$$\begin{bmatrix} x-2 & 3 & z \\ y & x & 2y \end{bmatrix} = \begin{bmatrix} y & z & 3 \\ 3z & y+2 & 6z \end{bmatrix}.$$

6. Carry out the following vector computations:

(i) $-(2, -2)$;

(ii) $3(3, 6, 1)$;

(iii) $3(2, 3) - 2(1, 4)$;

(iv) $2(-1, -1, 2) - 2(2, -1, -1)$;

(v) $3(1, -2, 2) + 2(2, 3, -1)$;

(vi) $2(4, -1, 2, 3) - 3(1, 6, -2, -3)$.

7. Suppose

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Calculate the following, or say why they do not exist:

(i) $2A - D$; (ii) $C - 4F$;

(iii) $A + 2B - E$; (iv) $C - D$;

(v) $C - A$; (vi) $2D + E - 3B$.

Exercises 5.5 B

1. An experimenter has 40 albino rats (A) and 64 regular rats (R). In each category, half the rats are in the control group (C) and half in the experimental group (E). Construct a matrix that shows the number of rats in each combination of category and group.

2. An investor has both certificates of deposit (CDs) and municipal bonds (MBs) at two banks, First National (FN) and Twentyfirst Century (TC). On January 1st, 2003 she had \$20000 in CDs at each bank, \$30000 in MBs at FN and \$10000 in MBs at TC.

(i) Represent the data in a matrix.

(ii) Assume she receives 5% annual interest on each account. Construct a matrix that shows how much interest she received on each account in 2003.

(iii) She puts all her interest into the account where it was earned. Construct a matrix that shows how much money she has in the various bank accounts on January 1st, 2004.

3. This week, a pet sitter has 12 clients with dogs and 14 with cats. Last week, she had 11 clients with dogs and seven with cats. How many clients with dogs, and how many with cats, did she have over the two weeks? Write down a vector equation that shows how you calculated the answer.

4. Trains travel between the four cities: Allentown (A), Baxter (B), Carterville (C), and Dromney (D). The distance from Allentown to Baxter is 24 miles, from Allentown to Carterville 17 miles, and from Allentown to Dromney 18 miles. From Baxter to Carterville is 14 miles, from Baxter to Dromney is 17 miles, and from Carterville to Dromney is 12 miles.

(i) Represent these distances in a matrix.

(ii) The railways charge \$2 for trips of 12 miles or shorter, \$3 for 13 to 20 miles, and \$4 for 21 to 35. Write down a matrix that shows the fares between the four towns. (Write 0 for the "fare" from a town to itself.)

5. Carry out the following matrix computations:

(i)
$$3\begin{bmatrix} 1 & -1 & -1 \\ -2 & 0 & 10 \end{bmatrix};$$

(ii)
$$\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix};$$

(iii)
$$3\begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix} + 2\begin{bmatrix} 4 & 2 \\ 1 & -1 \end{bmatrix};$$

(iv) $$2\begin{bmatrix} -1 & -1 & -2 \\ 3 & 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \end{bmatrix};$$

(v) $$2\begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ 2 & -1 \\ -3 & 0 \end{bmatrix};$$

(vi) $$5\begin{bmatrix} 2 & -1 \\ -3 & -2 \\ -2 & 1 \end{bmatrix} + 4\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ -1 & 0 \end{bmatrix};$$

(vii) $$3\begin{bmatrix} 2 & -1 \\ 4 & 2 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -1 & 0 \end{bmatrix};$$

(viii) $$2\begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}^T - \begin{bmatrix} 2 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix}.$$

6. In each case find x, y, and z so that the equation is true, or show that no such values exist:

(i) $$\begin{bmatrix} x+y & y+z \\ 4 & 2x-1 \end{bmatrix} = \begin{bmatrix} z+x & x+y \\ 2x & 3 \end{bmatrix};$$

(ii) $$\begin{bmatrix} x & y+2 & x \\ 5 & y & z \end{bmatrix} = \begin{bmatrix} y+1 & z & x \\ x+y & x-1 & 2y \end{bmatrix};$$

(iii) $$\begin{bmatrix} x & y+z \\ 4 & 2y \end{bmatrix} = \begin{bmatrix} 2 & x+y \\ 4 & x+z \end{bmatrix}.$$

7. Carry out the following vector computations:

(i) $4(2, -2)$; (ii) $2(5, 1, -1)$;

(iii) $(2, 3) + (1, 4)$; (iv) $(1, 0, 3) + 3(4, 4, 4)$;

(v) $3(-1, 2, 3) + 2(1, 1, -1)$; (vi) $3(1, 0, 1, 0) - 4(2, 0, -1, -1)$.

8. Carry out the following vector computations:

(i) $3(2, -1)$; (ii) $-4(-1, 1, -2)$;

(iii) $(1, -1) + 3(2, 2)$; (iv) $(3, 4, 3) - 2(1, -1, 4)$;

(v) $(2, 4, 1) + 2(2, 1, -1)$; (vi) $(1, 3, 3) - 4(1, 1, -1)$;

(vii) $3(-1, 2, 3) + 2(1, 1, -1)$; (viii) $2(1, 3, 2, 1) - 3(1, 1, -1, -1)$.

9. Suppose

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$
$$D = \begin{bmatrix} 3 & 3 \\ -2 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Calculate the following, or say why they do not exist:

(i) $2A + B$; (ii) $C + D + E$; (iii) $A + D - E$;
(iv) $3C - 2F$; (v) $C - 3F$; (vi) $3A - 3A$.

10. Suppose

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$
$$D = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Calculate the following, or say why they do not exist:

(i) $A + B$; (ii) $C - D + 2E$; (iii) $A - 2D + E$;
(iv) $2C + F$; (v) $2C + 3F$; (vi) $2A - B - D$.

5.6 Vector and Matrix Products

Lines and the Dot Product

The equation of a straight line in coordinate geometry has the form

$$ax + by = c,$$

where a, b, and c are numbers and x, y are the usual variables. The equation involves two vectors, the vector (a, b) of coefficients and the vector (x, y) of variables. For this reason it is natural to associate $ax + by$ with the two vectors (a, b) and (x, y).

We define the *dot product* (also called the *scalar product*) of two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ to be

$$\boldsymbol{u} \cdot \boldsymbol{v} = (u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) = \sum_{k=1}^{n} u_k v_k.$$

In this notation, a typical straight line in two-dimensional geometry has an equation of the form

$$\boldsymbol{a} \cdot \boldsymbol{x} = c,$$

where $\boldsymbol{a}$ is some vector of two real numbers, $\boldsymbol{x}$ is the vector of variables (x, y), and c is a constant. If $\boldsymbol{a}$ and $\boldsymbol{x}$ are of length three, then $\boldsymbol{a} \cdot \boldsymbol{x} = c$ could be the equation of a plane in three-dimensional space.

Sample Problem 5.27. *Suppose* $\boldsymbol{t} = (1, 2, 3)$, $\boldsymbol{u} = (-1, 3, 0)$ *and* $\boldsymbol{v} = (2, -2, 2)$. *Calculate* $\boldsymbol{u} \cdot \boldsymbol{v}$, $(\boldsymbol{t} - \boldsymbol{u}) \cdot \boldsymbol{v}$, *and* $3(\boldsymbol{v} \cdot \boldsymbol{t})$.

Solution. $\boldsymbol{u} \cdot \boldsymbol{v} = -2 - 6 + 0 = -8$; $(\boldsymbol{t} - \boldsymbol{u}) \cdot \boldsymbol{v} = (2, -1, 3) \cdot (2, -2, 2) = 4 + 2 + 6 = 12$; $3(\boldsymbol{v} \cdot \boldsymbol{t}) = 3 \cdot 4 = 12$.

Your Turn. Calculate $\boldsymbol{u} \cdot \boldsymbol{t}$ and $(2\boldsymbol{u} - 3\boldsymbol{v}) \cdot \boldsymbol{t}$.

It is not hard to see that the dot product is commutative. There is no need to discuss the associative law because dot products involving three vectors are not defined. For example, consider $\boldsymbol{t} \cdot (\boldsymbol{u} \cdot \boldsymbol{v})$. Since $(\boldsymbol{u} \cdot \boldsymbol{v})$ is a scalar, not a vector, we cannot calculate its dot product with anything.

As an example of the dot product, suppose a movie theater charges \$7 for adults, \$5 for students, and \$3 for children. There are 25 adults, 22 students, and 13 children in the theater. The total paid was $25 \cdot \$7$, for the adults, $22 \cdot \$5$, for the students, and $13 \cdot \$3$, for the children. The total is $25 \cdot \$7 + 22 \cdot \$5 + 13 \cdot \$3$, or \$324. Let us define two vectors, $\boldsymbol{u} = (25, 22, 13)$, the vector of attendees, and $\boldsymbol{v} = (7, 5, 3)$, the vector of charges. The amount paid was \$ $\boldsymbol{u} \cdot \boldsymbol{v}$.

Another important example: the dot product can be used to sum the elements of a vector. The number in attendance at the theater was $25 + 22 + 13$, the sum of the entries in $\boldsymbol{u}$. This could be written as

$$(1, 1, 1) \cdot \boldsymbol{v} = 25 + 22 + 13 = 60.$$

Matrix Product

We define the *product* of two matrices as a generalization of the scalar product of vectors. Suppose the rows of the matrix A are $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots$, and the columns of the matrix B are $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots$. Then AB is the matrix with (i, j) entry $\boldsymbol{a}_i \cdot \boldsymbol{b}_j$. These entries will only exist if the number of columns of A equals the number of rows of A, so this is a necessary condition for the product AB to exist.

Sample Problem 5.28. *Suppose*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Find AB *and* BA.

Solution. First, we find AB. The rows of A are $\boldsymbol{a}_1 = (1, 2)$ and $\boldsymbol{a}_2 = (1, -1)$. The columns of B are $\boldsymbol{b}_1 = (-1, 2)$ and $\boldsymbol{b}_2 = (1, 0)$. (Since we are treating them as vectors, it doesn't matter whether we write them as row or column vectors.) Then $\boldsymbol{a}_1 \cdot \boldsymbol{b}_1 = -1+4 = 3$, and similarly $\boldsymbol{a}_1 \cdot \boldsymbol{b}_2 = 1$, $\boldsymbol{a}_2 \cdot \boldsymbol{b}_1 = -3$, and $\boldsymbol{a}_2 \cdot \boldsymbol{b}_2 = 1$. So

$$AB = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{b}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{b}_2 \\ \boldsymbol{a}_2 \cdot \boldsymbol{b}_1 & \boldsymbol{a}_2 \cdot \boldsymbol{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix}.$$

Similarly we find

$$BA = \begin{bmatrix} 0 & -3 \\ 2 & 4 \end{bmatrix}.$$

The entries in AB will only exist if the number of columns in A equals the number of rows in B. For example, if A were 3×2 and B were 4×4, the product would not exist. In general, we can say the following.

Theorem 18. *Suppose A is an $m \times n$ matrix and B is an $r \times s$ matrix. If $n = r$, then AB exists and is an $m \times s$ matrix. If $n \neq r$, then AB does not exist.*

If A were 2×3 and B were 3×4, then AB would be a 2×4 matrix but BA would not exist. It is also possible that AB and BA might both exist but might be of different shapes; for example, if A and B have shapes 2×3 and 3×2, respectively, then AB is 2×2 and BA is 3×3. And we observe from the preceding example that, even when AB and BA both exist and are the same shape, they need not be equal. *There is no commutative law for matrix multiplication.*

Matrices and Linear Equations

The dot product of vectors gives us a way of writing linear equations. For example, the equation

$$3x - 4y + 2z = 3$$

can be written

$$(3, -4, 2) \cdot (x, y, z) = 3.$$

The system of two equations

$$3x - 4y + 2z = 3,$$
$$2x + 2y - 3z = 2$$

can be written as the pair of vector equations

$$(3, -4, 2) \cdot (x, y, z) = 3,$$
$$(2, 2, -3) \cdot (x, y, z) = 2.$$

But this is exactly the same as the matrix equation

$$\begin{bmatrix} 3 & -4 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

In the same way, any system of linear equations can be expressed in the form *matrix* $\times$ *column vector* $=$ *column vector.*

Sample Problem 5.29. *Express the following sets of equations in matrix form. (They come from Sample Problem 5.8.)*

$$\begin{aligned} 3x - 2y &= 4, \\ 3x - 2y &= -1; \end{aligned} \qquad \begin{aligned} 2x - y &= 1, \\ -4x + 2y &= -2. \end{aligned}$$

Solution.

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Your Turn. Express the following sets of equations in matrix form. (They are from Your Turn problem 5.8.)

$$\begin{aligned} 2x + 2y &= 3, \\ 2x + 2y &= 4; \end{aligned} \qquad \begin{aligned} x - 2y &= 1, \\ 2x - 4y &= 2. \end{aligned}$$

There are many cases where the matrix form of a system of equations is more appropriate. To illustrate this, we return (again!) to the movie theater manager. Recall that the theater charges \$7 for adults, \$5 for students, and \$3 for children. When 25 adults, 22 students, and 13 children attended, the total paid was

$$(25, 22, 13) \cdot (7, 5, 3).$$

Now suppose you have the ticket sales for a second session: say 41 adults, 12 students, and 11 children, sales total

$$(41, 12, 11) \cdot (7, 5, 3).$$

These two pieces of information can be put into a single matrix form:

$$\boldsymbol{r} = \begin{bmatrix} 25 & 22 & 13 \\ 41 & 12 & 11 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$$

is a column vector whose successive entries are the receipts for the successive sessions. Finally, we can calculate the total by adding the elements of $\boldsymbol{r}$. In other words, we calculate

$$(1, 1) \cdot \boldsymbol{r}.$$

This notation can be expanded by adding further rows as further sessions are held.

Sample Problem 5.30. *A department store sells men's shirts, pants, and jackets in its menswear department. The profit from a shirt is* \$6*, from pants* \$12*, and from a jacket* \$20*. Write down a matrix model to calculate the total profit for three months, if during month i the department sells s_i shirts, p_i pants, and j_i jackets.*

Solution.

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 & p_1 & j_1 \\ s_2 & p_2 & j_2 \\ s_3 & p_3 & j_3 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \\ 20 \end{bmatrix}.$$

Your Turn. A repair shop does welding, panelbeating, and painting. The profit from an hour of welding is \$15, from an hour of panelbeating \$18, and from an hour of painting \$20. Write down a matrix model to calculate the total profit for four weeks, if during week i the department does w_i hours of welding, b_i hours of panelbeating, and p_i hours of painting.

Zero and Identity Elements

The zero matrix behaves under multiplication the way you would expect: provided zero matrices of appropriate size are used,

$$OA = O \quad \text{and} \quad AO = O.$$

This is not just one rule, but an infinite set of rules. If we write in the subscripts, then the full statement is

> If A is any $r \times s$ matrix, then $O_{m,r}A = O_{m,s}$ for any positive integer m, and $AO_{s,n} = O_{r,n}$ for any positive integer n.

There are also matrices that act like the number 1: multiplicative identity elements. We define I_n to be the $n \times n$ matrix with its $(1, 1), (2, 2), \ldots, (n, n)$ entries 1 and all other entries 0. For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If A is any $r \times s$ matrix, then $I_r A = A = AI_s$.

We call I_n an *identity matrix* of order n.

Commutativity

The commutative law does not hold for matrices in general. Even if AB and BA are both defined and are the same size, it is possible for the two products to be different (see Sample Problem 5.28, above). On the other hand, some pairs of matrices have the same product in either order. If $AB = BA$ we say that A and B *commute*, or A *commutes with* B. For example, any 3×3 matrix commutes with I_3. There are many other examples.

Sample Problem 5.31. *Show that the following matrices commute.*

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Solution.

$$AB = BA = \begin{bmatrix} 3 & -4 \\ 2 & -1 \end{bmatrix}.$$

Your Turn. Show that the following matrices commute.

$$C = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Suppose A has shape $m \times n$ and B is $r \times s$. If both AB and BA exist, then necessarily $n = r$ and $m = s$; then AB is $m \times m$ and BA is $n \times n$. In order for A and B to commute, we must have $m = n$. Both A and B must have the same number of rows as columns. Such a matrix is called *square*, and the common dimension is called its *order*.

If A is square, we can evaluate the product AA. We call this A *squared*, and write it as A^2, just as with powers of numbers. We define other positive integer powers similarly: $A^3 = AAA = AA^2$, and in general $A^{n+1} = AA^n$.

Exercises 5.6 A

1. Carry out the following vector calculations:

(i) $(1, -1) \cdot (2, 3)$;

(ii) $(1, 3, 3) \cdot (1, 0, -2)$;

(iii) $(1, -1, -2) \cdot (3, 2, -1)$;

(iv) $(-1, 2, -1, 3) \cdot (2, 4, -3, -1)$;

(v) $(1, -1, 3, 1) \cdot (-2, -1, 1, 4)$;

(vi) $(1, -1, 3, 2) \cdot (2, -2, -1, 1)$.

2. A is a 2×4 matrix; B is 2×4; C is 1×3; D is 4×2; E is 3×4; F is 4×3; G is 4×4. Say whether the indicated matrix exists. If it does exist, what is its shape?

(i) $2A - B$; (ii) AD; (iii) CF;
(iv) CF^T; (v) DA; (vi) BFE;
(vii) FF; (viii) CEF; (ix) AGF.

3. Carry out the following matrix computations:

(i)
$$\begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix};$$

(ii)
$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix};$$

(iii)
$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix};$$

(iv)
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix};$$

(v)
$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}.$$

4. In this exercise,

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & -1 \\ -1 & 4 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$
$$D = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 3 & 3 \\ -2 & 2 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix},$$
$$G = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 2 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & 2 \end{bmatrix}.$$

Carry out the matrix computations, or explain why they are impossible:

(i) BF; (ii) AC;
(iii) CF; (iv) BG;
(v) GD; (vi) $EK + KB$.

5. Suppose the first quarter's menswear sales, in the department store of Sample Problem 5.30, are as shown. In each case, use matrices to calculate the profit for the quarter.

(i)

	Shirts	Pants	Jackets
January	150	50	20
February	130	40	40
March	140	50	15

(ii)

	Shirts	Pants	Jackets
January	60	30	12
February	15	20	10
March	35	20	25

6. In each case, find the products AB and BA. Do the two matrices commute?

(i) $$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix};$$

(ii) $$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 4 \\ 3 & 2 \end{bmatrix};$$

(iii) $$A = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}.$$

7. The matrix A is given. Find A^2 and A^3.

(i) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$;

(iii) $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$; (iv) $\begin{bmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$.

8. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

(i) Find A^2 and A^3.

(ii) Evaluate $A^3 + A^2 - 3A$.

(iii) Show that $A^2 - 3A + 3I = O$.

Exercises 5.6 B

1. Carry out the following vector calculations:

(i) $(2, 3) \cdot (1, -1)$; (ii) $(1, 3) \cdot (2, -1)$;

(iii) $(2, 1, -1) \cdot (3, 0, 1)$; (iv) $(1, 1, -1) \cdot (2, 0, 3)$;

(v) $(4, 2, 1) \cdot (1, 2, 4)$; (vi) $(2, 2, -1) \cdot (2, -2, -3)$;

(vii) $(0, 1, 0, 1) \cdot (3, -1, 2, 2)$; (viii) $(0, 2, 1, 1) \cdot (3, 4, 2, 1)$;

(ix) $(3, -1, 3, 2) \cdot (-1, -1, 2, 1)$; (x) $(1, -2, 5, 2) \cdot (2, 2, 3, 1)$.

2. A is a 2×4 matrix; B is 2×4; C is 1×3; D is 4×2; E is 3×4; F is 4×3; G is 4×4. Say whether the indicated matrix exists. If it does exist, what is its shape?

(i) $A + B$; (ii) CE; (iii) $D(A + B)$;
(iv) F^T; (v) $2FC$; (vi) $AD + DA$;
(vii) GG; (viii) $DA + 3G$; (ix) CFE.

3. A is a 2×4 matrix; B is 3×4; C is 1×3; D is 4×2; E is 3×4; F is 4×2; G is 4×4. Say whether the indicated matrix exists. If it does exist, what is its shape?

(i) $A + F$; (ii) AF; (iii) $A + F^T$;
(iv) AF^T; (v) FC; (vi) ADA;
(vii) $(B + E)F$; (viii) AGF; (ix) AGE.

4. Carry out the following matrix computations:

(i) $$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix};$$

(ii) $$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^T\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix};$$

(iii) $$\begin{bmatrix} 3 & -4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -7 & 4 \end{bmatrix};$$

(iv) $$\begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix};$$

(v) $$\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix};$$

(vi) $$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}^T.$$

5. In this exercise,

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$
$$D = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -3 \\ -2 & 2 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix},$$
$$G = \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & 2 \end{bmatrix}.$$

Carry out the matrix computations, or explain why they are impossible:

(i) BD;
(ii) EF;
(iii) $D - EB$;
(iv) KB;
(v) CE^T;
(vi) $BK^T + AH^T$.

6. A manufacturer sells three products: hard drives, zip drives and flash drives. The number that were sold in its US and Canada markets in April were given by the following matrix M.

	Hard	Zip	Flash
USA	5000	1000	9000
Canada	1000	300	2000

Prices, in dollars, for the three products are

$$S = \begin{bmatrix} 150 \\ 100 \\ 50 \end{bmatrix},$$

respectively, while the manufacturing costs are

$$T = \begin{bmatrix} 100 \\ 70 \\ 30 \end{bmatrix}.$$

(i) Calculate the matrix quantities MS, MT, and $M(S - T)$.

(ii) Interpret the matrix quantities MS, MT, and $M(S - T)$.

(iii) What was the total profit in April?

7. An automobile dealership sells three models of new cars. The profit from a sale is \$1200 per economy sedan, \$2000 per family sedan, and \$2200 per coupe. Write down a matrix product that gives the profit for the first three months of the year, given that the dealership sells e_i economy sedans, f_i family sedans, and c_i coupes in month i. Use the product to calculate the profit in the following cases:

(i)

	Economy	Family	Coupe
January	15	8	2
February	11	14	4
March	14	6	3

(ii)

	Economy	Family	Coupe
January	25	8	2
February	10	8	5
March	5	11	7

8. In each case, find the products AB and BA. Do the two matrices commute?

(i) $$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix};$$

(ii) $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix};$$

(iii) $$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -5 \\ 10 & 4 \end{bmatrix};$$

(iv) $$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix};$$

(v) $$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & 0 \\ 2 & 3 & 2 \end{bmatrix};$$

(vi) $$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

9. The matrix A is given. Find A^2 and A^3.

(i) $\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$; (iv) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$;

(v) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; (vi) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$;

(vii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$; (viii) $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$.

10. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}.$$

(i) Find A^2 and A^3.

(ii) Evaluate $A^3 - 2A - I$.

(iii) Show that $A^2 - 4A - 12I = O$.

11. A and B are any two matrices such that AB exists. Prove that $B^T A^T$ exists, and that

$$B^T A^T = (AB)^T.$$

5.7 Inverses

If the matrices A and B satisfy $AB = BA = I$, we say that B is an *inverse* of A.

Which matrices have inverses? In the real numbers, everything but 0 has an inverse. In the integers, only 1 and -1 have integer inverses, but we know that we can obtain inverses of other non-zero integers by going to the rational numbers. The situation is more complicated for matrices.

Suppose A is an $r \times c$ matrix. Then AB has r rows and BA has c columns. If the two are to be equal, then AB is an $r \times c$ matrix; and if this is to equal an identity matrix, it must be square. So $r = c$, and A is also a square matrix. Only a square matrix can have an inverse.

Moreover, there are non-zero square matrices without inverses. Even if we restrict our attention to the 2×2 case, there are examples.

Sample Problem 5.32. *Show that the matrix*

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

has no inverse.

Solution. Suppose A has inverse

$$B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}.$$

Then $AB = I$, so

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 2x+z & 2y+t \\ 2x+z & 2y+t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The (1, 1) entries of the two matrices must be equal, so $2x+z = 1$; but the (2, 1) entries must also be equal, so $2x + z = 0$. This is impossible.

Your Turn. Show that the matrices

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have no inverses.

A matrix that has an inverse will be called *invertible* or *non-singular*; a square matrix without an inverse is called *singular*.

We used the phrase "an inverse" above. However, we shall show that, if a matrix has an inverse, it is unique. We normally write it as A^{-1}, just as with numbers, but the notation $\frac{1}{A}$ is never used for matrices.

Theorem 19. *If matrices A, B, C satisfy $AB = BA = I$ and $AC = CA = I$, then $B = C$.*

Proof. Suppose A, B and C satisfy the given equations. Then

$$C = CI = C(AB) = (CA)B = IB = B$$

so B and C are equal. □

In fact, it can be shown that either of the conditions $AB = I$ or $BA = I$ is enough to determine that B is the inverse of A. However, this requires more algebra than we shall cover in this book.

Sample Problem 5.33. *Suppose*

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

What is the inverse of A if it exists?

Solution. Suppose the inverse is

$$C = \begin{bmatrix} x & z \\ y & t \end{bmatrix}.$$

Then $AC = I$ means

$$\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is equivalent to the four equations

$$\begin{aligned} 3x + 2y &= 1, \qquad & 3z + 2t &= 0, \\ 4x + 3y &= 0, \qquad & 4z + 3t &= 1. \end{aligned}$$

The left-hand pair of equations is easily solved to give $x = 3$ and $y = -4$, while the right-hand pair give $z = -2$ and $t = 3$. So the inverse exists, and is

$$A^{-1} = C = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}.$$

Your Turn. What is the inverse of B if it exists?

The above procedure can be used to invert square matrices of any order; if there is no inverse, then the equations will have no solution. We shall now show how to reduce the number of computations required.

Calculating the Inverse

The inverse of a square matrix can be calculated by the algorithm used for solving equations. Suppose A is an $n \times n$ matrix with inverse B. Write $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ for the columns of B, and write $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ for the columns of the identity matrix of order n. Consider the equation $AB = I$. Column j of the left-hand side is $A\boldsymbol{b}_j$. So the equation is equivalent to the set of n systems

$$A\boldsymbol{b}_1 = \boldsymbol{u}_1, \qquad A\boldsymbol{b}_2 = \boldsymbol{u}_2, \qquad \ldots, \qquad A\boldsymbol{b}_n = \boldsymbol{u}_n.$$

If all these systems have solutions, then the inverse is formed by putting the solution vectors next to each other in order. If any system has no solution, there is no inverse.

To solve $A\boldsymbol{b}_j = \boldsymbol{u}_j$, we reduce the augmented matrix $[A \mid \boldsymbol{u}_j]$ to reduced row echelon form. The same steps will produce this result, no matter what vector is on the right-hand side. It follows that we can carry out the reduction simultaneously for all n systems of equations. So we have the following technique for inverting an $n \times n$ matrix A.

Row reduce the matrix $[A \mid I_n]$. If the resulting matrix has form $[I_n \mid B]$ then A is invertible, and B is A^{-1}. Otherwise, A is singular.

It follows from this that if a matrix has a row with every entry zero, it must be singular. This is also true if the matrix has a column with every entry zero. For example, if A has every entry of its first column zero, then BA has every element zero in *its* first column for any choice of B, so the equation $A^{-1}A = I$ cannot possibly be true—it must fail in the (1, 1) position.

Sample Problem 5.34. *For the following matrices, find the inverse or show that the matrix is singular:*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 2 \\ 3 & -1 & 2 \\ 1 & 7 & 2 \end{bmatrix}.$$

Solution. For A, we have

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -2 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right].$$

So A has inverse

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix}.$$

For B, we get

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 2 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 1 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 7 & 2 & 0 & 0 & 1 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 2 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 7 & 2 & 0 & 0 & 1 \\ 0 & -11 & -2 & 0 & 1 & -2 \\ 0 & -22 & -4 & 1 & 0 & -3 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 7 & 2 & 0 & 0 & 1 \\ 0 & -11 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

and the zero row on the left tells us that B is singular.

Your Turn. In the above calculations, identify the steps that have been taken at each stage. (For example, for A, the first step was $R3 \leftarrow R3 - R1$.)

This method can be used to get a general solution for the inverse of a 2×2 matrix.

Theorem 20. *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is singular if $ad - bc = 0$*. Otherwise it is invertible, with inverse*

$$\frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{5.3}$$

Proof. If $a = c = 0$, then $ad - bc = 0$. Moreover A has a zero column, so as we saw it has no inverse. So we need only consider cases where a, c, or both are non-zero.

First, suppose a and c are both non-zero. The inverse procedure is

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ ac & ad & 0 & a \end{array}\right] \quad \begin{array}{l} R1 \leftarrow cR1, \\ R2 \leftarrow aR2, \end{array}$$
$$\Rightarrow \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & ad - bc & -c & a \end{array}\right] \quad R2 \leftarrow R2 - R1,$$
$$\Rightarrow \left[\begin{array}{cc|cc} ac & 0 & (1 + \frac{bc}{ad-bc})c & -\frac{abc}{ad-bc} \\ 0 & ad - bc & -c & a \end{array}\right] \quad R1 \leftarrow R1 - \frac{bc}{ad-bc} R2.$$

If $ad - bc = 0$, then we are finished and there is no inverse. Otherwise, notice that

$$1 + \frac{bc}{ad - bc} = \frac{ad - bc + bc}{ad - bc} = \frac{ad}{ad - bc},$$

so we have

$$\left[\begin{array}{cc|cc} ac & 0 & \frac{acd}{ad-bc} & -\frac{abc}{ad-bc} \\ 0 & ad - bc & -c & a \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \quad \begin{array}{l} R1 \leftarrow \frac{1}{ac}R1, \\ R1 \leftarrow \frac{1}{ad-bc}R2 \end{array}$$

as required.

If $a \neq 0$ and $c = 0$ the calculations are simpler. We obtain the inverse

$$\begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} \\ 0 & \frac{1}{d} \end{bmatrix}$$

and this is the form taken by (5.3) when $c = 0$.

The case where $a = 0$, $c \neq 0$ is similar to the latter case. □

The Determinant

The number $ad-bc$ is called the *determinant* of the matrix A, written $\det(A)$. Determinants may be defined for square matrices of any order, and it is a general theorem that a matrix is invertible if and only if its determinant is non-zero. However, we need only consider the 2×2 case.

Sample Problem 5.35. *Find the determinants of the following matrices and use them to find their inverses, if possible:*

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}.$$

Solution. $\det(A) = 3 \cdot 2 - 1 \cdot 2 = 4$, so

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

$\det(B) = 2 \cdot 1 - (-2) \cdot (-1) = 0$, so B has no inverse.

Your Turn. Repeat for

$$C = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

Exercises 5.7 A

1. Show that the given matrices are inverses:

(i) $$\begin{bmatrix} \frac{3}{2} & -1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & 1 \end{bmatrix};$$

(ii) $$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

2. Find matrices A and B such that

(i) $$A\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix};$$

(ii) $$B\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}.$$

3. Use row reduction either to find the inverse of the given matrix or to show that the matrix is singular:

(i) $\begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$; (ii) $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$;

(iii) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$; (iv) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;

(v) $\begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$; (vi) $\begin{bmatrix} 0 & 1 & 1 \\ 5 & 1 & -2 \\ 2 & -3 & -3 \end{bmatrix}$;

(vii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$; (viii) $\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 7 \\ 3 & 2 & -8 \end{bmatrix}$.

4. Find the determinant of the matrix, and use it to invert the matrix or show that it is singular:

(i) $\begin{bmatrix} 7 & 4 \\ 2 & 3 \end{bmatrix}$; (ii) $\begin{bmatrix} -2 & 2 \\ 2 & 3 \end{bmatrix}$;

(iii) $\begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$; (iv) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

Exercises 5.7 B

1. Show that the given matrices are inverses:

(i) $$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix};$$

(ii) $$\begin{bmatrix} \frac{1}{2} & -1 \\ 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 2 \\ -2 & 1 \end{bmatrix}.$$

2. Find matrices A, B, and C such that

(i) $$A\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix};$$

(ii) $$B\begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix};$$

(iii) $$\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} C = \begin{bmatrix} 7 & 3 \\ 0 & 1 \end{bmatrix}.$$

3. Use row reduction either to find the inverse of the given matrix or to show that the matrix is singular:

(i) $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 0.5 \end{bmatrix}$;

(iii) $\begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix}$; (iv) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$;

(v) $\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$; (vi) $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$;

(vii) $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$; (viii) $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$;

(ix) $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$; (x) $\begin{bmatrix} 1 & -2 & 3 \\ 3 & 5 & 1 \\ 6 & 4 & 2 \end{bmatrix}$;

(xi) $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 3 \\ 2 & 7 & -6 \end{bmatrix}$; (xii) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$;

(xiii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 7 & 4 \end{bmatrix}$; (xiv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 8 \end{bmatrix}$;

(xv) $\begin{bmatrix} 3 & 2 & 2 \\ 4 & 2 & 3 \\ 5 & 4 & 3 \end{bmatrix}$; (xvi) $\begin{bmatrix} 1 & 2 & 1 \\ -3 & -5 & -3 \\ 1 & 3 & 2 \end{bmatrix}$;

(xvii) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$; (xviii) $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}$;

(xix) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$; (xx) $\begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & -1 \\ 3 & 1 & 3 \end{bmatrix}$.

4. Find the determinant of the matrix, and use it to invert the matrix or show that it is singular:

(i) $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$; (ii) $\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$;

(iii) $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$; (iv) $\begin{bmatrix} 6 & 4 \\ 7 & 5 \end{bmatrix}$;

(v) $\begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$; (vi) $\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$;

(vii) $\begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix}$; (viii) $\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$;

(ix) $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$; (x) $\begin{bmatrix} 6 & -2 \\ -11 & 4 \end{bmatrix}$.

5.8 More About Inverses

Some Properties of Inverses

As we said, the usual notation for the inverse of A (if it exists) is A^{-1}. If we define $A^0 = I$ whenever A is square, then powers of matrices satisfy the usual index laws

$$A^m A^n = A^{m+n}, \qquad \left(A^m\right)^n = A^{mn}$$

for all non-negative integers m and n, and for negative values also provided that A^{-1} exists. If x and y are non-zero reals, then $(xy)^{-1} = x^{-1}y^{-1}$. The fact that matrices do not necessarily commute means that we have to be a little more careful, and prove the following small theorem.

Theorem 21. *If A and B are invertible matrices of the same order, then AB is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. We need to show that both $(B^{-1}A^{-1})(AB)$ and $(AB)(B^{-1}A^{-1})$ equal the identity. But $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I = AA^{-1} = AIA^{-1} = A(BB^{-1})A^{-1} = (AB)(B^{-1}A^{-1})$. □

There are two cancellation laws for matrix multiplication. If A is an invertible $r \times r$ matrix and B and C are $r \times s$ matrices such that $AB = AC$, then

$$AB = AC \quad \Rightarrow \quad A^{-1}(AB) = A^{-1}(AC)$$
$$\Rightarrow \quad (A^{-1}A)B = (A^{-1}A)C \quad \Rightarrow \quad IB = IC \quad \Rightarrow \quad B = C,$$

so $B = C$. Similarly, if A is an invertible $s \times s$ matrix and B and C are $r \times s$ matrices such that $BA = CA$, then $B = C$.

The requirement that A be invertible is necessary. We can find matrices A, B, and C such that AB and AC are the same size, A is non-zero and $AB = AC$, but B and C are different. One very easy example is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}.$$

Some other examples are given in the exercises.

Moreover, we can only cancel on one side of an equation; we cannot mix the two sides. Even if A is invertible it is possible that $AB = CA$ but $B \neq C$ (see the exercises for examples of this, also).

Linear Systems and Inverses

Consider the system of equations $A\boldsymbol{x} = \boldsymbol{b}$, where A is an invertible matrix. Multiplying by A^{-1}, $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$, so the equations have the unique solution $A^{-1}\boldsymbol{b}$. This could be used to solve the equations. This technique is not usually helpful in practical situations because the process of finding the inverse takes at least as long as solving the equations, but it is useful when there are several sets of equations with the same left-hand sides, or when the inverse is already known. It is also important in theoretical studies.

Sample Problem 5.36. *Solve the systems:*

$$\begin{aligned} x + 2y - z &= 3, \\ -2y + z &= 1, \\ x + y - z &= 0; \end{aligned} \qquad \begin{aligned} x + 2y - z &= -1, \\ -2y + z &= -2, \\ x + y - z &= 2. \end{aligned}$$

Solution. We saw in Sample Problem 5.34 that the matrix of coefficients has inverse

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix},$$

so the first system has solution $x = 4$, $y = 3$, $z = 7$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -8 \end{bmatrix},$$

and the second has solution $x = -3$, $y = -3$, $z = -8$.

Your Turn. Solve the systems:

$$\begin{aligned} x + 2y - z &= 2, \\ -2y + z &= 2, \\ x + y - z &= 1; \end{aligned} \qquad \begin{aligned} x + 2y - z &= 4, \\ -2y + z &= -1, \\ x + y - z &= 3. \end{aligned}$$

The Leontief Input–Output Model

Matrices have important applications in economics. One important example is the input–output model developed by Wassily Leontief in the 1950s. (He was awarded the Nobel Prize in economics for this work.)

We explain Leontief's method using a very much oversimplified example. We shall describe a company that has no outside needs other than its own products—no outside costs, etc. This is unrealistic, but illustrates the main principles.

Suppose a company produces three petroleum products—gasoline, heating oil, and lubricating oil. During production of gasoline, a certain amount of each product is required, and naturally the company uses its own products. Suppose that, in order to produce a barrel of gasoline, it is necessary to use 0.2 barrels of gasoline, 0.1 barrels of heating oil, and 0.2 barrels of lubricating oil. Similarly, producing a barrel of heating oil uses 0.2 barrels of gasoline, 0.3 barrels of heating oil, and 0.1 barrels of lubricating oil, and a barrel of lubricating oil requires 0.1 barrels of gasoline, 0.2 barrels of heating oil, and 0.2 barrels of lubricating oil. This information can be summarized in the following matrix, called the *technology matrix*, or *input–output matrix*, of the system.

$T =$

	Gas	Heat	Lub
Gas	0.2	0.2	0.1
Heat	0.1	0.3	0.2
Lub	0.2	0.1	0.2

Each column represents the demands of the system in producing one product. For example, in the "heating oil" column, the entry in the "gasoline" row says how much gasoline is needed to produce one barrel of heating oil.

Suppose the company produces 90 million barrels of gasoline, 60 million barrels of heating oil, and 70 million barrels of lubricating oil each year. Much of this product is consumed in production. To find out how much, we perform a matrix multiplication. First, we construct a column vector called the *production matrix* or *total output* matrix,

$$P = \begin{array}{lr} \hline \text{Gas} & 90 \\ \text{Heat} & 60 \\ \text{Lub} & 70 \\ \hline \end{array}$$

Then the internal demand is

$$D_0 = TP = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 90 \\ 60 \\ 70 \end{bmatrix} = \begin{bmatrix} 37 \\ 41 \\ 38 \end{bmatrix} \quad \begin{array}{l} \text{Gas,} \\ \text{Heat,} \\ \text{Lub.} \end{array}$$

So 37 million barrels of gasoline, 41 million barrels of heating oil, and 38 million barrels of lubricating oil is consumed within the company. This is called the internal demand and D_0 is the *internal demand matrix*.

We can now calculate how much of each product is available for outside users, how much the company can actually sell. This amount is

$$D = P - D_0 = \begin{bmatrix} 90 \\ 60 \\ 70 \end{bmatrix} - \begin{bmatrix} 37 \\ 41 \\ 38 \end{bmatrix} = \begin{bmatrix} 53 \\ 19 \\ 32 \end{bmatrix} \quad \begin{array}{l} \text{Gas} \\ \text{Heat} \\ \text{Lub} \end{array}$$

If the company is operating efficiently, this vector will show exactly how much is being bought by customers, so it is called the *external demand* matrix. The fundamental equation of input–output analysis is

$$P = TP + D. \tag{5.4}$$

Using the Leontief Model

Usually the values in P are not given to us. The typical problem encountered by a company is, given the external demands by our customers and the needs of our processes, how much of each product should we produce?

Equation (5.4) can be rewritten as

$$P - TP = D,$$

or

$$(I - T)P = D.$$

If the matrix $I - T$ has an inverse, then

$$P = (I - T)^{-1} D.$$

So we have the following result.

Theorem 22. *If a company is to meet its external demands, then the total production P satisfies*

$$P = (I - T)^{-1} D.$$

If the matrix $I - T$ has no inverse, the company cannot meet its demands.

In our ongoing example, the matrix was

$$I - T = \begin{bmatrix} 0.8 & -0.2 & -0.1 \\ -0.1 & 0.7 & -0.2 \\ -0.2 & -0.1 & 0.8 \end{bmatrix}.$$

This matrix has inverse

$$(I - T)^{-1} = \frac{1}{393} \begin{bmatrix} 540 & 170 & 110 \\ 120 & 620 & 170 \\ 150 & 120 & 540 \end{bmatrix}.$$

Notice that the elements in $(I - T)^{-1}$ are quite small numbers. This will often happen. All entries in $I - T$ will be less than 1, and they will usually be given in decimal form. In calculating the inverse, you will find it easier to invert $\frac{1}{10}(I - T)$ (or even $\frac{1}{100}(I - T)$), and multiply the resulting inverse by 10 (or 100).

Sample Problem 5.37. *Suppose our oil company has an external demand of* 50 *million barrels of gasoline,* 30 *million barrels of heating oil, and* 20 *million barrels of lubricating oil. How much of each product should be produced?*

Solution.

$$P = \frac{1}{393} \begin{bmatrix} 540 & 170 & 110 \\ 120 & 620 & 170 \\ 150 & 120 & 540 \end{bmatrix} \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

$$= \frac{1}{393} \begin{bmatrix} 34300 \\ 28000 \\ 21900 \end{bmatrix} = \begin{bmatrix} 87.277 \\ 71.247 \\ 55.725 \end{bmatrix} \begin{matrix} \text{Gas,} \\ \text{Heat,} \\ \text{Lub.} \end{matrix}$$

So the company should produce 87277000 barrels of gasoline, 71247000 barrels of heating oil, and 55725000 barrels of lubricating oil.

Leontief's original application was not to companies, but to the economies of countries. Rather than individual products, he dealt with whole *sectors* of the economy: the oil industry, the electric power industry, the electronics industry, and so on.

He identified 81 sectors of the U.S. economy, and applied his analysis to it. Such an analysis, applied to a small country, can identify areas where the economy cannot meet its demands, and is used to suggest which commodities must be imported, which trade agreements should therefore be strengthened, and so on.

Sample Problem 5.38. *A farming community produces two commodities, beef and vegetables. Beef production uses* 0.3 *pounds of beef and* 0.1 *pounds of vegetables, to feed animals and workers, for each pound produced. Vegetable production requires* 0.2 *pounds of beef and* 0.4 *pounds of vegetables. There is external demand for* 24000 *pounds of beef and* 16000 *pounds of vegetables. How much should be produced?*

Solution. The technology matrix is

$$T = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$$

and

$$(I - T) = \begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.6 \end{bmatrix}$$

which has determinant $0.42 - 0.02 = 0.4$. So

$$(I - T)^{-1} = 2.5 \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix},$$

so

$$P = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} 24000 \\ 16000 \end{bmatrix} = \begin{bmatrix} 44000 \\ 34000 \end{bmatrix}.$$

So the required production is 44000 pounds of beef and 34000 pounds of vegetables.

Your Turn. An economy produces coal and steel. Each ton of steel requires 0.3 tons of steel and 0.5 tons of coal in its production. Each ton of coal requires 0.2 tons of steel and 0.5 tons of coal. In order to supply an outside demand for 25 tons of steel and 50 tons of coal, how much should be produced?

Exercises 5.8 A

1. In each case show that $AB = AC$:

(i) $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 \\ 3 & 3 \end{bmatrix};$

(ii) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$

2. Show that A^{-1} exists, but $AB = CA$, even though $B \neq C$:

$$A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 7 \\ -2 & -3 \end{bmatrix}.$$

3. (i) Prove that the following matrices are inverses:

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix};$$

(ii) Use part (i) to solve the following systems:

$$\text{(a)} \quad \begin{aligned} 3x + 2y &= 4, \\ x + y &= 1; \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x - 2y &= -1, \\ -x + 3y &= 2. \end{aligned}$$

4. (i) Prove that the following matrices are inverses:

$$\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 & 1 \\ 1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix};$$

(ii) Use part (i) to solve the following systems.

$$\text{(a)} \quad \begin{aligned} 3x + 5y + 7z &= 1, \\ x + 2y + 3z &= 2, \\ 2x + 3y + 5z &= 1; \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x - 4y + z &= 2, \\ x + y - 2z &= 1, \\ -x + y + z &= -1. \end{aligned}$$

5. In each case the technology matrix T and external demand matrix D for a two-sector economy are shown. What is the associated production matrix P?

(i) $$T = \begin{bmatrix} 0.3 & 0.6 \\ 0.4 & 0.3 \end{bmatrix}, \qquad D = \begin{bmatrix} 30 \\ 20 \end{bmatrix};$$

(ii) $$T = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.6 \end{bmatrix}, \qquad D = \begin{bmatrix} 42 \\ 36 \end{bmatrix};$$

(iii) $$T = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 20 \\ 30 \end{bmatrix};$$

(iv) $$T = \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 200 \\ 120 \end{bmatrix}.$$

6. An economy produces two products, X and Y. To produce each kilogram of X, one must use 500 grams of X and 250 grams of Y. To produce each kilogram of Y, one must use 100 grams of X and 250 grams of Y. In a certain week, the external demand is 84 tonnes of X and 98 tonnes of Y. What is the required production?

Exercises 5.8 B

1. In each case show that $AB = AC$:

(i) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 3 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$;

(ii) $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 3 \\ 2 & 0 & 1 \end{bmatrix}$,

$C = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

2. Show that A^{-1} exists, but $AB = CA$, even though $B \neq C$, when

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

3. Find a matrix A such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

4. (i) Prove that the following matrices are inverses:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix};$$

(ii) Use part (i) to solve the following systems:

(a) $x + 2y + 3z = 3$, $x + y + z = 1$, $x + y + 2z = 4$;

(b) $-x + y + z = -1$, $x + y - 2z = 0$, $-y + z = 1$.

5. (i) Prove that the following matrices are inverses:

$$\begin{bmatrix} 2 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & -3 & -2 \\ 1 & -1 & -1 \\ -3 & 4 & 2 \end{bmatrix};$$

(ii) Use part (i) to solve the following systems:

(a) $2x - 2y + z = 3$, $x + z = 2$, $x - 3y = 1$;

(b) $3x - 3y - 2z = 2$, $x - y - z = -1$, $-3x + 4y + 2z = 2$.

6. In each case the technology matrix T and external demand matrix D for a two-sector economy are shown. What is the associated production matrix P?

(i) $$T = \begin{bmatrix} 0.4 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}, \qquad D = \begin{bmatrix} 30 \\ 45 \end{bmatrix};$$

(ii) $$T = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{bmatrix}, \qquad D = \begin{bmatrix} 42 \\ 36 \end{bmatrix};$$

(iii) $$T = \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 40 \\ 88 \end{bmatrix};$$

(iv) $$T = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}, \qquad D = \begin{bmatrix} 70 \\ 80 \end{bmatrix};$$

(v) $$T = \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}, \qquad D = \begin{bmatrix} 40 \\ 60 \end{bmatrix};$$

(vi) $$T = \begin{bmatrix} 0.4 & 0.7 \\ 0.4 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 44 \\ 28 \end{bmatrix};$$

(vii) $$T = \begin{bmatrix} 0.25 & 0.1 \\ 0.25 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 690 \\ 460 \end{bmatrix};$$

(viii) $$T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.6 \end{bmatrix}, \qquad D = \begin{bmatrix} 100 \\ 200 \end{bmatrix}.$$

7. In a three-sector economy the technology matrix T and external demand matrix D are

$$T = \begin{bmatrix} 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 40 \\ 30 \\ 45 \end{bmatrix}.$$

What is the associated production matrix P?

8. An economy produces three products, A, B, and C. To produce a hundred pounds of A requires 30 pounds of A, 30 pounds of B, and 10 pounds of C. To produce 100 pounds of B requires 10 pounds of A and 50 pounds of B (no C is used). To produce 100 pounds of C requires 50 pounds of A, 10 pounds of B and 70 pounds of C. What is the required production in the following cases?

(i) In the first quarter of the year, there was external demand for 7000 pounds of A, 14000 pounds of B, and 21000 pounds of C.

(ii) In the second quarter, there was external demand for 14000 pounds of A, 7000 pounds of B, and 14000 pounds of C.

9. Suppose M_x denotes the 2×2 matrix

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

where x may be any real number.

(i) Compute $M_x M_y$, and show that the matrices M_x and M_y commute for any real numbers x and y.

(ii) Find ${M_x}^2$, ${M_x}^3$, and ${M_x}^4$.

(iii) Find a formula for ${M_x}^n$, where n is any positive integer.

(iv) What is ${M_x}^{-1}$?

6

Linear Programming

6.1 Linear Programming Problems

Constrained Optimization

In many problems, the object is to *optimize*—either *maximize* or *minimize*—something. A company wishes to maximize profit or minimize cost; a traveler wishes to minimize the length of a journey; a factory manager wishes to maximize output. But it is not always straightforward. For example, a traveler might wish to minimize travel time, but cannot exceed her budget. In this case, the financial limitation is called a *constraint*.

Here is a simple example. Suppose a furniture manufacturer makes both dining suites and desk sets. Two types of timber are used: 12-inch wide oak for tabletops and chair seats, one-inch pine for legs and supports. Each dining suite uses 44 feet of oak and 80 feet of pine; each desk set uses 15 feet of oak and 20 feet of pine. Each dining suite returns a profit of $100 after all expenses (salaries, cost of materials, factory costs) have been subtracted; each desk set returns $35.

How many dining suites and desk sets should be produced per day? If you schedule x dining suites and y desk sets, the profit will be $\$(100x + 35y)$, and the objective is to maximize profit. So you need to find values for x and y that maximize $100x + 35y$. But you need to know what constraints apply. For example, it may be that only 8000 feet of pine can be supplied each day. The production of x dining suites and y desk sets requires $80x + 20y$ feet of pine, so the x and y values must satisfy

$$80x + 20y \leq 8000.$$

There will probably be another constraint, depending on the amount of oak available. A further constraint will probably be caused by staffing restrictions: each product will require a certain number of hours of work. If there are 50 employees each working

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1_6,

eight hours, at most 400 hours of work can be allocated. And, of course, x and y cannot be negative.

This particular problem is called a *linear* constrained optimization problem. The function to be maximized, $100x + 35y$, is a linear function of the two variables. The constraint on oak leads to a linear inequality, and the other constraints, due to the availability of oak and of manpower, will probably be linear also.

Constrained optimization problems involve some very difficult mathematics. But linear problems are easier than most. There are *algorithms*, or mathematical recipes, available to solve linear constrained optimizations. For this reason, we shall restrict our attention to the linear case. We shall explain the *geometrical method* of solution, which is very useful in problems involving two variables. Then we shall look in detail at one of the most widely used algorithms, the *simplex method*.

Linear Programming Models

The word "program" originally meant the same in mathematics as does "algorithm"—a mathematical recipe, a method for solving a problem that requires no input from the solver: once it is set up, the steps are mechanical. This precisely describes the operation of a computer, so "programming" nowadays usually means computer programming. But it is also used to describe algorithmic methods of solving constrained optimization problems. *Linear programming* is the algorithmic solution of linear constrained optimization problems.

By a *linear programming model* we mean a problem made ready for the application of such an algorithm. The main features of a linear programming model are:

1. There are several variables that we can control; the solution of the problem consists of finding the appropriate real number values of those variables.
2. The quantity to be maximized or minimized is a linear function, called the *objective function*.
3. The variables must satisfy certain linear inequalities, or *constraints* (often called *regular constraints*).
4. The variables are required to be non-negative (*non-negativity constraints*).

The inequalities will always be taken in the *weak* form, where equality is an option. For example, we write $x + y \leq 10$ rather than $x + y < 10$. This distinction is not usually important in practical problems.

In some problems, the variables are required to take integer values. Problems with this sort of constraint are called *integer programs*, and are sometimes very difficult. We normally ignore such requirements. Even though you cannot sell half a table, we can schedule 27.5 tables to be made in a day, with the unfinished table to be completed the next day. Similarly, there are some problems where some variables can be negative. There are special techniques to handle these unusual cases.

Setting up Linear Programs

Problems are hardly ever presented in the workplace in terms of equations and inequalities. It is important to practice interpreting plain English requirements and formulating the problem as a linear programming model.

The first part of the process is to decide what is the objective, the quantity to be optimized—profit (maximized), cost (minimized), and so on. Then the variables must be identified; this is usually done by looking at the objective. Then the objective function, the relation between the variables and the objective, must be formulated. Finally, you need to list the constraints.

Sample Problem 6.1. *Formulate the following problem as a linear programming model: An oil refinery produces both gasoline and heating oil. The refinery has the capacity to produce at most* 30000 *barrels per day. At least* 5000 *barrels of heating oil are required daily because of a contract. The profit is* \$12 *per barrel of gasoline and* \$8 *per barrel of heating oil. In order to satisfy EPA requirements, the amount of gasoline produced can be no greater than three times the amount of heating oil, to limit the noxious byproducts. To maximize profit, what should be produced daily?*

Solution. The quantity to be maximized is profit. The profit depends only on the number of barrels of the two products, so we define the following variables:

Let x denote the number of barrels of gasoline to be produced daily, and y the number of barrels of heating oil.

(This is sometimes called the *dictionary of variables.*)

Then the profit is \$$P$, where $P = 12x + 8y$.

There are three regular constraints. The maximum capacity means that $x + y$ cannot exceed 30000. The contractual requirement is that $y \geq 5000$. The EPA requires that $x \leq 3y$. Both variables must be non-negative. So the model is:

$$\begin{aligned} \text{Maximize} \quad & P = 12x + 8y \\ \text{subject to} \quad & x + y \leq 30000, \\ & y \geq 5000, \\ & x - 3y \leq 0, \\ & x \geq 0, \quad y \geq 0. \end{aligned}$$

(The requirement $y \geq 0$ is in fact redundant.)

In the next section, we shall discuss a geometric method for solving linear programs in two variables. The method involves graphical representation of the constraints, and for this reason it is convenient to denote the variables x and y. When there are three or more variables this method does not apply, and for the technique we use it will be more convenient to label the variables with subscripts: x_1, x_2, x_3, and so on.

Sample Problem 6.2. *Formulate the following problem as a linear programming model: a farmer wishes to plant potatoes, corn, and beans. Each acre of potatoes costs* \$400 *for seed and fertilizer*; *each acre of corn costs* \$160; *each acre of beans costs* \$280. *He has* 100 *acres available, and his capital is* \$20000. *The profit per acre* (*after replacing capital*) *is* \$120 *for potatoes,* \$40 *for corn, and* \$60 *for beans. He can sell any amount of corn and beans, but the markets will take at most* 30 *acres of potatoes. What should he plant to maximize profit?*

Solution. The quantity to be maximized is profit. The profit depends only on the number of acres of each crop, potatoes, corn, and beans. So we define variables:

$$\begin{aligned}
x_1 &= \text{number of acres of potatoes to be planted,}\\
x_2 &= \text{number of acres of corn,}\\
x_3 &= \text{number of acres of beans.}
\end{aligned}$$

The profit is $\$P$, where $P = 120x_1 + 40x_2 + 60x_3$.
The farmer's outlay will be $\$(400x_1 + 160x_2 + 280x_3)$, and this cannot exceed \$20000, and the total number of acres $(x_1 + x_2 + x_3)$ cannot exceed 100. If $x_1 > 30$, the excess potatoes will not be sold. So we have

$$\begin{array}{ll}
\text{Maximize} & P = 120x_1 + 40x_2 + 60x_3\\
\text{subject to} & 400x_1 + 160x_2 + 280x_3 \le 20000,\\
& x_1 + x_2 + x_3 \le 100,\\
& x_1 \le 30,\\
& x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.
\end{array}$$

Your Turn. Formulate the following problem as a linear programming model: a farmer wishes to plant potatoes, corn, and beans. The cost of seed and fertilizer is: \$500 for each acre of potatoes; \$250 for each acre of corn; \$400 for each acre of beans. He has 120 acres available, and his capital is \$30000. The net profit per acre is \$150 for potatoes, \$50 for corn, and \$80 for beans. He can sell any amount of corn and potatoes, but the markets will take at most 20 acres of beans. What should he plant, to maximize profit?

It is important to distinguish between *income* and *profit*. In the following example, the income is given. In order to maximize profit, it is first necessary to deduct the costs from the income to get the profit figure per item.

Sample Problem 6.3. *Formulate the following problem as a linear programming model: a window manufacturer produces two styles of windows, regular and deluxe. It costs* \$100 *to make each regular window, which the manufacturer sells* *for* \$150. *It costs* \$120 *to make each deluxe window, and these sell for* \$175 *each. The daily production capacity is* 110 *windows, and the daily cost cannot exceed* \$12000. *How many windows of each kind should be made daily in order to maximize profit?*

Solution. The quantity to be maximized, profit, depends only on the number of windows produced in each style. We define the following variables:

$$x_1 = \text{number of regular windows produced per day,}$$
$$x_2 = \text{number of deluxe windows produced per day.}$$

The profit for each regular window is \$(150 − 100) = \$50 (sales price − cost) and for each deluxe window it is \$(175 − 120) = \$55. So the daily profit is \$$P$, where

$$P = 50x_1 + 55x_2.$$

The daily outlay will be \$$(100x_1 + 120x_2)$, and this cannot exceed \$12000, and the day's window production, $(x_1 + x_2)$, cannot exceed 110. So we have

$$\begin{array}{ll} \text{Maximize} & P = 50x_1 + 55x_2 \\ \text{subject to} & 100x_1 + 120x_2 \le 12000, \\ & x_1 + x_2 \le 110, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{array}$$

(Since the problem has two variables, x and y could be used instead of x_1 and x_2.)

Sample Problem 6.4. *Formulate the following problem as a linear programming model: a cement company uses a process that produces* 2.2 *pounds of waste particles per barrel of cement produced. The EPA requires that the average waste be reduced to* 1.2 *pounds per barrel. The company installs two precipitators to remove waste; some barrels will be processed with the first machine, some with the second, and some left unprocessed. The first machine eliminates* 1.1 *pounds of waste per barrel and costs* 9 *cents per barrel processed. The second eliminates* 1.5 *pounds of waste per barrel and costs* 15 *cents per barrel processed. What percentage of barrels should be processed with each precipitator, if the cost is to be minimal?*

Solution. We use the following variables:

$$x = \text{percentage of barrels processed with machine 1,}$$
$$y = \text{percentage of barrels processed with machine 2.}$$

(There is no need for a variable representing the percentage not to be processed at all.) The cost will be $\$(0.09x + 0.15y)$. So we wish to minimize $\$C$, where

$$C = 9x + 15y.$$

Barrels processed with the first machine give 1.1 pounds of waste, those from the second machine give 0.7 pounds, and unprocessed barrels still give 2.2 pounds. So the average amount of particular waste will be

$$\big(1.1x + 0.7y + 2.2(100 - x - y)\big)/100.$$

So we require $(1.1x + 0.7y + 2.2(100 - x - y))/100 \leq 1.2$, or

$$1.1x + 1.5y \geq 100.$$

The number of unprocessed barrels cannot be negative, so

$$x + y \leq 100.$$

We have

$$\begin{array}{ll} \text{Minimize} & C = 9x + 15y \\ \text{subject to} & 1.1x + 1.5y \geq 100, \\ & x + y \leq 100, \\ & x \geq 0, \quad y \geq 0. \end{array}$$

Exercises 6.1 A

In each case, construct a linear programming model for the problem given.

1. A contractor makes two types of contract homes. The Fleetwood uses \$30000 worth of materials and requires 500 man-hours; the profit on a Fleetwood is \$20000. The Majestic needs \$40000 worth of materials and 600 man-hours, and returns profit \$25000. He has \$700000 worth of materials in stock and his employees can work at most 11000 hours. How many of each type should he build, to maximize profit?

2. A farmer's chickens need 70 units of carbohydrate, 50 units of protein, and 40 units of fat between them per month. He can buy two commercial feeds, A and B. A pound of feed A contains five units of carbohydrate, three units of protein, and four units of fat, and costs \$2.50. Feed B contains two units of carbohydrate, two units of protein, and one unit of fat per pound, and costs \$1.40 per pound. The farmer wants to feed his chickens a mixture of the two feeds. How many of each feed should he buy per month, in order to make the cheapest blend that supplies the required nutrition?

3. A company produces three types of wooden ornaments. Type I requires six board feet of lumber and takes two hours of labor to produce. Type II requires four board feet of lumber and takes three hours of labor to produce. Type III requires one board foot of lumber and takes one hour of labor to produce. The three types yield \$30, \$45, and \$15 profit, respectively. On a particular day there are 200 board feet of lumber and 100 hours of labor available. How many ornaments of each type should be made, to maximize profit?

4. A farmer wishes to plant up to 60 acres with some combination of corn, linseed, and oats. He has 160 hours of labor available, and \$6000 to spend on costs (seeds, fertilizer, and so on). Oats require three hours labor and cost \$80 per acre, and sell for \$350 per acre. Corn requires five hours labor and cost \$90 per acre, and sell for \$420 per acre. Linseed requires four hours labor and cost \$110 per acre, and sell for \$500 per acre. How many acres should be planted with each crop, to maximize profit? (Remember that to calculate the profit, you must subtract the cost from the income.)

5. A mutual fund has \$100000000 to invest. They estimate they can earn 8% in real estate, 5% in stocks, and 3% in treasury bills. To limit the more risky investments, they decide to invest at least twice as much in treasury bills as in stocks and real estate combined, and at least as much in stocks as in real estate. Uninvested money earns no income. How much should be invested in each, to maximize the anticipated income?

6. A manufacturer produces two kinds of clocks, standard and mantel. The standard clocks require five hours of labor and require 40 linear feet of wood, while the wall clocks need two hours of labor and require eight linear feet of wood. The profit is \$60 on each standard clock and \$11 on each mantel clock. Each week there are 5000 square feet of wood and 600 man-hours available. How many boxes of each kind should be made in order to maximize profit?

7. A carpentry company produces two different kinds of desk for State University. The Student desk uses 10 units of wood, 2 units of glue, and 1.5 hours of manpower. The Professor desk requires 12 units of wood, 2 units of glue, and 2 hours of manpower. The company has 8000 units of wood, 2000 units of glue, and 1800 man-hours available. If \$10 profit is made on each Student desk and \$12 on each Professor desk, how many of each should be made in order to maximize profit?

8. A farmer wishes to plant potatoes, corn, and peppers. Each acre of potatoes costs \$300 for seed; each acre of corn costs \$160; each acre of peppers costs \$280. He has 100 acres available for planting, and \$20000 available to spend. The profit per acre is \$80 for potatoes, \$40 for corn and \$70 for peppers. He can sell at most 30 acres of potatoes, but the markets will take any amount of corn and peppers. How many acres of each crop should he plant to maximize profit?

Exercises 6.1 B

In each case, construct a linear programming model for the problem given.

1. A manufacturer produces two kinds of wooden boxes, large and small. The large ones require nine minutes' labor and require ten square feet of wood, while the small ones need seven minutes of labor and require eight square feet of wood. The profit is \$3 on each large box and \$1 on each small box. Each week there are 100000 square feet of wood available. There are 40 workers, each of whom can work 40 hours per week. How many boxes of each kind should be made in order to maximize profit?

2. Your cat requires at least need 60 units of carbohydrate, 45 units of protein, and 30 units of fat each month. From each pound of feed A she receives five units of carbohydrate, three units of protein and three units of fat, and costs \$2.10. Feed B contains two units of carbohydrate, three units of protein, and one unit of fat per pound, and costs \$1.80 per pound. How many pounds of each feed should you buy per month in order to make the cheapest blend that supplies the required nutrition?

3. A clothing manufacturer has 100 square yards of cotton material, 100 square yards of wool, and 60 square yards of silk. A pair of slacks requires one square yard of cotton, two square yards of wool, and two square yards of silk. A skirt requires two square yards of cotton, one square yard of wool, and two square yards of silk. The net profit on a pair of slacks is \$3, while on a skirt the net profit is \$4. How many skirts and slacks should be made if net profit is to be maximized?

4. A company manufactures rackets for tennis, squash, and racketball. Each tennis racket requires two units of aluminum and one unit of nylon; each squash racket requires one unit of aluminum and two units of nylon; each racketball racket requires two units of aluminum and two units of nylon. The company has 1000 units of aluminum and 800 units of nylon. It cannot manufacture more than 550 rackets in total. The profit on a tennis, squash and racketball racket is \$7, \$8, and \$11, respectively. How many of each type should be made in order to maximize profit?

5. An oil company operates two refineries. Refinery I puts out 200, 100, and 80 barrels per day of low-, medium-, and high-grade oil, respectively. Refinery II puts out 100, 200, and 500 barrels per day of low-, medium-, and high-grade oil, respectively. To fill an order, the company must output 1000 barrels low-grade, 1500 barrels of medium-grade, and 3000 barrels of high-grade oil. If it costs \$200 per day to operate refinery I, and \$300 per day to operate refinery II, how many days should each refinery be operated in order to fill the order at minimum cost?

6. A toy company makes Bush, Rumsfeld, and Cheney masks using hair, plastic, and latex. Each Bush mask requires eight ounces of hair, eight ounces of plastic, and four ounces of latex. Each Rumsfeld mask requires nine ounces of hair, three ounces of plastic, and three ounces of latex. Each Cheney mask requires two ounces of hair, eight ounces of plastic, and five ounces of latex. Each Bush mask yields a profit of \$2.00, each Rumsfeld yields \$2.50, and each Cheney yields \$3.00. There are 600 ounces of hair, 400 ounces of plastic, and 400 ounces of latex available. How many copies of each mask should be made to maximize profit?

7. A company has 750 pounds of cashews and 1200 pounds of peanuts and wishes to market two kinds of 12-ounce packages of mixed nuts. The low-grade mixture contains four ounces of cashews and eight ounces of peanuts; the high-grade mixture contains eight ounces of cashews and four ounces of peanuts. The profit is \$0.25 on each low-grade package and \$0.45 on each high-grade package. How many packages of each kind should be made in order to maximize profit?

8. A mining company operates two mines, one in Green Valley and one at Andrew's Ridge. The Green Valley mine costs \$1500 per day to operate and produces 30 ounces of gold, 200 ounces of silver and 400 pounds of copper each day. Andrew's Ridge costs \$1200 per day to operate and produces 40 ounces of gold, 400 ounces of silver and 300 pounds of copper each day. The company contracts to supply 240 ounces of gold, 2000 ounces of silver and 1200 pounds of copper. How many days should it operate the mines, to fulfill the contract while maximizing profit?

9. Phillips Foundry makes two types of gargoyle. Small Evil gargoyles take three pounds of cast iron and 17 minutes of labor. Large Horrid gargoyles use four pounds of cast iron and need ten minutes of labor. The profit on each Small Evil gargoyle is \$12, while each Large Horrid gargoyle yields \$10 profit. There are 900 pounds of cast iron and 56 hours of labor available each day. To fulfill back orders, at least 100 Small Evil gargoyles must be made each day. How many gargoyles of each kind should be made each day, to maximize profit?

10. A garment factory makes sweaters and vests. Each sweater requires 15 ounces of wool, eight buttons, and 15 hours of work, while each vest requires 10 ounces of wool, six buttons, and 12 hours of work. The profit is \$8 on a sweater and \$6 on a vest. Each week the company has available 5000 ounces of wool and 1000 buttons. It has promised to provide at least 1200 hours of work per week to its employees. How many garments of each kind should be made each week, to maximize profit?

11. A toy manufacturer makes three different kinds of model cars: the Porsche, the Ferrari, and the Jaguar. They are all made of aluminum and steel. Each Porsche requires two units of steel and one unit of aluminum, each Ferrari requires three units of steel and four units of aluminum, and each Jaguar needs four units of steel and two units of aluminum. The company has 8000 units of steel and 12000 units of aluminum available. If \$10 profit is made on each Porsche, \$12 on each Ferrari, and \$5 on each Jaguar, how many of each model car should the manufacturer make in order to maximize its profit?

12. A farmer mixes three types of food, foods A, B, and C, for his livestock. The mixture must provide at least 300 units of nutrient X, 20 units of nutrient Y, and 100 units of nutrient Z. The cost of the foods, and the amounts of each type of nutrient supplied by each food, are shown in the table below. How much of each food should be used in order to provide at least the minimum amount of nutrient needed at the cheapest price?

Food	Cost per pound	Amount of nutrient per pound of food		
		X	*Y*	*Z*
A	\$0.40	100	5	20
B	\$0.20	60	8	10
C	\$0.50	100	12	30

13. A furniture shop makes tables, chairs, and cabinets. Each table requires ten units of wood and six units of paint, takes five hours to make and yields \$100 profit. Each chair requires four units of wood and three units of paint, takes two hours to make and yields \$15 profit. Each cabinet requires eight units of wood and nine units of paint, takes six hours to make and yields \$90 profit. Each table is sold with six chairs, so the number of chairs made cannot be smaller than six times the number of tables. (It could be greater; chairs can be sold separately.) If 1000 units of wood, 500 units of paint and 800 working hours are available per week, how many units of each kind should be produced?

14. A manufacturer produces three types of electrical appliances, models A1, A2, and A3. Each appliance must be processed by the machine shop and by the wiring shop. There are 900 hours of labor available in the machine shop each week, and 2000 in the wiring shop. The specifications of each appliance—number of hours' work required in each shop, and profit per unit—are shown below. There is a contract that requires them to produce at least 100 units of model A2 per week. How many units of each type should be produced each week to maximize the profit?

	A1	A2	A3
Machine shop	2	2	2
Wiring shop	6	4	2
Profit	\$20	\$15	\$12

15. Demiurg, the mythological Greek creator of the universe, is filling the universe with galaxies, black holes, and quasars. Each galaxy costs \$225, each black hole \$250, and each quasar \$200. The number of black holes and quasars combined cannot exceed twice the number of galaxies. There should be at least 50 black holes, and the number of quasars should be at most one fourth the number of galaxies. How many of each should Demiurg order to minimize the cost of constructing the universe?

16. A toy company manufactures pickups, dump trucks, and semi trucks from plastic, latex, and aluminum. A pickup truck requires four ounces of plastic, four ounces of latex, and two ounces of aluminum. A dump truck requires six ounces of plastic, five ounces of latex, and three ounces of aluminum, and a semi requires eight ounces of plastic, six ounces of latex, and three ounces of aluminum. A pickup truck sells for \$10.00, a dump truck for \$14.00, and a semi for \$16.00. There are 800 ounces plastic, 400 ounces latex, and 300 ounces aluminum available. Find the number of each truck to make in order to maximize revenue.

6.2 The Geometric Method

Planar Representation

If an optimization problem involves only two variables, we can call them X and Y and represent them in a planar diagram. The point (x, y) represents the situation where $X = x$ and $Y = y$. We choose the values x, y, so this choice could be called our *strategy*, and we sometimes refer to the point (x, y) as the strategy (x, y). With every point (x, y) of the plane we can associate several pieces of information—the value of the objective function when $X = x$ and $Y = y$, and whether the various constraints are true for these values of X and Y. The point is called *feasible* when all the constraints (including non-negativity) are true, and *infeasible* otherwise.

Sample Problem 6.5. *Consider the linear program*

$$\begin{aligned} \textit{Maximize} \quad & P = x + y \\ \textit{subject to} \quad & x + 2y \le 6, \quad \text{(A)} \\ & 2x + y \le 6, \quad \text{(B)} \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

Which of the points $(1, 1)$*,* $(0, 4)$*,* $(2, 2)$*,* $(1, -1)$*, and* $(3, 3)$ *are feasible? What is the value of* P *at these points?*

Solution. $(1, 1)$ is feasible, $P = 2$. $(0, 4)$ is not feasible ((A) is false), $P = 4$. $(2, 2)$ is feasible, $P = 4$. $(1, -1)$ is not feasible (y is negative), $P = 0$. $(3, 3)$ is not feasible ((A) and (B) are both false), $P = 6$.

Your Turn. Repeat this exercise for the points $(1, 0)$, $(4, 4)$, $(2, 1)$, $(-1, 1)$.

Graphing Inequalities

In two dimensions, the set of all points satisfying a linear constraint is easy to represent geometrically. The set of all points satisfying $ax + by \le c$ is shown as follows. First, draw the line $ax + by = c$, the *boundary line* of the inequality. The required set of points—the *solution set* of the inequality—consists of all points on this line together with all points on one side of the line. For example, the set of all points satisfying the inequality $y \geq 0$ consists of the line $y = 0$ (the x axis) together with all points above the axis. This sort of set of points is called a *half-plane*.

It is necessary to work out which side of the line is in the solution set. The easiest way is to find one point, not on the line, and check whether or not it satisfies the inequality. The easiest point to check is the origin, (0, 0); if the origin lies on the boundary line, some other point must be used.

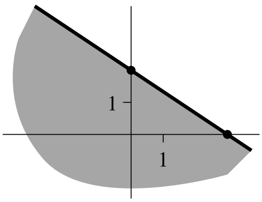
Sample Problem 6.6. *Graph the inequalities*

$$2x + 3y \leq 6, \qquad x - 3y \leq 0.$$

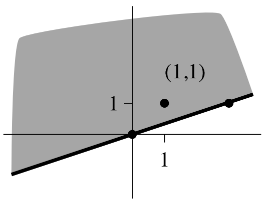
Solution. We draw the lines $2x + 3y = 6$ and $3x - y = 0$ on sets of axes. To draw $2x + 3y = 6$ we join the points (3, 0) and (0, 2) (remember, for a straight line, it is enough to find two points on the line and join them.) $x - 3y = 0$ joins (0, 0) to (3, 1). These four points are shown as dots in the diagrams.

For $2x + 3y \leq 6$, we test with (0, 0), and find that it satisfies the inequality; putting $x = 0$, $y = 0$ results in $0 \leq 6$, which is true. So we take the half-plane containing the origin.

For $x - 3y \leq 0$, (0, 0) is not a suitable test point, so we try (1, 1). This satisfies the inequality. The point (1, 1) is also shown as a dot in the diagram. Both solution-sets are shown in gray.



$2x + 3y \leq 6$



$x - 3y \leq 0$

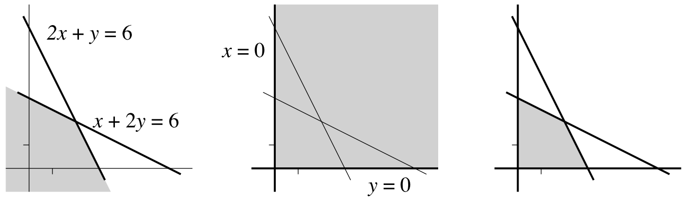
Your Turn. Graph the inequalities

$$2x + 3y \geq 6, \qquad x - y \leq 0.$$

The area of the plane that satisfies all the constraints of a constrained optimization problem is called the *feasible region*. It is the intersection of the solution sets of all the constraints. The points of the region are called *feasible solutions*.

Sample Problem 6.7. *Sketch the feasible region for the linear program in Sample Problem 6.5.*

Solution. The first diagram shows the points satisfying both constraints (A) and (B). The second diagram shows the points satisfying the non-negativity constraints. So the feasible region is the one shaded in the third diagram.

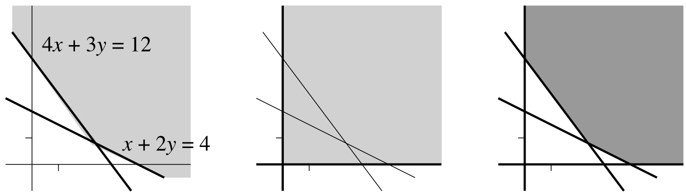


The feasible region we just constructed is called *closed* because it is a closed polygon. Some problems have an *open* feasible region that goes on to infinity in some direction. Here is an example.

Sample Problem 6.8. *Sketch the feasible region for the linear program:*

$$\begin{array}{ll} \textit{Minimize} & C = x + y \\ \textit{subject to} & x + 2y \geq 4, \\ & 4x + 3y \geq 12, \\ & x \geq 0, \quad y \geq 0. \end{array}$$

Solution. The first diagram shows the points satisfying the first two constraints, and the second diagram shows the points satisfying the non-negativity constraints. So the feasible region is the one shaded in the third diagram.

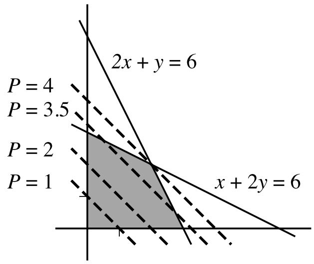


Your Turn. Sketch the feasible regions for the linear programs:

$$\begin{array}{llll} \text{(a)} & \text{Maximize} & P = x + 2y \\ & \text{subject to} & x + 3y \leq 6, \\ & & 4x - 2y \leq 6, \\ & & x \geq 0, \quad y \geq 0; \end{array} \qquad \begin{array}{lll} \text{(b)} & \text{Minimize} & C = 3x + 4y \\ & \text{subject to} & +3y \geq 6, \\ & & 3x + y \geq 6, \\ & & x \geq 0, \quad y \geq 0. \end{array}$$

Level Curves

Let us look again at the linear programming problem in worked Example 6.5. The function to be maximized was $P = x + y$. We saw, for example, that $x = y = 1$ is a feasible point, with $P = 2$. There are of course infinitely many other points that give $P = 2$. But all of them will lie on the same straight line. A straight line for which all the points give the same value of the objective function is called a *level curve* for that objective function, and the constant value is called the *level*.



The diagram shows the level curves for $P = 1$, 2, 3.5, and 4. They are parallel lines, and it is clear that, as you move further away from the origin, the value of P increases. The level curve for level 4 touches the feasible region at only one point, namely $x = y = 2$, and no level curve with $P > 4$ ever touches the feasible region. So the maximum value of P is 4, and the only way to attain it is to set $x = 2$ and $y = 2$.

This method of solving a linear programming problem is the *geometric method*, or *solution by diagram*, and may be used on any two-variable problem. To summarize, we proceed as follows.

1. Sketch the feasible region.
2. Draw some level curves.
3. By looking at the level curves, decide what is the *optimal* level (largest when maximizing, smallest when minimizing) for which the curve still touches the feasible region.

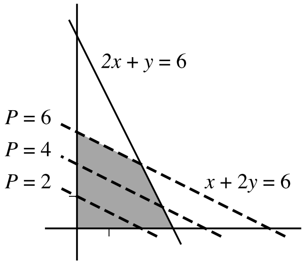
The different level curves can be found by shifting any one level curve so that the new line is parallel to the old one. Usually there is just one point of contact between the best line and the feasible region, and it will be a corner of the region. You can work out the value of x and y by solving the equations of the two lines that meet in the corner point.

Very occasionally there will be an infinitude of "best" points, as in the following example.

Sample Problem 6.9.

$$\begin{array}{ll} \textit{Maximize} & P = x + 2y \\ \textit{subject to} & x + 2y \le 6, \quad \text{(A)} \\ & 2x + y \le 6, \quad \text{(B)} \\ & x \ge 0, \quad y \ge 0. \end{array}$$

Solution. This is the same example as above, with a different objective function. The diagram, with level curves, is

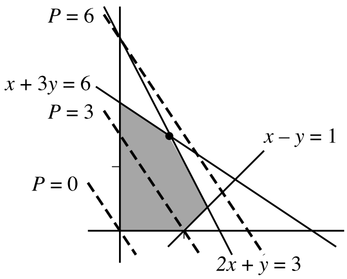


The boundary line $x + 2y = 6$ is one of the level curves. So the optimal value, $P = 6$, can be attained by any point (x, y) on the line segment from $(0, 3)$ to $(2, 2)$.

Sample Problem 6.10. *Solve the linear program:*

$$\begin{aligned} \textit{Maximize} \quad & P = 3x + 2y \\ \textit{subject to} \quad & 2x + y \le 3, \\ & x + 3y \le 6, \\ & x - y \le 1, \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

Solution. The following diagram shows the feasible region and the level curves for levels 0, 3, and 6. The maximum is clearly at the point marked with a dot.



To find the coordinates of the point, it is necessary to solve the simultaneous equations

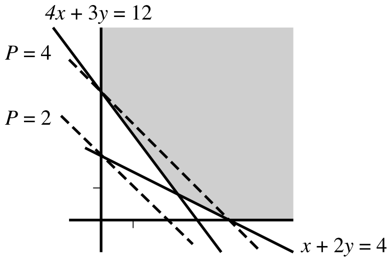
$$\begin{aligned} 2x + y &= 3, \\ x + 3y &= 6; \end{aligned}$$

the solution is $x = \frac{3}{5}$, $y = \frac{9}{5}$. The maximum is therefore $P = 3\frac{3}{5} + 2\frac{9}{5} = \frac{27}{5}$.

Your Turn. Solve the linear program of Your Turn problem 6.8(a).

Sample Problem 6.11. *Solve the linear program in Sample Problem 6.8.*

Solution. The following diagram shows the feasible region and the level curves for levels 2 and 4.



The maximum is clearly at the intersection point of

$$4x + 3y = 12,$$
$$x + 2y = 4;$$

the solution is $x = \frac{12}{5}$, $y = \frac{4}{5}$, $C = \frac{12}{5} + \frac{4}{5} = \frac{16}{5}$.

Your Turn. Solve the linear program of Your Turn problem 6.8(b).

Not every linear program has a solution. For example, suppose the previous Worked Example asked us to *maximize* $x + y$. There is no "last" level curve. The interpretation is that (for example) infinite profit is possible. Problems of this kind are called *unbounded*. If a real-world problem, this usually means there is some error in the data.

Exercises 6.2 A

1. In each case, sketch the region satisfying the inequality:

(i) $x + 4y \le 6$; (ii) $x + 2y \le 6$;

(iii) $2x + 3y \ge 4$; (iv) $4x - 4y \le 5$;

(v) $x - 2y \ge 6$; (vi) $3x + y \ge 12$;

(vii) $-2x + 3y \ge 3$; (viii) $x + 4y \ge 6$;

(ix) $2x + y \ge 4$; (x) $2x + 4y \le 10$;

(xi) $x - 2y \le 2$; (xii) $2x - y \ge -4$;

(xiii) $2x + y \le 4$; (xiv) $x + 2y \ge 4$;

(xv) $x + y \le 6$; (xvi) $4x + 2y \le 4$.

2. In each case, sketch the feasible region for the linear programming problem:

(i) Minimize $C = x + y$
subject to $2x + y \geq 4,$
$x + 2y \geq 4,$
$x \geq 0, \quad y \geq 0;$

(ii) Maximize $P = x - 2y$
subject to $x + 2y \leq 4,$
$4x + 4y \leq 12,$
$x \geq 0, \quad y \geq 0;$

(iii) Maximize $P = x + 3y$
subject to $x + 4y \leq 6,$
$x + 2y \leq 6,$
$x \geq 0, \quad y \geq 0;$

(iv) Minimize $C = 4x + 3y$
subject to $x + y \leq 3,$
$x + 2y \geq 4,$
$x + 4y \leq 6,$
$x \geq 0, \quad y \geq 0;$

(v) Minimize $P = x + y$
subject to $x + 2y \geq 6,$
$2x - y \geq -4,$
$2x + y \geq 4,$
$x \geq 0, \quad y \geq 0;$

(vi) Maximize $P = 3x + 4y$
subject to $x - y \geq -1,$
$x + y \leq 3,$
$x \geq 0, \quad y \geq 0;$

(vii) Minimize $C = 5x + 3y$
subject to $4x + 3y \geq 12,$
$x + 2y \geq 4,$
$x \geq 0, \quad y \geq 0;$

(viii) Maximize $P = 4x + 2y$
subject to $x + 4y \leq 16,$
$2x + 6y \leq 12,$
$x \geq 0, \quad y \geq 0;$

(ix)
$$\begin{aligned} \text{Maximize} \quad & P = 2x + 3y \\ \text{subject to} \quad & 8x + 2y \le 11, \\ & 2x + y \le 5, \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

3. Solve the linear programming problem by the geometric method in parts (i) to (v) of Exercise 2.

4. A company manufactures two types of ballpoint pen, the Deluxe and the Standard. It plans to produce x packs of Deluxe pens and y packs of Standard pens per week. (Clearly, x and y may not be negative. They need not be whole numbers; production can carry over to the next week.) The company has 8000 hours of labor available per week, and can spend at most \$10000 per week on materials.

(i) A pack of Deluxe pens requires \$2 in materials and a pack of Standard pens requires \$1. Write an inequality involving x and y that shows the effect of the \$10000 materials limitation.

(ii) Producing a pack of Deluxe pens requires 1 hour of labor, while a pack of Standard pens requires 2 hours. Write an inequality involving x and y that shows the effect of the 8000 hours restriction.

(iii) The profit from a pack of Deluxe pens is \$3, while the profit from a pack of Standard pens is \$5. Solve a linear programming problem to find out how many packs of each kind should be produced, in order to maximize profit.

Exercises 6.2 B

1. In each case, sketch the region satisfying the inequality:

(i) $x + 2y \le 12$;
(ii) $2x + y \le 4$;
(iii) $x + 3y \le 6$;
(iv) $4x + 4y \le 12$;
(v) $x - y \ge 8$;
(vi) $x + 2y \ge 6$;
(vii) $4x + 3y \le 12$;
(viii) $x - 4y \le 3$;
(ix) $-2x + y \ge 1$;
(x) $3x + 2y \le 6$;
(xi) $2x - 3y \ge 4$;
(xii) $2x + 4y \le 8$;
(xiii) $5x + 2y \le 20$;
(xiv) $x + 2y \ge 4$;
(xv) $5x + 3y \ge 5$;
(xvi) $x + y \le 7$;
(xvii) $2x + 3y \le 10$;
(xviii) $3x - y \ge 12$;
(xix) $4x - y \ge 4$;
(xx) $2x - 3y \ge 3$;
(xxi) $x + y \ge 3$;
(xxii) $x + y \le 2$;

(xxiii) $x + 2y \leq 4$; (xxiv) $x + 3y \geq 3$;

(xxv) $3x + 2y \geq 6$; (xxvi) $x + 2y \geq 3$;

(xxvii) $2x - 4y \leq 6$; (xxviii) $5x + 6y \leq 18$;

(xxix) $8x + 7y \geq 28$; (xxx) $-2x + 4y \leq 1$;

(xxxi) $-2x + 3y \geq 2$; (xxxii) $2x - 4y \leq 1$.

2. In each case, sketch the feasible region for the linear programming problem:

(i)
$$\begin{array}{ll} \text{Maximize} & P = 6x + y \\ \text{subject to} & x + y \leq 7, \\ & 3x + y \geq 12, \\ & x \geq 0, \quad y \geq 0; \end{array}$$

(ii)
$$\begin{array}{ll} \text{Maximize} & P = -4x + 2y \\ \text{subject to} & -2x + 3y \geq 3, \\ & 4x + 3y \leq 12, \\ & x \geq 0, \quad y \geq 0; \end{array}$$

(iii)
$$\begin{array}{ll} \text{Maximize} & P = 4x + y \\ \text{subject to} & -2x + y \geq 1, \\ & 5x + 2y \leq 20, \\ & x \geq 0, \quad y \geq 0; \end{array}$$

(iv)
$$\begin{array}{ll} \text{Minimize} & C = 3x - 2y \\ \text{subject to} & x + 2y \leq 4, \\ & 2x + y \geq 4, \\ & x \geq 0, \quad y \geq 0; \end{array}$$

(v)
$$\begin{array}{ll} \text{Maximize} & P = 3x + 4y \\ \text{subject to} & 2x + 3y \leq 10, \\ & x + 2y \leq 12, \\ & x + y \leq 7, \\ & x \geq 0, \quad y \geq 0; \end{array}$$

(vi)
$$\begin{array}{ll} \text{Minimize} & C = 3x + 6y \\ \text{subject to} & x + 3y \leq 6, \\ & 5x + 3y \geq 5, \\ & 2x + 3y \leq 10, \\ & x \geq 0, \quad y \geq 0; \end{array}$$

(vii) Maximize $P = x + y$
subject to
$$3x + 2y \leq 6,$$
$$2x + 4y \leq 8,$$
$$x - y \geq 8,$$
$$x \geq 0, \quad y \geq 0;$$

(viii) Maximize $P = 3x - 4y$
subject to
$$x + y \leq 2,$$
$$x \geq 0, \quad y \geq 0;$$

(ix) Maximize $P = 4x - 3y$
subject to
$$x - 2y \leq -2,$$
$$y \leq 3,$$
$$x \geq 0, \quad y \geq 0;$$

(x) Maximize $P = 4x + 5y$
subject to
$$x - y \leq 0,$$
$$3x + 7y \leq 21,$$
$$x \geq 0, \quad y \geq 0;$$

(xi) Maximize $P = 7x + 4y$
subject to
$$x + 2y \leq 6,$$
$$3x + 2y \leq 12,$$
$$x \geq 0, \quad y \geq 0;$$

(xii) Minimize $C = x + 2y$
subject to
$$3x + 4y \geq 12,$$
$$2x + y \geq 4,$$
$$x \geq 0, \quad y \geq 0;$$

(xiii) Maximize $P = 6x + y$
subject to
$$3x + y \leq 18,$$
$$x + y \leq 12,$$
$$2x - y \leq 0,$$
$$x \geq 0, \quad y \geq 0;$$

(xiv) Minimize $C = 3x + 4y$
subject to
$$x + 4y \geq 12,$$
$$2x + y \leq 10,$$
$$x - y \geq -1,$$
$$x \geq 0, \quad y \geq 0;$$

(xv)
$$\begin{aligned} \text{Maximize} \quad & P = 6x + 3y \\ \text{subject to} \quad & x + 5y \le 30, \\ & 2x + 2y \le 12, \\ & x \ge 0, \quad y \ge 0; \end{aligned}$$

(xvi)
$$\begin{aligned} \text{Maximize} \quad & P = 6x + 5y \\ \text{subject to} \quad & 2x + y \le 4, \\ & 2x + 3y \le 8, \\ & x \ge 0, \quad y \ge 0; \end{aligned}$$

(xvii)
$$\begin{aligned} \text{Maximize} \quad & P = 3x + 4y \\ \text{subject to} \quad & 2x + y \le 4, \\ & 3x + 2y \le 8, \\ & x \ge 0, \quad y \ge 0; \end{aligned}$$

(xviii)
$$\begin{aligned} \text{Maximize} \quad & P = 2x + 3y \\ \text{subject to} \quad & -x + 2y \le 1, \\ & 2x + y \le 4, \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

3. Solve the linear programming problem by the geometric method in parts (i) to (xiv) of Exercise 2.

4. The law firm of Grisham and Turow handles two types of lawsuits: medical malpractice suits against surgeons and group actions against pharmaceutical companies. Malpractice suits each require six months of law clerks' time for preparation and the hiring of five expert witnesses. Group actions each require ten months of preparation time and three expert witnesses. This year the firm can afford to hire at most 120 expert witnesses, and it has 240 months of clerks' time available.

(i) The profit from a surgical malpractice suit is \$2000000, while the profit from a group action is \$5000000. How many of each type of case should it handle to maximize profit? What profit will be achieved?

(ii) Suppose a law were introduced that limited profit from group actions to \$2500000, while malpractice suites were unchanged. How should the law firm react?

6.3 Linear Programming in Higher Dimensions

Generalizing to Higher Dimensions

In the previous section, we saw that the optimal solution, if one existed, always corresponded to a corner point of the feasible region, at the intersection of two or more boundary lines. The point could be found by solving the corresponding linear equations.

Suppose the problem involves three variables. Although we cannot draw it on a page, we could define a feasible region in a three-dimensional space. For example, in a simple case the region might be a cube. The boundary lines would be replaced by boundary planes, and the corner points would be the intersection of at least three boundary planes. The optimal solution would again occur at a boundary point.

These ideas can be extended to any number of variables, resulting in a multidimensional feasible region. Suppose there are n variables. By a *point* we shall mean an n-dimensional vector of real numbers. If

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \quad (\text{or} \geq b_i)$$

is one of the constraints in a linear programming problem, we define the corresponding *boundary hyperplane* to be any set of all points $(x_1, x_2, \ldots, x_n)$ such that

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i,$$

and the corresponding *half-space* is the set of all points satisfying the constraint.

We could now define a *corner point* to be a point satisfying n or more boundary hyperplanes. The corner point is again determined by solving the corresponding equations, and the optimal value will occur at a corner point.

All of our discussion will also apply to the two-dimensional case; many of our examples and most of the exercises will be taken from that case, for simplicity and so that they can be illustrated with diagrams. One minor complication, not evident in the two-dimensional case, is the fact that a set of n hyperplanes might not determine a unique point. For example, in three dimensions, three planes might meet in a common line. (The equivalent problem in two dimensions is when two constraints give the same line, but we would notice this immediately when setting up the problem.)

Slack Variables

The main problem is to determine which set of equations needs to be solved, in order to find the optimal solution. In two dimensions, this could be solved by sketching a feasible region and drawing level curves. A similar technique could be developed in three dimensions, but a more general method is needed.

To handle this problem, we introduce some new variables, called *slack* and *surplus variables*, one for each regular constraint, and assign these variables values at every point. The variable will equal 0 at points on the boundary of the constraint, be negative when the constraint is not satisfied, and in general provide a measure of how much the point varies from being on the boundary.

To each constraint with sign $\leq$ is added a new variable, called a *slack variable*. Thus the constraint $2x_1 + 2x_2 \leq 4$ becomes

$$2x_1 + 2x_2 + x_3 = 4$$

with the addition of the slack variable x_3. The original constraint will be satisfied whenever $x_3 \geq 0$.

From each constraint with sign $\geq$ is subtracted a new variable, called a *surplus variable*. Thus the constraint $x_1 + 5x_2 \geq 4$ becomes

$$x_1 + 5x_2 - x_4 = 4$$

with the addition of the slack variable x_4. The original constraint will be satisfied whenever $x_4 \geq 0$.

When we write the system, all variables are represented in the normal constraints. These constraints become equations, and the conditions that the new variables be non-negative are added to the non-negativity constraints. In this way, the linear programming problem is transformed into a set of equations which are to be satisfied by non-negative real numbers.

Sample Problem 6.12. *Give the representation of the linear programming problem*

$$\begin{aligned} \textit{Maximize} \quad & P = x_1 + x_2 \\ \textit{subject to} \quad & 2x_1 + 2x_2 \leq 4, \\ & x_1 + 5x_2 \geq 4, \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

with slack and surplus variables.

Solution.

$$\begin{aligned} \text{Maximize} \quad & P = x_1 + x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & 2x_1 + 2x_2 + x_3 + 0x_4 = 4, \\ & x_1 + 5x_2 + 0x_3 - x_4 = 4, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

Notice that every variable is included in every equation and in the objective function, even when the coefficient is 0. This will be useful later, when we recast the problems in matrix form.

Sample Problem 6.13. *Add slack and surplus variables to the following linear programming problem.*

$$\begin{aligned} \textit{Minimize} \quad & C = 3x_1 - 2x_2 \\ \textit{subject to} \quad & x_1 + 2x_2 \le 3, \\ & 3x_1 + x_2 \ge 24, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

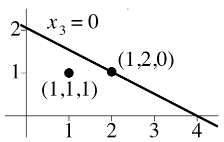
Solution.

$$\begin{aligned} \text{Minimize} \quad & C = 3x_1 - 2x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + 0x_4 = 3, \\ & 3x_1 + x_2 + 0x_3 - x_4 = 2, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0. \end{aligned}$$

Your Turn. Add slack and surplus variables to the following linear programming problem.

$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \le 2, \\ & 2x_1 - x_2 \ge 1, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

Consider the simple case of one constraint involving two variables. For example, consider $x_1 + 2x_2 \le 4$. With the addition of the slack variable x_3, this becomes $x_1 + 2x_2 + x_3 = 4$. The point $(2, 1)$ lies on the boundary line, so it receives x_3 coordinate 0, and its coordinates in the new system are $(2, 1, 0)$. The point $(1, 1)$ lies inside the feasible region, and from the equation we see that $x_3 = 1$, so its coordinates are $(1, 1, 1)$. These can be illustrated in a diagram, with x taking the role of x_1 and y taking the role of x_2. The boundary line is the line $x_1 + 2x_2 = 4$, or $x_3 = 0$.



Standard Form

It will later be important for every equation to have a non-negative right-hand side. After slack and surplus variables have been included, if any equation has negative right-hand side, negate the whole equation.

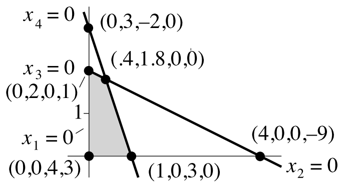
When the slack and surplus variables have been included, and any required negations have been made, we say the problem is *in standard form*.

Sample Problem 6.14. *Put the following linear programming problem in standard form. Sketch the feasible region, and label all the boundary lines and their points of intersection.*

$$\begin{aligned} \textit{Maximize} \quad & P = 3x_1 + 3x_2 \\ \textit{subject to} \quad & x_1 + 2x_2 \le 4, \\ & 3x_1 + x_2 \le 3, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

Solution.

$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 3x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + 0x_4 = 4, \\ & 3x_1 + x_2 + 0x_3 + x_4 = 3, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0. \end{aligned}$$



Your Turn. Put the following linear programming problem in standard form. Sketch the feasible region, and label all the boundary lines and their points of intersection.

$$\begin{aligned} \text{Maximize} \quad & P = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 6, \\ & 2x_1 + x_2 \le 8, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

Basic Feasible Solutions

In Sample Problem 6.14 all the corner points have two zero coordinates, corresponding to the fact that two equations of the form $x_j = 0$ are satisfied. In general, a problem with m regular constraints and n variables, the standard form of the equation will have m equations and $m + n$ variables, and the corner points are those points with n zero coordinates. To remember this, recall that the origin is always a corner point, and it has n variables (the variables of the original problem) zero.

The solutions corresponding to corner points are called *basic*, and often the corner points themselves are called basic solutions. Other solutions and points are non-basic. Among the basic solutions, those that satisfy the constraints by having no negative coordinates are called *feasible*, and the others are infeasible. The basic principle for solving linear programming problems is this.

Theorem 23. *The optimal solution to a linear programming problem is a basic feasible solution.*

To find all basic feasible solutions, you could list all sets of n variables. For each set, a basic solution is obtained by setting those variables to zero, and solving the equations for the other variables. The n variables in the set are called *non-basic* for the particular solution, because they make zero contribution, while the m variables for which you solve are *basic variable*.

In small problems, the number of basic feasible solutions is small, and they can all be examined. Of course, it is still necessary to determine whether the problem is unbounded.

Sample Problem 6.15. *In the linear programming problem of Sample Problem 6.14, list all basic solutions. State which are feasible, and solve the problem.*

Solution.

Non-basic variables	Basic variables	Basic solution	Feasible?	P
x_1, x_2	x_3, x_4	$(0, 0, 4, 3)$	yes	0
x_1, x_3	x_2, x_4	$(0, 2, 0, 1)$	yes	6
x_1, x_4	x_2, x_3	$(0, 3, -2, 0)$	no	9
x_2, x_3	x_1, x_4	$(4, 0, 0, -9)$	no	12
x_2, x_4	x_1, x_3	$(1, 0, 3, 0)$	yes	3
x_3, x_4	x_1, x_2	$(0.4, 1.8, 0, 0)$	yes	6.6

The problem is bounded (from the diagram in Sample Problem 6.14), so the optimum value is $P = 6.6$, attained at $x_1 = 0.4$, $x_2 = 1.8$.

Your Turn. Perform the same calculations for Your Turn problem 6.14.

Exercises 6.3 A

1. In each case, put the linear programming problem into standard form:

(i)
$$\begin{aligned} \text{Maximize} \quad & P = 4x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 5, \\ & 2x_1 + 6x_2 \le 12, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(ii) Minimize $C = x_1 + x_2$
subject to $2x_1 + x_2 \geq 4,$
$x_1 + 2x_2 \geq 4,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iii) Maximize $P = x - 2x_2$
subject to $2x_1 + x_2 \leq 4,$
$4x_1 + 4x_2 \leq 12,$
$x \geq 0, \quad x_2 \geq 0;$

(iv) Maximize $P = 2x_1 + 3x_2$
subject to $8x_1 + 2x_2 \leq 11,$
$2x_1 + x_2 \leq 5,$
$x_1 \geq 0, \quad x_2 \geq 0.$

2. For Exercise 1, parts (i) and (ii), sketch the feasible region and label all boundary lines and corner points, using all slack or surplus variables.

3. For the following linear programming problems, list all basic solutions, say whether they are feasible, and solve the problem.

(i) Exercise 1(i);

(ii) Exercise 1(ii);

(iii) Maximize $P = 3x_1 + 4x_2$
subject to $2x_1 + x_2 \leq 4,$
$3x_1 + 2x_2 \leq 8,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iv) Maximize $P = 2x_1 + 3x_2$
subject to $-x_1 + 2x_2 \leq 2,$
$2x_1 + x_2 \leq 4,$
$x_1 \geq 0, \quad x_2 \geq 0.$

Exercises 6.3 B

1. In each case, put the linear programming problem into standard form.

(i) Maximize $P = 3x_1 + 3x_2$
subject to $2x_1 + 3x_2 \leq 6,$
$4x_1 + 2x_2 \leq 8,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(ii) Maximize $P = x_1 + 4x_2$
subject to
$$-2x_1 + 3x_2 \geq -6,$$
$$3x_1 + 4x_2 \leq 12,$$
$$x \geq 0, \quad x_2 \geq 0;$$

(iii) Maximize $P = 5x_1 + 7x_2$
subject to
$$2x_1 + x_2 \leq 4,$$
$$x_1 + x_2 \geq 3,$$
$$x_1 \geq 0, \quad x_2 \geq 0;$$

(iv) Minimize $C = 3x_1 + 4x_2$
subject to
$$2x_1 + x_2 \leq 6,$$
$$x_1 + 3x_2 \geq 3,$$
$$x_1 \geq 0, \quad x_2 \geq 0;$$

(v) Maximize $P = 4x_1 + x_2$
subject to
$$-2x_1 + x_2 \geq 1,$$
$$5x_1 + 2x_2 \leq 20,$$
$$x_1 \geq 0, \quad x_2 \geq 0;$$

(vi) Maximize $P = 3x_1 + 4x_2$
subject to
$$2x_1 + 3x_2 \leq 10,$$
$$x_1 + 2x_2 \leq 12,$$
$$x_1 + x_2 \leq 7,$$
$$x_1 \geq 0, \quad x_2 \geq 0.$$

2. For each part of Exercise 1, sketch the feasible region and label all boundary lines and corner points, using all slack or surplus variables.

3. For each part of Exercise 1, list all basic solutions, say whether they are feasible, and solve the problem.

4. In each case, put the linear programming problem into standard form.

(i) Minimize $C = 3x_1 - 2x_2$
subject to
$$x_1 + 2x_2 \leq 4,$$
$$2x_1 + x_2 \geq 4,$$
$$x_1 \geq 0, \quad x_2 \geq 0;$$

(ii) Minimize $C = 3x_1 + 6x_2$
subject to
$$x_1 + 3x_2 \leq 6,$$
$$5x_1 + 3x_2 \geq 5,$$
$$2x_1 + 3x_2 \leq 10,$$
$$x_1 \geq 0, \quad x_2 \geq 0;$$

(iii) Maximize $P = x_1 + x_2$
subject to
$$\begin{aligned} 3x_1 + 2x_2 &\le 6, \\ 2x_1 + 4x_2 &\le 8, \\ x_1 - x_2 &\ge 8, \\ x_1 \ge 0, \quad x_2 &\ge 0; \end{aligned}$$

(iv) Maximize $P = 6x_1 + 3x_2$
subject to
$$\begin{aligned} x_1 + 5x_2 &\le 30, \\ 2x_1 + 2x_2 &\le 12, \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

5. For Exercise 1, parts (i) and (ii), sketch the feasible region and label all boundary lines and corner points, using all slack or surplus variables.

6. For the following linear programming problems, list all basic solutions, say whether they are feasible, and solve the problem.

(i) Exercise 4(i);

(ii) Exercise 4(ii);

(iii) Maximize $P = 8x_1 + 5x_2 + 6x_3$
subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 &\le 27, \\ -x_1 + x_3 &\le 0, \\ -2x_1 + x_2 - x_3 &\le 0, \\ x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 &\ge 0. \end{aligned}$$

6.4 Pivoting in Linear Programming

Exhibiting a Solution

When a linear programming problem is in standard form, there corresponds to each equation exactly one variable with coefficient 1 in that equation and 0 in the others, namely the slack or surplus variable for the corresponding constraint. For example, in Sample Problem 6.14, the equations were

$$\begin{aligned} x_1 + 2x_2 + x_3 + 0x_4 &= 4, \\ 3x_1 + x_2 + 0x_3 + x_4 &= 3, \end{aligned}$$

and x_3 corresponds to the first equation while x_4 corresponds to the second. If all other variables are set to 0, we can read off the values $x_3 = 4$ and $x_4 = 3$. We say that the equations *exhibit the basic solution* (0, 0, 4, 3).

The system contains two equations in four unknowns, so its solution set is infinite. It is possible to find another system with exactly the same solution set, which exhibits a different basic solution. In fact, there will be one such formulation for each basic solution.

Sample Problem 6.16. *For Sample Problem 6.14, find a formulation that exhibits the basic solution* (0, 2, 0, 1).

Solution. We want to find a form in which x_2 has coefficient 1 in one equation and 0, in the other, while x_4 plays the same role with the equations reversed. This could be achieved by adding some multiple of the first equation to the second. (If you add a multiple of the second equation to the first, you change the coefficient of x_4.) Adding $-\frac{1}{2}$ times equation 1 to equation 2, the resulting equations are

$$x_1 + 2x_2 + x_3 + 0x_4 = 4,$$
$$\frac{5}{2}x_1 + 0x_2 - \frac{1}{2}x_3 + x_4 = 1.$$

If we divide the first equation by 2, the resulting set

$$\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 + 0x_4 = 2,$$
$$\frac{5}{2}x_1 + 0x_2 - \frac{1}{2}x_3 + x_4 = 1$$

has the required form.

Your Turn. In the same problem, exhibit the basic solution (0, 3, −2, 0).

When a system exhibits a basic solution, the variables corresponding to the equations will be the basic variables in the solution. Each equation must exhibit one basic variable for the system to exhibit a basic solution. The set of basic variables is called the *basis*. As you might expect, variables outside the basis are called *non-basic* for the formulation.

Sample Problem 6.17. *Do the following systems exhibit basic solutions?*

(i)
$$x_1 + x_2 + x_3 = 4,$$
$$x_1 + 2x_2 + 0x_3 = 3;$$

(ii)
$$x_1 + 2x_2 + 0x_3 + 2x_4 = 3,$$
$$0x_1 + 2x_2 + x_3 + 2x_4 = 2.$$

Solution. The first system does not exhibit a basic solution because the second equation has no basic variable. The second exhibits the basic solution (3, 0, 2, 0) which is in fact a basic feasible solution.

Sometimes the same formulation can exhibit a basic solution in more than one way. For example, in

$$x_1 + x_2 + x_3 + 0x_4 = 2,$$
$$0x_1 + 0x_2 + 2x_3 - x_4 = 1,$$

one can consider either x_1 or x_2 to be the basic variable in the first equation.

Pivoting

Refer back to Section 5.4. We referred to *dependent* and *independent* variables. The augmented matrix of a system of equations exhibits the solution is such a way that each equation shows how to express one dependent variable in terms of the independent variables. This is exactly what is happening when a linear programming problem is put in standard form. The basic variables are the dependent variables, and the non-basic variables are the independent variables. The basic solution associated with the representation is found by setting all the independent variables to zero. So *pivoting* is the process of moving from one formulation to another in which the new basis is obtained by deleting exactly one basis element (and introducing exactly one new one). This is what was done in Sample Problem 6.16. In this section, we shall examine pivoting techniques in which both the basic solution exhibited originally and the new one are feasible.

To understand the pivoting process in terms of linear programming problems, we consider again the example of Sample Problem 6.16. The diagram is shown below in Figure 6.1. The process of moving from basic feasible solution (0, 0, 4, 3) to basic feasible solution (0, 2, 0, 1) corresponds to moving along the line $x_1 = 0$ from (0, 0, 4, 3) to (0, 2, 0, 1). As this happens, the value of the non-basic variable x_2 increases and the value of the basic variable x_3 decreases until x_3 becomes non-basic (reaches zero). x_2 is then basic.

If we were to cross the line $x_3 = 0$, we would reach an infeasible region, where x_3 is negative. We can only move along $x_1 = 0$ until the first time we reach another basic point.

The process can be summarized as follows.

1. Select a non-basic variable to become basic.
2. Increase that variable while keeping all other non-basic variables zero.
3. Proceed until you reach a basic solution, that is, until you first meet another boundary.
4. Rewrite the equations so that they exhibit this new basic feasible solution.

In the example, this worked as follows.

1. We selected the non-basic variable x_2 to become basic.

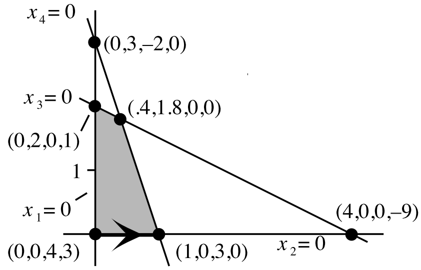


Fig. 6.1. Diagram for Sample Problem 6.16

2. In the example, the only other non-basic variable was x_1, so we proceeded along the line $x_1 = 0$. If there were another non-basic variable, x_5 say, the problem would be three-dimensional, and each boundary would be a plane. We would need to keep both x_1 and x_5 zero, so we would proceed along the line of intersection of the two planes $x_1 = 0$ and $x_5 = 0$. In general, we always need to proceed along the line of intersection of the relevant boundaries.
3. We reached the basic solution (0, 2, 0, 1) on the boundary $x_3 = 0$.
4. The rewritten equations were

$$\frac{1}{2}x_1 + x_2 + \frac{1}{2}x_3 + 0x_4 = 2,$$
$$\frac{5}{2}x_1 + 0x_2 - \frac{1}{2}x_3 + x_4 = 1.$$

Matrix Form

By the *constraint matrix* of a linear programming problem we mean the augmented matrix for the system of constraint equations. It is convenient to label each row with the basic variable exhibited by the corresponding equation, and the columns with the variable names (we use b for the right-hand side). The matrix for Sample Problem 6.16 is

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_3 & 1 & 2 & 1 & 0 & 4 \\ x_4 & 3 & 1 & 0 & 1 & 3 \end{array}\right].$$

With this labeling, we shall write a_{ij} for the entry in row x_i and column x_j, and b_i for the entry in row x_i and column b.

We shall assume that all entries in the b column are positive. If there were a negative entry, the basic solution being exhibited would not be feasible. In this case, we say that pivoting is impossible. (We could pivot the equations, but not for the purpose of solving the linear programming problem.)

Choosing the Pivot Element

Suppose we plan to pivot in such a way that basic variable x_i becomes non-basic and non-basic variable x_j becomes basic. Then a_{ij} is called the *pivot element*, and row x_i and column x_j are the *pivot row* and *pivot column*, respectively, just as before. The pivoting process involves adding a suitable multiple of the pivot row to every other row so that the other entries in the pivot column become 0, and then dividing the pivot row by the pivot entry.

In linear programming, we must choose a pivot element *such that the new basic solution will be feasible.*

The first requirement is that the pivot element be positive. If it is zero, we cannot use that row to eliminate the other entries in the pivot column; if it is negative, when we divide the pivot row by the pivot entry, the new right-hand side will be negative, and the new basic variable will be infeasible.

Now suppose column x_j has been chosen as the pivot column. It may be any column with at least one positive entry. To choose the pivot row, we calculate the ratio $\frac{b_i}{a_{ij}}$ for each row x_i such that a_{ij} is positive, and select the smallest value. This determines the pivot row. If any other row were chosen, pivoting would result in an infeasible solution (in the row that should have been used as pivot row, the right-hand side will be negative).

In the event of a tie, either candidate may be used. (You will find that in the resulting basic solution, one of the basic variables will equal zero.)

Sample Problem 6.18. *In each case, the pivot column is column x_2. Determine the pivot element.*

(i)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 2 & 0 & 5 & 5 \\ x_3 & 0 & 3 & 1 & 6 & 6 \end{array}\right];$$

(ii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & -2 & 0 & 5 & 1 \\ x_3 & 0 & 3 & 1 & 6 & 6 \end{array}\right];$$

(iii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 2 & 0 & 5 & -1 \\ x_3 & 0 & 3 & 1 & 6 & 6 \end{array}\right].$$

Solution.

(i) a_{32} is used because the ratio $\frac{6}{3} = 2$ is smaller than $\frac{5}{2}$.

(ii) a_{32} must be used because the ratio $\frac{b_1}{a_{12}}$ is negative.

(iii) No pivot is possible because the system does not exhibit a basic feasible solution—b_1 is negative.

Your Turn. In each case, the pivot column is column x_2. Determine the pivot element.

(i)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 2 & 0 & 4 & 4 \\ x_3 & 0 & 1 & 1 & 4 & 6 \end{array}\right];$$

(ii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 2 & 0 & 5 & 4 \\ x_3 & 0 & -1 & 1 & 4 & 6 \end{array}\right].$$

When pivoting, we shall underline the pivot element. After pivoting, it is necessary to update the BV column, to reflect the change in basic variables: if the pivot element is a_{ij}, then the pivot row previously exhibited basic variable x_i, but in the new matrix it represents basic variable x_j.

Sample Problem 6.19. *Carry out the pivot in Sample Problem* 6.18(i).

Solution. Originally the basic feasible solution (5, 0, 6, 0) was exhibited.

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 2 & 0 & 5 & 5 \\ x_3 & 0 & \underline{3} & 1 & 6 & 6 \end{array}\right]$$

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 0 & -\frac{2}{3} & 1 & 1 \\ x_2 & 0 & 1 & \frac{1}{3} & 2 & 2 \end{array}\right] \quad \begin{array}{l} R1 \to R1 - \frac{2}{3}R2, \\ R2 \to \frac{1}{3}R2. \end{array}$$

The basic feasible solution (1, 2, 0, 0) is now exhibited.

Your Turn. Carry out the pivot in Your Turn problem 6.18(i) .

Exercises 6.4 A

1. In each case, write the constraints in standard form, adding slack and surplus variables. Do the constraints exhibit a basic feasible solution? If so, write down that solution.

(i)
$$x_1 + 2x_2 \le 3,$$
$$2x_1 + 3x_2 \le 4;$$

(ii)
$$2x_1 + 2x_2 \le 3,$$
$$x_1 + 3x_2 \ge 5;$$

(iii)
$$2x_1 + x_2 \le 5,$$
$$x_1 + 3x_2 \le 8;$$

(iv)
$$3x_1 + 2x_2 \le 3,$$
$$x_1 - 2x_2 \ge 5;$$

(v)
$$3x_1 + 2x_2 \le 6,$$
$$x_1 + 3x_2 \ge 4;$$

(vi)
$$x_1 - x_2 + 2x_3 \le 1,$$
$$2x_1 + x_2 + x_3 \ge 3;$$

(vii)
$$x_1 + x_2 + 2x_3 \le 1,$$
$$3x_1 - 3x_2 + 2x_3 \le 1,$$
$$-x_1 + 2x_2 + 3x_3 \le 3;$$

(viii)
$$3x_1 + 2x_2 \le 3,$$
$$x_1 + 5x_2 \le 2,$$
$$2x_1 - 2x_2 \le 1.$$

2. For each problem in Exercise 1 where the constraints exhibit a basic feasible solution, write down the constraint matrix.

3. In the following problems, you wish to pivot on column x_3. Is there a pivot row available? If so, which row is it? If not, why not?

(i)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 0 & 4 & 1 & 2 \\ x_2 & 0 & 1 & 1 & 4 & 1 \end{array}\right];$$

(ii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_4 & 0 & 4 & 2 & 1 & 2 \\ x_1 & 1 & 1 & 1 & 0 & 2 \end{array}\right];$$

(iii)
$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ x_1 & 1 & 2 & 0 & 0 & 4 & 2 \\ x_4 & 0 & 3 & -2 & 1 & 2 & 6 \end{array}\right];$$

(iv) $$\left[\begin{array}{c|cccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & 1 & -1 & 0 & 0 & 4 & 2 \\ x_5 & 0 & 2 & 1 & 0 & 1 & 1 & 2 \\ x_4 & 0 & -2 & 3 & 1 & 0 & 2 & 6 \end{array}\right].$$

4. For those problems in Exercise 3 where a pivot is available, perform the pivot. Write down the basic feasible solution exhibited originally and the new basic feasible solution.

5. In the following problems, pivot on the indicated element:

(i) $$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_3 & 0 & \underline{1} & 1 & 1 & 1 \\ x_1 & 1 & 2 & 0 & 3 & 4 \end{array}\right];$$

(ii) $$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_4 & 2 & 0 & \frac{2}{3} & 1 & 4 \\ x_2 & 1 & 1 & \underline{1} & 0 & 3 \end{array}\right];$$

(iii) $$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_4 & 0 & \underline{1} & 1 & 1 & 2 \\ x_1 & 1 & 2 & 1 & 0 & 4 \end{array}\right];$$

(iv) $$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_4 & \underline{2} & 0 & 6 & 1 & 2 \\ x_2 & 2 & 1 & 4 & 0 & 3 \end{array}\right];$$

(v) $$\left[\begin{array}{c|cccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_2 & 0 & 1 & 4 & 0 & 1 & 4 & 3 \\ x_4 & 0 & 0 & -1 & 1 & 2 & -1 & 1 \\ x_1 & 1 & 0 & \underline{2} & 0 & 1 & 1 & 1 \end{array}\right];$$

(vi) $$\left[\begin{array}{c|cccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 \\ x_4 & 0 & 0 & 0 & 1 & 3 & 4 & 3 \\ x_2 & 0 & 1 & \underline{1} & 0 & 2 & -1 & 1 \end{array}\right].$$

6. In the constraint system

$$x_1 + \frac{1}{2}x_2 + 0x_3 - \frac{1}{2}x_4 = 2,$$
$$0x_1 + \frac{1}{2}x_2 + x_3 + \frac{1}{2}x_4 = 1,$$

the basic variables are x_1 and x_3, while x_2 and x_4 are non-basic.

(i) What are the values of x_1 and x_3? What is the basic feasible solution exhibited?

(ii) What is the maximum amount by which x_2 can be increased before x_1 becomes zero?

(iii) What is the maximum amount by which x_2 can be increased before x_3 becomes zero?

(iv) What is the maximum amount by which x_2 can be increased if the point under consideration is to remain feasible?

(v) Suppose we stop at a basic feasible solution for which x_2 is basic. What are the coordinates of the point?

Exercises 6.4 B

1. In each case, write the constraints in standard form, adding slack and surplus variables. Do the constraints exhibit a basic feasible solution?

(i)
$$7x_1 + 2x_2 \le 5,$$
$$5x_1 - x_2 \le 1;$$

(ii)
$$3x_1 + 2x_2 \le 6,$$
$$4x_1 + 3x_2 \ge 1;$$

(iii)
$$2x_1 + x_2 \ge 3,$$
$$3x_1 - x_2 \le 1;$$

(iv)
$$5x_1 + 7x_2 + 2x_3 \le 8,$$
$$3x_1 + 3x_2 - x_3 \le 4;$$

(v)
$$4x_1 + 2x_2 \le 1,$$
$$2x_1 + 4x_2 \le 5;$$

(vi)
$$3x_1 + 4x_2 + 3x_3 \le 5,$$
$$2x_1 + 3x_2 + x_3 \ge 1;$$

(vii)
$$2x_1 + 2x_2 \le 6,$$
$$x_1 + 3x_2 \le 4;$$

(viii)
$$\begin{aligned} x_1 - 3x_2 - 2x_3 &\geq 1, \\ 3x_1 + 2x_2 + 3x_3 &\leq 3; \end{aligned}$$

(ix)
$$\begin{aligned} 3x_1 - 5x_2 &\leq 1, \\ x_1 + 2x_2 &\leq 2; \end{aligned}$$

(x)
$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\geq 1, \\ 2x_1 + 3x_2 + x_3 &\geq 3; \end{aligned}$$

(xi)
$$\begin{aligned} 2x_1 - x_2 + x_3 &\leq 1, \\ 3x_1 - 2x_2 + 2x_3 &\leq 1, \\ -x_1 + x_2 + 2x_3 &\leq 3; \end{aligned}$$

(xii)
$$\begin{aligned} x_1 - 3x_2 &\leq 3, \\ x_1 + 2x_2 &\leq 2, \\ 2x_1 - 3x_2 &\leq 1; \end{aligned}$$

(xiii)
$$\begin{aligned} 4x_1 + x_2 &\leq 7, \\ 3x_1 + 3x_2 &\leq 6, \\ 2x_1 - x_2 &\geq 2; \end{aligned}$$

(xiv)
$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 3, \\ 2x_1 + 3x_2 + x_3 &\leq 3, \\ 3x_1 + x_2 - x_3 &\leq 2. \end{aligned}$$

2. For each problem in Exercise 1 where the constraints exhibit a basic feasible solution, write down the constraint matrix.

3. In the following problems, you wish to pivot on column x_2. Is there a pivot row available? If so, which row is it? If not, why not?

(i)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 2 & 0 & 5 & 2 \\ x_3 & 0 & 3 & 1 & 3 & 4 \end{array}\right];$$

(ii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 0 & 0 & 3 & 1 \\ x_3 & 0 & -1 & 1 & 2 & 1 \end{array}\right];$$

(iii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_4 & 1 & 1 & 0 & 1 & 3 \\ x_3 & 3 & 2 & 1 & 0 & 2 \end{array}\right];$$

(iv)
$$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 0 & 4 & 0 & 7 \\ x_4 & 0 & 0 & 1 & 1 & 2 \end{array}\right];$$

(v)
$$\left[\begin{array}{c|ccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ x_1 & 1 & 2 & 0 & 2 & 0 & 4 \\ x_5 & 0 & 4 & 0 & 3 & 1 & 4 \\ x_3 & 0 & 1 & 1 & 8 & 0 & 2 \end{array}\right];$$

(vi)
$$\left[\begin{array}{c|ccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ x_1 & 1 & -2 & 0 & 2 & 0 & 2 \\ x_5 & 0 & 0 & 0 & 3 & 1 & 2 \\ x_3 & 0 & 0 & 1 & 2 & 0 & 4 \end{array}\right];$$

(vii)
$$\left[\begin{array}{c|cccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & -1 & 0 & 2 & 0 & 1 & 4 \\ x_5 & 0 & 1 & 0 & 2 & 1 & 2 & 1 \\ x_3 & 0 & 1 & 1 & -3 & 0 & 1 & 2 \end{array}\right];$$

(viii)
$$\left[\begin{array}{c|cccccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_3 & 1 & 1 & 1 & 2 & 0 & 0 & 2 \\ x_6 & 2 & 3 & 0 & 2 & 0 & 1 & 6 \\ x_5 & 3 & 2 & 0 & -4 & 1 & 0 & 2 \end{array}\right].$$

4. For those problems in Exercise 3 where a pivot is available, perform the pivot. Write down the basic feasible solution exhibited originally and the new basic feasible solution.

5. In the following problems, you wish to pivot on column x_4. Is there a pivot row available? If so, which row is it? If not, why not?

(i)
$$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 0 & 2 & 1 & 2 \\ x_2 & 0 & 1 & 1 & 4 & 5 \end{array}\right];$$

(ii)
$$\left[\begin{array}{c|cccc|c} \mathrm{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_3 & 0 & 4 & 1 & -1 & 2 \\ x_1 & 1 & 1 & 0 & 0 & 2 \end{array}\right];$$

(iii)
$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ x_1 & 1 & 2 & 0 & 2 & 4 & 2 \\ x_3 & 0 & 3 & 1 & 1 & 2 & 6 \end{array}\right];$$

(iv)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & 1 & -1 & 0 & 0 & 0 & 2 \\ x_5 & 0 & 2 & 1 & 0 & 1 & 0 & 2 \\ x_6 & 0 & -2 & 3 & 0 & 0 & 1 & 3 \end{array}\right];$$

(v)
$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ x_1 & 1 & 2 & 0 & 2 & 0 & 2 \\ x_5 & 0 & 3 & -2 & 2 & 1 & 6 \end{array}\right];$$

(vi)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & 0 & -1 & 1 & 0 & 0 & 2 \\ x_5 & 0 & 1 & 1 & 2 & 1 & 0 & 2 \\ x_6 & 0 & 0 & 3 & 1 & 0 & 1 & 3 \end{array}\right].$$

6. For those problems in Exercise 5 where a pivot is available, perform the pivot. Write down the basic feasible solution exhibited originally and the new basic feasible solution.

7. In the following problems, pivot on the indicated element:

(i)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_3 & 0 & \underline{1} & 2 & 2 & 1 \\ x_1 & 1 & 2 & 0 & 4 & 6 \end{array}\right];$$

(ii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_4 & 1 & 0 & 4 & 1 & 9 \\ x_2 & 1 & 1 & \underline{1} & 0 & 2 \end{array}\right];$$

(iii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_3 & 0 & \underline{3} & 1 & 4 & 2 \\ x_1 & 1 & 1 & 0 & 3 & 2 \end{array}\right];$$

(iv)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_1 & 1 & 0 & 2 & 1 & 4 \\ x_2 & 0 & 1 & \underline{1} & 2 & 1 \end{array}\right];$$

(v)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_2 & 0 & 1 & 2 & 0 & 2 & 4 & 4 \\ x_4 & 0 & 0 & -1 & 1 & 3 & -1 & 2 \\ x_1 & 1 & 0 & \underline{1} & 0 & 1 & 1 & 1 \end{array}\right];$$

(vi)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & 0 & 2 & 0 & 2 & 2 & 2 \\ x_4 & 0 & 0 & 0 & 1 & 2 & 3 & 3 \\ x_2 & 0 & 1 & \underline{1} & 0 & 4 & 2 & 1 \end{array}\right];$$

(vii)
$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ x_3 & \underline{4} & 0 & 1 & 4 & 6 \\ x_2 & 3 & 1 & 0 & 3 & 8 \end{array}\right];$$

(viii)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_2 & 4 & 1 & 1 & 0 & 1 & 0 & 3 \\ x_4 & 2 & 0 & \underline{1} & 1 & 3 & 0 & 2 \\ x_6 & 1 & 0 & 1 & 0 & 4 & 1 & 4 \end{array}\right];$$

(ix)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_2 & 0 & 1 & 1 & 0 & 0 & 4 & 2 \\ x_1 & 1 & 0 & -1 & 1 & 3 & -1 & 1 \\ x_5 & 0 & 0 & \underline{1} & 0 & 1 & 2 & 1 \end{array}\right];$$

(x)
$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ x_1 & 1 & 3 & 1 & 0 & 1 & 0 & 2 \\ x_4 & 0 & 2 & 2 & 1 & 2 & 0 & 4 \\ x_6 & 0 & 1 & \underline{1} & 0 & 2 & 1 & 1 \end{array}\right].$$

8. In the constraint system

$$\begin{aligned} x_1 + x_2 + x_3 + 0x_4 &= 3, \\ 0x_1 + 2x_2 + x_3 + x_4 &= 4, \end{aligned}$$

the basic variables are x_1 and x_4, while x_2 and x_3 are non-basic.

(i) What are the values of x_1 and x_4? What is the basic feasible solution exhibited?

(ii) What is the maximum amount by which x_2 can be increased before x_1 becomes zero?

(iii) What is the maximum amount by which x_2 can be increased before x_4 becomes zero?

(iv) What is the maximum amount by which x_2 can be increased if the point under consideration is to remain feasible?

(v) Suppose we stop at a basic feasible solution for which x_2 is basic. What are the coordinates of the point?

6.5 The Simplex Method

Real World Problems

So far we have seen that a linear programming problem can be solved by listing all the basic solutions, testing them for feasibility, and finding the basic feasible solution for which the objective function has the best value. This method is fine for small problems, but impractical in the real world. In a problem with three variables and two constraints, there are at most 10 basic solutions. However, with 10 variables and 10 constraints, the number could be as high as 184756.

In general, if there are m constraints and n variables in the original problem, there will be $m + n$ variables after slacks and surpluses are taken into account. Each selection of n variables leads to a basic solution. If there are no duplicates, there will be $\binom{m+n}{n}$ basic solutions. A typical scheduling problem could easily have 50 original variables and 50 constraints, so there would be about a million million million million million basic solutions.

In the 1940s, George Dantzig developed a technique, called the *simplex method*, for solving linear programs. His algorithm involves pivoting from one basic feasible solution to another, in such a way that the new solution gives a better value of the objective function. He proved that, if his method cannot yield an improvement, then the best possible value of the objective function has been achieved. We shall not discuss the proof here, but it can be found in advanced books on linear programming.

In practice, the simplex method is nearly always very fast. It is possible to concoct problems where it will examine a great many of the basic feasible solutions, but in most cases only a small number will be treated. Moreover, once a basic feasible solution is found, no infeasible solutions need ever be considered.

In order to implement the simplex method, we need to do three things. First, we must find a convenient way to keep track of the value of the objective function. This is done by adding a row to the constraint matrix. The new row is called the *objective row*, and the enhanced matrix is called a *tableau*. A new variable is also added, to represent the value of the objective function. Second, we need a way to choose a pivot that will improve the objective function. And finally, we need to find a basic feasible solution for a starting-point. In this section, we discuss the first two points, and restrict our attention to maximization problems where the basic solution found by putting all the original variables to zero ("the origin") is feasible. We discuss more general problems in the later sections.

The Initial Tableau

Suppose the objective function is $P = k_1x_1 + k_2x_2 + \cdots + k_nx_n$. We add slack and surplus variables and move all variables to the left-hand side of the equation, obtaining

$$-k_1x_1 - k_2x_2 - \cdots - k_nx_n + 0x_{n+1} + \cdots + 0x_{m+n} + P = 0.$$

This line is added to the bottom of the constraint matrix. A new column is added for P, before the b column, with every entry 0 except the last.

Remember: the elements in the objective row are the *negatives* of the coefficients in the expression for P.

Sample Problem 6.20. *Construct the initial tableau for the problem with standard representation:*

$$\begin{aligned} &\textit{Maximize} \quad P = 3x_1 + 4x_2 + 0x_3 + 0x_4 \\ &\textit{subject to} \quad 2x_1 + 2x_2 + x_3 + 0x_4 = 6, \\ &\qquad\qquad x_1 + 3x_2 + 0x_3 + x_4 = 6, \\ &\qquad\qquad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

Solution.

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_3 & 2 & 2 & 1 & 0 & 0 & 6 \\ x_4 & 1 & 3 & 0 & 1 & 0 & 6 \\ \hline & -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

Your Turn. Construct the initial tableau for the problem with standard representation

$$\begin{aligned} &\text{Maximize} \quad P = 2x_1 + 3x_2 + 0x_3 + 0x_4 \\ &\text{subject to} \quad 4x_1 + 2x_2 + x_3 + 0x_4 = 6, \\ &\qquad\qquad 2x_1 + 4x_2 + 0x_3 + x_4 = 6, \\ &\qquad\qquad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

Choosing a Pivot Element

The next question is, which pivot to use? In the Sample Problem 6.20, either a_{31} or a_{42} would be suitable. However, consider the problem with initial tableau

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_3 & 2 & 2 & 1 & 0 & 0 & 6 \\ x_4 & 1 & 3 & 0 & 1 & 0 & 6 \\ \hline & -3 & 2 & 0 & 0 & 1 & 0 \end{array}\right].$$

(In this case the original objective function was $P = 3x_1 - 2x_2$.) If we pivot on a_{42} we obtain

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_3 & \frac{4}{3} & 0 & 1 & -\frac{2}{3} & 0 & 2 \\ x_2 & \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 2 \\ \hline & -\frac{11}{3} & 0 & 0 & -\frac{2}{3} & 1 & -4 \end{array}\right].$$

We have moved to the basic feasible solution (0, 2, 2, 0), and the value of the objective function has changed from 0 to −4. It has decreased.

The solution is straightforward. *An entry is a suitable pivot element only if the corresponding entry in the objective row is negative.* When we pivot, we are increasing the value of one non-basic variable. For this to increase the objective function, that variable must have a positive coefficient in the equation for P, which corresponds to a negative entry in the objective row.

The usual practice is to choose the column with the largest negative entry because experience suggests that this tends to help you reach the end of the process in fewer steps. However, this is not essential, and you can use any negative entry.

Sample Problem 6.21. *In each case, an objective row is shown. Which pivot column should be chosen?*

(i) $$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & P & b \\ -2 & -1 & 0 & 0 & 0 & 0 \end{array}\right];$$

(ii) $$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & P & b \\ -1 & 1 & 0 & 0 & 0 & 0 \end{array}\right];$$

(iii) $$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & P & b \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solution. In (i), either x_1 or x_2 can be used, but x_1 would be the usual choice. In case (ii), x_1 must be chosen. In case (iii), no pivot is available.

Sample Problem 6.22. *Pivot on row x_4 and column x_2 in Sample Problem 6.20.*

Solution.

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_3 & \frac{4}{3} & 0 & 1 & -\frac{2}{3} & 0 & 2 \\ x_2 & \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 2 \\ \hline & -\frac{5}{3} & 0 & 0 & \frac{4}{3} & 1 & 8 \end{array}\right].$$

Your Turn. Pivot on row x_3 and column x_1 in Your Turn problem 6.20.

Sample Problem 6.23. *Pivot on entry x_3 and column x_1 in Sample Problem 6.20.*

Solution.

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_1 & 1 & 1 & \frac{1}{2} & 0 & 0 & 3 \\ x_4 & 0 & 2 & -\frac{1}{2} & 1 & 0 & 3 \\ \hline & 0 & -1 & \frac{3}{2} & 0 & 1 & 9 \end{array}\right].$$

After pivoting, examine the new tableau and choose another pivot. Continue in this way until you reach a tableau with no pivot, as in case (iii) of Sample Problem 6.21. When this happens, you have reached the optimum solution.

Sample Problem 6.24. *Finish Sample Problem 6.20.*

Solution. We continue from Sample Problem 6.22. We pivot on x_3 and column x_1, obtaining

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_1 & 1 & 0 & \frac{3}{4} & -\frac{1}{2} & 0 & \frac{3}{2} \\ x_2 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & \frac{3}{2} \\ \hline & 0 & 0 & \frac{5}{4} & \frac{1}{2} & 1 & \frac{21}{2} \end{array}\right].$$

The solution is $P = \frac{21}{2}$, obtained by putting $x_1 = \frac{3}{2}$, $x_2 = \frac{3}{2}$.

Your Turn. Finish Your Turn problem 6.20.

Unbounded Problems

It can happen that a pivot column is available, with a negative entry in the objective row, but there is no positive entry in the column. In this case, it is possible to increase the objective function by increasing the value of the relevant variable, but no limit is found. The feasible region continues without bound. In this case, we say the objective function is unbounded. In real terms, the usual interpretation is that the problem has not been stated properly. Typically, one or more conditions have been omitted.

Minimization Problems

The simplex method, as described above, applies only to maximization problems. Moreover, we have assumed that all the constraints are in the form "some expression $\leq$ constant", so that a basic feasible solution will be exhibited.

For convenience, we shall continue to write all problems as maximization problems. If, for example, we need to minimize the cost $C = 3x_1 - 2x_2$, we shall simply define $P = -C$, and maximize $P = -3x_1 + 2x_2$. No other changes are involved.

If there are any constraints of the form "$\geq$", we can also rewrite them by negation. For example, $2x_1 - x_2 \geq 2$ becomes $-2x_1 + x_2 \leq -2$. However, a negative quantity on the right-hand side means that the basic solution being exhibited is not feasible.

Sample Problem 6.25. *Convert the following problem to a maximization problem.*

$$\begin{aligned} \textit{Minimize} \quad & C = 3x_1 - 2x_2 \\ \textit{subject to} \quad & x_1 + 2x_2 \leq 2, \\ & 2x_1 - x_2 \geq -1, \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Solution.

$$\begin{aligned} \text{Maximize} \quad & P = -3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 2, \\ & -2x_1 + x_2 \leq 1, \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Your Turn. Convert the following problem to a maximization problem.

$$\begin{aligned} \text{Minimize} \quad & C = 2x_1 - 3x_2 \\ \text{subject to} \quad & 2x_1 - 2x_2 \geq -5, \\ & 3x_1 + x_2 \leq 4, \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Summary of the Simplex Method

We now summarize the simplex method.

Step 1 *Write the problem in standard form as a maximization, with slack and surplus variables.* If a basic feasible solution is not exhibited, see the methods in the next section.

Step 2 *Construct the initial tableau.* To the constraint matrix, add an objective row and a column for P.

Step 3 *Is there a possible pivot column?* If there is no negative entry in the objective row (other than the b column), the process is finished and you can read off the optimum value. Otherwise go on.

Step 4 *Pivot.* Then go back to step 3.

Sample Problem 6.26. *Solve the following linear programming problem.*

$$\begin{aligned} \textit{Maximize} \quad & P = 3x_1 + 2x_2 \\ \textit{subject to} \quad & x_1 + 2x_2 \le 2, \\ & 2x_1 + 3x_2 \le 5, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

Solution.

Step 1. The problem is a maximization. We insert slack variables, obtaining

$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 2x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + 0x_4 = 2, \\ & 2x_1 + 3x_2 + 0x_3 + x_4 = 5, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0. \end{aligned}$$

Step 2. The initial tableau is

BV	x_1	x_2	x_3	x_4	P	b
x_3	1	2	1	0	0	2
x_4	2	3	0	1	0	5
	−3	−2	0	0	1	0

Step 3. Column x_1 is a suitable pivot column.

Step 4. We pivot on entry x_3 and column x_1, obtaining

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	2	1	0	0	2
x_4	0	−1	−2	1	0	1
	0	4	3	0	1	6

There is now no pivot. The maximum value is $P = 6$, obtained by putting $x_1 = 2, x_2 = 0$.

Your Turn. Solve the following linear programming problem.

$$\begin{aligned} \text{Maximize} \quad & P = 5x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + 2x_2 \le 4, \\ & 3x_1 + x_2 \le 7, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

Exercises 6.5 A

1. In each of the following problems, add slack variables and write down the initial tableau.

(i)
$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 5x_2 \\ \text{subject to} \quad & x_1 + 4x_2 \le 2, \\ & x_1 + 2x_2 \le 4, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(ii)
$$\begin{aligned} \text{Maximize} \quad & P = x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + 2x_2 \le 4, \\ & 3x_1 + 5x_2 \le 1, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(iii)
$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 2x_2 \\ \text{subject to} \quad & 3x_1 + 3x_2 \le 2, \\ & x_1 + 4x_2 \le 3, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(iv)
$$\begin{aligned} \text{Minimize} \quad & C = 4x_1 - 3x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 3, \\ & 3x_1 + 2x_2 + 3x_3 \le 4, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(v)
$$\begin{aligned} \text{Maximize} \quad & C = 2x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 6, \\ & 3x_1 + 3x_2 \le 7, \\ & x_1 + 2x_2 \le 5, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(vi)
$$\begin{aligned} \text{Minimize} \quad & C = 6x_1 - x_2 \\ \text{subject to} \quad & 5x_1 + 3x_2 \le 8, \\ & 3x_1 + x_2 \le 4, \\ & x_1 + 4x_2 \le 5, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

2. In the following tableaus, perform a pivot on the indicated element.

(i)
$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_2 & 2 & 1 & 0 & 1 & 0 & 3 \\ x_3 & \underline{1} & 0 & 1 & 2 & 0 & 1 \\ \hline & -2 & 0 & 0 & -1 & 1 & 2 \end{array}\right];$$

(ii)

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	1	$\underline{2}$	1	0	2
x_1	1	1	1	0	0	1
	0	−1	−3	0	1	3

;

(iii)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_2	0	1	0	0	1	0	2
x_1	1	0	$\underline{1}$	0	1	0	3
x_4	0	0	1	1	3	0	4
	0	0	−1	0	1	1	2

;

(iv)

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_4	0	0	2	1	$\underline{1}$	1	0	3
x_1	1	0	−2	0	−1	−2	0	4
x_2	0	1	1	0	1	−1	0	4
	0	0	−1	0	−4	−1	1	5

.

3. In the following tableaus, perform one operation:
if a pivot is required, perform it;
if the problem is unbounded, say so;
if the problem is finished, state the optimum value and the values of the variables that attain it.

(i) Maximize $P = 12x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_3	4	2	1	0	0	12
x_4	6	4	0	1	0	8
	−16	−24	0	0	1	0

;

(ii) Maximize $P = 4x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	−3	1	0	−3	0	15
x_3	2	0	1	0	0	20
	0	0	0	−4	1	45

;

(iii) Maximize $P = 12x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_3	2	1	1	0	0	6
x_4	3	2	0	1	0	4
	−8	−12	0	0	1	0

;

(iv) Maximize $P = 3x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	2	−1	0	3
x_1	1	0	−1	1	0	4
	0	0	1	1	1	18

;

(v) Maximize $P = 3x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_3	0	2	1	−1	0	4
x_1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	4
	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	1	12

;

(vi) Maximize $P = 3x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	2	−1	0	2
x_1	1	0	−1	1	0	1
	0	0	1	1	1	7

;

(vii) Maximize $P = 3x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	5
x_1	1	$\frac{5}{2}$	2	0	0	$\frac{3}{2}$
P	2	$\frac{3}{2}$	3	0	1	$\frac{9}{2}$

;

(viii) Maximize $P = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	−1	1	1	0	0	2
x_4	−1	0	2	1	0	5
	−5	0	3	0	1	6

;

(ix) Maximize $P = 4x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	0	0	0	1	2	0	2
x_2	0	1	−2	0	1	0	6
x_1	1	0	−1	0	1	0	3
	0	0	−3	0	0	1	30

;

(x) Maximize $P = 12x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	1	1	1	0	0	3
x_4	1	0	−1	1	0	1
	−8	0	−3	0	1	15

;

(xi) Maximize $P = 2x_1 + 4x_2 + 3x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_2	0	1	5	3	1	0	19
x_1	1	0	3	−1	2	0	10
	0	0	5	10	8	1	96

;

(xii) Maximize $P = 2x_1 + 8x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	3	0	1	0	8
x_3	0	1	1	−1	0	2
	0	−2	0	4	1	16

;

(xiii) Maximize $P = 4x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	−3	1	2	0	0	1
x_4	2	0	1	1	0	2
	0	0	0	2	1	3

;

(xiv) Maximize $P = 2x_1 + 4x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	−1	2	0	0	6
x_4	0	−2	2	1	0	4
	0	−1	3	0	1	12

;

(xv) Maximize $P = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	−3	0	1	0	2	0	2
x_4	−1	0	0	1	1	0	7
x_2	−1	1	0	0	1	0	2
	−5	0	0	0	3	1	6

;

(xvi) Maximize $P = 2x_1 + 2x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_3	0	0	1	1	1	3	0	3
x_1	1	0	0	2	−2	2	0	2
x_2	0	1	0	0	−1	−1	0	4
	0	0	0	−3	−2	1	1	15

.

4. Solve the following problems using the simplex method:

(i)
$$\begin{aligned} \text{Maximize} \quad & P = 6x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 5x_2 \le 30, \\ & 2x_1 + 2x_2 \le 12, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(ii)
$$\begin{aligned} \text{Maximize} \quad & P = 6x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 4, \\ & 2x_1 + 3x_2 \le 8, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(iii)
$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \le 12, \\ & 3x_1 + 2x_2 \le 12, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(iv)
$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 \le 2, \\ & 2x_1 + x_2 + x_3 \le 3, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(v)
$$\begin{aligned} \text{Maximize} \quad & P = 8x_1 + 5x_2 + 6x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \le 54, \\ & -x_1 + x_3 \le 0, \\ & -2x_1 + x_2 - x_3 \le 0, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

Exercises 6.5 B

1. In each of the following problems, add slack variables and write down the initial tableau:

(i) Maximize $P = 3x_1 + 5x_2$
subject to $5x_1 + 4x_2 \leq 2,$
$x_1 + 2x_2 \leq 3,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(ii) Maximize $P = x_1 + 3x_2$
subject to $x_1 - 2x_2 \leq 2,$
$2x_1 + 3x_2 \leq 6,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iii) Maximize $P = 3x_1 + 2x_2$
subject to $4x_1 + 2x_2 \leq 2,$
$5x_1 + 3x_2 \leq 3,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iv) Maximize $P = 3x_1 + 6x_2$
subject to $x_1 + 4x_2 \leq 2,$
$x_1 + 2x_2 \leq 4,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(v) Maximize $P = 2x_1 + 5x_2$
subject to $x_1 - 4x_2 \leq 4,$
$5x_1 + 2x_2 \leq 3,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(vi) Maximize $P = 4x_1 + 3x_2$
subject to $2x_1 + 3x_2 \leq 3,$
$x_1 + 2x_2 \geq 4,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(vii) Minimize $C = 4x_1 - 3x_2 + 2x_3$
subject to $x_1 - x_2 + x_3 \geq 3,$
$2x_1 + x_2 + 3x_3 \geq 8,$
$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0;$

(viii)
$$\begin{aligned} \text{Minimize} \quad & C = 4x_1 + 3x_2 + 2x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \le 4, \\ & 2x_1 + 4x_2 + 3x_3 \le 9, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(ix)
$$\begin{aligned} \text{Minimize} \quad & C = 5x_1 + 4x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 \ge 11, \\ & 3x_1 - x_2 + 2x_3 \ge 8, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(x)
$$\begin{aligned} \text{Maximize} \quad & C = 2x_1 + 5x_2 \\ \text{subject to} \quad & 3x_1 + x_2 \le 5, \\ & 2x_1 + 3x_2 \le 7, \\ & x_1 + 2x_2 \le 4, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(xi)
$$\begin{aligned} \text{Minimize} \quad & C = 6x_1 - x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \le 10, \\ & 3x_1 - 4x_2 \le 4, \\ & x_1 + 4x_2 \le 9, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(xii)
$$\begin{aligned} \text{Minimize} \quad & C = 50x_1 + 10x_2 + 20x_3 \\ \text{subject to} \quad & 5x_1 - 5x_2 + 5x_3 \ge 5, \\ & 10x_1 + 5x_2 + 5x_3 \ge 10, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(xiii)
$$\begin{aligned} \text{Maximize} \quad & C = 4x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 4x_2 \le 4, \\ & 3x_1 - x_2 \le 3, \\ & x_1 + 3x_2 \le 5, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(xiv)
$$\begin{aligned} \text{Minimize} \quad & C = 6x_1 - x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \ge 8, \\ & 3x_1 + 4x_2 \le 14, \\ & 5x_1 + 3x_2 \le 15, \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

2. In the following tableaus, perform a pivot on the indicated element:

(i)

BV	x_1	x_2	x_3	x_4	P	b
x_2	2	1	0	1	0	8
x_3	$\underline{2}$	0	4	2	0	6
	−2	0	0	−2	1	3

;

(ii)

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	$\underline{2}$	2	0	2
x_1	1	0	1	2	0	3
	0	0	−4	0	1	3

;

(iii)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_1	1	0	$\underline{2}$	0	1	0	6
x_2	0	1	1	0	1	0	4
x_4	0	0	1	1	3	0	6
	0	0	−2	0	3	1	4

;

(iv)

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	1	$\underline{1}$	1	0	2
x_1	1	1	1	0	0	3
	0	−1	−4	0	1	3

;

(v)

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	3	0	1	0	4
x_3	0	$\underline{2}$	1	2	0	2
	−4	0	0	−2	1	3

;

(vi)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_2	0	1	0	0	1	0	4
x_3	$\underline{1}$	0	1	0	1	0	2
x_4	2	0	0	1	2	0	5
	0	0	−2	0	2	1	3

;

(vii)

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_1	1	0	2	0	−1	−2	0	5
x_4	0	0	4	1	$\underline{1}$	2	0	3
x_2	0	1	1	0	1	−1	0	6
	0	0	−1	0	−2	−1	1	8

.

3. In the following tableaus, perform one operation:
if a pivot is required, perform it;
if the problem is unbounded, say so;
if the problem is finished, state the optimum value and the values of the variables that attain it.

(i) Maximize $P = 12x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	1	2	1	0	2
x_1	1	1	1	0	0	1
	0	−1	−3	0	1	3

;

(ii) Maximize $P = 4x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\frac{7}{2}$
x_1	1	$\frac{3}{2}$	2	0	0	3
	0	$\frac{5}{2}$	3	0	1	12

;

(iii) Maximize $P = x_1 + 7x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{5}{2}$
x_3	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$
	0	$-\frac{11}{2}$	0	$\frac{1}{2}$	1	$\frac{5}{2}$

;

(iv) Maximize $P = x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	4	1	0	1	0	45
x_3	2	0	1	−4	0	25
P	4	0	0	7	1	90

;

(v) Maximize $P = 12x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	3	0	2	0	4
x_3	0	6	1	3	0	9
	0	−6	0	−5	1	25

;

(vi) Maximize $P = 2x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	2	0	1	0	30
x_3	0	1	1	−1	0	20
	0	2	0	5	1	60

;

(vii) Maximize $P = 3x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	0	0	−2	1	0	0	6
x_2	0	1	−4	0	1	0	1
x_1	1	0	−1	0	2	0	2
	0	0	−3	0	1	1	8

;

(viii) Maximize $P = \frac{3}{2}x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_3	0	1	1	2	0	1
x_1	1	2	0	1	0	2
P	0	2	0	1	1	3

;

(ix) Maximize $P = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	−3	0	2	1	0	2
x_2	−1	1	1	0	0	2
	−5	0	3	0	1	6

;

(x) Maximize $P = 3x_1 + 3x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	$\frac{5}{2}$	2	0	0	$\frac{3}{2}$
x_4	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	5
	0	$-\frac{3}{2}$	4	0	1	$\frac{9}{2}$

;

(xi) Maximize $P = 3x_1 + 3x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{5}{2}$
x_3	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
	0	$-\frac{11}{2}$	0	$\frac{1}{2}$	1	$\frac{5}{2}$

;

(xii) Maximize $P = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	0	0	1	−2	5	0	15
x_1	1	0	0	−1	1	0	1
x_2	0	1	0	−1	2	0	5
P	0	0	0	−5	8	1	17

;

(xiii) Maximize $P = 3x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	−8	0	3	1	0	15
x_2	−3	1	1	0	0	3
	−4	0	1	0	1	3

;

(xiv) Maximize $P = 3x_1 - x_2 + 2x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	$\frac{5}{2}$	$\frac{3}{2}$	0	1	$\frac{1}{2}$	0	$\frac{3}{2}$
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{5}{2}$
	−2	2	0	0	1	1	5

;

(xv) Maximize $P = 5x_1 - 3x_2 + 3x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	5	4	1	0	−2	0	2
x_4	−3	2	0	1	−1	0	7
	2	0	0	0	−3	1	6

;

(xvi) Maximize $P = 3x_1 + 2x_2 + 3x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	1	−1	1	1	0	0	5
x_5	2	0	−1	0	1	0	10
	−1	−2	−1	0	0	1	0

;

(xvii) Maximize $P = 5x_1 + x_2 + 3x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	0	$\frac{8}{15}$	1	$\frac{2}{3}$	$-\frac{1}{15}$	0	$\frac{100}{3}$
x_1	1	$-\frac{1}{15}$	0	$-\frac{1}{3}$	$\frac{2}{15}$	0	$\frac{100}{3}$
	0	$\frac{4}{15}$	0	$\frac{1}{3}$	$\frac{7}{15}$	1	$\frac{800}{3}$

;

(xviii) Maximize $P = 3x_1 + 2x_2 + 3x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	1	1	1	1	0	0	10
x_5	0	1	3	0	1	0	20
	−2	2	−3	0	0	1	0

;

(xix) Maximize $P = 8x_1 + x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_3	1	1	1	0	0	2
x_4	2	1	0	1	0	6
	−8	1	0	0	1	0

;

(xx) Maximize $P = 2x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_3	0	1	1	1	0	3
x_1	1	2	0	1	0	8
	0	−1	0	1	1	16

;

(xxi) Maximize $P = 2x_1 + x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	2	3	0	2
x_1	1	0	−1	1	0	4
	0	0	4	3	1	10

;

(xxii) Maximize $P = 3x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	−5	1	1	0	2
x_1	1	−4	2	0	0	2
P	2	−2	2	0	1	6

;

(xxiii) Maximize $P = 2x_1 + 10x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	P	b
x_4	0	2	−4	1	0	2
x_1	1	4	2	0	0	5
P	2	−2	3	0	1	10

;

(xxiv) Maximize $P = 2x_1 + 4x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	0	0	1	3	4	0	3
x_1	1	0	0	1	−1	0	4
x_2	0	1	0	2	1	0	2
	0	0	0	2	3	1	16

;

(xxv) Maximize $P = 3x_1 + 5x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_1	1	1	0	1	1	0	5
x_3	0	1	1	−1	2	0	1
	0	−1	0	1	3	1	16

;

(xxvi) Maximize $P = 6x_1 + 2x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_2	2	1	1	3	1	0	4
x_3	2	0	4	−2	2	0	2
	−1	0	0	4	8	1	9

;

(xxvii) Maximize $P = 2x_1 + 4x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	0	0	1	3	4	0	3
x_1	1	0	0	1	−1	0	4
x_2	0	1	0	2	1	0	2
	0	0	0	2	3	1	16

;

(xxviii) Maximize $P = 3x_1 + 5x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	0	−2	1	0	4	0	3
x_1	1	0	0	0	−1	0	2
x_4	0	−1	0	1	1	0	2
	0	−1	0	0	3	1	10

;

(xxix) Maximize $P = 2x_1 + 8x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_3	0	3	1	3	2	0	0	7
x_1	1	2	0	−1	−1	0	0	6
x_6	0	1	0	2	1	1	0	2
	0	−1	0	2	5	0	1	19

;

(xxx) Maximize $P = 3x_1 + 8x_2 + 6x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_4	20	4	4	1	0	0	0	6
x_5	8	8	4	0	1	0	0	10
x_6	8	4	2	0	0	1	0	4
	−3	−8	−6	0	0	0	1	0

;

(xxxi) Maximize $P = 2x_1 + 2x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_2	0	1	5	3	1	3	0	3
x_3	0	0	1	−1	1	3	0	3
x_1	1	0	0	2	−2	2	0	4
	0	0	0	3	2	1	1	17

.

4. Solve the following problems using the simplex method.

(i) Maximize $P = 3x_1 + 4x_2$
subject to $2x_1 + x_2 \leq 4,$
$3x_1 + 2x_2 \leq 8,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(ii) Maximize $P = 2x_1 + 3x_2$
subject to $-x_1 + 2x_2 \leq 1,$
$2x_1 + x_2 \leq 4,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iii) Maximize $P = 3x_1 + 5x_2$
subject to $x_1 + x_2 \leq 6,$
$2x_1 + x_2 \leq 10,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iv)
$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 7x_1 + x_3 \le 6, \\ & 3x_2 + 4x_3 \le 30, \\ & x_1 + 2x_2 \le 20, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(v)
$$\begin{aligned} \text{Maximize} \quad & P = x_1 + 2x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 \le 4, \\ & 4x_1 + 3x_2 + x_3 \le 3, \\ & 2x_1 + 2x_2 + x_3 \le 3, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(vi)
$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 - 3x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 + x_2 \le 16, \\ & x_2 + x_3 \le 10, \\ & x_1 + x_2 + x_3 \le 20, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

5. Consider the following problem:

$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + x_2 + x_3 \\ \text{subject to} \quad & -x_1 + x_3 \le 1, \\ & -x_2 + x_3 \le 2, \\ & 2x_1 - x_2 \le 3, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

It is possible to derive the final tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_3	0	0	1	3	−1	1	0	3
x_1	1	0	0	2	−1	1	0	2
x_2	0	1	0	3	−2	1	0	1
	0	0	0	10	−5	5	1	8

,

and from column x_5 we deduce that the problem is unbounded.

(i) Verify that this tableau can be achieved (pivot on x_3, then x_1, then x_2).

(ii) Show that if we fix $x_3 = 3$, we can make P as large as we please by increasing x_1 and x_2, and that the problem remains feasible provided $x_2 \ge 3 + 2x_1$ and $x_1 \ge 2$.

(iii) Find conditions on x_2 and x_3 such that P increases without bound, provided $x_1 = 2$.

6.6 The Two Phase Simplex Method

Initially Infeasible Problems

So far we only know how to apply the simplex method when the initial tableau exhibits a basic feasible solution—when "the origin is feasible". In theory, we could start from any basic feasible solution, but we do not know how to find such a starting point.

We shall now show how to find a basic feasible solution to a problem that is initially infeasible. The technique adds further variables, called *artificial variables* in order to form a new problem (the *artificial problem*) that is initially feasible. Solving this problem—called the first phase—leads to a feasible solution to the original problem. In the second phase, the original is solved by the simplex method.

Problems with Inequalities

Consider a problem with constraints

$$\begin{aligned} x_1 + 2x_2 &\le 3, \\ 2x_1 - 3x_2 &\ge 2. \end{aligned}$$

If we add slack and surplus variables, the constraints become

$$\begin{aligned} x_1 + 2x_2 + x_3 + 0x_4 &= 3, \\ 2x_1 - 3x_2 + 0x_3 - x_4 &= 2. \end{aligned}$$

This does not exhibit a basic solution because the second equation does not exhibit a basic variable. Constraints that use $\ge$ do not exhibit basic variables because they take a surplus variable, with a negative sign.

We could change the second constraint to

$$-2x_1 + 3x_2 \le -2,$$

but then the constraint matrix becomes

$$\begin{aligned} x_1 + 2x_2 + x_3 + 0x_4 &= 3, \\ -2x_1 + 3x_2 + 0x_3 + x_4 &= -2. \end{aligned}$$

This exhibits the basic solution $(0, 0, 3, -2)$, which is not feasible.

In general, we can change a constraint with sign $\ge$ to one with sign $\le$, but if the original had a positive constant term then the basic solution exhibited by the new system will not be feasible.

To avoid this difficulty, we simply add a new variable which we call an *artificial variable*. Later we shall need to distinguish artificial variables from other variables, so we shall write artificial variables as A_1, A_2, and so on. The new constraint matrix would look like

$$\begin{aligned} x_1 + 2x_2 + x_3 + 0x_4 + 0A_1 &= 3, \\ 2x_1 - 3x_2 + 0x_3 - x_4 + A_1 &= 2. \end{aligned}$$

Problems with Equalities

Another problem arises if any of the constraints are equalities rather than inequalities. For example, suppose a farmer has \$20000 to spend on corn and wheat seed. If she spends \$$x_1$ on corn and \$$x_2$ on wheat, we could express the constraint as

$$x_1 + x_2 \leq 20000,$$

but perhaps the amount of money left over will enter into our calculations. For example, she may get 3% interest from the bank on any surplus. This could be expressed by adding a figure $0.03 \cdot (20000 - x_1 - x_2)$ to the profit, but often it is convenient to use a variable x_3 to represent the remaining cash. The constraint

$$x_1 + x_2 + x_3 = 20000$$

does not receive a slack variable, so it does not exhibit a basic variable. Again we can add an artificial variable.

The Artificial Problem

Sample Problem 6.27. *Add slack, surplus and artificial variables to the problem:*

$$\begin{aligned} \textit{Maximize} \quad & P = 4x_1 - 3x_2 \\ \textit{subject to} \quad & x_1 + 2x_2 \leq 6, \\ & 2x_1 - x_2 \geq 2, \\ & x_1 + x_2 + x_3 = 5, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

Solution. We add a slack variable to the first constraint, a surplus variable, and an artificial variable to the second constraint, and an artificial variable to the third constraint. The equations are

$$\begin{aligned} & x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 + 0A_1 + 0A_2 = 6, \\ & 2x_1 - x_2 + 0x_3 + 0x_4 - x_5 + A_1 + 0A_2 = 2, \\ & x_1 + x_2 + x_3 + 0x_4 + 0x_5 + 0A_1 + A_2 = 5, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0, \quad A_1 \geq 0, \quad A_2 \geq 0. \end{aligned}$$

Your Turn. Add slack, surplus, and artificial variables to the problem

$$\begin{aligned} \text{Maximize} \quad & C = 4x_1 - 3x_2 \\ \text{subject to} \quad & 2x_1 - 2x_2 \le 4, \\ & x_1 + 4x_2 \ge 4, \\ & x_1 + 2x_2 + x_3 = 7, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

The constraint system we have just constructed exhibits a basic feasible solution for some linear programming problem, but what is the problem? We need to find a problem that involves the artificial variables as well as the originals, slacks and surpluses. To do this, we define a new objective function. Instead of maximizing $P = 4x_1 - 3x_2$, we maximize $P = 4x_1 - 3x_2 - MA_1 - MA_2$, where M is some very large positive number. It is not usual to specify a value for M, but it is assumed to be much larger than anything else that arises in the problem.

Sample Problem 6.28. *Write down the initial tableau for Sample Problem 6.27.*

Solution.

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	1	2	0	1	0	0	0	0	6
A_1	2	−1	0	0	−1	1	0	0	2
A_2	1	1	1	0	0	0	1	0	5
	−4	3	0	0	0	M	M	1	0

When the initial tableau is written down, we wish to pivot in a way that involves M. This is not initially possible because M only occurs in the objective row with a positive sign. So the first step in solving an artificial problem is to change the objective row. This is called *updating* the tableau. Simply subtract M times row A_i from the objective row for each artificial variable A_i.

Sample Problem 6.29. *Write down the updated tableau for Sample Problem 6.27.*

Solution. We subtract $M\times$ the A_1 and A_2 rows from the objective row.

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	1	2	0	1	0	0	0	0	6
A_1	2	−1	0	0	−1	1	0	0	2
A_2	1	1	1	0	0	0	1	0	5
	−4	3	0	0	0	0	0	1	0
	$-3M$	0	$-M$	0	M	0	0	0	$-7M$

Your Turn. Write down the initial and updated tableaus for Your Turn Problem 6.27.

Notice that we write the objective row in two lines, with all multiples of M separated out. The entries in the x_1 column, for example, mean that the coefficient of x_1 in the objective row is $-4-3M$. This separation will be useful in later calculations.

The Solution Process

We proceed to solve the artificial problem. Because M is large and the objective function contains each artificial variable multiplied by $-M$, all artificial variables will necessarily be zero—that is, non-basic—in the optimal solution.

Sample Problem 6.30. *Solve the artificial problem in Sample Problem 6.27.*

Solution. We pivot on row A_1, column x_1. In order to add $(3M+4)$ times the new x_1 row to the objective row, we actually add $3M$ times the new row to the lower part of the objective row and 4 times the row to the upper part:

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	0	$\frac{5}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	5
x_1	1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1
A_2	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	4
	0	1	0	0	-2	2	0	1	4
	0	$-\frac{3}{2}M$	$-M$	0	$-\frac{1}{2}M$	$\frac{3}{2}M$	0	0	$-4M$

The next pivot is row A_2, column x_3:

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	0	$\frac{5}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	5
x_1	1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1
x_3	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	4
	0	1	0	0	-2	2	0	1	4
	0	0	0	0	0	M	M	0	0

The final pivot is row x_3, column x_5:

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	0	1	−1	1	0	0	−1	0	1
x_1	1	1	1	0	0	0	0	0	5
x_5	0	3	2	0	1	−1	2	0	8
	0	7	4	0	0	0	4	1	20
	0	0	0	0	0	M	M	0	0

The basic feasible solution has $x_1 = 5$, $x_4 = 1$, $x_5 = 8$, $x_2 = x_3 = A_1 = A_2 = 0$.

Your Turn. Solve the artificial problem in Your Turn Problem 6.27.

The solution of the artificial problem is not necessarily optimal for the original problem. However, the basic solution that was found is also a basic solution for the original problem because all the artificial variables are zero, and it is feasible. So we can go on from there to solve the original problem. We simply delete the artificial variable columns and the second objective row.

Sample Problem 6.31. *Complete Sample Problem 6.27.*

Solution. The tableau becomes

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	0	1	−1	1	0	0	1
x_1	1	1	1	0	0	0	5
x_5	0	3	2	0	1	0	8
	0	7	4	0	0	1	20

We have finished; no further pivoting is necessary. The basic feasible solution has $P = 20$, $x_1 = 5$, $x_2 = 0$, $x_3 = 0$. The slack variable and surplus variable are $x_4 = 1$, $x_5 = 8$.

Your Turn. Complete Your Turn Problem 6.27.

Of course, our small example could have been solved easily by diagram or by enumerating all basic points. But the two-phase method works quite quickly on larger problems, where the other methods mentioned would be too slow.

Notes on the Solution Process

1. To choose a pivot column, it is only necessary to look at the second objective row, the part with the multiples of M. A negative M will always outweigh any positive number in the other part of the objective row.

2. In the updated tableau—the starting point for the first phase—all the artificial variables are basic. After one of them is eliminated from the basis, it will never return. So you could delete the column for an artificial variable as soon as you have pivoted on its row.

3. It can happen that the first phase ends with an artificial variable remaining in the basis. If this variable has value 0 in the b column, it can still be eliminated. But if it is positive, this means that the original problem has no feasible solutions. There were no points in the feasible region. In this case, you simply report that the problem is infeasible.

Summary of the Two-Phase Method

First phase:

Step 1. *Change a minimization problem to a maximization problem if necessary. Add slack and surplus variables.* If a basic feasible solution is exhibited, go to Step 6.

Step 2. *Add artificial variables to all constraints that do not exhibit a feasible basic variable.*

Step 3. *Form the updated tableau.* Add $-M$ times each artificial variable to the objective function (this forms the initial tableau); then, for each row exhibiting an artificial variable, add M times that row to the objective function.

Step 4. *Solve the artificial problem.* If a positive artificial variable remains in the basis, the problem is infeasible. Otherwise go to Step 5.

Second phase:

Step 5. *Delete the artificial variable columns.*

Step 6. *Solve the problem by the simplex method.*

Exercises 6.6 A

1. For each of the following constraints, add all necessary slack, surplus or artificial variables:

(i) $4x_1 + 3x_2 \geq 5$;

(ii) $x_1 + x_2 \geq 2$;

(iii) $7x_1 - 2x_2 + 3x_3 = 7$;

(iv) $4x_1 + 5x_2 + x_3 \geq 4$;

(v) $4x_1 - 4x_2 + 2x_3 \leq 5$;

(vi) $7x_1 + 3x_2 - 3x_3 + 7x_4 = 5;$

(vii) $3x_1 - 2x_2 + 4x_3 - 3x_4 \geq 1;$

(viii) $11x_1 + 9x_2 + 5x_3 + 2x_4 \leq 3.$

2. Set up the initial and updated tableaus for the following linear programming problems:

(i)
$$\begin{aligned} \text{Minimize} \quad & C = 2x_1 + 3x_2 \\ \text{subject to} \quad & 4x_1 + 2x_2 \geq 5, \\ & 3x_1 + 2x_2 = 10, \\ & x_1 \geq 0, \quad x_2 \geq 0; \end{aligned}$$

(ii)
$$\begin{aligned} \text{Minimize} \quad & C = 2x_1 + 5x_2 \\ \text{subject to} \quad & 4x_1 + 2x_2 \leq 8, \\ & 2x_1 + 10x_2 \geq 11, \\ & 3x_1 - x_2 = 7, \\ & x_1 \geq 0, \quad x_2 \geq 0; \end{aligned}$$

(iii)
$$\begin{aligned} \text{Minimize} \quad & C = x_1 + 2x_2 + x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 \leq 6, \\ & x_1 + x_2 - x_3 \geq 4, \\ & x_2 + x_3 = 2, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0; \end{aligned}$$

(iv)
$$\begin{aligned} \text{Minimize} \quad & C = 2x_1 + 3x_2 + x_3 + x_4 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 + 2x_4 \leq 6, \\ & 2x_1 + 3x_2 + x_3 + x_4 \geq 9, \\ & 2x_1 + 2x_2 - x_3 - x_4 = 4, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

3. In the following tableaus, perform one operation:
if a pivot is required, perform it;
if the problem is infeasible, say so;
if the artificial problem is finished, set up the new tableau.

(i) Maximize $P = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_2	P	b
x_3	-1	0	1	-2	0	0	2
A_1	2	-1	0	3	1	0	4
	2	0	0	-3	0	1	0
	$-M$	M	0	$2M$	0	0	$-4M$

;

(ii) Maximize $P = x_1 + 2x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	0	0	3
A_1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	1	0	1
	3	0	1	0	0	1	6
	$\frac{3}{2}M$	0	$\frac{M}{2}$	M	0	0	$-M$

;

(iii) Minimize $C = 3x_1 + 4x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	2	-1	0	0	0	4
A_1	0	-3	1	-1	1	0	2
	0	-2	3	0	0	1	-12
	0	$3M$	$-M$	M	0	0	$-2M$

;

(iv) Minimize $C = 4x_1 + 6x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	1	1	0	0	0	2
A_1	0	-1	-4	-1	1	0	4
	0	2	-4	0	0	2	-4
	0	$2M$	$8M$	$2M$	0	0	$-12M$

;

(v) Maximize $C = x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	3	-1	0	0	0	4
A_1	0	-3	1	-1	1	0	2
P	0	-2	4	0	0	1	20
	0	$3M$	$-M$	M	0	0	$-2M$

;

(vi) Minimize $P = 2x_1 + 2x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
A_1	1	1	0	-1	0	1	0	0	1
x_5	2	1	1	0	1	0	0	0	3
A_2	1	1	2	0	0	0	1	0	2
	2	2	1	0	0	0	0	1	0
	$-2M$	$-3M$	$-4M$	M	0	0	0	0	$-6M$

;

(vii) Maximize $P = 3x_1 + 2x_2 + 2x_3$
subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	2	2	−1	1	0	0	0	0	4
x_5	3	1	2	0	1	0	0	0	6
A_1	2	−2	1	0	0	1	0	0	1
A_2	−1	3	3	0	0	0	1	0	5
	−3	−2	−2	0	0	0	0	1	0
	$-M$	$-M$	$-4M$	0	0	0	0	0	$-6M$

.

4. Solve the following linear programming problems, or show that they are unbounded or infeasible.

(i) Maximize $P = 2x_1 + x_2$
subject to $x_1 + 2x_2 = 4,$
$x_1 + 3x_2 \geq 3,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(ii) Minimize $C = 2x_1 - 3x_2 + x_3$
subject to $x_1 - 2x_2 - 2x_3 \geq 8,$
$-x_1 + 3x_2 + x_3 \geq 1,$
$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0;$

(iii) Maximize $P = 3x_1 + 2x_2 + x_3$
subject to $x_1 + 2x_2 + x_3 \leq 8,$
$2x_1 + 3x_2 + 3x_3 = 6,$
$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0;$

(iv) Minimize $C = 7x_1 + 5x_2 + 6x_3$
subject to $x_1 + 2x_2 + 3x_3 \leq 19,$
$2x_1 + 3x_2 \geq 21,$
$x_1 + x_2 + x_3 = 10,$
$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0;$

(v) Maximize $P = 2x_1 + 3x_2$
subject to $x_1 + x_2 \leq 4,$
$2x_1 + x_2 \geq 10,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(vi)
$$\begin{aligned} \text{Maximize} \quad & P = 7x_1 + 9x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 7, \\ & x_1 + 2x_2 - x_3 \le 3, \\ & x_2 - x_3 \le 5, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(vii)
$$\begin{aligned} \text{Minimize} \quad & C = 2x_1 + 5x_2 + 3x_3 - x_4 \\ \text{subject to} \quad & x_1 + x_2 + 6x_3 + x_4 \ge 7, \\ & -x_2 + 3x_3 + x_4 = 0, \\ & x_1 + x_2 + x_3 \ge 5, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

Exercises 6.6 B

1. For each of the following constraints, add all necessary slack, surplus, or artificial variables:

(i) $x_1 + 3x_2 \ge 2$;

(ii) $x_1 + 3x_2 \le 2$;

(iii) $3x_1 + 2x_2 \ge 5$;

(iv) $x_1 + 3x_2 = 2$;

(v) $5x_1 - 4x_2 + 3x_3 = 11$;

(vi) $3x_1 + 3x_2 + 2x_3 \le 4$;

(vii) $4x_1 - 3x_2 + 5x_3 \ge 3$;

(viii) $2x_1 - 2x_2 + x_3 = 6$;

(ix) $4x_1 + 7x_2 + x_3 \le 6$;

(x) $7x_1 + 4x_2 + 8x_3 \ge 5$;

(xi) $x_1 + 3x_2 - 3x_3 + x_4 = 1$;

(xii) $2x_1 - 2x_2 + 3x_3 + x_4 \ge 2$;

(xiii) $x_1 - 9x_2 + 2x_3 - 2x_4 \le 3$;

(xiv) $3x_1 - 4x_2 + 2x_3 + 4x_4 = 3$;

(xv) $2x_1 - 2x_2 - 5x_3 + 5x_4 \le 4$;

(xvi) $4x_1 - 4x_2 + 3x_3 - 2x_4 \ge 2$.

2. Set up the initial and updated tableaus for the following linear programming problems.

(i) Minimize $C = 3x_1 - x_2$
subject to $2x_1 + 3x_2 \leq 12,$
$x_1 + 2x_2 \geq 12,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(ii) Minimize $C = 2x_1 + 5x_2$
subject to $x_1 + x_2 \leq 9,$
$4x_1 + x_2 \geq 8,$
$2x_1 + 7x_2 = 28,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iii) Minimize $C = 2x_1 + x_2$
subject to $2x_1 + x_2 \leq 6,$
$x_1 + 3x_2 \geq 8,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(iv) Minimize $C = 2x_1 + 3x_2 + x_3$
subject to $2x_1 + 3x_2 + x_3 \leq 9,$
$2x_1 + 2x_2 + 3x_3 = 7,$
$3x_1 + 2x_2 + 2x_3 \geq 8,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(v) Maximize $P = 4x_1 + 3x_2 + 2x_3$
subject to $2x_1 + 2x_2 - x_3 \leq 4,$
$3x_1 + x_2 + 4x_3 \leq 8,$
$4x_1 - 2x_2 + 2x_3 = 9,$
$-2x_1 + 3x_2 + 3x_3 = 10,$
$x_1 \geq 0, \quad x_2 \geq 0;$

(vi) Minimize $C = 2x_1 + 2x_2 + x_3$
subject to $2x_1 + x_2 + x_3 = 8,$
$x_1 + x_2 \geq 6,$
$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$

3. In the following tableaus, perform one operation:
if a pivot is required, perform it;
if the problem is infeasible, say so;
if the artificial problem is finished, set up the new tableau.

(i) Minimize $C = x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_2	3	1	1	0	0	0	3
A_1	−4	0	−3	−1	1	0	6
	−8	0	−3	0	0	1	−9
	$4M$	0	$3M$	M	0	0	$-6M$

;

(ii) Minimize $C = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_3	1	1	1	0	0	0	3
A_1	2	1	0	−1	1	0	5
P	2	3	0	0	0	1	0
	$-2M$	$-M$	0	M	0	0	$-5M$

;

(iii) Minimize $C = 2x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	1	1	0	0	0	2
A_1	0	−1	−2	−1	1	0	1
	0	1	−2	0	0	1	−4
	0	M	$2M$	M	0	0	$-M$

;

(iv) Minimize $C = 3x_1 + 4x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	2	1	0	0	0	4
A_1	0	−3	−2	−1	1	0	2
	0	−2	−3	0	0	1	−12
	0	$3M$	$2M$	M	0	0	$-2M$

;

(v) Minimize $P = x_1 + x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_2	3	1	1	0	0	0	0	0	3
A_1	−1	0	−1	−1	0	1	0	0	2
A_2	−8	0	−3	0	−1	0	1	0	1
	−5	0	−2	0	0	0	0	1	−6
	$9M$	0	$4M$	M	M	0	0	0	$-3M$

;

(vi) Minimize $P = 2x_1 + 2x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
A_1	1	1	0	−1	0	1	0	0	6
x_5	2	1	1	0	1	0	0	0	8
A_2	1	2	4	0	0	0	1	0	12
	2	2	1	0	0	0	0	1	0
	$-2M$	$-3M$	$-4M$	M	0	0	0	0	$-18M$

;

(vii) Minimize $P = 2x_1 + 3x_2 + 6x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	P	b
x_4	1	3	1	1	0	0	0	20
A_1	2	3	1	0	−1	1	0	12
	2	3	6	0	0	0	1	0
	$-2M$	$-3M$	$-M$	0	M	0	0	$-12M$

;

(viii) Maximize $P = 4x_1 - 3x_2 + 3x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	0	1	−1	1	0	0	−1	0	1
A_1	1	1	1	0	0	0	1	0	5
A_2	0	3	2	0	1	−1	2	0	8
	0	7	1	0	0	0	4	1	20
	0	0	0	0	0	M	M	0	0

;

(ix) Minimize $P = 2x_1 + x_2 + 2x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	A_1	A_2	P	b
x_4	−1	0	1	1	0	2	0	−2	0	2
A_1	1	0	2	0	−1	0	1	0	0	4
A_2	1	1	1	0	0	−1	0	1	0	5
	1	0	1	0	0	1	0	−1	1	−5
	$-M$	0	$-2M$	0	M	0	0	M	0	$-4M$

;

(x) Minimize $C = 4x_1 - 3x_2 + x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	1	2	1	1	0	0	0	0	6
A_1	2	−1	0	0	−1	1	0	0	2
A_2	1	1	1	0	0	0	1	0	5
	−4	3	−1	0	0	0	0	1	0
	$-3M$	0	$-M$	0	M	0	0	0	$-7M$

;

(xi) Minimize $C = 4x_1 + 3x_2$

subject to the tableau

BV	x_1	x_2	x_3	x_4	A_1	A_2	P	b
	1	1	1	0	0	0	0	9
	3	1	0	−1	1	0	0	8
	2	3	0	0	0	1	0	8
	4	3	0	0	M	M	1	0

;

(xii) Minimize $P = 2x_1 + x_2 + 2x_3$

subject to the tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	A_1	A_2	P	b
x_4	2	2	2	1	0	0	0	0	0	11
A_1	1	0	2	0	−1	0	1	0	0	4
A_2	1	1	1	0	0	−1	0	1	0	5
	2	1	2	0	0	0	0	0	1	0
	$-2M$	$-M$	$-3M$	0	M	M	0	0	0	$-9M$

.

4. Solve the following linear programming problems:

(i)
$$\begin{aligned} \text{Minimize} \quad & C = 7x_1 + 3x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \le 6, \\ & 5x_1 + 10x_2 \ge 26, \\ & x_1 \ge 0, \quad x_2 \ge 0; \end{aligned}$$

(ii)
$$\begin{aligned} \text{Minimize} \quad & C = 2x_1 + 2x_2 - 3x_3 \\ \text{subject to} \quad & x_1 - x_2 - 2x_3 \ge 8, \\ & -2x_1 + x_2 + 6x_3 \ge 2, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0; \end{aligned}$$

(iii)
$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6, \\ & 5x_1 + 2x_2 \geq 18, \\ & 7x_1 + 4x_2 \geq 7, \\ & x_1 \geq 0, \quad x_2 \geq 0; \end{aligned}$$

(iv)
$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + 5x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \leq 6, \\ & x_1 + x_2 \geq 4, \\ & x_1 \geq 0, \quad x_2 \geq 0; \end{aligned}$$

(v)
$$\begin{aligned} \text{Minimize} \quad & C = 3x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 12, \\ & 2x_1 + x_2 \geq 12, \\ & x_1 \geq 0, \quad x_2 \geq 0; \end{aligned}$$

(vi)
$$\begin{aligned} \text{Minimize} \quad & C = 5x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & x_1 - x_2 + x_3 \geq 2, \\ & 2x_1 + x_2 + x_3 \geq 4, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0; \end{aligned}$$

(vii)
$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + 3x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 4, \\ & 2x_1 + x_2 - x_3 \geq 1, \\ & x_2 - x_3 \geq 1, \\ & x_1 \geq 0, \quad x_2 \geq 0; \end{aligned}$$

(viii)
$$\begin{aligned} \text{Minimize} \quad & C = x_1 + 3x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \geq 4, \\ & 5x_1 + 10x_2 + 15x_3 \geq 28, \\ & 3x_1 - x_2 + 3x_3 \leq 0, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0; \end{aligned}$$

(ix)
$$\begin{aligned} \text{Minimize} \quad & C = 4x_1 - 2x_2 + 2x_3 \\ \text{subject to} \quad & x_1 - x_2 - 2x_3 \geq 1, \\ & -x_1 - 2x_2 + 2x_3 \geq 4, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0; \end{aligned}$$

(x)
$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 2x_2 - 2x_3 \\ \text{subject to} \quad & 3x_1 + 3x_2 + 2x_3 \geq 8, \\ & 6x_1 + 4x_2 + 2x_3 = 18, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

7

Theory of Games

7.1 Fully-Determined Games

The "Theory of Games" is an area of mathematics that models conflicts. Many of the ideas are expressed in terms of games between two or more players—that is where the name originated—but the study also covers war and economic competition. In these cases, we assume that each player tries to win as much as possible. However, the methods developed can also be used in problems where we wish to know the best, or least risky, way to proceed in the face of uncertainty; we act as if life is a game and nature is a capricious and unpredictable player.

Two-Person, Zero-Sum Games

Suppose you bet with a friend. Usually the result is that the loser pays the winner, and no other payments occur. Therefore, the amount that the winner gains is the same as the amount his or her adversary loses. This is called a *zero-sum* situation, and the game is a *zero-sum game*.

A good, simple example of a zero-sum game is *matching pennies*. Each player puts a penny on the table, with either the head or the tail upward, but covers it so that the other player cannot see it. Then they reveal their choices. If the two coins match, Player I wins. If they are different, Player II wins.

Say the players decide to bet \$5 on each play. We shall represent the game by a matrix. We shall call Player I the *row Player*, and represent her possible choices by the rows of the matrix. The first row corresponds to Player I choosing to place her penny heads up, and the second to placing it heads down. Player II is the *column Player*. In each cell of the matrix, we write the outcome if the corresponding choices are made. By convention, we shall write the outcome for Player I. If Player I chooses heads and Player II chooses tails, then Player I loses \$5, and we put -5 in the row 1, column 2 position. The negative sign represents loss, so Player II's gains are shown

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
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by negative numbers and Player I's gains are shown by positives. Then our game is represented by the following matrix:

		Player II	
		H	*T*
Player I	*H*	5	−5
	T	−5	5

Sample Problem 7.1. *In the game of rock–paper–scissors, each Player chooses one of the three named items and they reveal them simultaneously. Rock beats ("breaks") scissors, scissors beat ("cut") paper, paper beats ("covers") rock. In the case of a tie, no money changes hands. Players I and II play rock–paper–scissors for \$2 per game. Write down the matrix.*

Solution.

		Player II		
		R	*S*	*P*
Player I	*R*	0	2	−2
	S	−2	0	2
	P	2	−2	0

Because of this representation, two-person, zero-sum games are often called *matrix games*. The matrix is called a *payoff table*, and the entries are the *payoffs* to Player I.

For convenience, we shall always allocate the rows of the matrix to Player I and the columns to Player II. (With this convention, there is no need to write the players' identities on the payoff matrix. We shall call them "the row player" and "the column player.") The rows of the payoff matrix are called the row player's *pure strategies*, and the columns are the column player's pure strategies. To play a game, each player selects a pure strategy at their turn. (The word *strategy* is used to describe a player's plan as to which row to choose at each turn.) We assume that the choice is made without knowledge of the other player's decision.

Domination

In some cases, there is only one sensible choice for a player. For example,

	1	2
1	2	3
2	2	−3

.

Player I will always choose row 1—strategy 1—because, in every case, it gives at least as good a result as row 2, and at least once the outcome is better. In this example, we say Player I's strategy is $(1, 0)$ to mean that she will always play row 1 and never play row 2.

When there is a dominant row, Player II knows which row will be chosen, so he ignores the other rows, and chooses the column whose entry in the dominant row is largest. He does not have a dominant column, but he will definitely choose column 1. The outcome of the game will be a payment of 2 to Player 1. We say the game has *value* 2. It is convenient to refer to the players' choice and the outcome as R1C1, value 2.

This process of finding the players' strategies and the value of the game is called *solving* the game. Later in this chapter, we shall generalize the ideas of strategy, value and solving the game to those cases where there are not unique pure strategies to be followed.

Sample Problem 7.2. *In the following matrix games, is there a dominant row or column? If so what are the players' strategies?*

(i)

	1	2
1	−2	3
2	2	−2

;

(ii)

	1	2	3
1	2	1	3
2	2	0	−1

.

Solution. (i) has no dominant row or column. In (ii) row 1 is dominant; the row player's strategy is (1, 0) and the column player's strategy is (0, 1, 0).

Your Turn. In the following matrix games, is there a dominant row or column? If so what are the players' strategies?

(i)

	1	2
1	1	3
2	2	1

;

(ii)

	1	2
1	2	1
2	2	−3

.

Consider the game

	1	2	3
1	2	1	0
2	1	0	2

.

There is no dominating column. However, every element of column 2 is smaller than the corresponding element—the one in the same row—in column 1. In row 1, $1 < 2$, and in row 2, $0 < 1$. (Remember, for the column player, smaller is better.) So the column player would never consider column 1; he would play as though the game were

	2	3
1	1	0
2	0	2

.

In this case, we say column 1 of the original game *is dominated by* column 2. A row or column that is dominated by another is called *recessive* and need not be considered in solving the game.

Sometimes a game can be reduced more than once by discovering dominance. For example, in the game

	1	2	3
1	−3	1	2
2	2	0	1
3	1	−2	0

there is no dominated row. However, column 3 is dominated by column 2. So Player II would consider the game

	1	2
1	−3	1
2	2	0
3	1	−2

.

Now row 3 is dominated by row 2, so it can be deleted, and both players try to solve the game

	1	2
1	−3	1
2	2	0

.

Saddle Points

From now on, we shall save space by omitting the names of the pure strategies. Assume each player's strategies are called 1, 2, ...

Consider the game

1	2	9
5	3	5
8	2	0

,

which has no dominated rows or columns. How should the row player proceed?

Suppose she is pessimistic. If she chooses row 1, it is possible to win as little as 1 unit. If row 3 is chosen, the gain might be as small as 0. But if she chooses row 2, a win of 3 or more is guaranteed. This reasoning suggests choosing row 2. This is called a *maximin* strategy: for each row, identify the minimum element, then choose rows so as to *maximize* this *minimum*. We say the (2, 2) element is the *maximin*.

If the column player thinks similarly, he sees that the maximum loss is 8 in column 1, 3 in column 2, and 9 in column 3. The *minimax* is again the (2, 2) element 3.

These calculations may be represented in the following way:

			MIN
1	2	9	1
5	3	5	∗3∗
8	2	0	0
MAX 8	∗3∗	9	

If the minimax over the columns (the pessimistic column player's choice) is equal to the maximin over the rows (the pessimistic row player's choice), and there is a unique element that achieves both values, then both players will choose this element. This is called a *minimax strategy*. If this occurs in a game, the matrix element commonly selected is called a *saddle point*. The example has saddle point (2, 2) with value 3. It is again convenient to refer to this solution by its position and value: we say R2C2, value 3.

Sometimes ties occur. For example, in the game

1	0
1	2

both entries in column 1 achieve the column maximum. However, only the (2, 1) entry is also a row minimum, so it is a saddle point.

Not all matrix games have saddle points. But if there is a saddle point, the value of the entry is the value of the game. The row player can be guaranteed at least this much profit and the column player can be guaranteed at most this much loss, so sensible play guarantees exactly this outcome. Games are called *strictly determined* if the exact outcome of each play is fixed provided the players choose an appropriate strategy. So games with saddle points are strictly determined.

Sample Problem 7.3. *Find a saddle point, if any, in*

2	1	−2
4	2	4
−2	1	0

.

Solution.

			MIN
2	1	−2	−2
4	2	4	∗3∗
−2	1	0	−2
MAX 4	∗3∗	4	

so there is a saddle point R2C2 with value 3.

Your Turn. Find a saddle point, if any, in

2	2	3
3	1	2
−2	0	1

Multiple Saddle-Points

Suppose a game has two saddle-points. For convenience, suppose they are the (1, 1) and (2, 2) entries. (The simpler case, where two saddle points occur in the same row, is left as an exercise.) The game looks like

a	b	...
c	d	...
	...	

As a and d are minimal in their rows, $b \geq a$ and $c \geq d$. From the column maximality, $a \geq c$ and $d \geq b$. So we have

$$a \geq c \geq d \geq b \geq a$$

and all four entries are equal. The game is still strictly determined. The row player can choose row 1 or row 2, and the column player can choose column 1 or column 2, but the payoff is the same.

This argument obviously applies whichever rows and columns contain the saddle points, and extends to three or more saddle points.

Exercises 7.1 A

1. Two players match fingers. Each shows one or two fingers, simultaneously. If the numbers of fingers shown by the two players is equal, the row player wins; otherwise the column player wins. The loser pays the winner the number of fingers shown (in dollars). Write down a matrix for this game.
2. You are playing a wargame. You plan to attack any one of Slovakia (1), Slovenia (2), or the Czech Republic (3). Your opponent can choose to defend any one of the three. If you choose to attack the same country that he chooses to defend, you will lose (and he will win) 200 points. If you choose different countries, you will gain (and he will lose) 100. Write down a matrix for this game, with you as the row player.
3. In each case, solve the game by domination:

(i)

−1	0
0	1

;

(ii)

3	−2
2	1

;

(iii) $\begin{array}{|ccc|}\hline 2 & 1 & 3 \\ 1 & -2 & 3 \\ \hline\end{array}$; (iv) $\begin{array}{|ccc|}\hline 1 & 2 & 0 \\ 1 & 0 & -1 \\ \hline\end{array}$;

(v) $\begin{array}{|ccc|}\hline 1 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ \hline\end{array}$; (vi) $\begin{array}{|ccc|}\hline 1 & -2 & 1 \\ 2 & 0 & 4 \\ 1 & 0 & 1 \\ \hline\end{array}$;

(vii) $\begin{array}{|ccc|}\hline 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \\ \hline\end{array}$; (viii) $\begin{array}{|ccc|}\hline 1 & 0 & 3 \\ -2 & 1 & -1 \\ 2 & 1 & 4 \\ \hline\end{array}$.

4. In each case, use domination to reduce the game to a 2×2 game:

(i) $\begin{array}{|ccc|}\hline -5 & -3 & 1 \\ 2 & -1 & 2 \\ -2 & 3 & 4 \\ \hline\end{array}$; (ii) $\begin{array}{|ccc|}\hline 1 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \\ \hline\end{array}$;

(iii) $\begin{array}{|ccc|}\hline 1 & 1 & -3 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \\ \hline\end{array}$; (iv) $\begin{array}{|ccc|}\hline 2 & 3 & 4 \\ 1 & 1 & 1 \\ 3 & -1 & 0 \\ \hline\end{array}$;

(v) $\begin{array}{|ccc|}\hline 2 & 1 & -2 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \\ \hline\end{array}$; (vi) $\begin{array}{|ccc|}\hline 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline\end{array}$;

(vii) $\begin{array}{|ccc|}\hline -2 & -1 & 1 \\ -1 & -2 & 1 \\ -2 & -2 & 0 \\ \hline\end{array}$; (viii) $\begin{array}{|ccc|}\hline 0 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \\ \hline\end{array}$.

5. In each case, solve the game by finding a saddle point:

(i) $\begin{array}{|ccc|}\hline -3 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & -1 & -2 \\ \hline\end{array}$; (ii) $\begin{array}{|ccc|}\hline 0 & 2 & -3 \\ -2 & 3 & 1 \\ 1 & 2 & 4 \\ \hline\end{array}$;

(iii) $\begin{array}{|ccc|}\hline 6 & -4 & -2 \\ 1 & -1 & 3 \\ -4 & -3 & 4 \\ \hline\end{array}$; (iv) $\begin{array}{|cccc|}\hline 0 & 1 & -2 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & -2 & 0 & 4 \\ \hline\end{array}$.

6. Suppose a game has two saddle-points in positions $(1, 1)$ and $(1, 2)$. Show that the game is strictly determined.

Exercises 7.1 B

1. Two players match fingers. Each shows one, two, or three fingers, simultaneously. If the total number of fingers shown is even, the row player wins; if it is odd the column player wins. The loser pays the winner the number of fingers shown (in dollars). Write down a matrix for this game.

2. Two players match fingers as in the preceding exercise. Each shows one, two, or three fingers, simultaneously, and the loser pays the winner the number of fingers shown (in dollars). If the total number of fingers shown is even, the player showing the greater number wins; if it is odd the player showing the smaller number wins; if the two show the same number, it is a tie. Write down a matrix for this game.

3. In each case, solve the game by domination:

(i) $\begin{array}{|cc|}\hline 3 & 2 \\ 0 & 2 \\ \hline\end{array}$; (ii) $\begin{array}{|cc|}\hline 1 & -1 \\ 2 & -2 \\ \hline\end{array}$;

(iii) $\begin{array}{|ccc|}\hline 1 & -1 & 0 \\ 1 & 2 & 2 \\ \hline\end{array}$; (iv) $\begin{array}{|ccc|}\hline 2 & 1 & 2 \\ 1 & 0 & 3 \\ \hline\end{array}$;

(v) $\begin{array}{|ccc|}\hline 1 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \\ \hline\end{array}$; (vi) $\begin{array}{|ccc|}\hline 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ \hline\end{array}$;

(vii) $\begin{array}{|ccc|}\hline 0 & 0 & 1 \\ -2 & 1 & -2 \\ -2 & 0 & -1 \\ \hline\end{array}$; (viii) $\begin{array}{|ccc|}\hline 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \\ \hline\end{array}$.

4. In each case, use domination to reduce the game to a 2×2 game:

(i) $\begin{array}{|ccc|}\hline 3 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \\ \hline\end{array}$; (ii) $\begin{array}{|ccc|}\hline 1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & -3 & -2 \\ \hline\end{array}$;

(iii) $\begin{array}{|ccc|}\hline 3 & 1 & 2 \\ 1 & 1 & -1 \\ -2 & -1 & 3 \\ \hline\end{array}$; (iv) $\begin{array}{|ccc|}\hline -1 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ \hline\end{array}$;

(v) $\begin{array}{|ccc|}\hline 1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 1 \\ \hline\end{array}$; (vi) $\begin{array}{|ccc|}\hline 1 & -2 & 2 \\ 2 & -1 & 3 \\ 0 & 3 & 1 \\ \hline\end{array}$;

(vii) $\begin{array}{|ccc|}\hline 1 & 2 & -2 \\ -3 & -2 & -1 \\ 1 & 3 & 0 \\ \hline\end{array}$; (viii) $\begin{array}{|ccc|}\hline 0 & -1 & 1 \\ 1 & 0 & -3 \\ 2 & 1 & 0 \\ \hline\end{array}$.

5. In each case, solve the game by finding a saddle point:

(i) $\begin{array}{|ccc|} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{array}$; (ii) $\begin{array}{|ccc|} 4 & 2 & -1 \\ 5 & 0 & 3 \\ 4 & 3 & 6 \end{array}$;

(iii) $\begin{array}{|ccc|} 4 & 8 & 3 \\ 7 & 5 & 4 \\ 2 & 1 & 4 \end{array}$; (iv) $\begin{array}{|cccc|} -5 & 2 & 15 & 3 \\ 0 & 6 & 10 & 5 \\ 5 & 5 & 5 & 6 \end{array}$.

7.2 2 × 2 Games

Most games do not have saddle points. This adds complications. For example, consider the game with matrix

$$\begin{array}{|cc|} 1 & -1 \\ -2 & 3 \end{array},$$

which has no saddle point.

How should the players proceed? The row player can guarantee a loss of at worst 1 unit—a payoff of at least −1—by selecting row 1. The column player can guarantee a loss of at worst 1 by choosing column 1. So we might expect to see row 1 and column 1 chosen, and a payoff of 1 to the row player.

But suppose the column player thought about his opponent's decision. Assuming row 1 is chosen, it will be better to choose column 2. So column 2 is chosen. But then the row player would do better to choose row 2 ... When there is no saddle point, the situation does not stabilize.

The point is that if one player's actions can be predicted, the other player can take advantage of this fact. The only solution is for each player to choose a row or column at random each time. However, this choice can be made in accordance with some probability distribution.

Suppose a game has m row strategies and n column strategies, and has matrix A with typical entry a_{ij}. The matrix A is of size $m \times n$. The row player chooses m non-negative numbers $x_1, x_2, \ldots, x_m$ whose sum is 1, and decides that at each turn she will select row i with probability x_i. Then $(x_1, x_2, \ldots, x_m)$ is called the *row strategy*; if two or more of the x_i are non-zero, this is called a *mixed strategy*. Similarly, the column player selects a strategy $(z_1, z_2, \ldots, z_n)$.

Once two strategies have been chosen, they can be interpreted as a probability distribution. The players act independently, so the probability that row i and column j will be chosen is $x_i z_j$. Therefore, the expected value of the payoff is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} z_j.$$

Mixed Strategies

We begin our general discussion with the simplest case, where both players have two pure strategies.

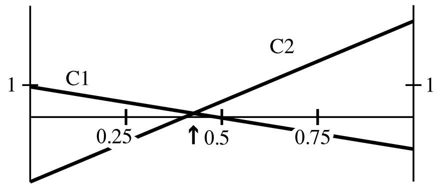
As a first example, consider the game

$$\boxed{\begin{array}{rr} 1 & -2 \\ -1 & 3 \end{array}}\,.$$

Suppose the row player uses strategy $(1 - x, x)$. If the column player selects column 1 (the pure strategy $(1, 0)$), the expected payoff is $1(1-x)+(-1)x$, or $1-2x$. If the second column—strategy (0, 1)—is used, the expected payoff is $(-2)(1-x)+3x$, or $5x - 2$. The expected payoff for column strategy $(1 - z, z)$ is

$$(1 - z)(1 - 2x) + z(5x - 2) = (7x - 3)z + (1 - 2x).$$

These facts are represented in the following diagram. The first vertical axis represents pure strategy "choose row 1"—the case $x = 0$—and the second represents pure strategy "choose row 2." The horizontal axis shows the value of x. The other lines correspond to the two pure column strategies and show the expected payoff, y say, to the row player plotted against x.



If you draw a vertical line at any given value of x, it is clear that the expected payout for that x must lie between the two values where the vertical line crosses the two column-strategy lines. In particular, if the row player chooses the x value corresponding to the arrow, the x value where the column-strategy lines intersect, the expected payoff can never be smaller than the y value at the intersection. If any other x is chosen, a smaller expected payoff is possible.

In the example, the lines meet when $1-2x = 5x-2$, or $x = \frac{3}{7}$. The corresponding expected payoff is $y = 1 - 2x = 1 - 2\frac{3}{7} = \frac{1}{7} = v$, say.

The column player wishes to minimize the expected payoff. He cannot make it smaller than v. He wants the payout to be v even when the row player chooses a pure strategy: that is, $x = 0$ or 1. This gives two equations:

$$\text{case } x = 0 \quad (7x - 3)z + (1 - 2x) = 1 - 3z = \frac{1}{7},$$
$$\text{case } x = 1 \quad (7x - 3)z + (1 - 2x) = 4z - 1 = \frac{1}{7},$$

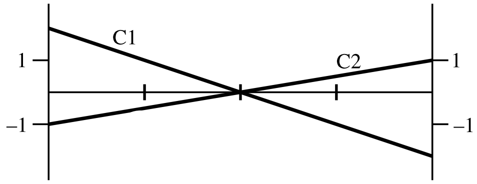
which both yield $z = \frac{2}{7}$. It is easy to check that the column strategy $(1 - \frac{2}{7}, \frac{2}{7}) = (\frac{5}{7}, \frac{2}{7})$ always gives expected payoff v.

Since both players can force an expected payoff no worse for them than v, this v is called the *value* of the game.

Sample Problem 7.4. *Use the above method to solve*

$$\begin{array}{|rr|} 2 & -1 \\ -2 & 1 \end{array}.$$

Solution. The diagram for this game is



The lines meet at $(\frac{2}{3}, 0)$. So the value is 0 and the row player's strategy is $(\frac{1}{3}, \frac{2}{3})$. For the column player, $2(1 - z) - 1z = 0$ so $z = \frac{2}{3}$ and the column strategy is also $(\frac{1}{3}, \frac{2}{3})$.

The lines meet at $(\frac{2}{3}, 0)$. So the value is 0 and the row player's strategy is $(\frac{1}{2}, \frac{1}{2})$. For the column player, the strategy is also $(\frac{1}{2}, \frac{1}{2})$.

Your Turn. Use the above method to solve

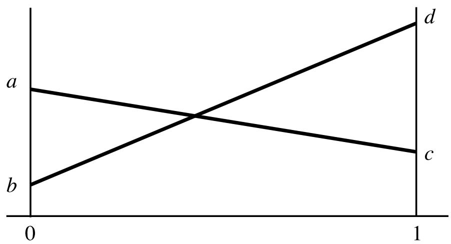
$$\begin{array}{|rr|} 2 & 0 \\ -2 & 4 \end{array}.$$

The General 2 × 2 Game

These principles can be applied to any 2×2 game. Consider the game

$$\begin{array}{|cc|} a & b \\ c & d \end{array}.$$

The corresponding diagram is



The x axis represents the row player's strategy $(1-x, x)$, ranging from $x = 0$ (the left vertical line, pure strategy row 1) to $x = 1$ (pure strategy, row 2). The y value is the payoff. The line from $(0, a)$ to $(1, c)$, whose equation is $y = a + (c-a)x$, shows the payoffs when column 1 is chosen; the line from $(0, b)$ to $(1, d)$, whose equation is $y = b + (d-b)x$, shows the payoffs when column 2 is chosen.

These two lines intersect when

$$a + (c-a)x = b + (d-b)x,$$

which simplifies to

$$x = \frac{a-b}{a+d-b-c},$$
$$y = \frac{ad-bc}{a+d-b-c}.$$

So the game has value

$$v = \frac{ad-bc}{a+d-b-c},$$

and the row player's best strategy is

$$\left(\frac{d-c}{a+d-b-c}, \frac{a-b}{a+d-b-c}\right).$$

Suppose the column player wants to ensure that the expected payoff is v. If he chooses strategy $(1-z, z)$ and the row player has chosen $(1-x, x)$, the expected payoff is $(1-z)(a+(c-a)x) + z(b+(d-b)x)$. We set this equal to v. In the case $x = 0$, we have

$$\frac{ad-bc}{a+d-b-c} = (1-z)a + zb,$$

and one can show that the solution is $z = \frac{a-c}{a+d-b-c}$, and the column player's best strategy is

$$\left(\frac{d-b}{a+d-b-c}, \frac{a-c}{a+d-b-c}\right).$$

Moreover, it can be seen that when z has this value, then $(1-z)(a+(c-a)x) + z(b+(d-b)x) = \frac{ad-bc}{a+d-b-c}$ for every possible x.

This method will not work when the game has a saddle point, and may give wrong answers. This is because one cannot assume in that case that the two lines will meet in the range $0 \le x \le 1$.

Sample Problem 7.5. *Solve the game*

1	−2
−1	3

by using the formulas.

Solution. For this game $a = 1, b = -2, c = -1, d = 3$. So $a + d - b - c = 1 + 3 + 2 + 1 = 7$. To find the row strategy, observe that $d - c = 4$, $a - b = 3$, so the row strategy is

$$\left(\frac{d-c}{a+d-b-c}, \frac{a-b}{a+d-b-c}\right) = \left(\frac{4}{7}, \frac{3}{7}\right).$$

As $d - b = 5$, $a - c = 2$, the column strategy is

$$\left(\frac{d-b}{a+d-b-c}, \frac{a-c}{a+d-b-c}\right) = \left(\frac{5}{7}, \frac{2}{7}\right).$$

The value is

$$v = \frac{ad-bc}{a+d-b-c} = \frac{1}{7}.$$

(Observe that this agrees with our earlier result.)

Your Turn. Solve the game

2	−1
−2	1

by using the formulas.

The formula method does not work when $a+d-b-c = 0$ because the quotients are not defined. However, this is no problem. If $a + d - b - c = 0$, the matrix has form

a	b
c	$b+c-a$

.

Suppose $a \geq b$. Then $c \geq b + c - a$. Therefore, column 2 dominates column 1. Therefore, the game has a saddle point. A similar result holds if $a \leq b$ (the proof is left as an exercise).

More generally, it is possible to show that a 2×2 game is not strictly determined *if and only if* each of the elements on one diagonal is greater than each of the elements on the other diagonal. Otherwise the game has a saddle point (Again, the proof is left as an exercise.)

Sample Problem 7.6. *For what values of a does the game*

a	2
1	−1

have a saddle point?

Solution. In order for the game to have no saddle point, each element of one diagonal must be greater than each element of the other diagonal. In this example, the possibility is that each of 1 and 2 be greater than each of a and -1. This requires $a < 1$. So there is a saddle point when $a \geq 1$.

Exercises 7.2 A

1. For the following games, either find a saddle point or solve by the diagram method:

(i) $\begin{array}{|rr|} 2 & -1 \\ -2 & 3 \end{array}$; (ii) $\begin{array}{|rr|} 3 & 4 \\ 2 & 1 \end{array}$;

(iii) $\begin{array}{|rr|} 3 & -2 \\ 1 & 2 \end{array}$; (iv) $\begin{array}{|rr|} 4 & -6 \\ -6 & 4 \end{array}$;

(v) $\begin{array}{|rr|} -2 & 2 \\ 0 & -2 \end{array}$; (vi) $\begin{array}{|rr|} 2 & -1 \\ -2 & -3 \end{array}$;

(vii) $\begin{array}{|rr|} 2 & -1 \\ -2 & 1 \end{array}$; (viii) $\begin{array}{|rr|} 3 & -1 \\ 1 & 2 \end{array}$;

(ix) $\begin{array}{|rr|} 2 & 4 \\ 4 & 2 \end{array}$; (x) $\begin{array}{|rr|} 2 & 6 \\ 4 & 1 \end{array}$;

(xi) $\begin{array}{|rr|} 4 & -1 \\ -2 & 2 \end{array}$; (xii) $\begin{array}{|rr|} 1 & -1 \\ 2 & -2 \end{array}$;

(xiii) $\begin{array}{|rr|} -2 & 3 \\ 1 & 2 \end{array}$; (xiv) $\begin{array}{|rr|} 2 & 1 \\ 0 & 3 \end{array}$;

(xv) $\begin{array}{|rr|} 2 & 3 \\ 3 & -4 \end{array}$; (xvi) $\begin{array}{|rr|} 4 & -1 \\ -5 & 2 \end{array}$.

2. For those games in the preceding Exercise that did not have saddle points, solve by the formula method.

3. For what values of a do the following games have saddle points?

(i) $\begin{array}{|rr|} a & 3 \\ 1 & 4 \end{array}$; (ii) $\begin{array}{|rr|} 3 & 3 \\ 1 & a \end{array}$;

(iii) $\begin{array}{|rr|} 4 & 3 \\ a & 2 \end{array}$; (iv) $\begin{array}{|rr|} 0 & a \\ -2 & 1 \end{array}$.

4. Verify that the solution of

$$\frac{ad-bc}{a+d-b-c} = (1-z)a + zb$$

is $z = \frac{a-c}{a+d-b-c}$.

5. Prove that if $z = \frac{a-c}{a+d-b-c}$ then

$$(1-z)\big(a+(c-a)x\big) + z\big(b+(d-b)x\big) = \frac{ad-bc}{a+d-b-c}$$

for every x with $0 \le x \le 1$.

Exercises 7.2 B

1. For the following games, either find a saddle point or solve by the diagram method:

(i) $\begin{array}{|rr|} 4 & -2 \\ -1 & 2 \end{array}$; (ii) $\begin{array}{|lr|} 3 & -1 \\ 0 & 2 \end{array}$;

(iii) $\begin{array}{|rr|} -3 & 2 \\ 1 & 0 \end{array}$; (iv) $\begin{array}{|rr|} 2 & -1 \\ -2 & 5 \end{array}$;

(v) $\begin{array}{|rr|} 4 & -2 \\ -6 & 4 \end{array}$; (vi) $\begin{array}{|lr|} 2 & 3 \\ 5 & -6 \end{array}$;

(vii) $\begin{array}{|lr|} 2 & 3 \\ 1 & -4 \end{array}$; (viii) $\begin{array}{|lr|} 3 & 2 \\ 1 & 4 \end{array}$;

(ix) $\begin{array}{|lr|} 1 & -1 \\ 0 & 3 \end{array}$; (x) $\begin{array}{|rr|} 2 & 1 \\ -1 & 3 \end{array}$;

(xi) $\begin{array}{|rr|} 3 & -2 \\ -1 & 2 \end{array}$; (xii) $\begin{array}{|lr|} 7 & 3 \\ 3 & 4 \end{array}$;

(xiii) $\begin{array}{|lr|} 2 & 1 \\ 0 & 3 \end{array}$; (xiv) $\begin{array}{|rr|} 2 & 1 \\ -1 & 4 \end{array}$;

(xv) $\begin{array}{|lr|} 2 & 3 \\ 5 & -4 \end{array}$; (xvi) $\begin{array}{|lr|} 2 & 3 \\ 0 & 4 \end{array}$.

2. For those games in the preceding Exercise that did not have saddle points, solve by the formula method.

3. For what values of a do the following games have saddle points?

(i) $\begin{array}{|cc|}\hline -4 & a \\ -4 & 2 \\ \hline\end{array}$; (ii) $\begin{array}{|cc|}\hline -2 & a \\ 3 & -1 \\ \hline\end{array}$;

(iii) $\begin{array}{|cc|}\hline 1 & -1 \\ 2 & a \\ \hline\end{array}$; (iv) $\begin{array}{|cc|}\hline a & 1 \\ 2 & 0 \\ \hline\end{array}$.

4. The game

$$\begin{array}{|cc|}\hline a & b \\ c & b+c-a \\ \hline\end{array}$$

satisfies $a+d-b-c=0$ and $a \le b$. Show that column 2 dominates column 1.

5. Prove that a 2×2 game has a saddle point if and only if the following is *not* true: each of the elements on one diagonal is greater than each of the elements on the other diagonal.

7.3 2 × *n* Games

The methods of the preceding section can be applied to games in which one player has two strategies but the other has more than two. We shall discuss $2 \times n$ games, that is, the row player has two choices, while the column player has n (and usually $n > 2$).

The Diagram Method

We generalize the diagram method, from the preceding section, to the case where there are more than two columns. Again, there are two vertical lines, distance 1 apart. The line $x = 0$ represents the case where the row player always chooses row 1, and the line $x = 1$ corresponds to column 2. The points with a given x value show expected payoffs when the row strategy is $(1-x, x)$.

The y coordinate shows the expected payoff. We draw a line for each pure column strategy: if the line passes through (x, y) that means the expected payoff, when the row player uses strategy $(1-x, x)$ and the column player always chooses the column in question, is y. If a column has first entry a and second entry d, the line for that column will pass through points $(0, a)$ and $(1, d)$. The diagram might look something like Figure 7.1(i).

Let's call a point (x, y) *reachable* if it is possible for the expected payoff to be y when the row strategy is $(1-x, x)$. In Figure 7.1(ii), the set of all reachable points is shaded. It follows that, no matter how she plays, the row player can always attain the expected payoff on the heavy line of Figure 7.2(i). Using the same reasoning as we did in the previous section, the row player should choose the x coordinate of the point indicated by the arrow, and she can guarantee an expected payoff at least as

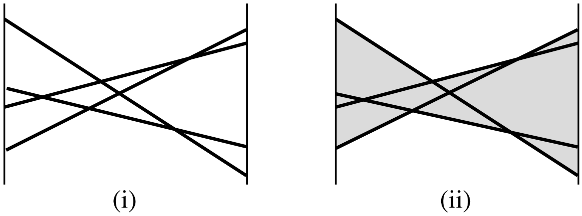


Fig. 7.1. The reachable points

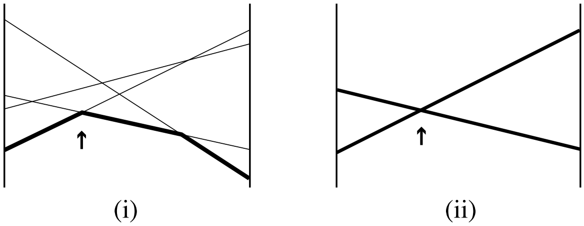


Fig. 7.2. Solving the game

great as the y-coordinate of that point. This highest point on the lower boundary of the reachable set will be called the *value point* of the game.

The value point was the intersection of two of the "column" lines. It follows that the calculations will be exactly the same as those for the game that contains only those two columns—the one shown in Figure 7.2(ii). The other columns (and the corresponding pure strategies) are called *recessive*.

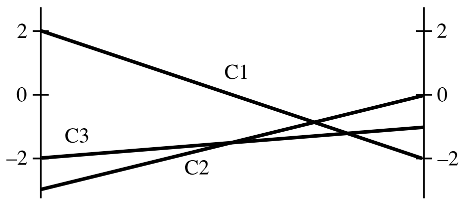
So here is how to solve a $2 \times n$ game. First, draw the diagram. Then eliminate the recessive columns. Finally, solve the resulting 2×2 game.

A dominated column is always recessive, so it is always worthwhile to check for those before drawing the diagram. And, of course, you should also check for saddle points.

Sample Problem 7.7. *Solve the following matrix game:*

$$\begin{array}{|ccc|} 2 & -3 & -2 \\ -2 & 0 & -1 \end{array}.$$

Solution. From the diagram



it is clear that column 2 can be deleted. The solution to

$$\begin{array}{|rr|} 2 & -2 \\ -2 & -1 \end{array}$$

is found (from the formula) to be row strategy $(\frac{1}{5}, \frac{4}{5})$, column strategy $(\frac{1}{5}, \frac{4}{5})$, value $-\frac{6}{5}$, so the original game has solution

$$\text{row strategy } \left(\frac{1}{5}, \frac{4}{5}\right), \quad \text{column strategy } \left(\frac{1}{5}, 0, \frac{4}{5}\right), \quad \text{value } -\frac{6}{5}.$$

Your Turn. Solve the following matrix game:

$$\begin{array}{|ccc|} 1 & 2 & 4 \\ 3 & 2 & 1 \end{array}\,.$$

$m \times 2$ Games

Suppose two players, Alice and Bob, play a game with the following matrix. Alice is row player, Bob is column player.

$$\begin{array}{|cc|} a & b \\ c & d \\ e & f \end{array}\,.$$

If the players had chosen the opposite roles—Alice chooses columns, and Bob chooses rows—the matrix representing the game would have two rows and three columns. Moreover, the payoffs need to be negated: in the original game, if Alice chose strategy 1 and Bob chose strategy 2, Alice would win b, the entry in row 1 and column 2, and Bob would win $-b$. In the new game, when Alice chooses strategy 1 and Bob chooses strategy 2, the result will be shown in the (2, 1) entry of the matrix. As Bob is the row player, the matrix should show his win, namely $-b$, as its (2, 1) entry. Applying this to all entries, we obtain the matrix

$$\begin{array}{|ccc|} -a & -c & -e \\ -b & -d & -f \end{array}\,.$$

To summarize:

> If the roles of row and column player in a matrix game are reversed, the matrix of the game is transposed and negated.

Suppose you need to solve a game with matrix A, size $m \times 2$. You could proceed as follows. First, solve the game with matrix A^T. The row player's strategy for A is the same as the column player's strategy for A^T; the column player's strategy for A is the same as the row player's strategy for A^T; and the value is the *negative* of the value for A^T (because the value for A^T is the expected payoff for the *column* player in the original game).

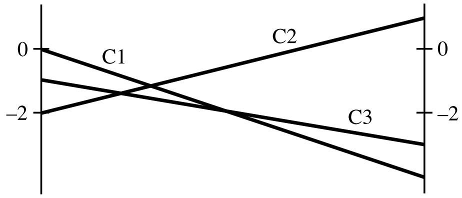
Sample Problem 7.8. *Solve the game*

$$\begin{bmatrix} 0 & 4 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

Solution. We start by solving

$$\begin{bmatrix} 0 & -2 & -1 \\ -4 & 1 & -3 \end{bmatrix}.$$

This game has diagram



so we delete column 1 and solve

$$\begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix}$$

by formula, obtaining row strategy $(\frac{4}{5}, \frac{1}{5})$, column strategy $(\frac{2}{5}, \frac{3}{5})$ and value $-\frac{7}{5}$. The 2×3 game has this row strategy and value, and column strategy $(0, \frac{2}{5}, \frac{3}{5})$. So the original had solution

$$\text{row strategy } \left(0, \frac{2}{5}, \frac{3}{5}\right), \quad \text{column strategy } \left(\frac{4}{5}, \frac{1}{5}\right), \quad \text{value } \frac{7}{5}.$$

Your Turn. Solve the following matrix game:

$$\begin{bmatrix} -2 & 2 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

Dominated Rows and Columns

Sometimes a game can be reduced to $2 \times n$ or $m \times 2$ form by deleting dominated rows and columns:

Sample Problem 7.9. *Solve the game*

$$\begin{bmatrix} 3 & 0 & 4 \\ 5 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Solution. Column 2 dominates column 1, so we reduce the game to

$$\begin{array}{|rr|} 0 & 4 \\ 2 & -1 \\ 1 & 3 \end{array}\,.$$

We solved this in Sample Problem 7.8; the solution was

$$\text{row strategy } \left(0, \frac{2}{5}, \frac{3}{5}\right), \quad \text{column strategy } \left(\frac{4}{5}, \frac{1}{5}\right), \quad \text{value } \frac{7}{5}.$$

Column 1 will never be used in the original game, so the solution is

$$\text{row strategy } \left(0, \frac{2}{5}, \frac{3}{5}\right), \quad \text{column strategy } \left(0, \frac{4}{5}, \frac{1}{5}\right), \quad \text{value } \frac{7}{5}.$$

Exercises 7.3 A

1. Solve the following games:

(i) $\begin{array}{|rrr|} 1 & 2 & -1 \\ -3 & 2 & 1 \end{array}$; (ii) $\begin{array}{|rrrr|} 1 & -2 & 3 & 1 \\ 4 & 1 & -2 & 0 \end{array}$;

(iii) $\begin{array}{|rrr|} -1 & 2 & -2 \\ 1 & -3 & 2 \end{array}$; (iv) $\begin{array}{|rrrr|} 2 & 3 & 1 & 2 \\ 1 & 0 & 4 & 3 \end{array}$.

2. Solve the following games:

(i) $\begin{array}{|rr|} 3 & -5 \\ -3 & 4 \\ 0 & -2 \end{array}$; (ii) $\begin{array}{|rr|} 2 & -2 \\ -3 & 0 \\ 1 & -1 \end{array}$;

(iii) $\begin{array}{|rrrr|} 0 & 6 & 10 & 2 \\ 8 & 0 & -2 & 4 \end{array}$; (iv) $\begin{array}{|rrrr|} 0 & 3 & -3 & 2 \\ 1 & -2 & 2 & 0 \end{array}$.

3. Solve the following games:

(i) $\begin{array}{|rrr|} 1 & 2 & 1 \\ -3 & 1 & 3 \end{array}$; (ii) $\begin{array}{|rrr|} 6 & -4 & 1 \\ -2 & 3 & 2 \end{array}$;

(iii) $\begin{array}{|rrr|} 0 & -1 & -2 \\ -6 & 2 & 8 \\ -1 & -1 & -4 \end{array}$; (iv) $\begin{array}{|rrr|} 3 & -3 & -2 \\ -2 & 1 & 2 \\ -1 & 1 & 1 \end{array}$.

4. Two players match coins. They can show a penny, a nickel, or a dime. If the total number of cents shown is even, the row player wins; if it is odd, the column player wins. The payoff is the value of the loser's coin.

(i) What is the matrix for this game? (Write the payoff in cents.)

(ii) Solve the game.

Exercises 7.3 B

1. Solve the following games:

(i) $\begin{array}{|ccc|} 1 & -3 & 4 \\ 2 & 4 & 1 \end{array}$; (ii) $\begin{array}{|ccc|} -2 & 4 & 3 \\ 6 & -3 & -1 \end{array}$;

(iii) $\begin{array}{|cccc|} 3 & -1 & -3 & -2 \\ -3 & -2 & 1 & -1 \end{array}$; (iv) $\begin{array}{|cccc|} -1 & 5 & -2 & 1 \\ 1 & -3 & 5 & -2 \end{array}$.

2. Solve the following games:

(i) $\begin{array}{|ccc|} 6 & 4 & -2 \\ -2 & -6 & 4 \end{array}$; (ii) $\begin{array}{|ccc|} 0 & -2 & 2 \\ -2 & 3 & -3 \end{array}$;

(iii) $\begin{array}{|ccc|} 1 & -2 & -3 \\ -3 & -1 & 3 \end{array}$; (iv) $\begin{array}{|cccc|} 3 & 6 & -1 & -3 \\ 1 & -2 & 3 & 4 \end{array}$.

3. Solve the following games:

(i) $\begin{array}{|cc|} 1 & -2 \\ -3 & 1 \\ -2 & 3 \end{array}$; (ii) $\begin{array}{|cc|} -1 & 4 \\ 1 & 2 \\ 3 & 1 \end{array}$;

(iii) $\begin{array}{|cc|} 0 & 3 \\ 2 & 0 \\ 4 & -2 \\ 0 & 1 \end{array}$; (iv) $\begin{array}{|cc|} 2 & -3 \\ -1 & 0 \\ -4 & 1 \\ 1 & -1 \end{array}$.

4. Solve the following games:

(i) $\begin{array}{|ccc|} 1 & -2 & 0 \\ 2 & -1 & 1 \\ -4 & 3 & 2 \end{array}$; (ii) $\begin{array}{|ccc|} 2 & -3 & -4 \\ -2 & 0 & -3 \\ 0 & 1 & 1 \end{array}$.

5. A game proceeds as follows. The row player selects one of two cards, labeled 1 and 2. The column player selects one of six cards, labeled 1, 2, 3, 4, 5, and 6. They show the cards simultaneously. If the sum is odd, the row player receives the sum of the two numbers shown (in dollars). If it is even, the column player receives the sum.

(i) Write down the matrix for this game.

(ii) Solve the game.

6. A game proceeds as follows. The row player selects one of two cards, labeled "odd" and "even". The column player selects one of six cards, labeled 1, 2, 3, 4, 5, and 6. They show the cards simultaneously. If the number is odd and the row player chose "odd", or if the sum is even and the row player chose "even", the row player receives the number shown (in dollars); if the row player chose the wrong parity, the column player receives the number (in dollars).

(i) Write down the matrix for this game.

(ii) Solve the game.

(iii) Suppose the players decide to pay the amount shown *plus one dollar*. (For example, if 3 is shown, the winner receives \$4.) Write down the matrix and solve the game.

7.4 General Matrix Games

In the general case, we consider a game with m row strategies and n column strategies. Suppose this game has matrix A with typical entry a_{ij}. The matrix A has size $m \times n$. The row player chooses m non-negative numbers $x_1, x_2, \dots, x_m$ whose sum is 1, and decides that at each turn she will select row i with probability x_i. Then $(x_1, x_2, \dots, x_m)$ is called the *row strategy*. Similarly, the column player selects a strategy $(z_1, z_2, \dots, z_n)$. As we said in Section 7.2, if two or more of the probabilities are non-zero, the strategy is called *mixed*.

Once two strategies have been chosen, they can be interpreted as a probability distribution. The players act independently, so the probability that row i and column j will be chosen is $x_i z_j$. Therefore, the expected value of the payoff is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} z_j.$$

The row player would like to find a strategy $(x_1, x_2, \dots, x_m)$ and a value $\underline{v}$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} z_j \ge \underline{v}$$

for every possible choice of column strategy. The column player would like to find a strategy $(z_1, z_2, \dots, z_n)$ and a value $\overline{v}$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} z_j \le \overline{v}$$

for every possible choice of row strategy.

Both players can be satisfied. The following remarkable result is called the *minimax theorem* of game theory.

Theorem 24. *For any matrix game A, there exists a value v such that*

(i) *there is a row strategy $(x_1, x_2, \ldots, x_m)$ such that $\sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} z_j \geq v$ for every column strategy $(z_1, z_2, \ldots, z_n)$;*

(ii) *there is a column strategy $(z_1, z_2, \ldots, z_n)$ such that $\sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} z_j \leq v$ for every row strategy $(x_1, x_2, \ldots, x_m)$.*

Solving a Game by Linear Programming

We shall look at a typical matrix game, with matrix

$$\boxed{\begin{matrix} 6 & 3 & 2 \\ 1 & 2 & 4 \end{matrix}}\,.$$

This game has only two rows, so we could solve it by diagram, but the methods we explore here could be used with a game of any size. Write v be the value of this game. All the elements are positive, so v will be positive also.

Suppose the column player uses strategy (z_1, z_2, z_3). If the row player uses a pure strategy, the payoff will be as follows:

$$\begin{aligned} &\text{for row 1,} \quad 6z_1 + 3z_2 + 2z_3; \\ &\text{for row 2,} \quad z_1 + 2z_2 + 4z_3. \end{aligned}$$

If the row player follows strategy (x_1, x_2) then the payoff is

$$(6z_1 + 3z_2 + 2z_3)x_1 + (z_1 + 2z_2 + 4z_3)x_2.$$

The column player wishes to choose his strategy so that

$$\begin{aligned} 6z_1 + 3z_2 + 2z_3 &\leq v, \\ z_1 + 2z_2 + 4z_3 &\leq v, \end{aligned}$$

as this will ensure a payoff of at most v. Of course, he doesn't yet know the value of v, but wishes it to be as small as possible.

We solve this by setting up some new variables, y_1, y_2, y_3, where $y_i = z_i/v$ write $\boldsymbol{v}$ for the column vector $(y_1, y_2, y_3)^T$. Then $y_1+y_2+y_3 = \frac{1}{v}$ because $z_1+z_2+z_3 = 1$, and in order to minimize v it suffices to maximize $\frac{1}{v}$. Notice that we need $v > 0$ here in order to divide through by v; if it were negative, then maximizing and minimizing would be mixed up, and if it were zero then division is impossible.

But what we have is a linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & P = y_1 + y_2 + y_3 \\ \text{subject to} \quad & 6y_1 + 3y_2 + 2y_3 \le 1, \\ & y_1 + 2y_2 - 4y_3 \le 1, \\ & y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0 \end{aligned}$$

(the non-negativity conditions arise because the z_i are non-negative, and v is positive).

So we solve this linear programming problem, and obtain the value of the game and the column player's optimal strategy. We add two slack variables, forming the tableau:

$$\left[\begin{array}{c|cccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & P & b \\ \hline y_4 & 6 & 3 & 2 & 1 & 0 & 0 & 1 \\ y_5 & 1 & 2 & 4 & 0 & 1 & 0 & 1 \\ \hline & -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right].$$

We pivot on row y_4, column y_2, obtaining

$$\left[\begin{array}{c|cccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & P & b \\ \hline y_2 & 2 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ y_5 & -3 & 0 & \frac{8}{3} & -\frac{2}{3} & 1 & 0 & \frac{1}{3} \\ \hline & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & \frac{1}{3} \end{array}\right].$$

We next pivot on row y_5, column y_3, obtaining

$$\left[\begin{array}{c|cccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & P & b \\ \hline y_2 & \frac{11}{4} & 1 & 0 & \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{4} \\ y_3 & -\frac{9}{8} & 0 & 1 & -\frac{1}{4} & \frac{3}{8} & 0 & \frac{1}{8} \\ \hline & \frac{5}{8} & 0 & 0 & \frac{1}{4} & \frac{1}{8} & 1 & \frac{3}{8} \end{array}\right],$$

and the problem is finished. The linear program had value $\frac{3}{8}$, so $v = \frac{8}{3}$ is the value of the game. The solution was achieved at $(y_1, y_2, y_3) = (0, \frac{1}{4}, \frac{1}{8})$, so the column player's strategy is

$$(z_1, z_2, z_3) = v(y_1, y_2, y_3) = \frac{8}{3}\left(0, \frac{1}{4}, \frac{1}{8}\right) = \left(0, \frac{2}{3}, \frac{1}{3}\right).$$

Now consider the bottom line of the last tableau. It may be interpreted as meaning that

$$\frac{5}{8}y_1 + \frac{1}{4}y_4 + \frac{1}{8}y_5 + P = \frac{3}{8}$$

is always true. In particular, at the optimal point, when $\boldsymbol{y} = (0, \frac{1}{4}, \frac{1}{8})$ and $P = \frac{3}{8}$,

$$\frac{1}{4}y_4 + \frac{1}{8}y_5 = 0.$$

Let us write A_i for row i of the game matrix A. From the original problem, $y_4 = 1 - A_1\boldsymbol{y}$ and $y_5 = 1 - A_2\boldsymbol{y}$. So

$$\frac{1}{4}y_4 + \frac{1}{8}y_5 = \left(\frac{1}{4}1 - A_1\boldsymbol{y}\right) + \frac{1}{8}(y_5 = 1 - A_2\boldsymbol{y}) = \frac{3}{8}\left[1 - \left(\frac{1}{4}, \frac{1}{8}\right)A\boldsymbol{y}\right],$$

so $(\frac{2}{3}, \frac{1}{3})A\boldsymbol{y}$ attains the optimal value $\frac{8}{3}$ for any value of $\boldsymbol{y}$, including the optimal strategy. So $(\frac{2}{3}, \frac{1}{3})$ is the optimal row strategy.

Games with Negative Entries

In the above discussion, it was essential that the game have positive value because we needed to divide by the value. If the game matrix contains zero or negative elements, the value might not be positive. For example, suppose we need to solve

$$\begin{array}{|ccc|} \hline 2 & -1 & -2 \\ -3 & -2 & 0 \\ \hline \end{array}\,.$$

First, we add a constant to every entry in order to make a new game, the *revised game*, with all numbers positive. This is an easy way to ensure a positive value. In the example, we add 4 throughout, obtaining revised game

$$\begin{array}{|ccc|} \hline 6 & 3 & 2 \\ 1 & 2 & 4 \\ \hline \end{array}$$

—the game we just solved. The optimal strategies for the original game are the same as those for the revised game: row strategy $(\frac{2}{3}, \frac{1}{3})$, column strategy $(0, \frac{2}{3}, \frac{1}{3})$. To find the value of the original game, subtract 4 from the value of the revised game: the original has value $\frac{8}{3} - 4 = -\frac{4}{3}$.

Sample Problem 7.10. *Solve the following game*

$$\begin{array}{|ccc|} \hline 1 & -1 & 0 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \\ \hline \end{array}\,.$$

Solution. We first add 2, and solve the revised game

$$\begin{array}{|ccc|} \hline 3 & 1 & 2 \\ 1 & 3 & 1 \\ 3 & 1 & 4 \\ \hline \end{array}\,.$$

For this, we start with the tableau

BV	y_1	y_2	y_3	y_4	y_5	y_6	P	b
y_4	3	1	2	1	0	0	0	1
y_5	1	3	1	0	1	0	0	1
y_6	3	1	4	0	0	1	0	1
	−1	−1	−1	0	0	0	1	0

We pivot on row y_6, column y_3:

BV	y_1	y_2	y_3	y_4	y_5	y_6	P	b
y_4	$\frac{3}{2}$	$\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
y_5	$\frac{1}{4}$	$\frac{11}{4}$	0	0	1	$-\frac{1}{4}$	0	$\frac{3}{4}$
y_3	$\frac{3}{4}$	$\frac{1}{4}$	1	0	1	$\frac{1}{4}$	0	$\frac{1}{4}$
	$-\frac{1}{4}$	$-\frac{3}{4}$	0	0	0	$\frac{1}{4}$	1	$\frac{1}{4}$

The next pivot is the (y_4, y_1) position:

BV	y_1	y_2	y_3	y_4	y_5	y_6	P	b
y_1	1	$\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$
y_5	0	$\frac{8}{3}$	0	$-\frac{1}{6}$	1	$-\frac{1}{6}$	0	$\frac{2}{3}$
y_3	0	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0
	0	$-\frac{2}{3}$	0	$\frac{1}{6}$	0	$-\frac{1}{6}$	0	$\frac{1}{3}$

The final pivot is (y_5, y_2):

BV	y_1	y_2	y_3	y_4	y_5	y_6	P	b
y_1	1	0	0	$\frac{11}{16}$	$-\frac{1}{8}$	$-\frac{5}{16}$	0	$\frac{1}{4}$
y_2	0	1	0	$-\frac{1}{16}$	$\frac{3}{8}$	$-\frac{1}{16}$	0	$\frac{1}{4}$
y_3	0	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0
	0	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{2}$

So the revised game has value 2, and the original game has value 0. The column player's strategy is $2(\frac{1}{4}, \frac{1}{4}, 0) = (\frac{1}{2}, \frac{1}{2}, 0)$, and the row player's strategy is $2(\frac{1}{8}, \frac{1}{4}, \frac{1}{8}) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

Your Turn. Solve the following matrix game by the simplex method.

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}.$$

The General Solution

In general we proceed as follows. Given a game with matrix A, of size $m \times n$, with every entry positive, solve the following linear program by the simplex method:

$$
\begin{aligned}
\text{Maximize} \quad & P = y_1 + y_2 + \cdots + y_n \\
\text{subject to} \quad & a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \le 1, \\
& a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \le 1, \\
& \vdots \\
& a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \le 1, \\
& y_1 \ge 0, \quad y_2 \ge 0, \quad \ldots, \quad y_n \ge 0.
\end{aligned}
$$

Suppose the maximum value of P is P^*, attained by $y_1 = y_1^*$, $y_2 = y_2^*, \ldots, y_n = y_n^*$, and suppose that in the final tableau the element in the last row (objective row) in column y_{n+i}, the column for the ith slack variable, is c_i. Then the game has value $1/P^*$, the row player has optimal strategy $(c_1/P^*, c_2/P^*, \ldots, y_m/P^*)$ and the column player has optimal strategy $(y_1^*/P^*, y_2^*/P^*, \ldots, y_n^*/P^*)$. If the original game had smallest entry $-m \le 0$, solve the game with matrix $A + m + 1$; the value of the original game was $\frac{1}{P^*} - m - 1$.

In order for the general solution method to work, it is necessary that the proposed "row player's strategy" $(c_1/P^*, c_2/P^*, \ldots, y_m/P^*)$ satisfies two conditions: the entries must be non-negative and they must add to 1. Non-negativity is obvious: P^* is positive, and if any of the c_i were negative there would be another pivot available. To see that the entries add to 1, observe that if the pivoting process yields value c_i in column y_{n+i}, then we have added $c_i \times$ row i to the objective row, so we have added c_i to the last element of that row (remember, every row has right-hand entry 1). So the entry on the right of the objective row is $\sum c_i$. So $\sum c_i = p^*$, and $\sum c_i/p^* = 1$.

Exercises 7.4 A

1. Solve the games in Exercise 7.2A1 using linear programming.
2. Solve the games in Exercises 7.3A1, 7.3A2, and 7.3A3 using linear programming.
3. Solve the following games using linear programming:

(i) $\begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}$; (ii) $\begin{bmatrix} 3 & -4 & 3 \\ 3 & 2 & -6 \\ -5 & -2 & 4 \end{bmatrix}$;

(iii) $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$; (iv) $\begin{bmatrix} 1 & -4 & 2 \\ 2 & -3 & -2 \\ 1 & 2 & 0 \end{bmatrix}$.

4. Solve the following games using linear programming:

(i) $\begin{array}{|rrr|} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{array}$; (ii) $\begin{array}{|rrrr|} 1 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{array}$;

(iii) $\begin{array}{|rrr|} 3 & 1 & -5 \\ -1 & 1 & 2 \\ -1 & -2 & 2 \end{array}$; (iv) $\begin{array}{|rrrr|} 2 & 1 & -2 & -1 \\ -1 & 2 & 1 & 2 \\ -1 & -2 & 3 & -3 \end{array}$.

5. Solve the following games using linear programming:

(i) $\begin{array}{|rrrr|} -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 2 \\ -1 & 2 & 0 & -1 \\ 1 & 2 & 2 & -2 \end{array}$; (ii) $\begin{array}{|rrr|} 3 & 1 & 0 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{array}$;

(iii) $\begin{array}{|rrrr|} 7 & 1 & -1 & 3 \\ 2 & 5 & -2 & -1 \\ 3 & 2 & 4 & -4 \\ -3 & 4 & -3 & 1 \end{array}$; (iv) $\begin{array}{|rrrr|} 1 & 1 & -1 & 2 \\ 2 & -2 & -2 & -1 \\ 1 & 2 & 2 & -1 \\ -3 & -4 & 3 & 1 \end{array}$.

Exercises 7.4 B

1. Solve the games in Exercise 7.2B1 using linear programming.

2. Solve the games in Exercises 7.3B1, 7.3B2, 7.3B3, and 7.3B4 using linear programming.

3. Solve the following games using linear programming:

(i) $\begin{array}{|rrr|} -1 & 2 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & -1 \end{array}$; (ii) $\begin{array}{|rrr|} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{array}$;

(iii) $\begin{array}{|rrr|} -2 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & 2 & -1 \end{array}$; (iv) $\begin{array}{|rrr|} 0 & -2 & 3 \\ -2 & 2 & -1 \\ 3 & 1 & -3 \end{array}$.

4. Solve the following games using linear programming:

(i) $\begin{array}{|rrr|} 3 & -4 & 3 \\ 3 & 2 & -2 \\ -2 & -2 & 4 \end{array}$; (ii) $\begin{array}{|rrr|} -1 & 2 & -2 \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{array}$;

(iii) $\begin{array}{|rrr|}\hline 3 & -3 & -2 \\ -1 & 4 & -1 \\ 1 & -2 & 2 \\ \hline\end{array}$; (iv) $\begin{array}{|rrr|}\hline 1 & -2 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -1 \\ \hline\end{array}$.

5. Solve the following games using linear programming:

(i) $\begin{array}{|rrrr|}\hline 0 & -1 & 3 & 2 \\ -1 & 1 & -3 & -1 \\ 1 & 4 & 0 & -1 \\ \hline\end{array}$; (ii) $\begin{array}{|rrrr|}\hline 1 & 3 & -1 & 2 \\ -2 & 0 & 2 & -1 \\ 3 & 2 & -3 & -2 \\ \hline\end{array}$;

(iii) $\begin{array}{|rrrr|}\hline 1 & -4 & 2 & 3 \\ 2 & -3 & -2 & -1 \\ 1 & 2 & 0 & -1 \\ \hline\end{array}$; (iv) $\begin{array}{|rrrr|}\hline 1 & -1 & 2 & 2 \\ 2 & 1 & -2 & -1 \\ 1 & 3 & 1 & -1 \\ \hline\end{array}$.

6. Solve the following games using linear programming:

(i) $\begin{array}{|rrrr|}\hline -1 & 3 & 1 & 0 \\ -1 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \\ 1 & -2 & 2 & -2 \\ \hline\end{array}$; (ii) $\begin{array}{|rrrr|}\hline 2 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 4 \\ \hline\end{array}$;

(iii) $\begin{array}{|rrrr|}\hline 3 & -2 & 1 & -3 \\ -1 & 1 & 1 & -1 \\ 0 & -2 & 2 & 2 \\ 2 & -2 & -2 & 1 \\ \hline\end{array}$; (iv) $\begin{array}{|rrrr|}\hline 2 & 1 & -1 & 3 \\ 2 & 3 & -2 & -1 \\ 3 & 2 & 2 & -4 \\ -3 & 4 & -3 & 1 \\ \hline\end{array}$.

8

Financial Mathematics

8.1 Simple Interest

In this chapter, we study the mathematics of finance: interest, investments and loans. There are some very deep mathematical ideas related to finance, and a full study is well beyond our scope. For example, many topics require the use of calculus. In our brief overview, we can only touch on a few elementary topics. Even for this, a calculator will be essential.

In all our financial calculations, we use the idea of percentages. Recall that "per cent" means "out of 100". Therefore, $R\%$ is just another way of saying $\frac{R}{100}$.

How Interest Works

The idea of simple interest is well-known. Suppose you put \$100 in a bank account, and at the end of one year the bank pays you back your \$100 plus \$6. This is called *interest* on your investment, and the rate is 6%.

This interest rate is always stated in terms of the annual interest. Suppose you put your \$100 in the bank for three months, and receive \$1.50 in interest. This is 1.5% of your original investment, but the rate is still quoted as 6%, the equivalent rate if the money had been kept for a year.

Sometimes you reinvest your money; in the first example, you would have \$106 in your account at the beginning of year 2. If this process is carried out automatically by the bank, it is called *compounding*. We shall discuss compounding and compound interest in the next section. *Simple interest* is just the case where compounding does not occur. The obvious model is where you take out the interest and spend it as, for example, when somebody retires and lives on the interest from their savings.

The arithmetic of borrowing money (loans, mortgages) is similar to that for investing. Typically, you do not wait until the end of a loan period to pay back a loan.

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1_8,

The usual practice is to pay equal amounts each month (or each week or ...). For this reason, most loans involve compound interest. However, some loans use simple interest. We shall give examples at the end of this section.

The original amount you borrow is called the *principal*, or *present value* of an investment or loan. Suppose you draw simple interest on a principal $\$P$ for n years at R\$ interest. The total amount you would receive is $\$A$, where

$$A = P\left(1 + n\frac{R}{100}\right) = P(1 + nr).$$

(The fraction $r = R/100$ is sometimes more convenient to use.) The total interest is $\$I$, where

$$I = Pn\frac{R}{100} = Pnr.$$

When dealing in periods shorter than a year, it is common to calculate as though the year consisted of 12 months, each of 30 days. This 360 day "year" is called a "standard" year. The regular 365 day year is called an "exact" year. Simple interest may be calculated for part of a year—n need not be an integer—and in that case, standard years are used. In the case of an investment, simple interest is usually paid at the end of each year, but sometimes at the end of the loan period; for a loan, both interest and principal are typically paid at the end of the whole loan period. (Compound interest is normally used in cases where parts are paid throughout the loan period.)

Sample Problem 8.1. *You borrow* \$1600 *at* 12% *simple interest for four months. How much must you pay at the end of the period? How much would you pay if the loan were for two years?*

Solution. We use the formula $A = P(1+nR/100)$ with $P = 1600$ and $R = 12$. In the first case, n is $\frac{1}{3}$ (four months is one-third of a year), so

$$A = 1600 \times (1 + 0.12/3) = 1600 \times (1.04) = 1664$$

and you repay \$1664. In the second case, n is 2, so

$$A = 1600 \times (1 + 0.12 \times 2) = 1600 \times (1.24) = 1984$$

and you repay \$1984.

Your Turn. You borrow \$1200 at 10% simple interest for three months. How much must you pay at the end of the period? How much would you pay if the loan were for three years?

Using the "Simple Interest" Formula

The simple interest formula can be inverted. Given the final amount, interest rate and term, you can calculate the principal:

$$P = A\left(1 + n\frac{R}{100}\right).$$

Similarly, you can calculate the interest rate from the other data.

Sample Problem 8.2. *You borrow* \$1800 *at simple interest for six months. At the end of the period you owe* \$1863*, including the principal. What was the interest rate?*

Solution. Again we use $A = P(1 + nR/100)$. We know $A = 1863$, $P = 1800$, and $n = \frac{1}{2}$. So

$$1863 = 1800 \cdot \left(1 + \frac{R}{200}\right) = 1800 + 9 \cdot R,$$

and therefore $9 \cdot R = 63$, $R = 7$. So the interest rate was 7%.

Your Turn. You borrow \$1400 at simple interest for four months. At he end of the period you owe \$1428, including the principal. What was the interest rate?

Sample Problem 8.3. *You need to borrow some money for three months. Your lender offers a rate of* 12%*. At the end of the period you repay* \$824*. How much was the principal?*

Solution. Using the formula with $R = 12$ and $n = \frac{1}{4}$, we get

$$824 = P(1 + 12/400), \qquad A = \frac{824}{1.03} = 800.$$

The principal was \$800.

Sample Problem 8.4. *You borrow* \$1400 *at simple interest for three years. At the end of that period, your interest is* \$210*. What was the interest rate?*

Solution. In this case $P = 1400$, $I = 210$, and $n = 3$. So

$$210 = 1400 \cdot 3 \cdot \frac{R}{100} = 42R; \qquad R = \frac{210}{42} = 5.$$

The rate was 5%.

Your Turn. You borrow \$2200 at simple interest for two years. At the end of that period, you owe a total of \$2233. What was the interest rate?

Add-on Loans

One example of a loan calculated with simple interest is the *add-on loan*. In an add-on loan, you add the whole amount of simple interest to the principal and pay off that total. If the principal is $\$P$, the interest is $R\%$, and the period is n years, then the total to be paid back is $\$P(1 + \frac{nR}{100})$.

Very often these loans are for short periods, and in those cases the interest is high.

Sample Problem 8.5. *What is your monthly payment on an add-on loan if you borrow* \$12000 *over five years at* 8% *per year?*

Solution. Simple interest is \$960 per year, so the total (simple) interest is for five years is \$4800. Therefore, the total to be paid is \$16800. There are 60 monthly payments, so your monthly payment will be \$16800/60, which is \$280.

Your Turn. What is your monthly payment on an add-on loan if you borrow \$500 over three months at 36% per year?

Discounted Loans

If the interest is $r\%$, then in a *discounted loan* you subtract the interest from the amount borrowed. Suppose your loan says you will borrow $\$P$ at an interest rate of $R\%$, and the period is n years. Instead of $\$P$ you receive $\$P(1 - Rn/100)$. For example, if your loan has principal \$12000 over 5 years at 8%, you only receive

$$\$12000 \times (1 - 0.4) = \$7200.$$

At the end of the period you repay the original principal, \$12000 in the example.

These loans are sometimes used by auto sales companies, for lease agreements with an option to buy.

Sample Problem 8.6. *You need to pay* \$12000. *What will be your payments for a five-year discounted loan at* 8% *per year?*

Solution. If you need \$12000 then your principal will be $\$A$ where

$$A \times (1 - 0.4) = 12000$$

so $\$A = \$(12000/0.6) = 20000$. Your monthly payments total \$20000 over 60 months, so they equal \$333.33 per month, to the nearest cent. (In actual fact, you would probably pay \$333.34 per month, with a last payment of \$332.94, or maybe \$334 per month, with the last payment adjusted down.)

Observe the difference between the payments in Sample Problems 8.5 and 8.6. This is not an isolated example. An add-on loan is always better than a discounted loan at the same (non-zero) interest rate.

To see this, suppose you need \$100. If you borrow \$100 for n years at $R\%$ interest, using an add-on loan, you eventually pay $\$100(1+\frac{nR}{100}) = \$(100+nR)$. In order to obtain \$100 using a discounted loan at $R\%$, your "principal" is $\$P$, where $P(1-nR/100) = 100$.

Suppose the discounted loan were as good a deal as the add-on. Then $P \le 100 + nR$. Then

$$\begin{aligned}100 &= P(1-nR/100) \le (100+nR)(1-nR/100)\\ &= (100+nR)(100-nR)/100\end{aligned}$$

from which

$$10000 \le (100+nR)(100-nR) = 10000 - n^2R^2.$$

This would mean $n^2R^2 \le 0$. This is never true.

In any case, for a discounted loan or an add-on loan to be worthwhile, the interest rate must be low. They are better for short-term loans.

Exercises 8.1 A

1. You borrow \$5000 at 7% simple interest. What is the total you must pay if the loan is for a period of:

(i) One year; (ii) Three years; (iii) Five years?

2. You borrow \$4800 at 4% simple interest. How much is the interest if the loan is for a period of:

(i) Two years; (ii) Five years; (iii) Eight years?

3. You borrow \$12000 at 4% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the total you must pay if the loan is for a period of:

(i) One month; (ii) Three months; (iii) Five months?

4. You borrow \$2500 at 6% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the total you must pay if the loan is for a period of:

(i) One month; (ii) Two months; (iii) Six months?

5. You borrow \$500 at simple interest for two years. At the end of the loan, you owe \$80 in interest. What was the interest rate?

6. You borrow \$1200 at simple interest for six months. At the end of the loan, you must repay a total of \$1260. What was the interest rate?

7. You borrow some money at 7% simple interest. (Assume the interest is calculated on a standard (360-day) year.) How much did you borrow, if:

(i) Your total repayment at the end of two years is \$2508;

(ii) Your total repayment at the end of eight months is \$1570;

(iii) The interest you owe at the end of six months is \$42?

8. You borrowed \$1000 from a loan company at simple interest. After three months, your total debt was \$1060. What interest did they charge?

9. You buy a treasury bill for \$980. After three months, you sell it back to the Government for \$1000. What was the interest rate?

10. What is your monthly payment on an add-on loan if you borrow \$3000 over three years at 12% per year?

11. What is your monthly payment on an add-on loan if you borrow \$1200 over six months at 36% per year?

12. You need to pay \$24000. How much must you borrow for a five-year discounted loan at 6%? What will be your monthly payments?

13. You need to pay \$16000. How much must you borrow for a three-year discounted loan at 9%? What will be your monthly payments?

Exercises 8.1 B

1. You borrow \$4000 at 8% simple interest. What is the total you must pay if the loan is for a period of:

(i) One year; (ii) Three years; (iii) Four years?

2. You borrow \$8000 at 3% simple interest. What is the total interest if the loan is for a period of:

(i) Two years; (ii) Three years; (iii) Six years?

3. You borrow \$2500 at 5% simple interest. What is the total you must pay if the loan is for a period of:

(i) Three years; (ii) Four years; (iii) Seven years?

4. You borrow \$3000 at 5% simple interest. What is the total interest if the loan is for a period of:

(i) Two years; (ii) Four years; (iii) Five years?

5. You borrow \$2000 at 3% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the total you must pay if the loan is for a period of:

(i) One month; (ii) Four months; (iii) Six months?

6. You borrow \$4800 at 5% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the interest if the loan is for a period of:

(i) One month; (ii) Three months; (iii) Six months?

7. You borrow \$400 at simple interest for three years. At the end of the loan, you owe \$120 in interest. What was the interest rate?

8. You borrow \$600 at simple interest for two years. At the end of the loan, you must repay \$690 in total. What was the interest rate?

9. You borrow \$1000 at simple interest for six months. At the end of the loan, you owe \$60 in interest. What was the interest rate?

10. You borrow \$5400 at simple interest for nine months. At the end of the loan, you must repay a total of \$5562. What was the interest rate?

11. You borrow some money at 8% simple interest. How much did you borrow, if:

(i) The interest you owe at the end of four years is \$1344;

(ii) Your total repayment at the end of eight months is \$13272;

(iii) The interest you owe at the end of six months is \$44?

12. You borrow some money at 6% simple interest, calculated on a standard (360-day) year. How much did you borrow, if:

(i) Your total repayment at the end of two years and four months is \$3648;

(ii) Your total repayment at the end of eight months is \$2912;

(iii) The interest you owe at the end of six months is \$198?

13. A loan of \$30000 is repaid after six years; the total repayment is \$48000. What was the interest rate?

14. A company borrows \$17300 at 6.8% simple interest to cover short-term costs, and pays interest of \$882.30. How long was the period of the loan?

15. What is your monthly payment on an add-on loan if you borrow \$500 over four months at 30% per year?

16. What is your monthly payment on an add-on loan if you borrow \$1200 over five years at 6% per year?

17. You need to pay \$15000. How much must you borrow for a five-year discounted loan at 5%? What will be your monthly payments (to the nearest cent)?

18. You need to pay \$12000. How much must you borrow for a four-year discounted loan at 9%? What will be your monthly payments?

8.2 Compound Interest

In this section, students will find it very useful if their calculator enables them to calculate powers of numbers: for example, if it has a key (possibly one marked x^y) that enables the user to input two numbers and automatically calculate the result of raising the first number to the power of the second number. (Only positive whole number powers will occur.)

Compounding

Say, you have $100 and every year you double your money.

One year from now you have $200.

2 years from now you have $400.

3 years from now you have $800.

So in 1 year you gain 100%; in three years you gain 700%—much more than $3 \times 100\%$. This process is called *compounding*. Tt also happens for interest less than 100%.

Sample Problem 8.7. *Suppose you put* $1000 *in the bank for five years at* 10% *interest paid annually. If you take your interest out of the bank at the end of each year, how much do you have at the end of five years? If you allow it to compound, how much do you have at the end of five years?*

Solution. If you take your interest out of the bank at the end of each year, you get $100 each year. After five years you have a total of $1500, a profit of $500.

If you put your interest back in the bank at 10%:

- After year 1 you get $100, so you have a total of $1100 in the bank.
- After year 2 you get $110 (10% of $1100), so you have a total of $1210 in the bank.
- After year 3 you get $121, for a total of $1331.
- After year 4 get $133.10, for a total of $1464.10.
- After year 5 get $146.41, for a total of $1610.51.

So after five years you have $1610.51, a profit of $610.

Let's look at this in general. Say you put $\$P$ in the bank at $R\%$ for N years, and reinvest all the interest. You end up with

$$P\left(1+\frac{R}{100}\right)^N.$$

This process is called *geometric growth.*

On the other hand, simple interest is the same as "we'll take the interest out each year". After N years at $R\%$ you would finish with

$$P\left(1+\frac{RN}{100}\right).$$

This is called *arithmetic growth.*

Sample Problem 8.8. *Suppose you invest* \$1200 *at* 10% *interest for three years with interest paid each year. How much interest is earned in total, if you take the interest out each year? How much if you reinvest the interest each year?*

Solution. We use the two formulas. For arithmetic growth, you end with

$$1200\left(1+\frac{10\cdot 3}{100}\right)=1200\cdot 1.3=1560$$

and the interest is $\$(1560-1200)=\$360$. Under geometric growth, the amount received after three years is

$$1200\left(1+\frac{10}{100}\right)^3=1200\cdot 1.1^3=1200\cdot 1.331=1597.2$$

and the interest is \$397.20.

Your Turn. What are the results for the above problem if your period of investment is four years?

Interest Periods

Very often a bank pays interest more frequently than once a year. Say you invest at R% per annum for one year but interest is paid four times per year. The bank pays $\frac{R}{4}\%$ every 3 months; this is called the *interest period*, or *term*. (Unfortunately, *term* is also used to denote the life of the loan.) If you invest $\$A$, your capital after a year is

$$\$P\left(1+\frac{R}{400}\right)^4,$$

just as if the interest rate was divided by 4 and the number of years was multiplied by 4.

(Notice that we used the simplifying assumption that three months is a quarter of a year. In fact, some quarters can be 92 days, others 91 or 90, but banks seldom take this into account. Similarly, when interest is calculated every month, it is usual to assume that each month is one-twelfth of a year.)

Sample Problem 8.9. *Say your bank pays $R = 8\%$ annual interest, and interest is paid four times per year. What interest rate, compounded annually, would give you the same return?*

Solution. $(1 + \frac{R}{400})^4 = (1.02)^4 = 1.0824\ldots$ so the effect is the same as compounding annually with $8.24\ldots\%$ interest rate.

Say interest is added t times per year. The result after one year is the same as if the interest rate were $(R/t)\%$ and the investment had been held for t years: after one year,

$$P\left(1 + \frac{R}{t \cdot 100}\right)^t,$$

so after N years,

$$P\left(1 + \frac{R}{t \cdot 100}\right)^{tN}.$$

The calculations are exactly the same when you borrow money as they are when you invest money.

Sample Problem 8.10. *You borrow* \$50000 *at* 10% *annual interest, compounded every three months, for ten years. Assuming you make no payments until the end of the period, how much will you owe* (*to the nearest dollar*)*?*

Solution. We have $P = 50000$, $R = 10$, $t = 4$, and $N = 10$. So

$$P\left(1 + \frac{R}{t \cdot 100}\right)^{tN}$$

becomes

$$50000 \times \left(1 + \frac{10}{400}\right)^{40} = 50000 \cdot (1.025)^{40} = 50000 \cdot 2.68506$$

which comes to $134253.19\ldots$ and you owe \$134253.

Your Turn. You borrow \$40000 at 12% annual interest, compounded every three months, for 15 years. Assuming you make no payments until the end of the period, how much will you owe?

In Sample Problem 8.9, 8% is called the *nominal rate* of interest. The nominal rate doesn't tell you how often compounding takes place. 8.24... % per annum is the *effective rate*. (These terms are established in the Truth in Savings Act.)

Note: when discussing a loan, the nominal rate per annum is called the *annual percentage rate* or APR.

When the period is a year ("per annum") the effective rate is called the *annual percentage yield* (APY) or *effective annual rate* (EAR). To avoid confusion, we shall refer to the APY, both for investments and loans.

Banks and other lenders like to tell you the APR, but what you really want to know is the APY. To calculate the APY, one works out how much would be owed on $100 at the end of a year, if no payments were made. This is not a real-world calculation: credit card companies and mortgage-holders normally require some minimum payment, or a penalty is charged. In calculating the APY, act as though all penalties are waived.

Sample Problem 8.11. *Your credit card company charges an APR of* 18%. *Payments are required monthly, and interest is charged each month. What is the corresponding APY?*

Solution. The amount owing from a $100 loan at the end of one year is

$$\begin{aligned} A &= 100\left(1+\frac{18}{12\cdot 100}\right)^{12} \\ &= 100\cdot 1.015^{12} \\ &= 119.56, \end{aligned}$$

so the APY is 19.56%.

Your Turn. Suppose your credit card company charges an APR of 12%, under the above conditions. What is the corresponding APY?

Exercises 8.2 A

1. You borrow $1000 at 12% interest, and repay it after one year. What is the total payment if the interest is compounded

(i) Every three months;

(ii) Every six months;

(iii) Once a year?

2. You borrow $100 at 6% interest for four years. What is the total interest if compounding takes place

(i) Every month;

(ii) Every three months;

(iii) Once a year?

3. You invest \$3250 at 2% interest compounded annually. What is its value after five years?

4. You borrow \$625 at 8% interest compounded quarterly, and repay it after 12 years. How much interest must you pay?

5. You invest \$200 for one year at interest rate 12%, compounded monthly. What is the value of your investment at the end of the year?

6. You invest \$2000 for three years at interest rate 6%, compounded every six months. What is the value of your investment at the end of the period?

7. You invest a sum for twelve years at interest rate 12%, compounded quarterly. At the end of the period your investment is worth \$10000. How much did you invest initially?

8. You wish to deposit a sum at 6% interest, compounded every six months, in order to pay \$10000 due in five years. How much must you deposit?

9. You deposit \$4000 at 7% interest, compounded monthly. How many years will it take until your investment exceeds \$9000?

10. You invest your money at 12% compound interest paid quarterly. When will it double in value?

11. What is the APY on a loan:

(i) At 5% APR, compounded monthly;

(ii) At 6% APR, compounded quarterly;

(iii) At 3% APR, compounded monthly?

12. What is the APY corresponding to compound interest at an annual rate of 12%, compounded:

(i) Annually;

(ii) Quarterly;

(iii) Monthly;

(iv) Daily (assume a 360-day year);

(v) Daily (assume a 365-day year)?

Round your answers to one-hundredth of one percent.

Exercises 8.2 B

1. You borrow \$2500 at 8% interest, and repay it after one year. What is the total payment if the interest is compounded

(i) Every three months;

(ii) Every six months;

(iii) Once a year?

2. You borrow \$5000 at 5% interest, compounded monthly. What is the total payment if the period is

   (i) Two years;

   (ii) Four years?

3. You invest \$2575 at 4% interest compounded quarterly, for two years. How much interest do you receive?

4. You borrow \$460 at 6% interest compounded quarterly, and repay it after seven years. How much must you repay, in total?

5. You invest \$80000 for 25 years at interest rate 6%, compounded annually. What is the value of your investment at the end of the period?

6. You invest \$1000 for two months at interest rate 24%, compounded each month. What is the value of your investment at the end of the period?

7. You invest a sum for fifteen years at interest rate 6%, compounded every two months. At the end of the period your investment is worth \$18000. How much did you invest initially?

8. You make a deposit at 6% interest, compounded every six months, in order to pay a promissory note for \$5000 that falls due in ten years. How much should you deposit?

9. You invest at 6% interest, compounded annually. How many years must you wait to double your money?

10. You invest your money at 10% compound interest paid quarterly. When will it double in value?

11. What is the APY on a loan:

    (i) At 7% APR, compounded monthly;

    (ii) At 6% APR, compounded monthly;

    (iii) At 8% APR, compounded quarterly?

12. Which investment would give a better return: one at 5.8% APR, compounded monthly, or one at 5.95% APR, compounded annually?

8.3 Regular Deposits

As we pointed out in Sect. 8.1, you do not usually wait until the end of a loan period to pay back a loan. The usual practice is to pay equal amounts each month (or each week or . . .). Another situation where equal deposits are made is the periodic savings account, such as a Christmas club or retirement account, where a fixed amount is deposited into savings each period.

Regular Savings

Consider a periodic savings account. Suppose you deposit $\$D$ each month. The interest each month is $M\%$; write $m = M/100$. Assume the account is empty to start, and you pay in for n months. (Often $n = 11$ or 12 because people use these accounts to save for vacations or Christmas shopping.)

The calculations to find the amount at the end of the nth month might start:

Principal, start of month 1	$\$D$
Interest earned in month 1	$\$mD$
Principal, end of month 1	$\$(1+m)D$
Add to principle	$\$D$
Principal, start of month 2	$\$D + \$(1+m)D$ $= \$(2+m)D$
Interest earned in month 2	$\$m(2+m)D$
Principal, end of month 2	$\$(2+m)D + \$m(2+m)D$ $= \$(1+m)(2+m)D$
Add to principle	$\$D$
Principal, start of month 3	$\$D + \$(1+m)(2+m)D,$

and so on.

This soon becomes complicated. An easier way is to calculate the effect of putting each new payment in a new bank account. The total in all the accounts at the end of n months will be the required amount.

Payment 1 draws interest for n months, so the amount in that account at the end is $\$D(1+m)^n$; payment 2 draws interest for $n-1$ months, so the amount in that account at the end is $\$D(1+m)^{n-1}$. The total of accounts 1 and 2 is

$$\$D(1+m)^n + \$D(1+m)^{n-1} = \$D\big[(1+m)^n + (1+m)^{n-1}\big].$$

If we proceed in this way, the total after n months (n accounts) is

$$\$D\big[(1+m)^n + (1+m)^{n-1} + \cdots + (1+m)\big].$$

If $m = 0$ then the total is simply $\$nD$. We assume $m \neq 0$ and evaluate

$$(1+m)^n + (1+m)^{n-1} + \cdots + (1+m).$$

Write

$$X = (1+m)^n + a^{n-1} + \cdots + (1+m)^2 + (1+m).$$

Then

$$(1+m)X = (1+m)^{n+1} + (1+m)^n + \cdots + (1+m)^3 + (1+m)^2.$$

Subtracting, $mX = (1+m)^{n+1} - (1+m)$ so $X = \frac{(1+m)^{n+1}}{m}$ (this is where we need to assume $m \neq 0$).

So the amount in the account after n months is

$$\frac{\$D \cdot [(1+m)^{n+1} - (1+m)]}{(1+m) - 1} = \frac{\$D \cdot [(1+m)^{n+1} - (1+m)]}{m}.$$

We call this amount the *accumulation*.

Sample Problem 8.12. *At the beginning of each month you put* \$100 *into an account that pays* 6% *annual interest. How much have you accumulated at the end of the year?*

Solution. 6% annual interest is 0.5% per month. So $m = 0.005$, $n = 12$, $D = 100$, and you get

$$\begin{aligned}
&\$100(1.005^{13} - 1.005)/0.005 \\
&= \$100(1.066986 - 1.005) \cdot 200 \\
&= \$20000(0.061986) \\
&= \$1239.72.
\end{aligned}$$

Your Turn. In the above example, suppose you started saving in April, so that you only made nine payments. How much will you accumulate?

Some investment funds are set up so that you make your payment at the *end* of the payment period, rather than the beginning. In these cases, it is usual to add the last payment to the accumulation, even though it accrues no interest. In that case, the accumulation is

$$\frac{\$D \cdot [(1+m)^n - 1]}{m}.$$

For example, in a Christmas club, you might make your first payment on January 31st and withdraw the money early in December. There are ten payments. If the annual interest is again 6%, your accumulation is

$$\frac{\$D \cdot [(1.005)^{10} - 1]}{0.005}.$$

Compound Interest Loans

When you borrow money at compound interest and make regular repayments, interest is normally calculated on the amount you owe. Usually the interest is calculated at the time your payment is due; when you buy a house on a mortgage and make monthly payments, the interest is compounded monthly. There is a penalty for late payments, in addition to the interest on the missed payment. Usually there is no reward for paying early in the month. We shall refer to this arrangement as a *standard compound interest loan*, or simply a *compound interest loan*.

To clarify this, suppose your house payment of \$2000 is due on the first of the month, and on June 1st this year your total indebtedness is \$100000. For simplicity, say your annual interest rate is 12%, so the interest for one month is 1%. On July 1st, interest of \$1000 (1% of \$100000) is added to your debt. Then payments are credited. Assuming you made the standard \$2000 payment, this is subtracted from your debt, which becomes (\$100000 + \$1000 − \$2000) = \$99000. When August 1st comes around, the new interest will be \$990 (1% of \$99000). It does not matter when you made the payment, provided it is on or before July 1st; even if you paid on June 2nd, it is applied on July 1st. If you made a payment greater than \$2000 during June, the total would be subtracted from your debt on July 1st.

If you miss a payment, or pay less than \$2000, some penalty is exacted. The arrangements differ from loan to loan. Some lenders charge a higher interest rate on the amount in arrears; some charge a fee; many do both.

Payments

The amount of payment required to pay off a loan can be calculated from the data about the loan.

The calculation proceeds as follows. Say you borrow $\$P$ at $M\%$ monthly interest (compounded monthly), and pay it back at the end of Y years. The arithmetic is the same as if you put $\$D$ into a savings account each month at $M\%$ interest compounded monthly, and at the end of Y years you have exactly enough money to pay off your loan. This amount is $\$P(1+m)^n$, where $n = 12Y$ and $m = M/100$. Then the required monthly payment is $\$D$.

The only difference between this and the example of accumulated savings is that loan repayments usually start at the *end* of the first month, so for Y years the total number of months for which your money accumulates is $12Y - 1$, not $12Y$, and you gain nothing during the first month. If we continue to write n for the number of months in the life of the loan, we use $n-1$ in place of n in the accumulation formula. This actually simplifies the formula: we want to sum

$$\$D\big[(1+m)^{n-1} + (1+m)^{n-2} + \cdots + 1\big].$$

The required payment $\$D$ for a loan is calculated from

$$P(1+m)^n = \frac{D \cdot [(1+m)^n - 1]}{m}.$$

Sample Problem 8.13. *You take out a compound interest loan of* \$100000 *at* 6% *annual interest to pay off your house. The period is* 30 *years. What payment is required each month?*

Solution. Suppose the monthly payment is $\$D$. The interest rate is $0.5\% = 0.005$ per month. There are 360 months in thirty years. So $P(1+m)^n$ is

$$\$100000 \times (1.005)^{360}$$

or \$602257.52. In this example

$$\frac{D \times [(1+m)n - 1]}{m} = \frac{\$D \times [(1.005)^{360} - 1)]}{0.005}$$
$$= \$D \times 5.0225752 \times 200 = \$D \times 1004.515$$

so $\$D \times 1004.515 = 602257.52$ and $D = 599.55$. You would pay \$599.55 per month.

Your Turn. To buy your car, you borrow \$12000 over five years at 8% interest, compounded monthly. To pay it off you pay $\$D$ per month. What is D?

It is interesting to observe the differences that follow from small changes in a long-term loan. Suppose we change the annual rate of interest in the preceding example from 6% to 8%, leaving the principal and period unchanged. The interest rate is $0.00666\ldots$ or $1/150$ per month. So the accumulation after 30 years at 8% is

$$\$100000 \cdot (151/150)^{360},$$

or \$1093572.96. In the "regular savings" model, depositing $\$D$ per month, your accumulated savings would be

$$\frac{\$D \cdot [(151/150)^{360} - 1)]}{1/150} = \$D \cdot 9.9357296 \cdot 150 = \$D \cdot 1490.36.$$

So your monthly payment is $D = 733.76$.

On the other hand, changing the period of the loan makes less difference than you might think. Reducing the period by 20%, to 24 years, adds less than 10% to your monthly payment:

Sample Problem 8.14. *You take out a compound interest loan of* \$100000 *at* 6% *annual interest to pay off your house. The period is* 24 *years. What payment is required each month?*

Solution. As in Sample Problem 8.14, the interest rate is $0.5\% = 0.005$ per month. There are 288 monthly payments. So in this case $P(1+m)^n$ is

$$\$100000 \times (1.005)^{288}$$

or \$420557.89, and

$$\begin{aligned}\frac{D \times [(1+m)^n - 1]}{m} &= \frac{\$D \times [(1.005)^{288} - 1)]}{0.005} \\ &= \$D \times 3.2055789 \times 200 = \$D \times 641.116.\end{aligned}$$

So $D \times 641.116 = 420557.89$. You would pay \$655.98 per month.

Your Turn. Repeat the preceding calculation when the interest rate is 8%.

Here is another way of interpreting the above calculation: suppose you contracted to buy your \$100000 house at 6% annual interest over 30 years. Your bank requires a monthly payment of \$600 (actually \$599.55 per month, but the bank rounds up slightly; the last payment would be reduced a little). If you decided to pay an extra \$56 each month, you would finish paying for your house six years ahead of schedule.

We have calculated D from A (answering the question "What payment must I make?"). We can also calculate A from D ("Given the maximum payment I can make, how much can I afford?").

Sample Problem 8.15. *You want to buy a car. You can get an* 8% *loan over five years. You can pay* \$200 *per month. How much can you afford to pay for the car?*

Solution. $m = 8/12\% = 1/150$, $D = 200$, $n = 60$. So

$$mP(1+m)^n = D \cdot \big[(1+m)^n - 1\big]$$

becomes

$$P(151/150)^{60}/150 = 200\big[(151/150)^{60} - 1\big].$$

So $P \cdot 1.49646 = 150 \cdot 200 \cdot 0.49646$, implying $P = 9952.69$, and you can afford about \$9950.

Your Turn. In the above example, suppose you negotiated a loan at 7.5%. How much could you then afford?

One can use this method to compare compound interest loans with other sorts of loans. For example, an add-on loan of \$1000 at 5% interest for a four-year period requires monthly payments of \$25. If you took out a \$1000 loan for four years at

6% compound interest, and found your monthly payment to be \$25, your principal was \$$P$, where

$$0.005 \cdot P \cdot 1.005^{48} = 25 \cdot \left[1.005^{48} - 1\right].$$

Then

$$\begin{aligned} P &= \frac{25 \cdot [1.005^{48} - 1]}{0.005 \cdot 1.005^{48}} \\ &= \frac{6.76223}{0.0063525} \\ &= 1064.50. \end{aligned}$$

So the APY for the four-year 5% add-on loan is greater than 6%.

Equity

Suppose you have finished 3 years' payment on a 5-year loan of \$9952 at 8% annual interest for a car. As we saw in Sample Problem 8.14, your payments were \$200 per month.

Think about the remaining 24 months. Your situation is as though you had just taken a loan of \$$A$ at 8% per annum, where

$$A(151/150)^{24}/150 = 200\left[(151/150)^{24} - 1\right],$$

so $1.17288A = 150 \cdot 200 \cdot 0.17288$, $A = 4421.94$.

We say your *equity in the loan* is \$9952 − \$4421.94, about \$5530.

You might think that, after making payments for three-fifths of the payment period, you would own 60% of your car. However, your equity is a little less than that amount: around 55.5%.

The difference is *much* greater on longer-term loans. For example, suppose you take out a 30-year house loan for \$100000 at 8% per annum, with equal monthly payments of \$733.76 (as we calculated above). After three-quarters of the term—270 of the 360 payments have been made—it is as if you had just taken a loan at 8% per annum with principal \$$A$, where

$$A(151/150)^{90}/150 = 733.76\left[(151/150)^{24} - 1\right],$$

that is,

$$A \cdot 0.012123295 = 600.578$$

so $A = 49539.20$ and your equity is \$50460.80.

After three-quarters of your payments, you own about half of your house.

Exercises 8.3 A

1. You invest \$200 every quarter for 20 years in an annuity that pays 5% interest compounded quarterly. What is the final value of the annuity?

2. You invest \$4000 every year for five years in an annuity that pays 10.5% interest compounded annually. What is the final value of the annuity?

3. Credit card interest is 18% interest compounded monthly. How much must be paid each month to eliminate a debt of \$1000 in one year?

4. \$200 is invested per month in an fund that pays 9% interest compounded monthly. The first payment is made at the end of the first month. What is the value of the annuity after:

(i) 1 year; (ii) 3 years; (iii) 5 years; (iv) 8 years?

5. A house mortgage is set at 9%, compounded monthly. If the house costs \$200000, what is the monthly payment if the term of the mortgage is

(i) 15 years; (ii) 30 years?

6. You take out a compound interest loan of \$120000 at 6.5% annual interest to pay off your house. The period is 30 years. What payment is required each month?

7. You want to buy a car for \$8000. The dealer offers you a 5% add-on loan for four years, with monthly payments. You can borrow \$8000 for four years from your credit union at 7.5% interest. What would be your monthly payment in each case? Which is the better deal?

8. You take out a compound interest loan of \$100000 at 6% annual interest to pay off your house. The period is 30 years. (We saw earlier that your monthly payment is \$599.55.) What is your equity after:

(i) 15 years; (ii) 20 years; (iii) 22.5 years?

Exercises 8.3 B

1. You invest \$160 every quarter for four years in an annuity that pays 4% interest compounded quarterly. What is the final value of the annuity?

2. You invest \$2000 every six months for 10 years in an annuity that pays 8% interest compounded twice yearly. What is the final value of the annuity?

3. You need to have \$100000 in ten years, so you set aside a fixed sum every three months in a savings account. How much should you set aside each quarter if the interest is:

(i) 6%; (ii) 8%; (iii) 10%?

4. You take out a compound interest loan of \$200000 at 6% annual interest to pay off your house. The period is 30 years. What payment is required each month?

5. You wish to accumulate \$200000 over a 30 year period by making monthly payments into a fund that pays 9% annual interest. how much must you pay each month?

6. You buy a car costing \$16995 (after down-payment) and are offered a loan at 1.6% annual interest over 60 months. What payment is required each month?

7. You take out a compound interest loan of \$100000 at 6% annual interest to pay off your house. The period is 24 years. As we saw, your monthly payment is \$655.98. What is your equity after:

(i) 15 years; (ii) 20 years?

8.4 Exponential Growth

In this section, we examine another aspect of compounding, in which either the compounding period is very short or the number of periods is very great. Again you may wish to have a calculator that enables you to raise numbers to integer powers while studying this section. It is also useful to have the constant e available on your calculator.

Continuous Compounding

In some cases, compounding takes place after a very short interest. For example, in some bank accounts, interest is calculated every day. Over one year, there are 365 compounding periods. We saw in Exercise 8.2A.12 that the interest rate differs very little between the final interest if the standard year is used instead of the exact year.

In another case, even though the period is longer, there are still a large number of compounding periods. For example, suppose a company invests some of its funds and loses track of the investment for 100 years. It is found that there is little difference whether compounding took place monthly or quarterly.

For our calculations it will be convenient to work in terms of the fraction $r = R/100$ rather than the percentage R. If your bank applies interest to your account n times per year, the interest is $R\%$ and your initial investment ("capital") was $\$A$, then your ending capital after N years is

$$\$A(1 + r/n)^{nN}.$$

Consider the example of 100% interest, the case $r = 1$. The following table shows the values of $(1 + 1/n)^n$ and $(1 + 1/n)^n - 1$ for several values of n. The right-hand number is the APY corresponding to an APR of 100% with compounding n times annually.

n	$(1 + 1/n)^n$	
1	2.0000000	1.0000000
2	2.2500000	1.2500000
5	2.4883200	1.4883200
10	2.5937424	1.5937424
100	2.6915880	1.6915880
1000	2.7169239	1.7169239
10000	2.7181459	1.7181459
100000	2.7182682	1.7182682

Eventually the right-hand number gets very close to a fixed value, usually denoted e, about $2.7182818\ldots$

A similar calculation can be made for other values of r. For large n, $(1 + r/n)^n$ gets very close to e^r. For example, if $r = 0.1$, then $e^r = 0.10517\ldots$

We define *continuous compounding* to be a process where, after N years, the ending capital is

$$\$Pe^{rN}.$$

The calculations above show that continuous compounding is a very good approximation to daily interest, and many lenders use it instead of daily interest because it is easy to calculate.

Sample Problem 8.16. *You borrow* \$50000 *at* 10% *annual interest, compounded continuously, for ten years. Assuming you make no payments until the end of the period, how much will you owe* (*to the nearest dollar*)*?*

Solution. We have $P = 50000$, $R = 10$ and $N = 10$. So $r = 0.1$ and

$$Pe^{rN} = 50000 \cdot e = 135914.09\ldots$$

and you owe \$135914. Compare this with the result of Sample Problem 8.10, where compounding was quarterly and the answer was \$134253. The difference is not great.

Your Turn. You borrow \$40000 at 12% annual interest, compounded continuously, for 15 years. Assuming you make no payments until the end of the period, how much will you owe?

One advantage of continuous compounding is that one can easily calculate the interest when principal and interest are known. For example, suppose you invest \$100 for two years and receive \$21 interest. Your money grew from \$100 to \$121 with continuous compounding, so the rate r satisfied $100e^{2r} = 121$, so $e^{2r} = 1.21$ and $e^r = \sqrt{1.21} = 1.1$. From this, r will equal the *natural logarithm*, or logarithm to base e, of 1.1, which equals $0.0953\ldots$, so the interest rate was 9.53%. (The natural logarithm function, written *ln*, is available on most scientific calculators.)

However, you don't even need to calculate the value of r in order to find the interest; all you need is e^r. In this example, after n years, your capital will be $\$100 \cdot (1.1)^n$ and your interest will be $\$100 \cdot [(1.1)^n - 1]$. Of course, you can easily calculate the APY — in the example, your money would grow from \$100 to \$110 in the first year, so the APY is 10%. If you don't want to deal with natural logarithms, there are tables to calculate the value of R (and r) from the APY.

Sample Problem 8.17. *You borrow* \$50000 *at continuous compounding for ten years. At the end of the period, you owe* \$100000. *What was the approximate APY?*

Solution. Suppose the rate is r. Then $50000e^{10r} = 100000$ so $e^{10r} = 2$. Therefore, e^r equals the tenth root of 2, which equals 1.0718 approximately. The APY is approximately 7.18%.

Your Turn. You borrow \$40000 for seven years. At the end of the period, you owe \$80000. What was the approximate APY?

Observe that the figures of \$50000 and \$100000 were not an essential part of the above Sample Problem. The main point was the *ratio*: the debt at the end was double the principal. The answer would be the same for any value of P, provided the final amount owed was twice P.

If k is any constant, the function $f(x) = k^x$ is called an *exponential function*, and the situation where a quantity changes over x units of time from A to Ak^x is called *exponential growth*; x is the *exponent*. So continuous compounding is an example of exponential growth.

Inflation and the Consumer Price Index

There is a tendency for the purchasing power of money to decrease over time; this is called *inflation*. Sometimes the rate of inflation will change rapidly, but often it stays roughly constant for several years. Inflation follows the same model as continuous compounding.

For example, let's suppose inflation rate is 5% from 2007 to 2012. If something costs \$100 in January 2007, then five years later, in January 2012, we expect it to cost $\$100 \cdot (1.05)^5 = \$127.63$. Of course, individual items do not increase at a uniform rate, but this would be a useful approximate guide to the cost of living.

As inflation is not constant, governments often calculate tables to show the purchasing power of today's dollar in earlier years. In the United States, the Consumer Price Index (CPI) is calculated each month by finding the cost of a standard set of items (food, housing, vehicles, and so on). There are in fact several CPIs constructed. We shall always refer to the CPI-U, an index that reflects the cost of living in urban areas (about 80% of America). There is also a CPI-W, for wage-earners, and there are other indices. Tables of the CPI-U are available online at http://stats/bls/gov/cpi/#tables.

The total CPI is divided by the average for 1982–1984 (the *base period*), and multiplied by 100. For example, the CPI for February 2006 was 198.7, so a collection of goods that cost \$198.70 in February 2006 would have cost an average of about \$100 in the base period. The average for a year is also published; the average for 1988 was 118.3; the figures for May and June were 118.0 and 18.5, respectively.

This can be used to compare two different years. The CPI for June 2002 was 179.9. The ratio $\frac{179.9}{118.5} = 1.518\ldots$ provides a comparison between June 2002 and June 1988 prices: if something cost \$1000 in 1988, our best guess is that it would cost about \$1518 in 2002. These figures are approximate because the prices of different items do not increase at the same rate. However, it is reasonable to say "the cost of living was about 50% higher in 2002 than in 1988." A speaker in 2002 might say: "A dollar today is worth about two-thirds of what it was worth in 1988."

Suppose the cost of a major item at time A is $\$X_A$. Suppose the CPI at time A is C_A, and at time B it is C_B. Then your estimate of the cost at time B is

$$\frac{C_B}{C_A} X_A.$$

Sample Problem 8.18. *A house cost* \$150000 *in June* 1988. *What would you expect a similar house to cost in February* 2006*?*

Solution. If the house cost \$150000 in mid-1988, it is equivalent to a house that cost $\frac{198.7}{118.5} \cdot \$150000 = \$251518.99$ in February 2006. Your realistic answer might be "about \$250,000."

Your Turn. A house costs \$225000 in February 2006. What would a similar house have cost in June 2002?

Animal Populations

Another example of exponential growth is the growth of an animal population. Given two animals (male and female), we know how frequently they will reproduce on average, and how many offspring will be produced. These numbers are not precise, but with large numbers the errors average out. If the animals reproduce an average of three offspring per year, and on average two die per year, the end result is as if the number of animals grows by 50% annually.

Of course, the animals do not all reproduce at the same time. The process is more like continuous compounding. In the example, the appropriate model is continuous compounding with an APY of 50%.

This model is more accurate with shorter breeding periods. When studying microscopic creatures, that reproduce within hours, reasonable predictions can be made of the population growth over periods of shorter than a day. For insects, a few days is often long enough for an accurate model. With humans, we need decades or even centuries. The "continuous compounding" model of a human population is used only for predicting the population movement in large cities, states, or whole countries.

Sample Problem 8.19. *A fish population doubles every year. At present it is* 10000. *Approximately when will it reach* 100000*? When will it reach* 1000000*?*

Solution. After n years, the total population is $10000 \cdot 2^n$, so the questions are, *"when is* $2^n = 10$*?"* and *"when is* $2^n = 100$*?"*

Now $2^3 = 8$, $2^4 = 16$, $2^6 = 64$, $2^7 = 128$, so the answers are

$$\begin{aligned} 100000 &: \quad \text{during the 4th year;} \\ 1000000 &: \quad \text{during the 7th year.} \end{aligned}$$

Radioactive Decay

Radioactive decay works like continuous compounding in reverse. A radioactive material will dissipate with time, its molecules breaking down into molecules of other substances. If you have a certain amount present at a given time, it is found that the proportion that is lost depends only on the type of material and the time elapsed.

If you start with 100 grams, then at the end of one day there will remain $100k$ grams, where k is a constant, between 0 and 1, depending only on the material. After n days, the amount remaining is $100k^n$ grams. This is exactly the same formula as continuous compounding, with $e^r = k$.

Of course, if e^r is to be smaller than 1, r must be negative. So radioactive decay is an example of exponential growth with a negative component.

The *half-life* of an element is the time it takes for the amount of it present to halve. For example, if you have 500 grams with half-life 1 year, there will be 250 grams after one year, 125 after two years, and so on. After n half-lives amount A decays to $A/2^n$.

Sample Problem 8.20. *An artificial element has a half-life of one hour. You have* 450 *grams. Approximately how long will it take until only* 50 *grams is left?*

Solution. You want $450/2^n = 50$.

$$\begin{aligned} n = 3 &: 450/23 = 450/8 = 56.25, \\ n = 4 &: 450/24 = 450/16 = 28.125, \end{aligned}$$

so the approximate answer is: a little over 3 hours.

Your Turn. An artificial element has a half-life of one day. You start with 240 grams. How much will be left after four days?

Exercises 8.4 A

1. \$1000 is invested at 10% annual interest, compounded continuously. What is the value of the investment after:

(i) Three months; (ii) Two years; (iii) Ten years?

2. You invest \$10000 with continuous compounding. After two years, your investment is worth \$11200. What is its value after

(i) Four years; (ii) Ten years?

3. A house was bought for \$120000 in March, 1991, when the CPI was 135.0. What is its expected value in February 2006?

4. In January 2000 you have the choice of investing in houses for five years, or investing in a five-year certificate of deposit that pays 3% compounded annually. The CPI for January 2000 was 168.8, and for January 2005 it was 190.7. Assuming the housing market shows the same growth as the CPI, which is the better investment? Why?

5. The 2000 census shows the population of Cook County, IL as 5376741. If the population grows at the rate of 1.2% per year, what is the expected population in:

(i) 2010; (ii) 2050?

6. A fish hatchery has 2550 fish at the beginning of 1998. Each year the population grows by 50% at the end of the year, 30% of the fish are sold. How many fish are there at the beginning of 1999? How many at the beginning of 2002 ?

7. A certain insect doubles in population every week. There are 3400 in a colony on Monday March 1st, and the numbers are checked every Monday. When do the numbers exceed 100000?

8. There are 1280 grams of an isotope present at noon on Monday. If the half-life is 12 hours, how much is left at noon on Thursday?

Exercises 8.4 B

1. \$1000 is invested at 6% annual interest, compounded continuously. What is the value of the investment after:

(i) One year; (ii) Two years; (iii) Four years?

2. \$1000 is invested, and after a year the value of the investment is \$1050. Assuming continuous compounding, what is the value after:

(i) Two years; (ii) Four years; (iii) Six years?

3. You invest \$20000 with continuous compounding. After two years, your investment is worth \$23000. What is its value after:

(i) Four years; (ii) Six years?

4. In January, 1995, you have the choice of investing in houses for five years, or investing in a five-year certificate of deposit that pays 3% compounded annually. The CPI for January 1995 was 150.3, and for January 2000 was 168.8. Assuming the housing market shows the same growth as the CPI, which is the better investment? Why?

5. A house was bought for \$86000 in May, 1989, when the CPI was 123.8. What is its expected value in February 2006?

6. You bought your house for \$167000 in January, 1999, when the CPI was 164.3. In December, 2005, when the CPI was 196.4, you were offered \$195000. Is this a good deal? Why?

7. The 2000 census shows the population of Cook County, Illinois as 59612. What is the expected population in 2050:

(i) If the population grows at the rate of 1.2% per year;

(ii) If the population grows at the rate of 1.5% per year?

8. A colony of birds had 4400 members in April 2001. In April, 2003, there are 5234 birds. What is the annual growth rate?

9. A certain class of bacteria double in population in a day if unchecked. How many days will it take for a colony of 10000 to exceed 1000000?

10. A radioactive material has a half-life of 11 hours. It is safe to move quantities of 125 grams or less. If you start with 1 kilogram, how long will it be until the material is safe to move?

Your Turn Solutions

Chapter 1

1.1 $\{1, 2, 3\}$, $\{x : x^3 - 5x^2 + x + 6 = 0\}$, the set of the first three positive integers, $\{x : x \in \mathbb{Z}, 1 \le x \le 3\}$, $\{x : x \in \mathbb{Z}, 0 < x < 4\}$ are some answers.

1.2 $(0, \infty)$, $(0, 1)$, $(1, 4]$, $(-\infty, 3)$.

1.3 24 has factors 1, 2, 3, 4, 6, 8, 12, 24; 15 has 1, 3, 5, 15; 13 has 1, 13. The common factors of 24 and 15 are 1, 3, so $(24, 15) = 3$.

1.4 $1, 0, 1, x^3, -0.5, 25$.

1.5 $t^{-2}/t^{-3} = t^3/t^2 = t$, $y^{5-2} = y^3$, $(4x^{-2})(3x^4) = 12x^{4-2} = 12x^2$.

1.6 $2, -2, 5, -4, 3.1, 4.4$.

1.7 If we replace each of x and y by 2, the equation becomes $6 - 4 = 4$, which is false. The other two suggested solutions lead respectively to $12 - 8 = 4$, which is true, and $9 - 2 = 4$, which is false. So (ii) is a solution, but (i) and (iii) are not.

1.8 First move the term x:

$$3x + 5 - x = 7.$$

Then move the 5:

$$3x - x = 7 - 5.$$

Gather terms:

$$2x = 2.$$

Finally, divide by 2:

$$x = 1.$$

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1,

1.9 First move $\frac{2x}{3}$ to the left and -5 to the right:

$$\frac{3x}{2} - \frac{2x}{3} = 5.$$

Next, multiply by the common denominator 6:

$$6 \times \frac{3x}{2} - 6 \times \frac{2x}{3} = 6 \times 5,$$
$$9x - 4x = 30.$$

Finally, divide by 5:

$$x = 6.$$

1.10 We move x to the right-hand side:

$$-2y = 2y - 8 - x^2.$$

All y terms belong on the left:

$$-4y = -8 - x^2.$$

Multiply both sides by $-\frac{1}{4}$:

$$y = 2 + \tfrac{1}{4}x^2.$$

1.11 We rewrite the inequality as:

$$2x - x \leq 6 - 3$$

or

$$x \leq 3.$$

Solution set $(-\infty, 3]$.

1.12 We first rewrite the inequality as:

$$2x - 4x \leq 5 - 3$$

or

$$-2x \leq 2.$$

Dividing both sides by -2, we obtain $x \geq -1$, solution set $[-1, \infty)$.

1.14 $\sum_{i=3}^{5} i(i-1) = 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 = 6 + 12 + 20 = 38$; $\sum_{i=2}^{6} i = 2 + 3 + 4 + 5 + 6 = 20$.

1.15 $1 + 3 + 5 + 7 + 9 = \sum_{i=1}^{5} 2i - 1$; $8 + 27 + 64 + 125 = \sum_{i=2}^{5} i^3$.

1.17 $3 + 7 + \cdots + 43 = \sum_{i=1}^{11} 4i - 1 = 4 \cdot \sum_{i=1}^{11} i - 11 = 4 \cdot \frac{12 \cdot 11}{2} - 11 = 264 - 11 = 253$.

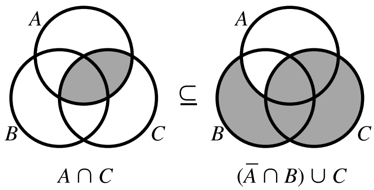
1.19 $\mathbb{Z}\backslash\mathbb{Z}^*$ consists of all the negative integers. $\mathbb{N} \subseteq \mathbb{Z}$, so $\mathbb{Z} \cap \mathbb{N} = \mathbb{N}$. $(\mathbb{N}\backslash\mathbb{E})$ contains just the odd positive integers, and all members of Π other than 2 are odd, so $(\mathbb{N}\backslash\mathbb{E}) \cup \Pi$ contains all the odd positive integers and 2. In symbols,

$$\begin{aligned} \mathbb{Z}\backslash\mathbb{Z}^* &= \{-1, -2, -3, -4, -5, -6, \ldots\}, \\ \mathbb{Z} \cap \mathbb{N} &= \mathbb{N}, \\ (\mathbb{N}\backslash\mathbb{E}) \cup \Pi &= \{1, 2, 3, 5, 7, 9, 11, \ldots\}. \end{aligned}$$

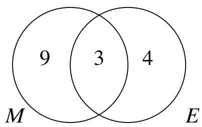
1.20 (i) The sets are not disjoint; their common element is 2. (ii) These sets are disjoint.

1.22 $\{(1, 1), (1, 4), (1, 5), (2, 1), (2, 4), (2, 5)\}$.

1.23

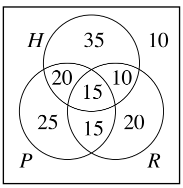


1.24 The figures are represented by the following diagram. As only readers were surveyed, there is no need for any "outside area" outside $M \cup E$.



From the diagram we see that 16 readers were surveyed.

1.26



20 like horror and police procedural movies, but do not like romances. 20 like romances only. 10 like none of these three types.

1.27 U has sum 49, so the mean is $49/7 = 7$. The mode is 7, the only reading to appear twice. To find the median, first write the readings in ascending order: 2, 3, 6, 7, 7, 9, 15. The "center" reading is 7, so the median is 7 also.

V has sum 117, so its mean is 19.5. Its median is 20 (after the readings are reordered as 14, 17, 19, 21, 22, 24, the two "center" values are 19 and 21, and we take the average of the two). There is no mode.

1.28 We define Y by the rule "$y_i = x_i - 830$". Then $Y = \{3, 4, 6, 7, 11, 11\}$; $m_Y = 42/6 = 7$, so $m_X = 7 + 830 = 837$.

1.29 The sum of the terms $x_i f_i$ is

$$2 \times 1 + 3 \times 3 + 4 \times 7 + 5 \times 11 + 6 \times 12 + 7 \times 4 + 8 \times 1 = 202$$

while the sum of the frequencies is

$$1 + 3 + 7 + 11 + 12 + 4 + 1 = 39$$

so the mean is $202/39 = 5.179\ldots$ or about 5.18. We would say the mean is "a little greater than 5".

1.31 The data add to 17, so the mean is $\frac{17}{7} = 2.\overline{428571}$. Using the formula, we find

$$\begin{aligned} 7s_X^2 &= \left[1^2 + 1^2 + 2^2 + 2^2 + 3^2 + 4^2 + 4^2\right] - 7\left(\frac{17}{7}\right)^2 \\ &= [1 + 1 + 4 + 4 + 9 + 16 + 16] - \frac{289}{7} \\ &= 51 - 41.2857 = 9.7143 \end{aligned}$$

so $s_X^2 = 1.38776$, and $s_X = 1.18$ (correct to two decimal places).

Chapter 2

2.2 Equation (2.1) yields

$$|S \cup T| = |S| + |T| - |S \cap T| = 42 + 32 - 22 = 52.$$

From (2.2) we get

$$|S \backslash T| = |S| - |S \cap T| = 42 - 22 = 20.$$

2.3 Each digit can be chosen in 10 ways. So there are 10^4 possibilities.

2.4 For the boys we now get $10 \times 9 \times 8 = 720$. For the girls we get $13 \times 12 \times 11 = 1716$. So the total is $720 \times 1716 = 1235520$.

2.5 $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

2.6 There are $9 \times 8 \times 7 = 504$ committees.

2.7 The boys can be ordered in $5! = 120$ ways. The girls can be ordered in $4! = 24$ ways. So there are $120 \times 24 = 2880$ arrangements.

2.8 $P(6, 4) = 6 \times 5 \times 4 \times 3 = 360$.

2.9 $P(12, 3)$, or 1320.

2.10 The boys can be ordered in $3! = 6$ ways, and the girls can be ordered in $4! = 24$ ways. As the table is circular, it doesn't matter whether the boys are to the left or to the right of the girls. So there are $3! \times 4! = 6 \times 24 = 144$ arrangements.

2.13 There are three A's, two N's and one A, for a total of six letters. So the number of orderings is $6!/(3! \times 2!) = 60$.

2.14 If unlimited numbers of each color were available, there would be $3^4 = 81$ solutions. It is necessary to exclude the solution with four blue marbles, $BBBB$. So the answer is $81 - 1 = 80$.

2.15 $C(9, 5) = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$, $\binom{6}{0} = \frac{6!}{6!\times0!} = 1$.

2.16 She must choose 4 of the last 7 questions, so $\binom{7}{4} = 35$ ways.

2.17 There are $C(8, 4)$ choices of where to place the 1's, so the answer is $C(8, 4)$, or 70.

2.18 You can choose the mysteries in $\binom{5}{2}$ ways and the westerns in $\binom{7}{3}$ ways. So you can choose in $\binom{5}{2} \times \binom{7}{3} = 10 \times 35 = 350$ ways.

2.19 The three consonants can be chosen in $\binom{5}{3} = 10$ ways, and the vowels in $\binom{3}{2} = 3$ ways. After the choice is made, the letters can be arranged in $5! = 120$ ways. So there are $10 \times 3 \times 120 = 3600$ "words."

2.20 The committee contains one man or no men. With one man, the number of choices is $3 \times \binom{7}{2} = 63$ (3 ways to choose the man, $\binom{7}{2}$ to choose the women). With no men, there are $\binom{7}{3} = 35$ possibilities. So there are $63 + 35 = 98$ possibilities.

2.22 $\binom{81}{79} = \binom{81}{2} = \frac{81\times80}{1\times2} = 3240$.

2.23 $(x-y)^6 = \binom{6}{0}x^6 - \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 - \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 - \binom{6}{5}xy^5 + \binom{6}{6}y^6 = 1 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.

2.24 The term involving y^2 is $\binom{4}{2}(3x)^2(2y)^2$, so the numerical coefficient is

$$\binom{4}{2}3^2 2^2 = 6 \times 9 \times 4 = 216$$

and the coefficient of y^2 is $216x^2$.

2.25 We write $0.99^8 = (1 - 10^{-2})^8$. Then it equals

$$1 - 8 \times 10^{-2} + 28 \times 10^{-4} + 56 \times 10^{-6} + \cdots.$$

Every subsequent term in at most one-tenth of the one before it. So the approximate value is

$$1 - 0.08 + 0.0028 - 0.000056 + \cdots = 0.923 \text{ approx.}$$

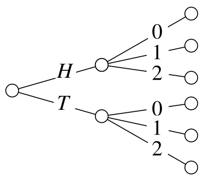
2.26 There are 2^{10} possible ways to choose a subset of the ten books. However, the subsets with 8, 9 or 10 elements are not allowed. So the number is

$$2^{10} - \binom{10}{10} - \binom{10}{9} - \binom{10}{8} = 1024 - 1 - 10 - 45 = 964.$$

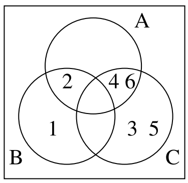
Chapter 3

3.1 $\{3, 4, 5, 6\}$.

3.3 Write H and T for heads and tails on the quarter, and 0, 1, 2 for the number of heads on the pennies. The sample space is $\{H0, H1, H2, T0, T1, T2\}$. The experiment consists of two parts. In the first, the outcomes are H and T; in the second, there are three outcomes, 1, 2, 3. (The two pennies are not flipped separately, so in the case of one head we don't need to worry about which penny got the head and which the tail. Only the number of heads was recorded.) The tree diagram is



3.4



3.5 $A \cap C = \{HHT, HTH\}$, $B \cup C = \{HHH, HHT, HTH, HTT, THT, TTH\}$, and $\overline{A} = \{HHH, HTT, THT, TTH\} = B$.

3.7 E consists of two possible rolls, so $P(E) = \frac{1}{3}$. similarly $P(F) = \frac{1}{2}$. G consists of the rolls 3 and 5, so $P(G) = \frac{1}{3}$.

3.8 Write HT to mean a head on the quarter and a tail on the nickel, and so on—the result for the quarter is shown first. Then there are four equally likely outcomes, HH, HT, TH, and TT, and two of them (HT and TH) are in the event. So

$$P(E) = \frac{|E|}{|S|} = \frac{2}{4} = \frac{1}{2}.$$

3.9 12 of the 52 cards are picture cards, so

$$P(E) = \frac{|E|}{|S|} = \frac{12}{52} = \frac{3}{13}.$$

3.10 The sample space is the set of five outcomes S_2, S_3, S_4, S_5, S_6, where S_j means that the sum is j. In terms of the slightly different experiment, in which we distinguish the two dice and record ab to mean a on die 1 and b on die 2, we have

$$\begin{aligned} S_2 &= \{11\}, \\ S_3 &= \{12, 21\}, \\ S_4 &= \{13, 22, 31\}, \\ S_5 &= \{23, 32\}, \\ S_6 &= \{33\}, \end{aligned}$$

and the new experiment has nine equally likely outcomes, so

$$P(S_2) = P(S_6) = \frac{1}{9}, \qquad P(S_3) = P(S_5) = \frac{2}{9}, \qquad P(S_4) = \frac{3}{9} = \frac{1}{3}.$$

3.11 Four of the marbles are blue and five are not blue, so

$$P(\text{not blue}) = \frac{|E|}{|S|} = \frac{5}{9}.$$

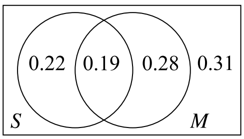
3.12

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\ &= 0.7 + 0.3 - 0.1 \\ &= 0.9, \\ P(\overline{E}) &= 1 - P(E) \\ &= 1 - 0.7 \\ &= 0.3. \end{aligned}$$

$E \cup \overline{F} = E \cup (\overline{E} \cap \overline{F}) = E \cup \overline{(E \cup F)}$, using De Morgan's laws. These two events are disjoint, so

$$\begin{aligned} P(E \cup \overline{F}) &= P(E) + P\big(\overline{(E \cup F)}\big) \\ &= P(E) + \big[1 - P(E \cup F)\big] \\ &= 0.7 + [1 - 0.9] \\ &= 0.8. \end{aligned}$$

3.13 The data are represented by



Since the percentages add to 100%, we can interpret "female" as "not male" (not always true in the insect kingdom) and the answers are (a) $P(S) = 41\%$; (b) $P(\overline{M}) = 53\%$; (c) $P(S \cup \overline{M}) = 72\%$.

3.14 There are three ways in which exactly one success can occur: the sequences SFF, FSF, and FFS. Now

$$P(SFF) = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27},$$
$$P(FSF) = \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{27},$$
$$P(FFS) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}.$$

So the probability of exactly two successes is

$$\frac{4}{27} + \frac{4}{27} + \frac{4}{27} = \frac{12}{27} = \frac{4}{9}.$$

3.15 In this case, each call is a Bernoulli trial with $p = \frac{1}{2}$, so the five calls in a day can be thought of as a binomial experiment with $p = \frac{1}{2}$, $n = 5$. So

$$P(3 \text{ successes}) = C(5,3)\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = 10 \times \frac{1}{32} = \frac{10}{32},$$
$$P(4 \text{ successes}) = C(5,4)\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = 5 \times \frac{1}{32} = \frac{5}{32},$$
$$P(5 \text{ successes}) = C(5,5)\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{32} = \frac{1}{32},$$

and the probability of at least two successes is the sum of these:

$$\frac{10+5+1}{32} = \frac{16}{32} = \frac{1}{2},$$

or 50%.

3.16 In this case, the salesman wants

$$\frac{3^{n+1}(3+n)}{4^n} < \frac{3}{10}.$$

We have already seen that eight calls are not enough. But:

$$n = 9 \frac{3^{n-1}(3+n)}{4^n} = \frac{6561 \times 12}{262144} = 0.3003\ldots$$

which is very slightly more than 0.30. Strictly speaking 10 calls are needed, but the salesman would probably be satisfied with nine calls.

3.17 There are 11 marbles, so the number of ways of selecting three is $C(11, 3)$. So $|S| = C(11, 3) = 165$. The number of ways of selecting one red marble is 5, the number of ways of selecting one blue is 4, and the number of ways of selecting one green is 2. So, by the multiplication principle, $|E| = 5 \times 4 \times 2 = 40$. So

$$P(E) = \frac{40}{C(11, 3)} = \frac{8}{33}.$$

3.18 There are $C(10, 2) = 45$ ways to choose two balls. In $C(4, 2) = 6$ of these cases, there are no white balls—both are blue—so there are 39 selections with at least one white, so

$$P(E) = \frac{|E|}{|S|} = \frac{39}{45} = \frac{13}{15}.$$

3.19 Again, there are $C(52, 5)$ different hands possible. To count all the possible flushes, observe that four suits are possible. If one is chosen—Spades, say—there are $C(13, 5)$ possible flushes. So there ate $4 \times C(13, 5)$ flushes in total. So the probability is

$$P(E) = \frac{|E|}{|S|} = \frac{4 \times C(13, 5)}{C(52, 5)} = \frac{33}{16660},$$

or about 0.2%.

3.20 There are 18 class members, so the number of possible committees is $C(18, 3)$. The number of committees with all Math majors is $C(6, 3)$; there are $C(5, 3)$ with all Economics majors; and $C(7, 3)$ have all computer Science majors. The number of committees with all members in the same major is $C(6, 3) + C(5, 3) + C(7, 3)$, and this is $|E|$. So

$$P(E) = \frac{|E|}{|S|} = \frac{C(6, 3) + C(5, 3) + C(7, 3)}{C(18, 3)} = \frac{6 \cdot 5 \cdot 4 + 5 \cdot 4 \cdot 3 + 7 \cdot 6 \cdot 5}{18 \cdot 17 \cdot 16},$$

and this comes to $65/816$, or about 8%.

3.21 Suppose you took a pencil and labeled the faces with a 1 on die B as $1a$, $1b$, $1c$ and those with a 6 as $6a$, $6b$, $6c$. If Xy means die X was rolled and the face showing was a y, then there are 12 equally likely outcomes, $A1$, $A2$, $A3$, $A4$, $A5$, $A6$, $B1a$, $B1b$, $B1c$, $B6a$, $B6b$, $B6c$. Each has probability $\frac{1}{12}$. Four rolls, namely $A6$, $B6a$, $B6b$, $B6c$, result in a 6, so the probability of a 6 is $P(6) = \frac{4}{12} = \frac{1}{3}$, while 3 results from roll $A3$ only, and $P(3) = \frac{1}{12}$.

3.22 Given that the first card is a Spade, the second card is a random selection from 51 equally likely possibilities. Twelve of these outcomes are Spades and thirteen are Hearts. So

(i) $P(S \mid S) = \frac{|E|}{|S|} = \frac{12}{51}$;

(ii) $P(H \mid S) = \frac{|E|}{|S|} = \frac{13}{51}$.

3.23 We write S and N for “a spade” and “a card other than a spade”. Then

$$P(S \mid S) = \frac{12}{51},$$
$$P(K \mid N) = \frac{13}{51}.$$

Given that the first card is a spade, the second card is either a spade or not, so

$$P(S \mid S) + P(N \mid S) = 1,$$

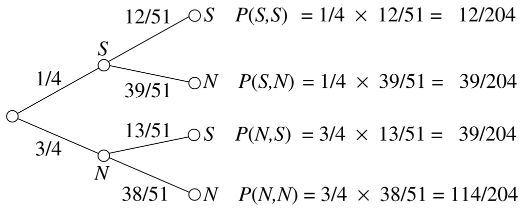
therefore

$$P(N \mid S) = \frac{39}{51},$$

and similarly

$$P(N \mid N) = \frac{38}{51}.$$

So the diagram is



(where, for example, “S, N” means “spade first, non-spade second”).

3.24 Write D for the event that at least one committee member is a Democrat, and B for the event that both are Democrats. There are $C(8, 2) = 28$ possible committees, and the number containing two Democrats is $C(3, 2) = 3$. So

$$P(B) = \frac{|B|}{|S|} = \frac{3}{28}.$$

Next, observe that there are $C(5, 2) = 10$ committees with both Republicans, so there are 15 possible committees with one member from each party. We can now proceed in two ways. Using the conditional probability formula, we observe that $P(D) = \frac{18}{28}$ and $P(B \cap D) = P(B) = \frac{3}{28}$ (if you think for a moment, you will realize that $B \cap D$ and B are the same event), so

$$P(B \mid D) = \frac{P(B \cap D)}{P(D)} = \frac{3}{18}.$$

Alternatively we can restrict ourselves to the new experiment “choose a committee of two at random from those committees with at least one Democrat”, so now $|S| = 18$, $|B| = 3$, and

$$P(B) = \frac{|B|}{|E|} = \frac{3}{18}.$$

3.25 In the obvious notation, $P(D) = \frac{2}{5}$ and $P(R \mid D) = \frac{1}{3}$, so

$$P(R \cap D) = P(R \mid D) \times P(D) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}.$$

3.26 $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{2}$. We expect these events to be independent because the two rolls are unconnected, and they are: when considered as one experiment, there are 36 outcomes; there are three outcomes in $A \cap B$, namely 6 on the red and 2, 4 and 6 on the green, so $P(A \cap B) = \frac{3}{36} = \frac{1}{12}$, and this equals $P(A) \times P(B)$.

3.27 From the data,

$$\begin{aligned} P(A) &= \frac{3}{8}, \\ P(B) &= \frac{1}{8}, \\ P(C) &= \frac{1}{2}, \\ P(A \cap B) &= 0, \\ P(A \cap C) &= \frac{2}{8}, \\ P(B \cap C) &= 0. \end{aligned}$$

No two are independent.

3.28 We write F for faulty, and X and Y to indicate the suppliers. Then $P(X) = 0.4$, $P(F \mid X) = 0.04$, $P(Y) = 0.6$, $P(F \mid Y) = 0.02$, and

$$\begin{aligned} P(F) = P(F \cap X) + P(F \cap Y) &= P(F \mid X)P(X) + P(F \mid Y)P(Y) \\ &= 0.04 \times 0.4 + 0.02 \times 0.6 \\ &= 0.016 + 0.012 = 0.028. \end{aligned}$$

3.29 Use B, F, H, T for *biased, fair, heads, tails.* Then

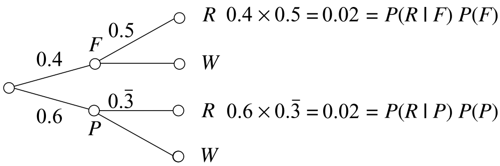
$$\begin{aligned} P(H) &= P(H \mid F)P(F) + P(H \mid B)P(B) \\ &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) \\ &= \frac{1}{3} + \frac{2}{9} = \frac{5}{9}; \\ P(B \cap H) &= P(H \mid B)P(B) = \frac{2}{9}. \end{aligned}$$

So $P(B \mid H) = \frac{2}{9} / \frac{5}{9} = \frac{2}{5}$.

3.30 Write E for the event that a blue pen is chosen. We know that $P(A) = P(B) = 0.5$. From Bayes' formula,

$$\begin{aligned} P(E)P(A \mid E) &= P(A)P(E \mid A), \\ 0.5 \times P(A \mid E) &= 0.5 \times 0.4, \\ P(A \mid E) &= 0.4. \end{aligned}$$

3.31



From the diagram, the denominator is 0.04. So

$$P(F \mid R) = \frac{P(R \mid F)P(F)}{0.04} = \frac{0.02}{0.04} = 0.5.$$

3.32 We want $P(D \mid V)$. From the diagram,

$$P(D \mid V) = \frac{0.112}{0.653} = 0.17.$$

3.33

	H	T	
B	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
U	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{5}{9}$	$\frac{4}{9}$	

3.34 We use the abbreviations T (tests positive), N (tests negative), D (has the disease) and H (is healthy). We want to find $P(D \mid T)$. From the data, we know

$$\begin{aligned} P(T \mid D) &= 0.95, & P(N \mid D) &= 0.05, \\ P(T \mid H) &= 0.05, & P(N \mid H) &= 0.95, \\ P(D) &= 0.05, & P(H) &= 0.95. \end{aligned}$$

So

$$\begin{aligned} P(D \mid T) &= \frac{P(T \mid D)P(D)}{P(T \mid H)P(H) + P(T \mid D)P(D)} \\ &= \frac{(0.95)(0.05)}{(0.05)(0.95) + (0.95)(0.05)} \\ &= \frac{0.0475}{0.0475 + 0.0475} = \frac{0.0475}{0.095} = 0.5. \end{aligned}$$

This example shows that, in some cases, the test gives no useful information.

3.35 Write E_i for the event that i boys are chosen and define the value of E_i to be i. What is required is the expected value of this experiment. If S is the sample space, then

$$|S| = C(8, 3) = 56.$$

If a committee contains i boys, it must contain $3 - i$ girls. There are $C(3, i)$ ways of choosing i boys from the three, and $C(5, 3 - i)$ ways of selecting $3 - i$ girls, so

$$|E_i| = C(4, i)C(6, 3 - i).$$

Therefore,

$$\begin{aligned}|E_0| &= C(3, 0) \times C(5, 3) = 10,\\ |E_1| &= C(3, 1) \times C(5, 2) = 30,\\ |E_2| &= C(3, 2) \times C(5, 1) = 15,\\ |E_3| &= C(3, 3) \times C(5, 0) = 1,\end{aligned}$$

so

$$p_0 = \frac{10}{56}, \qquad p_1 = \frac{30}{56}, \qquad p_2 = \frac{15}{56}, \qquad p_3 = \frac{1}{56},$$

and

$$\begin{aligned}\mu &= \frac{10 \times 0 + 30 \times 1 + 15 \times 2 + 1 \times 3}{56}\\ &= \frac{63}{56} = 1.125.\end{aligned}$$

So the expected number is 1.125.

3.36 Let E_1 denote the event that total 3 or 9 is rolled, E_2 denote the event that an even total (2, 4, 6, 8, 10 or 12) is rolled, and E_0 the event that some other total is rolled. We write x_i for the payoff to gambler B when E_i occurs, so $x_1 = 5$, $x_2 = -2$, and $x_0 = 0$. Out of every 36 rolls of the dice, we expect total 3 twice, 9 four times, and an even total 18 times. We have

$$\begin{aligned}&E_0, \text{probability } \frac{1}{3}, \text{value } x_0 = 0,\\ &E_1, \text{probability } \frac{1}{6}, \text{value } x_1 = 5,\\ &E_2, \text{probability } \frac{1}{2}, \text{value } x_2 = -2.\end{aligned}$$

So the expected value of the payout is

$$\frac{1}{3} \times 0 + \frac{1}{6} \times 5 - \frac{1}{2} \times 2 = -\frac{1}{6}.$$

So A expects to win in the long run, at an average of $\$\frac{1}{6}$ per play.

3.37 With no information to the contrary, we assume that whether or not she bowls a strike in one frame does not affect her performance in other frames. So we can model her performance as a binomial experiment, in which each frame is a Bernoulli trial and success equates with a strike. We have $n = 40$, $p = \frac{1}{5}$, and $\mu = 40\frac{1}{5} = 8$.

Chapter 4

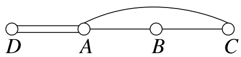
4.1 $s \le t$ corresponds to $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4)\}$. $s \ge t$ corresponds to $\{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$.

4.2 Both $=$ and $\ge$ are reflexive. $=$ is symmetric while $\ge$ antisymmetric. Both are transitive. So $=$ is an equivalence relation, but $\ge$ is not.

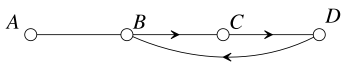
4.3 $f_1(f_1(x)) = f_1(x)$ so $f_1(f_1) = f_1$, $f_2(f_4(x)) = f_2(1/(1-x)) = 1/[1/(1-x)] = 1-x = f_3(x)$ so $f_2(f_4) = f_3$, $f_5(f_6(x)) = f_5((x-1)/x) = ((x-1)/x)/[((x-1)/x)-1] = ((x-1)/x)/(-1/x) = (x-1)/(-1) = 1-x = f_3(x)$ so $f_5(f_6) = f_3$ also.

4.4 $f_3(x) = 1-x$. If $f_3(x) = y$ then $y = 1-x$ so $x = 1-y$ and $f_3^{-1}(y) = 1-y$. So $f_3^{-1} = f_3$.

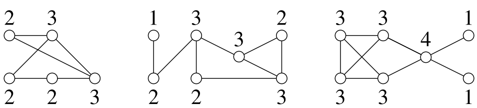
4.5



4.6



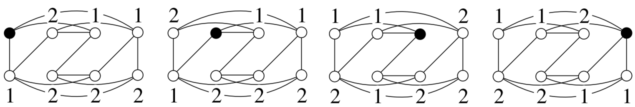
4.8



4.9 Suppose such a graph existed. Let x and y be the vertices of degrees 5 and 4, respectively. Then x is adjacent to all five other vertices, y being one of them. Therefore, y is adjacent to three vertices other than x. Of the remaining four vertices, at least three are adjacent both to x and to y, so they have degree at least 2. This is impossible.

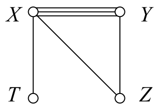
4.11 In a city's road network, it represents an intersection such that all traffic between two areas has to pass through that intersection. As such, it might get a traffic officer assigned in peak hours, or special priority might be given to servicing its traffic lights.

4.12 There are four different-looking vertices, shown in black. The distance of every other vertex from the black one is shown in each case. (The top left and second from left vertices are actually equivalent, but this is not easy to see.)



4.13 Look at the above diagram. The graph has $D = R = 2$.

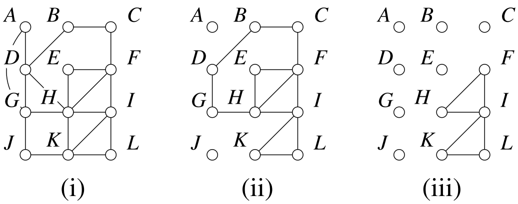
4.14 The walk must start (or finish) at T because there is only one bridge to that island. With a little experimentation we find the solution $TXYZXYX$.



4.15 The original network is shown in Figure (i). We start from A, and randomly choose the walk $AGJKHDA$. After these edges are deleted, Figure (ii) remains.

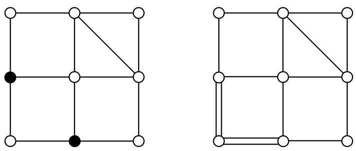
We now start at D (alphabetically, the first vertex remaining that was in the first walk and is not yet isolated). One walk is $DBCFEHGD$, and its deletion leaves Figure (iii).

Finally, walk $FILKIHF$ uses up the remaining edges.



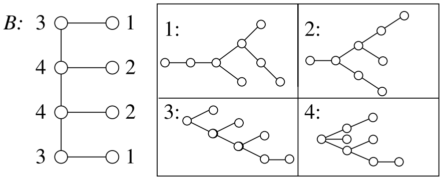
Putting these together, we get $AGJKHDBCFILKIHFEHGDA$.

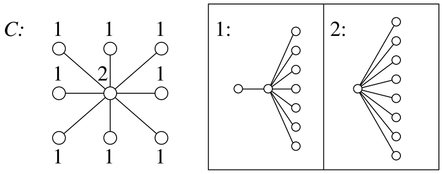
4.16 The original is shown on the left. The two black vertices need another edge. They are not joined, so one edge will not suffice. So $eu(H) > 1$. The right-hand shows an Eulerization that requires just two edges. So $eu(H) = 2$, and the right-hand picture is a good Eulerization.



4.18 Graph (a) is not a tree because it is not connected; it has an isolated vertex. Graph (b) is a tree. Graph (c) contains a cycle, so it is not a tree.

4.19 There are four different-looking diagrams for the second tree and two for the third. We show which trees correspond to the different starting vertices using the same notation as in the sample problem.





4.20 If a tree has four vertices, then the largest possible degree is 3. Moreover, there are three edges (by Theorem 13), so from Theorem 10 the sum of the four degrees is 6. As there are no vertices of degree 0 and at least two vertices of degree 2, the list of degrees must be one of 3, 1, 1, 1 or 2, 2, 1, 1. In the first case, the only solution is the star $K_{1,3}$. The only case with the second degree list is the path P_4. So there are two trees.

4.22 Candidates (i) and (iv) are Hamiltonian. Candidate (ii) is not Hamiltonian because it contains a repeated vertex, b. Candidate (iii) is not Hamiltonian because the graph contains no edge cd.

4.23 To traverse vertex a, a Hamiltonian cycle in G must contain one of the paths bad, bae or ead. If bad is included, the other edge through d might be dc or de; in the former case, neither bc nor de can be edges, and the only cycle is $badcfe$, while the latter case bars be, and the only cycle is $badefc$. If bae is included, ad is not an edge, so cd and de are edges, so we have the path $baedc$, and the cycle is $baedcf$. If ead is included, de is not an edge, so dc is an edge, and there are two possibilities, $adcbfe$ and $adcfbe$. So there are five Hamiltonian cycles.

Any Hamiltonian cycle in H must contain edges ab and ad, because a has degree 2. This means bd is not an edge (it would form a triangle), so de must be in the cycle. There are two ways to finish a cycle: $bcfe$ or $bfce$. So there are two cycles: $dabcfe$ and $dabfce$.

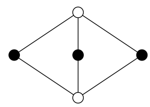
4.24 The nearest neighbor algorithm, starting from Evansville, begins with EM, because it has the least cost of the three edges incident with E. The next edge must have M as an endpoint, and ME is not allowed (one cannot return to E, it has already been used), so the cheaper of the remaining edges is chosen, namely MN. The only available edge from N is NS, as E and M have already been visited, and the route is EMNSE, with cost \$430.

Starting at Nashville, the first edge selected is NE, with cost \$90. The next choice is EM, then MS, then SN, and the resulting cycle NEMSN costs \$410. If you start at St. Louis, the first stop will be Memphis (\$110 is the cheapest flight from St. Louis), then Evansville, then Nashville, costing \$410. From Memphis, the cheapest leg is to Evansville, then Nashville, and finally St. Louis, for \$410. So both St. Louis and Memphis yield the same cycle as the Nashville case (with different starting points, and in reverse).

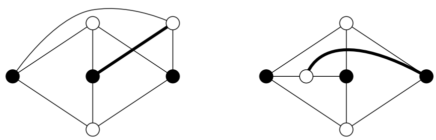
To apply the sorted edges algorithm, first sort the edges in order of increasing cost: EM(\$80), EN(\$90), NM(\$100), MS(\$110), ES (\$120), NS(\$130). Edge EM is included, and so is EN. The next choice would be MN, but this is not allowed because its inclusion would complete a cycle of length 3 (too short), so the only other choices are MS and NS, forming route EMSNE (or ENSME) at a cost of \$410.

In this example, the route ENMSE, with cost \$420, does not arise from either algorithm.

4.25 The graph $K_{2,3}$ can be drawn without crossings, as the illustration shows, and any representation of $K_{2,3}$ without crossings will look like this picture. Now $K_{3,3}$ can be constructed from $K_{2,3}$ by adding one vertex adjacent to the three black vertices. In the representation, this new vertex could be placed outside the diagram or inside one of the enclosed areas.

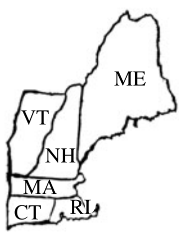


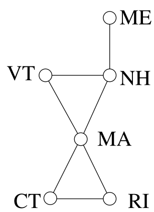
If it is placed outside, then the edge joining it to the central vertex must cross another edge. If it is placed inside one enclosed area, then the edge joining it to the vertex not on that enclosed area must cross an edge. That is, one of the following cases must occur (the thick line represents the offending edge). In either case, there is a crossing, so $\nu(K_{3,3}) > 0$.



These representations show $K_{3,3}$ with one crossing, so $\nu(K_{3,3}) = 1$.

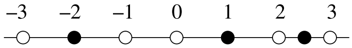
4.26



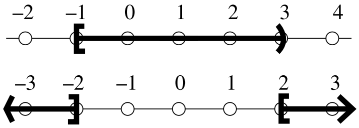


Chapter 5

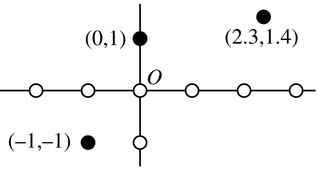
5.1



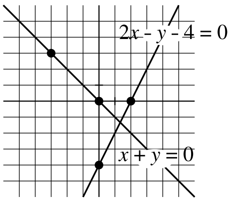
5.2



5.3



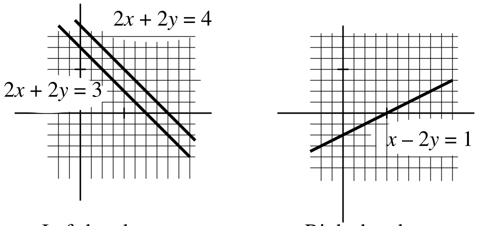
5.5



5.6 The line $2x + y - 4 = 0$ has slope -2. Its slope-intercept form is $y = -2x + 4$. The line $x + y = 0$ has slope -1 and slope-intercept form $y = -x$.

5.7 From $3x - y = 1$ we get $y = 3x - 1$. Substituting, $2x + (3x - 1) = 4$, or $5x = 5$, so $x = 1$. Therefore, $y = 3(1) - 1 = 2$. The solution is $x = 1$, $y = 2$; the point of intersection is $(1, 2)$.

5.8 In the first case, $2x+2y$ cannot equal both 3 and 4, so there are no solutions, and the solution set is empty. In the second system, whenever the first equation is true, the second will be true also: each side of the second equation is 2 times the corresponding side of the first one. The solution set could be written $\{(x, y) \mid x - 2y = 1\}$ or $\{(2y + 1, y) \mid y \in \mathbb{R}\}$. The graphs are



Left-hand system Right-hand system

5.9 We would like to eliminate y from the first equation. The coefficient of y in the second equation is -1, so we multiply both sides of the second equation by 3 and proceed to solve the equations

$$4x + 3y = 11,$$
$$3x - 3y = 3.$$

Adding the equations, we get

$$7x = 14,$$

so $x = 2$. Substituting this back into the second equation,

$$2 - y = 1,$$

so $y = 1$, and the solution is $(x, y) = (2, 1)$.

5.13 We eliminate x from the first and third equations. We add $-3\times$ the second equation to the first equation and $-2\times$ the second equation to the third equation; those two equations become

$$-y - z = -2,$$
$$-3y - 3z = -6,$$

so $y + z = 2$. The equations are equivalent to

$$y + z = 2,$$
$$x + y = 3.$$

The solution is $x = 3 - y$, $z = 2 - y$, for any real number y, or

$$(x, y, z) \in \{(3 - y, y, 2 - y) \mid y \in \mathbb{R}\}.$$

5.14 We eliminate z from the second equation by adding $-1\times$ the first equation to the second equation, obtaining

$$x - 5y = -2,$$

or $x = 5y - 2$.The first equation yields $z = x + 2y - 5 = 7y - 7$, and the solution is

$$(x, y, z) \in \{(5y - 2, y, 7y - 7) \mid y \in \mathbb{R}\}.$$

5.15 The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 3 & -2 & 1 \\ 3 & -2 & 4 & 6 \\ 2 & -3 & 2 & 8 \end{array}\right].$$

Its (2, 2) element equals −2.

5.18

$$\left[\begin{array}{cccc|c} 2 & -3 & -1 & -2 & 1 \\ 1 & 2 & -2 & 1 & 1 \\ 1 & -1 & 1 & 1 & 4 \end{array}\right],$$

$$\left[\begin{array}{cccc|c} 0 & -7 & 3 & -4 & -1 \\ 1 & 2 & -2 & 1 & 1 \\ 0 & -3 & 3 & 0 & 3 \end{array}\right] \quad \begin{array}{l} R1 \leftarrow R1 - 2R2 \quad \text{(using E3)}, \\ \\ R3 \leftarrow R3 + R2 \quad \text{(using E3)}, \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -7 & 3 & -4 & -1 \end{array}\right] \quad \begin{array}{l} R1 \leftarrow R2 \quad \text{(using E1)}, \\ R2 \leftarrow -\frac{1}{3}R3 \quad \text{(using E1, E2)}, \\ R3 \leftarrow R1 \quad \text{(using E1)}, \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & -4 & -4 & -8 \end{array}\right] \quad \begin{array}{l} R1 \leftarrow R1 - 2R2 \quad \text{(using E3)}, \\ \\ R3 \leftarrow R3 + 7R2 \quad \text{(using E3)}, \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow R2 - \frac{1}{4}R3 \quad \text{(using E3)}, \\ R3 \leftarrow -\frac{1}{4}R3 \quad \text{(using E2)}. \end{array}$$

So the solution (in terms of t) is

$$\begin{aligned} x &= 3 - t, \\ y &= 1 - t, \\ z &= 2 - t, \end{aligned}$$

with t any real number.

To solve in terms of x, pivot on position $(1, 4)$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ -1 & 1 & 0 & 1 & -2 \\ -1 & 0 & 1 & 0 & -1 \end{array}\right] \quad \begin{aligned} y &= -2 + x, \\ z &= -1 + x, \\ t &= \ \ 3 - x. \end{aligned}$$

To solve in terms of y, pivot on position $(2, 4)$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 \end{array}\right] \quad \begin{aligned} x &= 2 + y, \\ z &= 1 + y, \\ t &= 1 - y. \end{aligned}$$

To solve in terms of z, pivot on position (3, 4):

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \quad \begin{aligned} x &= 1 + z, \\ y &= -1 + z, \\ t &= 2 - z. \end{aligned}$$

5.19 Write S for sedans and P for pickups.

	S	P
Apr	32	16
May	44	12

5.20 The matrix has shape 4×3. Its third row is

$$\boxed{\begin{matrix} -1 & 4 & 6 \end{matrix}}$$

and its first column is

$$\boxed{\begin{matrix} 1 \\ 2 \\ -1 \\ 1 \end{matrix}}\,.$$

5.21 We have the two equations $2x = y$ and $2 = x$. The second gives $x = 2$, so the first gives $y = 2 \cdot 2 = 4$.

5.22 $-A = \left[\begin{smallmatrix} -1 & -3 \\ 1 & -2 \end{smallmatrix}\right]$, $3A - B = \left[\begin{smallmatrix} 5 & 9 \\ -4 & 2 \end{smallmatrix}\right]$, $B + C$ is not defined, as B and C are of different sizes.

5.24

$$B^T = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}.$$

5.25 $4(2, 0, -1) = (8, 0, -4)$, so $4(2, 0, -1) + (1, 4, -3) = (8, 0, -4) + (1, 4, -3) = (9, 4, -7)$.

5.27 $\boldsymbol{u} \cdot \boldsymbol{t} = (-1, 3, 0) \cdot (1, 2, 3) = (-1 \cdot 1) + (3 \cdot 2) + (0 \cdot 3) = (-1) + 6 + 0 = 5$; $(2\boldsymbol{u} - 3\boldsymbol{v}) \cdot \boldsymbol{t} = [(-2, 6, 0) + (-6, 6, -6)] \cdot (1, 2, 3) = (-8, 12, -6) \cdot (1, 2, 3) = (-8 \cdot 1) + (24 \cdot 2) + (-6 \cdot 3) = -8 + 12 - 18 = -2$.

5.29

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}; \qquad \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

5.30

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 & b_1 & p_1 \\ w_2 & b_2 & p_2 \\ w_3 & b_3 & p_3 \\ w_4 & b_4 & p_4 \end{bmatrix} \begin{bmatrix} 15 \\ 18 \\ 20 \end{bmatrix}.$$

5.31

$$CD = DC = \begin{bmatrix} 5 & 3 \\ -3 & -1 \end{bmatrix}.$$

5.32 Suppose C has inverse

$$E = \begin{bmatrix} x & y \\ z & t \end{bmatrix}.$$

Then $CE = I$, so

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The (2, 2) entries of the two matrices cannot be equal ($0 \neq 1$), so no such E exists. Similarly, if D has inverse

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix},$$

then

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} z & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and again we have the impossible equation $0 = 1$.

5.33 Suppose the inverse is

$$C = \begin{bmatrix} x & z \\ y & t \end{bmatrix}.$$

Then $BC = I$ means

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is equivalent to the four equations

$$\begin{aligned} 2x + y &= 1, & 2z + t &= 0, \\ x + y &= 0, & z + t &= 1. \end{aligned}$$

The left-hand pair of equations are easily solved to give $x = 1$ and $y = -1$, while the right-hand pair give $z = -1$ and $t = 2$. So the inverse exists, and is

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

5.34 For A: $R3 \leftarrow R3 - R1$; $R2 \leftrightarrow R3$; $R1 \leftarrow R1 - 2R2$, $R3 \leftarrow R3 + 2R2$; $R1 \leftarrow R1 + R3$. For B: $R1 \leftrightarrow R3$; $R2 \leftarrow R2 - 2R1$, $R3 \leftarrow R3 - 3R1$; $R3 \leftarrow R3 - 2R2$.

5.35

$$\det(C) = 1 \cdot 4 - 1 \cdot 3 = 1; \qquad C^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix},$$
$$\det(D) = 2 \cdot 2 - 4 \cdot 1 = 0; \qquad \text{no inverse.}$$

5.36 As observed in the Sample Problem, the matrix of coefficients has inverse

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix},$$

so the first system has solution $x = 4$, $y = 1$, $z = 4$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},$$

and the second system has solution $x = 3$, $y = 1$, $z = 1$.

5.38 The technology matrix is

$$T = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$

and

$$(I - T) = \begin{bmatrix} 0.7 & -0.2 \\ -0.5 & 0.5 \end{bmatrix}$$

which has determinant $0.35 - 0.10 = 0.25$. So

$$(I - T)^{-1} = 4 \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.7 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0.8 \\ 2 & 2.8 \end{bmatrix}$$

so

$$P = \begin{bmatrix} 2 & 0.8 \\ 2 & 2.8 \end{bmatrix} \begin{bmatrix} 25 \\ 50 \end{bmatrix} = \begin{bmatrix} 90 \\ 190 \end{bmatrix}.$$

So the required production is 90 tons of steel and 190 tons of vegetables.

Chapter 6

6.2 The quantity to be maximized is profit. The profit depends only on the number of acres of each crop, potatoes, corn, and beans. So we define variables:

$$x_1 = \text{number of acres of potatoes to be planted},$$
$$x_2 = \text{number of acres of corn},$$
$$x_3 = \text{number of acres of beans}.$$

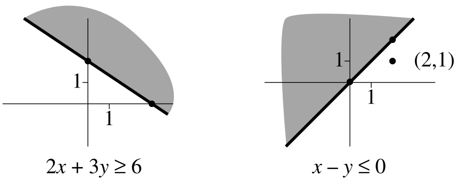
The profit is $\$P$, where $P = 150x_1 + 50x_2 + 80x_3$.
The farmer's outlay will be $\$(500x_1 + 250x_2 + 400x_3)$, and this cannot exceed \$30 000, and the total number of acres $(x_1 + x_2 + x_3)$ cannot exceed 120. If $x_3 > 20$, the excess beans will not be sold. So we have

$$\begin{aligned} \text{Maximize} \quad & P = 150x_1 + 50x_2 + 80x_3 \\ \text{subject to} \quad & 500x_1 + 250x_2 + 400x_3 \le 30000, \\ & x_1 + x_2 + x_3 \le 120, \\ & x_3 \le 20, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

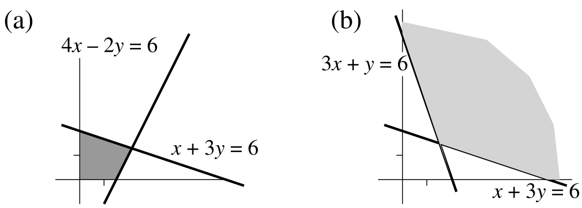
6.5 $(1, 0)$ is feasible, $P = 1$. $(4, 4)$ is not feasible ((A) and (B) are both false), $P = 8$. $(2, 1)$ is feasible, $P = 3$. $(-1, 1)$ is not feasible (x is negative), $P = 0$.

6.6 For $2x + 3y \ge 6$, we test with $(0, 0)$, and find that it does not satisfy the inequality; putting $x = 0$, $y = 0$ results in $0 \ge 6$, which is false. So we take the half-plane not containing the origin.

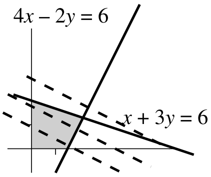
For $x - y \le 0$, $(0, 0)$ is not a suitable test point, so we try $(2, 1)$. This does not satisfy the inequality. The point $(2, 1)$ is also shown as a dot in the diagram. Both solution-sets are shown in gray.



6.8

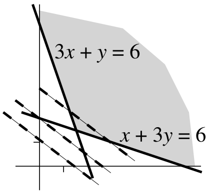


6.10



Some level curves are shown. The maximum clearly occurs at the intersection point, $x = \frac{15}{7}$, $y = \frac{9}{7}$, where $P = \frac{33}{7}$.

6.11



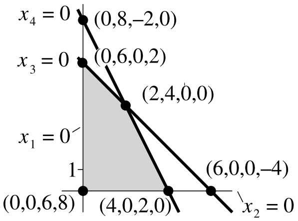
Some level curves are shown. The minimum clearly occurs at the intersection point, $x = \frac{3}{2}$, $y = \frac{3}{2}$, where $C = \frac{21}{2}$.

6.13

$$\begin{aligned} \text{Maximize} \quad & P = 3x_1 + 3x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & x_1 + 3x_2 + x_3 + 0x_4 = 2, \\ & 2x_1 - x_2 + 0x_3 - x_4 = 1, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

6.14

$$\begin{aligned} \text{Maximize} \quad & P = x_1 + 2x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + 0x_4 = 6, \\ & 2x_1 + x_2 + 0x_3 + x_4 = 8, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$



6.15

Non-basic variables	Basic variables	Basic solution	Feasible?	P
x_1, x_2	x_3, x_4	$(0, 0, 6, 8)$	Yes	0
x_1, x_3	x_2, x_4	$(0, 6, 0, 2)$	Yes	12
x_1, x_4	x_2, x_3	$(0, 8, -2, 0)$	No	16
x_2, x_3	x_1, x_4	$(6, 0, 0, -4)$	No	6
x_2, x_4	x_1, x_3	$(4, 0, 2, 0)$	Yes	4
x_3, x_4	x_1, x_2	$(2, 4, 0, 0)$	Yes	10

The problem is bounded (from the diagram in Problem 6.14, above), so the optimum value is $P = 12$, attained at $x_1 = 0$, $x_2 = 6$.

6.16 We need a form in which x_2 has coefficient 1 in one equation and 0, in the other, while x_3 plays the same role with the equations reversed. This can be achieved by adding a multiple of the second equation to the first. Adding $-2\times$ equation 2 to equation 1, the resulting equations are

$$-5x_1 + 0x_2 + x_3 - 2x_4 = -2,$$
$$3x_1 + x_2 + 0x_3 + x_4 = 3.$$

6.18 (i) The entry a_{12} is chosen because the ratio $\frac{4}{2} = 2$ is smaller than $\frac{6}{1} = 6$.

(ii) a_{12} is used because the ratio is negative.

6.19 Originally the basic feasible solution (4, 0, 6, 0) was exhibited.

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ \hline x_1 & 1 & \underline{2} & 0 & 4 & 4 \\ x_3 & 0 & 1 & 1 & 4 & 6 \end{array}\right],$$

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ \hline x_2 & \frac{1}{2} & 1 & 0 & 2 & 2 \\ x_3 & -\frac{1}{2} & 0 & 1 & 2 & 4 \end{array}\right] \qquad \begin{array}{l} R1 \to \frac{1}{2}R1, \\ R2 \to R2 - \frac{1}{2}R1. \end{array}$$

The basic feasible solution (0, 2, 4, 0) is now exhibited.

6.20

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_3 & 4 & 2 & 1 & 0 & 0 & 6 \\ x_4 & 2 & 4 & 0 & 1 & 0 & 6 \\ \hline & -2 & -3 & 0 & 0 & 1 & 0 \end{array}\right].$$

6.22

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_1 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{3}{2} \\ x_4 & 0 & 3 & -\frac{1}{2} & 1 & 0 & 3 \\ \hline & 0 & -2 & \frac{1}{2} & 0 & 1 & 3 \end{array}\right].$$

6.24 We pivot on a_{42}.

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_1 & 1 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 1 \\ x_2 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & 0 & 1 \\ \hline & 0 & 0 & \frac{1}{6} & \frac{2}{3} & 1 & 5 \end{array}\right].$$

The solution is $P = 5$, obtained by putting $x_1 = 1, x_2 = 1$.

6.25

$$\begin{aligned} \text{Maximize} \quad & P = -2x_1 + 3x_2 \\ \text{subject to} \quad & -2x_1 + 2x_2 \leq 5, \\ & 3x_1 + x_2 \leq 4, \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

6.26

Step 1. The problem is a maximization. We insert slack variables, obtaining

$$\begin{aligned} \text{Maximize} \quad & P = 5x_1 + 3x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & 2x_1 + 2x_2 + x_3 + 0x_4 = 4, \\ & 3x_1 + x_2 + 0x_3 + x_4 = 7, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

Step 2. The initial tableau is

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_3 & 2 & 2 & 1 & 0 & 0 & 4 \\ x_4 & 3 & 1 & 0 & 1 & 0 & 7 \\ \hline & -5 & -3 & 0 & 0 & 1 & 0 \end{array}\right].$$

Step 3. Column x_1 is a suitable pivot column.

Step 4. We pivot on entry a_{31}, obtaining

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & P & b \\ \hline x_1 & 1 & 1 & \frac{1}{2} & 0 & 0 & 2 \\ x_4 & 0 & -2 & -\frac{3}{2} & 1 & 0 & 1 \\ \hline & 0 & 2 & \frac{5}{2} & 0 & 1 & 10 \end{array}\right].$$

There is now no pivot. The maximum value is $P = 10$, obtained by putting $x_1 = 2, x_2 = 0$.

6.27

$$2x_1 - 2x_2 + 0x_3 + x_4 + 0x_5 + 0A_1 + 0A_2 = 4,$$
$$x_1 + 4x_2 + 0x_3 + 0x_4 - x_5 + A_1 + 0A_2 = 4,$$
$$x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 + 0A_1 + A_2 = 7,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0, \quad A_1 \geq 0, \quad A_2 \geq 0.$$

6.29

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	2	−2	0	1	0	0	0	0	4
A_1	1	4	0	0	−1	1	0	0	4
A_2	1	2	1	0	0	0	1	0	7
	4	−3	0	0	0	M	M	1	0

,

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	2	−2	0	1	0	0	0	0	4
A_1	1	4	0	0	−1	1	0	0	4
A_2	1	2	1	0	0	0	1	0	7
	4	−3	0	0	0	0	0	1	0
	$-2M$	$-6M$	$-M$	0	M	0	0	0	$-11M$

.

6.30 We pivot on row A_1, column x_2. We add $6M$ times the new row to the lower part of the objective row and 3 times the row to the upper part.

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	$\frac{5}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	6
x_2	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	1
A_2	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	5
	$\frac{19}{4}$	0	0	0	$-\frac{3}{4}$	$\frac{3}{4}$	0	1	3
	$-\frac{1}{2}M$	0	$-M$	0	$-\frac{1}{2}M$	$\frac{3}{2}M$	0	0	$-5M$

.

Next we pivot on row A_2, column x_3:

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	$\frac{5}{2}$	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	6
x_2	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	1
x_3	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	5
	$\frac{19}{4}$	0	0	0	$-\frac{3}{4}$	$\frac{3}{4}$	0	1	3
	0	0	0	0	0	M	M	0	0

.

6.31 The new tableau is

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	$\frac{5}{2}$	0	0	1	$-\frac{1}{2}$	0	6
x_2	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	0	1
x_3	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	5
	$\frac{19}{4}$	0	0	0	$-\frac{3}{4}$	1	3

We pivot on row x_3, column x_5, obtaining

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	3	0	1	1	0	0	11
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	$\frac{7}{2}$
x_3	1	0	2	0	1	0	10
	$\frac{11}{2}$	0	$\frac{3}{2}$	0	0	1	$\frac{21}{2}$

We have $P = \frac{21}{2}$, so $C = -\frac{21}{2}$, at $x_1 = 0, x_2 = \frac{7}{2}$.

Chapter 7

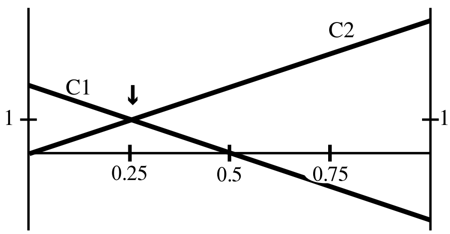
7.2 Game (i) has no dominant row or column. In (ii), column 2 is dominant—remember, a *smaller* result is better for the column player. The column player's strategy is (0, 1) and the row player's strategy is (1, 0).

7.3 The array is

				MIN
	2	2	3	∗2∗
	3	1	2	1
	−2	0	1	−2
MAX	3	∗2∗	3	

so there is a saddle point R1C2 with value 2.

7.4 The diagram is



The lines meet at $(\frac{1}{4}, 1)$. So the value is 1 and the row player's strategy is $(\frac{3}{4}, \frac{1}{4})$. For the column player, $(1 - z) + 0z = 1$ so $z = \frac{1}{2}$ and the column strategy is $(\frac{1}{2}, \frac{1}{2})$.

7.5 For this game $a = 2, b = -1, c = -2, d = 1$. So $a + d - b - c = 2 + 1 + 1 + 2 = 6$, $d - c = 3$, $a - b = 3$, $d - b = 2$, $a - c = 4$, $ad - bc = 0$. So the row strategy is

$$\left(\frac{d-c}{a+d-b-c}, \frac{a-b}{a+d-b-c}\right) = \left(\frac{3}{6}, \frac{3}{6}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

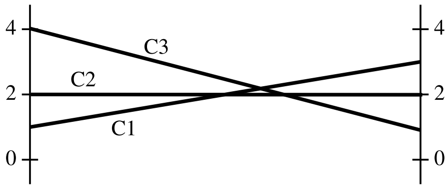
The column strategy is

$$\left(\frac{d-b}{a+d-b-c}, \frac{a-c}{a+d-b-c}\right) = \left(\frac{2}{6}, \frac{4}{6}\right) = \left(\frac{1}{3}, \frac{2}{3}\right).$$

The value is

$$v = \frac{ad - bc}{a + d - b - c} = \frac{0}{6} = 0.$$

7.7 From the diagram



it is clear that column 2 can be deleted. The solution to

1	4
3	1

is found from the formula to be row strategy $(\frac{2}{5}, \frac{3}{5})$, column strategy $(\frac{3}{5}, \frac{2}{5})$, value $(\frac{11}{5})$, so the original game has solution

$$\text{row strategy} \left(\frac{2}{5}, \frac{3}{5}\right), \qquad \text{column strategy} \left(\frac{3}{5}, 0, \frac{2}{5}\right), \qquad \text{value } \frac{11}{5}.$$

7.8 We first solve

2	−3	−2
−2	0	−1

.

This game was solved in Sample Problem 7.7; it has solution

$$\text{row strategy} \left(\frac{1}{5}, \frac{4}{5}\right), \qquad \text{column strategy} \left(\frac{1}{5}, 0, \frac{4}{5}\right), \qquad \text{value } -\frac{6}{5}.$$

So the original has solution

$$\text{row strategy } \left(\frac{1}{5}, 0, \frac{4}{5}\right), \qquad \text{column strategy } \left(\frac{1}{5}, \frac{4}{5}\right), \qquad \text{value } \frac{6}{5}.$$

7.10 We add 3; the revised game is

$$\begin{array}{|ccc|} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{array}.$$

For this, we start with the tableau

$$\left[\begin{array}{c|ccccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & P & b \\ \hline y_4 & 4 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \\ y_5 & 2 & 4 & 2 & 0 & 1 & 0 & 0 & 1 \\ y_6 & 1 & 2 & 4 & 0 & 0 & 1 & 0 & 1 \\ \hline & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

We pivot on row y_5, column y_2:

$$\left[\begin{array}{c|ccccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & P & b \\ \hline y_4 & 3 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ y_2 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ y_6 & 0 & 0 & 3 & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \hline & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{4} & 0 & 1 & \frac{1}{4} \end{array}\right].$$

The next pivot is the (y_4, y_1) position:

$$\left[\begin{array}{c|ccccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & P & b \\ \hline y_1 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & \frac{1}{6} \\ y_2 & 0 & 1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 & \frac{1}{12} \\ y_6 & 0 & 0 & 3 & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \hline & 0 & 0 & -\frac{1}{2} & \frac{1}{6} & \frac{1}{6} & 0 & 1 & \frac{1}{3} \end{array}\right].$$

The final pivot is (y_6, y_3):

$$\left[\begin{array}{c|ccccccc|c} \text{BV} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & P & b \\ \hline y_1 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & \frac{1}{6} \\ y_2 & 0 & 1 & 0 & -\frac{1}{6} & \frac{5}{12} & -\frac{1}{6} & 0 & \frac{1}{12} \\ y_3 & 0 & 0 & 1 & 0 & -\frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ \hline & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{12} & \frac{1}{6} & 0 & \frac{5}{12} \end{array}\right].$$

So the revised game has value $\frac{12}{5}$, and the original game has value $-\frac{3}{5}$. The column player's strategy is $\frac{12}{5}(\frac{1}{6}, \frac{1}{12}, \frac{1}{6}) = (\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$, and the row player's strategy is also $(\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$.

Chapter 8

8.1 Use the formula $A = P(1 + nR/100)$ with $P = 1200$ and $R = 10$. In the first case, n is $\frac{1}{4}$, so

$$P = 1200 \times (1 + 0.10/4) = 1200 \times (1.025) = 1230$$

and you repay \$1230. In the second case, n is 3, so

$$P = 1200 \times (1 + 0.10 \times 3) = 1200 \times (1.3) = 1560,$$

and you repay \$1560.

8.2 From $A = P(1 + nR/100)$ with $A = 1428$, $P = 1400$ and $n = \frac{1}{3}$, we get

$$1428 = 1400 \times \left(1 + \frac{R}{300}\right) = 1400 + \frac{14}{3} \times R,$$

and therefore $14 \times R = 28 \times 3$, $R = 6$. So the interest rate was 6%.

8.4 We have $P = 2200$, $A = 2530$ and $n = 2$. So

$$2530 = 2200\left(1 + 2 \times \frac{R}{100}\right) = 2200 + 44R; \quad R = \frac{330}{44} = 7.5.$$

The rate was 7.5%.

8.5 Simple interest is 3% per month, or 9% for the whole period. This comes to \$54, so the total to be paid is \$654. There are three monthly payments. Therefore, the monthly payment will be \$654/3, or \$218.

8.8 For arithmetic growth,

$$1200\left(1 + \frac{10 \times 4}{100}\right) = 1200 \times 1.4 = 1680$$

so the interest is \$480. Under geometric growth, the amount is

$$1200\left(1 + \frac{10}{100}\right)^4 = 1200 \times 1.1^4 = 1200 \times 1.4641 = 1756.92$$

and the interest is \$556.92.

8.10 In this case $P = 40000$, $R = 12$, $t = 4$ and $N = 15$. So

$$P\left(1 + \frac{R}{t \times 100}\right)^{tN}$$

becomes

$$40000 \times \left(1 + \frac{12}{400}\right)^{60} = 40000 \times (1.03)^{60} = 0 \times 5.8916$$

which comes to 235664.12..., and you owe \$235664.

8.11 The amount owing from a \$100 loan at the end of one year is

$$\begin{aligned} A &= 100\left(1 + \frac{12}{12 \times 100}\right)^{12} \\ &= 100 \times 1.01^{12} \\ &= 112.68, \end{aligned}$$

so the APY is 12.68%.

8.12 Again, 6% annual interest is 0.5% per month, so $m = 0.005$, $D = 100$, but in this case $n = 12$, so you get

$$\begin{aligned} &\$100(1.005^{10} - 1.005)/0.005 \\ &\quad = \$100(1.05114 - 1.005) \times 200 \\ &\quad = \$20000(0.04614) \\ &\quad = \$922.80. \end{aligned}$$

8.13 We use the formula

$$P(1+m)^n = \frac{D \times [(1+m)^n - 1]}{m}$$

with $P = 12000$, $n = 60$ and $M = \frac{2}{3}$, $(1+m) = \frac{151}{150}$. Now

$$\left(\frac{151}{150}\right)^{60} = 1.4858457\ldots,$$

so

$$12000 \times \left(\frac{151}{150}\right)^{60} = \frac{D \times [(\frac{151}{150})^{60} - 1]}{\frac{1}{150}}$$

becomes

$$12000 \times 1.4858457 = 150 \times D \times 0.4858457;$$

that is,

$$D = 12000 \times 1.4858457/(150 \times 0.4858457) = 244.676,$$

and your payment is \$244.68.

8.14 The interest rate is $1/150$ per month. So the accumulation after 24 years at 8% is

$$\$100000 \times (151/150)^{288}$$

or \$677763.55. If you deposit $\$D$ per month, your accumulated payments would be

$$\frac{\$D \times [(151/150)^{288} - 1)]}{1/150} = \$D \times 5.77763554 \times 150 = \$D \times 866.645.$$

So your monthly payment is $D = 782.05$.

8.15 $m = 7.5/12\% = 5/8\% = 1/160$, $D = 200$, $n = 60$. So

$$mA(1+m)^n = D \times [(1+m)n - 1]$$

becomes

$$A(161/160)^{60}/160 = 200[(161/160)^{60} - 1].$$

So $A \times 1.45329 = 160 \times 200 \times 0.45329$, $A = 9980.99$, and you can afford about \$9981.

8.16 In this case $P = 40000$, $R = 12$, $N = 15$ and $r = 0.12$. So

$$Pe^{rN} = 40000 \times e^{1.8} = 241985.89\ldots,$$

and you owe \$241986. Compare this with the result of Your Turn problem 8.10, where compounding was quarterly and the answer was \$235664.

8.17 Suppose the rate is r. Then $40000e^{7r} = 80000$ so $e^{10r} = 2$. Therefore, e^r equals the seventh root of 2, which equals 1.1041 approximately. The APR is approximately 10.4%.

8.18

$$\frac{118.3}{198.7} \times \$225000 = \$133958.23.$$

8.20 After one day, you have 120 grams; after two, 60; after three, 30. So after four days there are 15 grams remaining.

Answers to Exercises A

Section 1.1

1. (i) T; (ii) F; (iii) F; (iv) T; (v) F; (vi) F; (vii) T; (viii) T; (ix) T; (x) F. **2.** (i) 2, 4, 6, 8, 10; (ii) Red, White, Blue; (iii) Sunday, Saturday. **3.** (i) Only. **5.** (i) $\mathbb{Q}$, $\mathbb{R}$; (ii) $\mathbb{Q}$, $\mathbb{R}$; (iii) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$; (iv) $\mathbb{R}$; (v) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$; (vi) $\mathbb{R}$. **7.** (i) 1, 2, 3, 4, 6, 9, 12, 18, 36; (ii) 1, 2, 3, 4, 6, 8, 12, 16, 24, 48; (iii) 1, 2, 5, 10, 25, 50; (iv) 1, 3, 9, 27, 81; (v) 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72; (vi) 1, 61. **8.** (i) $2^2 \cdot 3^2 \cdot 29$; (ii) $2^2 \cdot 67$; (iii) 2^8; (iv) $3^2 \cdot 37$. **9.** (i) $2^3 \cdot 11, 2^2 \cdot 3 \cdot 11, 44$; (ii) $2^8, 2^5 \cdot 7, 32$; (iii) $2^3 \cdot 3^3 \cdot 5, 3^2 \cdot 5 \cdot 19, 45$; (iv) $2^3 \cdot 3 \cdot 7, 3 \cdot 7 \cdot 11, 21$. **10.** (i) 1; (ii) $4x^4/y^6$; (iii) xy; (iv) $2 \cdot f^3$ or 250; (v) $2b$; (vi) $41/(x+y)^3$; (vii) a^2; (viii) $\frac{1}{2}q$. **11.** (i) 117; (ii) 45; (iii) 28; (iv) -107; (v) 11.4; (vi) 2; (vii) 27; (viii) -73; (ix) 1; (x) -8.

Section 1.2

1. (i) Yes; (ii) No; (iii) No; (iv) Yes; (v) Yes; (vi) No. **2.** (i) 6; (ii) 0; (iii) 1; (iv) 5; (v) -2; (vi) 2; (vii) 6; (viii) $\frac{24}{17}$; (ix) $-\frac{1}{2}$; (x) -1; (xi) 2; (xii) $\frac{13}{2}$. **3.** (i) $\frac{2}{3}x - \frac{3}{2}$; (ii) $3 - x$; (iii) $x - 2$; (iv) $\frac{1}{3}x^2 + 1$; **4.** (i) $x > -\frac{3}{5}$; (ii) $x > 1$; (iii) $x \le 2$; (iv) $x \ge \frac{6}{5}$; (v) $x \le -1$; (vi) $x < \frac{5}{2}$; (vii) $x \ge \frac{1}{2}$; (viii) $x > 0$. **5.** (i) $y \le 2 - 2x$; (ii) $y \le 3 - 2x$; (iii) $y > 4x$; (iv) $y > \frac{1}{2}x - \frac{3}{2}$; (v) $y \ge \frac{1}{2}x - 2$; (vi) $y > 1 + x$.

Section 1.3

1. (i) $2+5+10+17+26+37 = 97$; (ii) $0.1 + 0.01 + 0.001 = 0.111$; (iii) $0 + 4 + 10 + 18 + 28 + 40 + 54 = 154$; (iv) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$; (v) $4 - 8 + 16 = 12$; (vi) $0+2+0+2 = 4$; (vii) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$; (viii) $4+7+10+13+16+19 = 69$. **2.** (i) $\sum_{i=1}^{5} 3i - 2$; (ii) $\sum_{i=1}^{3} 4i - 2$; (iii) $\sum_{i=0}^{4} (-1)^i (3i+1)$; (iv) $\sum_{i=0}^{4} (-1)^i i^2$; (v) $\sum_{i=1}^{6} 5 + 2(-1)^i$; (vi) $\sum_{i=1}^{4} i^2 - 1$. **3.** (i) $\sum_{i=1}^{n} b_i = (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + ((n-1)^3 - (n-2)^3) + (n^3 - (n-1)^3) = n^3 - 0^3 = n^3$;

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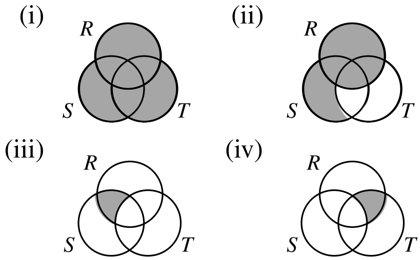
(ii) $b_i = i^3-(i-1)^3 = i^3-i^3+3i^2-3i+1 = 3i^2-3i+1$, so $\sum_{i=1}^n b_i = 3\sum_{i=1}^n i^2 - 3\sum_{i=1}^n i + \sum_{i=1}^n 1 = 3\sum_{i=1}^n i^2 - 3\sum_{i=1}^n i + n$; (iii) We know $\sum_{i=1}^n b_i = n^3$, so $n^3 = 3\sum_{i=1}^n i^2 - 3\sum_{i=1}^n i + n$. Using the formulae for $\sum_{i=1}^n i$, $n^3 = 3\cdot\sum_{i=1}^n i^2 - 3\cdot\frac{1}{2}n(n+1)+n$ so $\cdot \sum_{i=1}^n i^2 = \frac{1}{3}[n^3+3\cdot\frac{1}{2}n(n+1)-n]$. Simplifying gives the result. **4.** (i) $\sum_{i=1}^n c_i = \sum_{i=1}^n (2a_i+1) = \sum_{i=1}^n 2a_i + \sum_{i=1}^n 1 = 2A+n$; (ii) $3A-B$; (iii) $A+B+2n$; (iv) $A+B-1$ if n is odd; $A+B$ if n is even. **5.** (i) 39; (ii) 971; (iii) 124; (iv) $\frac{1}{6}n(n^2+3n+7)$; (v) $\frac{1}{2}(3^{n+1}-3)$; (vi) $n^2(2n+3)$; (vii) $n(n+1)(2n+1)/6-2\times n(n+1)/2 = n(n+1)(2n-5)/6$; (viii) $(1-\frac{1}{2}^n)/(1-\frac{1}{2})-1 = 1-1/2^n$.

Section 1.4

1. (i) 3, 5, 7, 9; (ii) Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday; (iii) 12, 15, 18; (iv) 2, 3, 5, 7; (v) 2; (vi) 1, 2, 3, 4, 6, 9, 12, 18, 36. **2.** (i) Set of multiples of 3 between 1 and 10; (ii) Set of all positive multiples of 3; (iii) Solution set of $x^2+9x=10$; (iv) Set of perfect squares less than 10; (v) Solution set of $x^2-4x+3=0$. **3.** (i) a, b, c, d, e, g, i; (ii) c, e; (iii) b, d; (iv) a, b, c, d, e, g. **4.** (i) 1, 2, 3, 4, 5, 6, 7, 8, 9; (ii) 1, 2, 4, 5, 6, 7, 9; (iii) 1, 5; (iv) 1, 5. **5.** (i) $\{1, 3, 5\}$; (ii) $\{5, 6, 7\}$; (iii) $\{1, 3, 6, 7\}$; (iv) $\{1, 2, 3, 4, 5, 6, 7, 8\}$; (v) $\emptyset$; (vi) $\{1, 2, 3, 4\}$. **6.** (i) $S_3 \subseteq S_1$, $S_4 \subseteq S_1$, $S_4 \subseteq S_3$; (ii) $S_1 \cap S_2 = \{2\}$, $S_1 \cap S_3 = \{2, 3, 5, 7\}$, $S_1 \cap S_4 = \{2, 3, 5\}$, $S_1 \cap S_5 = \{2\}$, $S_2 \cap S_3 = \{2\}$, $S_2 \cap S_4 = \{2\}$, $S_2 \cap S_5 = \{2, 4, 8\}$, $S_3 \cap S_4 = \{2, 3, 5\}$, $S_3 \cap S_5 = \{2\}$, $S_4 \cap S_5 = \{2\}$. **7.** (i) No; set of all squares of multiples of 5; (ii) No; S. **8.** (i) (a, a), (a, c), (a, f), (b, a), (b, c), (b, f), (c, a), (c, c), (c, f); (ii) (2, 2), (2, 3), (2, 4), (2, 5), (−2, 2), (−2, 3), (−2, 4), (−2, 5); (iii) (2, 4), (2, 5), (3, 4), (3, 5); **9.** (i) (1, 3, 5), (2, 3, 5), (1, 4, 5), (2, 4, 5), (1, 3, 6), (2, 3, 6), (1, 4, 6), (2, 4, 6); (ii) $(x, z, 1)$, $(y, z, 1)$, $(x, z, 2)$, $(y, z, 2)$.

Section 1.5

1.



4. (i) 367; (ii) 189; (iii) 871. **5.** 25. **6.** 2. **7.** (i) 5; (ii) 10; (iii) 15; (iv) no. **8.** (i) 10; (ii) 24; (iii) 10. **9.** (i) 5; (ii) 20; (iii) 18; (iv) 17.

Section 1.6

1. (i) 22.25; 21; (ii) 16.4; 13; (iii) 19.5; 20.5. **2.** (i) 5.5; 2; 3.5; (ii) 3.125; 6; 5.5; (iii) 6.5; no mode; 8. **3.** 5; −3, −2, 0, −3, 2, 4, 2. **4.** (i) 4.796; (ii) 4.256; (iii) 4.272. **5.** 3.692; 1.727.

Section 2.1

1. 22; 11. **2.** 6; 16. **3.** 18; 36. **4.** 12; 10. **5.** 7; 5. **6.** (i) 350; (ii) 475. **7.** 50; 150. **8.** $2^8 = 256$. **9.** (i) 12; (ii) 14. **10.** (i) $26^3 \times 10^3 = 17576000$; (ii) $24^3 \times 10^3 = 13824000$. **11.** ABC, ACB, BAC, BCA, CAB, CBA. **12.** 144. **13.** (i) $3^8 = 6561$; (ii) $4^8 = 65536$. **14.** $5! = 120$. **15.** $5! = 120$. **16.** $10 \times 9 \times 8 = 720$. **17.** (i) $10! = 3628800$; (ii) $6! \times 4! = 17280$.

Section 2.2

1. (i) 336; (ii) 24; (iii) 120; (iv) 72. **2.** $26 \times 26 \times 26 = 17576$. **3.** (i) $4! = 24$; (ii) $3! \times 2 = 12$. **4.** (i) $5! = 120$; (ii) $4! \times 2 = 48$. **5.** (i) $5! \times 3! = 720$; (ii) $4! \times 4! = 576$. **6.** (i) $9!/2!2!2! = 45360$; (ii) $9!/2!2! = 90720$; (iii) $6!/2! = 360$; (iv) $10!/3!2!2! = 151200$. **7.** (i) 15!; (ii) $3! \times 4! \times 5! \times 6! = 24883200$. **8.** (i) $8 \times 10^5 = 800000$; (ii) $8 \times P(9, 5) = 120960$.

Section 2.3

1. $\{A, B, C\}$, $\{A, B, D\}$, $\{A, B, E\}$, $\{A, C, D\}$, $\{A, C, E\}$, $\{A, D, E\}$, $\{B, C, D\}$, $\{B, C, E\}$, $\{B, D, E\}$, $\{C, D, E\}$. **2.** (i) 56; (ii) 126; (iii) 20; (iv) 35; (v) 1; (vi) 28. **3.** $C(16, 4) = 1820$. **4.** $C(6, 2) \times C(12, 3) = 3300$. **5.** $C(12, 5) = 792$; $C(9, 3) = 84$. **6.** (i) $C(49, 5)$; (ii) 44. **7.** $C(5, 3) \times C(7, 3) = 350$. **8.** 20.

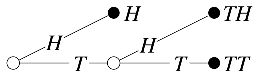
Section 2.4

2. (i) 60; (ii) $40y^2$; (iii) $-84x^2$; (iv) $-6x$. **3.** (i) $x^4 - 4x^3 + 6x^2 - 4x + 1$; (ii) $1 - 10z + 40z^2 - 80z^3 + 80z^4 - 32z^5$; (iii) $x^4 + 4x^2 + 6 + 4x^{-2} + x^{-4}$; (iv) $x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3xz^2 + 3y^2z + 3yz^2 + 6xyz$; (v) $x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$. **4.** 1.030301. **5.** 1.05101. **6.** $2^7 = 128$.

Section 3.1

1. (i) $\{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$; (ii) $\{2H, 2T, 4H, 4T, 6H, 6T\}$. **2.** (i) $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$; (ii)(a) $E = \{BBB, BBG, BGB, BGG\}$, $F = \{BBG, BGB, GBB\}$. **3.** (ii) 4; (iii) $\{F, PF\}$. **4.** (i) {22, 21, 12, 20, 11, 02, 10, 01, 00}; (ii) $E = \{21, 20, 10\}$, $F = \{20, 11, 02\}$,

$G = \{22, 20, 11, 02, 00\}$; (iii) $E \cup F = \{21, 20, 10, 11, 02\}$, $E \cap F = \{20\}$, $E \cap G = \{20\}$, $\overline{F} \cap G = \{22, 00\}$; (iv) $E \cap F$: there are exactly two heads on quarters and none on nickels, $\overline{F} \cap G$: the tosses are either all heads or all tails. **5.** (i)



(ii) $\{H, TH, TT\}$; (iii) $\{TH, TT\}$. **6.** (i) $\overline{E}$; (ii) $E \cap F$; (iii) $(E \cup F) \cap \overline{(E \cap F)}$; (iv) $E \cup F$; (v) $\overline{E} \cap \overline{F}$; (vi) $E \backslash F$. **7.** (i) Yes; (ii) No; (iii) No; (iv) Yes. **8.** Yes.

Section 3.2

1. (i) Yes; (ii) No (total $>$ 1); (iii) Yes; (iv) No (there is a negative probability). **2.** (i) $\frac{1}{2}$; (ii) $\frac{1}{2}$; (iii) $\frac{1}{4}$. **3.** (i) $\frac{9}{19}$; (ii) $\frac{9}{19}$; (iii) $\frac{1}{19}$; (iv) $\frac{1}{38}$; (v) $\frac{6}{19}$. **4.** (i) $\frac{1}{9}$; (ii) $\frac{1}{2}$; (iii) $\frac{1}{2}$. **5.** $\frac{5}{12}$. **6.** 0.1. **7.** (i) $\frac{1}{2}$; (ii) $\frac{1}{10}$; (iii) $\frac{9}{100}$. **8.** (i) 12, 13, 14, 15, 22, 23, 24, 25, 32, 33, 34, 35, 42, 43, 44, 45; (ii) $\frac{3}{16}$; (iii) $\frac{1}{2}$. **9.** (i) $\frac{1}{5}$; (ii) $\frac{2}{5}$; (iii) $\frac{2}{5}$. **10.** (i) $\{R, R, R\}$, $\{R, R, B\}$, $\{R, B, B\}$, $\{B, B, B\}$; (ii) $\frac{1}{56}$, $\frac{15}{56}$, $\frac{30}{56}$, $\frac{10}{56}$, respectively.

Section 3.3

1. 0.3; 0.8; 0.2. **2.** 0.3, 0.3, 0.1, 0.7. **3.** 0.4; 0.9; 0.9. **4.** (i) 0.3, 0.8, 0.8; (ii) 0.2; 0.3. **5.** (i) 0.225; (ii) 0.325. **6.** 0.4. **7.** (i) 0.625; (ii) 0.35; (iii) 0.125. **8.** (i) 0.5; (ii) 0.25; (iii) 0.2. **9.** (i) $\frac{8}{81}$; (ii) $\frac{48}{81}$. **10.** $\binom{20}{16}/2^{20}$ (about 1 chance in 200). **11.** (i) $8(\frac{7}{8})^7\frac{1}{8}$ (about 39%); (ii) $8(\frac{7}{8})^7\frac{1}{8}+(\frac{7}{8})^8$ (about 74%). **12.** $\frac{160}{729}$ (about 22%). **13.** $\frac{5}{16}$. **14.** (i) $\frac{8}{27}$; (ii) $\frac{1}{27}$.

Section 3.4

1. (i) $\frac{1}{15}$; (ii) $\frac{11}{15}$. **2.** (i) $\frac{1}{55}$; (ii) $\frac{14}{55}$. **3.** (i) $\frac{1}{3}$; (ii) $1-\frac{9\cdot 9\cdot 8\cdot 7}{9000} = 0.496$. **4.** $\frac{3}{10}$. **5.** (i) $\frac{2}{5}$; (ii) $\frac{8}{15}$. **6.** $10 \times 4^5/C(52, 5)$. **7.** (i) $\frac{1}{26}$; (ii) $\frac{63}{130}$; (iii) $\frac{1}{2}$. **8.** $C(5, 2)\cdot C(4, 2)/[C(9, 4)-6] = \frac{1}{2}$. **9.** (i) $\frac{1}{6}$; (ii) $\frac{1}{3}$. **10.** $[\binom{6}{3} + \binom{4}{3}]/\binom{10}{3} = \frac{1}{5}$. **11.** (i) $\frac{1}{10}$; (ii) $\frac{9}{10}^{5} = 0.59049$ (about 60%). **12.** (i) $\frac{1}{30}$; (ii) $\frac{1}{30}$; (iii) $\frac{1}{6}$. **13.** $\frac{57}{115}$ (about 50%).

Section 3.5

1. (ii) $\frac{2}{5}$, $\frac{4}{15}$, $\frac{4}{15}$, $\frac{1}{15}$, respectively; (iii) $\frac{8}{15}$. **2.** (ii) $P(WM, WF) = \frac{1}{10}$, $P(WM, GM) = \frac{1}{15}$, $P(WF, WM) = \frac{1}{10}$, $P(WF, WF) = \frac{1}{5}$, $P(WF, GM) = \frac{1}{5}$, $P(GM, WM) = \frac{1}{15}$, $P(GM, WF) = \frac{1}{5}$, $P(GM, GM) = \frac{1}{15}$; (iii) $\frac{1}{5}$; (iv) $\frac{8}{15}$; (v) $\frac{1}{3}$. **3.** (i) $P(HHH) = \frac{28}{1105}$, $P(HHM) = P(HMH) = P(MHH) = \frac{72}{1105}$, $P(HMM) = P(MHM) = P(MMH) = \frac{168}{1105}$, $P(MMM) = \frac{357}{1105}$; (ii) $\frac{244}{1105}$. **4.** (ii) $\frac{13}{18}$. **5.** (i) $\frac{17}{35}$; (ii) $\frac{11}{35}$; (iii) $\frac{1}{5}$. **6.** $\frac{1}{3}$. **7.** $\frac{1}{3}$, 1, $\frac{2}{11}$, $\frac{1}{3}$, respectively. **8.** $\frac{25}{102}$; $\frac{15}{34}$; $\frac{25}{51}$; $\frac{25}{77}$; $\frac{32}{51}$; $\frac{45}{77}$. **9.** (i) $\frac{1}{4}$; (ii) $\frac{3}{14}$. **10.** (i) $\frac{2}{5}$; (ii) $\frac{1}{5}$.

Section 3.6

1. (i) $\frac{3}{4}$; $\frac{3}{7}$; (ii) 1; $\frac{3}{7}$; (iii) $\frac{2}{3}$; $\frac{2}{3}$; (iv) $\frac{1}{3}$; $\frac{1}{3}$. **2.** (i) 0.9; 0.4; (ii) 0.85; 0.35; (iii) 0.97; 0.63. **3.** (i) (b) $\frac{1}{4}$; $\frac{1}{4}$; $\frac{12}{51}$; $\frac{12}{51}$; $\frac{3}{51}$; (c) No; (ii) (b) $\frac{1}{4}$; $\frac{1}{4}$; $\frac{1}{4}$; $\frac{1}{4}$; $\frac{1}{16}$; (c) Yes. **4.** (i) $\frac{2}{5}$; (ii) $\frac{3}{7}$; (iii) $\frac{5}{7}$; (iv) $\frac{9}{25}$. **5.** (i) 0; (ii) 0.5. **6.** (i) $\frac{1}{3}$; (ii) $\frac{2}{3}$; (iii) $\frac{1}{7}$; (iv) $\frac{6}{7}$; $\frac{1}{2}$. **7.** (i) $\frac{4}{9}$; (ii) Yes. **8.** (ii) 48%; (iii) 50%. **9.** (i) Yes; (ii) Yes. **10.** $\frac{19}{42}$.

Section 3.7

1. 0.5, 0.5. **2.** (i) $\frac{1}{6}$; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{3}{4}$; $\frac{1}{4}$; 0; (ii) $\frac{1}{5}$; $\frac{1}{5}$; $\frac{3}{5}$; $\frac{3}{10}$; $\frac{3}{10}$; $\frac{2}{5}$; (iii) $\frac{2}{9}$; 0; $\frac{7}{9}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{1}{12}$; (iv) $\frac{4}{15}$; 0; $\frac{11}{15}$; $\frac{8}{15}$; $\frac{1}{5}$; $\frac{4}{15}$. **3.** (i) 2; (ii) $\frac{1}{3}$. **4.** (i)

	Mini	Caprice	
On Time	0.72	0.14	0.86
Late	0.08	0.06	0.14
	0.8	0.2	

(ii) $\frac{7}{43}$. **5.** (i) $\frac{1}{97}$; (ii) $\frac{1}{17}$. **6.** (i) 0.011; (ii) $\frac{27}{44}$. **7.** $\frac{2}{3}$. **8.** $\frac{8}{15}$. **9.** $\frac{9}{16}$. **10.** $\frac{30}{52}$.

Section 3.8

1. $\frac{3}{8}$, $\frac{1}{4}$. **2.** $\frac{7}{31}$. **3.** $\frac{3}{143}$. **4.** (i) $\frac{1}{10}$ (10%); (ii) $\frac{1}{5}$ (20%). **5.** (i) $\frac{19}{117}$ (about 16.2%); (ii) $\frac{1}{883}$ (about 0.1%).

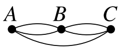
Section 3.9

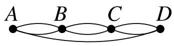
1. (i) $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$; (ii) 1.5. **2.** \$2.20. **3.** \$5. **4.** 0. **5.** \$17000. **6.** 5. **7.** 2.5. **8.** (i) $\frac{1}{9}$; (ii) $\frac{64}{81}$, $\frac{16}{81}$, $\frac{1}{81}$; (iii) $\frac{18}{81}$. **9.** 40*c*. **10.** Lose 50*c*.

Section 4.1

1. (i) $\{(1,1), (4,2), (9,3)\}$; (ii) $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (3,9), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8), (9,9)\}$; (iii) $\{(2,8), (3,3)\}$. **2.** $\alpha : \{(1,1)\}$, $\beta : \{(1,2), (2,3), (3,4)\}$. **3.** (i) α; (ii) γ; (iii) α, β; (iv) α; (v) none; (vi) α. **4.** (i) S; (ii) S; (iii) ST; (iv) RST. **5.** α in Exercise 4.1A.3; none in Exercise 4.1A.4. **6.** (i) No; (ii) Yes; (iii) Yes; (iv) Yes. **7.** (i) One-to-one and onto; (ii) Not one-to-one, but onto; (iii) Not one-to-one ($f(-1) = f(1)$); onto. **8.** (i) f_4; (ii) $f^{-1}(x) = 2 - x$ if $x \geq 1$; $f^{-1}(x) = 1/x$ if $0 < x < 1$.

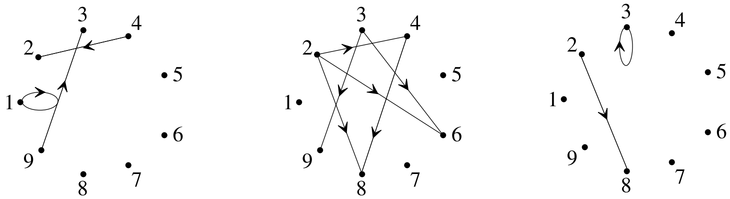
Section 4.2

1. (i) 

(ii) 

2. (i) $sa, st, as, at, bs, bt, tb$. $A(s) = \{a, t\}$, $B(s) = \{a, b\}$, $A(a) = \{s, t\}$, $B(a) = \{s\}$, $A(b) = \{s, t\}$, $B(b) = \{t\}$, $A(t) = \{b\}$, $B(t) = \{a, b, s\}$; (ii) sb, as, bc, ca, ce, dc, et, td. $A(s) = \{b\}$, $B(s) = \{a\}$, $A(a) = \{s\}$, $B(a) = \{c\}$, $A(b) = \{c\}$, $B(b) = \{s\}$, $A(c) = \{a, e\}$, $B(c) = \{b, d\}$, $A(d) = \{c\}$, $B(d) = \{t\}$, $A(e) = \{t\}$, $B(e) = \{c\}$, $A(t) = \{d\}$, $B(t) = \{e\}$; (iii) sa, sc, se, ab, ac, bd, ce, dc, dt, et. $A(s) = \{a, c, e\}$, $B(s) = \emptyset$, $A(a) = \{b, c\}$, $B(a) = \{s\}$, $A(b) = \{d\}$, $B(b) = \{a\}$, $A(c) = \{d, e\}$, $B(c) = \{s, a\}$, $A(d) = \{t\}$, $B(d) = \{b, c\}$, $A(e) = \{t\}$, $B(e) = \{s, c\}$, $A(t) = \emptyset$, $B(t) = \{d, e\}$.

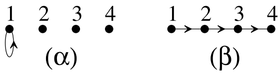
3.



(α) (β) (γ)

In addition to the edges shown, β has loops on all vertices and an edge directed from 1 to each other vertex. None are graphs; all are directed, and all have loops.

4.



(α) (β)

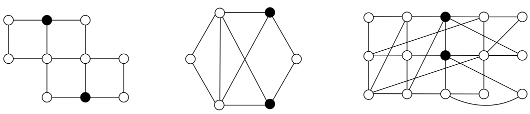
Neither is a graph; both are directed. α has a loop on 1, β is a digraph. (There is no edge from 4 to 3 in α, because $4 \notin S$.)

Section 4.3

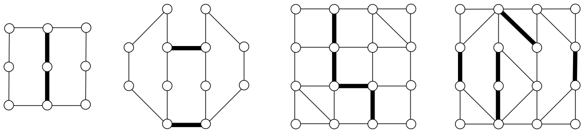
1 (i) 323303242; (ii) 1124222; (iii) 2422233; (iv) 2332332442. **2.** 4; 5. **3.** Yes. **4.** No. **5.** $2n$; n once, 3 n times. **6.** (ii) Has two bridges and one cutpoint. **7.** 21. **8.** $\frac{1}{2}n(n-1)$. **9.** $D = R = 2$. **10.** $d(s, t) = 3$. **11.** $sabct, safbt, saect, sdbct, sdbft$ (length 4); $saecbft, sdbaect$ (length 6); $d(s, t) = 4$.

Section 4.4

1. Graphs (ii), (v) and (viii) each have two odd vertices.

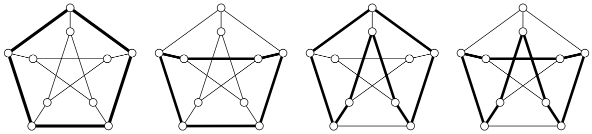


2. (i) Yes; Yes; (ii) Yes; No; (iii) Yes; Yes; (iv) No; (v) Yes; No; (vi) Yes; Yes; (vii) Yes; Yes; (viii) Yes; No. **3.** All odd n except $n = 1$. **4.** (i) 2; (ii) 2; (iii) 4; (iv) 5

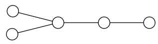


Section 4.5

1. $D = R = n$ **2.** The diagrams show cycles of lengths 5, 6, 8, and 9, respectively.

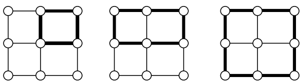


3. One example:

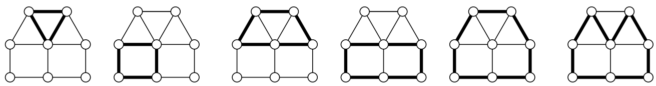


4.

(i)



(ii)

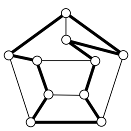


5. Say there is no cycle. Start at the vertex of degree 1. Select a (new) vertex adjacent to it. Continue in this way, never repeating a vertex. The process must stop, since the graph is finite, but it can only stop when all vertices adjacent to the new vertex have already been chosen. But if this is so, you must have a cycle. **6.** In (i), you get eight different solutions—start at any vertex, they will all be different. Case (ii) has seven solutions (the two top vertices as roots give the same diagram). In (iii), there are four different diagrams—for example, start with each vertex in the left-hand branch, including the bottom vertex. **7.** (i) Three trees (delete the edges of the triangle in turn). (ii) Three trees (delete the edges of the triangle in turn). (iii) Nine trees (delete one edge from the left triangle and one edge from the right triangle).

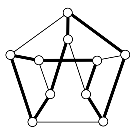
Section 4.6

1.

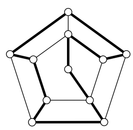
(i)



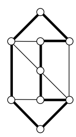
(ii)



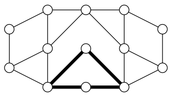
(iii)



(iv)



2. If there is a vertex of degree 2, any Hamiltonian cycle must contain both edges touching that vertex. So we would need to include all four heavy edges. But they form a 4-cycle, which is not allowed.



3. (i) $abcdef$, $abdcef$; (ii) $abcehgfd$, $abcfdgeh$, $abdfcegh$, $abdgfceh$, $abecfdgh$, $adbcfgeh$, $adbecfgh$, $adfcbegh$, $adgfcbeh$.

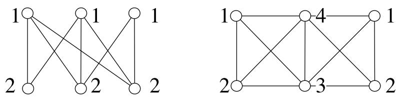
4. (i) NN: a 117, b 117, c 122, d 117, e 117, SE: 117; (ii) NN: a 112, b 113, c 112, d 113, e 116, SE: 113; (iii) NN: a 74, b 76, c 76, d 74, e 76, SE: 76; (iv) NN: a 105, b 105, c 105, d 100, e 100, SE: 105; (v) NN: a 119, b 122, c 122, d 122, e 122, SE: 122; (vi) NN: a 245, b 245, c 253, d 252, e 245, SE: 245.

Section 4.7

1. (i), (iii), (v), and (vi).

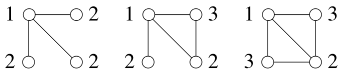
2. (i) 2; color the three top vertices in one color and the others in a second color; (ii) 4; color the leftmost and rightmost vertices the same.

3. (i) 2; (ii) 4.



4. 3 if n is even, 4 if n is odd.

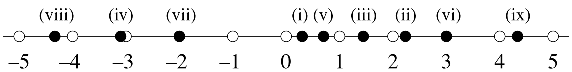
5.



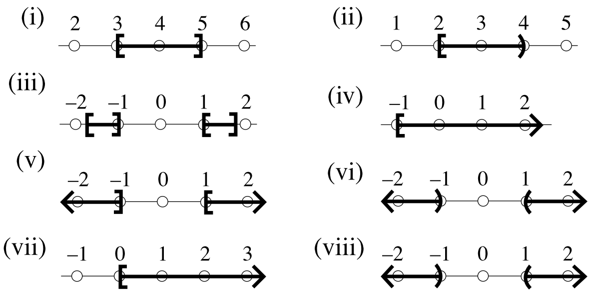
6. There are $v-1$ vertices, all connected, so at least $v-1$ colors are needed. (Consider all vertices except one endpoint of the deleted edge.) But $v-1$ colors suffice (color the endpoints of the deleted edge the same).

Section 5.1

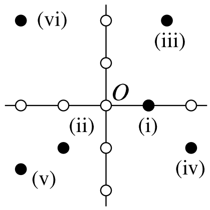
1.



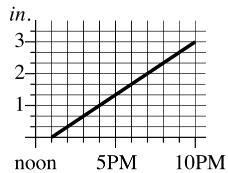
2.



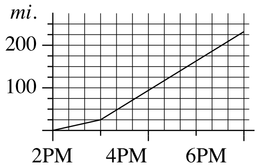
3.



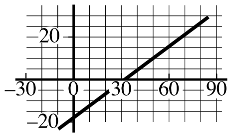
4.



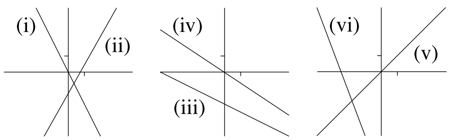
5. (i) 25 miles;(ii)



6. (i) $y = \frac{5}{9}(x - 32)$; (ii) 20°C; (iii) 77°F; (iv)



7.



8. (i) -2; $y = -2x$; (ii) 2; $y = 2x - 1$; (iii) $\frac{1}{2}$; $y = \frac{1}{2}x - 2$; (iv) $-\frac{2}{3}$; $y = -\frac{2}{3}x$; (v) 1; $y = x$; (vi) 2; $y = 2x - 3$. **9.** $y = x - 1$.

Section 5.2

1. (i) $x = 5, y = 2$; (ii) $x = -4, y = 3$; (iii) $x = \frac{1}{2}, y = \frac{3}{2}$; (iv) $x = 2, y = \frac{1}{2}$;

(v) $x = 3, y = 8$; (vi) $x = \frac{1}{2}, y = 1$; (vii) $x = 2, y = 1$; (viii) $x = 1, y = -\frac{1}{2}$; (ix) $x = -2, y = 6$; (x) $x = -1, y = 1$. **3.** (i) Inconsistent; (ii) Independent; $x = 1, y = \frac{3}{2}$; (iii) Dependent; $x = 2 - y/2$, any $y \in \mathbb{R}$; (iv) Independent; $x = 5$, $y = 4$; (v) Independent; $x = -1, y = \frac{5}{2}$; (vi) Dependent; $x = 1 - y/2$, any $y \in \mathbb{R}$; (vii) Inconsistent; (viii) Independent; $x = \frac{48}{5}, y = \frac{6}{5}$; (ix) Independent; $x = \frac{17}{21}$, $y = \frac{44}{21}$; (x) Dependent; $x = \frac{1}{5} + \frac{8}{5}y$. **4.** (i) $x = 3, y = 2, z = -1$; (ii) $x = 9$, $y = -8, z = \frac{1}{3}$; (iii) $x = 5z - 2, y = 4z - 3$, any $z \in \mathbb{R}$; (iv) Inconsistent; (v) $x = 2$, $y = -1, z = 1$; (vi) $x = 5z - 2, y = 4z - 3$, any $z \in \mathbb{R}$; (vii) $x = 2, y = -2$, $z = 1$; (viii) Inconsistent; (ix) $x = z - 3, y = z - 4$, any $z \in \mathbb{R}$; (x) $x = z + 1$, $y = 3z + 1$, any $z \in \mathbb{R}$; (xi) Independent; $x = 11, y = -5, z = 4$; (xii) Independent; $x = 4, y = 4, z = 0$; (xiii) Independent; $x = 0, y = 0, z = 2$; (xiv) Inconsistent.

Section 5.3

1. (i) $x = 3, y = 1$, any real z; (ii) $x = 2 - z, y = 1 + z$, any real z; (iii) No solutions; (iv) No solutions; (v) No solutions; (vi) $x = 2, y = 2$, any real z; (vii) $x = 4, y = 3$, $z = -3$; (viii) $x = 2 - z, y = 1 + z$, any real z; (ix) No solutions; (x) $x = 0$, $y = -1, z = 1$. **2.** (i) $x = -3, y = 2$; (ii) $x = \frac{1}{2}(5 - 3y)$, any real y; (iii) No solutions; (iv) $x = 2, y = -1$. **3.** (i) $x = 1 + 5z, y = 1 - 3z$, any real z; (ii) $x = \frac{2}{3}$, $y = \frac{1}{6}, z = \frac{1}{3}$; (iii) $x = 3, y = -1, z = -2$; (iv) $x = \frac{5}{2}, y = -\frac{1}{2}, z = 0$. **4.** (i) $x = 1 + 2z - t, y = 2 - z + 2t$, any real z, any real t; (ii) No solutions.

Section 5.4

1.

$M =$

	I	G
P	4	24
C	6	12

.

2. (i)

$M =$

	A	B	C
I	200	100	250
II	250	150	350

;

(ii) 1150 units A, 650 units B, 1550 units C.

3. (i) $\begin{bmatrix} 7 & -2 \\ -3 & 2 \end{bmatrix}$; (ii) $\begin{bmatrix} 28 & -1 \\ 8 & 19 \end{bmatrix}$;

(iii) $\begin{bmatrix} 15 & -5 & 18 \\ 10 & 11 & -13 \end{bmatrix}$; (iv) $\begin{bmatrix} -3 & 9 \\ -3 & 3 \end{bmatrix}$.

4. $x = 2$, $y = 1$. **5.** $x = 11$, $y = 9$, $z = 3$. **6.** (i) $(-2, 2)$; (ii) $(9, 18, 3)$; (iii) $(4, 1)$; (iv) $(-6, 0, 6)$; (v) $(7, 0, 4)$; (vi) $(5, -20, 10, 15)$.

7. (i) $\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$; (ii) $\begin{bmatrix} -5 \\ -14 \end{bmatrix}$; (iii) $\begin{bmatrix} 0 & -1 \\ -9 & 9 \end{bmatrix}$;

(iv) No (C, D not the same shape); (v) No (C, A not the same shape);

(vi) $\begin{bmatrix} 3 & 5 \\ 5 & -14 \end{bmatrix}$.

Section 5.5

1. (i) -1; (ii) -5; (iii) 3; (iv) 6; (v) 6; (vi) 3.

2. (i) 2×4; (ii) 2×2; (iii) No; (iv) 1×4; (v) 4×4; (vi) 2×4; (vii) No; (viii) 1×3; (ix) 2×3.

3. (i) $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$; (ii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 8 & 3 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & 7 & 5 \end{bmatrix}$;

(iv) $\begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$; (v) $\begin{bmatrix} 9 & -7 \\ 6 & 2 \end{bmatrix}$.

4. (i) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$; (ii) $\begin{bmatrix} 4 \\ -6 \end{bmatrix}$; (iii) No; (iv) No;

(v) $\begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$; (vi) No.

5. (i) \$5700; (ii) \$2440.

6. (i) $AB = \begin{bmatrix} -1 & 1 \\ -1 & 7 \end{bmatrix}$, $BA = \begin{bmatrix} 4 & 7 \\ 2 & 2 \end{bmatrix}$, No;

(ii) $AB = \begin{bmatrix} -4 & 8 \\ -5 & 2 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & -4 \\ 8 & -2 \end{bmatrix}$, No;

(iii) $AB = BA = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix}$, Yes.

7. (i) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$; (ii) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & -3 & 9 \\ 1 & 10 & -4 \\ -3 & -4 & -1 \end{bmatrix}$, $\begin{bmatrix} -7 & -18 & 15 \\ 6 & 34 & -11 \\ -5 & -11 & -6 \end{bmatrix}$;

(iv) $\begin{bmatrix} 7 & 0 & -2 \\ 6 & 1 & -1 \\ 4 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 17 & 0 & -5 \\ 17 & 1 & -4 \\ 10 & 0 & -3 \end{bmatrix}$.

8. (i) $\begin{bmatrix} 0 & 3 \\ -3 & 3 \end{bmatrix}$, $\begin{bmatrix} -3 & 6 \\ -6 & 3 \end{bmatrix}$, $\begin{bmatrix} -6 & 6 \\ -6 & 0 \end{bmatrix}$.

Section 5.6

2. $\begin{bmatrix} 5 & -7 \\ -4 & 7 \end{bmatrix}$

3. (i) $\begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$; (ii) $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$; (iii) No inverse;

(iv) $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; (v) $\begin{bmatrix} 4/7 & -1 & 2/7 \\ 3/7 & 0 & -2/7 \\ -5/7 & 1 & 1/7 \end{bmatrix}$;

(vi) $\begin{bmatrix} 3/2 & 0 & 1/2 \\ -11/6 & 1/3 & -5/6 \\ 17/6 & -1/3 & 5/6 \end{bmatrix}$; (vii) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$;

(viii) No inverse.

4. (i) 13; $\begin{bmatrix} 3/13 & -4/13 \\ -2/13 & 7/13 \end{bmatrix}$; (ii) -10; $\begin{bmatrix} -3/10 & 1/5 \\ 1/5 & 1/5 \end{bmatrix}$;

(iii) 0; No inverse; (iv) 1; $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

Section 5.7

3. (ii) (a) $x = 2$, $y = -1$; (b) $x = 1$, $y = 1$.

4. (ii) (a) $x = -6$, $y = 1$, $z = 2$; (b) $x = 4$, $y = 1$, $z = 2$.

5. (i) $\begin{bmatrix} 132 \\ 104 \end{bmatrix}$; (ii) $\begin{bmatrix} 68 \\ 124 \end{bmatrix}$; (iii) $\begin{bmatrix} 190 \\ 310 \end{bmatrix}$.

6. 208 tonnes of X, 20 tonnes of Y.

Section 6.1

1. x = number of Fleetwoods, y = number of Majestics. We figure in thousands of dollars.

$$\begin{aligned} \text{Maximize} \quad & P = 20x + 25y \\ \text{subject to} \quad & 500x + 600y \le 11000, \\ & 30x + 40y \le 700, \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

2. x = number of pounds of feed A per month, y = number of pounds of feed B per month.

$$\begin{aligned} \text{Minimize} \quad & C = 2.5x + 1.4y \\ \text{subject to} \quad & 5x + 2y \geq 70, \\ & 3x + 2y \geq 50, \\ & 4x + y \geq 40, \\ & x \geq 0, \quad y \geq 0. \end{aligned}$$

3. x_1, x_2, x_3 are the number of ornaments of types I, II, III, respectively.

$$\begin{aligned} \text{Maximize} \quad & P = 30x_1 + 45x_2 + 15x_3 \\ \text{subject to} \quad & 6x_1 + 4x_2 + x_3 \leq 200, \\ & 2x_1 + 3x_2 + x_3 \leq 100, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

4. x_1, x_2, x_3 are the acres of corn, linseed and oats, respectively.

$$\begin{aligned} \text{Maximize} \quad & P = 330x_1 + 340x_2 + 270x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 60, \\ & 5x_1 + 4x_2 + 3x_3 \leq 160, \\ & 90x_1 + 110x_2 + 80x_3 \leq 6000, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

5. x_1, x_2 and x_3 are amounts (in millions of dollars) to be invested in real estate, stocks and treasury bills, respectively. P is the expected income in *millions of cents.*

$$\begin{aligned} \text{Maximize} \quad & P = 8x_1 + 5x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 100, \\ & x_1 - x_2 \leq 0, \\ & 2x_1 + 2x_2 - x_3 \leq 0, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

6. x = standard clocks, y = number of mantel clocks.

$$\begin{aligned} \text{Maximize} \quad & P = 60x + 11y \\ \text{subject to} \quad & 40x + 8y \leq 5,000, \\ & 5x + 2y \leq 600, \\ & x \geq 0, \quad y \geq 0. \end{aligned}$$

7. x = number of student desks, y = number of faculty desks.

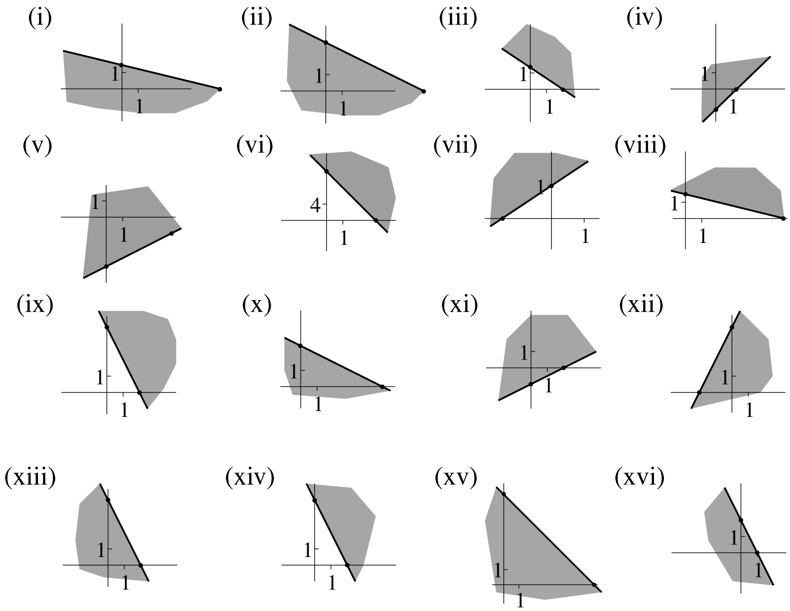
$$\begin{aligned} \text{Maximize} \quad & P = 10x + 12y \\ \text{subject to} \quad & 10x + 12y \le 8000, \\ & 2x + 2y \le 2000, \\ & 1.5x + 2y \le 1800 \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

8. x_1, x_2, x_3 are the number of acres of types potatoes, corn, and peppers, respectively.

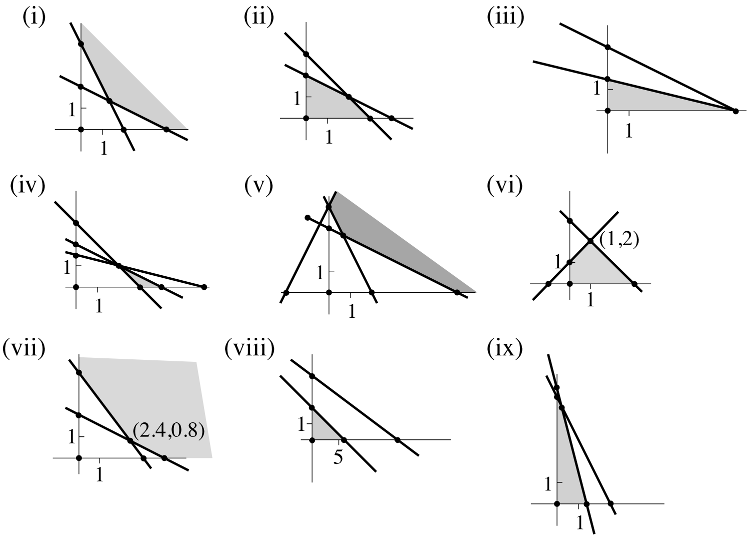
$$\begin{aligned} \text{Maximize} \quad & P = 80x_1 + 40x_2 + 70x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 100, \\ & x_1 \le 30, \\ & 300x_1 + 160x_2 + 280x_3 \le 20000, \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

Section 6.2

1.



2.



3. (i) $C = \frac{8}{3}$ at $x = \frac{4}{3}$, $y = \frac{4}{3}$; (ii) $P = 3$ at $x = 3$, $y = 0$; (iii) $P = 6$ at $x = 6$, $y = 0$; (iv) $C = 11$ at $x = 2$, $y = 1$; (v) $C = \frac{10}{3}$ at $x = \frac{2}{3}$, $y = \frac{8}{3}$.

4. (i) $2x + y \leq 10000$; (ii) $x + 2y \leq 8000$; (iii) The problem is to maximize $P = 3x + 5y$ subject to the given constraints (including $x, y \geq 0$). The maximum is 22000, achieved at $x = 4000$, $y = 2000$.

Section 6.3

1. (i)

$$\begin{array}{ll} \text{Maximize} & P = 4x_1 + 2x_2 + 0x_3 + 0x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 + 0x_4 = 5, \\ & 2x_1 + 6x_2 + 0x_3 - x_4 = 12, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{array}$$

(ii)

$$\begin{array}{ll} \text{Minimize} & C = x_1 + x_2 + 0x_3 + 0x_4 \\ \text{subject to} & 2x_1 + x_2 - x_3 + 0x_4 = 4, \\ & x_1 + 2x_2 + 0x_3 - x_4 = 4, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{array}$$

(iii)

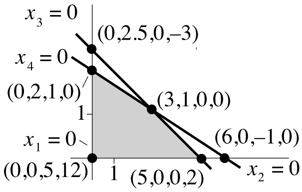
$$\begin{array}{ll} \text{Maximize} & C = x_1 + x_2 + 0x_3 + 0x_4 \\ \text{subject to} & 2x_1 + x_2 + x_3 + 0x_4 = 4, \\ & 4x_1 + 4x_2 + 0x_3 + x_4 = 12, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{array}$$

(iv)

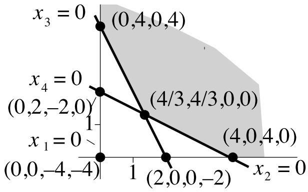
$$\begin{aligned} \text{Maximize} \quad & P = 2x_1 + 3x_2 + 0x_3 + 0x_4 \\ \text{subject to} \quad & 8x_1 + 2x_2 + x_3 + 0x_4 = 11, \\ & 2x_1 + x_2 + 0x_3 + x_4 = 5, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

2.

(i)



(ii)



3. (i)

Non-basic variables	Basic variables	Basic solution	Feasible?	P
x_1, x_2	x_3, x_4	$(0, 0, 5, 12)$	Yes	0
x_1, x_3	x_2, x_4	$(0, \frac{5}{2}, 0, -3)$	No	5
x_1, x_4	x_2, x_3	$(0, 2, 1, 0)$	Yes	4
x_2, x_3	x_1, x_4	$(5, 0, 0, 2)$	Yes	20
x_2, x_4	x_1, x_3	$(6, 0, -1, 0)$	No	24
x_3, x_4	x_1, x_2	$(3, 1, 0, 0)$	Yes	14

Maximum $P = 20$ at $x_1 = 5, x_2 = 0$.

(ii)

Non-basic variables	Basic variables	Basic solution	Feasible?	C
x_1, x_2	x_3, x_4	$(0, 0, -4, -4)$	No	0
x_1, x_3	x_2, x_4	$(0, 4, 0, 4)$	Yes	4
x_1, x_4	x_2, x_3	$(0, 2, -2, 0)$	No	2
x_2, x_3	x_1, x_4	$(2, 0, 0, -2)$	No	2
x_2, x_4	x_1, x_3	$(4, 0, 4, 0)$	Yes	4
x_3, x_4	x_1, x_2	$(\frac{4}{3}, \frac{4}{3}, 0, 0)$	Yes	$\frac{8}{3}$

Minimum $C = 4$ at $x_1 = 4, x_2 = 0$ or $x_1 = 0, x_2 = 4$.

(iii)

Non-basic variables	Basic variables	Basic solution	Feasible?	C
x_1, x_2	x_3, x_4	$(0, 0, 4, 8)$	Yes	0
x_1, x_3	x_2, x_4	$(0, 4, 0, 0)$	Yes	16
x_1, x_4	x_2, x_3	$(0, 4, 0, 0)$	Yes	16
x_2, x_3	x_1, x_4	$(2, 0, 0, 2)$	Yes	6
x_2, x_4	x_1, x_3	$(\frac{8}{3}, 0, -\frac{4}{3}, 0)$	No	8
x_3, x_4	x_1, x_2	$(0, 4, 0, 0)$	Yes	16

Maximum $P = 16$ at $x_1 = 0, x_2 = 4$. (Note: three boundary lines have a common point. There are actually only four boundary points.)

(iv)

Non-basic variables	Basic variables	Basic solution	Feasible?	C
x_1, x_2	x_3, x_4	$(0, 0, 2, 4)$	Yes	0
x_1, x_3	x_2, x_4	$(0, 1, 0, 3)$	Yes	3
x_1, x_4	x_2, x_3	$(0, 4, -6, 0)$	No	12
x_2, x_3	x_1, x_4	$(-2, 0, 0, 8)$	No	-4
x_2, x_4	x_1, x_3	$(2, 0, 4, 0)$	Yes	4
x_3, x_4	x_1, x_2	$(\frac{6}{5}, \frac{8}{5}, 0, 0)$	Yes	$\frac{36}{5}$

Maximum $C = \frac{36}{5}$ at $x_1 = \frac{6}{5}, x_2 = \frac{8}{5}$.

Section 6.4

1. (i)
$$\begin{aligned} x_1 + 2x_2 + x_3 + 0x_4 &= 3, \\ 2x_1 + 3x_2 + 0x_3 + x_4 &= 4, \end{aligned} \quad (0, 0, 3, 4);$$

(ii)
$$\begin{aligned} 2x_1 + 2x_2 + x_3 + 0x_4 &= 3, \\ x_1 + 3x_2 + 0x_3 - x_4 &= 5, \end{aligned} \quad \text{no};$$

(iii)
$$\begin{aligned} 2x_1 + x_2 + x_3 + 0x_4 &= 5, \\ x_1 + 3x_2 + 0x_3 + x_4 &= 8, \end{aligned} \quad (0, 0, 5, 8);$$

(iv)
$$\begin{aligned} 3x_1 + 2x_2 + x_3 + 0x_4 &= 3, \\ x_1 - 2x_2 + 0x_3 - x_4 &= 5, \end{aligned} \quad \text{no};$$

(v)
$$\begin{aligned} 3x_1 + 2x_2 + x_3 + 0x_4 &= 6, \\ x_1 + 3x_2 + 0x_3 - x_4 &= 4, \end{aligned} \quad \text{no};$$

(vi)

$$\begin{aligned} x_1 - x_2 + 2x_3 + x_4 + 0x_5 &= 1, \\ 2x_1 + x_2 + x_3 - 0x_4 - x_5 &= 3, \end{aligned} \quad \text{no;}$$

(vii)

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 + 0x_5 + 0x_6 &= 1, \\ 3x_1 - 3x_2 + 2x_3 + 0x_4 + x_5 + 0x_6 &= 1, \\ -x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 + x_6 &= 3, \end{aligned} \quad (0, 0, 0, 1, 1, 3);$$

(viii)

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 &= 3, \\ x_1 + 5x_2 + 0x_3 + x_4 + 0x_5 &= 2, \\ 2x_1 - 2x_2 + 0x_3 + 0x_4 + x_5 &= 1, \end{aligned} \quad (0, 0, 3, 2, 1).$$

2. (i)

BV	x_1	x_2	x_3	x_4	b
x_3	1	2	1	0	3
x_4	2	3	0	1	4

;

(iii)

BV	x_1	x_2	x_3	x_4	b
x_3	2	1	1	0	5
x_4	1	3	0	1	8

;

(vii)

BV	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	1	1	2	1	0	0	1
x_5	3	−3	2	0	1	0	1
x_6	−1	2	3	0	0	1	3

;

(viii)

BV	x_1	x_2	x_3	x_4	x_5	b
x_4	3	2	1	0	0	3
x_5	1	5	0	1	0	2
x_6	2	−2	0	0	1	1

.

3. (i) x_1; (ii) x_4; (iii) No positive entry in column x_2; (iv) x_5 *or* x_4; (v) x_1; (vi) x_5.

4. (i) (2, 1, 0, 0)

BV	x_1	x_2	x_3	x_4	b
x_3	$\frac{1}{4}$	0	1	$\frac{1}{4}$	$\frac{1}{2}$
x_2	$-\frac{1}{4}$	1	0	$\frac{15}{4}$	$\frac{1}{2}$

$(0, \frac{1}{2}, \frac{1}{2}, 0)$;

(ii) (2, 0, 0, 2)

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ \hline x_3 & 0 & 2 & 1 & \frac{1}{2} & 1 \\ x_1 & 1 & -1 & 0 & -\frac{1}{2} & 1 \end{array}\right]$$

(1, 0, 1, 0);

(iv) (2, 0, 0, 6, 2)

$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ \hline x_1 & 1 & 3 & 0 & 0 & 1 & 5 & 4 \\ x_3 & 0 & 2 & 1 & 0 & 1 & 1 & 2 \\ x_4 & 0 & -8 & 0 & 1 & -3 & -1 & 0 \end{array}\right]$$

or

$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ \hline x_5 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{14}{3} & 4 \\ x_3 & 0 & \frac{8}{3} & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ x_1 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 & \frac{2}{3} & 2 \end{array}\right]$$

(4, 0, 2, 0, 0);

(v) (2, 0, 0, 0, 6)

$$\left[\begin{array}{c|ccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline x_4 & \frac{1}{2} & 1 & 0 & 1 & 0 & 1 \\ x_5 & -1 & 1 & -2 & 0 & 1 & 4 \end{array}\right]$$

(0, 0, 0, 1, 4).

(vi) (2, 0, 0, 0, 2, 3)

$$\left[\begin{array}{c|cccccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \\ \hline x_1 & 1 & -\frac{1}{2} & -\frac{3}{2} & 0 & -\frac{1}{2} & 0 & 1 \\ x_4 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 1 \\ x_6 & 0 & -\frac{1}{2} & \frac{5}{2} & 0 & -\frac{1}{2} & 1 & 2 \end{array}\right].$$

(1, 0, 0, 1, 0, 2)

5. (i)

$$\left[\begin{array}{c|cccc|c} \text{BV} & x_1 & x_2 & x_3 & x_4 & b \\ \hline x_2 & 0 & 1 & 1 & 1 & 1 \\ x_1 & 1 & 0 & -2 & 1 & 2 \end{array}\right];$$

(ii)

BV	x_1	x_2	x_3	x_4	b
x_4	$\frac{4}{3}$	$-\frac{2}{3}$	0	1	2
x_3	1	1	1	0	3

;

(iii)

BV	x_1	x_2	x_3	x_4	x_5	x_6	b
x_2	-2	1	0	0	-1	2	1
x_4	$\frac{1}{2}$	0	0	1	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
x_3	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

;

(iv)

BV	x_1	x_2	x_3	x_4	x_5	x_6	b
x_1	1	-1	0	0	-1	3	1
x_4	0	0	0	1	3	4	3
x_3	0	1	1	0	2	-1	1

;

(v)

BV	x_1	x_2	x_3	x_4	x_5	x_6	b
x_2	-2	1	0	0	-1	2	1
x_4	$\frac{1}{2}$	0	0	1	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
x_3	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

;

(iv)

BV	x_1	x_2	x_3	x_4	x_5	x_6	b
x_1	1	-1	0	0	-1	3	1
x_4	0	0	0	1	3	4	3
x_3	0	1	1	0	2	-1	1

.

6. (i) 2, 1, (2, 0, 1, 0); (ii) 4; (iii) 2; (iv) 2; (v) (1, 2, 0, 0).

Section 6.5

1. (i)

BV	x_1	x_2	x_3	x_4	P	b
x_3	1	4	1	0	0	2
x_4	1	2	0	1	0	4
	-3	-5	0	0	1	0

;

(ii)

BV	x_1	x_2	x_3	x_4	P	b
x_3	2	2	1	0	0	4
x_4	3	5	0	1	0	1
	−1	−3	0	0	1	0

;

(iii)

BV	x_1	x_2	x_3	x_4	P	b
x_3	3	3	1	0	0	2
x_4	1	4	0	1	0	3
	−3	−2	0	0	1	0

;

(iv)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_4	1	1	1	1	0	0	3
x_5	3	2	3	0	1	0	4
	4	−3	2	0	0	1	0

;

(v)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	2	1	1	0	0	0	6
x_4	3	3	0	1	0	0	7
x_5	1	2	0	0	1	0	5
	−2	−5	0	0	0	1	0

;

(vi)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_3	5	3	1	0	0	0	8
x_4	3	1	0	1	0	0	4
x_5	1	4	0	0	1	0	5
	6	−1	0	0	0	1	0

.

2. (i)

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	−2	−3	0	1
x_1	1	0	1	2	0	1
	0	0	2	3	1	4

;

(ii)

BV	x_1	x_2	x_3	x_4	P	b
x_3	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1
x_1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	6

;

(iii)

BV	x_1	x_2	x_3	x_4	x_5	P	b
x_2	0	1	0	0	1	0	2
x_3	1	0	1	0	1	0	3
x_4	−1	0	0	1	2	0	1
	1	0	0	0	2	1	5

;

(iv)

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_5	0	0	2	1	1	1	0	3
x_1	1	0	0	1	0	−1	0	7
x_2	0	1	−1	−1	0	−2	0	1
	0	0	7	4	0	3	1	17

.

3. (i)

BV	x_1	x_2	x_3	x_4	P	b
x_3	1	0	1	$-\frac{1}{2}$	0	8
x_2	$\frac{3}{2}$	1	0	$\frac{1}{4}$	0	2
	20	0	0	6	1	48

;

(ii) Unbounded;

(iii)

BV	x_1	x_2	x_3	x_4	P	b
x_3	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	4
x_2	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	2
	10	0	0	6	1	24

;

(iv) Finished: $P = 18$ at $x_1 = 4, x_2 = 3$;

(v)

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_1	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	3
	0	0	$\frac{1}{4}$	$\frac{5}{4}$	1	13

;

(vi) Finished: $P = 7$ at $x_1 = 1, x_2 = 2$;
(vii) Finished: $P = \frac{9}{2}$ at $x_1 = \frac{3}{2}, x_2 = 0$; (viii) unbounded; (ix) unbounded;

(x)

BV	x_1	x_2	x_3	x_4	P	b
x_2	0	1	2	−1	0	2
x_1	1	0	−1	1	0	1
	0	0	−11	8	1	23

;

(xi) Finished: $P = 96$ at $x_1 = 10, x_2 = 19, x_3 = 0$;

(xii)

BV	x_1	x_2	x_3	x_4	P	b
x_1	1	0	−3	4	0	2
x_2	0	1	1	−1	0	2
	0	0	3	2	1	20

;

(xiii) Finished: $P = 3$ at $x_1 = 0, x_2 = 1$;

(xiv) Unbounded;
(xv) Unbounded;

(xvi)

BV	x_1	x_2	x_3	x_4	x_5	x_6	P	b
x_3	$-\frac{1}{2}$	0	1	0	2	2	0	2
x_1	$\frac{1}{2}$	0	0	1	−1	1	0	1
x_2	0	1	0	0	−1	−1	0	4
	$\frac{3}{2}$	0	0	0	−5	4	1	18

.

4. (i) $x_1 = 6, x_2 = 0, P = 36$; (ii) $x_1 = 1, x_2 = 2, P = 16$; (iii) $x_1 = 0, x_2 = 4, P = 20$; (iv) $x_1 = 0, x_2 = 0, x_3 = 2, P = 6$; (v) $x_1 = 9, x_2 = 27, x_3 = 9, P = 261$.

Section 6.6

1. (i) $4x_1 + 3x_2 - x_3 + A_1 = 5$; (ii) $x_1 + x_2 - x_3 + A_1 = 2$; (iii) $7x_1 - 2x_2 + 3x_3 + A_1 = 7$; (iv) $4x_1 + 5x_2 + x_3 - x_4 + A_1 = 4$; (v) $4x_1 - 4x_2 + 2x_3 + x_5 = 5$; (vi) $7x_1 + 3x_2 - 3x_3 + 7x_4 + A_1 = 5$; (vii) $3x_1 - 2x_2 + 4x_3 - 3x_4 - x_5 + A_1 = 1$; (viii) $11x_1 + 9x_2 + 5x_3 + 2x_4 + x_5 = 3$.

2. (i)

BV	x_1	x_2	x_3	A_1	A_2	P	b
	4	2	−1	1	0	0	5
	3	2	0	0	1	0	10
	2	3	0	M	M	1	0

;

BV	x_1	x_2	x_3	A_1	A_2	P	b
A_1	4	2	−1	1	0	0	5
A_2	3	2	0	0	1	0	10
	2	3	0	0	0	1	0
	$-7M$	$-4M$	M	0	0	0	$-15M$

;

(ii)

BV	x_1	x_2	x_3	x_4	A_1	A_2	P	b
	4	2	1	0	0	0	0	8
	2	10	0	−1	1	0	0	11
	3	−1	0	0	0	1	0	7
	2	5	0	0	M	M	1	0

BV	x_1	x_2	x_3	x_4	A_1	A_2	P	b
x_3	4	2	1	0	0	0	0	8
A_1	2	10	0	−1	1	0	0	11
A_2	3	−1	0	0	0	1	0	7
	2	5	0	0	0	0	1	0
	$-5M$	$-9M$	0	M	0	0	0	$-18M$

;

(iii)

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
	1	2	3	1	0	0	0	0	6
	2	1	−1	0	−1	1	0	0	4
	0	1	1	0	0	0	1	0	2
	1	2	1	0	0	M	M	1	0

;

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_3	1	2	3	1	0	0	0	0	6
A_1	2	1	−1	0	−1	1	0	0	4
A_2	0	1	1	0	0	0	1	0	2
	1	2	1	0	0	0	0	1	0
	$-2M$	$-2M$	0	0	M	0	0	0	$-6M$

;

(iv)

BV	x_1	x_2	x_3	x_4	x_5	x_6	A_1	A_2	P	b
	1	2	3	2	1	0	0	0	0	6
	2	3	1	1	0	−1	1	0	0	9
	2	2	−1	−1	0	0	0	1	0	4
	2	3	1	1	0	0	M	M	1	0

;

BV	x_1	x_2	x_3	x_4	x_5	x_6	A_1	A_2	P	b
x_5	1	2	3	2	1	0	0	0	0	6
A_1	2	3	1	1	0	−1	1	0	0	9
A_2	2	2	−1	−1	0	0	0	1	0	4
	2	3	1	1	0	0	0	0	1	0
	$-4M$	$-5M$	0	0	0	M	0	0	0	$-13M$

.

3. (i) Pivot A_1, x_1

BV	x_1	x_2	x_3	x_4	A_2	P	b
x_3	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	4
x_1	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{1}{2}$	0	2
	0	1	0	−6	−1	1	0
	0	$\frac{1}{2}M$	0	$\frac{7}{2}M$	$\frac{1}{2}M$	0	$-2M$

;

(ii) Infeasible;

(iii) Pivot A_1, x_3

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	−1	0	−1	1	0	6
x_3	0	−3	1	−1	1	0	2
	0	7	0	3	−3	1	−18
	0	0	0	0	M	0	0

;

(iv) Infeasible;

(v) Pivot A_1, x_3

BV	x_1	x_2	x_3	x_4	A_1	P	b
x_1	1	0	0	−1	1	0	6
x_3	0	−3	1	−1	1	0	2
P	0	10	4	4	−4	1	28
	0	0	0	0	M	0	0

;

(vi) Pivot A_2, x_3

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
A_1	1	1	0	-1	0	1	0	0	1
x_5	$\frac{3}{2}$	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	0	2
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	0	1
	$\frac{3}{2}$	$\frac{3}{2}$	0	0	0	0	$-\frac{1}{2}$	1	-1
	0	$-M$	0	M	0	0	$2M$	0	$-4M$

;

(vii) Pivot x_3, A_1

BV	x_1	x_2	x_3	x_4	x_5	A_1	A_2	P	b
x_4	4	0	0	1	0	1	0	0	5
x_5	-1	5	0	0	1	-2	0	0	4
x_3	2	-2	1	0	0	1	0	0	1
A_2	-7	9	0	0	0	-3	1	0	2
	1	-6	0	0	0	2	0	1	2
	$7M$	$-9M$	0	0	0	$4M$	0	0	$-2M$

.

4. (i) $P = 8$ at $x_1 = 4$, $x_2 = 0$; (ii) unbounded; (iii) $P = 9$ at $x_1 = 3$, $x_2 = 0$, $x_3 = 0$; (iv) $C = 52$ at $x_1 = 1, x_2 = 9, x_3 = 0$; (v) infeasible; (vi) $C = \frac{101}{3}$ at $x_1 = 0, x_2 = \frac{10}{3}, x_3 = \frac{11}{3}$; (vii) $C = 14$ at $x_1 = 3, x_2 = 2, x_3 = 0, x_4 = 2$.

Section 7.1

1.

$$\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}.$$

2.

$$\begin{bmatrix} -200 & 100 & 100 \\ 100 & -200 & 100 \\ 100 & 100 & -200 \end{bmatrix}.$$

3. (i) Strategy R2C1, value 0; (ii) Strategy R2C2, value 1; (iii) Strategy R1C2, value 1; (iv) Strategy R1C3, value 0; (v) Strategy R1C2, value 0; (vi) Strategy R2C2, value 0; (vii) Strategy R3C3, value 1; (viii) Strategy R3C2, value 1.

4.

(i) $\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; (iii) $\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$;

(v) $\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$; (vi) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; (vii) $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$; (viii) $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$.

5. (i) Saddle point R2C2, value 0; (ii) Saddle point R3C1, value 1; (iii) Saddle point R2C2, value -1; (iv) Saddle point R2C3, value 1.

Section 7.2

1. (i) $R(\frac{5}{8}, \frac{3}{8})C(\frac{1}{2}, \frac{1}{2})$ value $\frac{3}{8}$; (ii) Saddle point: R1C1 value 3; (iii) $R(\frac{1}{6}, \frac{5}{6})C(\frac{2}{3}, \frac{1}{3})$ value $\frac{4}{3}$; (iv) $R(\frac{1}{2}, \frac{1}{2})C(\frac{1}{2}, \frac{1}{2})$ value -1; (v) $R(\frac{1}{3}, \frac{2}{3})C(\frac{2}{3}, \frac{1}{3})$ value $-\frac{2}{3}$; (vi) Saddle point: R1C2 value -1; (vii) $R(\frac{1}{2}, \frac{1}{2})C(\frac{1}{3}, \frac{2}{3})$ value 0; (viii) $R(\frac{1}{5}, \frac{4}{5})C(\frac{3}{5}, \frac{2}{5})$ value $\frac{7}{5}$; (ix) $R(\frac{1}{2}, \frac{1}{2})C(\frac{1}{2}, \frac{1}{2})$ value 3; (x) $R(\frac{3}{7}, \frac{5}{7})C(\frac{4}{7}, \frac{2}{7})$ value $\frac{22}{7}$; (xi) $R(\frac{4}{9}, \frac{5}{9})C(\frac{1}{3}, \frac{2}{3})$ value $\frac{2}{3}$; (xii) Saddle point: R1C2 value -1; (xiii) Saddle point: R2C1 value 1; (xiv) $R(\frac{3}{4}, \frac{1}{4})C(\frac{1}{2}, \frac{1}{2})$ value $\frac{3}{2}$; (xv) $R(\frac{7}{8}, \frac{1}{8})C(\frac{7}{8}, \frac{1}{8})$ value $\frac{17}{8}$; (xvi) $R(\frac{7}{12}, \frac{5}{12})C(\frac{1}{4}, \frac{3}{4})$ value $\frac{1}{4}$.

3. (i) $a \leq 3$; (ii) all a; (iii) all a; (iv) $a \geq 0$.

Section 7.3

1. (i) $R(\frac{2}{3}, \frac{1}{3})C(\frac{1}{3}, 0, \frac{2}{3})$ value $-\frac{1}{3}$; (ii) $R(\frac{3}{8}, \frac{5}{8})C(0, \frac{5}{8}, \frac{3}{8}, 0)$ value $-\frac{1}{8}$; (iii) $R(\frac{5}{9}, \frac{4}{9})C(0, \frac{4}{9}, \frac{5}{9})$ value $-\frac{2}{9}$; (iv) $R(\frac{3}{4}, \frac{1}{4})C(\frac{1}{4}, 0, \frac{3}{4}, 0)$ value $\frac{7}{4}$.

2. (i) $R(\frac{7}{15}, \frac{8}{15}, 0)C(\frac{3}{5}, \frac{2}{5})$ value $-\frac{1}{5}$; (ii) $R(0, \frac{2}{5}, \frac{3}{5})C(\frac{1}{5}, \frac{4}{5})$ value $-\frac{3}{5}$; (iii) $R(\frac{1}{2}, \frac{1}{2})C(0, \frac{1}{4}, 0, \frac{3}{4})$ value 3; (iv) $R(\frac{2}{5}, \frac{3}{5})C(0, \frac{1}{2}, \frac{1}{2}, 0)$ value 0.

3. (i) $R(\frac{6}{7}, \frac{1}{7})C(\frac{3}{7}, 0, \frac{4}{7})$ value $\frac{3}{7}$; (ii) $R(\frac{1}{3}, \frac{2}{3})C(\frac{7}{15}, \frac{8}{15}, 0)$ value $\frac{2}{3}$; (iii) $R(\frac{7}{8}, 0, \frac{1}{8})C(\frac{5}{8}, 0, \frac{3}{8})$ value $-\frac{3}{4}$; (iv) $R(\frac{1}{4}, 0, \frac{3}{4})C(\frac{1}{2}, \frac{1}{2}, 0)$ value 0.

4. For each player, say the strategies are "penny, nickel, dime" in that order.

1	5	-1
1	5	-5
-10	-10	10

$R(\frac{10}{11}, 0, \frac{1}{11})C(\frac{1}{2}, 0, \frac{1}{2})$ value 0.

Section 7.4

3. (i) $R(\frac{1}{2}, \frac{1}{2}, 0)C(\frac{1}{14}, \frac{4}{7}, \frac{5}{14})$ value 0; (ii) $R(\frac{1}{8}, \frac{7}{16}, \frac{7}{16})C(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$ value $-\frac{1}{2}$; (iii) $R(\frac{1}{2}, 0, \frac{1}{2})C(0, \frac{1}{2}, \frac{1}{2})$ value 0; (iv) $R(\frac{1}{4}, 0, \frac{3}{4})C(0, \frac{1}{4}, \frac{3}{4})$ value $\frac{1}{2}$.

4. (i) $R(\frac{3}{11}, \frac{5}{11}, \frac{3}{11})C(\frac{5}{22}, \frac{4}{11}, \frac{9}{22})$ value $-\frac{1}{11}$; (ii) $R(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}C(0, \frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ value $-\frac{1}{3}$; (iii) $R(\frac{3}{11}, \frac{8}{11}, 0)C(\frac{7}{11}, 0, \frac{4}{11})$ value $\frac{1}{11}$; (iv) $R(\frac{10}{21}, \frac{8}{21}, \frac{1}{7}C(\frac{1}{2}, 0, \frac{5}{14}, \frac{1}{7})$ value $\frac{1}{7}$.

5. (i) $R(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})C(\frac{5}{9}, \frac{2}{9}, 0, \frac{2}{9})$ value $-\frac{1}{3}$; (ii) $R(\frac{3}{20}, \frac{9}{20}, \frac{2}{5}, 0)C(0, \frac{1}{2}, \frac{1}{2})$ value $\frac{1}{2}$; (iii) $R(\frac{2}{3}, 0, \frac{1}{3}, 0)C0, 0, \frac{7}{12}, \frac{5}{12})$ value $\frac{2}{3}$; (iv) $R(\frac{5}{9}, 0, \frac{7}{18}, \frac{1}{18})C(\frac{2}{9}, 0, \frac{3}{9}, \frac{4}{9})$ value $\frac{7}{9}$.

Section 8.1

1. (i) \$5350; (ii) \$6050; (iii) \$6750. **2.** (i) \$384; (ii) \$960; (iii) \$1536. **3.** (i) \$12040; (ii) \$12120; (iii) \$12200. **4.** (i) \$ 2512.50; (ii) \$2525; (iii) \$2575. **5.** 8%. **6.** 10%. **7.** (i) \$2200; (ii) \$1500; (iii) \$1200. **8.** 24%. **9.** 8.16%. **10.** \$113.33. **11.** \$236. **12.** \$40000; \$666.67. **13.** \$21917.81; \$608.83.

Section 8.2

1. (i) \$1125.51; (ii) \$1123.60; (iii) \$1120. **2.** (i) \$27.05; (ii) \$26.90; (iii) \$26.25. **3.** \$3588.26. **4.** \$991.92. **5.** \$225.36. **6.** \$2338.10. **7.** \$2419.99. **8.** \$7440.94. **9.** 12 (actually about $11\frac{2}{3}$ years). **10.** After 6 years. **11.** (i) 5.095%; (ii) 6.136%; (iii) 3.042%. **12.** (i) 12%; (ii) 12.55%; (iii) 12.68%; (iv) 12.75%; (v) 12.75%.

Section 8.3

1. \$27223.76. **2.** \$24664.64. **3.** \$91.68. **4.** (i) \$2501.52; (ii) \$8230.54; (iii) \$15084.83; (iv) \$27971.23. **5.** (i) \$2028.53; (ii) \$1609.25. **6.** \$758.48. **7.** \$200; \$193.43; the credit union is better. **8.** (i) 28951.22 (about 29%); (ii) 45996.46 (about 46%); (iii) 56641.35 (about 57%).

Section 8.4

1. (i) \$1025.32; (ii) \$1221.40; (iii) \$2718.28. **2.** (i) \$12544; (ii) \$17623.42. **3.** \$176622.22. **4.** The CD (it returns \$1159.27 for each \$1000 invested; housing returns \$1129.74). **5.** (i) $5376741 \times (1.012)^{10} = 6057930$; (ii) $5376741 \times (1.012)^{50} = 9762135$. **6.** 2550×1.2; 2550×1.2^4. **7.** April 5th. **8.** 20 grams.

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