Chapter 13
The Cases $k = 0$ and $k = 1$

13.1 The Case of 0-Forms and of Closed 1-Forms

13.1.1 The Case of 0*-Forms*

We start with 0-forms. We begin our study with a local existence theorem.

Theorem 13.1. *Let $r \geq 1$ be an integer, $x_0 \in \mathbb{R}^n$ and f and g be C^r functions in a neighborhood of x_0 such that $f(x_0) = g(x_0)$,*

$$\nabla f(x_0) \neq 0 \quad \text{and} \quad \nabla g(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^r(U;\varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g)(x) = g(\varphi(x)) = f(x).$$

Furthermore, if

$$\frac{\partial f}{\partial x_i}(x_0) \cdot \frac{\partial g}{\partial x_i}(x_0) \neq 0$$

for a certain $1 \leq i \leq n$, then φ can be chosen of the form

$$\varphi(x) = (x_1, \ldots, x_{i-1}, \varphi^i(x), x_{i+1}, \ldots, x_n).$$

Proof. Without loss of generality we may assume that $x_0 = 0$. We split the proof into two steps.

Step 1. We prove the main statement. Since $\nabla f(0) \neq 0$ and $\nabla g(0) \neq 0$, we can find

$$A_2, \ldots, A_n, B_2, \ldots, B_n \in \mathbb{R}^n$$

such that letting

$$F(x) = (f(x), \langle A_2; x\rangle, \ldots, \langle A_n; x\rangle) \quad \text{and} \quad G(x) = (g(x), \langle B_2; x\rangle, \ldots, \langle B_n; x\rangle),$$

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then

$$\det \nabla F(0) \neq 0 \quad \text{and} \quad \det \nabla G(0) \neq 0.$$

Hence, since $F(0) = G(0)$, we deduce that

$$F \in \operatorname{Diff}^r(U; F(U)), \quad G \in \operatorname{Diff}^r(U; G(U)) \quad \text{and} \quad G^{-1} \circ F \in \operatorname{Diff}^r(U; (G^{-1} \circ F)(U))$$

for a neighborhood U of 0 small enough. Therefore, $\varphi = G^{-1} \circ F$ has all of the desired properties.

Step 2. We now prove the extra property. Define

$$F(x) = (x_1, \ldots, x_{i-1}, f(x), x_{i+1}, \ldots, x_n),$$

$$G(x) = (x_1, \ldots, x_{i-1}, g(x), x_{i+1}, \ldots, x_n)$$

and note that $\varphi = G^{-1} \circ F$ has all of the required properties. The proof is therefore complete. □

We now have the following global result.

Theorem 13.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Let $r \geq 1$ be an integer and f and $g \in C^r\left(\overline{\Omega}\right)$ with $f = g$ on $\partial\Omega$ and*

$$\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} > 0 \text{ in } \overline{\Omega} \tag{13.1}$$

for a certain $1 \leq i \leq n$. Then there exists a diffeomorphism $\varphi \in \operatorname{Diff}^r\left(\overline{\Omega}; \overline{\Omega}\right)$ of the form

$$\varphi(x) = (x_1, \ldots, x_{i-1}, \varphi^i(x), x_{i+1}, \ldots, x_n)$$

satisfying

$$\begin{cases} \varphi^*(g) = g \circ \varphi = f & \text{in } \Omega, \\ \varphi = \text{id} & \text{on } \partial\Omega. \end{cases}$$

Proof. Let e_i be the ith vector of the Euclidean basis of $\mathbb{R}^n$. We will find φ of the form $\varphi(x) = x + u(x)e_i$, where $u : \overline{\Omega} \to \mathbb{R}$. Since Ω is Lipschitz, we can extend (according to Theorem 16.11) f and g to $C^r(\mathbb{R}^n)$ functions. We therefore also have

$$\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} > 0 \quad \text{in a neighborhood of } \overline{\Omega}. \tag{13.2}$$

By compactness, for every $x \in \Omega$, there exist $s_x, t_x \in \mathbb{R}$ with $s_x < 0 < t_x$ such that

$$x + s_x e_i, x + t_x e_i \in \partial\Omega \quad \text{and} \quad (x + s_x e_i, x + t_x e_i) \subset \Omega.$$

Define $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$h(x, v) = g(x + v e_i) - f(x).$$

We claim that there exists $u \in C^r(\overline{\Omega})$ such that

$$h(x,u(x)) = 0, \text{ for } x \in \overline{\Omega}, \quad u = 0 \text{ on } \partial\Omega \quad \text{and} \quad 1 + \frac{\partial u}{\partial x_i}(x) > 0 \text{ for } x \in \overline{\Omega}.$$

(i) For every $x \in \partial\Omega$, let $u(x) = 0$ and note that $h(x,u(x)) = h(x,0) = g(x) - f(x) = 0$ since $f = g$ on $\partial\Omega$.

(ii) Let $x \in \Omega$. Since $f = g$ on $\partial\Omega$, we have

$$h(x,s_x) = f(x + s_x e_i) - f(x) \quad \text{and} \quad h(x,t_x) = f(x + t_x e_i) - f(x).$$

Hence, recalling that $\partial f(x)/\partial x_i \neq 0$ for every $x \in \overline{\Omega}$, we get

$$h(x,s_x) \cdot h(x,t_x) < 0.$$

Note that $v \to h(x,v)$ is monotone. Therefore, there exists a unique $u(x) \in (s_x, t_x)$ verifying $h(x,u(x)) = 0$.

(iii) Using the implicit function theorem and (13.2), we immediately deduce that $u \in C^r(\overline{\Omega})$ and that

$$1 + \frac{\partial u}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x) \left(\frac{\partial g}{\partial x_i}(x + u(x)) \right)^{-1} > 0 \quad \text{for every } x \in \overline{\Omega}.$$

This proves the claim. Finally, letting $\varphi(x) = x + u(x)e_i$, we get that $g \circ \varphi = f$ in $\overline{\Omega}$, $\varphi \in C^r(\overline{\Omega}; \mathbb{R}^n)$, $\det \nabla \varphi > 0$ in $\overline{\Omega}$ and $\varphi = \text{id}$ on $\partial\Omega$. Hence, using Theorem 19.12, we have $\varphi \in \text{Diff}^r(\overline{\Omega}; \overline{\Omega})$, which concludes the proof. □

13.1.2 The Case of Closed 1-Forms

We get as immediate corollaries similar results for closed 1-forms. Recall that 1-forms are written as

$$f = \sum_{i=1}^{n} f_i \, dx^i \quad \text{and} \quad g = \sum_{i=1}^{n} g_i \, dx^i \, .$$

We start first with the local version.

Corollary 13.3. *Let $r \geq 0$ be an integer, $x_0 \in \mathbb{R}^n$ and f and g be C^r closed 1-forms in a neighborhood of x_0 such that*

$$f(x_0) \neq 0 \quad \textit{and} \quad g(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \text{Diff}^{r+1}(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = f \quad \textit{in } U.$$

Furthermore, if

$$f_i(x_0) \cdot g_i(x_0) \neq 0$$

for a certain $1 \leq i \leq n$*, then* φ *can be chosen of the form*

$$\varphi(x) = (x_1, \ldots, x_{i-1}, \varphi^i(x), x_{i+1}, \ldots, x_n).$$

Remark 13.4. When $r = 0$, the fact that a 1-form ω is closed has to be understood in the sense of distributions.

Proof. Using Corollary 8.6, there exist a small ball V centered at x_0 and $F, G \in C^{r+1}(V)$ such that

$$dF = f \quad \text{and} \quad dG = g \quad \text{in } V.$$

Adding, if necessary, a constant, we can also assume that $F(x_0) = G(x_0)$. Note that if $f_i(x_0) \cdot g_i(x_0) \neq 0$ for a certain $1 \leq i \leq n$, then

$$\frac{\partial F}{\partial x_i}(x_0) \cdot \frac{\partial G}{\partial x_i}(x_0) \neq 0.$$

We are then in a position to apply Theorem 13.1 to get $U \subset V$, a neighborhood of x_0 and $\varphi \in \operatorname{Diff}^{r+1}(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(G) = F,$$

which implies

$$\varphi^*(dG) = dF$$

and concludes the proof. □

We now conclude with the global version obtained in Bandyopadhyay and Dacorogna [8].

Corollary 13.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded simply connected smooth open set. Let* $r \geq 0$ *be an integer and* $f, g \in C^r(\overline{\Omega}; \Lambda^1)$ *be closed and such that*

$$\nu \wedge f = \nu \wedge g \textit{ on } \partial\Omega \quad \textit{and} \quad f_i \cdot g_i > 0 \textit{ in } \overline{\Omega}$$

for a certain $1 \leq i \leq n$*. Then there exists a diffeomorphism* $\varphi \in \operatorname{Diff}^{r+1}(\overline{\Omega}; \overline{\Omega})$ *of the form*

$$\varphi(x) = (x_1, \ldots, x_{i-1}, \varphi^i(x), x_{i+1}, \ldots, x_n)$$

satisfying

$$\begin{cases} \varphi^*(g) = f & \textit{in } \Omega, \\ \varphi = \text{id} & \textit{on } \partial\Omega. \end{cases}$$

Proof. We first claim that there exist $F, G \in C^{r+1}(\overline{\Omega})$ such that $F, G \in C^{r+1}(\overline{\Omega})$ and

$$dF = f,\ dG = g \text{ in } \Omega \quad \text{and} \quad F = G \text{ on } \partial\Omega.$$

Indeed, by Theorem 8.16 and the remark following it and recalling that $\mathscr{H}_T(\Omega,\Lambda^1)=\{0\}$ since Ω is simply connected (see Remark 6.6), there exists $H\in C^{r+1}(\overline{\Omega};\Lambda^1)$ such that

$$dH=f-g \text{ in } \Omega \quad \text{and} \quad H=0 \text{ on } \partial\Omega.$$

Then, using Corollary 8.6, there exists $G\in C^{r+1}(\overline{\Omega})$ such that $dG=g$ in Ω. Letting $F=H+G$, we have the claim. In particular, note that

$$\frac{\partial F}{\partial x_i}\cdot\frac{\partial G}{\partial x_i}>0 \text{ in } \overline{\Omega}.$$

Finally, apply Theorem 13.2 to get $\varphi\in\operatorname{Diff}^{r+1}(\overline{\Omega};\overline{\Omega})$ of the desired form so that

$$\varphi^*(G)=F \quad \text{in } \Omega,$$

which implies

$$\varphi^*(dG)=dF \quad \text{in } \Omega.$$

The proof is therefore complete. □

13.2 Darboux Theorem for 1-Forms

13.2.1 Main Results

The following result is classical and due to Darboux [34]; see, for example Bryant et al. [18], Olver [80], or Sternberg [93]. This result is equivalent to the Darboux theorem (cf. the remark below) for closed 2-forms.

Theorem 13.6. *Let $r\geq 3$ and $2\leq 2m\leq n$ be integers. Let $0<\alpha<1$, $x_0\in\mathbb{R}^n$ and w be a $C^{r,\alpha}$ 1-form such that*

$$\operatorname{rank}[dw]=2m \quad \textit{in a neighborhood of } x_0.$$

Then there exist a neighborhood U of x_0 and

$$\varphi\in\begin{cases}\operatorname{Diff}^{r,\alpha}(U;\varphi(U)) & \text{if } 2m=n\\ \operatorname{Diff}^{r-1,\alpha}(U;\varphi(U)) & \text{if } 2m<n\end{cases}$$

such that $\varphi(x_0)=x_0$ and

$$\varphi^*(w)=\sum_{i=1}^{m}x_{2i-1}dx^{2i}+dS \quad \textit{in } U,$$

with

$$S \in \begin{cases} C^{r,\alpha}(U) & \text{if } 2m = n \\ C^{r-1,\alpha}(U) & \text{if } 2m < n. \end{cases}$$

Remark 13.7. (i) The above result is equivalent to the Darboux theorem for closed 2-forms. This last theorem reads (see Theorems 14.1 and 14.3) as follows. Let $n \geq 2m$, $x_0 \in \mathbb{R}^n$ and f be a $C^{r,\alpha}$ closed 2-form satisfying

$$\operatorname{rank}[f] = 2m \quad \text{in a neighborhood of } x_0 .$$

Then there exist a neighborhood U of x_0 and

$$\varphi \in \begin{cases} \operatorname{Diff}^{r+1,\alpha}(U;\varphi(U)) & \text{if } n = 2m \\ \operatorname{Diff}^{r,\alpha}(U;\varphi(U)) & \text{if } n > 2m \end{cases}$$

such that $\varphi(x_0) = x_0$ and

$$\varphi^*(f) = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{in } U.$$

The fact that the Darboux theorem for 2-forms implies the one for 1-forms is precisely the proof of Theorem 13.6 below. The other implication follows immediately, once it has been observed that we can choose, for example, U to be a ball so that, f being closed in U, we can find (cf. Theorem 8.3) $w \in C^{r+1,\alpha}(U;\Lambda^1)$ such that $f = dw$. We then apply the theorem to w, getting

$$\varphi^*(f) = \varphi^*(dw) = d\varphi^*(w) = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

(ii) The hypothesis $r \geq 3$ can be weakened if we use a weak version of the fourth statement of Theorem 3.10. Indeed, it is enough to assume $r \geq 1$ if $n = 2m$ and $r \geq 2$ if $n > 2m$ (cf. Csató [23]).

Proof. Using Theorem 14.1 if $2m = n$ or Theorem 14.3 if $2m < n$, there exist a neighborhood U of x_0 and

$$\varphi \in \begin{cases} \operatorname{Diff}^{r,\alpha}(U;\varphi(U)) & \text{if } 2m = n \\ \operatorname{Diff}^{r-1,\alpha}(U;\varphi(U)) & \text{if } 2m < n \end{cases}$$

such that $\varphi(x_0) = x_0$ and

$$\varphi^*(dw) = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{in } U.$$

Note that

$$d\left[\varphi^*(w) - \sum_{i=1}^{m} x_{2i-1} dx^{2i}\right] = 0 \quad \text{in } U$$

and

$$\varphi^*(w) - \sum_{i=1}^{m} x_{2i-1} dx^{2i} \in \begin{cases} C^{r-1,\alpha}(U;\Lambda^1) & \text{if } 2m = n \\ C^{r-2,\alpha}(U;\Lambda^1) & \text{if } 2m < n. \end{cases}$$

Thus, by Theorem 8.3, restricting U if necessary, there exists

$$S \in \begin{cases} C^{r,\alpha}(U) & \text{if } 2m = n \\ C^{r-1,\alpha}(U) & \text{if } 2m < n \end{cases}$$

such that

$$dS = \varphi^*(w) - \sum_{i=1}^{m} x_{2i-1} dx^{2i},$$

which concludes the proof. □

The next two theorems refine the above result (cf. Bryant et al. [18] or Sternberg [93]). In particular, the second one gives a sufficient condition ensuring that $S = 0$.

Theorem 13.8. *Let* $r \geq 3$ *and* $2 \leq 2m \leq n$ *be integers. Let* $0 < \alpha < 1$, $x_0 \in \mathbb{R}^n$ *and* w *be a* $C^{r,\alpha}$ 1*-form such that*

$$\operatorname{rank}[dw] = 2m \quad \textit{in a neighborhood of } x_0$$

and

$$w \wedge \underbrace{dw \wedge \cdots \wedge dw}_{m \text{ times}}(x_0) \neq 0.$$

Then there exist a neighborhood U *of* x_0 *and*

$$\varphi \in \operatorname{Diff}^{r-1,\alpha}(U;\varphi(U))$$

such that $\varphi(x_0) = x_0$ *and*

$$\varphi^*(w) = \sum_{i=1}^{m} x_{2i-1}\, dx^{2i} + dx^{2m+1} \quad \textit{in } U.$$

Remark 13.9. Since $w \wedge (dw)^m$ is a $(2m+1)$-form and since

$$w \wedge (dw)^m(x_0) \neq 0,$$

we necessarily have $2m < n$.

Proof. With no loss of generality, we can assume $x_0 = 0$. Since (according to Remark 13.9) we necessarily have $2m < n$, we get, using Theorem 13.6, a neighborhood V of 0 and $\varphi_1 \in \operatorname{Diff}^{r-1,\alpha}(V;\varphi_1(V))$ such that $\varphi_1(0) = 0$ and

$$\varphi_1^*(w) = \sum_{i=1}^{m} x_{2i-1}\, dx^{2i} + dS \quad \text{in } V,$$

with $S \in C^{r-1,\alpha}(V)$. Since, by hypothesis,

$$w \wedge (dw)^m(0) \neq 0,$$

we get that since $\varphi_1(0) = 0$,

$$\varphi_1^*(w) \wedge (d\varphi_1^*(w))^m(0) \neq 0,$$

which is equivalent to

$$dS \wedge dx^1 \wedge \cdots \wedge dx^{2m}(0) \neq 0.$$

Permuting, if necessary, the coordinates $x_{2m+1}, \ldots, x_n$, we can therefore assume with no loss of generality that

$$\frac{\partial S}{\partial x_{2m+1}}(0) \neq 0.$$

Now, define, for $x \in V$,

$$\varphi_2(x) = (x_1, \ldots, x_{2m}, S(x) - S(0), x_{2m+2}, \ldots, x_n).$$

Taking V smaller, if necessary, we obtain that $\varphi_2 \in \mathrm{Diff}^{r-1,\alpha}(V; \varphi_2(V))$, $\varphi_2(0) = 0$ and

$$\varphi_2^*(dx^i) = \begin{cases} dx^i & \text{if } i \neq 2m+1 \\ dS & \text{if } i = 2m+1. \end{cases}$$

Finally, letting $U = \varphi_2(V)$ and $\varphi = \varphi_1 \circ (\varphi_2)^{-1}$, we easily obtain that $\varphi \in \mathrm{Diff}^{r-1,\alpha}(U; \varphi(U))$, $\varphi(0) = 0$ and

$$\varphi^*(w) = \sum_{i=1}^{m} x_{2i-1} dx^{2i} + dx^{2m+1} \quad \text{in } U,$$

which ends the proof. □

Theorem 13.10. *Let $2 \leq 2m \leq n$ be an integer, $x_0 \in \mathbb{R}^n$ and w a C^∞ 1-form such that*

$$\operatorname{rank}[dw] = 2m \quad \textit{in a neighborhood of } x_0\,,$$

$w(x_0) \neq 0$ and

$$w \wedge \underbrace{dw \wedge \cdots \wedge dw}_{m \text{ times}} = 0 \quad \textit{in a neighborhood of } x_0\,.$$

Then there exist an open set U and

$$\varphi \in \mathrm{Diff}^\infty(U; \varphi(U))$$

such that $\varphi(U)$ is a neighborhood of x_0 and

$$\varphi^*(w) = \sum_{i=1}^{m} x_{2i-1}\, dx^{2i} \quad \textit{in } U.$$

Remark 13.11. (i) If $w \in C^r$, the following proof shows in fact that $\varphi \in C^{r-2m+1}$ if $2m = n$ and $\varphi \in C^{r-2m}$ if $2m < n$.

(ii) If we, moreover, want $\varphi(x_0) = x_0$, then the conclusion becomes

$$\varphi^*(w) = \sum_{i=1}^{m} (x_{2i-1} - c_{2i-1})\, dx^{2i} \quad \text{in } U$$

for some $c_{2i-1} \in \mathbb{R}$, $1 \le i \le m$. Note that the c_{2i-1} cannot be arbitrary. For example, the c_{2i-1} can never verify $c_{2i-1} = (x_0)_{2i-1}$ for every $1 \le i \le m$. Indeed,

$$\varphi^*(w)(x_0) = \sum_{i=1}^{m} ((x_0)_{2i-1} - c_{2i-1})\, dx^{2i}$$

and thus we have the assertion since, recalling that $\varphi(x_0) = x_0$,

$$\varphi^*(w)(x_0) \neq 0 \;\Leftrightarrow\; w(x_0) \neq 0.$$

Proof. We split the proof into two steps. With no loss of generality, we can assume that $x_0 = 0$.

Step 1 (simplification). Let us first prove that we can assume that $n = 2m$. Applying Theorem 14.3 to dw, we can find a neighborhood U of 0 and $\psi \in \text{Diff}^\infty(U; \psi(U))$ such that $\psi(0) = 0$ and

$$\psi^*(dw) = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{in } U. \tag{13.3}$$

Note that since $\psi(0) = 0$, we have, restricting U if necessary,

$$\psi^*(w)(0) \neq 0 \quad \text{and} \quad \psi^*(w) \wedge (d(\psi^*(w)))^m = 0 \quad \text{in } U. \tag{13.4}$$

The last equation being equivalent to

$$\psi^*(w) \wedge dx^1 \wedge \cdots \wedge dx^{2m} = 0 \quad \text{in } U,$$

we immediately deduce

$$[\psi^*(w)]_i = 0 \quad \text{in } U \quad \text{for every } 2m+1 \le i \le n$$

and, hence,

$$\psi^*(w)(x) = \sum_{i=1}^{2m} b_i(x)\, dx^i \quad \text{in } U.$$

Combining the previous equation with (13.3), we get that

$$b_i(x) = b_i(x_1, \ldots, x_{2m}) \quad \text{in a neighborhood of } 0, \quad \text{for every } 1 \le i \le 2m.$$

We thus have the claim, replacing $\psi^*(w)$ by w.

Step 2 (conclusion). Applying Lemma 13.12 to w, we can find a neighborhood V of 0 in $\mathbb{R}^{2m}$ and $\varphi_1 \in \mathrm{Diff}^\infty(V;\varphi_1(V))$ such that $\varphi_1(0) = 0$ and

$$\varphi_1^*(w) = \sum_{i=1}^{m} b_{2i-1}(x_1,\dots,x_{2m})\,dx^{2i-1} \quad \text{in } V \tag{13.5}$$

for some $b_{2i-1} \in C^\infty(V)$. Since

$$\mathrm{rank}[d(\varphi_1^*(w))(0)] = \mathrm{rank}[dw(0)] = 2m,$$

we know that $(d\varphi_1^*(w))^m(0) \neq 0$, which is equivalent to, using (13.5),

$$dx^1 \wedge dx^3 \wedge \cdots \wedge dx^{2m-1} \wedge db_1 \wedge db_3 \wedge \cdots \wedge db_{2m-1}(0) \neq 0. \tag{13.6}$$

Now, define, for $x \in V$,

$$\varphi_2(x) = (b_1(x), x_1, b_3(x), x_3, \dots, b_{2m-1}(x), x_{2m-1}).$$

Using (13.6), we obtain that $\varphi_2 \in \mathrm{Diff}^\infty(V;\varphi_2(V))$, taking V smaller if necessary. Finally, letting $U = \varphi_2(V)$ and $\varphi = \varphi_1 \circ (\varphi_2)^{-1}$, we easily obtain that $\varphi \in \mathrm{Diff}^\infty(U;\varphi(U))$ and

$$\varphi^*(w) = \sum_{i=1}^{m} x_{2i-1}\,dx^{2i} \quad \text{in } U,$$

which ends the proof. □

13.2.2 A Technical Result

We still need to prove the following lemma.

Lemma 13.12. *Let $m \geq 1$ be an integer, $x_0 \in \mathbb{R}^{2m}$ and w be a C^∞ 1-form defined in a neighborhood of x_0 such that $w(x_0) \neq 0$ and*

$$\mathrm{rank}[dw(x_0)] = 2m.$$

Then there exist a neighborhood U of x_0 and

$$\varphi \in \mathrm{Diff}^\infty(U;\varphi(U))$$

such that $\varphi(x_0) = x_0$ and

$$[\varphi^*(w)]_{2i} = 0 \quad \text{in } U \text{ for every } 1 \leq i \leq m. \tag{13.7}$$

Remark 13.13. If $w \in C^r$, then the following proof gives $\varphi \in C^{r-2(m-1)}$.

For the proof of the lemma we will need the two following elementary results, the first of which is purely algebraic.

Lemma 13.14. *Let $f \in \Lambda^2(\mathbb{R}^{2m})$ with* $\operatorname{rank}[f] = 2m$ *and*

$$a = \sum_{i=1}^{2m-1} a_i e^i \in \Lambda^1(\mathbb{R}^{2m})$$

with $a \neq 0$. Then there exists $A \in \mathrm{GL}(2m)$ of the form

$$A = \begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $B \in \mathrm{GL}(2m-1)$ and such that

$$\sum_{1 \leq i < j \leq 2m-1} (A^*(f))_{ij} e^i \wedge e^j = \sum_{i=1}^{m-1} e^{2i-1} \wedge e^{2i} \quad \textit{and} \quad \sum_{i=1}^{2m-2} (A^*(a))_i e^i \neq 0.$$

Proof. Step 1. Using Proposition 2.24(ii), there exists $\widetilde{A} \in \mathrm{GL}(2m)$ such that

$$\widetilde{A}^*(f) = \sum_{i=1}^{m} e^{2i-1} \wedge e^{2i} \quad \text{and} \quad \widetilde{A}^*(e^{2m}) = e^{2m}.$$

Note that the condition $\widetilde{A}^*(e^{2m}) = e^{2m}$ is equivalent to

$$\widetilde{A} = \begin{pmatrix} & & & \widetilde{A}^1_{2m} \\ & \widetilde{B} & & \vdots \\ & & & \widetilde{A}^{2m-1}_{2m} \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $\widetilde{B} \in \mathrm{GL}(2m-1)$ is given by $\widetilde{B}^i_j = \widetilde{A}^i_j$. Define

$$A = \begin{pmatrix} & & & 0 \\ & \widetilde{B} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and observe that for $1 \leq i < j \leq 2m-1$,

$$\begin{aligned} (A^*(f))_{ij} &= \left(\sum_{1 \leq p < q \leq 2m} f_{pq} A^p \wedge A^q \right)_{ij} = \sum_{1 \leq p < q \leq 2m} f_{pq} \left(A^p_i A^q_j - A^p_j A^q_i \right) \\ &= \sum_{1 \leq p < q \leq 2m} f_{pq} \left(\widetilde{A}^p_i \widetilde{A}^q_j - \widetilde{A}^p_j \widetilde{A}^q_i \right) = (\widetilde{A}^*(f))_{ij} \,. \end{aligned}$$

We therefore have

$$\sum_{1 \le i < j \le 2m-1} (A^*(f))_{ij} e^i \wedge e^j = \sum_{1 \le i < j \le 2m-1} (\widetilde{A}^*(f))_{ij} e^i \wedge e^j = \sum_{i=1}^{m-1} e^{2i-1} \wedge e^{2i}.$$

Note that the previous equation is equivalent to

$$A^*(f) = \sum_{i=1}^{m-1} e^{2i-1} \wedge e^{2i} + h \wedge e^{2m} \tag{13.8}$$

for a certain $h = \sum_{i=1}^{2m-1} h_i e^i \in \Lambda^1(\mathbb{R}^{2m})$.

Step 2. Since $a \neq 0$, we have $A^*(a) = \sum_{i=1}^{2m-1} A^*(a)_i e^i \neq 0$ and thus there exists $1 \le i \le 2m-1$ such that $A^*(a)_i \neq 0$. If $1 \le i \le 2m-2$, the matrix A has all of the required properties. If $A^*(a)_i = 0$ for $1 \le i \le 2m-2$, we proceed as follows. Define

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ 1 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \Leftrightarrow P^i_j = \begin{cases} 1 & \text{if } i = j \\ 1 & \text{if } i = 2m-1 \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and let us show that $AP \in \mathrm{GL}(2m)$ has all the claimed properties. Indeed, first note that AP has the desired form. Since

$$P^*(e^i) = \begin{cases} e^i & \text{if } i \neq 2m-1 \\ e^1 + e^{2m-1} & \text{if } i = 2m-1, \end{cases}$$

we deduce that, using (13.8),

$$(AP)^*(f) = P^*(A^*(f)) = \sum_{i=1}^{m-1} e^{2i-1} \wedge e^{2i} + P^*(h) \wedge e^{2m}.$$

We therefore get

$$\sum_{1 \le i < j \le 2m-1} ((AP)^*(f))_{ij} e^i \wedge e^j = \sum_{i=1}^{m-1} e^{2i-i} \wedge e^{2i}.$$

Note also that

$$((AP)^*(a))_1 = (P^*(A^*(a)))_1 = (A^*(a))_{2m-1} \neq 0.$$

The proof is therefore complete. □

We now give the second result.

Lemma 13.15. *Let $U \subset \mathbb{R}^n$ be an open set, $n \ge 2$ and $w \in C^\infty(U; \Lambda^1)$ be such that*

$$(dx^n) \lrcorner dw = w \quad \text{in } U. \tag{13.9}$$

Then

$$\begin{cases} w = e^{x_n} \sum_{i=1}^{n-1} b_{in}(x_1, \ldots, x_{n-1})\, dx^i, \\ dw = -e^{x_n} \sum_{1 \le i < j \le n} b_{ij}(x_1, \ldots, x_{n-1})\, dx^i \wedge dx^j \end{cases}$$

for some $b_{ij} \in C^\infty$.

Proof. We first write

$$dw = \sum_{1 \le i < j \le n} a_{ij}\, dx^i \wedge dx^j$$

and observe that, as a direct consequence of (13.9), we have

$$w = -\sum_{i=1}^{n-1} a_{in}\, dx^i. \tag{13.10}$$

We finally show that for every $1 \le i < j \le n$ and $x = (x_1, \ldots, x_n) \in U$,

$$a_{ij}(x) = -e^{x_n} b_{ij}(x_1, \ldots, x_{n-1})$$

for some $b_{ij} \in C^\infty$. For this, it is enough to prove that for every $1 \le i < j \le n$,

$$a_{ij} = \frac{\partial a_{ij}}{\partial x_n}.$$

Let $1 \le i < j \le n$. First, since $ddw = 0$ and hence, in particular, $(ddw)_{ijn} = 0$, we have (with the convention that $a_{nn} = 0$)

$$\frac{\partial a_{jn}}{\partial x_i} - \frac{\partial a_{in}}{\partial x_j} + \frac{\partial a_{ij}}{\partial x_n} = 0.$$

Using (13.10) and the previous equation, we obtain

$$a_{ij} = (dw)_{ij} = -\left(\frac{\partial a_{jn}}{\partial x_i} - \frac{\partial a_{in}}{\partial x_j} \right) = \frac{\partial a_{ij}}{\partial x_n},$$

which concludes the proof. □

Finally, we prove Lemma 13.12.

Proof. With no loss of generality we can assume $x_0 = 0$. In the sequel, U will be a generic neighborhood of 0. We prove the lemma by induction on m and we split the proof into three steps.

Step 1. We start by introducing some notations. Let

$$x = (x_1, \ldots, x_{2m-2}, x_{2m-1}, x_{2m}) \in \mathbb{R}^n.$$

For every $(x_{2m-1}, x_{2m}) \in \mathbb{R}^2$, define $i_{(x_{2m-1},x_{2m})} : \mathbb{R}^{2m-2} \to \mathbb{R}^{2m}$ by

$$i_{(x_{2m-1},x_{2m})}(x_1, \ldots, x_{2m-2}) = x.$$

Let $1 \leq k \leq n$ and

$$g = \sum_{1 \leq i_1 < \cdots < i_k \leq 2m} g_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in C^0\big(\mathbb{R}^{2m}; \Lambda^k(\mathbb{R}^{2m})\big).$$

Then for every $(x_{2m-1}, x_{2m}) \in \mathbb{R}^2$, we have

$$i^*_{(x_{2m-1},x_{2m})}(g) \in C^0\big(\mathbb{R}^{2m-2}; \Lambda^k(\mathbb{R}^{2m-2})\big)$$

and, explicitly,

$$i^*_{(x_{2m-1},x_{2m})}(g)(x_1, \ldots, x_{2m-2}) = \sum_{1 \leq i_1 < \cdots < i_k \leq 2m-2} g_{i_1 \cdots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Step 2 (the case $m=1$). In that case, we have

$$w(x) = w_1(x_1, x_2)\, dx^1 + w_2(x_1, x_2)\, dx^2.$$

Since, by hypothesis, $(w_1(0), w_2(0)) \neq (0,0)$, there exist, applying Remark 4.3(ii), a neighborhood U of 0 and $\varphi \in \mathrm{Diff}^\infty(U; \varphi(U))$ such that $\varphi(0) = 0$ and

$$\frac{\partial \varphi}{\partial x_2} = (-w_2(\varphi), w_1(\varphi)) \quad \text{in } U.$$

We thus get, using the above equation,

$$\begin{aligned}
\varphi^*(w) &= w_1(\varphi) d\varphi^1 + w_2(\varphi) d\varphi^2 \\
&= \left[w_1(\varphi) \frac{\partial \varphi^1}{\partial x_1} + w_2(\varphi) \frac{\partial \varphi^2}{\partial x_1} \right] dx^1 + \left[w_1(\varphi) \frac{\partial \varphi^1}{\partial x_2} + w_2(\varphi) \frac{\partial \varphi^2}{\partial x_2} \right] dx^2 \\
&= \left[w_1(\varphi) \frac{\partial \varphi^1}{\partial x_1} + w_2(\varphi) \frac{\partial \varphi^2}{\partial x_1} \right] dx^1,
\end{aligned}$$

which is the desired assertion.

Step 3 (induction). We assume that the lemma has been proved for $m-1$ and prove it for m.

Step 3.1 (preliminaries). In this step we show the existence of a neighborhood U of 0 and $\psi \in \mathrm{Diff}^\infty(U; \psi(U))$ with $\psi(0) = 0$ such that for every $x = (x_1, \ldots, x_{2m-2}, x_{2m-1}, x_{2m}) \in U$,

$$i^*_{(x_{2m-1},x_{2m})}(\psi^*(w))(x_1, \ldots, x_{2m-2}) \neq 0, \tag{13.11}$$

$$\operatorname{rank}\left[d(i^*_{(x_{2m-1},x_{2m})}(\psi^*(w)))(x_1, \ldots, x_{2m-2}) \right] = 2m-2, \tag{13.12}$$

$$\psi^*(w)(x) = e^{x_{2m}} \sum_{i=1}^{2m-1} c_i(x_1, \ldots, x_{2m-1}) dx^i \quad \text{in } U \tag{13.13}$$

for some $c_i \in C^\infty(U)$.

(i) Since $\operatorname{rank}[dw] = 2m$ in a neighborhood of 0 and Proposition 2.50 holds, we can find a neighborhood U of 0 and a unique $v \in C^\infty(U; \Lambda^1)$ such that

$$v \lrcorner dw = w \quad \text{in } U.$$

Note that $v(0) \neq 0$ since $w(0) \neq 0$. Hence, using Remark 4.3(ii), there exist a neighborhood U of 0 and $\chi \in \operatorname{Diff}^\infty(U; \chi(U))$ such that $\chi(0) = 0$ and

$$\frac{\partial \chi}{\partial x_{2m}} = v \circ \chi \quad \text{in } U.$$

Using Theorem 3.10 and Proposition 3.11, we thus get

$$\chi^*(w) = \chi^*(v \lrcorner dw) = dx^{2m} \lrcorner d\chi^*(w) \quad \text{in } U.$$

Therefore, applying Lemma 13.15, we have

$$d\chi^*(w)(x) = -e^{x_{2m}} \sum_{1 \leq i < j \leq 2m} b_{ij}(x_1, \ldots, x_{2m-1})\, dx^i \wedge dx^j \quad \text{for every } x \in U,$$

$$\chi^*(w)(x) = e^{x_{2m}} \sum_{i=1}^{2m-1} b_{i(2m)}(x_1, \ldots, x_{2m-1})\, dx^i \quad \text{for every } x \in U \tag{13.14}$$

for some $b_{ij} \in C^\infty$.

(ii) Apply Lemma 13.14 to

$$f = d\chi^*(w)(0) \in \Lambda^2(\mathbb{R}^{2m}) \quad \text{and} \quad a = \chi^*(w)(0) \in \Lambda^1(\mathbb{R}^{2m})$$

to get $A \in \mathrm{GL}(n)$ of the form

$$A = \begin{pmatrix} & & 0 \\ & B & \vdots \\ & & 0 \\ 0 & \cdots & 0 \; 1 \end{pmatrix}$$

such that

$$\begin{gathered} \sum_{1 \leq i < j \leq 2m-1} (A^*(f))_{ij} e^i \wedge e^j = \sum_{i=1}^{m-1} e^{2i-1} \wedge e^{2i}, \\ \sum_{i=1}^{2m-2} (A^*(a))_i e^i \neq 0. \end{gathered} \tag{13.15}$$

(iii) Let $\theta(x) = A \cdot x$. We now prove that $\psi = \chi \circ \theta$ has all of the desired properties claimed by Step 3.1. In the following, we will frequently use (cf. Remark 3.9) that for any $\varphi \in C^1\left(\mathbb{R}^M;\mathbb{R}^N\right)$, any k-form α, and any fixed $x \in \mathbb{R}^M$,

$$\varphi^*(\alpha)(x) = (\nabla\varphi(x))^*[\alpha(\varphi(x))].$$

First, note that $\psi(0) = 0$ since $\chi(0) = \theta(0) = 0$. We now show (13.11). Restricting if necessary U, it is enough to show the property for $x = 0$. Using the second statement in (13.15), we deduce

$$i^*_{(0,0)}(\psi^*(w))(0,\ldots,0) = \sum_{i=1}^{2m-2} A^*(a)_i e^i \neq 0,$$

which proves the claim. We next prove (13.12). As before, restricting if necessary U, it is enough to prove the assertion for $x = 0$. Using the first equality in (13.15), we obtain

$$\begin{aligned} di^*_{(0,0)}(\psi^*(w))(0,\ldots,0) &= i^*_{(0,0)}(\psi^*(dw))(0,\ldots,0) \\ &= \sum_{1\leq i<j\leq 2m-2} (A^*(f))_{ij} e^i \wedge e^j = \sum_{i=1}^{m-1} e^{2i-1} \wedge e^{2i}. \end{aligned}$$

This establishes the claim. Finally, using (13.14) and since

$$\theta(x) = (\theta^1(x_1,\ldots,x_{2m-1}),\ldots,\theta^{2m-1}(x_1,\ldots,x_{2m-1}),x_{2m}),$$

we have

$$\begin{aligned} &\theta^*(\chi^*(w))(x) \\ &= e^{x_{2m}} \sum_{i=1}^{2m-1} b_{i(2m)}\left[\theta^1(x_1,\ldots,x_{2m-1}),\ldots,\theta^{2m-1}(x_1,\ldots,x_{2m-1})\right] d\theta^i \\ &= e^{x_{2m}} \sum_{i=1}^{2m-1} c_i(x_1,\ldots,x_{2m-1})\, dx^i, \quad x \in U, \end{aligned}$$

for some $c_i \in C^\infty$; therefore, (13.13) is fulfilled.

Step 3.2 (conclusion). Using (13.11) and (13.12), we get that

$$i^*_{(x_{2m-1},x_{2m})}(\psi^*(w))$$

satisfies the induction hypothesis for $m-1$, for x_{2m-1}, x_{2m} small. Moreover, note that using (13.13),

$$i^*_{(x_{2m-1},x_{2m})}(\psi^*(w))(x_1,\ldots,x_{2m-2}) = e^{x_{2m}} \sum_{i=1}^{2m-2} c_i(x_i,\ldots,x_{2m-1})\, dx^i.$$

Hence, by the induction hypothesis and thanks to the special form of the coefficients of

$$i^*_{(x_{2m-1},x_{2m})}(\psi^*(w))$$

with respect to x_{2m}, there exist a neighborhood $\widehat{U}$ of 0 in $\mathbb{R}^{2m-2}$ and, for every x_{2m-1} small, $\phi_{x_{2m-1}} \in \operatorname{Diff}^\infty(\widehat{U};\phi_{x_{2m-1}}(\widehat{U}))$, verifying

$$\left((\phi_{x_{2m-1}})^*(i^*_{(x_{2m-1},x_{2m})}(\psi^*(w))\right)_{2i} = 0 \quad \text{in } U, \quad 1 \le i \le m-1. \tag{13.16}$$

Furthermore, since the construction is smooth in the parameters, we have in fact

$$(x_1,\ldots,x_{2m-1}) \to \phi_{x_{2m-1}}(x_1,\ldots,x_{2m-2}) \quad \text{is } C^\infty.$$

Define, for a neighborhood U of 0 small enough, $\phi \in \operatorname{Diff}^\infty(U;\phi(U))$ by

$$\phi(x) = \phi(x_1,\ldots,x_{2m}) = (\phi_{x_{2m-1}}(x_1,\ldots,x_{2m-2}),x_{2m-1},x_{2m}).$$

Since $\phi \circ i_{(x_{2m-1},x_{2m})} = i_{(x_{2m-1},x_{2m})} \circ \phi_{x_{2m-1}}$, we obtain

$$(\phi_{x_{2m-1}})^*(i^*_{(x_{2m-1},x_{2m})}(\psi^*(w))) = i^*_{(x_{2m-1},x_{2m})}(\phi^*(\psi^*(w))).$$

Note also that for every $1 \le s \le 2m-2$ and for every 1-form g,

$$\left[i^*_{(x_{2m-1},x_{2m})}(g)(x_1,\ldots,x_{2m-2})\right]_s = [g(x_1,\ldots,x_{2m-2},x_{2m-1},x_{2m})]_s\,.$$

Therefore, combining (13.16) with the above two equations, one gets

$$[\phi^*(\psi^*(w))]_{2i} = 0 \quad \text{in } U, \quad 1 \le i \le m-1.$$

Moreover, since the first $(2m-1)$ components of ϕ do not depend on x_{2m}, we obtain, using (13.13),

$$[\phi^*(\psi^*(w))]_{2m} = 0 \quad \text{in } U.$$

Finally, letting $\varphi = \psi \circ \phi$, we have indeed found the desired diffeomorphism. □