6

Stokes' Theorem

6.1 Cells and chains

Up until now, we have not been very specific as to the types of subsets of $\mathbb{R}^m$ on which one integrates a differential n-form. All we have needed is a subset that can be parameterized by a region in $\mathbb{R}^n$. To go further, we need to specify the types of regions.

Definition 1. *Let $I = [0,1]$. An n-cell, σ, is the image of a differentiable map, $\phi : I^n \to \mathbb{R}^m$, with a specified orientation. We denote the same cell with opposite orientation as $-\sigma$. We define a 0-cell to be an oriented point of $\mathbb{R}^m$.*

Example 34. Suppose $g_1(x)$ and $g_2(x)$ are functions such that $g_1(x) < g_2(x)$ for all $x \in [a,b]$. Let R denote the subset of $\mathbb{R}^2$ bounded by the graphs of the equations $y = g_1(x)$ and $y = g_2(x)$ and by the lines $x = a$ and $x = b$. In Example 12, we showed that R is a 2-cell (assuming the induced orientation).

We would like to treat cells as algebraic objects which can be added and subtracted. However, if σ is a cell, it may not at all be clear what "2σ" represents. One way to think about it is as two copies of σ, placed right on top of each other.

Definition 2. *An n-chain is a formal linear combination of n-cells.*

As one would expect, we assume the following relations hold:

$$\sigma - \sigma = \emptyset,$$
$$n\sigma + m\sigma = (n+m)\sigma,$$
$$\sigma + \tau = \tau + \sigma.$$

You may be able to guess what the integral of an n-form, ω, over an n-chain is. Suppose $C = \sum n_i \sigma_i$. Then we define

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$$\int_C \omega = \sum_i n_i \int_{\sigma_i} \omega.$$

6.1. If f is the 0-form x^2y^3, p is the point $(-1,1)$, q is the point $(1,-1)$, and r is the point $(-1,-1)$, then compute the integral of f over the following 0-chains:

1. $p-q$; $r-p$.
2. $p+q-r$.

Another concept that will be useful for us is the *boundary* of an n-chain. As a warm-up, we define the boundary of a 1-cell. Suppose σ is the 1-cell which is the image of $\phi : [0,1] \to \mathbb{R}^m$ with the induced orientation. Then we define the boundary of σ (which we will denote $\partial\sigma$) as the 0-chain, $\phi(1) - \phi(0)$. We can represent this pictorially as in Figure 6.1.

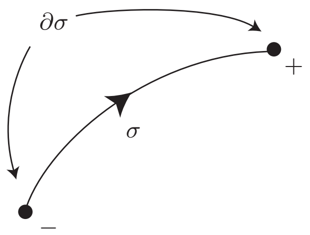


Fig. 6.1. Orienting the boundary of a 1-cell.

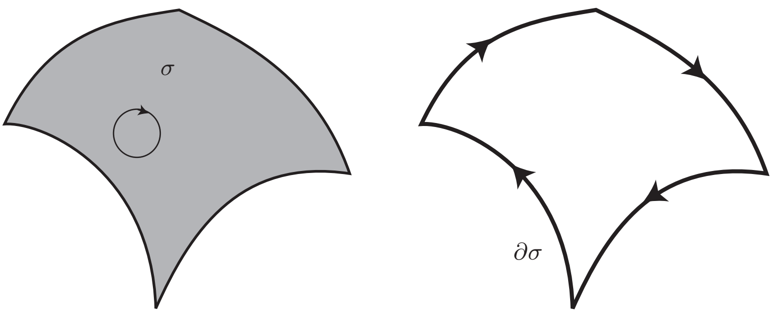


Fig. 6.2. The boundary of a 2-cell.

Figure 6.2 depicts a 2-cell and its boundary. Notice that the boundary consists of four individually oriented 1-cells. This hints at the general formula.

In general, if the n-cell σ is the image of the parameterization $\phi : I^n \to \mathbb{R}^m$ with the induced orientation, then

$$\partial\sigma = \sum_{i=1}^{n}(-1)^{i+1}\left(\phi|_{(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n)} - \phi|_{(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)}\right).$$

So, if σ is a 2-cell, then

$$\begin{aligned}\partial\sigma &= (\phi(1,x_2) - \phi(0,x_2)) - (\phi(x_1,1) - \phi(x_1,0)) \\ &= \phi(1,x_2) - \phi(0,x_2) - \phi(x_1,1) + \phi(x_1,0).\end{aligned}$$

The four terms on the right side of this equality are the four 1-cells depicted in Figure 6.2. The signs in front of these terms guarantee that the orientations are as pictured.

If σ is a 3-cell, then

$$\begin{aligned}\partial\sigma &= (\phi(1,x_2,x_3) - \phi(0,x_2,x_3)) - (\phi(x_1,1,x_3) - \phi(x_1,0,x_3)) \\ &\quad + (\phi(x_1,x_2,1) - \phi(x_1,x_2,0)) \\ &= \phi(1,x_2,x_3) - \phi(0,x_2,x_3) - \phi(x_1,1,x_3) + \phi(x_1,0,x_3) \\ &\quad + \phi(x_1,x_2,1) - \phi(x_1,x_2,0).\end{aligned}$$

An example will hopefully clear up the confusion this was sure to generate:

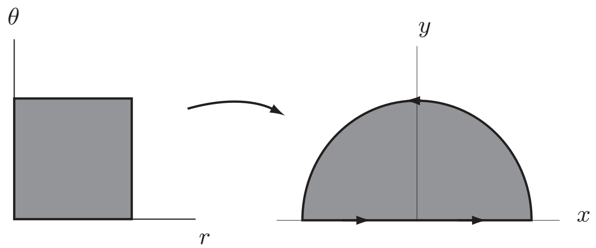


Fig. 6.3. Orienting the boundary of a 2-cell.

Example 35. Suppose $\phi(r,\theta) = (r\cos\pi\theta, r\sin\pi\theta)$. The image of ϕ is the 2-cell σ depicted in Figure 6.3. By the above definition,

$$\begin{aligned}\partial\sigma &= (\phi(1,\theta) - \phi(0,\theta)) - (\phi(r,1) - \phi(r,0)) \\ &= (\cos\pi\theta, \sin\pi\theta) - (0,0) + (r,0) - (-r,0).\end{aligned}$$

This is the 1-chain depicted in Figure 6.3.

Finally, we are ready to define what we mean by the boundary of an n-chain. If $C = \sum n_i\sigma_i$, then we define $\partial C = \sum n_i\partial\sigma_i$.

Example 36. Suppose

$$\phi_1(r,\theta) = (r\cos 2\pi\theta, r\sin 2\pi\theta, \sqrt{1-r^2}),$$

$$\phi_2(r,\theta) = (-r\cos 2\pi\theta, r\sin 2\pi\theta, -\sqrt{1-r^2}),$$

$\sigma_1 = \mathrm{Im}(\phi_1)$, and $\sigma_2 = \mathrm{Im}(\phi_2)$. Then $\sigma_1 + \sigma_2$ is a sphere in $\mathbb{R}^3$. One can check that $\partial(\sigma_1 + \sigma_2) = \emptyset$.

6.2. If σ is an n-cell, show that $\partial\partial\sigma = \emptyset$. (At least show this if σ is a 2-cell and a 3-cell. The 2-cell case can be deduced pictorially from Figures 6.1 and 6.2.)

6.3. If σ is given by the parameterization

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

for $0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{4}$, then what is $\partial\sigma$?

6.4. If σ is given by the parameterization

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$$

for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, then what is $\partial\sigma$?

6.2 The generalized Stokes' Theorem

In calculus, we learn that when you take a function, differentiate it, and then integrate the result, something special happens. In this section, we explore what happens when we take a form, differentiate it, and then integrate the resulting form over some chain. The general argument is quite complicated, so we start by looking at forms of a particular type integrated over very special regions.

Suppose $\omega = a\, dx_2 \wedge dx_3$ is a 2-form on $\mathbb{R}^3$, where $a : \mathbb{R}^3 \to \mathbb{R}$. Let R be the unit cube $I^3 \subset \mathbb{R}^3$. We would like to explore what happens when we integrate $d\omega$ over R. Note first that Problem 5.8 implies that $d\omega = \frac{\partial a}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3$.

Recall the steps used to define $\int\limits_R d\omega$:

1. Choose a lattice of points in R, $\{p_{i,j,k}\}$. Since R is a cube, we can choose this lattice to be rectangular.
2. Define $V^1_{i,j,k} = p_{i+1,j,k} - p_{i,j,k}$. Similarly, define $V^2_{i,j,k}$ and $V^3_{i,j,k}$.
3. Compute $d\omega_{p_{i,j,k}}(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k})$.
4. Sum over all i, j and k.
5. Take the limit as the maximal distance between adjacent lattice points goes to zero.

Let's focus on Step 3 for a moment. Let t be the distance between $p_{i+1,j,k}$ and $p_{i,j,k}$, and assume t is small. Then $\frac{\partial a}{\partial x_1}(p_{i,j,k})$ is approximately equal to $\frac{a(p_{i+1,j,k})-a(p_{i,j,k})}{t}$. This approximation gets better and better when we let $t \to 0$ in Step 5.

The vectors $V^1_{i,j,k}$ through $V^3_{i,j,k}$ form a little cube. If we say the vector $V^1_{i,j,k}$ is "vertical", and the other two are horizontal, then the "height" of this cube is t and the area of its base is $dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k})$, which makes its volume $t\ dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k})$. Putting all this together, we find that

$$\begin{aligned} d\omega_{p_{i,j,k}}(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k}) &= \frac{\partial a}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k}) \\ &\approx \frac{a(p_{i+1,j,k}) - a(p_{i,j,k})}{t} t\ dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k}) \\ &= \omega(V^2_{i+1,j,k}, V^3_{i+1,j,k}) - \omega(V^2_{i,j,k}, V^3_{i,j,k}). \end{aligned}$$

Let's move on to Step 4. Here we sum over all i, j and k. Suppose for the moment that i ranges between 1 and N. First, we fix j and k and sum over all i. The result is $\omega(V^2_{N,j,k}, V^3_{N,j,k}) - \omega(V^2_{1,j,k}, V^3_{1,j,k})$. Now notice that $\sum\limits_{j,k} \omega(V^2_{N,j,k}, V^3_{N,j,k})$ is a Riemann Sum for the integral of ω over the "top" of R and $\sum\limits_{j,k} \omega(V^2_{1,j,k}, V^3_{1,j,k})$ is a Riemann Sum for ω over the "bottom" of R. Finally, note that ω, evaluated on any pair of vectors which lie in the sides of the cube, gives zero. Hence, the integral of ω over a side of R is zero. Putting all this together, we conclude

$$\int\limits_R d\omega = \int\limits_{\partial R} \omega. \tag{6.1}$$

6.5. Prove that Equation 6.1 holds if $\omega = b\ dx_1 \wedge dx_3$ or if $\omega = c\ dx_1 \wedge dx_2$. Caution: Beware of signs and orientations.

6.6. Use the previous problem to conclude that if $\omega = a\ dx_2 \wedge dx_3 + b\ dx_1 \wedge dx_3 + c\ dx_1 \wedge dx_2$ is an arbitrary 2-form on $\mathbb{R}^3$, then Equation 6.1 holds.

6.7. If ω is an arbitrary $(n-1)$-form on $\mathbb{R}^n$ and R is the unit cube in $\mathbb{R}^n$, then show that Equation 6.1 still holds.

In general, if $C = \sum n_i \sigma_i$ is an n-chain, then

$$\boxed{\int\limits_{\partial C} \omega = \int\limits_C d\omega.}$$

This equation is called the *generalized Stokes' Theorem*. It is unquestionably the most crucial result of this text. In fact, everything we have done up to this point has been geared toward developing this equation and everything that follows will be applications of this equation. Technically, we have only established this theorem when integrating over cubes and their boundaries. We postpone the general proof to Section 7.1.

Example 37. Let $\omega = x\ dy$ be a 1-form on $\mathbb{R}^2$. Let σ be the 2-cell which is the image of the parameterization $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$, where $0 \le r \le R$ and $0 \le \theta \le 2\pi$. By the generalized Stokes' Theorem,

$$\int\limits_{\partial\sigma} \omega = \int\limits_{\sigma} d\omega = \int\limits_{\sigma} dx \wedge dy = \int\limits_{\sigma} dx\ dy = \text{Area}(\sigma) = \pi R^2.$$

6.8. Verify directly that $\int\limits_{\partial\sigma} \omega = \pi R^2$.

Example 38. Let $\omega = x\ dy + y\ dx$ be a 1-form on $\mathbb{R}^2$ and let σ be any 2-cell. Then $\int\limits_{\partial\sigma} \omega = \int\limits_{\sigma} d\omega = 0$.

6.9. Pick any 1-chain in $\mathbb{R}^2$, which bounds a 2-cell, and integrate the form $x\ dy + y\ dx$ over this curve.

6.10. Let ω be a differential $(n-1)$-form and σ a $(n+1)$-cell. Use the generalized Stokes' Theorem in *two* different ways to show $\int\limits_{\partial\sigma} d\omega = 0$.

Example 39. Let C be the curve in $\mathbb{R}^2$ parameterized by $\phi(t) = (t^2, t^3)$, where $-1 \le t \le 1$. Let f be the 0-form x^2y. We use the generalized Stokes' Theorem to calculate $\int\limits_C df$.

The curve C goes from the point (1,-1), when $t = -1$, to the point (1,1), when $t = 1$. Hence, ∂C is the 0-chain $(1,1) - (1,-1)$. Now we use Stokes' Theorem:

$$\int\limits_C df = \int\limits_{\partial C} f = \int\limits_{(1,1)-(1,-1)} x^2 y = 1 - (-1) = 2.$$

6.11. Calculate $\int\limits_C df$ directly.

6.12.

1. Let C be any curve in $\mathbb{R}^2$ which goes from the point $(1,0)$ to the point $(2,2)$. Calculate
$$\int\limits_C 2xy^3\ dx + 3x^2y^2\ dy.$$

2. Let C be any curve in $\mathbb{R}^3$ from $(0,0,0)$ to $(1,1,1)$. Calculate

$$\int_C y^2z^2\ dx + 2xyz^2\ dy + 2xy^2z\ dz.$$

6.13. Let ω be the 1-form on $\mathbb{R}^3$ given by

$$\omega = yz\ dx + xz\ dy + xy\ dz.$$

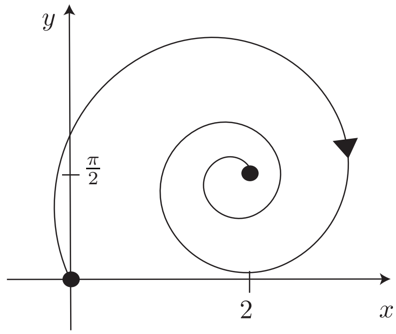
Let C be the curve parameterized by

$$\phi(t) = \left(\frac{4t}{\pi}\cos t, \frac{4t}{\pi}\sin t, \frac{4t}{\pi}\right), \quad 0 \le t \le \frac{\pi}{4},$$

with the induced orientation. Use the generalized Stokes' Theorem to calculate $\int\limits_C \omega$.

6.14. Let C be the curve pictured below. Calculate

$$\int_C \sin y\ dx + x\cos y\ dy.$$



Example 40. Let $\omega = (x^2+y)dx + (x-y^2)dy$ be a 1-form on $\mathbb{R}^2$. We wish to integrate ω over σ, the top half of the unit circle, oriented clockwise. First, note that $d\omega = 0$, so that if we integrate ω over the boundary of any 2-cell, we would get zero. Let τ denote the line segment connecting $(-1,0)$ to $(1,0)$. Then the 1-chain $\sigma - \tau$ bounds a 2-cell. So $\int\limits_{\sigma-\tau} \omega = 0$, which implies that $\int\limits_\sigma \omega = \int\limits_\tau \omega$. This latter integral is a bit easier to compute. Let $\phi(t) = (t,0)$ be a parameterization of τ, where $-1 \le t \le 1$. Then

$$\int_\sigma \omega = \int_\tau \omega = \int_{[-1,1]} \omega_{(t,0)}(\langle 1,0\rangle)\ dt = \int_{-1}^{1} t^2\ dt = \frac{2}{3}.$$

6.15. Let $\omega = -y^2\ dx + x^2\ dy$. Let σ be the 2-cell in $\mathbb{R}^2$ parameterized by the following:

$$\phi(u,v) = (2u - v, u + v),\ 1 \le u \le 2,\ 0 \le v \le 1.$$

Calculate $\int\limits_{\partial\sigma} \omega$.

6.16. Let $\omega = dx - \ln x\ dy$. Let σ be the 2-cell parameterized by the following:

$$\phi(u,v) = (uv^2, u^3 v),\ 1 \le u \le 2,\ 1 \le v \le 2.$$

Calculate: $\int\limits_{\partial\sigma} \omega$.

6.17. Let σ be the 2-cell given by the following parameterization:

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta),\ 0 \le r \le 1,\ 0 \le \theta \le \pi.$$

Suppose $\omega = x^2\ dx + e^y\ dy$.

1. Calculate $\int\limits_{\sigma} d\omega$ directly.
2. Let C_1 be the horizontal segment connecting $(-1,0)$ to $(0,0)$ and let C_2 be the horizontal segment connecting $(0,0)$ to $(1,0)$. Calculate $\int\limits_{C_1} \omega$ and $\int\limits_{C_2} \omega$ directly.
3. Use your previous answers to determine the integral of ω over the top half of the unit circle (oriented counterclockwise).

6.18. Let $\omega = (x + y^3)\ dx + 3xy^2\ dy$ be a differential 1-form on $\mathbb{R}^2$. Let Q be the rectangle $\{(x,y) | 0 \le x \le 3, 0 \le y \le 2\}$.

1. Compute $d\omega$.
2. Use the generalized Stokes' Theorem to compute $\int\limits_{\partial Q} \omega$.
3. Compute $\int\limits_{\partial Q} \omega$ directly, by integrating ω over each each edge of the boundary of the rectangle and then adding in the appropriate manner.
4. How does $\int\limits_{R-T-L} \omega$ compare to $\int\limits_{B} \omega$?
5. Let S be any curve in the upper half-plane (i.e., the set $\{(x,y) | y \ge 0\}$) that goes from the point $(3,0)$ to the point $(0,0)$. What is $\int\limits_{S} \omega$? Why?
6. Let S be *any* curve that goes from the point $(3,0)$ to the point $(0,0)$. What is $\int\limits_{S} \omega$? Why?

6.19. Calculate

$$\int\limits_{C} x^3\ dx + \left(\frac{1}{3}x^3 + xy^2\right)\ dy,$$

where C is the circle of radius 2, centered about the origin.

6.20. Suppose $\omega = x\ dx + x\ dy$ is a 1-form on $\mathbb{R}^2$. Let C be the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Determine the value of $\int_C \omega$ by integrating some 2-form over the region bounded by the ellipse.

6.21. Let $\omega = -y^2\ dx + x^2\ dy$. Let σ be the 2-cell in $\mathbb{R}^2$ parameterized by the following:

$$\phi(r, \theta) = (r \cosh\theta, r \sinh\theta),$$

where $0 \le r \le 1$ and $-1 \le \theta \le 1$. Calculate $\int_{\partial\sigma} \omega$.

$$\left(\text{Recall: } \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}, \quad \sinh\theta = \frac{e^\theta - e^{-\theta}}{2}.\right)$$

6.22. Suppose ω is a 1-form on $\mathbb{R}^2$ such that $d\omega = 0$. Let C_1 and C_2 be the 1-cells given by the following parameterizations:

$$C_1 : \phi(t) = (t, 0),\ 2\pi \le t \le 6\pi$$
$$C_2 : \psi(t) = (t\cos t, t\sin t),\ 2\pi \le t \le 6\pi.$$

Show that $\int_{C_1} \omega = \int_{C_2} \omega$. (Caution: Beware of orientations!)

6.23. Let S be the can-shaped surface in $\mathbb{R}^3$ whose side is the cylinder of radius 1 (centered on the z-axis), and whose top and bottom are in the planes $z = 1$ and $z = 0$, respectively. Let

$$\omega = z^2\ dx \wedge dy$$

Use the generalized Stokes' Theorem to calculate $\int_S \omega$ (assume the standard orientation on S).

6.24. Let V be the region between the cylinders of radii 1 and 2 (centered on the z-axis), in the positive octant, and below the plane $z = 2$. Let ω be the differential 2-form

$$\omega = -y^3\ dx \wedge dz + x^3\ dy \wedge dz.$$

Calculate $\int_{\partial V} \omega$.

6.25. Let ω be the following 2-form on $\mathbb{R}^3$:

$$\omega = (x^2 + y^2) dy \wedge dz + (x^2 - y^2) dx \wedge dz.$$

Let V be the region of $\mathbb{R}^3$ bounded by the graph of $y = \sqrt{1 - x^2}$, the planes $z = 0$ and $z = 2$, and the xz-plane (see Figure 6.4).

1. Parameterize V using cylindrical coordinates.
2. Determine $d\omega$.

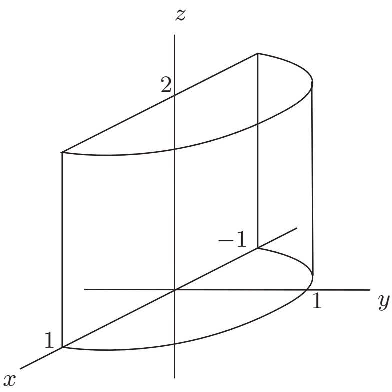


Fig. 6.4. The region V of Problem 6.25.

3. Calculate $\int\limits_V d\omega$.
4. The sides of V are parameterized as follows:
   a) Bottom: $\phi_B(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, where $0 \le r \le 1$ and $0 \le \theta \le \pi$.
   b) Top: $\phi_T(r,\theta) = (r\cos\theta, r\sin\theta, 2)$, where $0 \le r \le 1$ and $0 \le \theta \le \pi$.
   c) Flat side: $\phi_F(x,z) = (x, 0, z)$, where $-1 \le x \le 1$ and $0 \le z \le 2$.
   d) Curved side: $\phi_C(\theta,z) = (\cos\theta, \sin\theta, z)$, where $0 \le \theta \le \pi$ and $0 \le z \le 2$.

   Calculate the integral of ω over the top, bottom and flat side. (Do not calculate this integral over the curved side.)
5. If C is the curved side of ∂V, use your answers to the previous questions to determine $\int\limits_C \omega$.

6.26. Calculate the volume of a ball of radius 1, $\{(\rho,\theta,\phi)|\rho \le 1\}$, by integrating some 2-form over the sphere of radius 1, $\{(\rho,\theta,\phi)|\rho = 1\}$.

6.3 Vector calculus and the many faces of the generalized Stokes' Theorem

Although the language and notation may be new, you have already seen the generalized Stokes' Theorem in many guises. For example, let $f(x)$ be a 0-form on $\mathbb{R}$. Then $df = f'(x)dx$. Let $[a,b]$ be a 1-cell in $\mathbb{R}$. Then the generalized Stokes' Theorem tells us

$$\int\limits_a^b f'(x)\ dx = \int\limits_{[a,b]} f'(x)\ dx = \int\limits_{\partial[a,b]} f(x) = \int\limits_{b-a} f(x) = f(b) - f(a),$$

which is, of course, the "Fundamental Theorem of Calculus." If we let R be some 2-chain in $\mathbb{R}^2$, then the generalized Stokes' Theorem implies

$$\int_{\partial R} P\ dx + Q\ dy = \int_R d(P\ dx + Q\ dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx\ dy.$$

This is what we call "Green's Theorem" in calculus. To proceed further, we restrict ourselves to $\mathbb{R}^3$. In this dimension, there is a nice correspondence between vector fields and both 1- and 2-forms:

$$\begin{aligned}\mathbf{F} = \langle F_x, F_y, F_z\rangle &\leftrightarrow \omega^1_{\mathbf{F}} = F_x dx + F_y dy + F_z dz \\ &\leftrightarrow \omega^2_{\mathbf{F}} = F_x dy \wedge dz - F_y dx \wedge dz + F_z dx \wedge dy.\end{aligned}$$

On $\mathbb{R}^3$ there is also a useful correspondence between 0-forms (functions) and 3-forms:

$$f(x,y,z) \leftrightarrow \omega^3_f = f\ dx \wedge dy \wedge dz.$$

We can use these correspondences to define various operations involving functions and vector fields. For example, suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a 0-form. Then df is the 1-form $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. The vector field associated to this 1-form is then $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$. In calculus we call this vector field **grad** f, or ∇f. In other words, ∇f is the vector field associated with the 1-form, df. This can be summarized by the equation

$$df = \omega^1_{\nabla f}.$$

It will be useful to think of this as a diagram as well:

$$\begin{array}{ccc} f & \xrightarrow{\mathbf{grad}} & \nabla f \\ \| & & \uparrow \\ f & \xrightarrow[d]{} & df \end{array}$$

Example 41. Suppose $f = x^2y^3z$. Then $df = 2xy^3z\ dx + 3x^2y^2z\ dy + x^3y^3\ dz$. The associated vector field, **grad** f, is then $\nabla f = \langle 2xy^3z, 3x^2y^2z, x^3y^3\rangle$.

Similarly, if we start with a vector field $\mathbf{F}$, form the associated 1-form $\omega^1_{\mathbf{F}}$, differentiate it, and look at the corresponding vector field, then the result is called **curl F**, or $\nabla\times\mathbf{F}$. So, $\nabla\times\mathbf{F}$ is the vector field associated with the 2-form $d\omega^1_{\mathbf{F}}$. This can be summarized by the equation

$$d\omega^1_{\mathbf{F}} = \omega^2_{\nabla\times\mathbf{F}}.$$

This can also be illustrated by the following diagram:

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\mathbf{curl}} & \nabla \times \mathbf{F} \\ \downarrow & & \uparrow \\ \omega^1_{\mathbf{F}} & \xrightarrow[d]{} & d\omega^1_{\mathbf{F}} \end{array}$$

Example 42. Let $\mathbf{F} = \langle xy, yz, x^2 \rangle$. The associated 1-form is then

$$\omega^1_{\mathbf{F}} = xy\ dx + yz\ dy + x^2\ dz.$$

The derivative of this 1-form is the 2-form

$$d\omega^1_{\mathbf{F}} = -y\ dy \wedge dz + 2x\ dx \wedge dz - x\ dx \wedge dy.$$

The vector field associated to this 2-form is **curl F**, which is

$$\nabla \times \mathbf{F} = \langle -y, -2x, -x \rangle.$$

Finally, we can start with a vector field $\mathbf{F} = \langle F_x, F_y, F_z \rangle$ and then look at the 3-form $d\omega^2_{\mathbf{F}} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) dx \wedge dy \wedge dz$ (see Problem 5.13). The function $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ is called **div F**, or $\nabla \cdot \mathbf{F}$. This is summarized in the following equation and diagram:

$$d\omega^2_{\mathbf{F}} = \omega^3_{\nabla \cdot \mathbf{F}}$$

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\mathbf{div}} & \nabla \cdot \mathbf{F} \\ \downarrow & & \uparrow \\ \omega^2_{\mathbf{F}} & \xrightarrow[d]{} & d\omega^2_{\mathbf{F}} \end{array}$$

Example 43. Let $\mathbf{F} = \langle xy, yz, x^2 \rangle$. The associated 2-form is then

$$\omega^2_{\mathbf{F}} = xy\ dy \wedge dz - yz\ dx \wedge dz + x^2\ dx \wedge dy.$$

The derivative is the 3-form

$$d\omega^2_{\mathbf{F}} = (y + z)\ dx \wedge dy \wedge dz.$$

So **div F** is the function $\nabla \cdot \mathbf{F} = y + z$.

Two important vector identities follow from the fact that for a differential form, ω, calculating $d(d\omega)$ always yields zero (see Problem 5.9). For the first identity, consider the following diagram:

$$\begin{array}{ccccc} f & \xrightarrow{\mathbf{grad}} & \nabla f & \xrightarrow{\mathbf{curl}} & \nabla \times (\nabla f) \\ \Vert & & \uparrow & & \uparrow \\ f & \xrightarrow[d]{} & df & \xrightarrow[d]{} & ddf \end{array}$$

This shows that if f is a 0-form, then the vector field corresponding to ddf is $\nabla \times (\nabla f)$. However, $ddf = 0$, so we conclude

$$\nabla \times (\nabla f) = 0.$$

For the second identity, consider the diagram

$$\begin{array}{ccccc} \mathbf{F} & \xrightarrow{\mathbf{curl}} & \nabla \times \mathbf{F} & \xrightarrow{\mathbf{div}} & \nabla \cdot (\nabla \times \mathbf{F}) \\ \downarrow & & \downarrow & & \uparrow \\ \omega^1_{\mathbf{F}} & \xrightarrow[d]{} & d\omega^1_{\mathbf{F}} & \xrightarrow[d]{} & dd\omega^1_{\mathbf{F}}. \end{array}$$

This shows that if $dd\omega^1_{\mathbf{F}}$ is written as $g\ dx \wedge dy \wedge dz$, then the function g is equal to $\nabla \cdot (\nabla \times \mathbf{F})$. However, $dd\omega^1_{\mathbf{F}} = 0$, so we conclude

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

In vector calculus we also learn how to integrate vector fields over parameterized curves (1-chains) and surfaces (2-chains). Suppose, first, that σ is some parameterized curve. Then we can integrate the component of $\mathbf{F}$ which points in the direction of the tangent vectors to σ. This integral is usually denoted by $\int\limits_\sigma \mathbf{F} \cdot d\mathbf{s}$, and its definition is precisely the same as the definition we learned here for $\int\limits_\sigma \omega^1_{\mathbf{F}}$. A special case of this integral arises when $\mathbf{F} = \nabla f$ for some function f. In this case, $\omega^1_{\mathbf{F}}$ is just df, so the definition of $\int\limits_\sigma \nabla f \cdot d\mathbf{s}$ is the same as $\int\limits_\sigma df$.

6.27. Let C be any curve in $\mathbb{R}^3$ from $(0,0,0)$ to $(1,1,1)$. Let $\mathbf{F}$ be the vector field $\langle yz, xz, xy \rangle$. Show that $\int\limits_C \mathbf{F} \cdot d\mathbf{s}$ does not depend on C. (Hint: Use the generalized Stokes' Theorem.)

We also learn to integrate vector fields over parameterized surfaces. In this case, the quantity we integrate is the component of the vector field which is normal to the surface. This integral is often denoted by $\int\limits_S \mathbf{F} \cdot d\mathbf{S}$. Its definition is precisely the same as that of $\int\limits_S \omega^2_{\mathbf{F}}$ (see Problems 3.23 and 3.24). A special case of this is when $\mathbf{F} = \nabla \times \mathbf{G}$, for some vector field, $\mathbf{G}$. Then $\omega^2_{\mathbf{G}}$ is just $d\omega^1_{\mathbf{G}}$, so we see that $\int\limits_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$ must be the same as $\int\limits_S d\omega^1_{\mathbf{G}}$.

The most basic thing to integrate over a 3-dimensional region (i.e., a 3-chain) Ω in $\mathbb{R}^3$ is a function $f(x, y, x)$. In calculus we denote this integral by $\int_\Omega f\ dV$. Note that this is precisely the same as $\int_\Omega \omega_f^3$. A special case is when $f = \nabla \cdot \mathbf{F}$, for some vector field $\mathbf{F}$. In this case, $\int_\Omega f\ dV = \int_\Omega (\nabla \cdot \mathbf{F})\ dV$. However, we can write this integral with differential forms as $\int_\Omega d\omega_{\mathbf{F}}^2$.

We summarize the equivalence between the integrals developed in vector calculus and various integrals of differential forms in Table 6.1.

Table 6.1. The equivalence between the integrals of vector calculus and differential forms.

Vector Calculus	Differential Forms
$\int_\sigma \mathbf{F} \cdot d\mathbf{s}$	$\int_\sigma \omega_{\mathbf{F}}^1$
$\int_\sigma \nabla f \cdot d\mathbf{s}$	$\int_\sigma df$
$\int_S \mathbf{F} \cdot d\mathbf{S}$	$\int_S \omega_{\mathbf{F}}^2$
$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$	$\int_S d\omega_{\mathbf{F}}^1$
$\int_\Omega f\ dV$	$\int_\Omega \omega_f^3$
$\int_\Omega (\nabla \cdot \mathbf{F}) dV$	$\int_\Omega d\omega_{\mathbf{F}}^2$

Let us now apply the generalized Stokes' Theorem to various situations. First, we start with a parameterization, $\phi : [a, b] \to \sigma \subset \mathbb{R}^3$, of a curve in $\mathbb{R}^3$, and a function, $f : \mathbb{R}^3 \to \mathbb{R}$. Then we have

$$\int_\sigma \nabla f \cdot d\mathbf{s} \equiv \int_\sigma df = \int_{\partial\sigma} f = f(\phi(b)) - f(\phi(a)).$$

This shows the independence of path of line integrals of gradient fields. We can use this to prove that a line integral of a gradient field over any simple closed curve is zero, but for us there is an easier, direct proof, which again uses the generalized Stokes' Theorem. Suppose σ is a simple closed curve in $\mathbb{R}^3$ (i.e., $\partial\sigma = \emptyset$). Then $\sigma = \partial D$ for some 2-chain D. We now have

$$\int_\sigma \nabla f \cdot d\mathbf{s} \equiv \int_\sigma df = \int_D ddf = 0.$$

Now, suppose we have a vector field $\mathbf{F}$ and a parameterized surface S. Yet another application of the generalized Stokes' Theorem yields

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \equiv \int_{\partial S} \omega^1_{\mathbf{F}} = \int_S d\omega^1_{\mathbf{F}} \equiv \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

In vector calculus we call this equality "Stokes' Theorem." In some sense, $\nabla \times \mathbf{F}$ measures the "twisting" of $\mathbf{F}$ at points of S. So Stokes' Theorem says that the net twisting of $\mathbf{F}$ over all of S is the same as the amount $\mathbf{F}$ circulates around ∂S.

Example 44. Suppose we are faced with a problem phrased as "Use Stokes' Theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x + 1$, and $\mathbf{F}$ is the vector field $\langle -x^2y, xy^2, z^3 \rangle$."

We will solve this problem by translating to the language of differential forms and using the generalized Stokes' Theorem instead. To begin, note that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \omega^1_{\mathbf{F}}$ and $\omega^1_{\mathbf{F}} = -x^2y\ dx + xy^2\ dy + z^3\ dz$.

Now, to use the generalized Stokes' Theorem we will need to calculate

$$d\omega^1_{\mathbf{F}} = (x^2 + y^2)\ dx \wedge dy.$$

Let D denote the subset of the plane $z = x + 1$ bounded by C. Then $\partial D = C$. Hence, by the generalized Stokes' Theorem we have

$$\int_C \omega^1_{\mathbf{F}} = \int_D d\omega^1_{\mathbf{F}} = \int_D (x^2 + y^2)\ dx \wedge dy.$$

The region D is parameterized by $\Psi(r, \theta) = (r\cos\theta, r\sin\theta, r\cos\theta + 1)$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Using this, one can (and should!) show that $\int_D (x^2 + y^2)\ dx \wedge dy = \frac{\pi}{2}$.

6.28. Let C be the square with sides $(x, \pm 1, 1)$, where $-1 \le x \le 1$ and $(\pm 1, y, 1)$, where $-1 \le y \le 1$, with the indicated orientation (see Figure 6.5). Let $\mathbf{F}$ be the vector field $\langle xy, x^2, y^2z \rangle$. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.

6.29. Let $\mathbf{F}$ be the vector field $\langle 0, -z, 0 \rangle$.

1. Calculate $\nabla \times \mathbf{F}$.
2. Find a 2-form α such that for any surface P,

$$\int_P \alpha = \int_P (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

3. Let P be the portion of a paraboloid parameterized by

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta, r^2), \quad 0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Use the previous problem to calculate $\int_{\partial P} \mathbf{F} \cdot d\mathbf{s}$

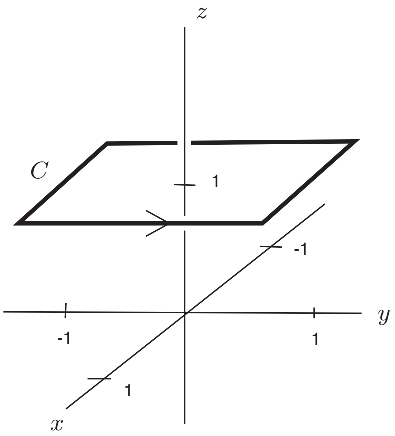


Fig. 6.5.

Suppose now that Ω is some volume in $\mathbb{R}^3$. Then we have

$$\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} \equiv \int_{\partial\Omega} \omega^2_{\mathbf{F}} = \int_{\Omega} d\omega^2_{\mathbf{F}} \equiv \int_{\Omega} (\nabla \cdot \mathbf{F}) dV.$$

This last equality is called "Gauss' Divergence Theorem." $\nabla \cdot \mathbf{F}$ is a measure of how much $\mathbf{F}$ "spreads out" at a point. So Gauss' Theorem says that the total spreading out of $\mathbf{F}$ inside Ω is the same as the net amount of $\mathbf{F}$ "escaping" through $\partial\Omega$.

6.30. Let Ω be the cube $\{(x, y, z) | 0 \le x, y, z \le 1\}$. Let $\mathbf{F}$ be the vector field $\langle xy^2, y^3, x^2y^2 \rangle$. Compute $\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$.

6.31. Let S be the portion of the graph of $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. Let V be the region of $\mathbb{R}^3$ that lies between S and the plane $z = 0$. Let $\mathbf{W} = \langle -y, x, z^2 \rangle$.

1. Compute $\nabla \cdot \mathbf{W}$.
2. Integrate $\nabla \cdot \mathbf{W}$ over V.
3. Use Gauss' Divergence Theorem to compute a surface integral over ∂V of something to get the same result as in Part 2.
4. Integrate $\mathbf{W}$ over ∂S.
5. Use Stokes' Theorem to compute a surface integral over S of something to get the same result as in Part 4.

6.4 Application: Maxwell's Equations

As a brief application, we show how the language of differential forms can greatly simplify the classical vector equations of Maxwell. Much of this material is taken from [MTW73], where the interested student can find many more applications of differential forms to physics.

Maxwell's Equations describe the relationship between electric and magnetic fields. Classically, both electricity and magnetism are described as a 3-dimensional vector field which varies with time:

$$\mathbf{E} = \langle E_x, E_y, E_z \rangle,$$

$$\mathbf{B} = \langle B_x, B_y, B_z \rangle,$$

where E_x, E_z, E_z, B_x, B_y and B_z are all functions of x, y, z and t.

Maxwell's Equations are then

$$\nabla \cdot \mathbf{B} = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0,$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho,$$

$$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -4\pi\mathbf{J}.$$

The quantity ρ is called the *charge density* and the vector $\mathbf{J} = \langle J_x, J_y, J_z \rangle$ is called the *current density*.

We can make all of this look much simpler by making the following definitions. First, we define a 2-form called the *Faraday*, which simultaneously describes both the electric and magnetic fields:

$$\begin{aligned} \mathbf{F} = {} & E_x \; dx \wedge dt + E_y \; dy \wedge dt + E_z \; dz \wedge dt \\ & + B_x \; dy \wedge dz + B_y \; dz \wedge dx + B_z \; dx \wedge dy. \end{aligned}$$

Next, we define the "dual" 2-form, called the *Maxwell*:

$$\begin{aligned} {}^*\mathbf{F} = {} & E_x \; dy \wedge dz + E_y \; dz \wedge dx + E_z \; dx \wedge dy \\ & + B_x \; dt \wedge dx + B_y \; dt \wedge dy + B_z \; dt \wedge dz. \end{aligned}$$

We also define the *4-current* $\mathbf{J}$ and its "dual" ${}^*\mathbf{J}$:

$$\begin{aligned} \mathbf{J} &= \langle \rho, J_x, J_y, J_z \rangle, \\ {}^*\mathbf{J} &= \rho \; dx \wedge dy \wedge dz, \\ & \quad - J_x \; dt \wedge dy \wedge dz, \\ & \quad - J_y \; dt \wedge dz \wedge dx, \\ & \quad - J_z \; dt \wedge dx \wedge dy. \end{aligned}$$

Maxwell's four vector equations now reduce to

$$d\mathbf{F} = 0,$$

$$d^*\mathbf{F} = 4\pi^*\mathbf{J}$$

6.32. Show that the equation $d\mathbf{F} = 0$ implies the first two of Maxwell's Equations.

6.33. Show that the equation $d^*\mathbf{F} = 4\pi^*\mathbf{J}$ implies the second two of Maxwell's Equations.

The differential form version of Maxwell's Equation has a huge advantage over the vector formulation: *It is coordinate-free!* A 2-form such as $\mathbf{F}$ is an operator that "eats" pairs of vectors and "spits out" numbers. The way it acts is completely geometric; that is, it can be defined without any reference to the coordinate system (t, x, y, z). This is especially poignant when one realizes that Maxwell's Equations are laws of nature that should not depend on a man-made construction such as coordinates.