Chapter 15
The Case $3 \leq k \leq n-1$

The results that will be discussed in this chapter are strongly based on Bandyopadhyay, Dacorogna and Kneuss [9]. For related results see Turiel [97–102].

15.1 A General Theorem for Forms of Rank $= k$

Our first result concerns k-forms of minimal nonzero rank.

Theorem 15.1. *Let $2 \leq k \leq n$, $r \geq 1$ be integers, $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let f and g be two $C^{r,\alpha}$ k-forms verifying, in a neighborhood of x_0,*

$$df = dg = 0 \quad \textit{and} \quad \operatorname{rank}[f] = \operatorname{rank}[g] = k.$$

Then there exist a neighborhood U of x_0 and

$$\varphi \in \begin{cases} \operatorname{Diff}^{r,\alpha}(U;\varphi(U)) & \textit{if } k < n \\ \operatorname{Diff}^{r+1,\alpha}(U;\varphi(U)) & \textit{if } k = n \end{cases}$$

such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = f \quad \textit{in } U.$$

In particular, if $g = dx^1 \wedge \cdots \wedge dx^k$, then

$$f = \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^k \quad \textit{in } U.$$

Remark 15.2. (i) The case $k = n-1$ is therefore completely solved (cf. Theorem 15.3).

(ii) We recall that the rank of a form is given in Definition 2.28 and Remark 2.31; see also Proposition 2.37.

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(iii) Throughout this chapter we will often use the following elementary fact. In order to solve $\varphi^*(g) = f$, it is enough to solve, for some h,

$$\varphi_1^*(h) = g, \quad \varphi_2^*(h) = f$$

and let $\varphi = \varphi_1^{-1} \circ \varphi_2$.

Proof. With no loss of generality, we can assume $x_0 = 0$ and (see Remark 15.2(iii)) $g = dx^1 \wedge \cdots \wedge dx^k$. We split the proof into two steps.

Step 1. We first prove the case $k = n$. Since $f = f_{1\cdots n}\, dx^1 \wedge \cdots \wedge dx^n$ and since $\operatorname{rank}[f] = n > 0$ in a neighborhood of 0, there exists a sufficiently small ball U centered at 0 such that $f_{1\cdots n}(x) \neq 0$ for every $x \in \overline{U}$. Using Theorem 10.1, there exists $\varphi_1 \in \operatorname{Diff}^{r+1,\alpha}(\overline{U};\overline{U})$ such that $\varphi_1 = \operatorname{id}$ on ∂U and

$$\varphi_1^*(c\, dx^1 \wedge \cdots \wedge dx^n) = f_{1\cdots n}\, dx^1 \wedge \cdots \wedge dx^n \quad \text{in } U,$$

where

$$c = \frac{1}{\operatorname{meas} U} \int_U f_{1\cdots n}.$$

Finally, let

$$\varphi_2(x) = x - \varphi_1(0)$$

and

$$\varphi_3(x) = \varphi_3(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, c x_n).$$

The map $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$ has all of the desired properties.

Step 2. We next suppose that $k < n$. Using Theorem 4.5, there exist a neighborhood V of 0 and $\varphi_1 \in \operatorname{Diff}^{r,\alpha}(V; \varphi_1(V))$ such that $\varphi_1(0) = 0$ and

$$\varphi_1^*(f)(x_1, \ldots, x_n) = a(x_1, \ldots, x_k)\, dx^1 \wedge \cdots \wedge dx^k \quad \text{in } V,$$

where $a \in C^{r-1,\alpha}$ in a neighborhood of 0 in $\mathbb{R}^k$. Using the fact that $\operatorname{rank}[f] = k$ and Proposition 17.1, there exists W, a sufficiently small ball in $\mathbb{R}^k$ centered at 0, such that $a(x) \neq 0$ for every $x \in \overline{W}$. Using Step 1, there exists $\varphi_2 \in \operatorname{Diff}^{r,\alpha}(\overline{W}; \varphi_2(\overline{W}))$ such that $\varphi_2(0) = 0$ and

$$\varphi_2^*(dx^1 \wedge \cdots \wedge dx^k) = a\, dx^1 \wedge \cdots \wedge dx^k.$$

Finally, defining $\widetilde{\varphi}_2 \in \operatorname{Diff}^{r,\alpha}(\overline{W} \times \mathbb{R}^{n-k}; \varphi_2(\overline{W}) \times \mathbb{R}^{n-k})$ by

$$\widetilde{\varphi}_2(x) = (\varphi_2(x_1, \ldots, x_k), x_{k+1}, \ldots, x_n),$$

we get that $\varphi = \widetilde{\varphi}_2 \circ \varphi_1^{-1}$ has all of the desired properties. This concludes the proof. $\square$

15.2 The Case of $(n-1)$-Forms

15.2.1 The Case of Closed $(n-1)$-Forms

The case of closed $(n-1)$-forms is a direct consequence of the results of Section 15.1 (see also Martinet [71]).

Theorem 15.3. *Let $r \geq 1$ be an integer, $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let f and g be two closed $C^{r,\alpha}$ $(n-1)$-forms verifying*

$$f(x_0) \neq 0 \quad \textit{and} \quad g(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(U;\varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = f \quad \textit{in } U.$$

In particular, if $g = dx^1 \wedge \cdots \wedge dx^{n-1}$, then

$$f = \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1} \quad \textit{in } U.$$

Proof. Recall first that a nonzero $(n-1)$-form has always (cf. Remark 2.38(i)) its rank equal to $n-1$. Therefore, the hypothesis

$$f(x_0) \neq 0 \quad \text{and} \quad g(x_0) \neq 0$$

is equivalent to

$$\operatorname{rank}[f] = \operatorname{rank}[g] = n-1 \text{ in a neighborhood of } x_0\,.$$

Applying Theorem 15.1, we have the result. □

Theorem 15.3 reads in a more analytical way when $k = n-1$ (cf. also Barbarosie [11]), since the exterior derivative of an $(n-1)$-form is then essentially the classical divergence operator.

Corollary 15.4. *Let $r \geq 1$ be an integer, $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let f be a $C^{r,\alpha}$ vector field satisfying*

$$f(x_0) \neq 0 \quad \textit{and} \quad \operatorname{div} f = 0 \textit{ in a neighborhood of } x_0\,.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(U;\varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$f = *\left(\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right) \quad \textit{in } U.$$

15.2.2 The Case of Nonclosed $(n-1)$-Forms

We conclude with the case of nonclosed $(n-1)$-forms.

Theorem 15.5. *Let $x_0 \in \mathbb{R}^n$ and f a C^∞ $(n-1)$-form verifying*

$$f(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and

$$\varphi \in C^\infty(U; \varphi(U))$$

such that

$$f = \varphi^n \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1} \quad \text{in } U.$$

If, moreover, $df(x_0) \neq 0$ then, up to restricting U, in addition to the previous equation, φ can be assumed in $\operatorname{Diff}^\infty(U; \varphi(U))$.

Remark 15.6. (i) If $f \in C^r$, then $\varphi^n \in C^{r-1}$ and $\varphi^i \in C^r$, $1 \le i \le n-1$. Moreover, another way to read the conclusion is

$$\varphi^*(x_n\, dx^1 \wedge \cdots \wedge dx^{n-1}) = f.$$

(ii) If $df = 0$ in a neighborhood of x_0, then we have a better result (cf. Theorem 15.3).

(iii) Note that we cannot, in general, ensure that $\varphi(x_0) = x_0$; for a similar result, see Remark 13.11(ii).

Proof. We split the proof into two steps. In the sequel, $*f \in C^\infty(\mathbb{R}^n; \Lambda^1)$ will sometimes be identified with a vector field (see Definition 2.9 for the notation).

Step 1. We prove the main assertion. Since $f(x_0) \neq 0$, using Remark 4.3(ii), there exist a neighborhood $V \subset \mathbb{R}^n$ of x_0 and $\varphi_1 \in \operatorname{Diff}^\infty(V; \varphi_1(V))$ such that $\varphi_1(x_0) = x_0$ and

$$\frac{\partial \varphi_1}{\partial x_n} = (*f) \circ \varphi_1 \quad \text{in } V. \tag{15.1}$$

Using Definition 2.11 and the fact that $(*f) \wedge (*f) = 0$ (since $*f$ is a 1-form), we deduce that $(*f) \lrcorner f = 0$. Thus, using (15.1), Theorem 3.10 and Proposition 3.11, we obtain

$$0 = \varphi_1^*((*f) \lrcorner f) = \varphi_1^\sharp(*f) \lrcorner \varphi_1^*(f) = dx^n \lrcorner \varphi_1^*(f).$$

From the previous equation we immediately deduce

$$\varphi_1^*(f)(x) = a(x_1, \ldots, x_n) dx^1 \wedge \cdots \wedge dx^{n-1}, \quad x \in V,$$

where $a \in C^\infty(V)$. Letting $U = \varphi_1(V)$ and

$$\varphi = (\varphi^1, \ldots, \varphi^n) = ((\varphi_1^{-1})^1, \ldots, (\varphi_1^{-1})^{n-1}, a \circ \varphi^{-1}),$$

we have the main assertion, namely

$$f = \varphi^n \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}.$$

Step 2. We prove the extra assertion. Let φ_1 be the diffeomorphism obtained in Step 1. It verifies, in particular,

$$\varphi_1^*(f)(x) = a(x_1, \ldots, x_n)\, dx^1 \wedge \cdots \wedge dx^{n-1}, \quad x \in V.$$

Since, by hypothesis, $df \neq 0$ in a neighborhood of x_0 and $\varphi_1(x_0) = x_0$, we have

$$d(\varphi_1^*(f)) = \varphi_1^*(df) \neq 0 \text{ in a neighborhood of } x_0$$

and, thus,

$$\frac{\partial a}{\partial x_n}(x_0) \neq 0.$$

Define $\varphi_2 : V \to \mathbb{R}^n$ by

$$\varphi_2(x) = (x_1, \ldots, x_{n-1}, a(x)).$$

Note that

$$\varphi_2^*(x_n\, dx^1 \wedge \cdots \wedge dx^{n-1}) = a(x)\, dx^1 \wedge \cdots \wedge dx^{n-1} \quad \text{in } V$$

and that, taking V smaller if necessary, $\varphi_2 \in \mathrm{Diff}^\infty(V; \varphi_2(V))$. Letting $\varphi = \varphi_2 \circ (\varphi_1)^{-1}$, it follows that $\varphi \in \mathrm{Diff}^\infty(\varphi_1(V); \varphi_2(V))$ and has the desired property. The proof is therefore complete. □

As before, the previous theorem can be seen in a more analytical way (cf. also Barbarosie [11]).

Corollary 15.7. *Let $x_0 \in \mathbb{R}^n$ and let f be a C^∞ vector field satisfying*

$$f(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and

$$\varphi \in C^\infty(U; \varphi(U))$$

such that

$$f = *\left(\varphi^n \nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right) \quad \text{in } U.$$

If, moreover, $\operatorname{div} f(x_0) \neq 0$, *then, up to restricting U, in addition to the previous equation, φ can be assumed in* $\mathrm{Diff}^\infty(U; \varphi(U))$.

15.3 Simultaneous Resolutions and Applications

15.3.1 Simultaneous Resolution for 1-Forms

We start with a simultaneous resolution of closed 1-forms; see also Cartan [21].

Proposition 15.8. *Let $r \geq 0$, $1 \leq m \leq n$ be integers and $x_0 \in \mathbb{R}^n$. Let $b^1, \ldots, b^m$ and $a^1, \ldots, a^m$ be C^r closed 1-forms verifying*

$$(b^1 \wedge \cdots \wedge b^m)(x_0) \neq 0 \quad \text{and} \quad (a^1 \wedge \cdots \wedge a^m)(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r+1}(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*\left(b^i\right) = a^i \quad \text{in } U \text{ and for every } 1 \leq i \leq m.$$

Remark 15.9. (i) When $r = 0$, the fact that a 1-form ω is closed has to be understood in the sense of distributions.

(ii) The result is also valid in Hölder spaces.

(iii) It is interesting to compare the above proposition and Theorem 15.1. In view of Proposition 2.43, we know that any m-form f with $\operatorname{rank}[f] = m$ is a product of 1-forms $a^1, \ldots, a^m$ so that

$$f = a^1 \wedge \cdots \wedge a^m;$$

however, we do not know, in general, that $a^1, \ldots, a^m$ are closed if f is closed (and even that $a^1, \ldots, a^m \in C^r$ if $f \in C^r$). But, Theorem 15.1 shows that there does exist a total decomposition with closed $a^1, \ldots, a^m$; however, we have lost one degree of regularity, namely $a^1, \ldots, a^m \in C^{r-1,\alpha}$ (unless $m = n$). Therefore, if we assume that $a^1, \ldots, a^m$ are closed, then the above proposition is better from the point of view of regularity than Theorem 15.1.

(iv) When $m = n$ and $f \in C^0$, it is, in general, impossible (according to Burago and Kleiner [19] and Mc Mullen [73]) to find closed 1-forms $a^1, \ldots, a^n \in C^0$ so that

$$f = a^1 \wedge \cdots \wedge a^n;$$

although, in view of Theorem 10.1, we can do so if $f \in C^{0,\alpha}$, finding even that $a^1, \ldots, a^n \in C^{0,\alpha}$.

Proof. We split the proof into two steps.

Step 1. With no loss of generality, we can assume $x_0 = 0$. Noticing that if $m < n$, we can choose $1 \leq k_{m+1} < \cdots < k_n \leq n$ and $1 \leq l_{m+1} < \cdots < l_n \leq n$ such that

$$(b^1 \wedge \cdots \wedge b^m \wedge dx^{k_{m+1}} \wedge \cdots \wedge dx^{k_n})(0) \neq 0,$$

$$(a^1 \wedge \cdots \wedge a^m \wedge dx^{l_{m+1}} \wedge \cdots \wedge dx^{l_n})(0) \neq 0.$$

We can therefore assume that $m = n$, letting $b^i = dx^{k_i}$ and $a^i = dx^{l_i}$ for $m+1 \le i \le n$. Using Corollary 8.6, we can find a neighborhood V of 0 and, for $1 \le i \le n$, $B^i, A^i \in C^{r+1}(V)$ such that

$$dA^i = a^i \quad \text{and} \quad dB^i = b^i \quad \text{in } V \text{ for every } 1 \le i \le n.$$

Moreover, adding, if necessary, a constant, we can assume that $A^i(0) = B^i(0) = 0$ for $1 \le i \le n$. Finally, define $A, B \in C^{r+1}(U; \mathbb{R}^n)$ by $A = (A^1, \ldots, A^n)$ and $B = (B^1, \ldots, B^n)$. Since $A(0) = B(0) = 0$ and since, identifying n-forms with 0-forms,

$$\det \nabla A(0) = (a^1 \wedge \cdots \wedge a^n)(0) \ne 0 \quad \text{and} \quad \det \nabla B(0) = (b^1 \wedge \cdots \wedge b^n)(0) \ne 0,$$

it follows that $A \in \operatorname{Diff}^{r+1}(U; A(U))$, $B \in \operatorname{Diff}^{r+1}(U; B(U))$ and

$$B^{-1} \circ A \in \operatorname{Diff}^{r+1}(U; (B^{-1} \circ A)(U))$$

for a neighborhood U of 0 small enough. Noticing that for $1 \le i \le n$,

$$A^*(dx^i) = a^i \quad \text{and} \quad B^*(dx^i) = b^i \quad \text{in } U,$$

we deduce that

$$(B^{-1} \circ A)^*(b^i) = A^*((B^{-1})^*(b^i)) = A^*(dx^i) = a^i \quad \text{in } U.$$

Therefore, $\varphi = B^{-1} \circ A$ has all of the desired properties and this concludes the proof. □

It is interesting to see that the above proposition can also be global.

Proposition 15.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν. Let $r \ge 0$ and $1 \le m \le n$ be integers. Let $b^1, \ldots, b^m \in C^r\left(\overline{\Omega}; \Lambda^1\right)$ be closed in Ω and such that*

$$b^1 \wedge \cdots \wedge b^m \wedge dx^{m+1} \wedge \cdots \wedge dx^n \ne 0 \quad \text{in } \overline{\Omega},$$

$$\nu \wedge b^i = \nu \wedge dx^i \quad \text{on } \partial\Omega \text{ for every } 1 \le i \le m,$$

$$\int_\Omega \langle b^i; \chi \rangle = \int_\Omega \langle dx^i; \chi \rangle \quad \text{for every } \chi \in \mathscr{H}_T\left(\Omega; \Lambda^1\right) \text{ and every } 1 \le i \le m.$$

Then there exists $\varphi \in \operatorname{Diff}^{r+1}\left(\overline{\Omega}; \overline{\Omega}\right)$ satisfying $\varphi = \operatorname{id}$ on $\partial\Omega$, and in Ω,

$$\begin{cases} \varphi^*\left(b^i\right) = dx^i, & 1 \le i \le m, \\ \varphi^*\left(dx^i\right) = dx^i, & m+1 \le i \le n. \end{cases}$$

Remark 15.11. If Ω is simply connected (cf. Remark 6.6), then $\mathscr{H}_T\left(\Omega; \Lambda^1\right) = \{0\}$ and hence the last condition on the b^i is automatically fulfilled.

Proof. Using Theorem 8.16 and the remark following it, we can find, for $1 \le i \le m$, $A^i \in C^{r+1}(\overline{\Omega})$ such that

$$\begin{cases} dA^i = b^i - dx^i & \text{in } \Omega, \\ A^i = 0 & \text{on } \partial\Omega. \end{cases}$$

Next, define $B \in C^{r+1}(\overline{\Omega};\mathbb{R}^n)$ by

$$B(x) = (x_1 + A^1(x), \dots, x_m + A^m(x), x_{m+1}, \dots, x_n).$$

Since $B = \text{id}$ on $\partial\Omega$ and since

$$\det \nabla B(x) = (b^1 \wedge \cdots \wedge b^m \wedge dx^{m+1} \wedge \cdots \wedge dx^n)_{1\cdots n}(x) \neq 0$$

for every $x \in \overline{\Omega}$, we immediately deduce from Theorem 19.12 that $B \in \text{Diff}^{r+1}(\overline{\Omega};\overline{\Omega})$. Note that for $1 \le i \le m$, $B^*(dx^i) = dB^i = d(x^i + A^i) = b^i$. Therefore, $\varphi = B^{-1} \in \text{Diff}^{r+1}(\overline{\Omega};\overline{\Omega})$ has all of the required properties. This concludes the proof. □

15.3.2 Applications to k-Forms

We next generalize Proposition 15.8 by mixing 1-forms and 2-forms.

Theorem 15.12. *Let $m, l \ge 0$ be integers and $x_0 \in \mathbb{R}^n$. Let $b^1, \dots, b^m$ and $a^1, \dots, a^m$ be closed C^∞ 1-forms. Let $g_1, \dots, g_l$ and $f_1, \dots, f_l$ be closed C^∞ 2-forms such that, in a neighborhood of x_0,*

$$\text{rank}\,[g_i] = \text{rank}\,[f_i] = 2s_i\,,\ 1 \le i \le l,$$

$$\begin{aligned} \text{rank}\left[g_1 \wedge \cdots \wedge g_l \wedge b^1 \wedge \cdots \wedge b^m\right] &= \text{rank}\left[f_1 \wedge \cdots \wedge f_l \wedge a^1 \wedge \cdots \wedge a^m\right] \\ &= 2\,(s_1 + \cdots + s_l) + m. \end{aligned}$$

Then there exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^\infty(U;\varphi(U))$ such that $\varphi(x_0) = x_0$ and, in U,

$$\begin{cases} \varphi^*(g_i) = f_i, & 1 \le i \le l, \\ \varphi^*\left(b^i\right) = a^i, & 1 \le i \le m. \end{cases}$$

Remark 15.13. (i) When $m = 0$, respectively $l = 0$, the theorem is to be understood as a statement only on 2-forms, respectively only on 1-forms (in this last case, see Proposition 15.8).

(ii) When $0 < \alpha < 1$, $g_i, f_i \in C^{r,\alpha}$ and $b^j, a^j \in C^{r,\alpha}$, the proof will give $\varphi \in \text{Diff}^{r-l+1,\alpha}$.

(iii) Of course, the theorem applies to k-forms, $k = 2l + m$, of the type

$$G = g_1 \wedge \cdots \wedge g_l \wedge b^1 \wedge \cdots \wedge b^m \quad \text{and} \quad F = f_1 \wedge \cdots \wedge f_l \wedge a^1 \wedge \cdots \wedge a^m.$$

We therefore obtain that there exists a diffeomorphism φ such that

$$\varphi^*(G) = F,$$

generalizing a result obtained by Bandyopadhyay and Dacorogna [8].

Proof. We establish the result by induction on l. When $l = 0$, we are in the situation of Proposition 15.8, which has already been proved. Let us suppose that the theorem is true for $l-1$ and prove it for l.

Step 1. Using Remark 15.2(iii), we can assume that

$$f_j = \sum_{i=(s_1+\cdots+s_{j-1})+1}^{s_1+\cdots+s_j} dx^{2i-1} \wedge dx^{2i}, \quad 1 \le j \le l,$$
$$a^i = dx^{2(s_1+\cdots+s_l)+i} \quad \text{for every } 1 \le i \le m.$$

Note that these particular f_j and a^i satisfy all of the hypotheses of the theorem. We find, using Theorem 14.3, a neighborhood U_1 of x_0 and $\varphi_1 \in \mathrm{Diff}^\infty(U_1; \varphi_1(U_1))$ such that $\varphi_1(x_0) = x_0$ and

$$\varphi_1^*(g_1) = f_1 = \sum_{i=1}^{s_1} dx^{2i-1} \wedge dx^{2i} \quad \text{in } U_1 .$$

Step 2. We claim that, in a neighborhood of x_0,

$$\begin{aligned}&\mathrm{rank}[\varphi_1^*(g_2) \wedge \cdots \wedge \varphi_1^*(g_l) \wedge dx^1 \wedge \cdots \wedge dx^{2s_1} \wedge \varphi_1^*(b^1) \wedge \cdots \wedge \varphi_1^*(b^m)] \\ &\quad = 2(s_2 + \cdots + s_l) + (2s_1 + m).\end{aligned} \tag{15.2}$$

Indeed, first note using Proposition 17.1 that, in a neighborhood of x_0,

$$\begin{aligned}&\mathrm{rank}\left[\varphi_1^*(g_1) \wedge \cdots \wedge \varphi_1^*(g_l) \wedge \varphi_1^*(b^1) \wedge \cdots \wedge \varphi_1^*(b^m)\right] \\ &\quad = \mathrm{rank}\left[\varphi_1^*(g_1 \wedge \cdots \wedge g_l \wedge b^1 \wedge \cdots \wedge b^m)\right] \\ &\quad = \mathrm{rank}\left[g_1 \wedge \cdots \wedge g_l \wedge b^1 \wedge \cdots \wedge b^m\right] = 2(s_1 + \cdots + s_l) + m.\end{aligned}$$

Setting

$$h = \varphi_1^*(g_2) \wedge \cdots \wedge \varphi_1^*(g_l) \wedge \varphi_1^*(b^1) \wedge \cdots \wedge \varphi_1^*(b^m)$$

and using Proposition 2.37(iv), we obtain

$$2(s_1 + \cdots + s_l) + m \le 2s_1 + \mathrm{rank}\,[h] - \dim\left(\Lambda^1_{\varphi_1^*(g_1)} \cap \Lambda^1_h\right).$$

On the other hand, a successive application of the same proposition gives

$$\mathrm{rank}\,[h] \le 2(s_2 + \cdots + s_l) + m.$$

Combining the two previous inequalities, we get

$$\mathrm{rank}\,[h] = 2(s_2 + \cdots + s_l) + m \quad \text{and} \quad \Lambda^1_{\varphi_1^*(g_1)} \cap \Lambda^1_h = \{0\}.$$

Finally, noticing that, in a neighborhood of x_0,

$$\Lambda^1_{\varphi_1^*(g_1)} = \operatorname{span}\{dx^1, \ldots, dx^{2s_1}\} = \Lambda^1_{dx^1 \wedge \cdots \wedge dx^{2s_1}},$$

we have the claim (15.2) using again Proposition 2.37(iv). Note also that

$$\begin{aligned} &\operatorname{rank}\left[f_2 \wedge \cdots \wedge f_l \wedge dx^1 \wedge \cdots \wedge dx^{2s_1} \wedge a^1 \wedge \cdots \wedge a^m\right] \\ &\quad = 2(s_2 + \cdots + s_l) + (2s_1 + m). \end{aligned}$$

Step 3. Therefore, using the induction hypothesis, there exist a neighborhood U_2 of x_0 and $\varphi_2 \in \operatorname{Diff}^\infty(U_2; \varphi_2(U_2))$ such that $\varphi_2(x_0) = x_0$ and for every $2 \leq i \leq l$, $1 \leq j \leq 2s_1$ and $1 \leq k \leq m$, the following identities hold in U_2:

$$\varphi_2^*(\varphi_1^*(g_i)) = f_i, \quad \varphi_2^*(dx^j) = dx^j \quad \text{and} \quad \varphi_2^*(\varphi_1^*(b^k)) = a^k.$$

Note, in particular, that $\varphi_2^*(\varphi_1^*(g_1)) = \varphi_2^*(f_1) = f_1$. Setting, choosing if necessary a smaller U_2,

$$\varphi = \varphi_1 \circ \varphi_2,$$

we have $\varphi \in \operatorname{Diff}^\infty(U_2; \varphi(U_2))$ with the claimed properties. □

It is interesting to contrast the algebraic result of Proposition 2.43(iii) with the analytical result of the above theorem, where it is essential to require that the 1-forms and the 2-forms be closed. Although every constant 3-form of rank $= 5$ is a linear pullback (combining Proposition 2.43(iii) and Proposition 2.24(ii)) of

$$(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge dx^5,$$

we have the following result.

Proposition 15.14. *There exists $F \in C^\infty(\mathbb{R}^5; \Lambda^3)$ with*

$$dF = 0 \quad \text{and} \quad \operatorname{rank}[F] = 5 \quad \text{in } \mathbb{R}^5,$$

which cannot be pulled back locally by a diffeomorphism to the canonical 3-form of rank 5:

$$(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge dx^5.$$

Proof. We will show that $F = f \wedge a$, where

$$f = \frac{1}{(x_3)^4 + 1} dx^1 \wedge dx^5 + dx^3 \wedge dx^4 \quad \text{and} \quad a = ((x_3)^2 + 1)dx^1 + ((x_3)^4 + 1)dx^2$$

has all of the desired properties. First, note that $dF = 0$ and $\operatorname{rank}[F] = 5$ in $\mathbb{R}^5$. We split the proof of the last assertion of the proposition into three steps.

Step 1. We claim that any 1-divisor c of F must be of the form $c = \lambda a$, where λ is a scalar function. Indeed, if this is not the case, we have that the 1-form $c(x_0)$ is linearly independent of $a(x_0)$ for a certain point $x_0 \in \mathbb{R}^5$. We therefore have

$$F(x_0) \wedge a(x_0) = F(x_0) \wedge c(x_0) = 0 \quad \text{and} \quad c(x_0) \wedge a(x_0) \neq 0.$$

Appealing to Theorem 2.42, we deduce that $F(x_0)$ is totally divisible and, hence (see again Proposition 2.43(ii)), $\operatorname{rank}[F(x_0)] = 3$, a contradiction.

Step 2. We show that if there exist an open set U and $\lambda \in C^1(U)$ such that

$$d(\lambda a) = 0 \quad \text{in } U,$$

then we necessarily have $\lambda \equiv 0$. Indeed, if $d(\lambda a) = 0$ in U, then, in particular,

$$(d(\lambda a))_{13} = (d(\lambda a))_{23} = 0$$

and, hence,

$$\frac{\partial(\lambda(x)(x_3^2+1))}{\partial x_3} = \frac{\partial(\lambda(x)(x_3^4+1))}{\partial x_3} = 0.$$

However, this implies the existence of $u, v \in C^1(U)$ with

$$u(x_1,x_2,x_3,x_4,x_5) = u(x_1,x_2,x_4,x_5),$$

$$v(x_1,x_2,x_3,x_4,x_5) = v(x_1,x_2,x_4,x_5)$$

such that

$$\lambda(x) = \frac{u(x_1,x_2,x_4,x_5)}{x_3^2+1} = \frac{v(x_1,x_2,x_4,x_5)}{x_3^4+1},$$

which is possible only if $u = v = 0$ in U, which proves the claim.

Step 3. We now conclude. If there exists a local diffeomorphism φ satisfying

$$F = \varphi^*((dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge dx^5) = \varphi^*(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge \varphi^*(dx^5),$$

it follows from Step 1 that

$$\varphi^*(dx^5) = \lambda a.$$

However, this leads to a contradiction, because the form on the left-hand side is closed and nonzero, whereas (cf. Step 2) the form on the right-hand side is either not closed or identically 0. □

We end this chapter with the following result, a particular case of which was proved in Bandyopadhyay and Dacorogna [8].

Theorem 15.15. *Let $4 \leq 2m \leq n$ be integers. Let $x_0 \in \mathbb{R}^n$, f and g be C^∞ closed 2-forms, and a and b be C^∞ closed 1-forms such that, in a neighborhood of x_0,*

$$\operatorname{rank}[f] = \operatorname{rank}[g] = 2m \quad \textit{and} \quad \operatorname{rank}[g \wedge b] = \operatorname{rank}[f \wedge a] = 2m-1.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^\infty(U;\varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = f \quad \textit{and} \quad \varphi^*(b) = a \quad \textit{in } U.$$

Remark 15.16. Note that if $\operatorname{rank}[g] = 2m = n \ge 4$ and $b \ne 0$, then $g \wedge b \ne 0$; otherwise by Theorem 2.42 there would exist c a 1-form such that

$$g = b \wedge c$$

and, hence, $\operatorname{rank}[g] = 2$, which is a contradiction. We therefore have, by Proposition 2.37(v), that

$$\operatorname{rank}[g \wedge b] = 2m - 1.$$

Proof. As usual, we may assume that $x_0 = 0$ and, using Remark 15.2(iii), that

$$f = \omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{and} \quad a = dx^1$$

(note that these particular f and a satisfy all of the hypotheses of the theorem, in view of Proposition 2.37(v)). We split the proof into three steps.

Step 1. Let us show that, with no loss of generality, we can assume

$$g = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} = \omega_m \quad \text{and} \quad b = \sum_{i=1}^{2m} b_i(x_1, \ldots, x_{2m}) dx^i$$

and, thus, we can assume that $2m = n$. Since $dg = 0$ and $\operatorname{rank}[g] = 2m$ in a neighborhood of 0, we can apply Theorem 14.3 to find a neighborhood U_1 of 0 and $\varphi_1 \in \operatorname{Diff}^\infty(U_1; \varphi(U_1))$ such that $\varphi_1(0) = 0$ and

$$\varphi_1^*(g) = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} = \omega_m \quad \text{in } U_1 .$$

We claim that

$$\varphi_1^*(b)(x_1, \ldots, x_n) = c(x_1, \ldots, x_{2m}) = \sum_{i=1}^{2m} c_i(x_1, \ldots, x_{2m})\, dx^i .$$

Once this is shown, we will have the assertion of Step 1. Let us prove the claim. Note that, in a neighborhood of 0,

$$\operatorname{rank}[\omega_m] = \operatorname{rank}[\varphi_1^*(g)] = \operatorname{rank}[g] = 2m,$$
$$\operatorname{rank}[c \wedge \omega_m] = \operatorname{rank}[\varphi_1^*(b \wedge g)] = \operatorname{rank}[b \wedge g] = 2m - 1.$$

Hence, using Proposition 2.37(v), we get, in a neighborhood of 0,

$$c \in \Lambda^1_{\omega_m} = \operatorname{span}\left\{dx^1, \ldots, dx^{2m}\right\}$$

and, thus,

$$c_i(x) = 0 \quad \text{for } 2m + 1 \le i \le n.$$

Finally, combining the previous equation with the fact that $dc = 0$, we immediately deduce that for every $1 \le i \le m$ and every x in a neighborhood of 0,

$$c_i(x_1,\ldots,x_n) = c_i(x_1,\ldots,x_{2m}),$$

which proves the claim.

Step 2. Using Theorem 8.1, we can find a C^∞ function (in a small ball B_ε centered at 0) ρ such that

$$d\rho = b(0) - b.$$

With no loss of generality, we can assume that $\rho(0) = 0$. Let $b_t(x) \in C^\infty\left([0,1] \times B_\varepsilon; \Lambda^1\right)$ be defined by

$$b_t(x) = (1-t)b(0) + tb(x).$$

Since for every $t \in [0,1]$, $b_t(0) = b(0) \neq 0$, there exist $1 \leq i \leq n$ and a neighborhood of 0 in which

$$[b_t \lrcorner\, \omega_m]_i = [\overline{\omega}_m b_t]_i \neq 0 \quad \text{for every } t \in [0,1].$$

Hence, we can apply Remark 8.21 and find a neighborhood U_2 of 0 and $w \in C^\infty\left([0,1] \times U_2; \Lambda^1\right)$, $w(t,x) = w_t(x)$ such that for every $t \in [0,1]$, $w_t(0) = 0$ and

$$dw_t = 0 \quad \text{and} \quad \langle w_t; \overline{\omega}_m b_t \rangle = \rho \quad \text{in } U_2 .$$

Finally, define $u \in C^\infty([0,1] \times U_2; \Lambda^1)$, $u = u(t,x) = u_t(x)$, as

$$u_t = \overline{\omega}_m^{-1} w_t \;\Leftrightarrow\; u_t \lrcorner\, \omega_m = w_t .$$

Note that for every $t \in [0,1]$, $u_t(0) = 0$ and in U_2, $d\left(u_t \lrcorner\, \omega_m\right) = dw_t = 0$ and since $\overline{\omega}_m \in O(n)$,

$$d(u_t \lrcorner\, b_t) = d(\langle u_t; b_t \rangle) = d\left(\langle w_t; \overline{\omega}_m b_t \rangle\right) = d\rho = -\frac{db_t}{dt} .$$

Hence, we deduce from Theorem 12.8 that for every $t \in [0,1]$, the solution ϕ_t of

$$\begin{cases} \dfrac{d}{dt}\phi_t = u_t \circ \phi_t, & 0 \leq t \leq 1, \\ \phi_0 = \mathrm{id} \end{cases}$$

exists in a neighborhood U_3 of 0 and verifies $\phi_t \in \mathrm{Diff}^\infty(U_3; \phi_t(U_3))$ and

$$\phi_t^*(\omega_m) = \omega_m , \quad \phi_t^*(b_t) = b(0) \quad \text{in } U_3 .$$

Step 3. Finally, recalling that $b(0) \in \Lambda^1_{\omega_m}$, there exists, using Proposition 2.24, $A \in \mathrm{GL}(n)$ such that

$$A^*(\omega_m) = \omega_m \quad \text{and} \quad A^*(b(0)) = dx^1.$$

Letting $\psi(x) = Ax$ and $\varphi = \phi_1 \circ \psi$, we get the result and this concludes the proof. □