Chapter 11
The Case Without Sign Hypothesis on *f*

11.1 Main Result

The aim of this chapter is to solve the problem

$$\begin{cases} g(\varphi(x))\det\nabla\varphi(x) = f(x), & x\in\Omega, \\ \varphi(x) = x, & x\in\partial\Omega, \end{cases}$$

equivalently written as

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \mathrm{id} & \text{on } \partial\Omega, \end{cases} \tag{11.1}$$

with $g > 0$ in $\mathbb{R}^n$ but with no sign restriction on f. Of course, the solution cannot be a diffeomorphism; nevertheless, if $f \geq 0$ and under further restrictions, it can be a homeomorphism (see Theorem 11.1(iii)).

The main result of this chapter, established by Cupini, Dacorogna and Kneuss [25], is the following. In the sequel, we denote by B_R the open ball of radius R centered at the origin.

Theorem 11.1. *Let $n \geq 2$ and $r \geq 1$ be integers and Ω a bounded open set in $\mathbb{R}^n$ such that $\overline{\Omega}$ is C^{r+1}-diffeomorphic to $\overline{B}_1$. Let $g \in C^r(\mathbb{R}^n)$ with $g > 0$ and $f \in C^r(\overline{\Omega})$ be such that*

$$\int_\Omega g = \int_\Omega f .$$

Then for every $\varepsilon > 0$, there exists $\varphi = \varphi_\varepsilon \in C^r(\overline{\Omega};\mathbb{R}^n)$ satisfying (11.1), namely

$$\begin{cases} \varphi^*(g) = f & \textit{in } \Omega, \\ \varphi = \mathrm{id} & \textit{on } \partial\Omega \end{cases}$$

and

$$\overline{\Omega} \subset \varphi\left(\overline{\Omega}\right) \subset \overline{\Omega} + B_\varepsilon .$$

G. Csató et al., *The Pullback Equation for Differential Forms*, Progress in Nonlinear Differential Equations and Their Applications 83, DOI 10.1007/978-0-8176-8313-9_11,

Moreover, the three following properties hold:

(i) If either $f > 0$ on $\partial\Omega$ or $f \geq 0$ in $\overline{\Omega}$, then ε can be taken to be 0. In other words, there exists $\varphi \in C^r(\overline{\Omega};\overline{\Omega})$ satisfying (11.1).

(ii) If $\operatorname{supp}(g-f) \subset \Omega$, then φ can be chosen such that

$$\varphi \in C^r(\overline{\Omega};\overline{\Omega}) \quad \textit{and} \quad \operatorname{supp}(\varphi - \mathrm{id}) \subset \Omega.$$

(iii) If $f \geq 0$ in $\overline{\Omega}$ and $f^{-1}(0) \cap \Omega$ is countable, then φ can be chosen such that

$$\varphi \in C^r(\overline{\Omega};\overline{\Omega}) \cap \operatorname{Hom}(\overline{\Omega};\overline{\Omega}).$$

Remark 11.2. (i) Note that, in view of (19.2), we always have $\overline{\Omega} \subset \varphi(\overline{\Omega})$ as soon as $\varphi = \mathrm{id}$ on $\partial\Omega$.

(ii) In general, without further hypothesis on f as the extra statement (i), it is not possible to find a solution that remains in $\overline{\Omega}$. In fact, if f is negative in some part of $\partial\Omega$, then any solution must go out of $\overline{\Omega}$ (cf. Proposition 11.3).

(iii) The above theorem is also valid in Hölder spaces.

The proof of the theorem will be discussed in Section 11.3, but we want to explain the two main steps. First, observe that the fact that f is not strictly positive precludes the use of either the flow method or the fixed point method developed in Chapter 10; the proof will be more constructive. Here are the main steps for Ω the unit ball. The idea is to look for radial solutions of the problem; however, to achieve this, we have to rearrange f in an appropriate way. We therefore will look for solutions of the form

$$\varphi = \psi \circ \chi^{-1}$$

with $\psi = \chi = \mathrm{id}$ on $\partial\Omega$.

— First, we rearrange f with a diffeomorphism χ, so that

$$f_1 = \chi^*(f)$$

satisfies $f_1(0) > 0$ and has nice symmetry properties, for instance, among others,

$$\int_0^r s^{n-1} f_1\left(s\frac{x}{|x|}\right) ds > 0 \quad \text{for every } x \neq 0 \text{ and } r \in (0,1].$$

This will be the most difficult part of our proof and will be achieved in Section 11.6 (with the help of Section 11.5). Note that in view of Proposition 11.6 the function f_1 cannot therefore be strictly positive if f is not strictly positive.

— We then find a map ψ so that

$$\psi^*(g) = f_1\,.$$

This will be achieved using Section 11.4 and Chapter 10. Note that the map ψ cannot be a diffeomorphism if f_1 vanishes even at a single point.

11.2 Remarks and Related Results

In this section Ω will be a bounded open set in $\mathbb{R}^n$. We start by showing that if $f < 0$ in some parts of $\partial\Omega$, then any solution of

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \text{id} & \text{on } \partial\Omega \end{cases} \tag{11.2}$$

must go out of $\overline{\Omega}$—more precisely,

$$\overline{\Omega} \underset{\neq}{\subset} \varphi\left(\overline{\Omega}\right).$$

We recall, using (19.2), that we necessarily have

$$\overline{\Omega} \subset \varphi(\overline{\Omega}).$$

Proposition 11.3. *Let Ω be a bounded open C^1 set in $\mathbb{R}^n$ and $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ with $\varphi = \text{id}$ on $\partial\Omega$. If there exists $\overline{x} \in \partial\Omega$ such that* $\det\nabla\varphi(\overline{x}) < 0$, *then*

$$\overline{\Omega} \underset{\neq}{\subset} \varphi(\overline{\Omega}). \tag{11.3}$$

Proof. We divide the proof into two steps.

Step 1 (simplification). Since Ω is C^1 (cf. Definition 16.5), there exists $\psi \in \text{Diff}^1(\overline{B}_1;\psi(\overline{B}_1))$ with $\psi(0) = \overline{x}$ and

$$\psi(\overline{B}_1 \cap \{x_n = 0\}) \subset \partial\Omega,$$
$$\psi(\overline{B}_1 \cap \{x_n > 0\}) \subset \Omega,$$
$$\psi(\overline{B}_1 \cap \{x_n < 0\}) \subset (\overline{\Omega})^c.$$

Therefore, using that $\varphi(\overline{x}) = \overline{x}$, we can choose $\varepsilon > 0$ small enough so that

$$\widetilde{\varphi} : \overline{B}_\varepsilon \cap \{x_n \geq 0\} \to \mathbb{R}^n \quad \text{with} \quad \widetilde{\varphi}(x) = \psi^{-1}(\varphi(\psi(x)))$$

is well defined. We observe that $\widetilde{\varphi}$ satisfies

$$\widetilde{\varphi} = \text{id on } \overline{B}_\varepsilon \cap \{x_n = 0\} \quad \text{and} \quad \det\nabla\widetilde{\varphi}(0) = \det\nabla\varphi(\overline{x}) < 0. \tag{11.4}$$

To prove (11.3) it is enough to show that

$$\widetilde{\varphi}(\overline{B}_{\varepsilon'} \cap \{x_n > 0\}) \subset \{x_n < 0\} \tag{11.5}$$

for a certain $0 < \varepsilon' \leq \varepsilon$.

Step 2. We finally show (11.5). Using (11.4), we immediately obtain

$$\frac{\partial \widetilde{\varphi}_n}{\partial x_n}(0) = \det \nabla \widetilde{\varphi}(0) < 0,$$

and therefore, by continuity, there exists $0 < \varepsilon' \leq \varepsilon$ such that

$$\frac{\partial \widetilde{\varphi}_n}{\partial x_n} < 0 \quad \text{in } B_{\varepsilon'} \cap \{x_n > 0\}. \tag{11.6}$$

Combining (11.6) and the fact that $\widetilde{\varphi}_n(0) = 0$ (by (11.4)), we get (11.5). □

We now discuss the special case $n = 1$ in the context $g > 0$ and with no sign restriction on f.

Proposition 11.4. *Let $n = 1$, $r \geq 0$, $\Omega = (a,b)$, $g \in C^r(\mathbb{R})$ with $g > 0$ and $f \in C^r([a,b])$. Let*

$$F(x) = \int_a^x f(t)\,dt \quad \text{and} \quad G(x) = \int_a^x g(t)\,dt.$$

Then there exists $\varphi \in C^{r+1}([a,b];\mathbb{R})$ a solution of (11.2) if and only if

$$F(b) = G(b) \quad \text{and} \quad F([a,b]) \subset G(\mathbb{R}).$$

Remark 11.5. Let F and G be as in the proposition with $F(b) = G(b)$. Then the following statements are verified:

(i) We always have

$$G([a,b]) \subset F([a,b]).$$

Moreover, when $f \geq 0$, the previous inclusion is an equality.

(ii) In general,

$$F([a,b]) \subsetneq G([a,b]).$$

This is for example always the case when $f(a) < 0$ or $f(b) < 0$.

(iii) The inclusion

$$F([a,b]) \subset G(\mathbb{R})$$

is not always fulfilled.

Proof. Step 1. First, note that Problem (11.2) becomes

$$\begin{cases} G(\varphi(x)) = F(x), & x \in (a,b), \\ G(b) = F(b). \end{cases}$$

Indeed, (11.2) is equivalent to

$$\begin{cases} [G(\varphi(x))]' = F'(x) & \text{if } x \in (a,b), \\ \varphi(a) = a \quad \text{and} \quad \varphi(b) = b. \end{cases}$$

We therefore get

$$G(\varphi(x)) = F(x) + c.$$

Since $\varphi(a) = a$ and $G(a) = F(a)$, we deduce that $c = 0$ and thus our claim.

Step 2. Since G is strictly monotone (because $g > 0$), the solution φ (if it exists) is given by

$$\varphi(x) = G^{-1}(F(x)).$$

Therefore, the conclusion easily follows. □

We now show that Problem (11.2) is not symmetric in g and f.

Proposition 11.6. *Let $g \in C^0(\mathbb{R}^n)$ with $g^{-1}(0) \cap \overline{\Omega} \neq \emptyset$ and $f \in C^0(\overline{\Omega})$ with $f > 0$ in $\overline{\Omega}$. Then no $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ can satisfy (11.2).*

Proof. We proceed by contradiction. Assume that $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ is a solution of (11.2). Since $\varphi = \mathrm{id}$ on $\partial\Omega$, then (see (19.2) below)

$$\varphi(\overline{\Omega}) \supset \overline{\Omega}.$$

Thus, there exists $z \in \overline{\Omega}$ such that $\varphi(z) \in \overline{\Omega}$ and $g(\varphi(z)) = 0$, which is the desired contradiction, since

$$g(\varphi(z)) \det \nabla\varphi(z) = f(z) > 0.$$

The proposition is therefore proved. □

In the following proposition, we state a necessary condition (see (11.7) below) for the existence of a one-to-one solution of (11.2). Moreover, we show that not all solutions of (11.2), verifying (11.7), are one-to-one.

Proposition 11.7. *Let*

$$g \in C^0(\mathbb{R}^n), \quad g > 0 \text{ in } \mathbb{R}^n, \quad f \in C^0(\overline{\Omega}) \quad \text{and} \quad \int_\Omega f = \int_\Omega g.$$

Then the following claims hold true:

(i) If $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ is a one-to-one solution of (11.2), then $\varphi \in \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$ and

$$f \geq 0 \quad \text{and} \quad \mathrm{int}(f^{-1}(0)) = \emptyset. \tag{11.7}$$

(ii) There exists f satisfying (11.7) such that not all solutions $\varphi \in C^1(\overline{\Omega};\mathbb{R}^n)$ of (11.2) are one-to-one.

Proof. (i) By Lemma 19.11, we have that $\varphi \in \mathrm{Hom}(\overline{\Omega};\overline{\Omega})$. Applying Proposition 19.14, we have the claim.

(ii) We provide a counterexample in two dimensions. Let $f \in C^1(\overline{B}_1)$ be such that $f \geq 0$,

$$f^{-1}(0) = \{(t,0) : t \in [1/2, 3/4]\}, \quad f \equiv 1 \text{ in a neighborhood of } 0$$

and, for every $x \neq 0$,

$$\int_0^1 s f\left(s\frac{x}{|x|}\right) ds = \frac{1}{2}.$$

Define next $\alpha : \overline{B}_1 \to [0,1]$, through $\alpha(0) = 0$ and, for $0 < |x| \leq 1$,

$$\frac{\alpha(x)^2}{2} = \int_0^{\alpha(x)} s\, ds = \int_0^{|x|} s f\left(s\frac{x}{|x|}\right) ds.$$

As in Step 2 of the proof of Lemma 11.11 (with $g = 1$), the map

$$\varphi(x) = \alpha(x)\frac{x}{|x|}$$

is in $C^1(\overline{B}_1; \overline{B}_1)$, with

$$\varphi^*(1) = f \quad \text{and} \quad \varphi = \mathrm{id} \text{ on } \partial B_1.$$

Since $\varphi(1/2,0) = \varphi(3/4,0)$, φ is not one-to-one. □

The next proposition can be proved with the same technique as the one developed in this chapter and we refer to [60] for details.

Proposition 11.8. *Let $r \geq 1$ and $n \geq 2$ be integers. Let $g \in C^r(\mathbb{R}^n)$ with $g > 0$ in $\mathbb{R}^n$, $f \in C^r(\overline{B}_1)$ satisfying*

$$\int_{B_1} g = \int_{B_1} f.$$

Then there exist $\delta = \delta(n,r,g,f)$ and $\gamma = \gamma(n,r,g,f)$ such that for every $g_1, g_2 \in C^r(\mathbb{R}^n)$, $f_1, f_2 \in C^r(\overline{B}_1)$ satisfying, for $i = 1,2$,

$$\int_{B_1} g_i = \int_{B_1} f_i, \quad \|f_i - f\|_{C^r(B_1)} \leq \delta \quad \text{and} \quad \|g_i - g\|_{C^r(B_2)} \leq \delta,$$

there exist $\varphi_i \in C^r(\overline{B}_1; B_2)$, $i = 1,2$, such that for every $0 \leq k \leq r-1$,

$$\varphi_i^*(g_i) = f_i \text{ in } B_1, \quad \varphi_i = \mathrm{id} \text{ on } \partial B_1,$$

$$\|\varphi_1 - \varphi_2\|_{C^k(\overline{B}_1)} \leq \gamma(\|f_1 - f_2\|_{C^k(\overline{B}_1)} + \|g_1 - g_2\|_{C^k(\overline{B}_2)}),$$

$$\|\varphi_i\|_{C^r(\overline{B}_1)} \leq \gamma.$$

Remark 11.9. We can make the conclusion of the proposition more precise. In the sense that for every $\varepsilon > 0$, by letting δ and γ depending of ε we can replace B_2 above by $B_{1+\varepsilon}$.

11.3 Proof of the Main Result

We can now discuss the proof of the main theorem. For the sake of simplicity, we will split it into two proofs. First, we establish the main statement of the theorem and then we show its three extra statements.

Proof. We divide the proof into five steps and we fix $\varepsilon > 0$.

Step 1 (transfer of the problem into the ball). Since $\overline{\Omega}$ is C^{r+1}-diffeomorphic to $\overline{B}_1$, there exists $\varphi_1 \in \operatorname{Diff}^{r+1}(\overline{B}_1;\overline{\Omega})$. With no loss of generality we can assume that $\det\nabla\varphi_1 > 0$. Indeed, if $\det\nabla\varphi_1 < 0$ (note that since φ_1 is a diffeomorphism, then $\det\nabla\varphi_1 \neq 0$ everywhere), then replace $\varphi_1(x)$ by $\varphi_1(-x_1,x_2,\dots,x_n)$. Using Corollary 16.15, we extend φ_1 and choose $\varepsilon_1 > 0$ small enough so that $\varphi_1 \in \operatorname{Diff}^{r+1}(\overline{B}_{1+\varepsilon_1};\varphi_1(\overline{B}_{1+\varepsilon_1}))$ with

$$\varphi_1(\overline{B}_{1+\varepsilon_1}) \subset \overline{\Omega} + B_\varepsilon\,.$$

Define

$$f_1 = \varphi_1^*(f) \in C^r(\overline{B}_1) \quad \text{and} \quad g_1 = \varphi_1^*(g) \in C^r(\overline{B}_{1+\varepsilon_1}).$$

By the change of variables formula, we have that

$$\int_{B_1} f_1 = \int_\Omega f = \int_\Omega g = \int_{B_1} g_1 > 0. \tag{11.8}$$

Step 2 (positive radial integration). Since (11.8) holds, we may apply Lemma 11.21 to f_1. Therefore, there exists $\varphi_2 \in \operatorname{Diff}^\infty(\overline{B}_1;\overline{B}_1)$ with

$$\operatorname{supp}(\varphi_2 - \mathrm{id}) \subset B_1$$

such that, letting $f_2 = \varphi_2^*(f_1) \in C^r(\overline{B}_1)$, we have $f_2(0) > 0$ and

$$\int_0^r s^{n-1} f_2\left(s\frac{x}{|x|}\right) ds > 0 \quad \text{for every } x \neq 0 \text{ and } r \in (0,1]$$

$$\int_r^1 s^{n-1} f_2\left(s\frac{x}{|x|}\right) ds > -\frac{\displaystyle\int_{B_{1+\varepsilon_1}} g_1 - \int_{B_1} g_1}{n\operatorname{meas}(B_1)} \quad \text{for every } x \neq 0 \text{ and } r \in [0,1].$$

The change of variables formula and (11.8) lead to

$$\int_{B_1} f_2 = \int_{B_1} \varphi_2^*(f_1) = \int_{B_1} f_1 = \int_{B_1} g_1\,. \tag{11.9}$$

Step 3 (radial solution). By the previous step, f_2 satisfies all of the hypotheses of Lemma 11.10 (with $m = \int_{B_{1+\varepsilon_1}} g_1$). Therefore, there exist $g_2 \in C^r(\mathbb{R}^n)$ with $g_2 > 0$ in $\mathbb{R}^n$ and

$$\int_{B_{1+\varepsilon_1}} g_2 = \int_{B_{1+\varepsilon_1}} g_1$$

and $\varphi_3 \in C^r(\overline{B}_1;\overline{B}_{1+\varepsilon_1})$ verifying

$$\begin{cases} \varphi_3^*(g_2) = f_2 & \text{in } B_1, \\ \varphi_3 = \text{id} & \text{on } \partial B_1. \end{cases}$$

Note that, using (19.3),

$$\int_{B_1} g_2 = \int_{B_1} f_2$$

and therefore, by (11.9),

$$\int_{B_1} g_2 = \int_{B_1} g_1 .$$

Step 4 (positive resolution). Since $g_1, g_2 \in C^r(\overline{B}_{1+\varepsilon_1})$, $g_1, g_2 > 0$ in $\overline{B}_{1+\varepsilon_1}$,

$$\int_{B_1} g_1 = \int_{B_1} g_2 \quad \text{and} \quad \int_{B_{1+\varepsilon_1}} g_1 = \int_{B_{1+\varepsilon_1}} g_2 ,$$

there exists, using Corollary 10.8, $\varphi_4 \in \operatorname{Diff}^r(\overline{B}_{1+\varepsilon_1};\overline{B}_{1+\varepsilon_1})$ such that

$$\begin{cases} \varphi_4^*(g_1) = g_2 & \text{in } B_{1+\varepsilon_1}, \\ \varphi_4 = \text{id} & \text{on } \partial B_1 \cup \partial B_{1+\varepsilon_1}. \end{cases}$$

Step 5 (conclusion). By the above steps, we have that

$$\varphi = \varphi_1 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2^{-1} \circ \varphi_1^{-1} \in C^r(\overline{\Omega};\mathbb{R}^n)$$

satisfies

$$\overline{\Omega} \subset \varphi(\overline{\Omega}) \subset \overline{\Omega} + B_\varepsilon,$$

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \text{id} & \text{on } \partial\Omega. \end{cases}$$

Indeed, for $x \in \partial\Omega$, since $\varphi_1(\partial B_1) = \partial\Omega$ (see Theorem 19.6) and $\varphi_i = \text{id}$ on ∂B_1, $i = 2,3,4$, we have

$$\begin{aligned} \varphi(x) &= \varphi_1 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2^{-1} \circ \varphi_1^{-1}(x) \\ &= \varphi_1(\varphi_1^{-1}(x)) = x. \end{aligned}$$

Thus, using (19.2), we have that $\overline{\Omega} \subset \varphi(\overline{\Omega})$. Noticing that

$$\varphi_1^{-1}(\overline{\Omega}) = \overline{B}_1, \quad \varphi_2^{-1}(\overline{B}_1) = \overline{B}_1, \quad \varphi_3(\overline{B}_1) \subset \overline{B}_{1+\varepsilon_1},$$

$$\varphi_4(\overline{B}_{1+\varepsilon_1}) = \overline{B}_{1+\varepsilon_1} \quad \text{and} \quad \varphi_1(\overline{B}_{1+\varepsilon_1}) \subset \overline{\Omega} + B_\varepsilon,$$

we have

$$\varphi(\overline{\Omega}) \subset \overline{\Omega} + B_\varepsilon .$$

Eventually, using several times the third statement in Theorem 3.10,

$$\begin{aligned}
\varphi^*(g) &= \left(\varphi_1 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2^{-1} \circ \varphi_1^{-1}\right)^*(g) \\
&= (\varphi_1^{-1})^*\left((\varphi_2^{-1})^*\left(\varphi_3^*\left(\varphi_4^*\left(\varphi_1^*(g)\right)\right)\right)\right) \\
&= (\varphi_1^{-1})^*\left((\varphi_2^{-1})^*\left(\varphi_3^*\left(\varphi_4^*(g_1)\right)\right)\right) \\
&= (\varphi_1^{-1})^*\left((\varphi_2^{-1})^*\left(\varphi_3^*(g_2)\right)\right) \\
&= (\varphi_1^{-1})^*\left((\varphi_2^{-1})^*(f_2)\right) \\
&= (\varphi_1^{-1})^*(f_1) = f,
\end{aligned}$$

which concludes the proof. □

We now prove the three extra statements of Theorem 11.1.

Proof. We divide the proof into seven steps.

Step 1 (transfer of the problem into the ball). Since $\overline{\Omega}$ is C^{r+1}-diffeomorphic to $\overline{B}_1$, there exists $\varphi_1 \in \operatorname{Diff}^{r+1}(\overline{B}_1;\overline{\Omega})$. With no loss of generality we can assume that $\det\nabla\varphi_1 > 0$. Indeed, if $\det\nabla\varphi_1 < 0$, then replace $\varphi_1(x)$ by $\varphi_1(-x_1,x_2,\ldots,x_n)$. Define

$$f_1 = \varphi_1^*(f) \in C^r(\overline{B}_1) \quad \text{and} \quad g_1 = \varphi_1^*(g) \in C^r(\overline{B}_1).$$

From the change of variables formula, we get

$$\int_{B_1} f_1 = \int_\Omega f = \int_\Omega g = \int_{B_1} g_1 > 0. \tag{11.10}$$

We notice the following facts:

(i) If $f > 0$ on $\partial\Omega$, then

$$f_1 > 0 \quad \text{on } \partial B_1 \tag{11.11}$$

since $\varphi_1(\partial B_1) = \partial\Omega$ by the invariance of domain theorem (see Theorem 19.6).

(ii) If $\operatorname{supp}(g-f) \subset \Omega$, then

$$\operatorname{supp}(g_1 - f_1) \subset B_1\,. \tag{11.12}$$

(iii) If $f \geq 0$ in $\overline{\Omega}$, then

$$f_1 \geq 0 \quad \text{in } \overline{B}_1 \tag{11.13}$$

since $\det\nabla\varphi_1 > 0$ in $\overline{B}_1$.

(iv) If $f \geq 0$ in $\overline{\Omega}$ and $f^{-1}(0)\cap\Omega$ is countable, then

$$f_1 \geq 0 \text{ in } \overline{B}_1 \quad \text{and} \quad f_1^{-1}(0)\cap B_1 \quad \text{is countable.} \tag{11.14}$$

Step 2 (positive radial integration). Applying Corollary 11.23 to f_1, which is justified by (11.10) and (11.11) if $f > 0$ on $\partial\Omega$ and by (11.10) and (11.13) if $f \geq 0$ in $\overline{\Omega}$, we can find $\varphi_2 \in \operatorname{Diff}^\infty(\overline{B}_1;\overline{B}_1)$ with

$$\operatorname{supp}(\varphi_2 - \mathrm{id}) \subset B_1$$

such that, letting $f_2 = \varphi_2^*(f_1) \in C^r(\overline{B}_1)$, we have $f_2(0) > 0$ and

$$\int_0^r s^{n-1} f_2\left(s\frac{x}{|x|}\right) ds > 0 \quad \text{for every } x \neq 0 \text{ and } r \in (0,1],$$

$$\int_r^1 s^{n-1} f_2\left(s\frac{x}{|x|}\right) ds \geq 0, \quad \text{for every } x \neq 0 \text{ and } r \in [0,1].$$

Moreover, using the change of variables formula and (11.10), we obtain

$$\int_{B_1} f_2 = \int_{B_1} \varphi_2^*(f_1) = \int_{B_1} f_1 = \int_{B_1} g_1 \,. \tag{11.15}$$

Finally, we notice the two following facts:

(i) If $\operatorname{supp}(g - f) \subset \Omega$, then by (11.12) and since $\operatorname{supp}(\varphi_2 - \mathrm{id}) \subset B_1$, we have

$$\operatorname{supp}(g_1 - f_2) \subset B_1 \,. \tag{11.16}$$

(ii) If $f \geq 0$ in $\overline{\Omega}$ and $f^{-1}(0) \cap \Omega$ is countable, then by (11.14), we get that

$$f_2 \geq 0 \text{ in } \overline{B}_1 \quad \text{and} \quad f_2^{-1}(0) \cap B_1 \quad \text{is countable.} \tag{11.17}$$

Step 3 (radial solution). Since f_2 satisfies all the hypotheses of Lemma 11.11, there exist $g_2 \in C^r(\overline{B}_1)$ with $g_2 > 0$ in $\overline{B}_1$ and $\varphi_3 \in C^r(\overline{B}_1; \overline{B}_1)$ verifying

$$\begin{cases} \varphi_3^*(g_2) = f_2 & \text{in } B_1, \\ \varphi_3 = \mathrm{id} & \text{on } \partial B_1 \,. \end{cases}$$

Note that using (19.3),

$$\int_{B_1} g_2 = \int_{B_1} f_2$$

and therefore, using (11.15),

$$\int_{B_1} g_2 = \int_{B_1} g_1 \,.$$

We, moreover, have the two following facts:

(i) If $\operatorname{supp}(g - f) \subset \Omega$ (which implies, in particular, by (11.16) that $f_2 > 0$ on ∂B_1), the first extra statement of Lemma 11.11 implies that g_2 and φ_3 can be chosen so that

$$\operatorname{supp}(g_2 - f_2) \subset B_1 \quad \text{and} \quad \operatorname{supp}(\varphi_3 - \mathrm{id}) \subset B_1 \,. \tag{11.18}$$

(ii) If $f \geq 0$ in $\overline{\Omega}$ and $f^{-1}(0) \cap \Omega$ is countable (which implies by (11.17) that $f_2 \geq 0$ in $\overline{B}_1$ and $f_2^{-1}(0) \cap B_1$ is countable), the second extra statement of Lemma 11.11 implies that φ_3 can be chosen so that

$$\varphi_3 \in \operatorname{Hom}(\overline{B}_1; \overline{B}_1). \tag{11.19}$$

Step 4 (positive resolution). Since $g_1, g_2 \in C^r(\overline{B}_1)$, $g_1, g_2 > 0$ in $\overline{B}_1$ and

$$\int_{B_1} g_1 = \int_{B_1} g_2\,,$$

using Theorem 10.7, we can find $\varphi_4 \in \mathrm{Diff}^r(\overline{B}_1; \overline{B}_1)$ such that

$$\begin{cases} \varphi_4^*(g_1) = g_2 & \text{in } B_1, \\ \varphi_4 = \mathrm{id} & \text{on } \partial B_1\,. \end{cases}$$

We, moreover, have the following fact: If $\mathrm{supp}(g-f) \subset \Omega$, then by (11.16) and (11.18) we get that $\mathrm{supp}(g_1 - g_2) \subset B_1$. Therefore, using Theorem 10.11 instead of Theorem 10.7, we can furthermore assume that

$$\mathrm{supp}(\varphi_4 - \mathrm{id}) \subset B_1\,. \tag{11.20}$$

Step 5 (conclusion). Using the above steps, we have that

$$\varphi = \varphi_1 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2^{-1} \circ \varphi_1^{-1} \in C^r(\overline{\Omega}; \overline{\Omega})$$

satisfies

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \mathrm{id} & \text{on } \partial\Omega. \end{cases}$$

Indeed, for $x \in \partial\Omega$, since $\varphi_1(\partial B_1) = \partial\Omega$ (see Theorem 19.6) and $\varphi_i = \mathrm{id}$ on ∂B_1, $i = 2, 3, 4$, we have

$$\begin{aligned} \varphi(x) &= \varphi_1 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2^{-1} \circ \varphi_1^{-1}(x) \\ &= \varphi_1(\varphi_1^{-1}(x)) = x. \end{aligned}$$

Since $\varphi_1^{-1}(\overline{\Omega}) = \overline{B}_1$, $(\varphi_2)^{-1}(\overline{B}_1) = \overline{B}_1$, $\varphi_4(\overline{B}_1) = \overline{B}_1$, $\varphi_3(\overline{B}_1) = \overline{B}_1$ (by (19.2)) and $\varphi_1(\overline{B}_1) = \overline{\Omega}$, we have that

$$\varphi(\overline{\Omega}) = \overline{\Omega}.$$

Finally, exactly as in Step 5 of the previous proof, we prove that

$$\varphi^*(g) = f \quad \text{in } \Omega,$$

which shows the first extra statement.

Step 6. We show the second extra assertion. If $\mathrm{supp}(g-f) \subset \Omega$, then (11.18) and (11.20) imply the result, since

$$\mathrm{supp}(\varphi_2 - \mathrm{id}),\ \mathrm{supp}(\varphi_3 - \mathrm{id}),\ \mathrm{supp}(\varphi_4 - \mathrm{id}) \subset B_1\,.$$

Step 7. Finally, we show the third extra assertion. If $f \geq 0$ in $\overline{\Omega}$ and $f^{-1}(0) \cap \Omega$ is countable, then (11.19) implies the assertion since φ_1, φ_2 and φ_4 are diffeomorphisms. $\square$

11.4 Radial Solution

In this section we give sufficient conditions on f in order to have a positive g and a radial solution φ of (11.2) in the unit ball (i.e., a solution of the form $\alpha(x)x/|x|$ with $\alpha : \overline{B}_1 \to \mathbb{R}$). For the sake of simplicity, we split the discussion into two lemmas.

Lemma 11.10. *Let $r \geq 1$ be an integer, $m > 0$ and $f \in C^r(\overline{B}_1)$ be such that*

$$f(0) > 0, \quad m > \int_{B_1} f,$$

$$\int_0^r s^{n-1} f\left(s\frac{x}{|x|}\right) ds > 0 \quad \textit{for every } x \neq 0 \textit{ and } r \in (0,1], \tag{11.21}$$

$$\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds > -\frac{m - \int_{B_1} f}{n\,\mathrm{meas}(B_1)} \quad \textit{for every } x \neq 0 \textit{ and } r \in [0,1]. \tag{11.22}$$

Then for every $\varepsilon > 0$, there exist $g = g_{m,\varepsilon} \in C^r(\mathbb{R}^n)$ with $g > 0$ in $\mathbb{R}^n$ and

$$\int_{B_{1+\varepsilon}} g = m$$

and $\varphi = \varphi_{m,\varepsilon} \in C^r(\overline{B}_1; B_{1+\varepsilon})$ such that

$$\begin{cases} \varphi^*(g) = f & \textit{in } B_1, \\ \varphi = \mathrm{id} & \textit{on } \partial B_1\,. \end{cases}$$

Proof. We split the proof into two steps. Fix $\varepsilon > 0$.

Step 1 (construction of g). In this step we construct a function $g \in C^r(\mathbb{R}^n)$ with the following properties:

$$g > 0 \text{ in } \mathbb{R}^n, \quad g = f \text{ in a neighborhood of } 0, \quad \int_{B_{1+\varepsilon}} g = m,$$

$$\int_0^1 s^{n-1} g\left(s\frac{x}{|x|}\right) ds = \int_0^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds \quad \text{for every } x \neq 0, \tag{11.23}$$

$$\int_0^{1+\varepsilon} s^{n-1} g\left(s\frac{x}{|x|}\right) ds > \int_0^r s^{n-1} f\left(s\frac{x}{|x|}\right) ds \quad \text{for every } x \neq 0 \text{ and } r \in [0,1]\,. \tag{11.24}$$

Step 1.1 (preliminaries). Since $f(0) > 0$ and (11.21) and (11.22) hold, there exists $\delta > 0$ small enough such that

$$f > 0 \quad \text{in } B_\delta\,, \quad \min_{x \neq 0} \int_\delta^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds > 0, \tag{11.25}$$

$$\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds > -\frac{m - \int_{B_1} f}{n\,\mathrm{meas}(B_1)} + \delta \quad \text{for every } x \neq 0 \text{ and } r \in [0,1]. \tag{11.26}$$

Let $\eta \in C^\infty([0,\infty);[0,1])$ be such that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \delta/2 \\ 0 & \text{if } \delta \leq s. \end{cases}$$

Define then $\overline{h} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$\overline{h}(x) = \frac{\int_0^1 s^{n-1}(1-\eta(s)) f\left(s\frac{x}{|x|}\right) ds}{\int_0^1 s^{n-1}(1-\eta(s))\, ds}.$$

It is easily seen that $\overline{h} \in C^r(\mathbb{R}^n \setminus \{0\})$,

$$\overline{h}(x) = \overline{h}(\lambda x) \text{ for every } \lambda > 0,$$

and, using (11.25),

$$\overline{h} > 0, \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Now define, for $x \in \mathbb{R}^n$,

$$h(x) = \eta(|x|) f(x) + (1 - \eta(|x|))\overline{h}(x).$$

Using the definition of $\overline{h}$ and η, we have that

$$\begin{cases} h \in C^r(\mathbb{R}^n), \quad h > 0 \text{ in } \mathbb{R}^n, \quad h = f \text{ in } B_{\delta/2}, \\ \int_0^1 s^{n-1} h\left(s\frac{x}{|x|}\right) ds = \int_0^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds \text{ for every } x \neq 0. \end{cases} \tag{11.27}$$

For every $0 < \mu < \varepsilon$, let $\rho_\mu \in C^\infty(\mathbb{R}^n;[0,1])$ be such that

$$\rho_\mu = \begin{cases} 1 & \text{in } \overline{B}_1, \\ 0 & \text{in } (B_{1+\mu})^c \end{cases}$$

and define

$$c_\mu = \frac{m - \int_{B_{1+\varepsilon}} \rho_\mu h}{\int_{B_{1+\varepsilon}} (1-\rho_\mu)}.$$

Integrating the last equation of (11.27) on the unit sphere, we obtain that

$$\int_{B_1} h = \int_{B_1} f$$

and, thus, we get

$$\lim_{\mu\to 0} c_\mu = \frac{m - \int_{B_1} h}{\operatorname{meas}(B_{1+\varepsilon}\setminus B_1)} = \frac{m - \int_{B_1} f}{\operatorname{meas}(B_{1+\varepsilon}\setminus B_1)} = \frac{m - \int_{B_1} f}{[(1+\varepsilon)^n - 1]\operatorname{meas}(B_1)}.$$

This implies

$$\lim_{\mu\to 0} \frac{(1+\varepsilon)^n - (1+\mu)^n}{n} c_\mu = \frac{(1+\varepsilon)^n - 1}{n} \lim_{\mu\to 0} c_\mu = \frac{m - \int_{B_1} f}{n\operatorname{meas}(B_1)}$$

and therefore, by (11.26) we can choose μ_1 small enough such that $c_{\mu_1} > 0$ and

$$\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds > -\frac{(1+\varepsilon)^n - (1+\mu_1)^n}{n} c_{\mu_1}, \quad \text{for every } x \neq 0 \text{ and } r \in [0,1]. \tag{11.28}$$

Step 1.2 (conclusion). Let us show that the function

$$g = \rho_{\mu_1} h + (1-\rho_{\mu_1}) c_{\mu_1} \in C^r(\mathbb{R}^n)$$

has all of the desired properties. Indeed, since $h > 0$ in $\mathbb{R}^n$ and $c_{\mu_1} > 0$, we have that $g > 0$ in $\mathbb{R}^n$. By definition of c_{μ_1}, we see that

$$\int_{B_{1+\varepsilon}} g = m.$$

Using the last equation of (11.27) and the fact that $g = h$ in $\overline{B}_1$, we get (11.23). We finally show (11.24). Using (11.23), this is equivalent to showing

$$\int_1^{1+\varepsilon} s^{n-1} g\left(s\frac{x}{|x|}\right) ds > -\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds \quad \text{for every } x \neq 0 \text{ and } r \in [0,1].$$

Let $x \neq 0$ and $r \in [0,1]$. We have, since $g = c_{\mu_1}$ in $\overline{B}_{1+\varepsilon}\setminus B_{1+\mu_1}$ and (11.28) holds,

$$\begin{aligned}\int_1^{1+\varepsilon} s^{n-1} g\left(s\frac{x}{|x|}\right) ds &> \int_{1+\mu_1}^{1+\varepsilon} s^{n-1} g\left(s\frac{x}{|x|}\right) ds = \int_{1+\mu_1}^{1+\varepsilon} s^{n-1} c_{\mu_1} ds \\ &= \frac{(1+\varepsilon)^n - (1+\mu_1)^n}{n} c_{\mu_1} > -\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds\end{aligned}$$

and therefore the assertion.

Step 2 (construction of φ). We will construct a solution φ of the form

$$\varphi(x) = \alpha(x)\frac{x}{|x|},$$

where $\alpha : \overline{B}_1 \to \mathbb{R}$.

Step 2.1 (definition of α). Let $\alpha : \overline{B}_1 \to \mathbb{R}$ be such that $\alpha(0) = 0$ and, for $0 < |x| \leq 1$,

$$\int_0^{\alpha(x)} s^{n-1} g\left(s\frac{x}{|x|}\right) ds = \int_0^{|x|} s^{n-1} f\left(s\frac{x}{|x|}\right) ds. \tag{11.29}$$

Since $g > 0$, using (11.21) and (11.24), we get, for every $x \in \overline{B}_1 \setminus \{0\}$, that $\alpha(x)$ is well defined and verifies $0 < \alpha(x) < 1 + \varepsilon$. Since $g = f$ in a neighborhood of 0, we obtain that

$$\alpha(x) = |x| \quad \text{in the same neighborhood of } 0.$$

By (11.23), we immediately have

$$\alpha(x) = 1 \quad \text{on } \partial B_1 .$$

Therefore, by the implicit function theorem, which can be used since $\alpha > 0$ and $g > 0$, we have that $\alpha \in C^r(\overline{B}_1 \setminus \{0\})$. Moreover, since $\alpha(x) = |x|$ in a neighborhood of 0, the function $x \to \alpha(x)/|x|$ is $C^r(\overline{B}_1)$.

Step 2.2 (conclusion). We finally show that

$$\varphi(x) = \frac{\alpha(x)}{|x|}\, x$$

is in $C^r(\overline{B}_1; B_{1+\varepsilon})$ and verifies

$$\begin{cases} \varphi^*(g) = f & \text{in } B_1, \\ \varphi = \text{id} & \text{on } \partial B_1 . \end{cases}$$

In fact, by the properties of α, it is obvious that $\varphi \in C^r(\overline{B}_1; B_{1+\varepsilon})$ and that $\varphi = \text{id}$ on ∂B_1 . Using Lemma 11.12, we obtain

$$\det \nabla\varphi(x) = \frac{\alpha^{n-1}(x)}{|x|^n} \sum_{i=1}^n x_i \frac{\partial \alpha}{\partial x_i}(x). \tag{11.30}$$

Computing the derivative of (11.29) with respect to x_i , we get

$$\begin{aligned} &\alpha^{n-1}(x) g(\varphi(x)) \frac{\partial \alpha}{\partial x_i}(x) + \sum_{j=1}^n \int_0^{\alpha(x)} s^n \frac{\partial g}{\partial x_j}\left(s\frac{x}{|x|}\right) \left(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2}\right) ds \\ &= |x|^{n-1} f(x) \frac{x_i}{|x|} + \sum_{j=1}^n \int_0^{|x|} s^n \frac{\partial f}{\partial x_j}\left(s\frac{x}{|x|}\right) \left(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2}\right) ds, \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Multiplying the above equality by x_i, adding up the terms with respect to i and using

$$\sum_{i=1}^n x_i \left(\frac{|x|\delta_{ij} - \frac{x_i x_j}{|x|}}{|x|^2}\right) = 0, \quad 1 \leq j \leq n,$$

we obtain

$$\alpha^{n-1}(x)g(\varphi(x))\sum_{i=1}^{n}x_i\frac{\partial\alpha}{\partial x_i}(x)=|x|^n f(x).$$

This equality, together with (11.30), implies $\varphi^*(g)=f$, which shows the assertion. □

Lemma 11.11. *Let $r\geq 1$ be an integer, $f\in C^r(\overline{B}_1)$ be such that $f(0)>0$ and*

$$\int_0^r s^{n-1}f\left(s\frac{x}{|x|}\right)ds>0 \quad \textit{for every } x\neq 0 \textit{ and } r\in(0,1], \tag{11.31}$$

$$\int_r^1 s^{n-1}f\left(s\frac{x}{|x|}\right)ds\geq 0 \quad \textit{for every } x\neq 0 \textit{ and } r\in[0,1]. \tag{11.32}$$

Then there exists $g\in C^r(\overline{B}_1)$ with $g>0$ in $\overline{B}_1$ and $\varphi\in C^r(\overline{B}_1;\overline{B}_1)$ such that

$$\begin{cases}\varphi^*(g)=f & \textit{in } B_1,\\ \varphi=\mathrm{id} & \textit{on } \partial B_1.\end{cases}$$

Furthermore, the following two extra properties hold:

(i) If $f>0$ on ∂B_1, then g and φ can be chosen so that

$$\operatorname{supp}(g-f)\subset B_1 \quad \textit{and} \quad \operatorname{supp}(\varphi-\mathrm{id})\subset B_1.$$

(ii) If $f\geq 0$ in $\overline{B}_1$ and

$$f^{-1}(0)\cap B_1 \quad \textit{is countable,}$$

then φ can be chosen in $\operatorname{Hom}(\overline{B}_1;\overline{B}_1)$.

Proof. The proof is essentially the same as the previous one. We split the proof into two steps.

Step 1 (construction of g). In this step we construct a function $g\in C^r(\overline{B}_1)$ with the following properties: $g>0$ in $\overline{B}_1$, $g=f$ in a neighborhood of 0 (and also $\operatorname{supp}(g-f)\subset B_1$ if $f>0$ on ∂B_1),

$$\int_0^1 s^{n-1}g\left(s\frac{x}{|x|}\right)ds=\int_0^1 s^{n-1}f\left(s\frac{x}{|x|}\right)ds \quad \text{for every } x\neq 0, \tag{11.33}$$

$$\int_0^1 s^{n-1}g\left(s\frac{x}{|x|}\right)ds\geq\int_0^r s^{n-1}f\left(s\frac{x}{|x|}\right)ds \quad \text{for every } x\neq 0 \text{ and } r\in[0,1]. \tag{11.34}$$

Step 1.1 (preliminaries). Since $f(0)>0$ and (11.31) holds, there exists $\delta>0$ small enough such that

$$f>0 \quad \text{in } B_\delta \quad \text{and} \quad \min_{x\neq 0}\int_\delta^1 s^{n-1}f\left(s\frac{x}{|x|}\right)ds>0. \tag{11.35}$$

Let $\eta \in C^\infty([0,\infty);[0,1])$ be such that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \le s \le \delta/2 \\ 0 & \text{if } \delta \le s. \end{cases}$$

If $f > 0$ on ∂B_1, we modify the definition of δ and η as follows. We assume that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \le s \le \delta/2 \text{ or } 1-\delta/2 \le s \le 1 \\ 0 & \text{if } \delta \le s \le 1-\delta, \end{cases}$$

where $\delta > 0$ small enough is such that

$$f > 0 \quad \text{in } B_\delta \cup (\overline{B}_1 \setminus B_{1-\delta}) \quad \text{and} \quad \min_{x \neq 0} \int_\delta^{1-\delta} s^{n-1} f\left(s\frac{x}{|x|}\right) ds > 0. \tag{11.36}$$

Define next $\overline{h} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$\overline{h}(x) = \frac{\int_0^1 s^{n-1}(1-\eta(s)) f(s\frac{x}{|x|})\, ds}{\int_0^1 s^{n-1}(1-\eta(s))\, ds}.$$

It is easily seen that $\overline{h} \in C^r(\mathbb{R}^n \setminus \{0\})$, that

$$\overline{h}(x) = \overline{h}(\lambda x) \text{ for every } \lambda > 0,$$

and, using (11.35) or (11.36), that

$$\overline{h} > 0.$$

Step 1.2 (conclusion). Let us show that g defined by

$$g(x) = \eta(|x|) f(x) + (1-\eta(|x|))\overline{h}(x), \quad x \in \overline{B}_1,$$

has all of the required properties. Using the definition of $\overline{h}$ and η, we see that $g \in C^r(\overline{B}_1)$ satisfies $g > 0$ in $\overline{B}_1$, (11.33) and $g = f$ in $B_{\delta/2}$ (if, moreover, $f > 0$ on ∂B_1, then $\operatorname{supp}(g-f) \subset B_1$). Finally, we show (11.34). Let $x \neq 0$ and $r \in [0,1]$. Using (11.32) and (11.33), we get

$$\begin{aligned}
\int_0^1 s^{n-1} g\left(s\frac{x}{|x|}\right) ds &= \int_0^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds \\
&= \int_0^r s^{n-1} f\left(s\frac{x}{|x|}\right) ds + \int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds \\
&\ge \int_0^r s^{n-1} f\left(s\frac{x}{|x|}\right) ds,
\end{aligned}$$

which ends the construction of g.

Step 2 (construction of φ). We will construct, as before, a solution φ of the form

$$\varphi(x) = \alpha(x)\,\frac{x}{|x|},$$

where $\alpha : \overline{B}_1 \to \mathbb{R}$.

Step 2.1 (definition of α). Let $\alpha : \overline{B}_1 \to \mathbb{R}$ be such that $\alpha(0) = 0$ and, for $0 < |x| \leq 1$,

$$\int_0^{\alpha(x)} s^{n-1} g\left(s\frac{x}{|x|}\right) ds = \int_0^{|x|} s^{n-1} f\left(s\frac{x}{|x|}\right) ds.$$

Since $g > 0$, using (11.31) and (11.34), we get for every $x \in \overline{B}_1 \setminus \{0\}$ that $\alpha(x)$ is well defined and verifies $0 < \alpha(x) \leq 1$. Since $g = f$ in a neighborhood of 0, we obtain that

$$\alpha(x) = |x| \quad \text{in the same neighborhood of } 0.$$

By (11.33), we immediately have

$$\alpha(x) = 1 \quad \text{on } \partial B_1\,.$$

Moreover, if $\operatorname{supp}(g - f) \subset B_1$, then α also verifies

$$\alpha(x) = |x| \quad \text{in a neighborhood of } \partial B_1\,. \tag{11.37}$$

Therefore, by the implicit function theorem, which can be used since $\alpha > 0$ and $g > 0$, we have that $\alpha \in C^r(\overline{B}_1 \setminus \{0\})$. Moreover, since $\alpha(x) = |x|$ in a neighborhood of 0, the map $x \to \alpha(x)x/|x|$ is $C^r(\overline{B}_1)$.

Step 2.2 (conclusion). We show that

$$\varphi(x) = \frac{\alpha(x)}{|x|}\,x$$

is in $C^r(\overline{B}_1;\overline{B}_1)$ and verifies

$$\begin{cases} \varphi^*(g) = f & \text{in } B_1, \\ \varphi = \mathrm{id} & \text{on } \partial B_1\,. \end{cases}$$

In fact, by the properties of α, it is obvious that $\varphi \in C^r(\overline{B}_1;\overline{B}_1)$ and that $\varphi = \mathrm{id}$ on ∂B_1. Finally, proceeding exactly as in Step 2.2 of the proof of Lemma 11.10, we obtain that

$$\varphi^*(g) = f \quad \text{in } B_1,$$

which concludes the proof of the main statement.

It remains to show the two extra statements.

(i) If $f > 0$ on ∂B_1, then we have $\operatorname{supp}(g - f) \subset B_1$. Hence, it follows from (11.37) that

$$\operatorname{supp}(\varphi - \mathrm{id}) \subset B_1,$$

which proves the first extra statement.

(ii) If $f \geq 0$ and

$$f^{-1}(0) \cap B_1 \quad \text{is countable,}$$

we immediately obtain

$$\alpha(x) \neq \alpha(rx) \quad \text{for every } x \in \overline{B}_1 \setminus \{0\} \text{ and } r \in [0,1)\,,$$

which implies that $\varphi \in \operatorname{Hom}(\overline{B}_1; \overline{B}_1)$ and establishes the second statement and ends the proof. □

In the proof of Lemmas 11.10 and 11.11, we used the following elementary result.

Lemma 11.12. *Let $\lambda \in C^1(\overline{B}_1)$ and $\varphi \in C^1(\overline{B}_1; \mathbb{R}^n)$ be such that $\varphi(x) = \lambda(x)x$. Then*

$$\det \nabla \varphi(x) = \lambda^n(x) + \lambda^{n-1}(x) \sum_{i=1}^{n} x_i \frac{\partial \lambda}{\partial x_i}(x).$$

In particular, if $\lambda(x) = \alpha(x)/|x|$ for some α, then

$$\det \nabla \varphi(x) = \frac{\alpha^{n-1}(x)}{|x|^n} \sum_{i=1}^{n} x_i \frac{\partial \alpha}{\partial x_i}(x).$$

Proof. Since $\nabla \varphi = \lambda \, \mathrm{Id} + x \otimes \nabla \lambda$ and $x \otimes \nabla \lambda$ is a rank-1 matrix, the first equality holds true. The second one easily follows. □

11.5 Concentration of Mass

We start with an elementary lemma.

Lemma 11.13. *Let $c \in C^0([0,1]; B_1)$. Then for every $\varepsilon > 0$ such that*

$$c([0,1]) + B_\varepsilon \subset B_1\,,$$

there exists $\varphi_\varepsilon \in \operatorname{Diff}^\infty(\overline{B}_1; \overline{B}_1)$ satisfying

$$\varphi_\varepsilon(c(0)) = c(1) \quad \textit{and} \quad \operatorname{supp}(\varphi_\varepsilon - \mathrm{id}) \subset c([0,1]) + B_\varepsilon\,.$$

Proof. Define $\eta_\varepsilon \in C^\infty(\mathbb{R}^n; [0,1])$ such that

$$\eta_\varepsilon = \begin{cases} 1 & \text{in } B_{\varepsilon/4} \\ 0 & \text{in } \left(B_{\varepsilon/2}\right)^c. \end{cases}$$

Set, for $a \in \mathbb{R}^n$,

$$\eta_{a,\varepsilon}(x) = \eta_\varepsilon(x-a).$$

We then have

$$\delta \|\nabla \eta_{a,\varepsilon}\|_{C^0} = \delta \|\nabla \eta_\varepsilon\|_{C^0} < 1/(2n) \tag{11.38}$$

for a suitable $\delta > 0$. Let $x_i \in B_1$, $1 \le i \le N$, with $x_1 = c(0)$ and $x_N = c(1)$, be such that

$$x_i \in c([0,1]) \text{ for } 1 \le i \le N \quad \text{and} \quad |x_{i+1} - x_i| < \delta \text{ for } 1 \le i \le N-1$$

and define

$$\varphi_i(x) = x + \eta_{x_i,\varepsilon}(x)(x_{i+1} - x_i), \quad 1 \le i \le N-1.$$

Since (11.38) holds and $\operatorname{supp}(\varphi_i - \mathrm{id}) \subset c([0,1]) + B_\varepsilon \subset B_1$, we have

$$\det \nabla \varphi_i > 0 \quad \text{and} \quad \varphi_i = \mathrm{id} \text{ on } \partial B_1.$$

Therefore, $\varphi_i \in \mathrm{Diff}^\infty(\overline{B}_1; \overline{B}_1)$ by Theorem 19.12. Moreover, $\varphi_i(x_i) = x_{i+1}$. Then the diffeomorphism

$$\varphi_\varepsilon = \varphi_{N-1} \circ \cdots \circ \varphi_1$$

has all of the required properties. □

Before stating the main result of this section, we need some notations and elementary properties of pullbacks and connected components.

Notation 11.14. *(i) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $f \in C^0(\overline{\Omega})$, we adopt the following notations:*

$$F^+ = f^{-1}((0,\infty)) \quad \textit{and} \quad F^- = f^{-1}((-\infty,0)).$$

Moreover, if $x \in F^\pm$, then

$$F_x^\pm \textit{ denotes the connected component of } F^\pm \textit{ containing } x.$$

(ii) Given a set $A \subset \mathbb{R}^n$, we let

$$1_A(x) = \begin{cases} 1 & \textit{if } x \in A \\ 0 & \textit{otherwise.} \end{cases}$$

In the following lemma we state an easy property of pullbacks.

Lemma 11.15. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f \in C^0(\overline{\Omega})$,*

$$\varphi \in \mathrm{Diff}^1(\overline{\Omega}; \overline{\Omega}) \quad \textit{with} \quad \det \nabla \varphi > 0,$$

$x \in F^+$, $y \in F^-$. If $\widetilde{f} = \varphi^*(f)$, then $\varphi^{-1}(F^+) = \widetilde{F}^+$, $\varphi^{-1}(F^-) = \widetilde{F}^-$,

$$\varphi^{-1}(F_x^+) = \widetilde{F}^+_{\varphi^{-1}(x)} \quad \textit{and} \quad \varphi^{-1}(F_y^-) = \widetilde{F}^-_{\varphi^{-1}(y)}.$$

The following lemma is a trivial result about the cardinality of the connected components of super (sub)-level sets of continuous functions and we state it for the sake of completeness.

Lemma 11.16. *Let $f \in C^0(\overline{B}_1)$. Let $\{F^+_{x_i}\}_{i\in I^+}$ and $\{F^-_{y_j}\}_{j\in I^-}$ be the connected components of F^+, respectively of F^-. Then I^+ and I^- are at most countable. Moreover, if $|I^+| = \infty$, respectively $|I^-| = \infty$, then*

$$\lim_{k\to\infty} \operatorname{meas}\left(F^+ \setminus \bigcup_{i=1}^{k} F^+_{x_i}\right) = 0, \quad \textit{respectively} \quad \lim_{k\to\infty} \operatorname{meas}\left(F^- \setminus \bigcup_{j=1}^{k} F^-_{y_j}\right) = 0.$$

We now give the first main result of the present section.

Lemma 11.17 (Concentration of the positive mass). *Let $r \geq 1$ be an integer, $f \in C^r(\overline{B}_1)$ and $z \in F^+$. Let also A_i, $1 \leq i \leq M$, be M closed sets pairwise disjoint of positive measure such that*

$$A_i \subset F^+_z \cap B_1\,, \quad 1 \leq i \leq M.$$

Then for every $\varepsilon > 0$ small enough, there exists $\varphi_{\varepsilon,f,\{A_i\}} \in \operatorname{Diff}^r(\overline{B}_1;\overline{B}_1)$ (which will be simply denoted φ_ε) satisfying the following properties:

$$\operatorname{supp}(\varphi_\varepsilon - \mathrm{id}) \subset F^+_z \cap B_1,$$

$$\varphi^*_\varepsilon(f) \geq \frac{\int_{F^+_z} f}{M \operatorname{meas}(A_i)} - \varepsilon \quad \textit{in } A_i\,, \quad 1 \leq i \leq M. \tag{11.39}$$

Remark 11.18. Indeed, the above lemma allows one to concentrate the positive mass of the connected component containing z into the union of the A_i. The conclusion of the lemma immediately implies that

$$\int_{F^+_z} f = \int_{F^+_z} \varphi^*_\varepsilon(f) \geq \sum_{i=1}^{M} \int_{A_i} \varphi^*_\varepsilon(f) \geq \int_{F^+_z} f - \varepsilon \sum_{i=1}^{M} \operatorname{meas}(A_i).$$

Proof. We split the proof into three steps.

Step 1 (simplification). Using Theorem 10.11, it is sufficient to prove the existence of $f_\varepsilon \in C^r(\overline{B}_1)$, such that

$$f_\varepsilon > 0 \quad \text{in } F^+_z\,, \quad \operatorname{supp}(f - f_\varepsilon) \subset F^+_z \cap B_1 \quad \text{and} \quad \int_{F^+_z} f_\varepsilon = \int_{F^+_z} f$$

satisfying also (11.39) with $\varphi^*_\varepsilon(f)$ replaced by f_ε.

Step 2 (definition of f_ε and conclusion). Let $K \subset F^+_z \cap B_1$ be a closed set with

$$\bigcup_{i=1}^{M} A_i \subset \operatorname{int} K \subset K \subset F^+_z \cap B_1$$

and let, for every $\varepsilon > 0$ and $1 \le i \le M$,

$$\eta_{i,\varepsilon} \in C^\infty(\overline{B}_1;[0,1]) \quad \text{and} \quad \xi_\varepsilon \in C^\infty(\overline{B}_1;[0,1])$$

be such that

$$\operatorname{supp}(\eta_{i,\varepsilon}) \cap \operatorname{supp}(\eta_{j,\varepsilon}) = \emptyset \quad \text{for } i \ne j, \tag{11.40}$$

$$A_i \subset \{x \in \overline{B}_1 : \eta_{i,\varepsilon}(x) = 1\} \subset \operatorname{supp} \eta_{i,\varepsilon} \subset \operatorname{int} K, \tag{11.41}$$

$$K \subset \{x \in \overline{B}_1 : \xi_\varepsilon(x) = 1\} \subset \operatorname{supp}(\xi_\varepsilon) \subset F_z^+ \cap B_1, \tag{11.42}$$

$$\lim_{\varepsilon \to 0} \xi_\varepsilon = 1_{F_z^+ \cap B_1} \quad \text{and} \quad \lim_{\varepsilon \to 0} \eta_{i,\varepsilon} = 1_{A_i}. \tag{11.43}$$

Define f_ε, ε small, as

$$f_\varepsilon = \begin{cases} \sum_{i=1}^M \eta_{i,\varepsilon} C_{i,\varepsilon}^+ + (1 - \sum_{i=1}^M \eta_{i,\varepsilon}) \cdot \varepsilon & \text{in } K \\ \xi_\varepsilon \cdot \varepsilon + (1 - \xi_\varepsilon) f & \text{elsewhere,} \end{cases}$$

where

$$C_{i,\varepsilon}^+ = \frac{\int_{F_z^+} f}{M \operatorname{meas}(A_i)}, \quad 1 \le i \le M-1, \tag{11.44}$$

and $C_{M,\varepsilon}^+$ is the unique constant defined implicitly by the equation

$$\int_{F_z^+} f_\varepsilon = \int_{F_z^+} f.$$

We claim that f_ε has, up to rescaling ε, all of the required properties. Using (11.41) and (11.42), we get that

$$f_\varepsilon \in C^r(\overline{B}_1), \quad \operatorname{supp}(f_\varepsilon - f) \subset F_z^+ \cap B_1, \quad \int_{F_z^+} f_\varepsilon = \int_{F_z^+} f.$$

We claim that

$$\lim_{\varepsilon \to 0} C_{i,\varepsilon}^+ = \frac{\int_{F_z^+} f}{M \operatorname{meas}(A_i)}, \quad 1 \le i \le M. \tag{11.45}$$

By (11.44), it is obviously sufficient to prove the assertion for $i = M$. Using (11.43), (11.44), and the dominated convergence theorem, we get

$$\begin{aligned} \int_{F_z^+} f = \lim_{\varepsilon \to 0} \int_{F_z^+} f_\varepsilon &= \sum_{i=1}^{M-1} \int_{F_z^+} 1_{A_i} \frac{\int_{F_z^+} f}{M \operatorname{meas}(A_i)} + \int_{F_z^+} 1_{A_M} \lim_{\varepsilon \to 0} C_{M,\varepsilon}^+ \\ &= \frac{M-1}{M} \int_{F_z^+} f + \operatorname{meas}(A_M) \lim_{\varepsilon \to 0} C_{M,\varepsilon}^+ \end{aligned}$$

and thus the assertion holds. By the definition of f_ε, (11.40) and (11.45), we get that, for ε small,

$$f_\varepsilon > 0 \quad \text{in } F_z^+ .$$

Finally, since, by (11.41),

$$f_\varepsilon = C_{i,\varepsilon}^+ \quad \text{in } A_i, \quad 1 \le i \le M,$$

(11.45) directly implies, up to rescaling ε, (11.39), which ends the proof. ☐

We now give a similar result for the negative mass.

Lemma 11.19 (Concentration of the negative mass). *Let $r \geq 1$ be an integer, $f \in C^r(\overline{B}_1)$ and $y \in F^-$. Let also A_i, $1 \le i \le M$, be M closed sets pairwise disjoint of positive measure such that*

$$A_i \subset F_y^- \cap B_1 \quad \textit{and} \quad \operatorname{meas}(\partial A_i) = 0, \quad 1 \le i \le M. \tag{11.46}$$

Then for every $\varepsilon > 0$ small enough, there exists $\varphi_{\varepsilon,f,\{A_i\}} \in \operatorname{Diff}^r(\overline{B}_1;\overline{B}_1)$ (simply denoted φ_ε) satisfying the following properties:

$$\operatorname{supp}(\varphi_\varepsilon - \mathrm{id}) \subset F_y^- \cap B_1,$$

$$\frac{\displaystyle\int_{F_y^-} f}{M \operatorname{meas}(A_i)} - \varepsilon \le \varphi_\varepsilon^*(f) < 0 \quad \textit{in } A_i, \quad 1 \le i \le M, \tag{11.47}$$

$$\int_0^1 s^{n-1} \left(1_{F_y^- \setminus \left(\cup_{i=1}^M A_i\right)} \varphi_\varepsilon^*(f)\right) \left(s \frac{x}{|x|}\right) ds \ge -\varepsilon, \quad x \neq 0. \tag{11.48}$$

Remark 11.20. Integrating the last inequality over the unit sphere, we indeed obtain that the negative mass of the connected component containing y is concentrated into the union of the A_i.

Proof. We split the proof into three steps.

Step 1 (simplification). Using Theorem 10.11, it is sufficient to prove the existence of $f_\varepsilon \in C^r(\overline{B}_1)$, such that

$$f_\varepsilon < 0 \quad \text{in } F_y^-, \quad \operatorname{supp}(f - f_\varepsilon) \subset F_y^- \cap B_1 \quad \text{and} \quad \int_{F_y^-} f_\varepsilon = \int_{F_y^-} f$$

satisfying also (11.47) and (11.48) with $\varphi_\varepsilon^*(f)$ replaced by f_ε.

Step 2 (preliminaries). It is easily seen that the family of closed sets K_ε, ε small, defined by

$$K_\varepsilon = \{x \in \overline{F_y^- \cap B_{1-\varepsilon}} : f(x) \le -\varepsilon\}$$

has the following properties:

$$K_\varepsilon \subset K_{\varepsilon'} \quad \text{if } \varepsilon' < \varepsilon \quad \text{and} \quad \bigcup_{\varepsilon > 0} K_\varepsilon = F_y^- \cap B_1, \tag{11.49}$$

$$f\,|_{(F_y^- \cap B_{1-\varepsilon}) \setminus K_\varepsilon} > -\varepsilon. \tag{11.50}$$

Let $\xi_\varepsilon \in C^\infty(\overline{B}_1;[0,1])$ be such that

$$\xi_\varepsilon = 1 \text{ in } K_\varepsilon \quad \text{and} \quad \operatorname{supp}\xi_\varepsilon \subset F_y^- \cap B_1 . \tag{11.51}$$

Using (11.49) and (11.51), we immediately deduce that

$$\lim_{\varepsilon\to 0}\xi_\varepsilon = 1_{F_y^-\cap B_1} \quad \text{and} \quad \bigcup_{i=1}^M A_i \subset \operatorname{int}(K_\varepsilon) \quad \text{for } \varepsilon \text{ small.} \tag{11.52}$$

Finally, for every $1 \le i \le M$ and ε small enough, let $\eta_{i,\varepsilon} \in C^\infty(\overline{B}_1;[0,1])$ be such that

$$\operatorname{supp}(\eta_{i,\varepsilon}) \subset \operatorname{int}(A_i) \quad \text{and} \quad \lim_{\varepsilon\to 0}\eta_{i,\varepsilon} = 1_{\operatorname{int}(A_i)}, \quad 1 \le i \le M. \tag{11.53}$$

Step 3 (definition of f_ε and conclusion). Define f_ε, ε small, as

$$f_\varepsilon = \begin{cases} \sum_{i=1}^M \eta_{i,\varepsilon}C_{i,\varepsilon}^- + (1-\sum_{i=1}^M \eta_{i,\varepsilon})\cdot(-\varepsilon) & \text{in } \bigcup_{i=1}^M A_i \\ \xi_\varepsilon\cdot(-\varepsilon) + (1-\xi_\varepsilon)f & \text{elsewhere,} \end{cases}$$

where

$$C_{i,\varepsilon}^- = \frac{\displaystyle\int_{F_y^-} f}{M\operatorname{meas}(A_i)}, \quad 1 \le i \le M-1, \tag{11.54}$$

and $C_{M,\varepsilon}^-$ is the unique constant defined implicitly by the equation

$$\int_{F_y^-} f_\varepsilon = \int_{F_y^-} f .$$

We claim that, up to rescaling ε, f_ε has all the required properties. Using (11.51)–(11.53), we obtain that

$$f_\varepsilon \in C^r(\overline{B}_1), \quad \operatorname{supp}(f-f_\varepsilon) \subset F_y^- \cap B_1 , \quad \int_{F_y^-} f_\varepsilon = \int_{F_y^-} f.$$

We assert that

$$\lim_{\varepsilon\to 0} C_{i,\varepsilon}^- = \frac{\displaystyle\int_{F_y^-} f}{M\operatorname{meas}(A_i)}, \quad 1 \le i \le M. \tag{11.55}$$

By (11.54), it is obviously sufficient to prove (11.55) for $i = M$. Using (11.52) and (11.53) and noticing (using (11.46))

$$\operatorname{meas}(A_i) = \operatorname{meas}(\operatorname{int}(A_i)),$$

we get, by the dominated convergence theorem,

$$\int_{F_y^-} f = \lim_{\varepsilon\to 0}\int_{F_y^-} f_\varepsilon = \sum_{i=1}^{M-1}\int_{F_y^-} 1_{\mathrm{int}(A_i)}\frac{\int_{F_y^-} f}{M\,\mathrm{meas}(A_i)} + \int_{F_y^-} 1_{\mathrm{int}(A_M)}\lim_{\varepsilon\to 0} C_{M,\varepsilon}^-$$
$$= \frac{M-1}{M}\int_{F_y^-} f + \mathrm{meas}(A_M)\lim_{\varepsilon\to 0} C_{M,\varepsilon}^-$$

and, thus, the assertion is verified. Equation (11.55) immediately implies $f_\varepsilon < 0$ in F_y^- for ε small and also, rescaling ε if necessary, (11.47).

It remains to prove (11.48). First, we claim that

$$f_\varepsilon\,|_{(F_y^-\cap B_{1-\varepsilon})\setminus\left(\cup_{i=1}^M A_i\right)} \geq -\varepsilon, \tag{11.56}$$

$$f_\varepsilon \geq -D \tag{11.57}$$

for some $D>0$ independent of ε. In fact, (11.56) is obtained combining the fact that (by definition of f_ε)

$$f_\varepsilon = -\varepsilon \quad \text{in} \quad K_\varepsilon \setminus \left(\cup_{i=1}^M A_i\right)$$

and, by (11.50) and the definition of f_ε,

$$f_\varepsilon\,|_{(F_y^-\cap B_{1-\varepsilon})\setminus K_\varepsilon} \geq -\varepsilon.$$

Equation (11.57) is an immediate consequence of (11.55) and the definition of f_ε. Using (11.56) and (11.57) we get, for ε small and every $x\neq 0$,

$$\int_0^1 s^{n-1}(1_{F_y^-\setminus(\cup_{i=1}^M A_i)} f_\varepsilon)\left(s\frac{x}{|x|}\right)ds \geq \int_0^1 (1_{F_y^-\setminus(\cup_{i=1}^M A_i)} f_\varepsilon)\left(s\frac{x}{|x|}\right)ds$$
$$= \int_0^{1-\varepsilon}(1_{F_y^-\setminus(\cup_{i=1}^M A_i)} f_\varepsilon)\left(s\frac{x}{|x|}\right)ds + \int_{1-\varepsilon}^1 (1_{F_y^-\setminus(\cup_{i=1}^M A_i)} f_\varepsilon)\left(s\frac{x}{|x|}\right)ds$$
$$\geq \int_0^{1-\varepsilon} -\varepsilon ds + \int_{1-\varepsilon}^1 (-D)ds \geq -\varepsilon - \varepsilon D = -(D+1)\varepsilon.$$

Replacing ε by $\varepsilon/(D+1)$, we have shown (11.48) while still conserving the inequality (11.47). This ends the proof. □

11.6 Positive Radial Integration

Lemma 11.21 is the central part of the proof of Theorem 11.1. We show how to modify the mass distribution $f\in C^0(\overline{B}_1)$ satisfying $\int_{B_1} f>0$, in order to have strictly positive integrals on every radius starting from 0 and almost positive integrals on every radius starting from any point of the boundary (see Lemma 11.21). Moreover, if f is strictly positive on the boundary or if $f\geq 0$ in $\overline{B}_1$, we will be able to modify the mass of f in order to have strictly positive integrals on every radius starting either from 0 or from any point of the boundary (see Corollary 11.23).

Lemma 11.21 (Positive radial integration). *Let $f \in C^0(\overline{B}_1)$ be such that*

$$\int_{B_1} f > 0. \tag{11.58}$$

Then for every $\sigma > 0$, there exists $\varphi = \varphi_\sigma \in \mathrm{Diff}^\infty(\overline{B}_1;\overline{B}_1)$ such that

$$\operatorname{supp}(\varphi - \mathrm{id}) \subset B_1\,, \quad \varphi^*(f)(0) > 0,$$

$$\int_0^r s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds > 0 \quad \textit{for every } x \neq 0 \textit{ and } r \in (0,1], \tag{11.59}$$

$$\int_r^1 s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds > -\sigma \quad \textit{for every } x \neq 0 \textit{ and } r \in [0,1]. \tag{11.60}$$

Remark 11.22. (i) If $f \geq 0$, the proof is straightforward (see Corollary 11.23).

(ii) If f_1 satisfies $\varphi^*(f_1)(0) > 0$, (11.59) and (11.60), for a certain φ as in the lemma, then every $f \geq f_1$ also satisfies $\varphi^*(f)(0) > 0$, (11.59) and (11.60) with the same φ. Indeed, we clearly have

$$\varphi^*(f_1)(x) = f_1(\varphi(x))\underbrace{\det\nabla\varphi(x)}_{>0} \leq f(\varphi(x))\det\nabla\varphi(x) = \varphi^*(f)(x).$$

(iii) Integrating (11.59) over the sphere with $r = 1$, we get $\int_{B_1}\varphi^*(f) > 0$ and, therefore, (11.58) is necessary using the change of variables formula.

(iv) In general, (11.60) cannot be assumed to be positive or 0 for every x and r. This is, for example, always the case when $f(\overline{x}) < 0$ for some $\overline{x} \in \partial B_1$. Indeed, noting that

$$\varphi^*(f)(\overline{x}) = f(\overline{x})\det\nabla\varphi(\overline{x}) < 0,$$

we have that (11.60) will be strictly negative for $x = \overline{x}$ and r sufficiently close to 1.

(v) We could have replaced, without any changes, the unit ball by any ball centered at 0.

As a corollary, we have the following result.

Corollary 11.23. *Let $f \in C^0(\overline{B}_1)$ be such that*

$$\int_{B_1} f > 0 \tag{11.61}$$

and

$$\textit{either} \quad f > 0 \;\; \textit{on } \partial B_1 \quad \textit{or} \quad f \geq 0 \;\; \textit{in } \overline{B}_1\,.$$

Then there exists $\varphi \in \operatorname{Diff}^\infty(\overline{B}_1;\overline{B}_1)$ *such that*

$$\operatorname{supp}(\varphi - \mathrm{id}) \subset B_1\,, \quad \varphi^*(f)(0) > 0,$$

$$\int_0^r s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds > 0 \quad \text{for every } x \neq 0 \text{ and } r \in (0,1], \tag{11.62}$$

$$\int_r^1 s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds \geq 0 \quad \text{for every } x \neq 0 \text{ and } r \in [0,1]. \tag{11.63}$$

Proof (Corollary 11.23). We split the proof into two parts.

Part 1. We prove the corollary when $f \geq 0$ in $\overline{B}_1$. By (11.61) there exists $a \in B_1$ with $f(a) > 0$. Using Lemma 11.13, there exists $\varphi \in \operatorname{Diff}^\infty(\overline{B}_1;\overline{B}_1)$ such that

$$\operatorname{supp}(\varphi - \mathrm{id}) \subset B_1 \quad \text{and} \quad \varphi(0) = a.$$

Since $\varphi^*(f)(0) = f(\varphi(0))\det\nabla\varphi(0) > 0$ and $\varphi^*(f) \geq 0$ in $\overline{B}_1$, it is immediate that φ has all of the required properties.

Part 2. We prove the corollary when $f > 0$ on ∂B_1.

Part 2.1. By (11.61), there exist $0 < \eta < 1$ and $\varepsilon > 0$ such that

$$\int_{B_\eta} f > 0 \quad \text{and} \quad f > \varepsilon \quad \text{on } \overline{B}_1 \setminus B_\eta\,.$$

Using Lemma 11.21 with B_η instead of B_1, there exists $\varphi \in \operatorname{Diff}^\infty(\overline{B}_\eta;\overline{B}_\eta)$ verifying

$$\operatorname{supp}(\varphi - \mathrm{id}) \subset B_\eta\,, \quad \varphi^*(f)(0) > 0,$$

$$\int_0^r s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds > 0 \quad \text{for every } x \neq 0 \text{ and } r \in (0,\eta], \tag{11.64}$$

$$\int_r^\eta s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds > -\frac{\varepsilon(1-\eta^n)}{n} \quad \text{for every } x \neq 0 \text{ and } r \in [0,\eta]. \tag{11.65}$$

Part 2.2. Let us show that φ (extended by the identity to $\overline{B}_1$) has all of the required properties. Trivially, $\varphi \in \operatorname{Diff}^\infty(\overline{B}_1;\overline{B}_1)$,

$$\varphi^*(f)(0) > 0 \quad \text{and} \quad \operatorname{supp}(\varphi - \mathrm{id}) \subset B_\eta \subset B_1\,.$$

Since $\varphi^*(f) = f > 0$ in $\overline{B}_1 \setminus B_\eta$, (11.64) directly implies (11.62). Finally, we show (11.63). Using again that $\varphi^*(f) = f > 0$ in $\overline{B}_1 \setminus B_\eta$, it is obvious that (11.63) is verified for every $r \in [\eta, 1]$. Suppose that $r \in [0, \eta)$. Combining the fact that $\varphi^*(f) = f > \varepsilon$ in $\overline{B}_1 \setminus B_\eta$ and (11.65), we obtain for every $x \neq 0$,

$$\begin{aligned}\int_r^1 s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds &= \int_\eta^1 s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds + \int_r^\eta s^{n-1}\varphi^*(f)\left(s\frac{x}{|x|}\right)ds \\ &> \int_\eta^1 s^{n-1}\varepsilon\, ds - \frac{\varepsilon(1-\eta^n)}{n} = 0.\end{aligned}$$

The proof is therefore complete. □

Finally, we give the proof of Lemma 11.21.

Proof. Since the proof is rather long, we divide it into five steps. The three following facts will be crucial.

(i) For fixed $a,b \in B_1$, there exists, from Lemma 11.13, $\varphi \in \operatorname{Diff}^{\infty}(\overline{B}_1;\overline{B}_1)$ such that $\varphi(a) = b$. This will be used in Step 1.3 and Step 3.1.

(ii) From Lemmas 11.17 and 11.19, we concentrate the mass contained in connected components of F^+ and F^- in sectors of cones. This will be achieved in Step 4.

(iii) From Remark 11.22(ii), it is sufficient to prove the result for a function $f_1 \leq f$. This will be used in Steps 1.1, 1.2 and 1.4.

Step 1. We show that we can, without loss of generality, assume that

$$f \in C^{\infty}(\overline{B}_1), \quad F^- \text{ connected}, \quad f(0) > 0 \quad \text{and} \quad \int_{B_1 \backslash F_0^+} f > 0, \tag{11.66}$$

recalling that F_0^+ is the connected component of $F^+ = f^{-1}((0,\infty))$ containing 0.

Step 1.1. We start by showing that we can assume $f \in C^{\infty}(\overline{B}_1)$. First, using Theorem 16.11, we extend f so that $f \in C^0(\mathbb{R}^n)$. Then we choose $\delta > 0$ small enough such that

$$\int_{B_1} f > \delta \operatorname{meas}(B_1).$$

By continuity of f, there exists $f_\delta \in C^{\infty}(\mathbb{R}^n)$ such that

$$f_\delta(x) < f(x) < f_\delta(x) + \delta \quad \text{for every } x \in \overline{B}_1.$$

Note that

$$\int_{B_1} f_\delta > \int_{B_1} f - \delta \operatorname{meas}(B_1) > 0.$$

Using Remark 11.22(ii), we have the assertion. From now on, we write f instead of f_δ and we can therefore assume that $f \in C^{\infty}(\overline{B}_1)$.

Step 1.2. We show that we can, without loss of generality, assume that F^- is connected.

Step 1.2.1 (preliminaries). For every $\varepsilon > 0$ there exist $M \in \mathbb{N}$, $a_1, \ldots, a_M \in B_1$ and $\delta_1, \ldots, \delta_M > 0$ (depending all on ε) such that

$$\bigcup_{i=1}^{M} \overline{B}_{\delta_i}(a_i) \subset F^+ \cap B_1$$

$$\overline{B}_{\delta_i}(a_i) \cap \overline{B}_{\delta_j}(a_j) = \emptyset \quad \text{for every } i \neq j,$$

$$\operatorname{meas}\left(F^+ \setminus \left(\cup_{i=1}^{M} B_{\delta_i}(a_i)\right)\right) < \varepsilon.$$

Using the last equation and since

$$\int_{B_1} f = \int_{F^+} f + \int_{F^-} f > 0,$$

we can choose $\varepsilon > 0$ (and, therefore, also M, a_i and δ_i) small enough so that

$$\int_{\cup_{i=1}^M B_{\delta_i}(a_i)} f + \int_{F^-} f > 0.$$

We then choose $\delta > 0$ small enough such that

$$\bigcup_{i=1}^{M} \overline{B}_{\delta_i+4\delta}(a_i) \subset F^+ \cap B_1,$$

$$\overline{B}_{\delta_i+4\delta}(a_i) \cap \overline{B}_{\delta_j+4\delta}(a_j) = \emptyset \quad \text{for every } i \neq j,$$

$$\int_{\cup_{i=1}^M B_{\delta_i}(a_i)} f + \int_{F^-} f > \delta \operatorname{meas}(B_1). \tag{11.67}$$

Let $\xi \in C^\infty(\overline{B}_1;[0,1])$ be such that

$$\xi = 1 \quad \text{in} \bigcup_{i=1}^{M} \left(\overline{B}_{\delta_i+3\delta}(a_i) \setminus B_{\delta_i+\delta}(a_i)\right),$$

$$\operatorname{supp} \xi \subset \bigcup_{i=1}^{M} \left(B_{\delta_i+4\delta}(a_i) \setminus \overline{B}_{\delta_i}(a_i)\right),$$

$$\{x \in \overline{B}_1 \setminus \left(\cup_{i=1}^M B_{\delta_i+2\delta}(a_i)\right) \,:\, \xi(x) < 1\} \quad \text{is connected.} \tag{11.68}$$

Using Theorem 16.11, we extend f so that $f \in C^\infty(\mathbb{R}^n)$. Define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(x) = \min\{f(x), 0\}.$$

By continuity of $\widetilde{f}$, there exists $h_\delta \in C^\infty(\mathbb{R}^n)$ such that

$$h_\delta(x) < \widetilde{f}(x) < h_\delta(x) + \delta \quad \text{for every } x \in \overline{B}_1\,. \tag{11.69}$$

In particular, note that

$$h_\delta < 0 \quad \text{in } \overline{B}_1\,.$$

Step 1.2.2 (conclusion). Let $f_\delta : \overline{B}_1 \to \mathbb{R}$ be defined by

$$f_\delta = \begin{cases} (1-\xi)f & \text{in } \bigcup_{i=1}^M B_{\delta_i+2\delta}(a_i) \\ (1-\xi)h_\delta & \text{in } \overline{B}_1 \setminus \bigcup_{i=1}^M B_{\delta_i+2\delta}(a_i). \end{cases} \tag{11.70}$$

It is easily seen that f_δ is of class C^∞ and satisfies the following properties:

$$f_\delta(x)\begin{cases} = h_\delta(x) < \min\{f(x),0\} \le f(x) & \text{if } x \in \overline{B}_1 \setminus \bigcup_{i=1}^M B_{\delta_i+4\delta}(a_i) \\ \le 0 < f(x) & \text{if } x \in \bigcup_{i=1}^M \left(B_{\delta_i+4\delta}(a_i) \setminus B_{\delta_i+3\delta}(a_i)\right) \\ = 0 < f(x) & \text{if } x \in \bigcup_{i=1}^M \left(B_{\delta_i+3\delta}(a_i) \setminus B_{\delta_i+\delta}(a_i)\right) \\ \le f(x) & \text{if } x \in \bigcup_{i=1}^M \left(B_{\delta_i+\delta}(a_i) \setminus B_{\delta_i}(a_i)\right) \\ = f(x) & \text{if } x \in \bigcup_{i=1}^M B_{\delta_i}(a_i). \end{cases}$$

In particular, $f_\delta \le f$. We, moreover, have, since $h_\delta < 0$ and

$$f_\delta \ge 0 \quad \text{in } \bigcup_{i=1}^M B_{\delta_i+2\delta}(a_i),$$

that

$$\begin{aligned} F_\delta^- &= \{x \in \overline{B}_1 : f_\delta(x) < 0\} = \{x \in \overline{B}_1 \setminus \cup_{i=1}^M B_{\delta_i+2\delta}(a_i) : f_\delta(x) < 0\} \\ &= \{x \in \overline{B}_1 \setminus \cup_{i=1}^M B_{\delta_i+2\delta}(a_i) : (1-\xi(x))h_\delta(x) < 0\} \\ &= \{x \in \overline{B}_1 \setminus \cup_{i=1}^M B_{\delta_i+2\delta}(a_i) : \xi(x) < 1\}, \end{aligned}$$

which is a connected set by (11.68). We thus have that

$$F_\delta^- \subset \overline{B}_1 \setminus \bigcup_{i=1}^M B_{\delta_i+2\delta}(a_i) \quad \text{and} \quad F_\delta^- \text{ is connected.}$$

Observe next that

$$\begin{aligned} \int_{F_\delta^-} f_\delta &= \int_{F_\delta^-} (1-\xi)h_\delta \ge \int_{F_\delta^-} h_\delta > \int_{F_\delta^-} \left(\widetilde{f} - \delta\right) \ge \int_{F_\delta^-} \widetilde{f} - \delta \operatorname{meas}(B_1) \\ &= \int_{F_\delta^- \cap F^-} \widetilde{f} + \int_{F_\delta^- \setminus F^-} \widetilde{f} - \delta \operatorname{meas}(B_1) = \int_{F_\delta^- \cap F^-} \widetilde{f} - \delta \operatorname{meas}(B_1) \\ &= \int_{F_\delta^- \cap F^-} f - \delta \operatorname{meas}(B_1) \ge \int_{F^-} f - \delta \operatorname{meas}(B_1). \end{aligned}$$

This leads to

$$\begin{aligned} \int_{B_1} f_\delta &= \int_{F_\delta^+} f_\delta + \int_{F_\delta^-} f_\delta \ge \int_{\cup_{i=1}^M B_{\delta_i}(a_i)} f_\delta + \int_{F_\delta^-} f_\delta = \int_{\cup_{i=1}^M B_{\delta_i}(a_i)} f + \int_{F_\delta^-} f_\delta \\ &> \int_{\cup_{i=1}^M B_{\delta_i}(a_i)} f + \int_{F^-} f - \delta \operatorname{meas}(B_1) > 0, \end{aligned}$$

where we have used (11.67) in the last inequality. From now on, we write f in place of f_δ, since $f_\delta \le f$ and Remark 11.22(ii) holds. We may therefore assume, in the remaining part of the proof, that $f \in C^\infty\left(\overline{B}_1\right)$ and F^- is connected.

Step 1.3. Let us prove that we can assume that $f(0) > 0$. In fact, suppose that $f(0) \le 0$. We prove that there exists a diffeomorphism φ_1 such that $\varphi_1^*(f)(0) > 0$. Indeed, since $\int_{B_1} f > 0$, there exists $a \in B_1$ such that $f(a) > 0$. By Lemma 11.13, there exists $\varphi_1 \in \mathrm{Diff}^\infty(\overline{B}_1; \overline{B}_1)$ such that

$$\operatorname{supp}(\varphi_1 - \mathrm{id}) \subset B_1 \quad \text{and} \quad \varphi_1(0) = a.$$

Since $\varphi_1^*(f)(0) = f(a) \det \nabla \varphi_1(0) > 0$, we have the result. Note that, using the change of variables formula,

$$\int_{B_1} \varphi_1^*(f) = \int_{B_1} f > 0.$$

Note also that $\varphi_1^*(f) \in C^\infty\left(\overline{B}_1\right)$ and, using Lemma 11.15,

$$(\varphi_1^*(f))^{-1}((-\infty, 0)) = \varphi_1^{-1}(F^-) \quad \text{is connected.}$$

From now on, we write f in place of $\varphi_1^*(f)$ and thus we can assume, without loss of generality, that $f \in C^\infty\left(\overline{B}_1\right)$, F^- is connected and $f(0) > 0$.

Step 1.4. We finally show that we can assume that

$$\int_{B_1 \setminus F_0^+} f > 0.$$

In fact, since $f(0) > 0$ and $\int_{B_1} f > 0$, if $\delta_1 > 0$ is small enough, we have that $B_{4\delta_1} \subset F_0^+$ and

$$\int_{B_1 \setminus B_{4\delta_1}} f > 0. \tag{11.71}$$

Let $\eta \in C^\infty([0,1];[0,1])$ be such that

$$\eta(r) = \begin{cases} 1 & \text{if } r \le \delta_1 \text{ or } 4\delta_1 \le r \le 1 \\ 0 & \text{if } 2\delta_1 \le r \le 3\delta_1 . \end{cases}$$

Let $h \in C^\infty(\overline{B}_1)$ defined by $h(x) = \eta(|x|) f(x)$. We then have

$$h(0) > 0, \quad H^- = F^- \text{ connected} \quad \text{and} \quad B_{\delta_1} \subset H_0^+ \subset B_{2\delta_1} .$$

Using (11.71), we get

$$\int_{B_1 \setminus H_0^+} h \ge \int_{B_1 \setminus B_{4\delta_1}} h = \int_{B_1 \setminus B_{4\delta_1}} f > 0.$$

Since $h \le f$, we may, according to Remark 11.22(ii), proceed replacing f with $h = \eta f$. The proof of Step 1 is therefore complete.

Step 2. In this step we start by selecting N connected components of $F^+ \setminus F_0^+$. Then we select an appropriate amount of points in each of them.

Step 2.1 (selection of N connected components of $F^+ \setminus F_0^+$). Let $F_{x_i}^+$, $i \in I^+$, $x_i \in B_1 \setminus F_0^+$, be the pairwise disjoint connected components of $F^+ \setminus F_0^+$. Notice that I^+ is not empty by Step 1.4 and is at most countable; see Lemma 11.16. We claim that there exists $N \in \mathbb{N}$ such that

$$\int_{\cup_{i=1}^N F_{x_i}^+} f + \int_{F^-} f > 0. \tag{11.72}$$

In fact, suppose that I^+ is infinite (otherwise the assertion is trivial because of the fourth statement in (11.66)). Since, by the fourth statement in (11.66),

$$\int_{F^+ \setminus F_0^+} f + \int_{F^-} f > 0$$

and since, using Lemma 11.16,

$$\lim_{N\to\infty} \int_{\cup_{i=1}^N F_{x_i}^+} f = \int_{F^+ \setminus F_0^+} f,$$

we have (11.72) for N large enough.

Step 2.2 (selection of M_i points in $F_{x_i}^+$, $1 \le i \le N$ and of $M_1 + \cdots + M_N - 1$ points in F^-). We start by defining the integers M_i. We claim that there exist $M_1, \ldots,$ $M_N \in \mathbb{N}$ such that

$$\frac{\int_{F_{x_i}^+} f}{M_i} + \frac{\int_{F^-} f}{(\sum_{j=1}^N M_j) - 1} > 0 \quad \text{for every } 1 \le i \le N. \tag{11.73}$$

In order to simplify the notations, let

$$m_i^+ = \int_{F_{x_i}^+} f, \quad 1 \le i \le N \quad \text{and} \quad m^- = \int_{F^-} f.$$

We claim that for an integer ν large enough,

$$M_1 = \nu \quad \text{and} \quad M_i = \left[\frac{m_i^+}{m_1^+}\nu\right], \quad 2 \le i \le N,$$

where $[x]$ stands for the integer part of x, satisfy (11.73). Indeed, let $1 \le i \le N$; then since

$$\frac{m_i^+}{m_1^+}\nu - 1 < M_i < \frac{m_i^+}{m_1^+}\nu + 1, \quad 1 \le i \le N,$$

we deduce

$$\frac{m_i^+}{M_i} + \frac{m^-}{(\sum_{j=1}^N M_j) - 1} \ge \frac{m_i^+}{\frac{m_i^+}{m_1^+}\nu + 1} + \frac{m^-}{\frac{\sum_{j=1}^N m_j^+}{m_1^+}\nu - N - 1}.$$

Therefore, since, by (11.72),

$$\sum_{j=1}^{N} m_j^+ + m^- > 0,$$

we get

$$\lim_{\nu\to\infty}\left[\nu\left(\frac{m_i^+}{M_i} + \frac{m^-}{(\sum_{j=1}^N M_j)-1}\right)\right] \geq m_1^+\left(1+\frac{m^-}{\sum_{j=1}^N m_j^+}\right) > 0.$$

This proves the assertion.

Finally, choose M_1 distinct points

$$z_1,\ldots,z_{M_1} \in F_{x_1}^+ .$$

Then choose M_2 distinct points

$$z_{M_1+1},\ldots,z_{M_1+M_2} \in F_{x_2}^+$$

and so on, and finally choose M_N distinct points

$$z_{M_1+\cdots+M_{N-1}+1},\ldots,z_{M_1+\cdots+M_N} \in F_{x_N}^+ .$$

Similarly, choose $M_1+\cdots+M_N-1$ distinct points

$$y_1,\ldots,y_{M_1+\cdots+M_N-1} \in F^- .$$

We define

$$M = M_1+\cdots+M_N .$$

In particular, we have

$$f(z_k) > 0, \quad 1\leq k\leq M \quad \text{and} \quad f(y_j) < 0, \quad 1\leq j\leq M-1.$$

Step 3. In this step we move the $2M-1$ points selected in the above step so that they are on the same radial axis and well ordered; moreover, we define some cone sectors.

Step 3.1 (displacement of the points z_k and y_j). Choose $(2M-1)$ points, $\widetilde{z}_1,\ldots,\widetilde{z}_M$ and $\widetilde{y}_1,\ldots,\widetilde{y}_{M-1}$ such that

$$0 < |\widetilde{z}_1| < |\widetilde{y}_1| < |\widetilde{z}_2| < |\widetilde{y}_2| < \cdots < |\widetilde{z}_{M-1}| < |\widetilde{y}_{M-1}| < |\widetilde{z}_M| < 1,$$

$$\frac{\widetilde{z}_k}{|\widetilde{z}_k|} = \frac{\widetilde{y}_j}{|\widetilde{y}_j|} \quad \text{for every } 1\leq k\leq M,\ 1\leq j\leq M-1.$$

Then choose ε_1 small enough and

$$c_l \in C^0([0,1];B_1),\ 1\leq l\leq 2M-1,$$

such that the sets

$$c_l([0,1]) + B_{\varepsilon_1} \quad \text{are pairwise disjoint and contained in } B_1 \setminus \{0\},$$
$$\begin{cases} c_l(0) = \widetilde{z}_l & c_l(1) = z_l & \text{if } 1 \le l \le M, \\ c_l(0) = \widetilde{y}_{l-M} & c_l(1) = y_{l-M} & \text{if } M+1 \le l \le 2M-1. \end{cases}$$

Applying, for $1 \le l \le 2M-1$, Lemma 11.13 with $\varepsilon = \varepsilon_1$ and $c = c_l$, we get $\psi_l \in \mathrm{Diff}^\infty(\overline{B}_1; \overline{B}_1)$ with

$$\psi_l(c_l(0)) = c_l(1) \quad \text{and} \quad \operatorname{supp}(\psi_l - \mathrm{id}) \subset c_l([0,1]) + B_{\varepsilon_1} \subset B_1 \setminus \{0\}.$$

Thus, defining $\varphi_2 = \psi_1 \circ \cdots \circ \psi_{2M-1}$, we get that $\operatorname{supp}(\varphi_2 - \mathrm{id}) \subset B_1 \setminus \{0\}$ (and thus, in particular $\varphi_2(0) = 0$) and

$$\varphi_2(\widetilde{z}_k) = z_k\,, \quad 1 \le k \le M \quad \text{and} \quad \varphi_2(\widetilde{y}_j) = y_j\,, \quad 1 \le j \le M-1.$$

To complete, we also define

$$\widetilde{x}_i = \varphi_2^{-1}(x_i) \quad 1 \le i \le N.$$

Step 3.2 (definition of cone sectors). For $0 < \delta < 1$, let K_δ be the closed cone having vertex 0 and axis $\mathbb{R}_+ \widetilde{y}_1$ and such that

$$\operatorname{meas}(K_\delta \cap B_1) = \delta \operatorname{meas} B_1\,.$$

This immediately implies that

$$\operatorname{meas}(K_\delta \cap B_r) = \delta \operatorname{meas} B_r \quad \text{for every } r > 0. \tag{11.74}$$

Define

$$\widetilde{f} = \varphi_2^*(f).$$

By the properties of f and φ_2 we get that

$$\widetilde{f}(0) > 0, \quad \widetilde{f}(\widetilde{z}_k) > 0,\ 1 \le k \le M \quad \text{and} \quad \widetilde{f}(\widetilde{y}_j) < 0,\ 1 \le j \le M-1.$$

Therefore, there exists $\delta > 0$ small enough such that

$$\begin{cases} \widetilde{f} > \delta \quad \text{in } B_\delta, \\ K_\delta \cap \left(\overline{B}_{|\widetilde{z}_k|+\delta} \setminus B_{|\widetilde{z}_k|-\delta}\right) \subset \widetilde{F}^+ \cap B_1, & 1 \le k \le M, \\ K_\delta \cap \left(\overline{B}_{|\widetilde{y}_j|+\delta} \setminus B_{|\widetilde{y}_j|-\delta}\right) \subset \widetilde{F}^- \cap B_1, & 1 \le j \le M-1; \end{cases}$$

in particular,

$$\delta < |\widetilde{z}_1| - \delta < |\widetilde{z}_1| + \delta < |\widetilde{y}_1| - \delta < |\widetilde{y}_1| + \delta < |\widetilde{z}_2| - \delta < |\widetilde{z}_2| + \delta$$
$$< \cdots < |\widetilde{y}_{M-1}| - \delta < |\widetilde{y}_{M-1}| + \delta < |\widetilde{z}_M| - \delta < |\widetilde{z}_M| + \delta < 1.$$

Using Lemma 11.15 and (11.72), we get that $\widetilde{f} \in C^\infty\left(\overline{B}_1\right)$ is such that $\widetilde{F}^-$ is connected and

$$\int_{\cup_{i=1}^N \widetilde{F}^+_{\widetilde{x}_i}} \widetilde{f} + \int_{\widetilde{F}^-} \widetilde{f} > 0.$$

From now on, we write f, x_i, z_k and y_j instead of $\widetilde{f} = \varphi_2^*(f)$, $\widetilde{x}_i$, $\widetilde{z}_k$ and $\widetilde{y}_j$, respectively. Define

$$\begin{cases} S_k^+ = K_\delta \cap \left(\overline{B}_{|z_k|+\delta} \setminus B_{|z_k|-\delta}\right), & 1 \le k \le M, \\ S_j^- = K_\delta \cap \left(\overline{B}_{|y_j|+\delta} \setminus B_{|y_j|-\delta}\right), & 1 \le j \le M-1, \end{cases}$$

in particular,

$$\begin{aligned} \delta < |z_1| - \delta < |z_1| + \delta < |y_1| - \delta < |y_1| + \delta < |z_2| - \delta < |z_2| + \delta \\ < \cdots < |y_{M-1}| - \delta < |y_{M-1}| + \delta < |z_M| - \delta < |z_M| + \delta < 1. \end{aligned}$$

Choosing δ even smaller, we can assume, without loss of generality, that

$$\frac{\delta^{n+1}}{n} < \sigma, \tag{11.75}$$

where σ is the σ in the statement of the lemma. Note that f has the following properties:

$$S_k^+ \subset F^+_{x_{t(k)}},$$

where $t(k)$ is defined by

$$t(k) = \begin{cases} 1 & \text{if } 1 \le k \le M_1 \\ \vdots & \vdots \\ N & \text{if } M_1 + \cdots + M_{N-1} + 1 \le k \le M, \end{cases}$$

$f > \delta$ in $B_\delta \subset F_0^+$, F^- is connected and

$$\int_{\cup_{i=1}^N F^+_{x_i}} f + \int_{F^-} f > 0. \tag{11.76}$$

Step 4. In this step we concentrate the positive and the negative mass in the cone sectors defined in the previous step.

Step 4.1 (concentration of the positive mass in S_k^+, $1 \le k \le M$). Using (11.73), we can find ε_1 small enough such that

$$\frac{\int_{F^+_{x_i}} f}{M_i} - 2\varepsilon_1 \operatorname{meas} B_1 + \frac{\int_{F^-} f}{M-1} > 0, \quad 1 \le i \le N. \tag{11.77}$$

Applying, for $1 \le i \le N$, Lemma 11.17 to f, $z = x_i$, $\varepsilon = \varepsilon_1$ and

$$A_1 = S^+_{1+\sum_{j=1}^{i-1} M_j}, \ldots, A_{M_i} = S^+_{\sum_{j=1}^{i} M_j},$$

we get $\psi_i \in \mathrm{Diff}^\infty(\overline{B}_1; \overline{B}_1)$ with $\mathrm{supp}(\psi_i - \mathrm{id}) \subset F^+_{x_i} \cap B_1$ and

$$(\psi_i)^*(f) \ge \frac{\displaystyle\int_{F^+_{x_i}} f}{M_i \,\mathrm{meas}\, S^+_k} - \varepsilon_1 \text{ in } S^+_k, \quad 1 + \sum_{j=1}^{i-1} M_j \le k \le \sum_{j=1}^{i} M_j .$$

Letting $\varphi_3 = \psi_1 \circ \cdots \circ \psi_N \in \mathrm{Diff}^\infty(\overline{B}_1; \overline{B}_1)$, we obtain that

$$\mathrm{supp}(\varphi_3 - \mathrm{id}) \subset \bigcup_{j=1}^{N} (F^+_{x_j} \cap B_1) \subset B_1 \setminus F^+_0 ,$$

$$\varphi_3^*(f) = f > \delta \quad \text{in } B_\delta,$$

and, for every $1 \le i \le N$,

$$\varphi_3^*(f) \ge \frac{\displaystyle\int_{F^+_{x_i}} f}{M_i \,\mathrm{meas}\, S^+_k} - \varepsilon_1 \;\text{ in } S^+_k, \quad 1 + \sum_{j=1}^{i-1} M_j \le k \le \sum_{j=1}^{i} M_j .$$

We define, for $1 \le k \le M$,

$$C^+_k = \frac{\displaystyle\int_{F^+_{x_i}} f}{M_i \,\mathrm{meas}\, S^+_k} - \varepsilon_1 \quad \text{if } 1 + \sum_{j=1}^{i-1} M_j \le k \le \sum_{j=1}^{i} M_j$$

and we replace $\varphi_3^*(f)$ by f. We therefore have, using (11.77) and the fact that $\mathrm{meas}(S^+_k) \le \mathrm{meas}(B_1)$,

$$\begin{cases} f \ge C^+_k \quad \text{in } S^+_k, \quad 1 \le k \le M, \qquad\qquad f > \delta \quad \text{in } B_\delta, \\ C^+_k \,\mathrm{meas}(S^+_k) + \dfrac{\displaystyle\int_{F^-} f}{M-1} - \varepsilon_1 \,\mathrm{meas}(B_1) > 0, \quad 1 \le k \le M. \end{cases} \tag{11.78}$$

We also have

$$\bigcup_{k=1}^{M} S^+_k \subset F^+ \setminus F^+_0 .$$

Step 4.2 (concentration of the negative mass in S^-_j, $1 \le j \le M-1$). Using Lemma 11.19, recalling that F^- is connected, with $A_j = S^-_j$, $1 \le j \le M-1$, and

$$\varepsilon = \min\{\varepsilon_1, \delta^{n+1}/n\},$$

where δ has been defined in Step 3.2, we get $\varphi_4 \in \mathrm{Diff}^\infty(\overline{B}_1;\overline{B}_1)$ with $\mathrm{supp}(\varphi_4 - \mathrm{id}) \subset F^- \cap B_1$ and

$$\begin{cases} \dfrac{\displaystyle\int_{F^-} f}{(M-1)\operatorname{meas} S_j^-} - \varepsilon_1 \le \varphi_4^*(f) < 0 \quad \text{in } S_j^-, & 1 \le j \le M-1, \\ \displaystyle\int_0^1 s^{n-1} (1_{F^- \setminus \left(\cup_{j=1}^{M-1} S_j^-\right)} \varphi_4^*(f)) \left(s\frac{x}{|x|}\right) ds \ge -\frac{\delta^{n+1}}{n}, & x \ne 0. \end{cases}$$

Defining

$$C_j^- = \frac{\displaystyle\int_{F^-} f}{(M-1)\operatorname{meas} S_j^-} - \varepsilon_1, \quad 1 \le j \le M-1,$$

we thus get, using the second inequality of (11.78),

$$\begin{cases} C_j^- \le \varphi_4^*(f) < 0 \quad \text{in } S_j^-, & 1 \le j \le M-1, \\ C_k^+ \operatorname{meas} S_k^+ + C_j^- \operatorname{meas} S_j^- > 0, & 1 \le j \le M-1,\ 1 \le k \le M, \\ \displaystyle\int_0^1 s^{n-1} (1_{F^- \setminus \left(\cup_{j=1}^{M-1} S_j^-\right)} \varphi_4^*(f)) \left(s\frac{x}{|x|}\right) ds \ge -\frac{\delta^{n+1}}{n}, & x \ne 0. \end{cases}$$

Note that $\varphi_4^*(f) = f$ in F^+. Finally, as usual, we replace $\varphi_4^*(f)$ by f. We therefore obtain, using (11.78) and recalling (by (11.75)) that $\frac{\delta^{n+1}}{n} < \sigma$,

$$\begin{cases} f > \delta \quad \text{in } B_\delta \subset F_0^+, & \\ f \ge C_k^+ \quad \text{in } S_k^+ \subset F^+ \setminus F_0^+, & 1 \le k \le M, \\ f \ge C_j^- \quad \text{in } S_j^- \subset F^-, & 1 \le j \le M-1, \\ C_k^+ \operatorname{meas} S_k^+ + C_j^- \operatorname{meas} S_j^- > 0, & 1 \le k \le M,\ 1 \le j \le M-1, \\ \displaystyle\int_0^1 s^{n-1} (1_{F^- \setminus \left(\cup_{j=1}^{M-1} S_j^-\right)} f) \left(s\frac{x}{|x|}\right) ds \ge -\frac{\delta^{n+1}}{n} > -\sigma, & x \ne 0. \end{cases} \tag{11.79}$$

Step 4.3 (summary of the properties of f). We claim that f has the following properties:

$$f > \delta \quad \text{in } B_\delta \subset F_0^+, \tag{11.80}$$

$$\bigcup_{k=1}^{M} S_k^+ \subset F^+ \setminus F_0^+, \quad \bigcup_{j=1}^{M-1} S_j^- \subset F^-, \tag{11.81}$$

$$\int_0^1 s^{n-1} (1_{F^- \setminus \left(\cup_{j=1}^{M-1} S_j^-\right)} f) \left(s\frac{x}{|x|}\right) ds \ge -\frac{\delta^{n+1}}{n} > -\sigma \quad \text{if } x \ne 0 \tag{11.82}$$

and for every $x \ne 0$ and $1 \le k \le M$, $1 \le j \le M-1$,

$$\int_0^1 s^{n-1} (1_{S_k^+} f) \left(s\frac{x}{|x|}\right) ds + \int_0^1 s^{n-1} (1_{S_j^-} f) \left(s\frac{x}{|x|}\right) ds > 0. \tag{11.83}$$

In fact, (11.80)–(11.82) are just the first, second and fifth inequalities of (11.79), respectively. Let us show (11.83). Fix $1 \le k \le M$ and $1 \le j \le M-1$. Recall that

$$S_k^+ = K_\delta \cap (\overline{B}_{|z_k|+\delta} \setminus B_{|z_k|-\delta}) \quad \text{and} \quad S_j^- = K_\delta \cap (\overline{B}_{|y_j|+\delta} \setminus B_{|y_j|-\delta}),$$

where K_δ is a cone with vertex 0 and aperture δ. Thus, according to (11.74),

$$\begin{aligned} \operatorname{meas} S_k^+ &= \delta\,[(|z_k|+\delta)^n - (|z_k|-\delta)^n] \operatorname{meas} B_1, \\ \operatorname{meas} S_j^- &= \delta\,[(|y_j|+\delta)^n - (|y_j|-\delta)^n] \operatorname{meas} B_1\,. \end{aligned}$$

Then, using (11.79), we get

$$\begin{aligned} &\int_0^1 s^{n-1}(1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^1 s^{n-1}(1_{S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\ &\ge \int_{|z_k|-\delta}^{|z_k|+\delta} s^{n-1} C_k^+\, ds + \int_{|y_j|-\delta}^{|y_j|+\delta} s^{n-1} C_j^-\, ds \\ &= C_k^+\, \frac{(|z_k|+\delta)^n - (|z_k|-\delta)^n}{n} + C_j^-\, \frac{(|y_j|+\delta)^n - (|y_j|-\delta)^n}{n} \\ &= C_k^+\, \frac{\operatorname{meas} S_k^+}{n\,\delta \operatorname{meas} B_1} + C_j^-\, \frac{\operatorname{meas} S_j^-}{n\,\delta \operatorname{meas} B_1} > 0, \end{aligned}$$

which is the claim.

Step 5 (conclusion). Let

$$\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \varphi_4\,.$$

Note that, by construction, $\operatorname{supp}(\varphi - \mathrm{id}) \subset B_1$. Because of all of the successive replacements of f in Steps 1–4 by a new f, the lemma has to be proved for $\varphi = \mathrm{id}$.

Step 5.1. First, note that $f(0) > 0$ by (11.80).

Step 5.2. We now show (11.59). We divide the discussion into three steps.

Step 5.2.1. If $r \le \delta$, (11.80) directly implies the assertion.

Step 5.2.2. We now suppose that either $x \notin K_\delta$ and $r \in (\delta, 1]$ or $x \in K_\delta$ and $r \in (\delta, |y_1| - \delta)$ and thus, in particular,

$$\left[0, r\frac{x}{|x|}\right] \bigcap \left(\bigcup_{j=1}^{M-1} S_j^-\right) = \emptyset.$$

Observe that (11.80) and (11.82) then imply

$$\begin{aligned} \int_0^r s^{n-1} f\left(s\frac{x}{|x|}\right) ds &\ge \int_0^r s^{n-1}(1_{F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^r s^{n-1}(1_{F^-} f)\left(s\frac{x}{|x|}\right) ds \\ &= \int_0^r s^{n-1}(1_{F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^r s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \end{aligned}$$

$$
\begin{aligned}
&> \int_0^{\delta} s^{n-1}\delta\, ds + \int_0^{r} s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\geq \int_0^{\delta} s^{n-1}\delta\, ds + \int_0^{1} s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \geq 0
\end{aligned}
$$

and the assertion is proved.

Step 5.2.3. It only remains to show the assertion when $x \in K_\delta$ and $r \in [\,|y_1| - \delta, 1]$. We get

$$
\begin{aligned}
&\int_0^{r} s^{n-1} f\left(s\frac{x}{|x|}\right) ds \\
&\quad = \int_0^{r} s^{n-1}(1_{F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{F^+ \setminus F_0^+} f)\left(s\frac{x}{|x|}\right) ds \\
&\qquad + \int_0^{r} s^{n-1}(1_{F^-} f)\left(s\frac{x}{|x|}\right) ds
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\int_0^{r} s^{n-1} f\left(s\frac{x}{|x|}\right) ds \\
&\quad = \int_0^{r} s^{n-1}(1_{F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\qquad + \int_0^{r} s^{n-1}(1_{F^+ \setminus F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{\cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds.
\end{aligned}
$$

Since $r \geq |y_1| - \delta \geq |z_1| + \delta \geq \delta$, (11.80) holds, and $f < 0$ in $F^- \setminus \cup_{j=1}^{M-1} S_j^-$, we obtain

$$
\begin{aligned}
&\int_0^{r} s^{n-1}(1_{F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\quad \geq \int_0^{\delta} s^{n-1}\delta\, ds + \int_0^{1} s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds.
\end{aligned}
$$

and hence, according to (11.82),

$$
\int_0^{r} s^{n-1}(1_{F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \geq 0.
$$

We therefore find, using (11.81), that

$$
\begin{aligned}
\int_0^{r} s^{n-1} f\left(s\frac{x}{|x|}\right) ds &\geq \int_0^{r} s^{n-1}(1_{F^+ \setminus F_0^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{\cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\geq \sum_{k=1}^{M-1} \left\{ \int_0^{r} s^{n-1}(1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^{r} s^{n-1}(1_{S_k^-} f)\left(s\frac{x}{|x|}\right) ds \right\}.
\end{aligned}
$$

Define

$$A = \sum_{k=1}^{M-1} \left\{ \int_0^r s^{n-1} (1_{S_k^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^r s^{n-1} (1_{S_k^-} f) \left(s \frac{x}{|x|} \right) ds \right\}.$$

In order to conclude the proof of Step 5.2.3 and thus of Step 5.2, it is sufficient to show that $A > 0$. We consider several cases.

Case 1: $r \in [\,|y_1| - \delta, |z_2| + \delta)$. We then have

$$\begin{aligned} A = &\int_0^r s^{n-1} (1_{S_2^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^r s^{n-1} (1_{S_1^+} f) \left(s \frac{x}{|x|} \right) ds \\ &+ \int_0^r s^{n-1} (1_{S_1^-} f) \left(s \frac{x}{|x|} \right) ds \end{aligned}$$

and thus, recalling that $r \geq |y_1| - \delta > |z_1| + \delta$,

$$\begin{aligned} A &\geq \int_0^r s^{n-1} (1_{S_1^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^r s^{n-1} (1_{S_1^-} f) \left(s \frac{x}{|x|} \right) ds \\ &\geq \int_0^1 s^{n-1} (1_{S_1^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^1 s^{n-1} (1_{S_1^-} f) \left(s \frac{x}{|x|} \right) ds, \end{aligned}$$

which is positive, according to (11.83).

Case 2: $r \in [\,|z_i| + \delta, |z_{i+1}| + \delta)$, $2 \leq i \leq M-1$. We therefore find

$$\begin{aligned} A &= \sum_{k=1}^{i+1} \int_0^r s^{n-1} (1_{S_k^+} f) \left(s \frac{x}{|x|} \right) ds + \sum_{k=1}^{i} \int_0^r s^{n-1} (1_{S_k^-} f) \left(s \frac{x}{|x|} \right) ds \\ &\geq \sum_{k=1}^{i} \left\{ \int_0^r s^{n-1} (1_{S_k^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^r s^{n-1} (1_{S_k^-} f) \left(s \frac{x}{|x|} \right) ds \right\} \\ &\geq \sum_{k=1}^{i} \left\{ \int_0^1 s^{n-1} (1_{S_k^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^1 s^{n-1} (1_{S_k^-} f) \left(s \frac{x}{|x|} \right) ds \right\} \end{aligned}$$

which is positive, in view of (11.83).

Case 3: $r \in [\,|z_M| + \delta, 1]$. We now have

$$A = \sum_{k=1}^{M-1} \left\{ \int_0^1 s^{n-1} (1_{S_k^+} f) \left(s \frac{x}{|x|} \right) ds + \int_0^1 s^{n-1} (1_{S_k^-} f) \left(s \frac{x}{|x|} \right) ds \right\},$$

which is positive, according to (11.83).

Step 5.3. We finally prove (11.60) and we divide the proof into two steps.

Step 5.3.1. First, suppose that either $x \notin K_\delta$ or

$$x \in K_\delta \quad \text{and} \quad r \in (\,|y_{M-1}| + \delta, 1]$$

and thus, in particular,

$$\left[r\frac{x}{|x|}, \frac{x}{|x|} \right] \bigcap \left(\bigcup_{j=1}^{M-1} S_j^- \right) = \emptyset.$$

Inequality (11.82) then implies

$$\begin{aligned}
\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds &\geq \int_r^1 s^{n-1} (1_{F^-} f)\left(s\frac{x}{|x|}\right) ds \\
&= \int_r^1 s^{n-1} (1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\geq \int_0^1 s^{n-1} (1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds > -\sigma,
\end{aligned}$$

which proves the assertion.

Step 5.3.2. It only remains to show the assertion when $x \in K_\delta$ and $r \in [0, |y_{M-1}| + \delta]$. We get, using the fact that $f < 0$ in F^-, (11.81) and $f > 0$ in F_0^+, that

$$\begin{aligned}
\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds &= \int_r^1 s^{n-1} (1_{F^-} f)\left(s\frac{x}{|x|}\right) ds + \int_r^1 s^{n-1} (1_{F^+} f)\left(s\frac{x}{|x|}\right) ds \\
&\geq \int_r^1 s^{n-1} (1_{F^- \setminus \cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\quad + \int_r^1 s^{n-1} (1_{\cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\quad + \int_r^1 s^{n-1} (1_{F^+ \setminus F_0^+} f)\left(s\frac{x}{|x|}\right) ds
\end{aligned}$$

and hence, appealing to (11.82) and since $f > 0$ in S_1^+,

$$\begin{aligned}
\int_r^1 s^{n-1} f\left(s\frac{x}{|x|}\right) ds &> -\sigma + \int_r^1 s^{n-1} (1_{\cup_{j=1}^{M-1} S_j^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\quad + \int_r^1 s^{n-1} (1_{F^+ \setminus F_0^+} f)\left(s\frac{x}{|x|}\right) ds \\
&\geq -\sigma + \sum_{k=2}^{M} \int_r^1 s^{n-1} (1_{S_{k-1}^-} f)\left(s\frac{x}{|x|}\right) ds \\
&\quad + \sum_{k=2}^{M} \int_r^1 s^{n-1} (1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds.
\end{aligned}$$

Define

$$B = \sum_{k=2}^{M} \left\{ \int_r^1 s^{n-1} (1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \int_r^1 s^{n-1} (1_{S_{k-1}^-} f)\left(s\frac{x}{|x|}\right) ds \right\}.$$

In order to obtain the claim, it remains to prove that $B > 0$. This is obtained exactly as in Step 5.2.3.

Case 1: $r \in [|z_{M-1}| - \delta, |y_{M-1}| + \delta]$. We then have

$$\begin{aligned} B = &\int_r^1 s^{n-1}(1_{S_M^+} f)\left(s\frac{x}{|x|}\right) ds + \int_r^1 s^{n-1}(1_{S_{M-1}^+} f)\left(s\frac{x}{|x|}\right) ds \\ &+ \int_r^1 s^{n-1}(1_{S_{M-1}^-} f)\left(s\frac{x}{|x|}\right) ds \end{aligned}$$

and thus, recalling that $r \le |y_{M-1}| + \delta < |z_M| - \delta$,

$$\begin{aligned} B &\ge \int_r^1 s^{n-1}(1_{S_M^+} f)\left(s\frac{x}{|x|}\right) ds + \int_r^1 s^{n-1}(1_{S_{M-1}^-} f)\left(s\frac{x}{|x|}\right) ds \\ &\ge \int_0^1 s^{n-1}(1_{S_M^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^1 s^{n-1}(1_{S_{M-1}^-} f)\left(s\frac{x}{|x|}\right) ds, \end{aligned}$$

which leads to $B > 0$, in view of (11.83).

Case 2: $r \in [|z_{i-1}| - \delta, |z_i| - \delta)$, $2 \le i \le M-1$. We thus deduce

$$\begin{aligned} B &= \sum_{k=i-1}^{M} \int_r^1 s^{n-1}(1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \sum_{k=i}^{M} \int_r^1 s^{n-1}(1_{S_{k-1}^-} f)\left(s\frac{x}{|x|}\right) ds \\ &\ge \sum_{k=i}^{M} \left\{ \int_r^1 s^{n-1}(1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \int_r^1 s^{n-1}(1_{S_{k-1}^-} f)\left(s\frac{x}{|x|}\right) ds \right\} \\ &\ge \sum_{k=i}^{M} \left\{ \int_0^1 s^{n-1}(1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^1 s^{n-1}(1_{S_{k-1}^-} f)\left(s\frac{x}{|x|}\right) ds \right\} \end{aligned}$$

and, using (11.83), we get that $B > 0$.

Case 3: $r \in [0, |z_1| - \delta)$. We therefore find

$$B = \sum_{k=2}^{M} \left\{ \int_0^1 s^{n-1}(1_{S_k^+} f)\left(s\frac{x}{|x|}\right) ds + \int_0^1 s^{n-1}(1_{S_{k-1}^-} f)\left(s\frac{x}{|x|}\right) ds \right\};$$

using once more (11.83), we get that $B > 0$. This concludes the proof of the lemma. □