Chapter 14
The Case $k = 2$

14.1 Notations

We recall, from Chapter 2, some notations that we will use throughout the present chapter. As usual, when necessary, we identify in a natural way 1-forms with vectors in $\mathbb{R}^n$.

(i) If $u \in \Lambda^1(\mathbb{R}^n)$ and $f \in \Lambda^2(\mathbb{R}^n)$, then (cf. Proposition 2.12)

$$u \lrcorner f = \sum_{j=1}^{n} \left[\sum_{i=1}^{n} f_{ij} u_i \right] dx^j \in \Lambda^1(\mathbb{R}^n).$$

(ii) Given $f \in \Lambda^2(\mathbb{R}^n)$, the matrix $\overline{f} \in \mathbb{R}^{n \times n}$ (denoted in Notation 2.30 as $\overline{f}_{\lrcorner,1}$) is defined, by abuse of notations, as

$$\overline{f}\, u = u \lrcorner f \quad \text{for every } u \in \Lambda^1(\mathbb{R}^n) \approx \mathbb{R}^n.$$

(iii) The *rank* of $f \in \Lambda^2(\mathbb{R}^n)$ is defined (cf. Proposition 2.32(i)) by

$$\operatorname{rank}[f] = \operatorname{rank}\left(\overline{f}\right).$$

We also recall that in the present chapter we denote by rank what was denoted by rank_1 in Chapter 2. In particular, if $\operatorname{rank}[f] = n$, then $\overline{f}$ is invertible and

$$v = u \lrcorner f \Leftrightarrow u = \left(\overline{f}\right)^{-1} v.$$

(iv) When n is even, identifying n-forms with 0-forms, we have (cf. Proposition 2.37(iii))

$$\left|\det \overline{f}\right|^{1/2} = \frac{1}{(n/2)\,!} \left| f^{n/2} \right|,$$

where $f^m = \underbrace{f \wedge \cdots \wedge f}_{m \text{ times}}$.

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(v) Let $r \geq 0$ be an integer and $0 \leq \alpha \leq 1$. Let $f \in \Lambda^2(\mathbb{R}^n)$ with rank $[f]=n$ (thus, in particular, n is even). In view of Corollary 16.30 and of the previous point, if $c>0$ is such that

$$\left\| \frac{1}{f^{n/2}} \right\|_{C^0}, \quad \|f\|_{C^0} \leq c,$$

then there exists a constant $C=C(c,r,\Omega)>0$ such that

$$\| (\overline{f})^{-1} \|_{C^{r,\alpha}} \leq C \|f\|_{C^{r,\alpha}}.$$

(vi) Finally, we recall the notion of harmonic fields with a vanishing tangential part (cf. Section 6.1). If $\Omega \subset \mathbb{R}^n$ is a bounded open smooth set, then

$$\mathscr{H}_T(\Omega;\Lambda^2) = \{\omega \in C^\infty(\overline{\Omega};\Lambda^2) : d\omega = 0,\ \delta\omega = 0 \text{ in } \Omega \text{ and } \nu \wedge \omega = 0 \text{ on } \partial\Omega\}.$$

Recall that if Ω is contractible, then

$$\mathscr{H}_T(\Omega;\Lambda^2) = \{0\} \quad \text{if } n \geq 3.$$

In terms of the components of

$$\omega = \sum_{1 \leq i < j \leq n} \omega_{ij}\, dx^i \wedge dx^j,$$

we have

$$d\omega = 0 \Leftrightarrow \frac{\partial \omega_{ij}}{\partial x_k} - \frac{\partial \omega_{ik}}{\partial x_j} + \frac{\partial \omega_{jk}}{\partial x_i} = 0,\ \forall 1 \leq i < j < k \leq n,$$

$$\delta\omega = 0 \Leftrightarrow \sum_{j=1}^n \frac{\partial \omega_{ij}}{\partial x_j} = 0,\ \forall 1 \leq i \leq n,$$

$$\nu \wedge \omega = 0 \Leftrightarrow \omega_{ij}\nu_k - \omega_{ik}\nu_j + \omega_{jk}\nu_i = 0,\ \forall 1 \leq i < j < k \leq n.$$

14.2 Local Result for Forms with Maximal Rank

The following result is the classical Darboux theorem for closed 2-forms but with optimal regularity. This is a delicate point and it has been obtained by Bandyopadhyay and Dacorogna [8]. The other existing results provide solutions that are only in $C^{r,\alpha}$, whereas in the theorem below we find a solution which belongs to $C^{r+1,\alpha}$.

Theorem 14.1 (Darboux theorem with optimal regularity). *Let $r \geq 0$ and $n = 2m \geq 4$ be integers. Let $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let ω_m be the standard symplectic form of rank $2m$,*

$$\omega_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a 2-form. The two following statements are then equivalent:

(i) The 2-form ω is closed, is in $C^{r,\alpha}$ in a neighborhood of x_0, and verifies

$$\operatorname{rank}\left[\omega\left(x_0\right)\right]=n.$$

(ii) There exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r+1,\alpha}(U;\varphi(U))$ such that

$$\varphi^*\left(\omega_m\right)=\omega \quad \text{in } U \quad \text{and} \quad \varphi\left(x_0\right)=x_0 .$$

Remark 14.2. (i) When $r=0$, the hypothesis $d\omega=0$ is to be understood in the sense of distributions.

(ii) The theorem is still valid when $n=2$, but it is then the result of Dacorogna and Moser [33] (cf. Theorem 10.3).

Proof. The necessary part is obvious and we discuss only the sufficient part. We divide the proof into four steps.

Step 1. Without loss of generality we take $x_0=0$. We can, according to Proposition 2.24(ii), also always assume that

$$\omega(0)=\omega_m .$$

Step 2. Our theorem will follow from Theorem 18.1. So we need to define the spaces and the operators and then check all of the hypotheses.

1) We choose V a sufficiently small ball centered at 0 and we define the sets

$$X_1=C^{1,\alpha}(\overline{V};\mathbb{R}^n) \quad \text{and} \quad Y_1=C^{0,\alpha}(\overline{V};\Lambda^2),$$

$$X_2=C^{r+1,\alpha}(\overline{V};\mathbb{R}^n) \quad \text{and} \quad Y_2=\{b\in C^{r,\alpha}(\overline{V};\Lambda^2): db=0 \text{ in } V\}.$$

Using Proposition 16.23, we immediately deduce that (H_{XY}) of Theorem 18.1 is fulfilled.

2) Define $L: X_2\to Y_2$ by

$$La=d[a\lrcorner\,\omega_m]=b.$$

We will show that there exists $L^{-1}: Y_2\to X_2$ a linear right inverse of L and a constant $C_1=C_1(r,\alpha,V)$ such that

$$\|L^{-1}b\|_{X_i}\le C_1\|b\|_{Y_i} \text{ for every } b\in Y_2 \text{ and } i=1,2.$$

Once shown this, (H_L) of Theorem 18.1 will be satisfied. First, using Theorem 8.3, find $w\in C^{r+1,\alpha}(\overline{V};\Lambda^1)$ and $C_1=C_1(r,\alpha,V)>0$ such that

$$dw=b \quad \text{in } V,$$

$$\|w\|_{C^{r+1,\alpha}}\le C_1\|b\|_{C^{r,\alpha}} \quad \text{and} \quad \|w\|_{C^{1,\alpha}}\le C_1\|b\|_{C^{0,\alpha}}.$$

Moreover, the correspondence $b \to w$ can be chosen to be linear. Next, define $a \in C^{r+1,\alpha}(\overline{V};\mathbb{R}^n)$ by

$$a_{2i-1} = w_{2i} \quad \text{and} \quad a_{2i} = -w_{2i-1}, \quad 1 \leq i \leq m,$$

and note that

$$a \lrcorner \omega_m = w.$$

Finally, defining $L^{-1} : Y_2 \to X_2$ by $L^{-1}(b) = a$, we easily check that L^{-1} is linear,

$$LL^{-1} = \text{id} \quad \text{on } Y_2$$

and

$$\|L^{-1}b\|_{X_i} \leq C_1 \|b\|_{Y_i} \text{ for every } b \in Y_2 \text{ and } i = 1,2.$$

So (H_L) of Theorem 18.1 is satisfied.

3) We then let Q be defined by

$$Q(u) = \omega_m - (\text{id}+u)^* \omega_m + d\left[u \lrcorner \omega_m\right].$$

Since

$$d\left[u \lrcorner \omega_m\right] = \sum_{i=1}^{m} \left[du^{2i-1} \wedge dx^{2i} + dx^{2i-1} \wedge du^{2i}\right],$$

$$\omega_m - (\text{id}+u)^* \omega_m = \sum_{i=1}^{m} \left[dx^{2i-1} \wedge dx^{2i} - \left(dx^{2i-1} + du^{2i-1}\right) \wedge \left(dx^{2i} + du^{2i}\right)\right],$$

we get

$$Q(u) = -\sum_{i=1}^{m} du^{2i-1} \wedge du^{2i}.$$

4) Note that $Q(0) = 0$ and $dQ(u) = 0$ in V. Appealing to Theorem 16.28 (a similar but more involved estimate can be found in Lemma 14.8), there exists a constant $C_2 = C_2(r,V)$ such that for every $u, v \in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$, the following estimates hold:

$$\begin{aligned} \|Q(u) - Q(v)\|_{C^{0,\alpha}} &\leq \sum_{i=1}^{m} \|du^{2i-1} \wedge du^{2i} - dv^{2i-1} \wedge dv^{2i}\|_{C^{0,\alpha}} \\ &\leq \sum_{i=1}^{m} \|du^{2i-1} \wedge \left(du^{2i} - dv^{2i}\right)\|_{C^{0,\alpha}} \\ &\quad + \sum_{i=1}^{m} \|\left(dv^{2i-1} - du^{2i-1}\right) \wedge dv^{2i}\|_{C^{0,\alpha}} \\ &\leq C_2(\|u\|_{C^{1,\alpha}} + \|v\|_{C^{1,\alpha}})\|u - v\|_{C^{1,\alpha}} \end{aligned}$$

and

$$\begin{aligned}\|Q(u)\|_{C^{r,\alpha}} &\leq \sum_{i=1}^{m} \|du^{2i-1} \wedge du^{2i}\|_{C^{r,\alpha}} \\ &\leq C \sum_{i=1}^{m} \left[\|du^{2i-1}\|_{C^{r,\alpha}} \|du^{2i}\|_{C^0} + \|du^{2i}\|_{C^{r,\alpha}} \|du^{2i-1}\|_{C^0}\right] \\ &\leq C_2 \|u\|_{C^{1,\alpha}} \|u\|_{C^{r+1,\alpha}}\,.\end{aligned}$$

We therefore see that property (H_Q) is valid for every ρ and we choose $\rho = 1/(2n)$,

$$c_1(r,s) = C_2(r+s) \quad \text{and} \quad c_2(r,s) = C_2\, r\, s.$$

5) Setting $\varphi = \mathrm{id} + u$, we can rewrite the equation $\varphi^*(\omega_m) = \omega$ as

$$\begin{aligned}Lu = d\left[u \lrcorner\, \omega_m\right] &= \omega - (\mathrm{id}+u)^* \omega_m + d\left[u \lrcorner\, \omega_m\right] \\ &= \omega - \omega_m + \left[\omega_m - (\mathrm{id}+u)^* \omega_m + d\left[u \lrcorner\, \omega_m\right]\right] \\ &= \omega - \omega_m + Q(u).\end{aligned}$$

Step 3. We may now apply Theorem 18.1 and get that there exists $\psi \in C^{r+1,\alpha}\left(\overline{V};\mathbb{R}^n\right)$ such that $\psi^*(\omega_m) = \omega$ in V with $\|\nabla\psi - I\|_{C^0} \leq 1/(2n)$, provided

$$\|\omega - \omega_m\|_{C^{0,\alpha}} \leq \frac{1}{2C_1 \max\{4C_1C_2\,, 1\}}\,. \tag{14.1}$$

Setting $\varphi(x) = \psi(x) - \psi(0)$, we have indeed proved that there exists $\varphi \in C^{r+1,\alpha}\left(\overline{V};\mathbb{R}^n\right)$ satisfying

$$\varphi^*(\omega_m) = \omega \text{ in } V, \quad \|\nabla\varphi - I\|_{C^0} \leq \frac{1}{2n} \quad \text{and} \quad \varphi(0) = 0.$$

Step 4. We may now conclude the proof of the theorem.

Step 4.1. Let $0 < \varepsilon < 1$ and define

$$\omega^\varepsilon(x) = \omega(\varepsilon x)\,.$$

Observe that $\omega^\varepsilon \in C^{r,\alpha}(\overline{V};\Lambda^2)$, $d\omega^\varepsilon = 0$, $\omega^\varepsilon(0) = \omega_m$ and

$$\|\omega^\varepsilon - \omega_m\|_{C^{0,\alpha}(\overline{V})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Choose ε sufficiently small so that

$$\|\omega^\varepsilon - \omega_m\|_{C^{0,\alpha}(\overline{V})} \leq \frac{1}{2C_1 \max\{4C_1C_2\,, 1\}}\,.$$

Apply Step 3 to find $\psi_\varepsilon \in C^{r+1,\alpha}\left(\overline{V};\mathbb{R}^n\right)$ satisfying

$$\psi_\varepsilon^*(\omega_m) = \omega^\varepsilon \text{ in } V, \quad \|\nabla\psi_\varepsilon - I\|_{C^0} \leq \frac{1}{2n} \quad \text{and} \quad \psi_\varepsilon(0) = 0.$$

Step 4.2. Let

$$\chi_\varepsilon(x) = \frac{x}{\varepsilon}$$

and define

$$\varphi = \varepsilon\, \psi_\varepsilon \circ \chi_\varepsilon\,.$$

Define $U = \varepsilon V$. It is easily seen that $\varphi \in C^{r+1,\alpha}(\overline{U};\mathbb{R}^n)$,

$$\varphi^*(\omega_m) = \omega \;\text{ in } U \quad \text{and} \quad \varphi(0) = 0.$$

Note in particular that

$$\|\nabla\varphi - I\|_{C^0(\overline{U})} = \|\nabla\psi_\varepsilon - I\|_{C^0(\overline{V})} \leq \frac{1}{2n}$$

and therefore $\det\nabla\varphi > 0$ in $\overline{U}$. Hence, restricting U, if necessary, we can assume that $\varphi \in \operatorname{Diff}^{r+1,\alpha}(\overline{U};\varphi(\overline{U}))$. This concludes the proof of the theorem. □

14.3 Local Result for Forms of Nonmaximal Rank

The main result of the present section is to obtain the Darboux theorem for degenerate closed 2-forms. We will provide, following Bandyopadhyay, Dacorogna and Kneuss [9], two proofs of the theorem. The standard proof uses the Frobenius theorem to reduce the dimension so that the forms have maximal rank and then apply the classical Darboux theorem. We will follow this path but using the more sophisticated Theorem 14.1. Our theorem will provide a solution in $C^{r,\alpha}$, whereas in the existing literature solutions are found only in $C^{r-1,\alpha}$.

We will also give a completely different proof; it will use an argument based on the flow method. Still a different proof can be found in [8] when $n = 2m+1$.

14.3.1 The Theorem and a First Proof

Theorem 14.3. *Let $n \geq 3$, $r, m \geq 1$ be integers and $0 < \alpha < 1$. Let $x_0 \in \mathbb{R}^n$ and ω_m be the standard symplectic form with $\operatorname{rank}[\omega_m] = 2m < n$, namely*

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a $C^{r,\alpha}$ closed 2-form such that

$$\operatorname{rank}[\omega] = 2m \text{ in a neighborhood of } x_0\,.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(U;\varphi(U))$ such that

$$\varphi^*(\omega_m) = \omega \text{ in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

Remark 14.4. The theorem is standard in the C^∞ case. In all proofs that we have seen, the regularity that is established is, at best, that if $\omega \in C^{r,\alpha}$, then $\varphi \in C^{r-1,\alpha}$. However, our result asserts that ω and φ have the same regularity in Hölder spaces. This is, of course, better but still not optimal, as in the nondegenerate case of Theorem 14.1.

Proof. Step 1. Without loss of generality, we can assume $x_0 = 0$. We first find, appealing to Theorem 4.5, a neighborhood $V \subset \mathbb{R}^n$ of 0 and $\psi \in \operatorname{Diff}^{r,\alpha}(V; \psi(V))$ with $\psi(0) = 0$ and

$$\psi^*(\omega)(x_1, \ldots, x_n) = \widetilde{\omega}(x_1, \ldots, x_{2m}) = \sum_{1 \le i < j \le 2m} \widetilde{\omega}_{ij}(x_1, \ldots, x_{2m})\, dx^i \wedge dx^j.$$

Therefore, $\psi^*(\omega) = \widetilde{\omega} \in C^{r-1,\alpha}$ in a neighborhood of 0 in $\mathbb{R}^{2m}$ and $\operatorname{rank} \widetilde{\omega} = 2m$ in a neighborhood of 0.

Step 2. We then apply Theorem 14.1 to $\widetilde{\omega}$ find a neighborhood $W \subset \mathbb{R}^{2m}$ of 0 and $\chi \in \operatorname{Diff}^{r,\alpha}(W; \chi(W))$, with $\chi(0) = 0$, such that

$$\chi^*(\omega_m) = \widetilde{\omega} \text{ in } W.$$

We set

$$\widetilde{\chi}(x) = \widetilde{\chi}(x_1, \ldots, x_{2m}, x_{2m+1}, \ldots, x_n) = (\chi(x_1, \ldots, x_{2m}), x_{2m+1}, \ldots, x_n).$$

We then choose V smaller, if necessary, so that

$$V \subset W \times \mathbb{R}^{n-2m}.$$

We finally have that $U = \psi(V)$ and $\varphi = \widetilde{\chi} \circ \psi^{-1}$ have all of the desired properties. □

14.3.2 A Second Proof

We now provide a second proof of Theorem 14.3 under the extra assumption that ω is in C^∞. It seems that the present proof is more appropriate if one wants to look for global results.

Proof. As usual, we consider, without loss of generality, that $x_0 = 0$.

Step 1. Define, for a sufficiently small neighborhood U_1 of 0,

$$h(t,x) = h_t(x) = \omega(tx).$$

Then the homotopy h is such that $h \in C^\infty([0,1] \times U_1; \Lambda^2)$ and for every $t \in [0,1]$, the following identities hold in U_1:

$$dh_t = 0, \quad h_t^m \neq 0 \quad \text{and} \quad h_t^{m+1} = 0 \tag{14.2}$$

(recall that the last two conditions are equivalent to rank $[h_t] = 2m$) and

$$h_0 = \omega(0) \quad \text{and} \quad h_1 = \omega.$$

Step 2. Since (14.2) holds and

$$h_t^m \wedge \frac{\partial h_t}{\partial t} = \frac{1}{m+1} \frac{\partial h_t^{m+1}}{\partial t} = 0,$$

we can apply Theorem 8.22. We can therefore find a neighborhood $U_2 \subset U_1$ of 0 and $w \in C^\infty([0,1] \times U_2; \mathbb{R}^n)$, $w(t,x) = w_t(x)$, satisfying, for every $t \in [0,1]$, $w_t(0) = 0$ and

$$dw_t = -\frac{\partial h_t}{\partial t} \quad \text{and} \quad w_t \wedge h_t^m = 0 \quad \text{in } U_2 .$$

We then apply Proposition 2.50 to find $u \in C^\infty([0,1] \times U_2; \mathbb{R}^n)$, $u(t,x) = u_t(x)$, with

$$u_t \lrcorner h_t = w_t \quad \text{and} \quad u_t(0) = 0.$$

Step 3. We next find the flow, associated to the vector field u_t ,

$$\begin{cases} \dfrac{d}{dt}\varphi_t = u_t \circ \varphi_t, & 0 \le t \le 1, \\ \varphi_0 = \text{id} . \end{cases}$$

Theorem 12.8 gives that φ_1 is a diffeomorphism in a neighborhood $U_3 \subset U_2$ of 0 such that

$$\varphi_1^*(h_1) = h_0 \text{ in } U_3 \quad \text{and} \quad \varphi_1(0) = 0 .$$

Step 4. Since h_0 is constant, we can use Proposition 2.24(ii) to find a diffeomorphism ψ of the form $\psi(x) = Ax$ with $A \in \text{GL}(n)$ so that

$$\psi^*(h_0) = \omega_m = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

Letting $\varphi = \psi^{-1} \circ \varphi_1^{-1}$, we have the claim. □

14.4 Global Result with Dirichlet Data

14.4.1 The Main Result

We now state our main theorem. It has been obtained under slightly more restrictive hypotheses by Bandyopadhyay and Dacorogna [8]; as stated, it is due to Dacorogna and Kneuss [32]. We will provide two proofs of the theorem in Sections 14.4.5 and 14.4.6.

Theorem 14.5. *Let $n>2$ be even and $\Omega\subset\mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν. Let $0<\alpha<1$ and $r\geq 1$ be an integer. Let $f,g\in C^{r,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ satisfying $df=dg=0$ in Ω,*

$$\nu\wedge f,\nu\wedge g\in C^{r+1,\alpha}\left(\partial\Omega;\Lambda^3\right)\quad\text{and}\quad \nu\wedge f=\nu\wedge g\text{ on }\partial\Omega,$$

$$\int_\Omega\langle f;\psi\rangle\,dx=\int_\Omega\langle g;\psi\rangle\,dx\quad\text{for every }\psi\in\mathscr{H}_T\left(\Omega;\Lambda^2\right)\tag{14.3}$$

and, for every $t\in[0,1]$,

$$\operatorname{rank}\left[tg+(1-t)f\right]=n\ \text{ in }\overline{\Omega}.$$

Then there exists $\varphi\in\operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ such that

$$\varphi^*(g)=f\text{ in }\Omega\quad\text{and}\quad\varphi=\operatorname{id}\text{ on }\partial\Omega.$$

Remark 14.6. (i) As already mentioned, we can consider, in a similar way, a general homotopy f_t with $f_0=f$, $f_1=g$,

$$df_t=0,\quad \nu\wedge f_t=\nu\wedge f_0\text{ on }\partial\Omega\quad\text{and}\quad\operatorname{rank}[f_t]=n\text{ in }\overline{\Omega},$$

$$\int_\Omega\langle f_t;\psi\rangle\,dx=\int_\Omega\langle f_0;\psi\rangle\,dx\quad\text{for every }\psi\in\mathscr{H}_T\left(\Omega;\Lambda^2\right).$$

Note that the nondegeneracy condition $\operatorname{rank}[f_t]=n$ implies (identifying, as usual, volume forms with functions)

$$f^{n/2}\cdot g^{n/2}>0\ \text{ in }\overline{\Omega}.$$

(ii) The nondegeneracy condition

$$\operatorname{rank}\left[tg+(1-t)f\right]=n\quad\text{for every }t\in[0,1]$$

is equivalent to the condition that the matrix $(\overline{g})\left(\overline{f}\right)^{-1}$ has no negative eigenvalues.

(iii) If Ω is contractible, then $\mathscr{H}_T\left(\Omega;\Lambda^2\right)=\{0\}$ and, therefore, (14.3) is automatically satisfied.

(iv) Note that the extra regularity on f and g holds only on the boundary and only for their tangential parts. More precisely, recall that for $x\in\partial\Omega$, we denote by $\nu=\nu(x)$ the exterior unit normal to Ω. By

$$\nu\wedge f\in C^{r+1,\alpha}\left(\partial\Omega;\Lambda^3\right)$$

we mean that the tangential part of f is in $C^{r+1,\alpha}$, namely the 3-form F defined by

$$F(x)=\nu(x)\wedge f(x)$$

is such that

$$F\in C^{r+1,\alpha}\left(\partial\Omega;\Lambda^3\right).$$

14.4.2 The Flow Method

We now state and prove a weaker version, from the point of view of regularity, of Theorem 14.5. It has, however, the advantage of having a simple proof. It has been obtained by Bandyopadhyay and Dacorogna [8].

Theorem 14.7. *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν. Let $r \geq 1$ be an integer, $0 < \alpha < 1$ and $f, g \in C^{r,\alpha}\left(\overline{\Omega}; \Lambda^2\right)$ satisfy*

$$df = dg = 0 \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega,$$

$$\int_{\Omega} \langle f; \psi \rangle\, dx = \int_{\Omega} \langle g; \psi \rangle\, dx \quad \text{for every } \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right),$$

$$\operatorname{rank}\left[tg + (1-t) f\right] = n \text{ in } \overline{\Omega} \text{ and for every } t \in [0,1].$$

Then there exists $\varphi \in \operatorname{Diff}^{r,\alpha}\left(\overline{\Omega}; \overline{\Omega}\right)$ such that

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \operatorname{id} \text{ on } \partial\Omega.$$

Furthermore, if $0 < \beta \leq \alpha < 1$ and if $c > 0$ is such that

$$\|f\|_{C^1}, \|g\|_{C^1}, \left\| \frac{1}{\left[tg + (1-t) f\right]^{n/2}} \right\|_{C^0} \leq c \quad \text{for every } t \in [0,1],$$

then there exists a constant $C = C(c, r, \alpha, \beta, \Omega) > 0$ such that

$$\|\varphi - \operatorname{id}\|_{C^{r,\alpha}} \leq C \left[\|f\|_{C^{r,\alpha}} + \|g\|_{C^{r,\alpha}}\right] \|f - g\|_{C^{0,\beta}} + C \|f - g\|_{C^{r-1,\alpha}}.$$

Proof. We solve (cf. Theorem 8.16)

$$\begin{cases} dw = f - g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and find $w \in C^{r+1,\alpha}\left(\overline{\Omega}; \Lambda^1\right)$ and $C_1 = C_1(r, \alpha, \beta, \Omega) > 0$ such that

$$\|w\|_{C^{r,\alpha}} \leq C_1 \|f - g\|_{C^{r-1,\alpha}} \quad \text{and} \quad \|w\|_{C^{1,\beta}} \leq C_1 \|f - g\|_{C^{0,\beta}}.$$

Since $\operatorname{rank}\left[tg + (1-t) f\right] = n$, we can find $u_t \in C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ so that

$$u_t \lrcorner \left[tg + (1-t) f\right] = w \Leftrightarrow u_t = \left[t\overline{g} + (1-t)\overline{f}\right]^{-1} w.$$

Moreover (in view of Notation (v) in Section 14.1, Theorem 16.28 and Corollary 16.30), we can find constants $C_i = C_i(c, r, \alpha, \beta, \Omega) > 0$, $i = 2, 3$, such that

$$\begin{aligned} \|u_t\|_{C^{r,\alpha}} &\leq C_2 \left[\|f\|_{C^{r,\alpha}} + \|g\|_{C^{r,\alpha}}\right] \|w\|_{C^0} + C_2 \|w\|_{C^{r,\alpha}} \\ &\leq C_3 \left[\|f\|_{C^{r,\alpha}} + \|g\|_{C^{r,\alpha}}\right] \|f - g\|_{C^{0,\beta}} + C_3 \|f - g\|_{C^{r-1,\alpha}} \end{aligned}$$

and $\|u_t\|_{C^1} \leq C_3$. We then apply Theorem 12.7 to u_t and $f_t = tg + (1-t)f$ to find φ satisfying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \text{id on } \partial\Omega.$$

The estimate follows from Theorem 12.1. The proof is therefore complete. □

14.4.3 The Key Estimate for Regularity

The following estimate will play a crucial role in getting the optimal regularity in Theorem 14.10. We have encountered a result of the same type in the much simpler case of volume forms (see Theorem 10.9) or in the local case (see Theorem 14.1). We will state the theorem for k-forms, although we will use it only when $k = 2$.

Lemma 14.8. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Let $r \geq 1$, $2 \leq k \leq n$ be integers, $c > 0$ and $0 \leq \gamma \leq \alpha \leq 1$. Let $g \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^k)$ be closed, $u, v \in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ and $c > 0$ with*

$$\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \leq c,$$

$$(\text{id}+tu)(\overline{\Omega}), (\text{id}+tv)(\overline{\Omega}) \subset \overline{\Omega}, \; \forall t \in [0,1].$$

Set

$$Q(u) = g - (\text{id}+u)^*(g) + d[u \lrcorner g].$$

Then there exists a constant $C = C(c, r, \Omega)$ such that the following estimates hold:

$$\|Q(u) - Q(v)\|_{C^{0,\gamma}} \leq C\|g\|_{C^{2,\gamma}}(\|u\|_{C^{1,\gamma}} + \|v\|_{C^{1,\gamma}})\|u - v\|_{C^{1,\gamma}},$$

$$\|Q(u)\|_{C^{r,\alpha}} \leq C\|g\|_{C^{r+1,\alpha}}\|u\|_{C^1} + C\|g\|_{C^1}\|u\|_{C^{r+1,\alpha}}\|u\|_{C^1}.$$

Remark 14.9. With essentially the same argument, we can replace the last estimate by the following one. In addition to the hypotheses of the lemma, let $0 \leq \alpha < \beta \leq 1$ and $g \in C^{r+1,\beta}(\overline{\Omega};\Lambda^k)$; then the last estimate takes the following form:

$$\|Q(u)\|_{C^{r,\alpha}} \leq C\|g\|_{C^{r+1,\beta}}\|u\|_{C^1}^{1+\beta-\alpha} + C\|g\|_{C^{r+1,\alpha}}\|u\|_{C^{r+1,\alpha}}\|u\|_{C^1}$$

for every $u, v \in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ with

$$\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \leq c,$$

$$(\text{id}+tu)(\overline{\Omega}), (\text{id}+tv)(\overline{\Omega}) \subset \overline{\Omega}, \; \forall t \in [0,1].$$

Proof. We divide the proof into four steps. Since we will apply the result only when $k = 2$, we will always single out the formulas for this case. We also will constantly use Theorem 16.28.

Step 1. We start with some notations. The form g will be written as

$$g = \sum_{I \in \mathscr{T}_k} g_I \, dx^I .$$

We first need to write $(\mathrm{id}+u)^*(g)$ in a different way. For this, we observe that we have, for $I \in \mathscr{T}_k$,

$$\begin{aligned} d\,(x+u)^I &= \left(dx^{i_1} + du^{i_1}\right) \wedge \cdots \wedge \left(dx^{i_k} + du^{i_k}\right) \\ &= dx^I + \sum_{\substack{(J,K)=I \\ 1 \le |K| \le k}} dx^J \wedge du^K \\ &= dx^I + \sum_{\substack{(J,i)=I \\ 1 \le i \le n}} dx^J \wedge du^i + \sum_{\substack{(J,K)=I \\ 2 \le |K| \le k}} dx^J \wedge du^K , \end{aligned}$$

where we have used the notation

$$\sum_{\substack{(J,i)=I \\ 1 \le i \le n}} dx^J \wedge du^i = \sum_{\gamma=1}^{k} (-1)^{k+\gamma} dx^{i_1} \wedge \cdots \wedge dx^{i_{\gamma-1}} \wedge dx^{i_{\gamma+1}} \wedge \cdots \wedge dx^{i_k} \wedge du^{i_\gamma}$$

and similarly for

$$\sum_{\substack{(J,K)=I \\ 2 \le |K| \le k}} dx^J \wedge du^K .$$

When $k=2$, we have

$$\begin{aligned} (dx+du)^{ij} &= (dx^i + du^i) \wedge (dx^j + du^j) \\ &= dx^i \wedge dx^j + \left[du^i \wedge dx^j + dx^i \wedge du^j\right] + du^i \wedge du^j . \end{aligned}$$

We can therefore write

$$\begin{aligned} (\mathrm{id}+u)^*(g) &= \sum_{I \in \mathscr{T}_k} g_I \,(\mathrm{id}+u)\, dx^I + \sum_{I \in \mathscr{T}_k} g_I \,(\mathrm{id}+u) \sum_{\substack{(J,K)=I \\ 1 \le |K| \le k}} dx^J \wedge du^K \\ &= g\,(\mathrm{id}+u) + \sum_{I \in \mathscr{T}_k} \sum_{\substack{(J,i)=I \\ 1 \le i \le n}} g_I \,(\mathrm{id}+u)\, dx^J \wedge du^i \\ &\quad + \sum_{I \in \mathscr{T}_k} \sum_{\substack{(J,K)=I \\ 2 \le |K| \le k}} g_I \,(\mathrm{id}+u)\, dx^J \wedge du^K \end{aligned}$$

so that when $k=2$, we find

$$\begin{aligned} (\mathrm{id}+u)^*(g) = g\,(\mathrm{id}+u) &+ \sum_{1 \le i < j \le n} g_{ij}(\mathrm{id}+u) \left[du^i \wedge dx^j + dx^i \wedge du^j\right] \\ &+ \sum_{1 \le i < j \le n} g_{ij}(\mathrm{id}+u)\, du^i \wedge du^j . \end{aligned}$$

We will also use, for $I \in \mathscr{T}_k$,

$$d\left[u \lrcorner dx^I\right] = \sum_{\substack{(J,i)=I \\ 1\le i\le n}} dx^J \wedge du^i,$$

which reads, when $k=2$, as

$$d\left[u \lrcorner dx^{ij}\right] = d\left[u \lrcorner \left(dx^i \wedge dx^j\right)\right] = du^i \wedge dx^j + dx^i \wedge du^j.$$

Step 2. We have, since g is closed and according to Lemma 5.4, that

$$\begin{aligned} d\left[u \lrcorner g\right] &= \sum_{I\in\mathscr{T}_k} g_I\, d\left[u \lrcorner dx^I\right] + \sum_{I\in\mathscr{T}_k} \langle \operatorname{grad} g_I; u\rangle\, dx^I \\ &= \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,i)=I \\ 1\le i\le n}} g_I\, dx^J \wedge du^i + \sum_{I\in\mathscr{T}_k} \langle \operatorname{grad} g_I; u\rangle\, dx^I \end{aligned}$$

and hence, when $k=2$,

$$d\left[u \lrcorner g\right] = \sum_{1\le i<j\le n} g_{ij}\left[du^i \wedge dx^j + dx^i \wedge du^j\right] + \sum_{1\le i<j\le n} \left\langle \operatorname{grad} g_{ij}; u\right\rangle dx^i \wedge dx^j.$$

In order to get the right estimates, we rewrite $Q(u)$, defined by

$$Q(u) = g - (\mathrm{id}+u)^*(g) + d\left[u \lrcorner g\right],$$

in the following way:

$$\begin{aligned} Q(u) = g - g(\mathrm{id}+u) &- \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,i)=I \\ 1\le i\le n}} g_I(\mathrm{id}+u)\, dx^J \wedge du^i \\ &- \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,K)=I \\ 2\le |K|\le k}} g_I(\mathrm{id}+u)\, dx^J \wedge du^K + d\left[u \lrcorner g\right] \end{aligned}$$

and thus

$$\begin{aligned} Q(u) = g - g(\mathrm{id}+u) &- \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,i)=I \\ 1\le i\le n}} g_I(\mathrm{id}+u)\, dx^J \wedge du^i \\ &- \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,K)=I \\ 2\le |K|\le k}} g_I(\mathrm{id}+u)\, dx^J \wedge du^K \\ &+ \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,i)=I \\ 1\le i\le n}} g_I\, dx^J \wedge du^i + \sum_{I\in\mathscr{T}_k} \langle \operatorname{grad} g_I; u\rangle\, dx^I. \end{aligned}$$

We then let

$$Q_1(u) = \sum_{I\in\mathscr{T}_k} \sum_{\substack{(J,i)=I \\ 1\le i\le n}} \left[g_I - g_I(\mathrm{id}+u)\right]\left[dx^J \wedge du^i\right],$$

$$Q_2(u) = \sum_{I \in \mathcal{T}_k} [g_I(\mathrm{id}+u) - g_I - \langle \operatorname{grad} g_I; u\rangle]\, dx^I,$$

$$Q_3(u) = \sum_{I \in \mathcal{T}_k} \sum_{\substack{(J,K)=I \\ 2 \le |K| \le k}} g_I\,(\mathrm{id}+u)\, dx^J \wedge du^K$$

so that

$$Q(u) = Q_1(u) - Q_2(u) - Q_3(u).$$

We therefore have, when $k = 2$, that

$$Q_1(u) = \sum_{1 \le i < j \le n} [g_{ij} - g_{ij}(\mathrm{id}+u)] \left[du^i \wedge dx^j + dx^i \wedge du^j\right],$$

$$Q_2(u) = \sum_{1 \le i < j \le n} [g_{ij}(\mathrm{id}+u) - g_{ij} - \langle \operatorname{grad} g_{ij}; u\rangle] dx^i \wedge dx^j,$$

$$Q_3(u) = \sum_{1 \le i < j \le n} g_{ij}(\mathrm{id}+u) du^i \wedge du^j.$$

Step 3. We now establish the first estimate for each of the Q_p, $p = 1,2,3$. So let $u, v \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ with

$$\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le c \quad \text{and} \quad (\mathrm{id}+tu)\left(\overline{\Omega}\right), (\mathrm{id}+tv)\left(\overline{\Omega}\right) \subset \overline{\Omega},\ \forall t \in [0,1].$$

In the sequel, C_i will denote constants that depend only on c and Ω. Since in all cases we will make the estimates component by component, we immediately drop the sum signs. Before starting, we recall (cf. Theorems 16.31 and 16.36) that there exists a constant $C_1 = C_1(c, \Omega)$ such that for every $f \in C^{1,\gamma}(\overline{\Omega})$ and every $w, \widetilde{w} \in C^1(\overline{\Omega}; \overline{\Omega})$ with $\|w\|_{C^1}, \|\widetilde{w}\|_{C^1} \le c$,

$$\|f \circ w\|_{C^{0,\gamma}} \le C_1 \|f\|_{C^{0,\gamma}},$$

$$\|f \circ w - f \circ \widetilde{w}\|_{C^0} \le C_1 \|f\|_{C^1} \|w - \widetilde{w}\|_{C^0},$$

$$\|f \circ w - f \circ \widetilde{w}\|_{C^{0,\gamma}} \le C_1 \|f\|_{C^{1,\gamma}} \|w - \widetilde{w}\|_{C^{0,\gamma}}.$$

Estimate for Q_1. We have

$$\begin{aligned}
&\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \\
&\quad = \|\,[g_I(\mathrm{id}) - g_I(\mathrm{id}+u)]\left[dx^J \wedge du^i\right] - [g_I(\mathrm{id}) - g_I(\mathrm{id}+v)]\left[dx^J \wedge dv^i\right]\|_{C^{0,\gamma}} \\
&\quad \le \|[g_I(\mathrm{id}+v) - g_I(\mathrm{id}+u)]\left[dx^J \wedge dv^i\right]\|_{C^{0,\gamma}} \\
&\qquad + \|[g_I(\mathrm{id}+u) - g_I(\mathrm{id})]\left[dx^J \wedge \left[dv^i - du^i\right]\right]\|_{C^{0,\gamma}}.
\end{aligned}$$

We therefore get

$$\begin{aligned}
\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \le\ & C_2 \|[g_I(\mathrm{id}+v) - g_I(\mathrm{id}+u)]\|_{C^0} \|v\|_{C^{1,\gamma}} \\
&+ C_2 \|[g_I(\mathrm{id}+v) - g_I(\mathrm{id}+u)]\|_{C^{0,\gamma}} \|v\|_{C^1} \\
&+ C_2 \|[g_I(\mathrm{id}+u) - g_I(\mathrm{id})]\|_{C^0} \|u - v\|_{C^{1,\gamma}} \\
&+ C_2 \|[g_I(\mathrm{id}+u) - g_I(\mathrm{id})]\|_{C^{0,\gamma}} \|u - v\|_{C^1}.
\end{aligned}$$

Hence (bearing in mind that $\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le c$), we get

$$\begin{aligned}
&\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \\
&\quad \le C_3 \|g\|_{C^1} \|v-u\|_{C^0} \|v\|_{C^{1,\gamma}} + C_3 \|g\|_{C^{1,\gamma}} \|v-u\|_{C^{0,\gamma}} \|v\|_{C^1} \\
&\qquad + C_3 \|g\|_{C^1} \|u\|_{C^0} \|u-v\|_{C^{1,\gamma}} + C_3 \|g\|_{C^{1,\gamma}} \|u\|_{C^{0,\gamma}} \|u-v\|_{C^1} .
\end{aligned}$$

We thus have

$$\|Q_1(u) - Q_1(v)\|_{C^{0,\gamma}} \le C \|g\|_{C^{1,\gamma}} (\|u\|_{C^{1,\gamma}} + \|v\|_{C^{1,\gamma}}) \|u-v\|_{C^{1,\gamma}} .$$

Estimate for Q_2. For Q_2 we proceed in the following way. We first observe that

$$\begin{aligned}
Q_2(u) &= \int_0^1 \frac{d}{dt} \left[(g_I(\mathrm{id}+tu) - t\langle \operatorname{grad} g_I(\mathrm{id}); u\rangle)\, dx^I \right] dt \\
&= \int_0^1 \left[\langle \operatorname{grad} g_I(\mathrm{id}+tu) - \operatorname{grad} g_I(\mathrm{id}); u\rangle\, dx^I \right] dt.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
&\|Q_2(u) - Q_2(v)\|_{C^{0,\gamma}} \\
&\quad \le \int_0^1 \|\langle \operatorname{grad} g_I(\mathrm{id}+tu) - \operatorname{grad} g_I(\mathrm{id}); u\rangle \\
&\qquad - \langle \operatorname{grad} g_I(\mathrm{id}+tv) - \operatorname{grad} g_I(\mathrm{id}); v\rangle\|_{C^{0,\gamma}}\, dt \\
&\quad \le \int_0^1 \{\|\langle \operatorname{grad} g_I(\mathrm{id}+tu) - \operatorname{grad} g_I(\mathrm{id}+tv); u\rangle\|_{C^{0,\gamma}} \\
&\qquad + \|\langle \operatorname{grad} g_I(\mathrm{id}+tv) - \operatorname{grad} g_I(\mathrm{id}); u-v\rangle\|_{C^{0,\gamma}}\}\, dt
\end{aligned}$$

and, hence,

$$\begin{aligned}
&\|Q_2(u) - Q_2(v)\|_{C^{0,\gamma}} \\
&\quad \le C_2 \int_0^1 \{\| \operatorname{grad} g_I(\mathrm{id}+tu) - \operatorname{grad} g_I(\mathrm{id}+tv)\|_{C^{0,\gamma}} \|u\|_{C^0} \\
&\qquad + \| \operatorname{grad} g_I(\mathrm{id}+tu) - \operatorname{grad} g_I(\mathrm{id}+tv)\|_{C^0} \|u\|_{C^{0,\gamma}} \\
&\qquad + \| \operatorname{grad} g_I(\mathrm{id}+tv) - \operatorname{grad} g_I(\mathrm{id})\|_{C^{0,\gamma}} \|u-v\|_{C^0} \\
&\qquad + \| \operatorname{grad} g_I(\mathrm{id}+tv) - \operatorname{grad} g_I(\mathrm{id})\|_{C^0} \|u-v\|_{C^{0,\gamma}}\}\, dt.
\end{aligned}$$

This leads to (recall that $\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le c$)

$$\begin{aligned}
&\|Q_2(u) - Q_2(v)\|_{C^{0,\gamma}} \\
&\quad \le C_3 \|g\|_{C^{2,\gamma}} \|u-v\|_{C^{0,\gamma}} \|u\|_{C^0} + C_3 \|g\|_{C^2} \|u-v\|_{C^0} \|u\|_{C^{0,\gamma}} \\
&\qquad + C_3 \|g\|_{C^{2,\gamma}} \|v\|_{C^{0,\gamma}} \|u-v\|_{C^0} + C_3 \|g\|_{C^2} \|v\|_{C^0} \|u-v\|_{C^{0,\gamma}}.
\end{aligned}$$

We therefore have the estimate

$$\|Q_2(u) - Q_2(v)\|_{C^{0,\gamma}} \le C \|g\|_{C^{2,\gamma}} (\|u\|_{C^{0,\gamma}} + \|v\|_{C^{0,\gamma}}) \|u-v\|_{C^{0,\gamma}}.$$

Estimate for Q_3. It remains to prove the estimate for Q_3. We get

$$\begin{aligned}&\|Q_3(u)-Q_3(v)\|_{C^{0,\gamma}}\\&\quad=\|g_I(\mathrm{id}+v)\,dx^J\wedge dv^K-g_I(\mathrm{id}+u)\,dx^J\wedge du^K\|_{C^{0,\gamma}}\\&\quad\leq\|g_I(\mathrm{id}+v)(dx^J\wedge\left(dv^K-du^K\right))\|_{C^{0,\gamma}}\\&\qquad+\|(g_I(\mathrm{id}+v)-g_I(\mathrm{id}+u))\,dx^J\wedge du^K\|_{C^{0,\gamma}},\end{aligned}$$

which leads to (recalling that $\|u\|_{C^{1,\gamma}},\|v\|_{C^{1,\gamma}}\leq c$ and $|K|\geq 2$, just as in (10.19))

$$\begin{aligned}\|Q_3(u)-Q_3(v)\|_{C^{0,\gamma}}\leq\ &C_3\|g\|_{C^{0,\gamma}}(\|u\|_{C^{1,\gamma}}+\|v\|_{C^{1,\gamma}})\|u-v\|_{C^{1,\gamma}}\\&+C_3\|g\|_{C^{1,\gamma}}\|u-v\|_{C^{0,\gamma}}\|u\|_{C^{1,\gamma}}\end{aligned}$$

and, thus,

$$\|Q_3(u)-Q_3(v)\|_{C^{0,\gamma}}\leq C\|g\|_{C^{1,\gamma}}(\|u\|_{C^{1,\gamma}}+\|v\|_{C^{1,\gamma}})\|u-v\|_{C^{1,\gamma}},$$

proving the estimate for Q_3.

Step 4. We next establish the second estimate for each of the Q_p, $p=1,2,3$. So let $u\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$ with

$$\|u\|_{C^{1,\gamma}}\leq c\quad\text{and}\quad(\mathrm{id}+tu)\left(\overline{\Omega}\right)\subset\overline{\Omega},\quad\forall\,t\in[0,1].$$

As before, C_i will denote constants that depend only on c, r and Ω. Since in all cases we will make the estimates component by component, we drop the sum signs. We recall (cf. Theorem 16.31) that there exists a constant $C_1=C_1\left(c,r,\Omega\right)$ such that for every $f\in C^{r,\alpha}(\overline{\Omega})$ and every $w\in C^{r,\alpha}(\overline{\Omega};\overline{\Omega})$ with $\|w\|_{C^1}\leq c$,

$$\|f\circ w\|_{C^{r,\alpha}}\leq C_1\|f\|_{C^{r,\alpha}}+C_1\|f\|_{C^1}\|w\|_{C^{r,\alpha}}.$$

We also claim that

$$\|g\circ(\mathrm{id}+u)-g\circ\mathrm{id}\|_{C^{r,\alpha}}\leq C_1\,\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^1}+C_1\,\|g\|_{C^1}\,\|u\|_{C^{r+1,\alpha}}$$

for every $u\in C^{r+1,\alpha}(\overline{\Omega};\mathbb{R}^n)$, with

$$\|u\|_{C^1}\leq c\quad\text{and}\quad(\mathrm{id}+u)\left(\overline{\Omega}\right)\subset\overline{\Omega}.$$

Indeed, from Theorem 16.36, we have

$$\begin{aligned}\|g\circ(\mathrm{id}+u)-g\circ\mathrm{id}\|_{C^{r,\alpha}}\leq\ &C_2\,\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^0}+C_2\,\|g\|_{C^2}\,[1+\|u\|_{C^{r,\alpha}}]\,\|u\|_{C^0}\\&+C_2\,\|g\|_{C^1}\,\|u\|_{C^{r,\alpha}},\end{aligned}$$

and from Corollary 16.27, we get

$$\|g\|_{C^2}\,\|u\|_{C^{r,\alpha}}\leq C_3\,[\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^1}+\|g\|_{C^1}\,\|u\|_{C^{r+1,\alpha}}]\,.\tag{14.4}$$

Combining the two estimates, we have our claim.

Estimate for Q_1. We have

$$\begin{aligned}\|Q_1(u)\|_{C^{r,\alpha}} &= \|\,[g_I(\mathrm{id})-g_I(\mathrm{id}+u)]\left[dx^J\wedge du^i\right]\|_{C^{r,\alpha}}\\ &\le C_2\|[g_I(\mathrm{id}+u)-g_I(\mathrm{id})]\|_{C^0}\|u\|_{C^{r+1,\alpha}}\\ &\quad + C_2\|[g_I(\mathrm{id}+u)-g_I(\mathrm{id})]\|_{C^{r,\alpha}}\|u\|_{C^1}\,.\end{aligned}$$

We therefore get (bearing in mind that $\|u\|_{C^{1,\gamma}}\le c$)

$$\begin{aligned}\|Q_1(u)\|_{C^{r,\alpha}} &\le C_3\|g\|_{C^1}\|u\|_{C^0}\|u\|_{C^{r+1,\alpha}}\\ &\quad + C_3\,\|g\|_{C^{r+1,\alpha}}\,\|u\|^2_{C^1} + C_3\,\|g\|_{C^1}\,\|u\|_{C^{r+1,\alpha}}\,\|u\|_{C^1}\end{aligned}$$

and, thus,

$$\|Q_1(u)\|_{C^{r,\alpha}} \le C\,\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^1} + C\,\|g\|_{C^1}\,\|u\|_{C^{r+1,\alpha}}\,\|u\|_{C^1}\,.$$

Estimate for Q_2. As before, we have that

$$\begin{aligned}Q_2(u) &= \int_0^1 \frac{d}{dt}\left[(g_I(\mathrm{id}+tu)-t\langle \operatorname{grad} g_I(\mathrm{id});u\rangle)\,dx^I\right]dt\\ &= \int_0^1 \left[\langle \operatorname{grad} g_I(\mathrm{id}+tu)-\operatorname{grad} g_I(\mathrm{id});u\rangle dx^I\right]dt.\end{aligned}$$

We therefore obtain

$$\begin{aligned}\|Q_2(u)\|_{C^{r,\alpha}} &\le C_2\int_0^1 \{\|\operatorname{grad} g_I(\mathrm{id}+tu)-\operatorname{grad} g_I(\mathrm{id})\|_{C^{r,\alpha}}\|u\|_{C^0}\\ &\quad + \|\operatorname{grad} g_I(\mathrm{id}+tu)-\operatorname{grad} g_I(\mathrm{id})\|_{C^0}\|u\|_{C^{r,\alpha}}\}dt\end{aligned}$$

and, hence,

$$\begin{aligned}\|Q_2(u)\|_{C^{r,\alpha}} &\le C_2\int_0^1 \{[\|\operatorname{grad} g_I(\mathrm{id}+tu)\|_{C^{r,\alpha}} + \|\operatorname{grad} g_I\|_{C^{r,\alpha}}]\,\|u\|_{C^0}\\ &\quad + \|\operatorname{grad} g_I(\mathrm{id}+tu)-\operatorname{grad} g_I(\mathrm{id})\|_{C^0}\|u\|_{C^{r,\alpha}}\}\,dt.\end{aligned}$$

This leads to (recall that $\|u\|_{C^{1,\gamma}}\le c$)

$$\|Q_2(u)\|_{C^{r,\alpha}} \le C_3\,[\|g\|_{C^{r+1,\alpha}} + \|g\|_{C^2}\|u\|_{C^{r,\alpha}}]\,\|u\|_{C^0} + C_3\|g\|_{C^2}\|u\|_{C^0}\|u\|_{C^{r,\alpha}}\,.$$

From (14.4) we get

$$\|Q_2(u)\|_{C^{r,\alpha}} \le C\,\|g\|_{C^{r+1,\alpha}}\,\|u\|_{C^1} + C_3\|g\|_{C^1}\|u\|_{C^{r+1,\alpha}}\|u\|_{C^1}\,.$$

Estimate for Q_3. We immediately have

$$\begin{aligned}\|Q_3(u)\|_{C^{r,\alpha}} &= \|g_I(\mathrm{id}+u)\,dx^J\wedge du^K\|_{C^{r,\alpha}}\\ &\le C_2\|g(\mathrm{id}+u)\|_{C^{r,\alpha}}\|\,du^K\|_{C^0} + C_2\|g\|_{C^0}\|\,du^K\|_{C^{r,\alpha}}\,.\end{aligned}$$

Since $|K| \geq 2$ and $\|u\|_{C^{1,\gamma}} \leq c$, we get

$$\|Q_3(u)\|_{C^{r,\alpha}} \leq C_3 \left[\|g\|_{C^{r,\alpha}} + \|g\|_{C^1} \|u\|_{C^{r,\alpha}}\right] \|u\|_{C^1}^{|K|} + C_3 \|g\|_{C^0} \|u\|_{C^1}^{|K|-1} \|u\|_{C^{r+1,\alpha}}$$

and, thus, since $\|u\|_{C^{1,\gamma}} \leq c$,

$$\|Q_3(u)\|_{C^{r,\alpha}} \leq C\|g\|_{C^{r,\alpha}} \|u\|_{C^1} + C\|g\|_{C^1} \|u\|_{C^{r+1,\alpha}} \|u\|_{C^1}\,.$$

The combination of the three estimates gives the proof of the lemma. □

14.4.4 The Fixed Point Method

The first proof of Theorem 14.5 relies on the following key theorem (obtained by Bandyopadhyay and Dacorogna [8] under more restrictive hypotheses; as stated, it is due to Dacorogna and Kneuss [32]).

Theorem 14.10. *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $r \geq 1$ be an integer and $0 < \gamma \leq \alpha < 1$. Let $g \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ and $f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ be such that*

$$df = dg = 0 \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega,$$

$$\int_\Omega \langle f;\psi\rangle\, dx = \int_\Omega \langle g;\psi\rangle\, dx \quad \text{for every } \psi \in \mathscr{H}_T\left(\Omega;\Lambda^2\right),$$

$$\operatorname{rank}[g] = n \text{ in } \overline{\Omega}.$$

Let $c > 0$ be such that

$$\|g\|_{C^0}\,, \left\|\frac{1}{[g]^{n/2}}\right\|_{C^0} \leq c$$

and define

$$\theta(g) = \frac{1}{\|g\|_{C^{1,\gamma}}^2} \min\left\{\|g\|_{C^{1,\gamma}}\,, \frac{1}{\|g\|_{C^{2,\gamma}}}\,, \frac{1}{\|g\|_{C^{r+1,\alpha}}}\right\}.$$

There exists $C = C(c,r,\alpha,\gamma,\Omega) > 0$ such that if

$$\|f-g\|_{C^{0,\gamma}} \leq C\theta(g) \quad \text{and} \quad \|f-g\|_{C^{0,\gamma}} \leq C\frac{\|f-g\|_{C^{r,\alpha}}}{\|g\|_{C^{1,\gamma}}\|g\|_{C^{r+1,\alpha}}}, \tag{14.5}$$

then there exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ verifying

$$\varphi^*(g) = f \text{ in } \Omega \quad \text{and} \quad \varphi = \operatorname{id} \text{ on } \partial\Omega. \tag{14.6}$$

Furthermore, there exists $\widetilde{C} = \widetilde{C}(c, r, \alpha, \gamma, \Omega) > 0$ *such that*

$$\|\varphi - \mathrm{id}\|_{C^{r+1,\alpha}} \leq \widetilde{C} \|g\|_{C^{r+1,\alpha}} \|f - g\|_{C^{r,\alpha}} .$$

Remark 14.11. (i) Note that since $g \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^2)$ and $\nu \wedge f = \nu \wedge g$ on $\partial\Omega$, then $\nu \wedge f \in C^{r+1,\alpha}(\partial\Omega; \Lambda^3)$.

(ii) With essentially the same proof, but replacing the last estimate of Lemma 14.8 by the corresponding one in Remark 14.9, we get the following result. In addition to the hypotheses of the theorem, let $0 < \gamma \leq \alpha < \beta < 1$, $g \in C^{r+1,\beta}\left(\overline{\Omega}; \Lambda^2\right)$, $f \in C^{r,\alpha}\left(\overline{\Omega}; \Lambda^2\right)$ and $c > 0$ be such that

$$\begin{gathered} df = dg = 0 \ \text{ in } \Omega, \quad \nu \wedge f = \nu \wedge g \ \text{ on } \partial\Omega, \\ \int_\Omega \langle f; \psi \rangle \, dx = \int_\Omega \langle g; \psi \rangle \, dx \quad \text{for every } \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right), \\ \operatorname{rank}[g] = n \ \text{ in } \overline{\Omega}, \\ \|g\|_{C^0}, \quad \left\| \frac{1}{[g]^{n/2}} \right\|_{C^0} \leq c. \end{gathered}$$

Define

$$\theta(g) = \frac{1}{\|g\|_{C^{1,\gamma}}} \min \left\{ \begin{array}{c} 1, \dfrac{1}{\|g\|_{C^{1,\gamma}} \|g\|_{C^{2,\gamma}}}, \dfrac{1}{\|g\|_{C^{r+1,\alpha}}^2}, \\ \left[\dfrac{1}{\|g\|_{C^{1,\gamma}} \|g\|_{C^{r+1,\beta}}} \right]^{\frac{1}{\beta - \alpha}} \end{array} \right\}.$$

There exist $C = C(c, r, \alpha, \beta, \gamma, \Omega) > 0$ and $\widetilde{C} = \widetilde{C}(c, r, \alpha, \beta, \gamma, \Omega) > 0$ such that if (compare with (14.5))

$$\|f - g\|_{C^{0,\gamma}} \leq C\theta(g),$$

then there exists $\varphi \in \mathrm{Diff}^{r+1,\alpha}\left(\overline{\Omega}; \overline{\Omega}\right)$ verifying

$$\varphi^*(g) = f \ \text{ in } \Omega \quad \text{and} \quad \varphi = \mathrm{id} \text{ on } \partial\Omega$$

and

$$\|\varphi - \mathrm{id}\|_{C^{r+1,\alpha}} \leq \widetilde{C} \|g\|_{C^{r+1,\alpha}} \|f - g\|_{C^{r,\alpha}} .$$

Proof. The theorem will follow from Theorem 18.1. We divide the proof into five steps; the first four to verify the hypotheses of the theorem and the last one to conclude.

Step 1. We define the spaces as follows:

$$\begin{gathered} X_1 = C^{1,\gamma}\left(\overline{\Omega}; \mathbb{R}^n\right) \quad \text{and} \quad Y_1 = C^{0,\gamma}\left(\overline{\Omega}; \Lambda^2\right), \\ X_2 = \left\{ a \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right) : a = 0 \text{ on } \partial\Omega \right\}, \\ Y_2 = \left\{ b \in C^{r,\alpha}\left(\overline{\Omega}; \Lambda^2\right) : \left[\begin{array}{c} db = 0 \text{ in } \Omega, \quad \nu \wedge b = 0 \text{ on } \partial\Omega, \\ \int_\Omega \langle b; \psi \rangle \, dx = 0, \ \forall \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right) \end{array} \right. \right\}. \end{gathered}$$

It is easily seen that they satisfy Hypothesis (H_{XY}) of Theorem 18.1 (see Proposition 16.23).

Step 2. Define $L : X_2 \to Y_2$ by

$$La = d[a \lrcorner g] = b.$$

We will show that there exist $L^{-1} : Y_2 \to X_2$, a linear right inverse of L and a constant $K_1 = K_1(c, r, \alpha, \gamma, \Omega)$ such that, defining

$$k_1 = K_1 \|g\|_{C^{1,\gamma}} \quad \text{and} \quad k_2 = K_1 \|g\|_{C^{r+1,\alpha}},$$

we get

$$\|L^{-1}b\|_{X_i} \le k_i \|b\|_{Y_i} \text{ for every } b \in Y_2 \text{ and } i = 1, 2.$$

Once this is shown, (H_L) of Theorem 18.1 will be satisfied.

Step 2.1. Indeed, we first solve, using Theorem 8.16, the equation

$$\begin{cases} dw = b & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and find $w \in C^{r+1,\alpha}\left(\overline{\Omega}; \Lambda^1\right)$ and $C_1 = C_1(r, \alpha, \gamma, \Omega) > 0$ such that

$$\|w\|_{C^{r+1,\alpha}} \le C_1 \|b\|_{C^{r,\alpha}} \quad \text{and} \quad \|w\|_{C^{1,\gamma}} \le C_1 \|b\|_{C^{0,\gamma}}.$$

Moreover, the correspondence $b \to w$ can be chosen to be linear.

Step 2.2. Since $\operatorname{rank}[g] = n$, we can find a unique $a \in C^{r+1,\alpha}\left(\overline{\Omega}; \mathbb{R}^n\right)$ so that

$$a \lrcorner g = w,$$

which is equivalent to

$$a = [\overline{g}]^{-1} w.$$

Define $L^{-1} : Y_2 \to X_2$ by $L^{-1}(b) = a$. First, note that L^{-1} is linear and that

$$LL^{-1} = \text{id} \quad \text{on } Y_2.$$

Moreover, using Theorem 16.28, Corollary 16.30 and Step 2.1, we can find constants $C_i = C_i(c, r, \alpha, \gamma, \Omega)$, $i = 2, 3, 4$, such that

$$\begin{aligned} \|a\|_{C^{r+1,\alpha}} &\le C_2 \left\|(\overline{g})^{-1}\right\|_{C^{r+1,\alpha}} \|w\|_{C^0} + C_2 \left\|(\overline{g})^{-1}\right\|_{C^0} \|w\|_{C^{r+1,\alpha}} \\ &\le C_3 \|g\|_{C^{r+1,\alpha}} \|b\|_{C^{0,\gamma}} + C_3 \|g\|_{C^0} \|b\|_{C^{r,\alpha}} \\ &\le C_4 \|g\|_{C^{r+1,\alpha}} \|b\|_{C^{r,\alpha}} \end{aligned}$$

and, similarly,

$$\|a\|_{C^{1,\gamma}} \le C_4 \|g\|_{C^{1,\gamma}} \|b\|_{C^{0,\gamma}}.$$

Thus, the claim of Step 2 is valid.

Step 3. We define

$$Q(u) = g - (\mathrm{id} + u)^* (g) + d [u \lrcorner g].$$

We will verify that Property (H_Q) of Theorem 18.1 holds with $\rho = 1/(2n)$. The fact that $Q(0) = 0$ is evident.

Step 3.1. According to Lemma 14.8, there exists a constant $K_2 = K_2(r, \Omega)$ such that the following estimates hold:

$$\|Q(u) - Q(v)\|_{C^{0,\gamma}} \le K_2 \|g\|_{C^{2,\gamma}} (\|u\|_{C^{1,\gamma}} + \|v\|_{C^{1,\gamma}}) \|u - v\|_{C^{1,\gamma}},$$

$$\|Q(u)\|_{C^{r,\alpha}} \le K_2 \|g\|_{C^{r+1,\alpha}} \|u\|_{C^1} + K_2 \|g\|_{C^1} \|u\|_{C^{r+1,\alpha}} \|u\|_{C^1}$$

for every $u, v \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$, with

$$\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} \le 1/(2n),$$

$$(\mathrm{id} + tu)\left(\overline{\Omega}\right), (\mathrm{id} + tv)\left(\overline{\Omega}\right) \subset \overline{\Omega},\ \forall t \in [0,1].$$

We finally let

$$c_1(t_1, t_2) = K_2 \|g\|_{C^{2,\gamma}} (t_1 + t_2),$$

$$c_2(t_1, t_2) = K_2 \|g\|_{C^{r+1,\alpha}} t_1 + K_2 \|g\|_{C^1} t_1 t_2.$$

Note that if

$$F(t, x) = x + tu(x) \quad \text{and} \quad \|u\|_{C^1} \le 1/(2n),$$

then for every $t \in [0,1]$,

$$\det \nabla_x F(t, x) = \det(I + t \nabla u(x)) > 0, \quad x \in \overline{\Omega}.$$

Therefore, if $u = 0$ on $\partial\Omega$, then, appealing to Theorem 19.12, we get that

$$F(t, x) \in \overline{\Omega} \quad \text{for every } (t, x) \in [0,1] \times \overline{\Omega}.$$

Thus, (18.1) and (18.2) have been verified.

Step 3.2. Let us check that

$$Q : \{u \in X_2 : \|u\|_{X_1} \le 1/(2n)\} \to Y_2$$

is well defined. We have to prove that

$$dQ(u) = 0 \text{ in } \Omega, \quad \nu \wedge Q(u) = 0 \text{ on } \partial\Omega,$$

$$\int_\Omega \langle Q(u); \psi \rangle \, dx = 0,\ \forall \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right).$$

(i) The first condition follows immediately since $dg = 0$ and

$$dQ(u) = dg - (\mathrm{id} + u)^* (dg) + dd [u \lrcorner g].$$

(ii) The second one is true since $u = 0$ on $\partial\Omega$. Indeed, clearly (using the notations Q_i used in the proof of Lemma 14.8),

$$Q_1(u) = Q_2(u) = 0 \text{ on } \partial\Omega.$$

Since $u = 0$ on $\partial\Omega$, each of $\operatorname{grad} u^i$ and $\operatorname{grad} u^j$ is parallel to the normal ν. Thus, $du^i \wedge du^j = 0$ on $\partial\Omega$ for every $i < j$, which implies that

$$Q_3(u) = 0 \text{ on } \partial\Omega.$$

Thus, we have, in fact, proved that $Q(u) = 0$ on $\partial\Omega$.

(iii) Choosing $F(t,x) = x + tu(x)$ in Remark 17.4, we find that there exists Φ such that

$$\begin{cases} d\Phi = g - (\mathrm{id}+u)^*(g) & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\Psi = \Phi + u \lrcorner g$ satisfies

$$\begin{cases} d\Psi = Q(u) & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega, \end{cases}$$

we have the claim, namely

$$\int_\Omega \langle Q(u); \psi \rangle \, dx = 0, \ \forall \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right).$$

Step 4. With the definition of L and Q in hand, we now rewrite (14.6) as follows. Setting $\varphi = \mathrm{id}+u$, the equation $\varphi^*(g) = f$ becomes

$$\begin{aligned} Lu = d[u \lrcorner g] &= f - (\mathrm{id}+u)^*(g) + d[u \lrcorner g] \\ &= f - g + [g - (\mathrm{id}+u)^*(g) + d[u \lrcorner g]] \\ &= f - g + Q(u). \end{aligned}$$

In order to apply Theorem 18.1, it remains to see how the hypotheses

$$\begin{gathered} 2k_1 \|f-g\|_{C^{0,\gamma}} \le 1/(2n), \\ 2k_1 c_1 (2k_1 \|f-g\|_{C^{0,\gamma}}, 2k_1 \|f-g\|_{C^{0,\gamma}}) \le 1, \\ c_2(2k_1 \|f-g\|_{C^{0,\gamma}}, 2k_2 \|f-g\|_{C^{r,\alpha}}) \le \|f-g\|_{C^{r,\alpha}} \end{gathered} \tag{14.7}$$

translate in our context.

(i) The first one leads to

$$\|f-g\|_{C^{0,\gamma}} \le \frac{1}{4nK_1 \|g\|_{C^{1,\gamma}}}.$$

(ii) The second one gives

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{8K_1^2K_2\|g\|_{C^{1,\gamma}}^2\|g\|_{C^{2,\gamma}}}.$$

(iii) The third condition reads as

$$\begin{aligned}&K_2\|g\|_{C^{r+1,\alpha}}(2K_1\|g\|_{C^{1,\gamma}}\|f-g\|_{C^{0,\gamma}})\\&+K_2\|g\|_{C^1}(2K_1\|g\|_{C^{1,\gamma}}\|f-g\|_{C^{0,\gamma}})(2K_1\|g\|_{C^{r+1,\alpha}}\|f-g\|_{C^{r,\alpha}})\\&\quad\leq\|f-g\|_{C^{r,\alpha}}.\end{aligned}$$

Note that the third condition is verified if

$$2K_1K_2\|g\|_{C^{r+1,\alpha}}\|g\|_{C^{1,\gamma}}\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{2}\|f-g\|_{C^{r,\alpha}}$$

and

$$4K_1^2K_2\|g\|_{C^1}\|g\|_{C^{1,\gamma}}\|g\|_{C^{r+1,\alpha}}\|f-g\|_{C^{0,\gamma}}\|f-g\|_{C^{r,\alpha}} \leq \frac{1}{2}\|f-g\|_{C^{r,\alpha}}.$$

The first one leads to

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{\|f-g\|_{C^{r,\alpha}}}{4K_1K_2\|g\|_{C^{r+1,\alpha}}\|g\|_{C^{1,\gamma}}}$$

and the second one is verified if

$$\|f-g\|_{C^{0,\gamma}} \leq \frac{1}{8K_1^2K_2\|g\|_{C^{r+1,\alpha}}\|g\|_{C^{1,\gamma}}^2}.$$

Combining the four conditions, we have just obtained, letting

$$\theta(g) = \frac{1}{\|g\|_{C^{1,\gamma}}^2}\min\left\{\|g\|_{C^{1,\gamma}}, \frac{1}{\|g\|_{C^{2,\gamma}}}, \frac{1}{\|g\|_{C^{r+1,\alpha}}}\right\},$$

that there exists $C = C(c, r, \alpha, \gamma, \Omega) > 0$ such that the inequalities (14.7) are satisfied if

$$\|f-g\|_{C^{0,\gamma}} \leq C\theta(g) \quad \text{and} \quad \|f-g\|_{C^{0,\gamma}} \leq C\frac{\|f-g\|_{C^{r,\alpha}}}{\|g\|_{C^{1,\gamma}}\|g\|_{C^{r+1,\alpha}}}.$$

Step 5. The hypotheses of Theorem 18.1 having been verified, we conclude that there exists $u \in C^{r+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$, with $\|u\|_{C^{1,\gamma}} \leq 1/(2n)$, satisfying $u = 0$ on $\partial\Omega$ and

$$Lu = d[u \lrcorner g] = f - g + Q(u) = f - (\mathrm{id} + u)^*(g) + d[u \lrcorner g].$$

Letting $\varphi = \mathrm{id} + u$, we therefore have found that

$$\varphi^*(g) = f \text{ in } \Omega.$$

Since $u=0$ on $\partial\Omega$, we have that $\varphi=\text{id}$ on $\partial\Omega$. Since $\|u\|_{C^1}\le 1/(2n)$, we deduce that

$$\det\nabla\varphi>0 \quad \text{in } \overline{\Omega},$$

and therefore, according to Theorem 19.12, we find that $\varphi\in \text{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$. Moreover, by construction (cf. (18.5)),

$$\|u\|_{C^{r+1,\alpha}}\le 2k_2\|f-g\|_{C^{r,\alpha}},$$

which implies the desired estimate, namely

$$\|\varphi-\text{id}\|_{C^{r+1,\alpha}}\le \widetilde{C}\|g\|_{C^{r+1,\alpha}}\|f-g\|_{C^{r,\alpha}}.$$

The proof is thus complete. □

14.4.5 A First Proof of the Main Theorem

We first prove Theorem 14.5 for special f and general g with extra regularity and under a smallness assumption.

Proposition 14.12. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open smooth set, $r\ge 1$, $0<\alpha<\beta<1$ and $g\in C^{r,\beta}\left(\overline{\Omega};\Lambda^2\right)$ with*

$$\nu\wedge g\in C^{r+1,\alpha}\left(\partial\Omega;\Lambda^3\right),\quad dg=0 \quad \text{and} \quad \text{rank}[g]=n \quad \text{in } \overline{\Omega}.$$

Then for every ε small, there exist $g_\varepsilon\in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ and $\varphi_\varepsilon\in \text{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ such that

$$\begin{cases}\varphi_\varepsilon^*(g_\varepsilon)=g & \text{in } \Omega,\\ \varphi_\varepsilon=\text{id} & \text{on } \partial\Omega,\end{cases}$$

$$dg_\varepsilon=0,\quad \nu\wedge g_\varepsilon=\nu\wedge g \ \text{ on } \partial\Omega,$$

$$\int_\Omega\langle g_\varepsilon;\psi\rangle=\int_\Omega\langle g;\psi\rangle,\ \forall\,\psi\in\mathcal{H}_T\left(\Omega;\Lambda^2\right),$$

$$\lim_{\varepsilon\to 0}\|g_\varepsilon-g\|_{C^{r,\alpha}(\overline{\Omega})}=0.$$

Moreover, there exists $C=C(r,\alpha,\beta,\Omega,\|g\|_{C^{1,\alpha}},\|1/g^{n/2}\|_{C^0})$, such that for every ε small,

$$\|\varphi_\varepsilon-\text{id}\|_{C^{r+1,\alpha}(\overline{\Omega})}\le C\,\frac{\varepsilon^{\beta-\alpha}}{\beta-\alpha}\|g\|_{C^{r,\beta}(\overline{\Omega})}+C\,\varepsilon\|\nu\wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}. \tag{14.8}$$

Proof. For the sake of alleviating the notations we will write in the present proof, for example, $\|g\|_{C^{r,\beta}}$ instead of $\|g\|_{C^{r,\beta}(\overline{\Omega})}$. However, when we will be considering norms on the boundary of Ω, we will keep the notation $\|g\|_{C^{r,\beta}(\partial\Omega)}$.

Step 1 (definition of g_ε). Apply Theorem 16.49 and Remark 16.50(v) and get, for every $\varepsilon \in (0,1]$, that there exist $g_\varepsilon \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^2)$ and a constant $C_1 = C_1(r,\alpha,\beta,\Omega)$ such that

$$dg_\varepsilon = 0 \quad \text{in } \Omega, \quad \nu \wedge g_\varepsilon = \nu \wedge g \quad \text{on } \partial\Omega, \tag{14.9}$$

$$\int_\Omega \langle g_\varepsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle \quad \text{for every } \psi \in \mathscr{H}_T(\Omega;\Lambda^2), \tag{14.10}$$

$$\|g_\varepsilon\|_{C^{r+1,\alpha}} \le \frac{C_1}{\varepsilon^{1+\alpha-\beta}}\|g\|_{C^{r,\beta}} + C_1\|\nu\wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}, \tag{14.11}$$

$$\|g_\varepsilon - g\|_{C^{r,\alpha}} \le C_1\varepsilon^{\beta-\alpha}\|g\|_{C^{r,\beta}}, \tag{14.12}$$

$$\left\|\frac{d}{d\varepsilon}g_\varepsilon\right\|_{C^{0,\alpha}} \le C_1\|g\|_{C^{1,\alpha}} \quad \text{and} \quad \left\|\frac{d}{d\varepsilon}g_\varepsilon\right\|_{C^{r,\alpha}} \le C_1\varepsilon^{\beta-\alpha-1}\|g\|_{C^{r,\beta}}. \tag{14.13}$$

Moreover, defining $G : (0,1]\times\overline{\Omega} \to \Lambda^2$ by $G(\varepsilon,x) = g_\varepsilon(x)$, we have

$$G \in C^{r+1,\alpha}\big((0;1]\times\overline{\Omega};\Lambda^2\big) \quad \text{and} \quad \frac{\partial}{\partial\varepsilon}G \in C^\infty\big((0;1]\times\overline{\Omega};\Lambda^2\big). \tag{14.14}$$

Since $\operatorname{rank}[g] = n$ in $\overline{\Omega}$ (which is equivalent to $g^{n/2}(x) \ne 0$ for every $x \in \overline{\Omega}$) and since (14.12) holds, there exists $\overline{\varepsilon} \le 1$ such that for every $\varepsilon \in (0,\overline{\varepsilon}]$,

$$\begin{aligned} &\|g_\varepsilon\|_{C^0} \le 2\|g\|_{C^0}, \quad \|g_\varepsilon\|_{C^1} \le 2\|g\|_{C^1}, \\ &\|1/(g_\varepsilon)^{n/2}\|_{C^0} \le 2\|1/g^{n/2}\|_{C^0}. \end{aligned} \tag{14.15}$$

Hence, combining (14.15) and Notation (v) in Section 14.1, we deduce that for every $\varepsilon \in (0,\overline{\varepsilon}]$,

$$\|(\overline{g}_\varepsilon)^{-1}\|_{C^1} \le C_2\|g_\varepsilon\|_{C^1} \quad \text{and} \quad \|(\overline{g}_\varepsilon)^{-1}\|_{C^{r+1,\alpha}} \le C_2\|g_\varepsilon\|_{C^{r+1,\alpha}}, \tag{14.16}$$

where $C_2 = C_2(r,\Omega,\|g\|_{C^0},\|1/g^{n/2}\|_{C^0})$.

Step 2. In this step we will show that for every $\varepsilon \in (0,\overline{\varepsilon}]$, there exist $u_\varepsilon \in C^{r+1,\alpha}(\overline{\Omega};\Lambda^1)$ and a constant $C_3 = C_3(r,\alpha,\beta,\Omega,\|g\|_{C^{1,\alpha}},\|1/g^{n/2}\|_{C^0})$ such that $u_\varepsilon = 0$ on $\partial\Omega$ and

$$d(u_\varepsilon \lrcorner\, g_\varepsilon) = -\frac{d}{d\varepsilon}g_\varepsilon \quad \text{in } \Omega, \tag{14.17}$$

$$\|u_\varepsilon\|_{C^{r+1,\alpha}} \le \frac{C_3}{\varepsilon^{1+\alpha-\beta}}\|g\|_{C^{r,\beta}} + C_3\|\nu\wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}, \tag{14.18}$$

$$\|u_\varepsilon\|_{C^1} \le C_3. \tag{14.19}$$

Moreover, defining $u : (0,\overline{\varepsilon}]\times\overline{\Omega} \to \Lambda^1$ by $u(\varepsilon,x) = u_\varepsilon(x)$, we will show that $u \in C^{r+1,\alpha}\big((0,\overline{\varepsilon}]\times\overline{\Omega};\Lambda^1\big)$.

Step 2.1. Since (14.9), (14.10) and (14.14) hold, using Theorem 8.16 we can find, for every $\varepsilon \in (0,\overline{\varepsilon}]$, $w_\varepsilon \in C^\infty(\overline{\Omega};\Lambda^1)$ and a constant $C_4 = C_4(r,\alpha,\Omega)$ such that

$$dw_\varepsilon = -\frac{d}{d\varepsilon}g_\varepsilon \quad \text{in } \Omega, \quad w_\varepsilon = 0 \quad \text{on } \partial\Omega$$

and, for every integer $q \le r$,

$$\|w_\varepsilon\|_{C^{q+1,\alpha}} \le C_4 \left\| \frac{d}{d\varepsilon} g_\varepsilon \right\|_{C^{q,\alpha}} . \tag{14.20}$$

Moreover, defining $w : (0, \overline{\varepsilon}] \times \overline{\Omega} \to \Lambda^1$ by $w(\varepsilon, x) = w_\varepsilon(x)$, we have, using (14.14), $w \in C^\infty\big((0, \overline{\varepsilon}] \times \overline{\Omega}; \Lambda^1\big)$.

Step 2.2. Since by (14.15), we have, for every $\varepsilon \in (0, \overline{\varepsilon}]$, $\operatorname{rank}[g_\varepsilon] = n$ in $\overline{\Omega}$, there exists a unique $u_\varepsilon : \overline{\Omega} \to \Lambda^1$ verifying

$$u_\varepsilon \lrcorner g_\varepsilon = w_\varepsilon .$$

Note that $u_\varepsilon \in C^{r+1,\alpha}\big(\overline{\Omega}; \Lambda^1\big)$ and that $u_\varepsilon = 0$ on $\partial\Omega$. Moreover, defining $u : (0, \overline{\varepsilon}] \times \overline{\Omega} \to \Lambda^1$ by $u(\varepsilon, x) = u_\varepsilon(x)$, we have $u \in C^{r+1,\alpha}\big((0, \overline{\varepsilon}] \times \overline{\Omega}; \Lambda^1\big)$.

Step 2.3. To show Step 2, it only remains to prove (14.18) and (14.19). Using Theorem 16.28, (14.15), (14.16) and (14.20), it follows that

$$\begin{aligned}
\|u_\varepsilon\|_{C^{r+1,\alpha}} &= \|(\overline{g}_\varepsilon)^{-1} w_\varepsilon\|_{C^{r+1,\alpha}} \\
&\le C_5 \|(\overline{g}_\varepsilon)^{-1}\|_{C^{r+1,\alpha}} \|w_\varepsilon\|_{C^0} + C_5 \|(\overline{g}_\varepsilon)^{-1}\|_{C^0} \|w_\varepsilon\|_{C^{r+1,\alpha}} \\
&\le C_6 \|g_\varepsilon\|_{C^{r+1,\alpha}} \|w_\varepsilon\|_{C^{1,\alpha}} + C_6 \|w_\varepsilon\|_{C^{r+1,\alpha}} \\
&\le C_7 \|g_\varepsilon\|_{C^{r+1,\alpha}} \left\| \frac{d}{d\varepsilon} g_\varepsilon \right\|_{C^{0,\alpha}} + C_7 \left\| \frac{d}{d\varepsilon} g_\varepsilon \right\|_{C^{r,\alpha}}
\end{aligned}$$

and thus, invoking (14.11) and (14.13),

$$\begin{aligned}
&\|u_\varepsilon\|_{C^{r+1,\alpha}} \\
&\quad \le C_8 \left(\frac{1}{\varepsilon^{1+\alpha-\beta}} \|g\|_{C^{r,\beta}} + \|\nu \wedge g\|_{C^{r+1,\alpha}(\partial\Omega)} \right) \|g\|_{C^{1,\alpha}} + \frac{C_8}{\varepsilon^{1+\alpha-\beta}} \|g\|_{C^{r,\beta}} \\
&\quad \le \frac{C_9}{\varepsilon^{1+\alpha-\beta}} \|g\|_{C^{r,\beta}} + C_9 \|\nu \wedge g\|_{C^{r+1,\alpha}(\partial\Omega)},
\end{aligned}$$

where $C_i = C_i(r, \alpha, \beta, \Omega, \|g\|_{C^{1,\alpha}}, \|1/g^{n/2}\|_{C^0})$. We similarly obtain

$$\begin{aligned}
\|u_\varepsilon\|_{C^1} &= \|(\overline{g}_\varepsilon)^{-1} w_\varepsilon\|_{C^1} \le C_{10} \|(\overline{g}_\varepsilon)^{-1}\|_{C^1} \|w_\varepsilon\|_{C^1} \\
&\le C_{11} \|g\|_{C^1} \left\| \frac{d}{d\varepsilon} g_\varepsilon \right\|_{C^{0,\alpha}} \le C_{12} \|g\|_{C^1} \|g\|_{C^{1,\alpha}} \le C_{13},
\end{aligned}$$

where $C_i = C_i(r, \alpha, \beta, \Omega, \|g\|_{C^{1,\alpha}}, \|1/g^{n/2}\|_{C^0})$. This shows the assertion.

Step 3. We can now conclude the proof.

Step 3.1. Since $u \in C^{r+1,\alpha}((0, \overline{\varepsilon}] \times \overline{\Omega} : \mathbb{R}^n)$, $u_\varepsilon = 0$ on $\partial\Omega$ and by (14.18),

$$\int_0^{\overline{\varepsilon}} \|u_\varepsilon\|_{C^{r+1,\alpha}} d\varepsilon < \infty,$$

we deduce, using Theorem 12.1, that the solution $\varphi : [0,\overline{\varepsilon}] \times \overline{\Omega} \to \overline{\Omega}$, $\varphi(\varepsilon,x) = \varphi_\varepsilon(x)$, of

$$\begin{cases} \dfrac{d}{d\varepsilon}\varphi_\varepsilon = u_\varepsilon \circ \varphi_\varepsilon, \quad 0 < \varepsilon \leq \overline{\varepsilon}, \\ \varphi_0 = \mathrm{id} \end{cases}$$

verifies

$$\varphi \in C^{r+1,\alpha}([0,\overline{\varepsilon}] \times \overline{\Omega};\overline{\Omega}) \tag{14.21}$$

and that for every $\varepsilon \in [0,\overline{\varepsilon}]$,

$$\varphi_\varepsilon \in \mathrm{Diff}^{r+1,\alpha}(\overline{\Omega};\overline{\Omega}) \quad \text{and} \quad \varphi_\varepsilon = \mathrm{id} \text{ on } \partial\Omega.$$

Finally, inserting (14.18) and (14.19) in (12.3), we immediately deduce (14.8).

Step 3.2. Since (14.17) holds, we deduce, using Theorem 12.7, that for every $0 < \varepsilon_1 \leq \varepsilon_2 \leq \overline{\varepsilon}$,

$$\varphi_{\varepsilon_2}^*(g_{\varepsilon_2}) = \varphi_{\varepsilon_1}^*(g_{\varepsilon_1}) \quad \text{in } \Omega.$$

Since, using (14.12) and (14.21),

$$\lim_{\varepsilon \to 0} \|g_\varepsilon - g\|_{C^0} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \|\varphi_\varepsilon - \varphi_0\|_{C^1} = 0,$$

we immediately infer that for every $\varepsilon \in (0,\overline{\varepsilon}]$,

$$\varphi_\varepsilon^*(g_\varepsilon) = \varphi_0^*(g) = g\,.$$

The proof is therefore complete. □

We can now go back to the first proof of Theorem 14.5 using an iteration scheme involving appropriate regularization.

Proof. We split the proof into three steps.

Step 1 (approximation of g and f). Choose $\gamma \in (0,\alpha)$ and $\delta > 0$ with $2\delta \leq \alpha - \gamma$ and $\alpha + 2\delta < 1$. We next regularize g and f with the help of Theorem 16.49 (and Remark 16.50(v)) and construct for every $\varepsilon \in (0,1]$, $g_\varepsilon, f_\varepsilon \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ such that

$$dg_\varepsilon = df_\varepsilon = 0, \quad \nu \wedge g_\varepsilon = \nu \wedge g = \nu \wedge f = \nu \wedge f_\varepsilon \text{ on } \partial\Omega,$$

$$\int_\Omega \langle g_\varepsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle = \int_\Omega \langle f;\psi\rangle = \int_\Omega \langle f_\varepsilon;\psi\rangle\,,\ \forall\, \psi \in \mathscr{H}_T\left(\Omega;\Lambda^2\right),$$

$$\|g_\varepsilon - g\|_{C^{0,\gamma}} \leq C\varepsilon^{r+\alpha-\gamma}\|g\|_{C^{r,\alpha}}\,,$$

$$\|g_\varepsilon - g\|_{C^{1,\gamma}} \leq C\varepsilon^{r-1+\alpha-\gamma}\|g\|_{C^{r,\alpha}}\,,$$

$$\|g_\varepsilon\|_{C^{r+1,\alpha}} \leq \frac{C}{\varepsilon}\|g\|_{C^{r,\alpha}} + C\|\nu \wedge g\|_{C^{r+1,\alpha}(\partial\Omega)}\,,$$

$$\|g_\varepsilon\|_{C^{r,\alpha+2\delta}} \leq \frac{C}{\varepsilon^{2\delta}}\|g\|_{C^{r,\alpha}} + C\|\nu \wedge g\|_{C^{r,\alpha+2\delta}(\partial\Omega)},$$

$$\|g_\varepsilon\|_{C^{2,\gamma}} \leq \frac{C}{\varepsilon}\|g\|_{C^{1,\gamma}} + C\|\nu \wedge g\|_{C^{2,\gamma}(\partial\Omega)},$$

where $C = C(r,\alpha,\gamma,\delta,\Omega) > 0$ and similarly for f and f_ε. Note that using the first inequality above, there exists $\overline{\varepsilon}$ such that for every $\varepsilon_1, \varepsilon_2 \leq \overline{\varepsilon}$, we have that

$$\operatorname{rank}[tg_{\varepsilon_1} + (1-t)f_{\varepsilon_2}] = n \quad \text{in } \overline{\Omega} \text{ and for every } t \in [0,1].$$

Step 2. In this step we show that there exist $\varepsilon_1, \varepsilon_2 \leq \overline{\varepsilon}$ and $\varphi_1, \varphi_3 \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ such that

$$\begin{cases} \varphi_1^*(g_{\varepsilon_1}) = g & \text{in } \Omega, \\ \varphi_1 = \text{id} & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \varphi_3^*(f_{\varepsilon_2}) = f & \text{in } \Omega, \\ \varphi_3 = \text{id} & \text{on } \partial\Omega. \end{cases}$$

For this we will use a combination of Theorem 14.10 and Proposition 14.12. We only show the assertion for g, the one with f being proved exactly in the same way.

Step 2.1. We start with some preliminary estimates. Using the second inequality in Step 1, we deduce that for every ε small enough, recalling that $r \geq 1$ and $\gamma < \alpha$,

$$\frac{1}{2}\|g_\varepsilon\|_{C^{1,\gamma}} \leq \|g\|_{C^{1,\gamma}} \leq 2\|g_\varepsilon\|_{C^{1,\gamma}} \quad \text{and} \quad \left\|\frac{1}{[g_\varepsilon]^{n/2}}\right\|_{C^0} \leq 2\left\|\frac{1}{[g]^{n/2}}\right\|_{C^0}.$$

In what follows, ε will always be assumed small enough. Combining the left-hand side of the previous inequality with the third and fifth inequalities in Step 1, we deduce that there exists $D_1 > 0$, a constant independent of ε, such that, defining

$$\theta(g_\varepsilon) = \frac{1}{\|g_\varepsilon\|_{C^{1,\gamma}}^2}\min\left\{\|g_\varepsilon\|_{C^{1,\gamma}}, \frac{1}{\|g_\varepsilon\|_{C^{2,\gamma}}}, \frac{1}{\|g_\varepsilon\|_{C^{r+1,\alpha}}}\right\},$$

we have

$$\theta(g_\varepsilon) \geq D_1\varepsilon.$$

Hence, since $\|g_\varepsilon - g\|_{C^{0,\gamma}} \leq C\varepsilon^{r+\alpha-\gamma}$, $r \geq 1$ and $\gamma < \alpha$, we immediately deduce

$$\lim_{\varepsilon\to 0}\frac{\|g_\varepsilon - g\|_{C^{0,\gamma}}}{\theta(g_\varepsilon)} = 0. \tag{14.22}$$

Note also that there exists $D_2 > 0$, a constant independent of ε, such that

$$\|g_\varepsilon\|_{C^0}, \left\|\frac{1}{[g_\varepsilon]^{n/2}}\right\|_{C^0} \leq D_2.$$

Step 2.2. Let $C = C(D_2, r, \alpha, \gamma, \Omega)$ be the constant given in (14.5) of Theorem 14.10. Due to (14.22), the first inequality of (14.5) is satisfied for every $\varepsilon \leq \widetilde{\varepsilon}$ and for some $\widetilde{\varepsilon} \leq \overline{\varepsilon}$. We show the assertion by considering two cases. In the first one, we use Theorem 14.10 to obtain the assertion and in the second one, we use Proposition 14.12.

(i) Suppose that for some $\varepsilon \leq \widetilde{\varepsilon}$, the second inequality of (14.5) is also satisfied, namely

$$\|g_\varepsilon - g\|_{C^{0,\gamma}} \leq C(D_2, r, \alpha, \gamma, \Omega) \frac{\|g_\varepsilon - g\|_{C^{r,\alpha}}}{\|g_\varepsilon\|_{C^{1,\gamma}} \|g_\varepsilon\|_{C^{r+1,\alpha}}}.$$

Hence, we have the claim of Step 2 using Theorem 14.10.

(ii) Suppose that the first case does not hold true. Hence, for all $\varepsilon \leq \widetilde{\varepsilon}$

$$\|g_\varepsilon\|_{C^{1,\gamma}} \|g_\varepsilon\|_{C^{r+1,\alpha}} \|g_\varepsilon - g\|_{C^{0,\gamma}} > C(D_2, r, \alpha, \gamma, \Omega) \|g_\varepsilon - g\|_{C^{r,\alpha}}.$$

Using the first and third inequality of Step 1, the fact that $\|g_\varepsilon\|_{C^{1,\gamma}} \leq 2\|g\|_{C^{1,\gamma}}$, we obtain, recalling that $r \geq 1$ and that $2\delta \leq \alpha - \gamma$,

$$\|g_\varepsilon - g\|_{C^{r,\alpha}} \leq D_3 \varepsilon^{2\delta} \quad \text{for every } 0 < \varepsilon \leq \widetilde{\varepsilon},$$

where D_3 is independent of ε. Combining the above equation with the fact that, by the fourth inequality in Step 1 (where $D_4 > 0$ is independent of ε),

$$\|g_\varepsilon\|_{C^{r,\alpha+2\delta}} \leq \frac{D_4}{\varepsilon^{2\delta}},$$

we immediately deduce from Proposition 16.45 that $g \in C^{r,\alpha+\delta}\left(\overline{\Omega}; \Lambda^2\right)$. The assertion then follows directly from Proposition 14.12 once noticed, using Remark 16.50(v), that the g_ε constructed in Proposition 14.12 are the same as the ones defined in Step 1.

Step 3. Since

$$\begin{cases} dg_{\varepsilon_1} = df_{\varepsilon_2} = 0 & \text{in } \Omega, \\ \nu \wedge g_{\varepsilon_1} = \nu \wedge f_{\varepsilon_2} & \text{on } \partial\Omega, \\ \int_\Omega \langle g_{\varepsilon_1}; \psi \rangle = \int_\Omega \langle f_{\varepsilon_2}; \psi \rangle & \text{for every } \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right), \\ \operatorname{rank}\left[t g_{\varepsilon_1} + (1-t) f_{\varepsilon_2}\right] = n & \text{in } \overline{\Omega} \text{ and for every } t \in [0,1], \end{cases}$$

we can apply Theorem 14.7 to find $\varphi_2 \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega}; \overline{\Omega}\right)$ such that

$$\begin{cases} \varphi_2^*(g_{\varepsilon_1}) = f_{\varepsilon_2} & \text{in } \Omega, \\ \varphi_2 = \operatorname{id} & \text{on } \partial\Omega. \end{cases}$$

The claimed solution is then given by

$$\varphi = \varphi_1^{-1} \circ \varphi_2 \circ \varphi_3.$$

This achieves the proof of the theorem. □

14.4.6 A Second Proof of the Main Theorem

We first show Theorem 14.5 for special f and general g with extra regularity only on the boundary and under a smallness assumption.

Proposition 14.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set, $r \geq 1$ and $0 < \delta < \alpha < \alpha + \delta < 1$. Let $g \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^2\right)$ with*

$$\nu \wedge g \in C^{r+1,\alpha+\delta}\left(\partial\Omega;\Lambda^3\right), \quad dg = 0 \quad \textit{and} \quad \operatorname{rank}[g] = n \quad \textit{in } \overline{\Omega}.$$

Then for every ε small, there exist

$$g_\varepsilon \in C^{r+1,\alpha+\delta}\left(\overline{\Omega};\Lambda^2\right) \quad \textit{and} \quad \varphi_\varepsilon \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$$

such that

$$\begin{cases} \varphi_\varepsilon^*(g_\varepsilon) = g & \textit{in } \Omega, \\ \varphi_\varepsilon = \mathrm{id} & \textit{on } \partial\Omega, \end{cases}$$

$$dg_\varepsilon = 0, \quad \nu \wedge g_\varepsilon = \nu \wedge g \;\; \textit{on } \partial\Omega,$$

$$\int_\Omega \langle g_\varepsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle \textit{ for every } \psi \in \mathscr{H}_T\left(\Omega;\Lambda^2\right),$$

$$\lim_{\varepsilon\to 0} \|g_\varepsilon - g\|_{C^{r,\alpha-\delta}(\overline{\Omega})} = 0.$$

Proof. We adopt the same simplification in the notations of the norms as in the proof of Proposition 14.12.

Step 1 (definition of g_ε). Apply Theorem 16.49 and Remark 16.50(v)–16.50(vi). Therefore, for every $\varepsilon \in (0,1]$, there exist $g_\varepsilon \in C^{r+1,\alpha+\delta}\left(\overline{\Omega};\Lambda^2\right)$ and a constant $C_1 = C_1(r,\alpha,\delta,\Omega)$ such that for every $\gamma \in [\alpha-\delta,\alpha+\delta]$,

$$dg_\varepsilon = 0 \quad \text{in } \Omega, \quad \nu \wedge g_\varepsilon = \nu \wedge g \quad \text{on } \partial\Omega, \tag{14.23}$$

$$\int_\Omega \langle g_\varepsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle \quad \text{for every } \psi \in \mathscr{H}_T(\Omega;\Lambda^2), \tag{14.24}$$

$$\|g_\varepsilon\|_{C^{r+1,\gamma}} \leq \frac{C_1}{\varepsilon^{1+\gamma-\alpha}}\|g\|_{C^{r,\alpha}} + C_1\|\nu\wedge g\|_{C^{r+1,\gamma}(\partial\Omega)}, \tag{14.25}$$

$$\|g_\varepsilon - g\|_{C^{r,\alpha-\delta}} \leq C_1\varepsilon^\delta \|g\|_{C^{r,\alpha}}, \tag{14.26}$$

$$\left\|\frac{d}{d\varepsilon}g_\varepsilon\right\|_{C^{0,\alpha}} \leq C_1\|g\|_{C^{1,\alpha}} \quad \text{and} \quad \left\|\frac{d}{d\varepsilon}g_\varepsilon\right\|_{C^{r,\gamma}} \leq \frac{C_1}{\varepsilon^{1+\gamma-\alpha}}\|g\|_{C^{r,\alpha}}. \tag{14.27}$$

Moreover, defining $G:(0,1]\times\overline{\Omega}\to\Lambda^2$ by $G(\varepsilon,x) = g_\varepsilon(x)$, we have

$$G \in C^{r+1,\alpha+\delta}\left((0;1]\times\overline{\Omega};\Lambda^2\right) \quad \text{and} \quad \frac{\partial}{\partial\varepsilon}G \in C^\infty\left((0;1]\times\overline{\Omega};\Lambda^2\right). \tag{14.28}$$

Since $\operatorname{rank}[g] = n$ in $\overline{\Omega}$ (which is equivalent to $g^{n/2}(x) \neq 0$ for every $x \in \overline{\Omega}$) and since (14.26) holds, there exists $\overline{\varepsilon} \leq 1$ such that for every $\varepsilon \in (0, \overline{\varepsilon}]$,

$$\|g_\varepsilon\|_{C^0} \leq 2\|g\|_{C^0} \quad \text{and} \quad \|1/(g_\varepsilon)^{n/2}\|_{C^0} \leq 2\|1/g^{n/2}\|_{C^0}\,. \tag{14.29}$$

Hence, combining (14.29) and Notation (v) in Section 14.1, we deduce that for every $\varepsilon \in (0, \overline{\varepsilon}]$ and every $\gamma \in [\alpha - \delta, \alpha + \delta]$,

$$\|(\overline{g}_\varepsilon)^{-1}\|_{C^{r+1,\gamma}} \leq C_2 \|g_\varepsilon\|_{C^{r+1,\gamma}}, \tag{14.30}$$

where $C_2 = C_2(r, \Omega, \|g\|_{C^0}, \|1/g^{n/2}\|_{C^0})$.

Step 2. In this step we will show that for every $\varepsilon \in (0, \overline{\varepsilon}]$, there exist $u_\varepsilon \in C^{r+1,\alpha+\delta}(\overline{\Omega}; \Lambda^1)$ and a constant $C_3 = C_3(r, \alpha, \delta, \Omega, \|g\|_{C^{1,\alpha}}, \|1/g^{n/2}\|_{C^0})$ such that $u_\varepsilon = 0$ on $\partial\Omega$ and

$$d(u_\varepsilon \lrcorner g_\varepsilon) = -\frac{d}{d\varepsilon} g_\varepsilon \quad \text{in } \Omega \tag{14.31}$$

and, for every $\gamma \in [\alpha - \delta, \alpha + \delta]$,

$$\|u_\varepsilon\|_{C^{r+1,\gamma}} \leq \frac{C_3}{\varepsilon^{1+\gamma-\alpha}} \|g\|_{C^{r,\alpha}} + C_3 \|\nu \wedge g\|_{C^{r+1,\alpha+\delta}(\partial\Omega)}\,. \tag{14.32}$$

Moreover, defining $u : (0, \overline{\varepsilon}] \times \overline{\Omega} \to \Lambda^1$ by $u(\varepsilon, x) = u_\varepsilon(x)$, we will show that $u \in C^{r+1,\alpha+\delta}\big((0, \overline{\varepsilon}] \times \overline{\Omega}; \Lambda^1\big)$.

Step 2.1. Since (14.23), (14.24) and (14.28) hold, using Theorem 8.16, we can find for every $\varepsilon \in (0, \overline{\varepsilon}]$, $w_\varepsilon \in C^\infty(\overline{\Omega}; \Lambda^1)$ and a constant $C_4 = C_4(r, \alpha, \delta, \Omega)$ such that

$$dw_\varepsilon = -\frac{d}{d\varepsilon} g_\varepsilon \quad \text{in } \Omega, \quad w_\varepsilon = 0 \quad \text{on } \partial\Omega$$

and, for every integer $q \leq r$ and every $\gamma \in [\alpha - \delta, \alpha + \delta]$,

$$\|w_\varepsilon\|_{C^{q+1,\gamma}} \leq C_4 \left\| \frac{d}{d\varepsilon} g_\varepsilon \right\|_{C^{q,\gamma}}. \tag{14.33}$$

Moreover, defining $w : (0, \overline{\varepsilon}] \times \overline{\Omega} \to \Lambda^1$ by $w(\varepsilon, x) = w_\varepsilon(x)$, we have, using (14.28), $w \in C^\infty\big((0, \overline{\varepsilon}] \times \overline{\Omega}; \Lambda^1\big)$.

Step 2.2. Since, by (14.29), we have for every $\varepsilon \in (0, \overline{\varepsilon}]$, $\operatorname{rank}[g_\varepsilon] = n$ in $\overline{\Omega}$, that there exists a unique $u_\varepsilon : \overline{\Omega} \to \Lambda^1$ verifying

$$u_\varepsilon \lrcorner g_\varepsilon = w_\varepsilon\,.$$

Note that $u_\varepsilon \in C^{r+1,\alpha+\delta}(\overline{\Omega}; \Lambda^1)$ and that $u_\varepsilon = 0$ on $\partial\Omega$. Moreover, defining $u : (0, \overline{\varepsilon}] \times \overline{\Omega} \to \Lambda^1$ by $u(\varepsilon, x) = u_\varepsilon(x)$, we have $u \in C^{r+1,\alpha+\delta}\big((0, \overline{\varepsilon}] \times \overline{\Omega}; \Lambda^1\big)$.

Step 2.3. To show Step 2, it only remains to prove (14.32). Using Theorem 16.28, (14.29), (14.30) and (14.33), it follows that

$$\begin{aligned}\|u_\varepsilon\|_{C^{r+1,\gamma}} &= \|(\overline{g}_\varepsilon)^{-1}w_\varepsilon\|_{C^{r+1,\gamma}} \\ &\le C_5\|(\overline{g}_\varepsilon)^{-1}\|_{C^{r+1,\gamma}}\|w_\varepsilon\|_{C^0} + C_5\|(\overline{g}_\varepsilon)^{-1}\|_{C^0}\|w_\varepsilon\|_{C^{r+1,\gamma}} \\ &\le C_6\|g_\varepsilon\|_{C^{r+1,\gamma}}\|w_\varepsilon\|_{C^{1,\alpha}} + C_6\|w_\varepsilon\|_{C^{r+1,\gamma}} \\ &\le C_7\|g_\varepsilon\|_{C^{r+1,\gamma}}\left\|\frac{d}{d\varepsilon}g_\varepsilon\right\|_{C^{0,\alpha}} + C_7\left\|\frac{d}{d\varepsilon}g_\varepsilon\right\|_{C^{r,\gamma}}\end{aligned}$$

and hence, appealing to (14.25) and (14.27),

$$\begin{aligned}&\|u_\varepsilon\|_{C^{r+1,\gamma}} \\ &\le C_8\left(\frac{1}{\varepsilon^{1+\gamma-\alpha}}\|g\|_{C^{r,\alpha}} + \|\nu\wedge g\|_{C^{r+1,\gamma}(\partial\Omega)}\right)\|g\|_{C^{1,\alpha}} + \frac{C_8}{\varepsilon^{1+\gamma-\alpha}}\|g\|_{C^{r,\alpha}} \\ &\le \frac{C_9}{\varepsilon^{1+\gamma-\alpha}}\|g\|_{C^{r,\alpha}} + C_9\|\nu\wedge g\|_{C^{r+1,\alpha+\delta}(\partial\Omega)},\end{aligned}$$

where $C_i = C_i(r,\alpha,\delta,\Omega,\|g\|_{C^{1,\alpha}},\|1/g^{n/2}\|_{C^0})$. This shows the assertion.

Step 3. We can now conclude the proof.

Step 3.1. Since $u \in C^{r+1,\alpha+\delta}((0,\overline{\varepsilon}]\times\overline{\Omega}:\mathbb{R}^n)$, $u_\varepsilon = 0$ on $\partial\Omega$ and (14.32) holds, we deduce, using Theorem 12.4, that the solution $\varphi : [0,\overline{\varepsilon}]\times\overline{\Omega}\to\overline{\Omega}$, $\varphi(\varepsilon,x) = \varphi_\varepsilon(x)$, of

$$\begin{cases}\dfrac{d}{d\varepsilon}\varphi_\varepsilon = u_\varepsilon\circ\varphi_\varepsilon, & 0<\varepsilon\le\overline{\varepsilon}, \\ \varphi_0 = \mathrm{id}\end{cases}$$

verifies

$$\varphi \in C^{r+1}([0,\overline{\varepsilon}]\times\overline{\Omega};\overline{\Omega}) \tag{14.34}$$

and that for every $\varepsilon\in[0,\overline{\varepsilon}]$,

$$\varphi_\varepsilon \in \mathrm{Diff}^{r+1,\alpha}(\overline{\Omega};\overline{\Omega}) \quad \text{and} \quad \varphi_\varepsilon = \mathrm{id} \text{ on } \partial\Omega.$$

Step 3.2. Since (14.31) holds, we deduce, using Theorem 12.7, that for every $0<\varepsilon_1\le\varepsilon_2\le\overline{\varepsilon}$,

$$\varphi_{\varepsilon_2}^*(g_{\varepsilon_2}) = \varphi_{\varepsilon_1}^*(g_{\varepsilon_1}) \quad \text{in } \Omega.$$

Since, using (14.26) and (14.34),

$$\lim_{\varepsilon\to 0}\|g_\varepsilon - g\|_{C^0} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\|\varphi_\varepsilon - \varphi_0\|_{C^1} = 0,$$

we immediately deduce that for every $\varepsilon\in(0,\overline{\varepsilon}]$,

$$\varphi_\varepsilon^*(g_\varepsilon) = \varphi_0^*(g) = g\,.$$

This concludes the proof. □

We now turn to our second proof of Theorem 14.5. We will do it under the stronger assumption that there exists $0 < \alpha < \beta < 1$ so that

$$\nu \wedge f,\ \nu \wedge g \in C^{r+1,\beta}\left(\partial\Omega;\Lambda^3\right).$$

Proof. Step 1. Let $\delta > 0$ small enough so that $[\alpha-\delta,\alpha+\delta] \subset (0,\beta)$. Applying Proposition 14.13 to f and g, there exist for every ε small,

$$f_\varepsilon, g_\varepsilon \in C^{r+1,\alpha+\delta}\left(\overline{\Omega};\Lambda^2\right) \quad \text{and} \quad \varphi_{1,\varepsilon}, \varphi_{2,\varepsilon} \in \mathrm{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$$

such that

$$\begin{cases} \varphi_{1,\varepsilon}^*(f_\varepsilon) = f, \quad \varphi_{2,\varepsilon}^*(g_\varepsilon) = g & \text{in } \Omega, \\ \varphi_{1,\varepsilon} = \varphi_{2,\varepsilon} = \mathrm{id} & \text{on } \partial\Omega, \end{cases}$$

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{C^{r,\alpha-\delta}} = \lim_{\varepsilon\to 0} \|g_\varepsilon - g\|_{C^{r,\alpha-\delta}} = 0.$$

Using the previous equation, there exists $\varepsilon_0 > 0$ small enough so that for every $t \in [0,1]$,

$$\mathrm{rank}[t g_{\varepsilon_0} + (1-t) f_{\varepsilon_0}] = n \quad \text{in } \overline{\Omega}.$$

Moreover, f_ε and g_ε satisfy

$$dg_\varepsilon = df_\varepsilon = 0, \quad \nu \wedge g_\varepsilon = \nu \wedge f_\varepsilon = \nu \wedge f = \nu \wedge g \ \text{ on } \partial\Omega,$$

$$\int_\Omega \langle g_\varepsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle = \int_\Omega \langle f_\varepsilon;\psi\rangle = \int_\Omega \langle f;\psi\rangle,\ \forall\,\psi \in \mathscr{H}_T\left(\Omega;\Lambda^2\right).$$

Step 2. Using Theorem 14.7, we find $\varphi_3 \in C^{r+1,\alpha+\delta}(\overline{\Omega})$ verifying

$$\begin{cases} \varphi_3^*(g_{\varepsilon_0}) = f_{\varepsilon_0} & \text{in } \Omega, \\ \varphi_3 = \mathrm{id} & \text{on } \partial\Omega. \end{cases}$$

Finally, the diffeomorphism $\varphi = \varphi_{2,\varepsilon_0}^{-1} \circ \varphi_3 \circ \varphi_{1,\varepsilon_0}$ has all of the required properties.

□