8

Differential Geometry via Differential Forms

In this chapter we present a brief introduction to some basic concepts of differential geometry. As the reader will see, the facility we have developed with differential forms will greatly aid us in this endeavor.

8.1 Covariant derivatives

In first-year calculus we learn how to measure the rate of change of a real-valued function of a single variable by taking its derivative. Later, we learn how to vary a real-valued function of several variables in the direction of a particular vector, thereby leading us to the definition of the *directional derivative.* Here, we go one step further and vary a vector field in the direction of a vector, leading us to the idea of a *covariant derivative.*

We will use a vector field on $\mathbb{R}^2$ for illustrative purposes, but the reader should remain aware that there is nothing special about two dimensions in this section. Suppose $\mathbf{W}$ is a vector field on $\mathbb{R}^2$ and $\alpha(t)$ is a parameterization of a curve.

At $t = 0$, we compute $\frac{d}{dt}\mathbf{W}(\alpha(t))$ (see Figure 8.1). Geometrically, what is happening is the following. As we walk along the curve $\alpha(t)$ we watch the vector field $\mathbf{W}$. From our perspective, we can think of ourselves as being stationary and the vector $\mathbf{W}(\alpha(t))$ as changing. The tip of this vector traces out a parameterized curve in $T\mathbb{R}^2$. Differentiating then gives a tangent vector U to this curve, which is precisely $\frac{d}{dt}\mathbf{W}(\alpha(t))$ (see Figure 8.2).

Example 53. Suppose $\mathbf{W} = \langle xy^2, x + y\rangle$ and $\alpha(t) = (t^2, t)$. Then

$$\mathbf{W}(\alpha(t)) = \langle t^4, t^2 + t\rangle$$

and, hence,

$$\frac{d}{dt}\mathbf{W}(\alpha(t))\Big|_{t=0} = \langle 4t^3, 2t + 1\rangle\big|_{t=0} = \langle 0, 1\rangle.$$

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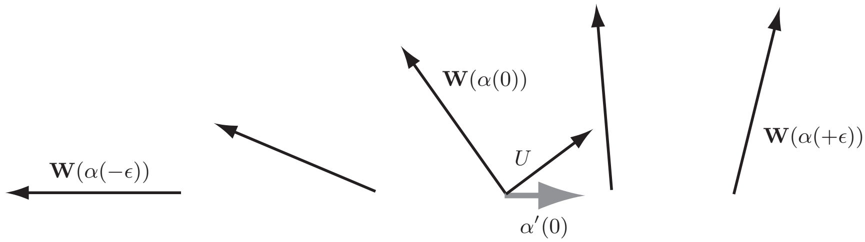


Fig. 8.1. The derivative of a vector field **W** at the point $p = \alpha(0)$, in the direction of the vector $\alpha'(0)$, is the vector U.

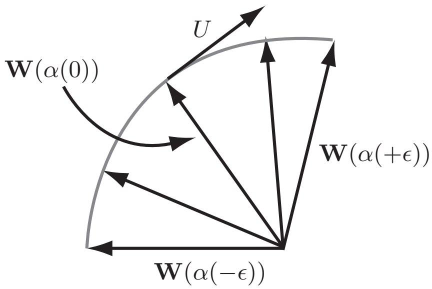


Fig. 8.2. Superimposing vectors from the field **W** creates a parameterized curve. The derivative vector U is then a tangent vector to this curve.

Abstractly, **W** is really a function from $\mathbb{R}^2$ to $T\mathbb{R}^2$, so we can write $\mathbf{W} = \langle w_1(x,y), w_2(x,y)\rangle$. The parameterization $\alpha : \mathbb{R}^1 \to \mathbb{R}^2$ can be written as $\alpha(t) = (\alpha_1(t), \alpha_2(t))$. So, in general, we can use the chain rule to obtain

$$\begin{aligned}
\left.\frac{d}{dt}\mathbf{W}(\alpha(t))\right|_{t=0} &= \left.\frac{d}{dt}\langle w_1(\alpha(t)), w_2(\alpha(t))\rangle\right|_{t=0} \\
&= \left.\left\langle \frac{dw_1(\alpha(t))}{dt}, \frac{dw_2(\alpha(t))}{dt}\right\rangle\right|_{t=0} \\
&= \left.\left\langle \frac{\partial w_1}{\partial x}\frac{d\alpha_1}{dt} + \frac{\partial w_1}{\partial y}\frac{d\alpha_2}{dt}, \frac{\partial w_2}{\partial x}\frac{d\alpha_1}{dt} + \frac{\partial w_2}{\partial y}\frac{d\alpha_2}{dt}\right\rangle\right|_{t=0} \\
&= \left\langle \frac{\partial w_1}{\partial x}\alpha_1'(0) + \frac{\partial w_1}{\partial y}\alpha_2'(0), \frac{\partial w_2}{\partial x}\alpha_1'(0) + \frac{\partial w_2}{\partial y}\alpha_2'(0)\right\rangle \\
&= \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} \begin{pmatrix} \alpha_1'(0) \\ \alpha_2'(0) \end{pmatrix}.
\end{aligned}$$

To make this a bit easier to write, we define the following matrix:

$$\nabla \mathbf{W} = \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix}.$$

Then we have

$$\frac{d}{dt}\mathbf{W}(\alpha(t))\bigg|_{t=0} = [\nabla \mathbf{W}]\,\alpha'(0).$$

Example 54. We continue with the previous example, where $\mathbf{W} = \langle xy^2, x+y\rangle$ and $\alpha(t) = (t^2, t)$. Then

$$\nabla \mathbf{W} = \begin{bmatrix} y^2 & 2xy \\ 1 & 1 \end{bmatrix}.$$

At the point $\alpha(0) = (0, 0)$, this becomes

$$\nabla \mathbf{W} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The vector $\alpha'(0) = \langle 2t, 1\rangle|_{t=0} = \langle 0, 1\rangle$. So,

$$\frac{d}{dt}\mathbf{W}(\alpha(t))\bigg|_{t=0} = [\nabla \mathbf{W}]\,\alpha'(0) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \langle 0, 1\rangle$$

as we saw before.

The previous example illustrates an important point. Our final answer doesn't depend on the curve α, only the tangent vector $\alpha'(0)$. Hence, if we define

$$\nabla_V \mathbf{W} = [\nabla \mathbf{W}]\, V$$

and α is *any* parameterized curve with $\alpha'(0) = V$, then

$$\frac{d}{dt}\mathbf{W}(\alpha(t))\bigg|_{t=0} = \nabla_V \mathbf{W}.$$

Before we end this section, we present one final way to think about $\nabla_V \mathbf{W}$ called the *covariant derivative of* $\mathbf{W}$ *in the* V *direction.* By definition,

$$\begin{aligned} \nabla_V \mathbf{W} = [\nabla \mathbf{W}]V &= \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \left\langle \frac{\partial w_1}{\partial x}v_1 + \frac{\partial w_1}{\partial y}v_2, \frac{\partial w_2}{\partial x}v_1 + \frac{\partial w_2}{\partial y}v_2 \right\rangle \\ &= \langle dw_1(V), dw_2(V)\rangle\,. \end{aligned}$$

Here, we are thinking of w_i as a 0-form and dw_i as the 1-form that is its derivative. We can simplify this a bit further and define $d\mathbf{W}$ to be the *vector of 1-forms* $\langle dw_1, dw_2\rangle$. Thus, we now have

$$\nabla_V \mathbf{W} = d\mathbf{W}(V).$$

This is completely analogous to the statement $\nabla_V f = df(V)$, where $\nabla_V f$ is the directional derivative of the real-valued function f.

Example 55. We continue with the previous example, where $\mathbf{W} = \langle xy^2, x+y\rangle$. Then $d\mathbf{W}$ is the vector of 1-forms $\langle y^2\, dx + 2xy\, dy, dx+dy\rangle$. At the point $(0,0)$, this becomes $\langle 0, dx+dy\rangle$. If we plug the vector $\langle 0,1\rangle$ into this, the result is the vector $\langle 0,1\rangle$, as before.

In this section we have encountered three ways to think about/compute the derivative of a vector field, at a point p, in the direction of a vector $V \in T_p\mathbb{R}^n$. The reader would be well advised to remember these, as we will be switching between them frequently to suit our needs. We summarize as follows:

1. $\nabla_V \mathbf{W} = \frac{d}{dt}\mathbf{W}(\alpha(t))\big|_{t=0}$, where $\alpha(t)$ is any parameterized curve such that $\alpha'(0) = V$.
2. $\nabla_V \mathbf{W} = [\nabla \mathbf{W}]V$, where $\nabla \mathbf{W}$ is the matrix $\begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix}$.
3. $\nabla_V \mathbf{W} = d\mathbf{W}(V)$, where $d\mathbf{W}$ is the vector of 1-forms whose components are the derivatives of the components of $\mathbf{W}$, thought of as 0-forms.

8.1. In advanced settings we often refer to the analog of the "product rule" for differentiation as a "Leibniz rule." For example, the Leibniz rule for directional derivatives is

$$\nabla_V(fg) = (\nabla_V f)g + f(\nabla_V g).$$

Prove the following Leibniz rules for covariant differentiation:

$$\nabla_V(\mathbf{X}\cdot\mathbf{Y}) = (\nabla_{\mathbf{V}}\mathbf{X})\cdot\mathbf{Y} + \mathbf{X}\cdot(\nabla_{\mathbf{V}}\mathbf{Y}),$$

$$\nabla_V(f\mathbf{W}) = (\nabla_{\mathbf{V}} f)\mathbf{W} + f(\nabla_{\mathbf{V}}\mathbf{W}).$$

8.2. Suppose $\mathbf{V}$ and $\mathbf{W}$ are vector fields. Then $\mathbf{V}\cdot\mathbf{W}$ is a real-valued function, which we can think of as a 0-form. We can then differentiate this to get a 1-form. Prove the following Leibniz rule for derivatives of dot products:

$$d(\mathbf{V}\cdot\mathbf{W}) = d\mathbf{V}\cdot\mathbf{W} + \mathbf{V}\cdot d\mathbf{W}.$$

8.3. It will be useful later to be able to take dot products of vectors of 1-forms and vectors of real numbers. Show that this commutes with evaluation on a third vector; that is, if $\mathbf{W}$ is a vector field and U and V are vectors in $T_p\mathbb{R}^n$ for some $p \in \mathbb{R}^n$, show that

$$d\mathbf{W}(V)\cdot U = d\mathbf{W}\cdot U(V).$$

8.4. Often we parameterize surfaces in $\mathbb{R}^3$ by starting with cylindrical or spherical coordinates and expressing one of the parameters in terms of the other two. So, for example, by starting with spherical coordinates and expressing ρ as a function of θ and ϕ, we may end up with a parameterization of the form $\Psi(\theta,\phi) = (x,y,z)$. However, we may also view $\theta(x,y,z)$ as a function on $\mathbb{R}^3$ that gives the θ coordinate of the point (x,y,z), when it is expressed in spherical coordinates. Hence, it is often the case that we have parameterizations of the form $\Psi(\theta, \phi)$ such that

$$\theta(\Psi(\theta, \phi)) = \theta \quad \text{and} \quad \phi(\Psi(\theta, \phi)) = \phi.$$

When we think of $\theta(x, y, z)$ as a 0-form on $\mathbb{R}^3$, we can differentiate to get a 1-form $d\theta$. Show that

$$d\theta\left(\frac{\partial \Psi}{\partial \theta}\right) = d\phi\left(\frac{\partial \Psi}{\partial \phi}\right) = 1 \quad \text{and} \quad d\theta\left(\frac{\partial \Psi}{\partial \phi}\right) = d\phi\left(\frac{\partial \Psi}{\partial \theta}\right) = 0.$$

8.2 Frame fields and Gaussian curvature

Our approach to the subject of differential geometry loosely follows Cartan's *method of moving frames*. Roughly speaking, this involves a choice of a particularly nice basis for the tangent space at each point of a surface S in $\mathbb{R}^n$, called a *frame field*. Geometric properties of S can then be derived from the way the frame field varies from one point to the next. This is where the language of differential forms comes in handy.

Definition 3. *Let S be a surface in $\mathbb{R}^3$. A frame field on S is a choice of vector fields $\{\mathbf{E_1}, \mathbf{E_2}\}$ such that at each point p of S, $\mathbf{E_1}(p)$ and $\mathbf{E_2}(p)$ form an orthonormal basis for T_pS.*

The orthonormality condition is equivalent to

$$\mathbf{E_1} \cdot \mathbf{E_2} = 0, \quad \mathbf{E_1} \cdot \mathbf{E_1} = \mathbf{E_2} \cdot \mathbf{E_2} = 1.$$

Geometric properties of S follow from the way $\mathbf{E_1}$ and $\mathbf{E_2}$ vary from one point of S to the next. To this end, we now examine the covariant derivatives of $\mathbf{E_1}$ and $\mathbf{E_2}$. Let V be a tangent vector to S at some point. We can then take the directional derivative of both sides of the equation

$$\mathbf{E_1} \cdot \mathbf{E_1} = 1$$

to obtain

$$\nabla_V(\mathbf{E_1} \cdot \mathbf{E_1}) = \nabla_V 1.$$

Using the appropriate Leibniz rule this becomes

$$(\nabla_V \mathbf{E_1}) \cdot \mathbf{E_1} + \mathbf{E_1} \cdot (\nabla_V \mathbf{E_1}) = 0.$$

and thus, $(\nabla_V \mathbf{E_1}) \cdot \mathbf{E_1} = 0$. We conclude that $\nabla_V \mathbf{E_1}$ is a vector that is perpendicular to $\mathbf{E_1}$ and is thus in the plane spanned by $\mathbf{E_2}$ and the normal vector N to S.

Now, suppose S is a plane. Then a normal vector N to S is constant (i.e., it does not depend on a choice of $p \in S$). Since $\mathbf{E_1}$ is tangent to S, we have $\mathbf{E_1} \cdot N = 0$. Again, taking the directional derivative of both sides of this equation and applying the appropriate Leibniz rule gives us

$$(\nabla_V \mathbf{E_1}) \cdot N + \mathbf{E_1} \cdot (\nabla_V N) = 0.$$

However, since N is constant, $\nabla_V N = 0$. We conclude $(\nabla_V \mathbf{E_1}) \cdot N = 0$, and, thus, when S is a plane, $\nabla_V \mathbf{E_1}$ is also perpendicular to N. We already knew $\nabla_V \mathbf{E_1}$ was perpendicular to $\mathbf{E_1}$, so we may now conclude that $\nabla_V \mathbf{E_1}$ points in the same direction as $\mathbf{E_2}$.

Now, suppose for some surface S that the vector $\nabla_V \mathbf{E_1}$ does *not* point in the same direction as $\mathbf{E_2}$ at a point $p \in S$. Then, near p, it must be the case that S does not look like a plane (i.e., it has some *curvature*). This motivates us to look toward the *projection of* $\nabla_V \mathbf{E_1}$ *onto* $\mathbf{E_2}$ as a measure of the curvature (or lack thereof) of S. This projection is a linear function of the vector V and is thus a 1-form. We define

$$\Omega(V) = (\nabla_V \mathbf{E_1}) \cdot \mathbf{E_2}.$$

Since $(\nabla_V \mathbf{E_1}) \cdot \mathbf{E_2} = d\mathbf{E_1}(V) \cdot \mathbf{E_2} = d\mathbf{E_1} \cdot \mathbf{E_2}(V)$, it is sometimes more convenient to write

$$\Omega = d\mathbf{E_1} \cdot \mathbf{E_2}.$$

We now define a numerical measure of the curvature of S at each point, which does not depend on a choice of tangent vector at that point.

Definition 4. *The Gaussian curvature at each point of S is defined to be the number*

$$K = -d\Omega(\mathbf{E_1}, \mathbf{E_2}).$$

The amazing thing about the Gaussian curvature is that it is independent of the choice of frame field. It is a number that is completely determined by the shape of S. We prove this now.

Theorem 3. *At each point of S, the Gaussian curvature is independent of the choice of frame field.*

Proof. Suppose $\mathbf{E_1} = \langle E_1^1, E_1^2, E_1^3 \rangle$ and $\mathbf{E_2} = \langle E_2^1, E_2^2, E_2^3 \rangle$. Then note that

$$\begin{aligned} d\Omega &= d(d\mathbf{E_1} \cdot \mathbf{E_2}) \\ &= d(\langle dE_1^1, dE_1^2, dE_1^3 \rangle \cdot \langle E_2^1, E_2^2, E_2^3 \rangle) \\ &= d(dE_1^1 E_2^1 + dE_1^2 E_2^2 + dE_1^3 E_2^3) \\ &= dE_1^1 \wedge dE_2^1 + dE_1^2 \wedge dE_2^2 + dE_1^3 \wedge dE_2^3. \end{aligned}$$

We can rewrite this more compactly using summation notation:

$$d\Omega = \sum_{i=1}^{3} dE_1^i \wedge dE_2^i.$$

Now, suppose $\{\mathbf{F_1}, \mathbf{F_2}\}$ is some other frame field on S. Let $\phi : S \to \mathbb{R}^1$ be the angle between $\mathbf{E_1}$ and $\mathbf{F_1}$. As any orthonormal basis can be obtained from any other by rotation, it must be the case that

$$\begin{aligned}\mathbf{F_1} &= \cos\phi\mathbf{E_1} + \sin\phi\mathbf{E_2},\\ \mathbf{F_2} &= -\sin\phi\mathbf{E_1} + \cos\phi\mathbf{E_2}.\end{aligned}$$

Let $\Omega' = d\mathbf{F_1} \cdot \mathbf{F_2}$. Then the Gaussian curvature computed by using the frame field $\{\mathbf{F_1}, \mathbf{F_2}\}$ is given by evaluation of the 2-form $d\Omega'$ on these two vectors. We now calculate

$$\begin{aligned}d\Omega' &= \sum_{i=1}^{3} dF_1^i \wedge dF_2^i\\ &= \sum_{i=1}^{3} d(\cos\phi E_1^i + \sin\phi E_2^i) \wedge d(-\sin\phi E_1^i + \cos\phi E_2^i)\\ &= \sum_{i=1}^{3}(-\sin\phi E_1^i d\phi + \cos\phi dE_1^i + \cos\phi E_2^i d\phi + \sin\phi dE_2^i)\\ &\qquad \wedge(-\cos\phi E_1^i d\phi - \sin\phi dE_1^i - \sin\phi E_2^i d\phi + \cos\phi dE_2^i)\\ &= \sum_{i=1}^{3} E_1^i d\phi \wedge dE_1^i + E_2^i d\phi \wedge dE_2^i + dE_1^i \wedge dE_2^i\\ &= d\phi \wedge \left(\sum_{i=1}^{3} E_1^i dE_1^i\right) + d\phi \wedge \left(\sum_{i=1}^{3} E_2^i dE_2^i\right) + \sum_{i=1}^{3} dE_1^i \wedge dE_2^i\\ &= d\phi \wedge (\mathbf{E_1} \cdot d\mathbf{E_1}) + d\phi \wedge (\mathbf{E_2} \cdot d\mathbf{E_2}) + d\Omega.\end{aligned}$$

Now, note that differentiating the equation $\mathbf{E_1} \cdot \mathbf{E_1} = 1$ tells us that $\mathbf{E_1} \cdot d\mathbf{E_1} = 0$. Identical reasoning leads us to conclude $\mathbf{E_2} \cdot d\mathbf{E_2} = 0$, and thus we have $d\Omega' = d\Omega$. Finally, note that since these 2-forms are the same, then evaluating either one on any pair of vectors that span a parallelogram of area one in a fixed plane will always produce the same number. Thus,

$$K' = d\Omega'(\mathbf{F_1}, \mathbf{F_2}) = d\Omega(\mathbf{E_1}, \mathbf{E_2}) = K,$$

as desired.

Example 56. We compute the Gaussian curvature of a sphere S of radius R. We begin by defining a frame field on it. As usual, a parameterization for S is given by

$$\Psi(\theta, \phi) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi).$$

The partial derivatives of this give tangent vectors to S:

$$\frac{\partial \Psi}{\partial \theta} = \langle -R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0\rangle,$$
$$\frac{\partial \Psi}{\partial \phi} = \langle R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi\rangle.$$

Since the dot product of these two vectors is zero, they are orthogonal. To get a frame field then, we must simply divide each vector by its magnitude to get unit vectors:

$$\mathbf{E_1} = \frac{1}{R\sin\phi}\frac{\partial \Psi}{\partial \theta} = \langle -\sin\theta, \cos\theta, 0\rangle,$$
$$\mathbf{E_2} = \frac{1}{R}\frac{\partial \Psi}{\partial \phi} = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle.$$

Now, we find the 1-form Ω. Since

$$d\mathbf{E_1} = \langle -\cos\theta\ d\theta, -\sin\theta\ d\theta, 0\rangle,$$

we have

$$\Omega = d\mathbf{E_1}\cdot\mathbf{E_2} = -\cos\phi\cos^2\theta\ d\theta - \cos\phi\sin^2\theta\ d\theta = -\cos\phi\ d\theta.$$

Thus,

$$-d\Omega = -\sin\phi\ d\phi\wedge d\theta.$$

To compute $K = -d\Omega(\mathbf{E_1}, \mathbf{E_2})$, we will need to know the following:

$$d\theta(\mathbf{E_1}) = d\theta\left(\frac{1}{R\sin\phi}\frac{\partial \Psi}{\partial \theta}\right) = \frac{1}{R\sin\phi}d\theta\left(\frac{\partial \Psi}{\partial \theta}\right) = \frac{1}{R\sin\phi},$$
$$d\phi(\mathbf{E_1}) = d\phi\left(\frac{1}{R\sin\phi}\frac{\partial \Psi}{\partial \theta}\right) = \frac{1}{R\sin\phi}d\phi\left(\frac{\partial \Psi}{\partial \theta}\right) = 0,$$
$$d\theta(\mathbf{E_2}) = d\theta\left(\frac{1}{R}\frac{\partial \Psi}{\partial \phi}\right) = \frac{1}{R}d\theta\left(\frac{\partial \Psi}{\partial \phi}\right) = 0,$$
$$d\phi(\mathbf{E_2}) = d\phi\left(\frac{1}{R}\frac{\partial \Psi}{\partial \phi}\right) = \frac{1}{R}d\phi\left(\frac{\partial \Psi}{\partial \phi}\right) = \frac{1}{R}.$$

We are now prepared to compute the Gaussian curvature:

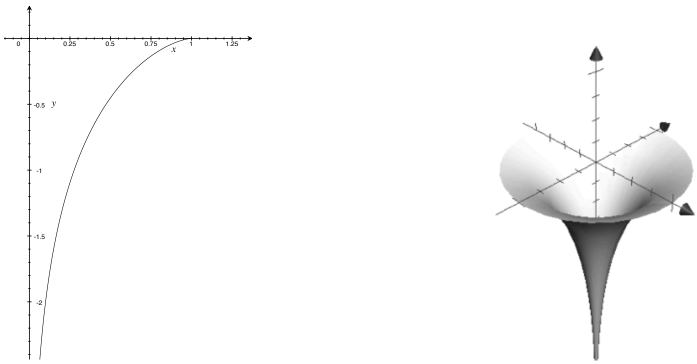
$$\begin{aligned} K &= -d\Omega(\mathbf{E_1}, \mathbf{E_2}) \\ &= -\sin\phi\ d\phi\wedge d\theta(\mathbf{E_1}, \mathbf{E_2}) \\ &= -\sin\phi\begin{vmatrix} d\phi(\mathbf{E_1}) & d\theta(\mathbf{E_1}) \\ d\phi(\mathbf{E_2}) & d\theta(\mathbf{E_2}) \end{vmatrix} \\ &= -\sin\phi\begin{vmatrix} 0 & \frac{1}{R\sin\phi} \\ \frac{1}{R} & 0 \end{vmatrix} \\ &= \sin\phi\frac{1}{R^2\sin\phi} \\ &= \frac{1}{R^2}. \end{aligned}$$

8.5. Let $(f(t), 0, g(t))$ be a unit speed curve in the xz-plane (thus, $f'(t)^2 + g'(t)^2 = 1$). Then $\Psi(\theta, t) = (f(t)\cos\theta, f(t)\sin\theta, g(t))$ is the surface obtained by revolving this curve about the z-axis.

1. Show that the Gaussian curvature of this surface is $-\frac{f''}{f}$.
2. Find the Gaussian curvature of a cylinder of radius R.
3. Find the Gaussian curvature of a right-angled cone.
4. Confirm that the Gaussian curvature of a sphere of radius R is $\frac{1}{R^2}$ by viewing it as a surface of revolution.
5. A unit-speed parameterization for the *tractrix* is given by

   $$(e^t, \sqrt{1 - e^{2t}} - \tanh^{-1}\sqrt{1 - e^{2t}}),$$

   where $t < 0$. The surface of revolution of a tractrix is called a *tractricoid* (see Figure 8.3). Calculate its Gaussian curvature. Why is this surface sometimes called a *pseudo-sphere*?



(a) (b)

Fig. 8.3. (a) The tractrix; (b) the tractricoid.

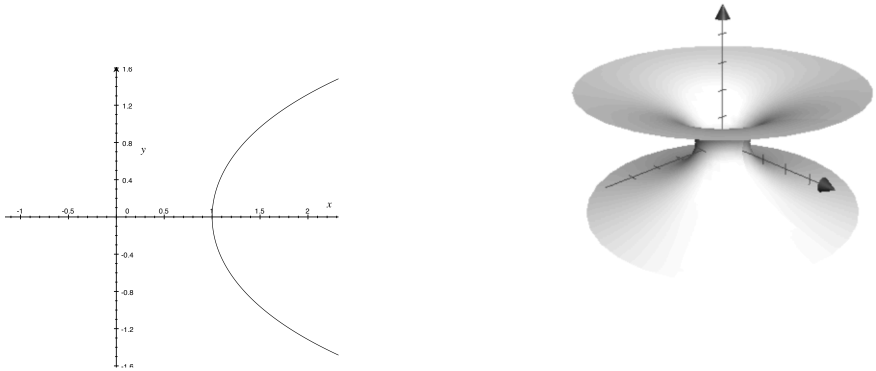
6. A unit speed parameterization for the *catenary*, the graph of $x = \cosh z$, is given by

   $$(\sqrt{1 + t^2}, \sinh^{-1} t).$$

   The surface of revolution of the catenary is called a *catenoid* (see Figure 8.4). Calculate its Gaussian curvature.

8.6. The *helicoid* (see Figure 8.5) is the surface parameterized by

$$\Psi(t, \theta) = (t\cos\theta, t\sin\theta, \theta).$$



(a) (b)

Fig. 8.4. (a) The catenary; (b) the catenoid.

Fig. 8.5. The helicoid.

1. Calculate the Gaussian curvature of the helicoid.
2. Show that there is a continuous function f from the helicoid to the catenoid such that the Gaussian curvature at p is the same as the Gaussian curvature at $f(p)$.

8.3 Parallel vector fields

Let $\alpha(t)$ be a parameterized curve in a surface S and let $\mathbf{Y}$ be a unit vector field defined on α, tangent to S, that turns *as little as possible*; that is, we assume that $\mathbf{Y}$ is a vector field which has been chosen so that there is no component of $\mathbf{Y}'$ that is tangent to S. Said another way, we assume that the projection of $\nabla_{\alpha'}\mathbf{Y}$ onto the tangent space to S is zero. Such a vector field is said to be *parallel* along α.

For each t, let $\phi(t)$ denote the angle between $\mathbf{Y}(t)$ and $\mathbf{E_1}$. Then

$$\cos\phi = \mathbf{Y}\cdot\mathbf{E_1}.$$

We now take the derivative of both sides of this equation in the direction of α':

$$-\sin\phi\nabla_{\alpha'}\phi = (\nabla_{\alpha'}\mathbf{Y})\cdot\mathbf{E_1} + \mathbf{Y}\cdot(\nabla_{\alpha'}\mathbf{E_1}).$$

Since we are assuming $\nabla_{\alpha'}\mathbf{Y}$ has no component tangent to S, its dot product with $\mathbf{E_1}$ (a tangent vector to S) must be zero. We now have

$$-\sin\phi\nabla_{\alpha'}\phi = \mathbf{Y}\cdot(\nabla_{\alpha'}\mathbf{E_1}).$$

As $\mathbf{Y}$ is tangent to S, we may compute the right-hand side above by first projecting $\nabla_{\alpha'}\mathbf{E_1}$ into the tangent space to S and then taking the dot product with $\mathbf{Y}$. Earlier we observed that for any tangent vector V, $\nabla_V\mathbf{E_1}$ is perpendicular to $\mathbf{E_1}$. Thus, to project this vector into the tangent space to S, we can take its dot product with $\mathbf{E_2}$, and then multiply by the vector $\mathbf{E_2}$. Hence, we will replace the expression $\nabla_{\alpha'}\mathbf{E_1}$ with $(\nabla_{\alpha'}\mathbf{E_1})\cdot\mathbf{E_2}\ \mathbf{E_2}$. This gives us

$$\begin{aligned}-\sin\phi\nabla_{\alpha'}\phi &= \mathbf{Y}\cdot[(\nabla_{\alpha'}\mathbf{E_1})\cdot\mathbf{E_2}\ \mathbf{E_2}]\\ &= \Omega(\alpha')\mathbf{Y}\cdot\mathbf{E_2}.\end{aligned}$$

Now, observe that since $\mathbf{E_1}$ and $\mathbf{E_2}$ are orthogonal, $\mathbf{Y}$ is in the tangent plane defined by these two vectors, and ϕ is the angle between $\mathbf{E_1}$ and $\mathbf{Y}$, it follows that $\mathbf{Y}\cdot\mathbf{E_2} = \sin\phi$. Incorporating this into the above equation gives

$$-\sin\phi\nabla_{\alpha'}\phi = \Omega(\alpha')\sin\phi$$

and, thus, $-\nabla_{\alpha'}\phi = \Omega(\alpha')$. Finally, note that $\nabla_{\alpha'}\phi$ is precisely the definition of ϕ', giving

$$-\phi' = \Omega(\alpha').$$

What is particularly striking now is what happens when we integrate both sides of this equation. Suppose t, the parameter for α, ranges from a to b. Then we get

$$\phi(a) - \phi(b) = \int_\alpha \Omega.$$

In other words, the integral of the 1-form Ω along α is precisely the net amount of turning done by $\mathbf{E_1}$, relative to a parallel vector field. This number is called the *holonomy* of α and will be denoted $\mathcal{H}(\alpha)$.

Example 57. Let S be the sphere of radius 1. Then S is parameterized by

$$\Psi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

Now, suppose α is a circle of constant latitude; that is, α is a loop for which ϕ is a constant. In Example 56 we computed $\Omega = -\cos\phi\ d\theta$. Thus, going once around the circle α will mean a parallel vector field will have to rotate in relation to $\mathbf{E_1}$ (which happens to be tangent to α) by a total angle of

$$\mathcal{H}(\alpha) = -\int_{\alpha} \Omega = \int_0^{2\pi} \cos\phi\ d\theta = 2\pi\cos\phi.$$

See Figure 8.6. Note that at the equator, where $\phi = \frac{\pi}{2}$ and thus $\cos\phi = 0$, parallel vector fields do not rotate at all. Near the poles, parallel vector fields will rotate close to a full circle (but in opposite directions at each pole.) You can physically observe this effect with a *Foucault Pendulum.*

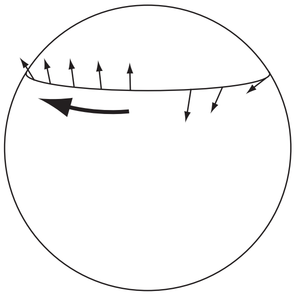


Fig. 8.6. A parallel vector field along a latitude rotates in relation to a tangent to the latitude.

8.7. We continue the study of surfaces of revolution begun in Problem 8.5. Recall that such a surface is parameterized by

$$(f(t)\cos\theta, f(t)\sin\theta, g(t)),$$

where we assume $f'(t)^2 + g'(t)^2 = 1$. Determine how much a parallel vector field rotates as you go once around a loop with t constant and $0 \leq \theta \leq 2\pi$.

8.8.

1. Suppose α is the boundary of a disk D in a surface S. Then show that the the holonomy $\mathcal{H}(\alpha)$ is

$$\mathcal{H}(\alpha) = \iint_D K\ dA.$$

It follows that if $\overline{K}$ is the average Gaussian curvature over the disk D, then

$$\mathcal{H}(\alpha) = \overline{K}\text{Area}(D).$$

2. Suppose p is a point of a surface S and D_r is a disk of radius r centered at p. Conclude that that Gaussian curvature K at p is given by

$$K = \lim_{r \to 0} \frac{\mathcal{H}(\partial D_r)}{\text{Area}(D_r)}.$$

8.9. For any vector V tangent to a surface S, let $V^\perp$ denote the orthogonal tangent vector obtained by rotating it clockwise (with respect to the orientation of S) by $\frac{\pi}{2}$. As above, let α be a curve in S. Let T be the surface defined by $\alpha(t) + \lambda\alpha'(t)^\perp$ (so that α is a curve on T as well). Show that the holonomy of α as a curve on S is the same as the holonomy of α on T.

8.10. Show that the holonomy around any curve on a cone is the same as the holonomy around the corresponding curve on the "unrolled" cone.

8.11. Combine the previous two problems to deduce the formula for the rotation of a parallel vector field around the latitude of a sphere by unrolling the cone that is tangent to it so that it is flat and using basic trigonometry. See Figure 8.7.

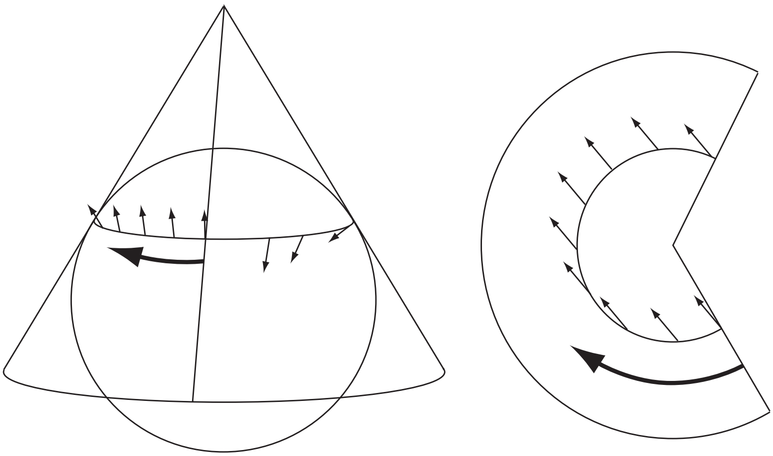


Fig. 8.7. The holonomy around the unrolled cone (right) gives the rotation of a parallel vector field around a latitude of the sphere (left).

8.12. A *geodesic* is a parameterized curve on a surface whose tangent vector field is parallel.

1. Show that the only circle of latitude on a sphere that is a geodesic is the equator.
2. Show that the only geodesics in a plane are lines.

8.4 The Gauss–Bonnet Theorem

In this section we prove the fundamental result that ties Differential Geometry to Topology. Here, S will always denote a compact, closed surface in $\mathbb{R}^3$. For our purposes, this just means S has finite area and $\partial S = \emptyset$. A *triangulation* of S is a decomposition into triangles. Each such triangle is called a *face*. Each face has three *edges* and three *vertices*. We denote the number of Faces, Edges and Vertices by F, E and V, respectively.

Definition 5. *The* Euler Characteristic *is the number*

$$\chi(S) = V - E + F.$$

It is a basic result of topology that the Euler Characteristic is a *homeomorphism invariant*; that is, if there is a continuous, 1-1 function from S to S' with continuous inverse, then $\chi(S) = \chi(S')$. It follows that $\chi(S)$ does not depend on the choice of triangulation of S.

8.13. Both the tetrahedron and the octahedron (see Figure 8.8) are homeomorphic to the sphere. Confirm that the Euler Characteristic of a sphere is 2 by computing $V - E + F$ for each.

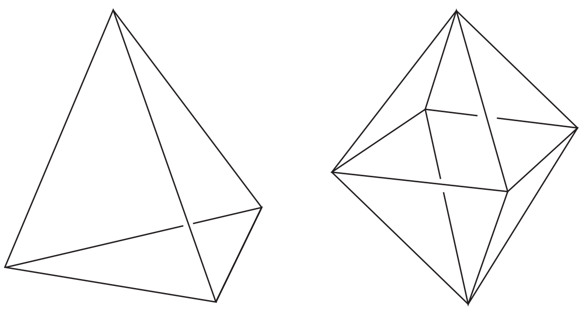


Fig. 8.8. A tetrahedron (left) and an octahedron (right) are homeomorphic to the sphere.

We now come to the most amazing theorem of Differential Geometry.

Theorem 4 (Gauss–Bonnet).

$$\iint\limits_S K \, dA = 2\pi\chi(S).$$

It follows that the total Gaussian curvature over a surface is a homeomorphism invariant. So, for example, if you try to increase the curvature of S in one spot by deforming it in some way, you must also decrease the curvature elsewhere by the same amount.

Proof. As the Euler characteristic is a homeomorphism invariant, we are free to choose any triangulation of S to compute it. We pick one so that each edge is a geodesic segment (see Problem 8.12). Furthermore, we will assume that each triangle is small enough so that the vector field $\mathbf{E_1}$ is *roughly* parallel on it. We do not have to be terribly precise about this. All we need is for Figure 8.9 to be close to correct. These are simplifying assumptions only and are not strictly necessary for the proof.

Recall that the Gaussian curvature K is defined by the equation $K = -d\Omega(\mathbf{E_1}, \mathbf{E_2})$. Hence, $-d\Omega$ is a 2-form on S which takes a parallelogram of area 1 and returns the number K. It follows that the integral of K over the surface S is equivalent to the integral of the 2-form $-d\Omega$ over S. In other words,

$$\iint\limits_S K \; dA = -\int\limits_S d\Omega.$$

Now, let T be a triangle of the triangulation. Then by the generalized Stokes' Theorem we have

$$\int\limits_T d\Omega = \int\limits_{\partial T} \Omega.$$

Let α, β and γ denote the three edges of T, where α goes from the vertex x to the vertex y, β goes from y to z, and γ goes from z back to x. Then we have

$$\begin{aligned}\int\limits_{\partial T} \Omega &= \int\limits_{\alpha\cup\beta\cup\gamma} \Omega \\ &= \int\limits_{\alpha} \Omega + \int\limits_{\beta} \Omega + \int\limits_{\gamma} \Omega \\ &= [\phi_\alpha(x) - \phi_\alpha(y)] + [\phi_\beta(y) - \phi_\beta(z)] + [\phi_\gamma(z) - \phi_\gamma(x)] \\ &= [\phi_\alpha(x) - \phi_\gamma(x)] + [\phi_\beta(y) - \phi_\alpha(y)] + [\phi_\gamma(z) - \phi_\beta(z)].\end{aligned}$$

Here, ϕ_α, ϕ_β and ϕ_γ are functions that measure the angles made between the parallel vector fields α', β' and γ' and the vector field $\mathbf{E_1}$. Let $\epsilon(v)$ denote the exterior angle at vertex v. Inspection of Figure 8.9 reveals that at vertex y, the difference $\phi_\beta(y) - \phi_\alpha(y)$ is $-\epsilon(y)$. Similarly, from Figure 8.9 we can see that at vertex z, the difference $\phi_\beta(y) - \phi_\alpha(y)$ is precisely $-\epsilon(z)$. However, at x, the difference $\phi_\gamma(z) - \phi_\beta(z)$ is $2\pi - \epsilon(x)$. Putting this all together allows us to rewrite the above equation as

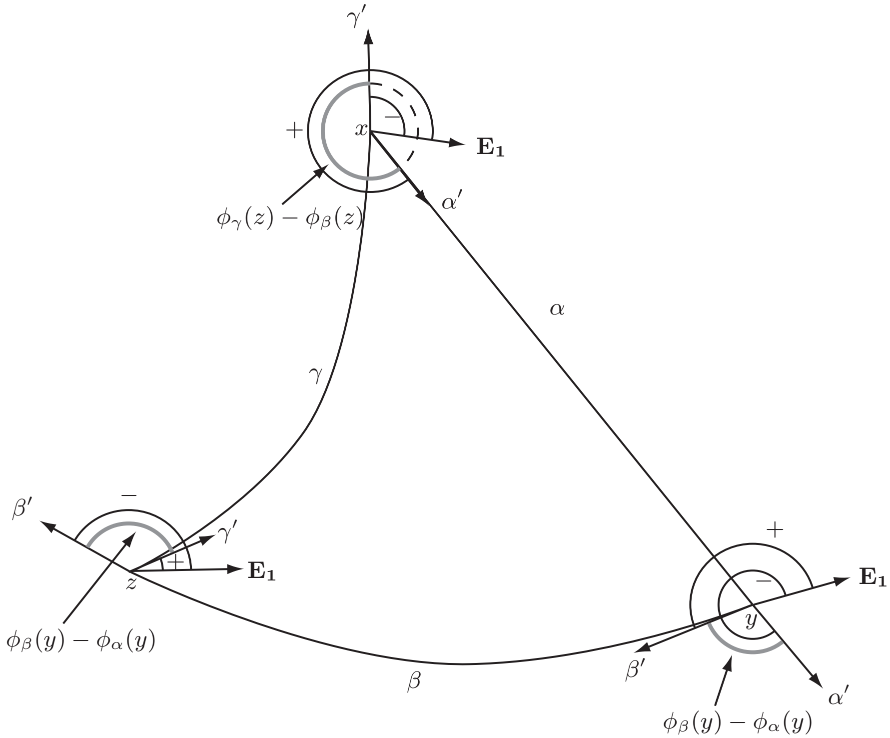


Fig. 8.9. At vertex y, the difference between the angle made by β' and $\mathbf{E_1}$ and the angle made by α' and $\mathbf{E_1}$ is the negative of the exterior angle $\epsilon(y)$. Similarly, at vertex z, the difference is $-\epsilon(z)$. However, at x, the difference is $2\pi - \epsilon(x)$.

$$\int_{\partial T} \Omega = 2\pi - \epsilon(x) - \epsilon(y) - \epsilon(z).$$

It will be more convenient to express this in terms of the interior angles. Let $\iota(v)$ denote the interior angle at v. Then $\epsilon(v) = \pi - \iota(v)$. Thus, we now have

$$\int_{\partial T} \Omega = -\pi + \iota(x) + \iota(y) + \iota(z).$$

Now, we add over all triangles. Around each vertex we end up adding all of the interior angles and therefore get 2π. Hence, when we sum over all triangles, the sum of all interior angles is just $2\pi V$. We also add $-\pi$ for each triangle, so this sum is $-\pi F$. Putting everything together thus gives us

$$\begin{aligned}\iint\limits_S K\ dA &= -\int\limits_S d\Omega \\ &= -\sum_{i=1}^{F}\int\limits_{T_i} d\Omega \\ &= -\sum_{i=1}^{F}\int\limits_{\partial T_i} \Omega \\ &= \pi F - 2\pi V.\end{aligned}$$

Now, notice that every triangle has three edges, but these edges are counted twice when we sum over all triangles. Hence, $E = \frac{3}{2}F$, or $2E = 3F$. This allows us to rewrite the above as

$$\begin{aligned}\iint\limits_S K\ dA &= \pi F - 2\pi V \\ &= -2\pi F + 3\pi F - 2\pi V \\ &= -2\pi F + 2\pi E - 2\pi V \\ &= -2\pi\chi(S).\end{aligned}$$

8.14. Show by direct computation that the integral of the Gaussian curvature over a sphere of radius R is -4π.

8.15. Derive the formula for the area of a sphere of radius R from the Gauss–Bonnet Theorem and the fact that the Euler Characteristic of a sphere is 2.

8.16. A *geodesic triangle* is a triangle in a surface whose sides are geodesics. Find formulas for the area of geodesic triangles on a sphere of radius 1 and a pseudo-sphere (whose Gaussian curvature is -1) in terms of their interior angles. Why can't you find such a formula for geodesic triangles in a plane? What can you conclude about the sum of the angles of a geodesic triangle, compared to π, on a sphere, plane and psuedo-sphere?

8.17. Let $\alpha(t)$ be a unit-speed parameterization of a simple, closed curve in $\mathbb{R}^3$, where $a \leq t \leq b$. Let $T = \alpha'$, N be a unit vector pointing in the direction of α'', and $B = T \times N$. Let S be the surface parameterized by

$$\Psi(t,\theta) = \alpha(t) + \cos\theta N(t) + \sin\theta B(t),$$

where $a \leq t \leq b$ and $0 \leq \theta \leq 2\pi$.

1. Describe S.
2. Show that at the point $\Psi(t,\theta)$, the vectors $\mathbf{E_1} = T(t)$ and $\mathbf{E_2} = \frac{\partial\Psi}{\partial\theta}$ are orthogonal unit tangent vectors to S and thus form a frame field.

3. Compute the Gaussian curvature of S.
4. Compute the holonomy of a loop on S where $a \leq t \leq b$ and θ is fixed.
5. Compute the total Gaussian curvature of S by integration.
6. What can you conclude about the Euler Characteristic of S?