Chapter 1
Introduction

1.1 Statement of the Problem

The aim of this book is the study of the *pullback equation*

$$\varphi^*(g) = f. \tag{1.1}$$

More precisely, we want to find a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$; preferably we want this map to be a diffeomorphism that satisfies the above equation, where f and g are differential k-forms, $0 \leq k \leq n$. Most of the time we will require these two forms to be closed. Before going further, let us examine the exact meaning of (1.1). We write

$$g(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and similarly for f. The meaning of (1.1) is that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} \circ \varphi\, d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where

$$d\varphi^i = \sum_{j=1}^{n} \frac{\partial \varphi^i}{\partial x_j} dx^j.$$

This turns out to be a *nonlinear* (if $2 \leq k \leq n$) homogeneous of degree k (in the derivatives) first-order system of $\binom{n}{k}$ partial differential equations. Let us see the form that the equation takes when $k = 0, 1, 2, n$.

Case: $k = 0$. Equation (1.1) reads as

$$g(\varphi(x)) = f(x)$$

while

$$dg = 0 \Leftrightarrow \operatorname{grad} g = 0.$$

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We will be, only marginally, interested in this elementary case, which is trivial for closed forms. In any case, (1.1) is *not*, when $k=0$, a differential equation.

Case: $k=1$. The form g, and analogously for f, can be written as

$$g(x)=\sum_{i=1}^{n} g_i(x)\,dx^i.$$

Equation (1.1) then becomes

$$\sum_{i=1}^{n} g_i(\varphi(x))\,d\varphi^i=\sum_{i=1}^{n} f_i(x)\,dx^i$$

while

$$dg=0 \Leftrightarrow \operatorname{curl} g=0 \Leftrightarrow \frac{\partial g_i}{\partial x_j}-\frac{\partial g_j}{\partial x_i}=0, \quad 1\le i<j\le n.$$

Writing

$$d\varphi^i=\sum_{j=1}^{n}\frac{\partial \varphi^i}{\partial x_j}dx^j$$

and substituting into the equation, we find that (1.1) is equivalent to

$$\sum_{j=1}^{n} g_j(\varphi(x))\frac{\partial \varphi^j}{\partial x_i}(x)=f_i(x), \quad 1\le i\le n.$$

This is a system of $\binom{n}{1}=n$ first-order *linear* (in the first derivatives) partial differential equations.

Case: $k=2$. The form g, and analogously for f, can be written as

$$g=\sum_{1\le i<j\le n} g_{ij}(x)\,dx^i\wedge dx^j$$

while

$$dg=0 \Leftrightarrow \frac{\partial g_{ij}}{\partial x_k}-\frac{\partial g_{ik}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}=0, \quad 1\le i<j<k\le n.$$

The equation $\varphi^*(g)=f$ becomes

$$\sum_{1\le p<q\le n} g_{pq}(\varphi(x))\,d\varphi^p\wedge d\varphi^q=\sum_{1\le i<j\le n} f_{ij}(x)\,dx^i\wedge dx^j.$$

We get, as before, that (1.1) is equivalent, for every $1\le i<j\le n$, to

$$\sum_{1\le p<q\le n} g_{pq}(\varphi(x))\left(\frac{\partial \varphi^p}{\partial x_i}\frac{\partial \varphi^q}{\partial x_j}-\frac{\partial \varphi^p}{\partial x_j}\frac{\partial \varphi^q}{\partial x_i}\right)(x)=f_{ij}(x),$$

which is a *nonlinear* homogeneous of degree 2 (in the derivatives) system of $\binom{n}{2}=\frac{n(n-1)}{2}$ first-order partial differential equations.

Case: $k=n$. In this case we always have $df=dg=0$. By abuse of notations, if we identify volume forms and functions, we get that the equation $\varphi^*(g)=f$ becomes

$$g(\varphi(x))\det\nabla\varphi(x)=f(x).$$

It is then a nonlinear homogeneous of degree n (in the derivatives) first-order partial differential equation.smallskip

The main questions that we will discuss are the following.

1) *Local existence*. This is the easiest question. We will handle fairly completely the case of closed 2-forms, which is the case of the Darboux theorem. The cases of 1 and $(n-1)$-forms as well as the case of n-forms will also be dealt with. It will turn out that the case $3\le k\le n-2$ is much more difficult and we will be able to handle only closed k-forms with special structure.
2) *Global existence*. This is a much more difficult problem. We will obtain results in the case of volume forms and of closed 2-forms.
3) *Regularity*. A special emphasis will be given on getting sharp regularity results. For this reason we will have to work with Hölder spaces $C^{r,\alpha}$, $0<\alpha<1$, not with spaces C^r. Apart from the linear problems considered in Part II, we will not deal with Sobolev spaces. In the present context the reason is that Hölder spaces form an algebra contrary to Sobolev spaces (with low exponents).

1.2 Exterior and Differential Forms

In Chapter 2 we have gathered some algebraic results about exterior forms that are used throughout the book.

1.2.1 Definitions and Basic Properties of Exterior Forms

Let $1\le k\le n$ be an integer. An exterior k-form will be denoted by

$$f=\sum_{1\le i_1<\cdots<i_k\le n} f_{i_1\cdots i_k}e^{i_1}\wedge\cdots\wedge e^{i_k}.$$

The set of exterior k-forms over $\mathbb{R}^n$ is a vector space and is denoted $\Lambda^k(\mathbb{R}^n)$ and its dimension is

$$\dim(\Lambda^k(\mathbb{R}^n))=\binom{n}{k}.$$

If $k=0$, we set

$$\Lambda^0(\mathbb{R}^n)=\mathbb{R}.$$

By abuse of notations, we will, when convenient and in order not to burden the notations, identify k-forms with vectors in $\mathbb{R}^{\binom{n}{k}}$.

(i) The *exterior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$, denoted by $f \wedge g$, is defined as usual (cf. Definition 2.2) and it belongs to $\Lambda^{k+l}(\mathbb{R}^n)$. The *scalar product* between two k-forms f and g is denoted by

$$\langle g; f \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} f_{i_1 \cdots i_k}.$$

The *Hodge star operator* (cf. Definition 2.9) associates to $f \in \Lambda^k(\mathbb{R}^n)$ a form $(*f) \in \Lambda^{n-k}(\mathbb{R}^n)$. We define (cf. Definition 2.11) the *interior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$ by

$$g \lrcorner f = (-1)^{n(k-l)} * (g \wedge (*f)).$$

These definitions are linked through the following elementary facts (cf. Proposition 2.16). For every $f \in \Lambda^k(\mathbb{R}^n)$, $g \in \Lambda^{k+1}(\mathbb{R}^n)$ and $h \in \Lambda^1(\mathbb{R}^n)$,

$$|h|^2 f = h \lrcorner (h \wedge f) + h \wedge (h \lrcorner f),$$
$$\langle h \wedge f; g \rangle = \langle f; h \lrcorner g \rangle.$$

(ii) Let $A \in \mathbb{R}^{n \times n}$ be a matrix and let $f \in \Lambda^k(\mathbb{R}^n)$ be given by

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

We define (cf. Definition 2.17) the *pullback of f by A,* denoted $A^*(f)$, by

$$A^*(f) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} A^{i_1} \wedge \cdots \wedge A^{i_k} \in \Lambda^k(\mathbb{R}^n),$$

where A^j is the jth row of A and is identified by

$$A^j = \sum_{k=1}^{n} A_k^j e^k \in \Lambda^1(\mathbb{R}^n).$$

If $k = 0$, we then let

$$A^*(f) = f.$$

The present definition is consistent with the one given at the beginning of the chapter; just set $\varphi(x) = Ax$ in (1.1).

(iii) We next define the notion of *rank* (also called rank of order 1 in Chapter 2) of $f \in \Lambda^k(\mathbb{R}^n)$. We first associate to the linear map

$$g \in \Lambda^1(\mathbb{R}^n) \to g \lrcorner f \in \Lambda^{k-1}(\mathbb{R}^n)$$

a matrix $\overline{f} \in \mathbb{R}^{\binom{n}{k-1} \times n}$ such that, by abuse of notations,

$$g \lrcorner f = \overline{f}\, g \quad \text{for every } g \in \Lambda^1 (\mathbb{R}^n).$$

In this case, we have

$$g \lrcorner f$$
$$= \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} \left(\sum_{\gamma=1}^{k} (-1)^{\gamma-1} \sum_{j_{\gamma-1} < i < j_\gamma} f_{j_1 \cdots j_{\gamma-1} i j_\gamma \cdots j_{k-1}}\, g_i \right) e^{j_1} \wedge \cdots \wedge e^{j_{k-1}}.$$

More explicitly, using the lexicographical order for the columns (index below) and the rows (index above) of the matrix $\overline{f}$, we have

$$(\overline{f})_i^{j_1 \cdots j_{k-1}} = f_{i\, j_1 \cdots j_{k-1}}$$

for $1 \leq i \leq n$ and $1 \leq j_1 < \cdots < j_{k-1} \leq n$. The rank of the k-form f is then the rank of the $\binom{n}{k-1} \times n$ matrix $\overline{f}$ (or similarly the rank of the map $g \to g \lrcorner f$). We then write (in Chapter 2, we write $\text{rank}_1 [f]$, but in the remaining part of the book we write only $\text{rank}[f]$)

$$\text{rank}[f] = \text{rank}\left(\overline{f}\right).$$

Note that only when $k = 2$ or $k = n$, the matrix $\overline{f}$ is a square matrix. We will get our best results precisely in these cases and when the matrix $\overline{f}$ is invertible.

We then have the following elementary result (cf. Proposition 2.37).

Proposition 1.1. *Let $f \in \Lambda^k (\mathbb{R}^n)$, $f \neq 0$.*

(i) If $k = 1$, then the rank of f is always 1.

(ii) If $k = 2$, then the rank of f is even. The forms

$$\omega_m = \sum_{i=1}^{m} e^{2i-1} \wedge e^{2i}$$

are such that $\text{rank}[\omega_m] = 2m$. *Moreover,* $\text{rank}[f] = 2m$ *if and only if*

$$f^m \neq 0 \quad \text{and} \quad f^{m+1} = 0,$$

where $f^m = \underbrace{f \wedge \cdots \wedge f}_{m \text{ times}}$.

(iii) If $3 \leq k \leq n$, then

$$\text{rank}[f] \in \{k, k+2, \ldots, n\}$$

and any of the values in $\{k, k+2, \ldots, n\}$ can be achieved by the rank of a k-form. In particular, if $k = n-1$, then $\text{rank}[f] = n-1$, *whereas if $k = n$, then* $\text{rank}[f] = n$.

Remark 1.2 (cf. Propositions 2.24 and 2.33). The rank is an invariant for the pullback equation. More precisely, if there exists $A \in \mathrm{GL}(n)$ (i.e., A is an invertible $n \times n$ matrix) such that

$$A^*(g) = f,$$

then

$$\mathrm{rank}[g] = \mathrm{rank}[f].$$

Conversely, when $k = 1, 2, n-1, n$, if $\mathrm{rank}[g] = \mathrm{rank}[f]$, then there exists $A \in \mathrm{GL}(n)$ such that

$$A^*(g) = f.$$

However, the converse is not true, in general, if $3 \leq k \leq n-2$. For example (cf. Example 2.36), when $k = 3$, the forms

$$f = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6,$$
$$g = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^5 \wedge e^6$$

have both rank $= 6$, but there is no $A \in \mathrm{GL}(6)$ so that

$$A^*(g) = f.$$

Similarly and more strikingly (cf. Example 2.35), when $k = 4$ and

$$f = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 \wedge e^6,$$

there is no $A \in \mathrm{GL}(6)$ such that

$$A^*(f) = -f.$$

1.2.2 Divisibility

We then discuss the notion of *divisibility* for exterior forms. Given two integers $1 \leq l \leq k \leq n$, a k-form f and a l-form g, we want to know if we can find a $(k-l)$-form u so that

$$f = g \wedge u.$$

This is an important question in the theory of Grassmann algebras. A well-known result is the so called Cartan lemma (cf. Theorem 2.42).

Theorem 1.3 (Cartan lemma). *Let $1 \leq k \leq n$ and $f \in \Lambda^k(\mathbb{R}^n)$ with $f \neq 0$. Let $1 \leq l \leq k$ and $g_1, \ldots, g_l \in \Lambda^1(\mathbb{R}^n)$ be such that*

$$g_1 \wedge \cdots \wedge g_l \neq 0.$$

Then there exists $u \in \Lambda^{k-l}(\mathbb{R}^n)$ verifying

$$f = g_1 \wedge \cdots \wedge g_l \wedge u$$

if and only if

$$f \wedge g_1 = \cdots = f \wedge g_l = 0.$$

Remark 1.4. In the same spirit, the following facts can easily be proved (cf. Proposition 2.43):

(i) The form $f \in \Lambda^k(\mathbb{R}^n)$ is totally divisible, meaning that there exist $f_1, \cdots, f_k \in \Lambda^1(\mathbb{R}^n)$ such that

$$f = f_1 \wedge \cdots \wedge f_k$$

if and only if

$$\operatorname{rank}[f] = k.$$

(ii) If k is odd and if $f \in \Lambda^k(\mathbb{R}^n)$ with $\operatorname{rank}[f] = k+2$, then there exist $u \in \Lambda^1(\mathbb{R}^n)$ and $g \in \Lambda^{k-1}(\mathbb{R}^n)$ such that

$$f = g \wedge u.$$

Our main result (cf. Theorem 2.45 for a more general statement) will be the following theorem obtained by Dacorogna–Kneuss [31]. It generalizes the Cartan lemma.

Theorem 1.5. *Let $0 \leq l \leq k \leq n$ be integers. Let $g \in \Lambda^l(\mathbb{R}^n)$ and $f \in \Lambda^k(\mathbb{R}^n)$. The following statements are then equivalent:*

(i) There exists $u \in \Lambda^{k-l}(\mathbb{R}^n)$ verifying

$$f = g \wedge u.$$

(ii) For every $h \in \Lambda^{n-k}(\mathbb{R}^n)$, the following implication holds:

$$[h \wedge g = 0] \quad \Rightarrow \quad [h \wedge f = 0].$$

1.2.3 Differential Forms

In Chapter 3 we have gathered the main notations concerning differential forms.

Definition 1.6. Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C^1\left(\Omega; \Lambda^k\right)$, namely

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

(i) The *exterior derivative* of f denoted df belongs to $C^0\left(\Omega; \Lambda^{k+1}\right)$ and is defined by

$$df = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{m=1}^{n} \frac{\partial f_{i_1 \cdots i_k}}{\partial x_m} dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

If $k = n$, then $df = 0$.

(ii) The *interior derivative* or *codifferential* of f denoted δf belongs to $C^0\left(\Omega;\Lambda^{k-1}\right)$ and is defined by

$$\delta f = (-1)^{n(k-1)} * (d(*f)) .$$

Remark 1.7. (i) If $k=0$, then the operator d can be identified with the gradient operator, while $\delta f = 0$ for any f.

(ii) If $k=1$, then the operator d can be identified with the curl operator and the operator δ is the divergence operator.

We next gather some well-known properties of the operators d and δ (cf. Theorems 3.5 and 3.7).

Theorem 1.8. *Let* $f \in C^2(\Omega;\Lambda^k)$. *Then*

$$ddf = 0, \quad \delta\delta f = 0 \quad \textit{and} \quad d\delta f + \delta d f = \Delta f.$$

We also need the following definition. In the sequel we will denote the exterior unit normal of $\partial\Omega$ by ν.

Definition 1.9. The *tangential component* of a k-form f on $\partial\Omega$ is the $(k+1)$-form

$$\nu \wedge f \in \Lambda^{k+1}.$$

The *normal component* of a k-form f on $\partial\Omega$ is the $(k-1)$-form

$$\nu \lrcorner f \in \Lambda^{k-1}.$$

We easily deduce the following properties (cf. Theorem 3.23).

Proposition 1.10. *Let* $0 \le k \le n$ *and* $f \in C^1\left(\overline{\Omega};\Lambda^k\right)$*; then*

$$\begin{aligned} \nu \wedge f = 0 \text{ on } \partial\Omega \ &\Rightarrow\ \nu \wedge df = 0 \text{ on } \partial\Omega, \\ \nu \lrcorner f = 0 \text{ on } \partial\Omega \ &\Rightarrow\ \nu \lrcorner \delta f = 0 \text{ on } \partial\Omega. \end{aligned}$$

We will constantly use the integration by parts formula (cf. Theorem 3.28).

Theorem 1.11. *Let* $1 \le k \le n$, $f \in C^1\left(\overline{\Omega};\Lambda^{k-1}\right)$ *and* $g \in C^1\left(\overline{\Omega};\Lambda^k\right)$. *Then*

$$\int_\Omega \langle df; g\rangle + \int_\Omega \langle f; \delta g\rangle = \int_{\partial\Omega} \langle \nu \wedge f; g\rangle = \int_{\partial\Omega} \langle f; \nu \lrcorner g\rangle.$$

We will adopt the following notations.

Notation 1.12. *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $r \ge 0$ *be an integer and* $0 \le \alpha \le 1 \le p \le \infty$. *Spaces with vanishing tangential or normal component will be denoted in the following way:*

$$\begin{aligned}
C_T^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) &= \{f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) : \nu \wedge f = 0 \quad \text{on } \partial\Omega\},\\
C_N^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) &= \{f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) : \nu \lrcorner f = 0 \quad \text{on } \partial\Omega\},\\
W_T^{r+1,p}\left(\Omega;\Lambda^k\right) &= \{f \in W^{r+1,p}\left(\Omega;\Lambda^k\right) : \nu \wedge f = 0 \quad \text{on } \partial\Omega\},\\
W_N^{r+1,p}\left(\Omega;\Lambda^k\right) &= \{f \in W^{r+1,p}\left(\Omega;\Lambda^k\right) : \nu \lrcorner f = 0 \quad \text{on } \partial\Omega\}.
\end{aligned}$$

The different sets of harmonic fields *will be denoted by*

$$\begin{aligned}
\mathscr{H}\left(\Omega;\Lambda^k\right) &= \{f \in W^{1,2}\left(\Omega;\Lambda^k\right) : df = 0 \text{ and } \delta f = 0 \text{ in } \Omega\},\\
\mathscr{H}_T\left(\Omega;\Lambda^k\right) &= \{f \in \mathscr{H}\left(\Omega;\Lambda^k\right) : \nu \wedge f = 0 \quad \text{on } \partial\Omega\},\\
\mathscr{H}_N\left(\Omega;\Lambda^k\right) &= \{f \in \mathscr{H}\left(\Omega;\Lambda^k\right) : \nu \lrcorner f = 0 \quad \text{on } \partial\Omega\}.
\end{aligned}$$

We now list (cf. Section 6.1) some properties of the harmonic fields.

Theorem 1.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$\mathscr{H}\left(\Omega;\Lambda^k\right) \subset C^\infty\left(\Omega;\Lambda^k\right).$$

Moreover if Ω is bounded and smooth, then the next statements are valid.

(i) The following inclusion holds:

$$\mathscr{H}_T\left(\Omega;\Lambda^k\right) \cup \mathscr{H}_N\left(\Omega;\Lambda^k\right) \subset C^\infty\left(\overline{\Omega};\Lambda^k\right).$$

Furthermore, if $r \geq 0$ is an integer and $0 \leq \alpha \leq 1$, then there exists $C = C(r,\Omega)$ such that for every $\omega \in \mathscr{H}_T\left(\Omega;\Lambda^k\right) \cup \mathscr{H}_N\left(\Omega;\Lambda^k\right)$,

$$\|\omega\|_{W^{r,2}} \leq C\|\omega\|_{L^2} \quad \text{and} \quad \|\omega\|_{C^{r,\alpha}} \leq C\|\omega\|_{C^0}.$$

(ii) The spaces $\mathscr{H}_T\left(\Omega;\Lambda^k\right)$ and $\mathscr{H}_N\left(\Omega;\Lambda^k\right)$ are finite dimensional and closed in $L^2\left(\Omega;\Lambda^k\right)$.

(iii) Furthermore, if Ω is contractible (cf. Definition 6.1), then

$$\begin{aligned}
\mathscr{H}_T\left(\Omega;\Lambda^k\right) &= \{0\} \quad \text{if } 0 \leq k \leq n-1,\\
\mathscr{H}_N\left(\Omega;\Lambda^k\right) &= \{0\} \quad \text{if } 1 \leq k \leq n.
\end{aligned}$$

(iv) If $k = 0$ or $k = n$ and $h \in \mathscr{H}\left(\Omega;\Lambda^k\right)$, then h is constant on each connected component of Ω. In particular, $\mathscr{H}_T\left(\Omega;\Lambda^0\right) = \{0\}$ and $\mathscr{H}_N\left(\Omega;\Lambda^n\right) = \{0\}$.

Remark 1.14. If $k = 1$ and assuming that Ω is smooth, then the sets $\mathscr{H}_T$ and $\mathscr{H}_N$ can be rewritten, as usual by abuse of notations, as

$$\mathscr{H}_T\left(\Omega;\Lambda^1\right) = \left\{ f \in C^\infty\left(\overline{\Omega};\mathbb{R}^n\right) : \begin{bmatrix} \operatorname{curl} f = 0 \text{ and } \operatorname{div} f = 0 \\ f_i\nu_j - f_j\nu_i = 0,\ \forall 1 \leq i < j \leq n \end{bmatrix} \right\},$$

$$\mathscr{H}_N\left(\Omega;\Lambda^1\right) = \left\{ f \in C^\infty\left(\overline{\Omega};\mathbb{R}^n\right) : \begin{bmatrix} \operatorname{curl} f = 0 \text{ and } \operatorname{div} f = 0 \\ \sum_{i=1}^n f_i\nu_i = 0 \end{bmatrix} \right\}.$$

Moreover, if Ω is simply connected, then

$$\mathscr{H}_T\left(\Omega;\Lambda^1\right) = \mathscr{H}_N\left(\Omega;\Lambda^1\right) = \{0\}.$$

1.3 Hodge–Morrey Decomposition and Poincaré Lemma

1.3.1 A General Identity and Gaffney Inequality

In the proof of Morrey of the Hodge decomposition, one of the key points to get compactness is the following inequality (cf. Theorem 5.16).

Theorem 1.15 (Gaffney inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\|\omega\|^2_{W^{1,2}} \le C\left(\|d\omega\|^2_{L^2} + \|\delta\omega\|^2_{L^2} + \|\omega\|^2_{L^2}\right)$$

for every $\omega \in W^{1,2}_T(\Omega;\Lambda^k) \cup W^{1,2}_N(\Omega;\Lambda^k)$.

Remark 1.16. When $k = 1$, the inequality says, identifying 1-forms with vector fields,

$$\|\omega\|^2_{W^{1,2}} \le C\left(\|\operatorname{curl}\omega\|^2_{L^2} + \|\operatorname{div}\omega\|^2_{L^2} + \|\omega\|^2_{L^2}\right)$$

for every $\omega \in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfying either one of the following two conditions:

$$\nu \wedge \omega = 0 \ \Leftrightarrow \ \omega_i \nu_j - \omega_j \nu_i = 0, \ \forall\, 1 \le i < j \le n,$$
$$\nu \lrcorner \omega = \langle \nu;\omega\rangle = \sum_{i=1}^n \omega_i \nu_i = 0.$$

The inequality, as stated above, has been proved by Morrey [76, 77], generalizing results of Gaffney [44, 45]. We will prove in Section 5.3 the inequality appealing to a very general identity (see Theorem 5.7) proved by Csató and Dacorogna [24].

Theorem 1.17 (A general identity). *Let $0 \le k \le n$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and with exterior unit normal ν. Then every $\alpha, \beta \in C^1(\overline{\Omega};\Lambda^k)$ satisfy the equation*

$$\begin{aligned}
&\int_\Omega \left(\langle d\alpha; d\beta\rangle + \langle \delta\alpha;\delta\beta\rangle - \langle \nabla\alpha;\nabla\beta\rangle\right) \\
&\quad = -\int_{\partial\Omega} \left(\langle \nu \wedge d(\nu \lrcorner \alpha); \nu \wedge \beta\rangle + \langle \nu \lrcorner \delta(\nu \wedge \alpha); \nu \lrcorner \beta\rangle\right) \\
&\qquad + \int_{\partial\Omega} \left(\langle L^\nu(\nu \wedge \alpha); \nu \wedge \beta\rangle + \langle K^\nu(\nu \lrcorner \alpha); \nu \lrcorner \beta\rangle\right).
\end{aligned}$$

The operators L^ν and K^ν (cf. Definition 5.1) can be seen as matrices acting on $(k+1)$-forms and $(k-1)$-forms respectively (identifying, as usual, a k-form with a $\binom{n}{k}$ vector). They depend only on the geometry of Ω and on the degree k of the form. They can easily be calculated explicitly for general k-forms and, when Ω is a ball of radius R (cf. Corollary 5.9), it turns out that

$$L^{\nu}(\nu \wedge \omega) = \frac{k}{R}\,\nu \wedge \omega \quad \text{and} \quad K^{\nu}(\nu \lrcorner \omega) = \frac{n-k}{R}\,\nu \lrcorner \omega$$

and, thus,

$$\langle L^{\nu}(\nu \wedge \omega); \nu \wedge \omega \rangle = \frac{k}{R}\,|\nu \wedge \omega|^2 \quad \text{and} \quad \langle K^{\nu}(\nu \lrcorner \omega); \nu \lrcorner \omega \rangle = \frac{n-k}{R}\,|\nu \lrcorner \omega|^2.$$

In the case of a 1-form and for general open sets Ω (cf. Proposition 5.11), it can be shown that K^{ν} is a scalar and it is a multiple of κ, the mean curvature of the hypersurface $\partial\Omega$, namely

$$K^{\nu} = (n-1)\,\kappa.$$

Summarizing the results for a 1-form ω in $\mathbb{R}^n$ (cf. Corollary 5.12) with vanishing tangential component (i.e., $\nu \wedge \omega = 0$ on $\partial\Omega$), we have

$$\int_{\Omega} \left(|\operatorname{curl} \omega|^2 + |\operatorname{div} \omega|^2 - |\nabla \omega|^2 \right) = (n-1) \int_{\partial\Omega} \kappa\, [\langle \nu; \omega \rangle]^2,$$

where κ is the mean curvature of the hypersurface $\partial\Omega$ and $\langle .;. \rangle$ denotes the scalar product in $\mathbb{R}^n$.

1.3.2 The Hodge–Morrey Decomposition

We now turn to the celebrated Hodge–Morrey decomposition (cf. Theorem 6.9).

Theorem 1.18 (Hodge–Morrey decomposition). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $0 \leq k \leq n$ and $f \in L^2(\Omega; \Lambda^k)$. Then there exist*

$$\alpha \in W_T^{1,2}(\Omega; \Lambda^{k-1}), \quad \beta \in W_T^{1,2}(\Omega; \Lambda^{k+1}),$$
$$h \in \mathcal{H}_T(\Omega; \Lambda^k) \quad \text{and} \quad \omega \in W_T^{2,2}(\Omega; \Lambda^k)$$

such that, in Ω,

$$f = d\alpha + \delta\beta + h, \quad \alpha = \delta\omega \quad \text{and} \quad \beta = d\omega.$$

Remark 1.19. (i) We have quoted only one of the three decompositions (cf. Theorem 6.9 for details). Another one, completely similar, is by replacing T by N and the other one mixing both T and N.

(ii) If $k \leq n-1$ and if Ω is contractible, then $h = 0$.

(iii) If $k = 0$, then the theorem reads as

$$f = \delta\beta = \delta d\omega = \Delta\omega \text{ in } \Omega \quad \text{with} \quad \omega = 0 \text{ on } \partial\Omega.$$

(iv) When $k = 1$ and $n = 3$, the decomposition reads as follows. Let ν be the exterior unit normal. For any $f \in L^2(\Omega;\mathbb{R}^3)$, there exist

$$\omega \in W^{2,2}\left(\Omega;\mathbb{R}^3\right) \quad \text{with } \omega_i \nu_j - \omega_j \nu_i = 0 \text{ on } \partial\Omega,\ \forall\, 1 \le i < j \le 3$$
$$\alpha \in W_0^{1,2}(\Omega) \quad \text{and} \quad \alpha = \operatorname{div}\omega,$$
$$\beta \in W^{1,2}\left(\Omega;\mathbb{R}^3\right) \quad \text{with } \beta = -\operatorname{curl}\omega \quad \text{and} \quad \langle \nu;\beta\rangle = 0 \text{ on } \partial\Omega$$
$$h \in \left\{ h \in C^\infty\left(\overline{\Omega};\mathbb{R}^3\right) : \left[\begin{array}{c} \operatorname{curl} h = 0 \text{ and } \operatorname{div} h = 0 \\ h_i \nu_j - h_j \nu_i = 0,\ \forall\, 1 \le i < j \le 3 \end{array} \right. \right\}$$

such that

$$f = \operatorname{grad}\alpha + \operatorname{curl}\beta + h \text{ in } \Omega.$$

Furthermore, if Ω is simply connected, then $h = 0$.

(v) If f is more regular than in L^2, then α, β and ω are in the corresponding class of regularity (cf. Theorem 6.12). More precisely if, for example, $r \ge 0$ is an integer, $0 < q < 1$ and $f \in C^{r,q}(\overline{\Omega};\Lambda^k)$, then

$$\alpha \in C^{r+1,q}\left(\overline{\Omega};\Lambda^{k-1}\right), \quad \beta \in C^{r+1,q}\left(\overline{\Omega};\Lambda^{k+1}\right) \quad \text{and} \quad \omega \in C^{r+2,q}\left(\overline{\Omega};\Lambda^{k}\right).$$

(vi) The proof of Morrey (cf. Theorem 6.7) uses the direct methods of the calculus of variations. One minimizes

$$D_f(\omega) = \int_\Omega \left(\frac{1}{2}|d\omega|^2 + \frac{1}{2}|\delta\omega|^2 + \langle f;\omega\rangle \right)$$

in an appropriate space, Gaffney inequality giving the coercivity of the integral.

1.3.3 First-Order Systems of Cauchy–Riemann Type

It turns out that the Hodge–Morrey decomposition is in fact equivalent (cf. Proposition 7.9) to solving the first-order system

$$\begin{cases} d\omega = f \quad \text{and} \quad \delta\omega = g & \text{in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 & \text{on } \partial\Omega \end{cases}$$

or the similar one,

$$\begin{cases} d\omega = f \quad \text{and} \quad \delta\omega = g & \text{in } \Omega, \\ \nu \lrcorner \omega = \nu \lrcorner \omega_0 & \text{on } \partial\Omega. \end{cases}$$

Both systems are discussed in Theorems 7.2 and 7.4. We here state a simplified version of the first one.

Theorem 1.20. *Let $r \geq 0$ and $1 \leq k \leq n-2$ be integers, $0 < q < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded contractible open smooth set and with exterior unit normal ν. Let $g \in C^{r,q}\left(\overline{\Omega};\Lambda^{k-1}\right)$ and $f \in C^{r,q}\left(\overline{\Omega};\Lambda^{k+1}\right)$ be such that*

$$\delta g = 0 \text{ in } \Omega, \quad df = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge f = 0 \text{ on } \partial\Omega.$$

Then there exists $\omega \in C^{r+1,q}\left(\overline{\Omega};\Lambda^k\right)$, such that

$$\begin{cases} d\omega = f \quad \text{and} \quad \delta\omega = g & \text{in } \Omega, \\ \qquad \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 1.21. (i) It turns out that the sufficient conditions are also necessary (cf. Theorems 7.2 and 7.4).

(ii) When $k = n-1$, the result is valid provided

$$\int_\Omega f = 0.$$

Note that in this case the conditions $df = 0$ and $\nu \wedge f = 0$ are automatically fulfilled.

(iii) Completely analogous results are given in Theorems 7.2 and 7.4 for Sobolev spaces.

(iv) If Ω is not contractible, then additional necessary conditions have to be added.

(v) When $k = 1$ and $n = 3$, the theorem reads as follows. Let $\Omega \subset \mathbb{R}^3$ be a bounded contractible smooth open set, $g \in C^{r,q}\left(\overline{\Omega}\right)$ and $f \in C^{r,q}\left(\overline{\Omega};\mathbb{R}^3\right)$ be such that

$$\operatorname{div} f = 0 \text{ in } \Omega \quad \text{and} \quad \langle f;\nu\rangle = 0 \text{ on } \partial\Omega.$$

Then there exists $\omega \in C^{r+1,q}\left(\overline{\Omega};\mathbb{R}^3\right)$ such that

$$\begin{cases} \operatorname{curl}\omega = f \quad \text{and} \quad \operatorname{div}\omega = g & \text{in } \Omega, \\ \omega_i\nu_j - \omega_j\nu_i = 0 \ \forall\, 1 \leq i < j \leq 3 & \text{on } \partial\Omega. \end{cases}$$

1.3.4 Poincaré Lemma

We start with the classical Poincaré lemma (cf. Theorem 8.1).

Theorem 1.22 (Poincaré lemma). *Let $r \geq 1$ and $0 \leq k \leq n-1$ be integers and $\Omega \subset \mathbb{R}^n$ be an open contractible set. Let $g \in C^r\left(\Omega;\Lambda^{k+1}\right)$ with $dg = 0$ in Ω. Then there exists $G \in C^r\left(\Omega;\Lambda^k\right)$ such that*

$$dG = g \quad \text{in } \Omega.$$

With the help of the Hodge–Morrey decomposition, the result can be improved (cf. Theorem 8.3) in two directions. First, one can consider general sets Ω, not only contractible sets. Moreover, one can get sharp regularity in Hölder and in Sobolev spaces. We quote here only the case of Hölder spaces. We also give the theorem with the d operator. Analogous results are also valid for the δ operator; see Theorem 8.4.

Theorem 1.23. *Let $r \geq 0$ and $0 \leq k \leq n-1$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $f : \overline{\Omega} \to \Lambda^{k+1}$. The following statements are equivalent:*

(i) Let $f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^{k+1}\right)$ be such that

$$df = 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega \langle f;\psi\rangle = 0 \text{ for every } \psi \in \mathscr{H}_N\left(\Omega;\Lambda^{k+1}\right).$$

(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^{k}\right)$ such that

$$d\omega = f \quad \text{in } \Omega.$$

Remark 1.24. (i) When $k = n-1$, there is no restriction on the solvability of $d\omega = f$.

(ii) Recall that if Ω is contractible and $0 \leq k \leq n-1$, then

$$\mathscr{H}_N\left(\Omega;\Lambda^{k+1}\right) = \{0\}.$$

We finally consider the boundary value problems

$$\begin{cases} d\omega = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \delta\omega = g & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

We give a result for the first one and for $\omega_0 = 0$ (cf. Theorem 8.16 for general ω_0), but a similar one (cf. Theorem 8.18) exists for the second problem. We only discuss the case of Hölder spaces, but the result is also valid in Sobolev spaces (see Theorems 8.16 and 8.18 for details).

Theorem 1.25. *Let $r \geq 0$ and $0 \leq k \leq n-1$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and with exterior unit normal ν. Then the following statements are equivalent:*

(i) Let $f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^{k+1}\right)$ satisfy

$$df = 0 \text{ in } \Omega, \quad \nu \wedge f = 0 \text{ on } \partial\Omega,$$

and, for every $\chi \in \mathscr{H}_T\left(\Omega;\Lambda^{k+1}\right)$,

$$\int_\Omega \langle f;\chi\rangle = 0.$$

(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^k\right)$ *such that*

$$\begin{cases} d\omega = f & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

1.4 The Case of Volume Forms

1.4.1 Statement of the Problem

In Part III, we will discuss the following problem. Given Ω a bounded open set in $\mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}$, we want to find $\varphi : \overline{\Omega} \to \mathbb{R}^n$ verifying

$$\begin{cases} g(\varphi(x)) \det \nabla\varphi(x) = f(x) & x \in \Omega, \\ \varphi(x) = x & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Writing the functions f and g as volume forms through the straightforward identification

$$g = g(x)dx^1 \wedge \cdots \wedge dx^n \quad \text{and} \quad f = f(x)dx^1 \wedge \cdots \wedge dx^n,$$

problem (1.2) can be written as

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \text{id} & \text{on } \partial\Omega, \end{cases}$$

where $\varphi^*(g)$ is the pullback of g by φ.

The following preliminary remarks are in order.

(i) The case $n = 1$ is completely elementary and is discussed in Section 1.4.2.

(ii) When $n \geq 2$, the equation in (1.2) is a nonlinear first-order *partial differential equation* homogeneous of degree n in the derivatives. It is *underdetermined*, in the sense that we have n unknowns (the components of φ) and only one equation. Related to this observation, we have that if there exists a solution to our problem, then there are infinitely many ones. Indeed, for example, if $n = 2$, Ω is the unit ball and $f = g = 1$, the maps φ_m (written in polar and in Cartesian coordinates) defined by

$$\begin{aligned} \varphi_m(x) = \varphi_m(x_1, x_2) &= \begin{pmatrix} r\cos\left(\theta + 2m\pi r^2\right) \\ r\sin\left(\theta + 2m\pi r^2\right) \end{pmatrix} \\ &= \begin{pmatrix} x_1\cos\left(2m\pi\left(x_1^2 + x_2^2\right)\right) - x_2\sin\left(2m\pi\left(x_1^2 + x_2^2\right)\right) \\ x_2\cos\left(2m\pi\left(x_1^2 + x_2^2\right)\right) + x_1\sin\left(2m\pi\left(x_1^2 + x_2^2\right)\right) \end{pmatrix} \end{aligned}$$

satisfy (1.2) for every $m \in \mathbb{Z}$.

(iii) An integration by parts, or, what amounts to the same thing, an elementary topological degree argument (see (19.3)), immediately gives the *necessary condition* (independently of the fact that φ is a diffeomorphism or not and of the fact that $\varphi(\Omega)$ contains strictly or not Ω)

$$\int_\Omega f = \int_\Omega g. \tag{1.3}$$

In most of our analysis, it will turn out that this condition is also sufficient.

(iv) We will always assume that $g > 0$. If g is not strictly positive, then hypotheses other than (1.3) are necessary; for example, f cannot be strictly positive. Indeed if, for example, $f \equiv 1$ and g is allowed to vanish even at a single point, then no C^1 solution of our problem exists (cf. Proposition 11.6). However, in a very special case (cf. Lemma 11.21), we will deal with functions f and g that *both* change sign.

(v) We will, however, allow f to change sign, but the analysis is very different if $f > 0$ or if f vanishes, even at a single point, let alone if it becomes negative. The first problem will be discussed in Chapter 10, whereas the second one will be dealt with in Chapter 11. One of the main differences is that in the first case, any solution of (1.2) is necessarily a diffeomorphism (cf. Theorem 19.12), whereas this is never true in the second case.

(vi) It is easy to see (cf. Corollary 19.4) that any solution of (1.2) satisfies

$$\varphi(\Omega) \supset \Omega \quad \text{and} \quad \varphi(\overline{\Omega}) \supset \overline{\Omega}. \tag{1.4}$$

If $f > 0$, we have, since φ is a diffeomorphism, that (cf. Theorem 19.12)

$$\varphi(\Omega) = \Omega \quad \text{and} \quad \varphi(\overline{\Omega}) = \overline{\Omega}.$$

If this is not the case, then, in general, the inclusions can be strict. We will discuss in Chapter 11 this matter in details.

(vii) Problem (1.2) admits a *weak formulation*. Indeed, if φ is a diffeomorphism, we can write (cf. Theorem 19.7) the equation $g(\varphi)\det\nabla\varphi = f$ as

$$\int_{\varphi(E)} g = \int_E f \quad \text{for every open set } E \subset \Omega$$

or, equivalently,

$$\int_\Omega g\,\zeta\left(\varphi^{-1}\right) = \int_\Omega f\,\zeta \quad \text{for every } \zeta \in C_0^\infty(\Omega).$$

We observe that both new writings make sense if φ is only a homeomorphism.

(viii) The problem can be seen as a question of *mass transportation*. Indeed, we want to transport the mass distribution g to the mass distribution f without moving the points of the boundary of Ω. In this context, the equation is usually written as

$$\int_E g = \int_{\varphi^{-1}(E)} f \quad \text{for every open set } E \subset \Omega.$$

The problem of *optimal* mass transportation has received considerable attention. We should point out that our analysis is not in this framework. The two main strong points of our analysis are that we are able to find smooth solutions, sometimes with the optimal regularity and to deal with fixed boundary data.

1.4.2 The One-Dimensional Case

As already stated, the case $n = 1$ is completely elementary (cf. Proposition 11.4), but it exhibits some striking differences with the case $n \geq 2$. However, it may shed some light on some issues that we will discuss in the higher-dimensional case. Let $\Omega = (a,b)$,

$$F(x) = \int_a^x f(t)\,dt \quad \text{and} \quad G(x) = \int_a^x g(t)\,dt.$$

Then problem (1.2) becomes

$$\begin{cases} G(\varphi(x)) = F(x) & \text{if } x \in (a,b), \\ \varphi(a) = a \quad \text{and} \quad \varphi(b) = b. \end{cases}$$

If G is invertible and this happens if, for example, $g > 0$ and if

$$F([a,b]) \subset G(\mathbb{R}), \tag{1.5}$$

and this happens if, for example, $g \geq g_0 > 0$, then the problem has the solution

$$\varphi(x) = G^{-1}(F(x)).$$

The necessary condition (1.3)

$$\int_a^b f = \int_a^b g$$

ensures that

$$\varphi(a) = a \quad \text{and} \quad \varphi(b) = b.$$

This very elementary analysis leads to the following conclusions:

1) Contrary to the case $n \geq 2$, the necessary condition (1.3) is not sufficient. We need the extra condition (1.5); see Proposition 11.4 for details.

2) The problem has a *unique* solution, contrary to the case $n \geq 2$.

3) If f and g are in the space C^r, then the solution φ is in C^{r+1}.

4) If $f > 0$, then φ is a diffeomorphism from $[a,b]$ onto itself.

5) If f is allowed to change sign, then, in general,

$$[a,b] \underset{\neq}{\subset} \varphi([a,b]).$$

For example, this always happens if $f(a) < 0$ or $f(b) < 0$.

1.4.3 The Case $f \cdot g > 0$

In Chapter 10 we will study problem (1.2) when $f \cdot g > 0$. It will be seen that (1.3) is sufficient to solve (1.2) and that any solution is in fact a diffeomorphism from $\overline{\Omega}$ to $\overline{\Omega}$ (see Theorem 19.12). This last observation implies, in particular, a symmetry in f and g and allows us to restrict ourselves, without loss of generality, to the case $g \equiv 1$. Our main result (cf. Theorem 10.3) will be the following.

Theorem 1.26 (Dacorogna–Moser theorem). *Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set. Then the two following statements are equivalent:*

(i) The function $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$, $f > 0$ in $\overline{\Omega}$ and satisfies

$$\int_\Omega f = \operatorname{meas} \Omega.$$

(ii) There exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$ satisfying

$$\begin{cases} \det \nabla\varphi(x) = f(x) & x \in \Omega, \\ \varphi(x) = x & x \in \partial\Omega. \end{cases}$$

Furthermore, if $c > 0$ is such that

$$\left\|\frac{1}{f}\right\|_{C^0}, \quad \|f\|_{C^{0,\alpha}} \leq c,$$

then there exists a constant $C = C(c,r,\alpha,\Omega) > 0$ such that

$$\|\varphi - \operatorname{id}\|_{C^{r+1,\alpha}} \leq C\,\|f-1\|_{C^{r,\alpha}}.$$

The study of this problem originated in the seminal work of Moser [78]. The above optimal theorem was obtained by Dacorogna and Moser [33]. Burago and Kleiner [19] and Mc Mullen [73], independently, proved that the result is false if $r = \alpha = 0$, suggesting that the gain of regularity is to be expected only when $0 < \alpha < 1$.

In Section 10.5 (cf. Theorem 10.11), we present a different approach proposed by Dacorogna and Moser [33] to solve our problem. This method is constructive and does not use the regularity of elliptic differential operators; in this sense, it is more elementary. The drawback is that it does not provide any gain of regularity, which is the strong point of the above theorem. However, the advantage is that it is much more flexible. For example, if we assume in (1.2) that

$$\operatorname{supp}(f-g) \subset \Omega,$$

then we will be able to find φ such that

$$\operatorname{supp}(\varphi - \mathrm{id}) \subset \Omega.$$

This type of result, unreachable by the method of elliptic partial differential equations, will turn out to be crucial in Chapter 11.

1.4.4 The Case with No Sign Hypothesis on f

In Chapter 11, we discuss the case where the function f is allowed to change sign and we will follow Cupini, Dacorogna and Kneuss [25]. As already pointed out, we will however (apart from a very special case) assume that $g > 0$. In fact, contrary to the case $f \cdot g > 0$, the problem is no longer symmetric in f and g.

We start by observing that if f vanishes even at a single point, then the solution φ cannot be a diffeomorphism, although it can be a homeomorphism. In any case, if f is negative somewhere, it can never be a homeomorphism (see Proposition 19.14). Furthermore, if f is negative in some parts of the boundary, then any solution φ must go out of the domain (see Proposition 11.3); more precisely,

$$\overline{\Omega} \underset{\neq}{\subset} \varphi(\overline{\Omega}).$$

A special case of our theorem (cf. Theorem 11.1) is the following.

Theorem 1.27. *Let $n \geq 2$ and $r \geq 1$ be integers. Let $B_1 \subset \mathbb{R}^n$ be the open unit ball. Let $f \in C^r(\overline{B}_1)$ be such that*

$$\int_{B_1} f = \operatorname{meas} B_1 .$$

Then there exists $\varphi \in C^r(\overline{B}_1; \mathbb{R}^n)$ satisfying

$$\begin{cases} \det \nabla \varphi(x) = f(x) & x \in B_1, \\ \varphi(x) = x & x \in \partial B_1 . \end{cases}$$

Furthermore, the following conclusions also hold:

(i) If either $f > 0$ on ∂B_1 or $f \geq 0$ in $\overline{B}_1$, then φ can be chosen so that

$$\varphi(\overline{B}_1) = \overline{B}_1 .$$

(ii) If $f \geq 0$ in $\overline{B}_1$ and $f^{-1}(0) \cap B_1$ is countable, then φ can be chosen as a homeomorphism from $\overline{B}_1$ onto $\overline{B}_1$.

1.5 The Case $0 \le k \le n-1$

Having dealt with the case $k = n$, we now discuss the equation

$$\varphi^*(g) = f$$

when $0 \le k \le n-1$. The cases $k = 0, 1, n-1$ are the simplest ones. The most important results of Part IV are for the case $k = 2$, where we obtain not only a local result but also a global one; we, moreover, obtain sharp regularity results for both cases. The case $3 \le k \le n-2$ is considerably harder, even at the algebraic level and we will be able to obtain results only for forms having a special structure.

We first point out the following necessary conditions (cf. Proposition 17.1).

Proposition 1.28. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and $\varphi \in \operatorname{Diff}^1\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$. Let $1 \le k \le n$, $f \in C^1\left(\overline{\Omega}; \Lambda^k\right)$ and $g \in C^1\left(\varphi\left(\overline{\Omega}\right); \Lambda^k\right)$ be such that*

$$\varphi^*(g) = f \text{ in } \Omega.$$

(i) For every $x \in \Omega$,

$$\operatorname{rank}[g(\varphi(x))] = \operatorname{rank}[f(x)] \quad \text{and} \quad \operatorname{rank}[dg(\varphi(x))] = \operatorname{rank}[df(x)].$$

In particular,

$$dg = 0 \text{ in } \varphi(\Omega) \Leftrightarrow df = 0 \text{ in } \Omega.$$

(ii) If $\varphi(x) = x$ for $x \in \partial\Omega$, then

$$\nu \wedge f = \nu \wedge g \text{ on } \partial\Omega,$$

where ν is the exterior unit normal to Ω.

If we drop the condition that φ is a diffeomorphism, then the rank is, in general, not conserved. We have already seen such a phenomenon when $k = n$ in Theorem 1.27.

1.5.1 The Flow Method

One of the simplest and most elegant tools that we will use for the pullback equation is Theorem 12.7 and it was first established by Moser in [78], who, however, dealt only with manifolds without boundary. Its main drawback is that it does not provide the expected gain in regularity.

Theorem 1.29. *Let $r \ge 1$ and $0 \le k \le n$ be integers, $0 \le \alpha \le 1$, $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Let*

$$u \in C^{r,\alpha}\left([0,T] \times \overline{\Omega};\mathbb{R}^n\right),\ u = u(t,x) = u_t(x),$$
$$f \in C^{r,\alpha}\left([0,T] \times \overline{\Omega};\Lambda^k\right),\ f = f(t,x) = f_t(x)$$

be such that for every $t \in [0,T]$,

$$u_t = 0 \text{ on } \partial\Omega, \quad df_t = 0 \text{ in } \Omega \quad \text{and} \quad d(u_t \lrcorner f_t) = -\frac{d}{dt} f_t \text{ in } \Omega.$$

Then for every $t \in [0,T]$, the solution φ_t of

$$\begin{cases} \dfrac{d}{dt}\varphi_t = u_t \circ \varphi_t, & 0 \le t \le T \\ \varphi_0 = \mathrm{id} \end{cases}$$

belongs to $\mathrm{Diff}^{r,\alpha}\left(\overline{\Omega};\overline{\Omega}\right)$, *satisfies* $\varphi_t = \mathrm{id}$ *on* $\partial\Omega$ *and*

$$\varphi_t^*(f_t) = f_0 \quad \text{in} \quad \Omega.$$

1.5.2 The Cases $k = 0$ and $k = 1$

We start with the case $k = 0$, which is particularly elementary. We have for example the following local result (cf. Theorem 13.1). For a global result, see Theorem 13.2.

Theorem 1.30. *Let $r \ge 1$ be an integer, $x_0 \in \mathbb{R}^n$ and $f, g \in C^r$ in a neighborhood of x_0 and such that $f(x_0) = g(x_0)$,*

$$\nabla f(x_0) \ne 0 \quad \text{and} \quad \nabla g(x_0) \ne 0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \mathrm{Diff}^r(U;\varphi(U))$ such that

$$\varphi^*(g) = f \text{ in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

The results for $k = 0$ extend in a straightforward way to the case of closed 1-forms (cf. Corollaries 13.3 and 13.5).

We now give a theorem (cf. Theorems 13.8 and 13.10) for nonclosed 1-forms. It can be considered as the 1-form version of the Darboux theorem. We will see below that it is equivalent to the Darboux theorem for closed 2-forms.

Theorem 1.31. *Let $2 \le 2m \le n$ be integers, $x_0 \in \mathbb{R}^n$ and ω be a C^∞ 1-form such that $\omega(x_0) \ne 0$ and*

$$\mathrm{rank}[d\omega] = 2m \quad \text{in a neighborhood of } x_0.$$

Then there exist an open set U and

$$\varphi \in \mathrm{Diff}^\infty(U;\varphi(U))$$

such that $\varphi(U)$ is a neighborhood of x_0 and

$$\varphi^*(\omega) = \begin{cases} \sum_{i=1}^{m} x_{2i-1} dx^{2i} & \text{if } \omega \wedge (d\omega)^m = 0 \text{ in a neighborhood of } x_0 \\ \sum_{i=1}^{m} x_{2i-1} dx^{2i} + dx^{2m+1} & \text{if } \omega \wedge (d\omega)^m \neq 0 \text{ in a neighborhood of } x_0 . \end{cases}$$

Remark 1.32. (i) In the theorem, we have adopted the notation

$$(d\omega)^m = \underbrace{d\omega \wedge \cdots \wedge d\omega}_{m \text{ times}} .$$

(ii) Note that if $n = 2m$, then $\omega \wedge (d\omega)^m \equiv 0$.

1.5.3 The Case $k = 2$

Our best results besides the ones for volume forms are in the case $k = 2$.

We start with two *local* results. The first one is the celebrated *Darboux theorem*, but as stated it is due to Bandyopadhyay and Dacorogna [8] (cf. Theorem 14.1). The difference between the following theorem and all of the classical ones is in terms of regularity of the diffeomorphism. We provide the optimal possible regularity in Hölder spaces; the other ones give only that if $\omega \in C^{r,\alpha}$, then $\varphi \in C^{r,\alpha}$.

Theorem 1.33. *Let $r \geq 0$ and $n = 2m \geq 4$ be integers. Let $0 < \alpha < 1$ and $x_0 \in \mathbb{R}^n$. Let ω_m be the* standard symplectic form *of* $\operatorname{rank}[\omega_m] = 2m = n$*,*

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a 2-form. The two following statements are then equivalent:

(i) The 2-form ω is closed, is in $C^{r,\alpha}$ in a neighborhood of x_0 and verifies

$$\operatorname{rank}[\omega(x_0)] = n.$$

(ii) There exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r+1,\alpha}(U; \varphi(U))$ such that

$$\varphi^*(\omega_m) = \omega \text{ in } U \quad \text{and} \quad \varphi(x_0) = x_0 .$$

One possible proof of the theorem could be to use Theorem 1.31 with $n = 2m$ (cf. Remark 13.7 for details). We, however, will go the other way around and prove Theorem 1.31 using Theorem 1.33.

We next discuss the case of forms of lower rank. This is also well known in the literature. However, our theorem (cf. Theorem 14.3, proved in [9] by Bandyopadhyay, Dacorogna and Kneuss) provides, as the previous theorem, one class higher degree of regularity than the other results. Indeed, in all other theorems it is proved that if $\omega \in C^{r,\alpha}$, then $\varphi \in C^{r-1,\alpha}$. It may appear that the theorem below is still not optimal, since it only shows that $\varphi \in C^{r,\alpha}$ when $\omega \in C^{r,\alpha}$. However, since there are some missing variables, it is probably the best possible regularity.

Theorem 1.34. *Let $n \ge 3$, $r, m \ge 1$ be integers and $0 < \alpha < 1$. Let $x_0 \in \mathbb{R}^n$ and ω_m be the standard symplectic form of $\operatorname{rank}[\omega_m] = 2m < n$, namely*

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

Let ω be a $C^{r,\alpha}$ closed 2-form such that

$$\operatorname{rank}[\omega] = 2m \text{ in a neighborhood of } x_0.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^{r,\alpha}(U; \varphi(U))$ such that

$$\varphi^*(\omega_m) = \omega \text{ in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

We now turn to a *global* result (cf. Theorem 14.5). It has been obtained under slightly more restrictive hypotheses by Bandyopadhyay and Dacorogna [8] and as stated by Dacorogna and Kneuss [32]. The theorem provides the first global result on manifolds with boundary. It is also nearly optimal.

Theorem 1.35. *Let $n > 2$ be even and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with exterior unit normal ν. Let $0 < \alpha < 1$ and $r \ge 1$ be an integer. Let $f, g \in C^{r,\alpha}(\overline{\Omega}; \Lambda^2)$ satisfying $df = dg = 0$ in Ω,*

$$\nu \wedge f, \nu \wedge g \in C^{r+1,\alpha}\left(\partial\Omega; \Lambda^3\right), \quad \nu \wedge f = \nu \wedge g \text{ on } \partial\Omega,$$

$$\int_\Omega \langle f; \psi \rangle = \int_\Omega \langle g; \psi \rangle \quad \text{for every } \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right) \tag{1.6}$$

and, for every $t \in [0,1]$,

$$\operatorname{rank}[tg + (1-t) f] = n \quad \text{in } \overline{\Omega}.$$

Then there exists $\varphi \in \operatorname{Diff}^{r+1,\alpha}\left(\overline{\Omega}; \overline{\Omega}\right)$ such that

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \operatorname{id} & \text{on } \partial\Omega. \end{cases}$$

Remark 1.36. (i) In a similar way, we can consider a general homotopy f_t with $f_0 = f$, $f_1 = g$, provided

$$df_t = 0, \quad \nu \wedge f_t = \nu \wedge f_0 \text{ on } \partial\Omega \quad \text{and} \quad \operatorname{rank}[f_t] = n \text{ in } \overline{\Omega},$$

$$\int_\Omega \langle f_t; \psi \rangle = \int_\Omega \langle f_0; \psi \rangle \quad \text{for every } \psi \in \mathscr{H}_T\left(\Omega; \Lambda^2\right).$$

(ii) If Ω is contractible, then $\mathscr{H}_T\left(\Omega;\Lambda^2\right)=\{0\}$ and, therefore, (1.6) is automatically satisfied.

1.5.4 The Case $3 \le k \le n-1$

The presentation in Chapter 15 follows closely the results of Bandyopadhyay, Dacorogna and Kneuss [9]. We start with the case $k=n-1$. We have as a consequence of Theorems 15.3 and 15.5 the following result.

Theorem 1.37. *Let $x_0 \in \mathbb{R}^n$ and f be a $(n-1)$-form such that $f \in C^\infty$ in a neighborhood of x_0 and $f(x_0) \neq 0$. Then there exist a neighborhood U of x_0 and*

$$\varphi \in \operatorname{Diff}^\infty(U;\varphi(U))$$

such that φ satisfies one of the two following equations in U:

(i) If $df=0$ in a neighborhood of x_0, then

$$f=\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}=\varphi^*\left(dx^1 \wedge \cdots \wedge dx^{n-1}\right).$$

(ii) If $df(x_0)\neq 0$, then

$$f=\varphi^n\left(\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right)=\varphi^*\left(x_n\, dx^1 \wedge \cdots \wedge dx^{n-1}\right).$$

Remark 1.38. (i) The present theorem, when $df=0$, is a consequence of Theorem 15.1, which is valid for k-forms of rank k.

(ii) With our usual abuse of notations, identifying a $(n-1)$-form with a vector field and observing that the d operator can then be essentially identified with the divergence operator, we can rewrite the theorem as follows (cf. Corollaries 15.4 and 15.7). For any C^∞ vector field f such that $f(x_0)\neq 0$, there exist an open set U and

$$\varphi \in \operatorname{Diff}^\infty(U;\varphi(U))$$

such that $\varphi(U)$ is a neighborhood of x_0 and

$$f=\begin{cases} *\left(\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right) & \text{if } \operatorname{div} f=0 \\ *\left(\varphi^n\left(\nabla\varphi^1 \wedge \cdots \wedge \nabla\varphi^{n-1}\right)\right) & \text{if } \operatorname{div} f\neq 0, \end{cases}$$

where $*$ denotes the Hodge $*$ operator.

We now turn to the case $3 \le k \le n-2$, which is, as already said, much more difficult. This is so already at the algebraic level, since there are no known canonical forms. Additionally, even when the algebraic setting is simple, the analytical situation is more complicated than in the cases $k=0,1,2,n-1,n$ (see Proposition 15.14 for such an example). The only cases that we will be able to study in Chapter 15 are those that are combinations of 1 and 2-forms that we can handle separately.

For 1-forms, we easily obtain local (cf. Proposition 15.8) as well as global results (cf. Proposition 15.10). We now give a simple theorem (a more general statement can be found in Theorem 15.15) that deals with 3-forms obtained by product of a 1-form and a 2-form (in the same spirit, Theorem 15.12 allows to deal with some k-forms that are product of 1 and 2-forms).

Theorem 1.39. *Let $n = 2m \geq 4$ be integers, $x_0 \in \mathbb{R}^n$ and f be a C^∞ symplectic (i.e., closed and with* $\operatorname{rank}[f] = n$*) 2-form and a be a nonzero closed C^∞ 1-form. Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^\infty(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and*

$$\varphi^*(\omega_m) = f \quad \text{and} \quad \varphi^*(dx^n) = a \quad \text{in } U,$$

where

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

In particular, if

$$G = \left[\sum_{i=1}^{m-1} dx^{2i-1} \wedge dx^{2i}\right] \wedge dx^n = \omega_m \wedge dx^n,$$

then

$$\varphi^*(G) = f \wedge a \quad \text{in } U.$$

1.6 Hölder Spaces

Throughout the book we have used very fine properties of Hölder continuous functions. Most of the results discussed in Chapter 16 are "standard," but they are scattered in the literature. There does not exist such a huge literature as the one for Sobolev spaces. Some of the best references are Fefferman [42], Gilbarg and Trudinger [49] and Hörmander [55].

1.6.1 Definition and Extension of Hölder Functions

We give here the definition of Hölder continuous functions (cf. Definition 16.2).

Definition 1.40. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \overline{\Omega} \to \mathbb{R}$ and $0 < \alpha \leq 1$. Let

$$[f]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \right\}.$$

(i) The set $C^{0,\alpha}\left(\overline{\Omega}\right)$ is the set of $f \in C^0\left(\overline{\Omega}\right)$ so that

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} = \|f\|_{C^0(\overline{\Omega})} + [f]_{C^{0,\alpha}(\overline{\Omega})} < \infty,$$

where

$$\|f\|_{C^0(\overline{\Omega})} = \sup_{x\in\Omega}\{|f(x)|\}.$$

If there is no ambiguity, we drop the dependence on the set $\overline{\Omega}$ and write simply

$$\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_{C^{0,\alpha}}\,.$$

(ii) If $r \geq 1$ is an integer, then the set $C^{r,\alpha}\left(\overline{\Omega}\right)$ is the set of functions $f \in C^r\left(\overline{\Omega}\right)$ so that

$$[\nabla^r f]_{C^{0,\alpha}(\overline{\Omega})} < \infty.$$

We equip $C^{r,\alpha}\left(\overline{\Omega}\right)$ with the following norm:

$$\|f\|_{C^{r,\alpha}(\overline{\Omega})} = \|f\|_{C^r(\overline{\Omega})} + [\nabla^r f]_{C^{0,\alpha}(\overline{\Omega})}\,,$$

where

$$\|f\|_{C^r(\overline{\Omega})} = \sum_{m=0}^{r} \|\nabla^m f\|_{C^0(\overline{\Omega})}\,.$$

Remark 1.41. (i) $C^{r,\alpha}\left(\overline{\Omega}\right)$ with its norm $\|\cdot\|_{C^{r,\alpha}}$ is a Banach space.

(ii) If $\alpha = 0$, we set

$$\|f\|_{C^{r,0}} = \|f\|_{C^r}\,.$$

(iii) If we assume that Ω is bounded and Lipschitz, then the norms

$$\|f\|_{C^{r,\alpha}} = \sum_{m=0}^{r} \|\nabla^m f\|_{C^{0,\alpha}}$$

and

$$\|f\|_{C^{r,\alpha}} = \begin{cases} \|f\|_{C^0} + [\nabla^r f]_{C^{0,\alpha}} & \text{if } 0 < \alpha \leq 1 \\ \|f\|_{C^0} + \|\nabla^r f\|_{C^0} & \text{if } \alpha = 0. \end{cases}$$

are equivalent to the one defined above. We should, however, point out that these norms are, in general, not equivalent for very wild sets.

(iv) When $\alpha = 1$, we note that $C^{0,1}\left(\overline{\Omega}\right)$ is in fact the set of *Lipschitz continuous* and bounded functions.

The following result (cf. Theorem 16.11) is a remarkable extension result due to Calderon [20] and Stein [92].

Theorem 1.42. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set. Then there exists a continuous linear extension operator*

$$E : C^{r,\alpha}\left(\overline{\Omega}\right) \to C_0^{r,\alpha}\left(\mathbb{R}^n\right)$$

for any integer $r \geq 0$ and any $0 \leq \alpha \leq 1$. More precisely, there exists a constant $C = C\left(r, \Omega\right) > 0$ such that for every $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$,

$$E(f)|_{\overline{\Omega}} = f, \quad \operatorname{supp}[E(f)] \text{ is compact,}$$
$$\|E(f)\|_{C^{r,\alpha}(\mathbb{R}^n)} \le C\,\|f\|_{C^{r,\alpha}(\overline{\Omega})}.$$

Remark 1.43. The extension is universal, in the sense that the same extension also leads to

$$\|E(f)\|_{C^{s,\beta}(\mathbb{R}^n)} \le C\,\|f\|_{C^{s,\beta}(\overline{\Omega})}$$

for any integer s and any $0 \le \beta \le 1$, with, of course, $C = C(s,\Omega)$ as far as $f \in C^{s,\beta}(\overline{\Omega})$. The same extension is also valid for Sobolev spaces.

1.6.2 Interpolation, Product, Composition and Inverse

We now state the interpolation theorem (cf. Theorem 16.26) that plays an essential role in our analysis.

Theorem 1.44. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set, $s \ge r \ge t \ge 0$ be integers and $0 \le \alpha, \beta, \gamma \le 1$ with*

$$t + \gamma \le r + \alpha \le s + \beta.$$

Let $\lambda \in [0,1]$ be such that

$$r + \alpha = \lambda(s+\beta) + (1-\lambda)(t+\gamma).$$

Then there exists a constant $C = C(s,\Omega) > 0$ such that

$$\|f\|_{C^{r,\alpha}} \le C\,\|f\|_{C^{s,\beta}}^{\lambda}\,\|f\|_{C^{t,\gamma}}^{1-\lambda}.$$

As a byproduct of the interpolation theorem, we get the following result (cf. Theorem 16.28).

Theorem 1.45. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set, $r \ge 0$ an integer and $0 \le \alpha \le 1$. Then there exists a constant $C = C(r,\Omega) > 0$ such that*

$$\|fg\|_{C^{r,\alpha}} \le C\left(\|f\|_{C^{r,\alpha}}\,\|g\|_{C^0} + \|f\|_{C^0}\,\|g\|_{C^{r,\alpha}}\right).$$

The next theorem (cf. Theorem 16.31) will also be intensively used.

Theorem 1.46. *Let $\Omega \subset \mathbb{R}^n$, $O \subset \mathbb{R}^m$ be bounded open Lipschitz sets, $r \ge 0$ an integer and $0 \le \alpha \le 1$. Let $g \in C^{r,\alpha}(\overline{O})$ and $f \in C^{r,\alpha}(\overline{\Omega};\overline{O}) \cap C^1(\overline{\Omega};\overline{O})$. Then*

$$\|g \circ f\|_{C^{0,\alpha}(\overline{\Omega})} \le \|g\|_{C^{0,\alpha}(\overline{O})}\,\|f\|_{C^1(\overline{\Omega})}^{\alpha} + \|g\|_{C^0(\overline{O})},$$

whereas if $r \geq 1$, *there exists a constant* $C = C(r, \Omega, O) > 0$ *such that*

$$\|g \circ f\|_{C^{r,\alpha}(\overline{\Omega})} \leq C \left[\|g\|_{C^{r,\alpha}(\overline{O})} \|f\|^{r+\alpha}_{C^1(\overline{\Omega})} + \|g\|_{C^1(\overline{O})} \|f\|_{C^{r,\alpha}(\overline{\Omega})} + \|g\|_{C^0(\overline{O})} \right].$$

We easily deduce, from the previous results, an estimate on the inverse (cf. Theorem 16.32).

Theorem 1.47. *Let* $\Omega, O \subset \mathbb{R}^n$ *be bounded open Lipschitz sets,* $r \geq 1$ *an integer and* $0 \leq \alpha \leq 1$. *Let* $c > 0$. *Let* $f \in C^{r,\alpha}\left(\overline{\Omega}; \overline{O}\right)$ *and* $g \in C^{r,\alpha}\left(\overline{O}; \overline{\Omega}\right)$ *be such that*

$$g \circ f = \mathrm{id} \quad \textit{and} \quad \|g\|_{C^1(\overline{O})}, \|f\|_{C^1(\overline{\Omega})} \leq c.$$

Then there exists a constant $C = C(c, r, \Omega, O) > 0$ *such that*

$$\|f\|_{C^{r,\alpha}(\overline{\Omega})} \leq C \|g\|_{C^{r,\alpha}(\overline{O})}.$$

1.6.3 Smoothing Operator

The next theorem (cf. Theorem 16.43) is about smoothing C^r or $C^{r,\alpha}$ functions. We should draw the attention that in order to get the conclusions of the theorem, one proceeds, as usual, by convolution. However, we have to choose the kernel very carefully.

Theorem 1.48. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open Lipschitz set. Let* $s \geq r \geq t \geq 0$ *be integers and* $0 \leq \alpha, \beta, \gamma \leq 1$ *be such that*

$$t + \gamma \leq r + \alpha \leq s + \beta.$$

Let $f \in C^{r,\alpha}\left(\overline{\Omega}\right)$. *Then, for every* $0 < \varepsilon \leq 1$, *there exist a constant* $C = C(s, \Omega) > 0$ *and* $f_\varepsilon \in C^\infty\left(\overline{\Omega}\right)$ *such that*

$$\|f_\varepsilon\|_{C^{s,\beta}} \leq \frac{C}{\varepsilon^{(s+\beta)-(r+\alpha)}} \|f\|_{C^{r,\alpha}},$$
$$\|f - f_\varepsilon\|_{C^{t,\gamma}} \leq C \varepsilon^{(r+\alpha)-(t+\gamma)} \|f\|_{C^{r,\alpha}}.$$

We also need to approximate closed forms in $C^{r,\alpha}\left(\overline{\Omega}; \Lambda^k\right)$ by smooth closed forms in a precise way (cf. Theorem 16.49).

Theorem 1.49. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open smooth set and* ν *be the exterior unit normal. Let* $s \geq r \geq t \geq 0$ *with* $s \geq 1$ *and* $1 \leq k \leq n-1$ *be integers. Let* $0 < \alpha, \beta, \gamma < 1$ *be such that*

$$t + \gamma \leq r + \alpha \leq s + \beta.$$

Let $g \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right)$ *with*

$$dg = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge g \in C^{s,\beta}\left(\partial\Omega;\Lambda^{k+1}\right).$$

Then for every $\varepsilon \in (0,1]$, *there exist* $g_\varepsilon \in C^\infty\left(\Omega;\Lambda^k\right) \cap C^{s,\beta}\left(\overline{\Omega};\Lambda^k\right)$ *and a constant* $C = C(s,\alpha,\beta,\gamma,\Omega) > 0$ *such that*

$$\begin{gathered} dg_\varepsilon = 0 \text{ in } \Omega, \quad \nu \wedge g_\varepsilon = \nu \wedge g \text{ on } \partial\Omega, \\ \int_\Omega \langle g_\varepsilon;\psi\rangle = \int_\Omega \langle g;\psi\rangle \quad \text{for every } \psi \in \mathscr{H}_T\left(\Omega;\Lambda^k\right), \\ \|g_\varepsilon\|_{C^{s,\beta}\left(\overline{\Omega}\right)} \leq \frac{C}{\varepsilon^{(s+\beta)-(r+\alpha)}}\|g\|_{C^{r,\alpha}\left(\overline{\Omega}\right)} + C\|\nu \wedge g\|_{C^{s,\beta}(\partial\Omega)}, \\ \|g_\varepsilon - g\|_{C^{t,\gamma}\left(\overline{\Omega}\right)} \leq C\varepsilon^{(r+\alpha)-(t+\gamma)}\|g\|_{C^{r,\alpha}\left(\overline{\Omega}\right)}. \end{gathered}$$

Remark 1.50. We recall that if Ω is contractible and since $1 \leq k \leq n-1$, then

$$\mathscr{H}_T\left(\Omega;\Lambda^k\right) = \{0\}.$$