4

Differential Forms

4.1 Families of forms

Let us now go back to the example in Chapter 1. In the last section of that chapter, we showed that the integral of a function, $f : \mathbb{R}^3 \to \mathbb{R}$, over a surface parameterized by $\phi : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ is

$$\int\limits_R f(\phi(r,\theta))\text{Area}\left[\frac{\partial \phi}{\partial r}(r,\theta), \frac{\partial \phi}{\partial \theta}(r,\theta)\right] dr\ d\theta.$$

This gave one motivation for studying differential forms. We wanted to generalize this integral by considering functions other than "Area$(\cdot,\cdot)$" that eat pairs of vectors and return numbers. However, in this integral, the point at which such a pair of vectors is based changes. In other words, Area$(\cdot,\cdot)$ does *not* act on $T_p\mathbb{R}^3 \times T_p\mathbb{R}^3$ for a *fixed* p. We can make this point a little clearer by reexamining the above integrand. Note that it is of the form $f(\star)\text{Area}(\cdot,\cdot)$. For a fixed point, $\star$, of $\mathbb{R}^3$, this is an operator on $T_\star\mathbb{R}^3 \times T_\star\mathbb{R}^3$, much like a 2-form is.

So far all we have done is to define 2-forms at fixed points of $\mathbb{R}^3$. To really generalize the above integral, we must start to consider entire families of 2-forms, $\omega_p : T_p\mathbb{R}^3 \times T_p\mathbb{R}^3 \to \mathbb{R}$, where p ranges over all of $\mathbb{R}^3$. Of course, for this to be useful, such a family must have some "niceness" properties. For one thing, it should be *continuous*; that is, if p and q are close, then ω_p and ω_q should be similar.

An even stronger property is that the family ω_p is *differentiable*. To see what this means, recall that for a fixed p, a 2-form ω_p can always be written as $a_p dx \wedge dy + b_p dy \wedge dz + c_p dx \wedge dz$, where a_p, b_p and c_p are constants. If we let our choice of p vary over all of $\mathbb{R}^3$, then so will these constants. In other words, a_p, b_p and c_p are all functions from $\mathbb{R}^3$ to $\mathbb{R}$. To say that the family ω_p is differentiable, we mean that each of these functions is differentiable. If ω_p is differentiable, then we will refer to it as a *differential* form. When there can be no confusion, we will suppress the subscript p.

D. Bachman, *A Geometric Approach to Differential Forms*,
DOI 10.1007/978-0-8176-8304-7_4,

Example 17. $\omega = x^2y\ dx \wedge dy - xz\ dy \wedge dz$ is a differential 2-form on $\mathbb{R}^3$. On the space $T_{(1,2,3)}\mathbb{R}^3$ it is just the 2-form $2dx \wedge dy - 3dy \wedge dz$. We will denote vectors in $T_{(1,2,3)}\mathbb{R}^3$ as $\langle dx, dy, dz\rangle_{(1,2,3)}$. Hence, the value of $\omega(\langle 4, 0, -1\rangle_{(1,2,3)}, \langle 3, 1, 2\rangle_{(1,2,3)})$ is the same as the 2-form $2dx \wedge dy + dy \wedge dz$, evaluated on the vectors $\langle 4, 0, -1\rangle$ and $\langle 3, 1, 2\rangle$, which we compute as follows:

$$\begin{aligned}\omega(\langle 4, 0, -1\rangle_{(1,2,3)},\ &\langle 3, 1, 2\rangle_{(1,2,3)})\\ &= 2dx \wedge dy - 3dy \wedge dz(\langle 4, 0, -1\rangle, \langle 3, 1, 2\rangle)\\ &= 2\begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 5.\end{aligned}$$

Suppose ω is a differential 2-form on $\mathbb{R}^3$. What does ω act on? It takes a pair of vectors at each point of $\mathbb{R}^3$ and returns a number. In other words, it takes two *vector fields* and returns a function from $\mathbb{R}^3$ to $\mathbb{R}$. A vector field is simply a choice of vector in $T_p\mathbb{R}^3$ for each $p \in \mathbb{R}^3$. In general, a differential n-form on $\mathbb{R}^m$ acts on n vector fields to produce a function from $\mathbb{R}^m$ to $\mathbb{R}$ (see Figure 4.1).

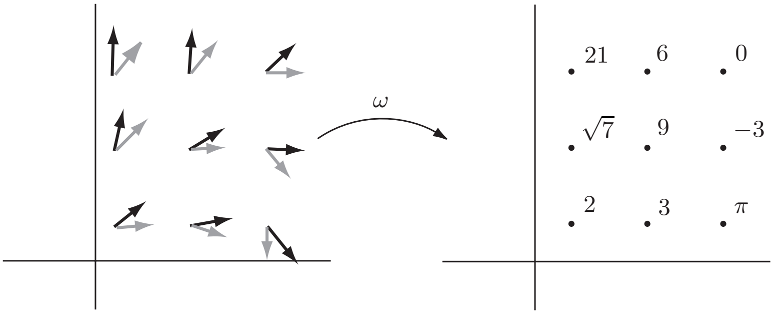


Fig. 4.1. A differential 2-form, ω, acts on a pair of vector fields and returns a function from $\mathbb{R}^n$ to $\mathbb{R}$.

Example 18. $V_1 = \langle 2y, 0, -x\rangle_{(x,y,z)}$ is a vector field on $\mathbb{R}^3$. For example, it contains the vector $\langle 4, 0, -1\rangle \in T_{(1,2,3)}\mathbb{R}^3$. If $V_2 = \langle z, 1, xy\rangle_{(x,y,z)}$ and ω is the differential 2-form $x^2y\ dx \wedge dy - xz\ dy \wedge dz$, then

$$\begin{aligned}\omega(V_1, V_2) &= x^2y\ dx \wedge dy - xz\ dy \wedge dz(\langle 2y, 0, -x\rangle_{(x,y,z)}, \langle z, 1, xy\rangle_{(x,y,z)})\\ &= x^2y\begin{vmatrix} 2y & z \\ 0 & 1 \end{vmatrix} - xz\begin{vmatrix} 0 & 1 \\ -x & xy \end{vmatrix} = 2x^2y^2 - x^2z,\end{aligned}$$

which is a function from $\mathbb{R}^3$ to $\mathbb{R}$.

Notice that V_2 contains the vector $\langle 3, 1, 2\rangle_{(1,2,3)}$. So, from the previous example we would expect that $2x^2y^2 - x^2z$ equals 5 at the point $(1, 2, 3)$, which is indeed the case.

4.1. Let ω be the differential 2-form on $\mathbb{R}^3$ given by

$$\omega = xyz\ dx \wedge dy + x^2 z\ dy \wedge dz - y\ dx \wedge dz.$$

Let V_1 and V_2 be the following vector fields:

$$V_1 = \langle y, z, x^2 \rangle_{(x,y,z)}, \quad V_2 = \langle xy, xz, y \rangle_{(x,y,z)}.$$

1. What vectors do V_1 and V_2 contain at the point $(1, 2, 3)$?
2. Which 2-form is ω on $T_{(1,2,3)}\mathbb{R}^3$?
3. Use your answers to the previous two questions to compute $\omega(V_1, V_2)$ at the point $(1, 2, 3)$.
4. Compute $\omega(V_1, V_2)$ at the point (x, y, z). Then plug in $x = 1$, $y = 2$ and $z = 3$ to check your answer against the previous question.

4.2 Integrating differential 2-forms

Let's now recall the steps involved with integration of functions on subsets of $\mathbb{R}^2$. Suppose $R \subset \mathbb{R}^2$ and $f : R \to \mathbb{R}$. The following steps define the integral of f over R:

1. Choose a lattice of points in R, $\{(x_i, y_j)\}$.
2. For each i and j, define $V^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$ and $V^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$ (see Figure 4.2). Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are both vectors in $T_{(x_i,y_j)}\mathbb{R}^2$.
3. For each i and j, compute $f(x_i, y_j)\text{Area}(V^1_{i,j}, V^2_{i,j})$, where $\text{Area}(V, W)$ is the function which returns the area of the parallelogram spanned by the vectors V and W.
4. Sum over all i and j.
5. Take the limit as the maximal distance between adjacent lattice points goes to zero. This is the number that we define to be the value of $\int\limits_R f\ dx\ dy$.

Let's focus on Step 3. Here we compute $f(x_i, y_j)\text{Area}(V^1_{i,j}, V^2_{i,j})$. Notice that this is exactly the value of the differential 2-form $\omega = f(x, y)dx \wedge dy$ evaluated on the vectors $V^1_{i,j}$ and $V^2_{i,j}$ at the point (x_i, y_j). Hence, in Step 4 we can write this sum as $\sum\limits_i \sum\limits_j f(x_i, y_j)\text{Area}(V^1_{i,j}, V^2_{i,j}) = \sum\limits_i \sum\limits_j \omega_{(x_i,y_j)}(V^1_{i,j}, V^2_{i,j})$. It is reasonable, then, to adopt the shorthand "$\int\limits_R \omega$" to denote the limit in Step 5. The upshot of all this is the following:

> *If* $\omega = f(x, y)dx \wedge dy$*, then* $\int\limits_R \omega = \int\limits_R f\ dx\ dy.$

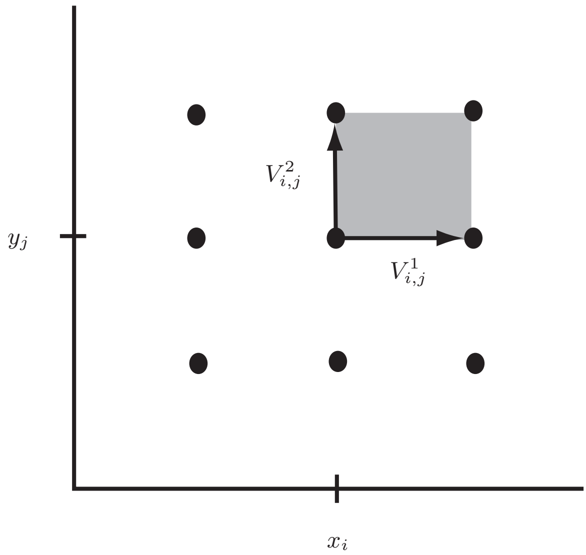


Fig. 4.2. The steps toward integration.

Since all differential 2-forms on $\mathbb{R}^2$ are of the form $f(x,y)dx \wedge dy$, we now know how to integrate them.

CAUTION: When integrating 2-forms on $\mathbb{R}^2$, it is tempting to always drop the "$\wedge$" and forget you have a differential form. This is only valid with $dx \wedge dy$. It is NOT valid with $dy \wedge dx$. This may seem a bit curious since Fubini's Theorem gives us

$$\int f\ dx \wedge dy = \int f\ dx\ dy = \int f\ dy\ dx.$$

All of these are equal to $-\int f\ dy \wedge dx$. We will revisit this issue in Example 27.

4.2. Let $\omega = xy^2\ dx \wedge dy$ be a differential 2-form on $\mathbb{R}^2$. Let D be the region of $\mathbb{R}^2$ where $0 \le x \le 1$ and $0 \le y \le 1$. Calculate $\int\limits_D \omega$.

What about integration of differential 2-forms on $\mathbb{R}^3$? As remarked at the end of Section 1.4, we do this only over those subsets of $\mathbb{R}^3$ which can be parameterized by subsets of $\mathbb{R}^2$. Suppose M is such a subset, like the top half of the unit sphere. To define what we mean by $\int\limits_M \omega$, we just follow the above steps:

1. Choose a lattice of points in M, $\{p_{i,j}\}$.
2. For each i and j, define $V^1_{i,j} = p_{i+1,j} - p_{i,j}$ and $V^2_{i,j} = p_{i,j+1} - p_{i,j}$. Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are both vectors in $T_{p_{i,j}}\mathbb{R}^3$ (see Figure 4.3).

3. For each i and j, compute $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$.
4. Sum over all i and j.
5. Take the limit as the maximal distance between adjacent lattice points goes to zero. This is the number that we define to be the value of $\int\limits_M \omega$.

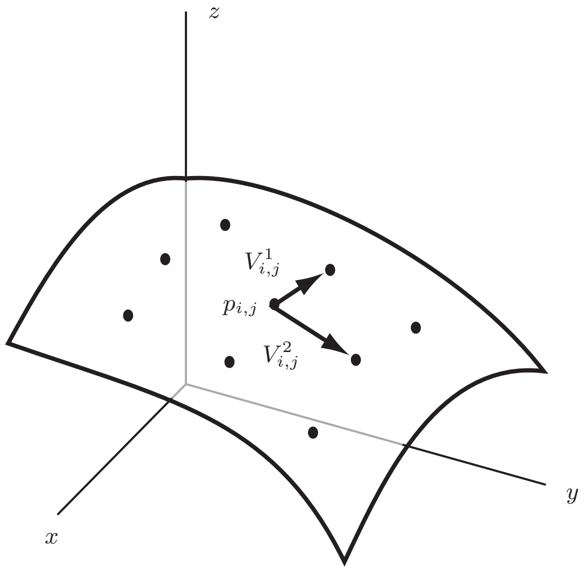


Fig. 4.3. The steps toward integrating a 2-form.

Unfortunately these steps are not so easy to follow. For one thing, it is not always clear how to pick the lattice in Step 1. In fact, there is an even worse problem. In Step 3, why did we compute $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ instead of $\omega_{p_{i,j}}(V^2_{i,j}, V^1_{i,j})$? After all, $V^1_{i,j}$ and $V^2_{i,j}$ are two randomly oriented vectors in $T\mathbb{R}^3_{p_{i,j}}$. There is no reasonable way to decide which should be first and which second. There is nothing to be done about this. At some point we just have to make a choice and make it clear which choice we have made. Such a decision is called an *orientation*. We will have much more to say about this later. For now, we simply note that a different choice will only change our answer by changing its sign.

While we are on this topic, we also note that we would end up with the same number in Step 5 if we had calculated $\omega_{p_{i,j}}(-V^1_{i,j}, -V^2_{i,j})$ in Step 4 instead. Similarly, if it turns out later that we should have calculated $\omega_{p_{i,j}}(V^2_{i,j}, V^1_{i,j})$, then we could have also arrived at the right answer by computing $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$. In other words, there are really only two possibilities:

Either $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ gives the correct answer or $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$ does. Which one will depend on our choice of orientation.

Despite all the difficulties with using the above definition of $\int\limits_M \omega$, all hope is not lost. Remember that we are only integrating over regions which can be parameterized by subsets of $\mathbb{R}^2$. The trick is to use such a parameterization to translate the problem into an integral of a 2-form over a region in $\mathbb{R}^2$. The steps are analogous to those in Section 1.4.

Suppose $\phi : R \subset \mathbb{R}^2 \to M$ is a parameterization. We want to find a 2-form $f(x,y)\ dx \wedge dy$, such that a Riemann Sum for this 2-form over R gives the same result as a Riemann Sum for ω over M. Let's begin:

1. Choose a rectangular lattice of points in R, $\{(x_i, y_j)\}$. This also gives a lattice, $\{\phi(x_i, y_j)\}$, in M.
2. For each i and j, define $V^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$, $V^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$, $\mathcal{V}^1_{i,j} = \phi(x_{i+1}, y_j) - \phi(x_i, y_j)$ and $\mathcal{V}^2_{i,j} = \phi(x_i, y_{j+1}) - \phi(x_i, y_j)$ (see Figure 4.4). Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are vectors in $T_{(x_i,y_j)}\mathbb{R}^2$ and $\mathcal{V}^1_{i,j}$ and $\mathcal{V}^2_{i,j}$ are vectors in $T_{\phi(x_i,y_j)}\mathbb{R}^3$.
3. For each i and j, compute $f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j})$ and $\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$.
4. Sum over all i and j.

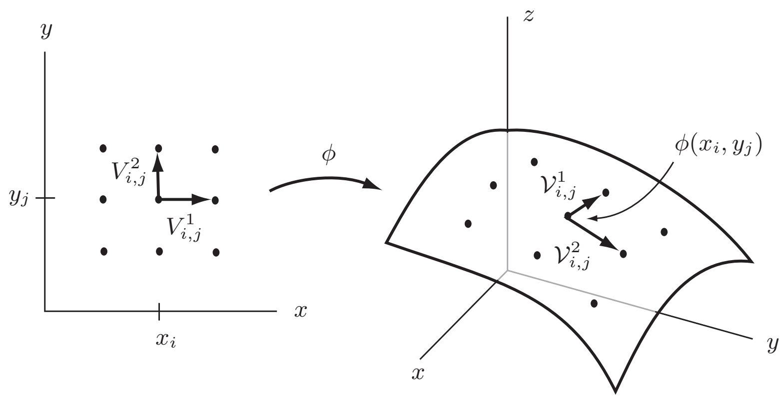


Fig. 4.4. Using ϕ to integrate a 2-form.

At the conclusion of Step 4 we have two sums: $\sum\limits_i \sum\limits_j f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j})$ and $\sum\limits_i \sum\limits_j \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$. In order for these to be equal, we must have

$$f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j}) = \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j}).$$

So,

$$f(x_i, y_j) = \frac{\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})}{dx \wedge dy(V^1_{i,j}, V^2_{i,j})}.$$

Since we are using a rectangular lattice in R, we know $dx \wedge dy(V^1_{i,j}, V^2_{i,j}) = \text{Area}(V^1_{i,j}, V^2_{i,j}) = |V^1_{i,j}| \cdot |V^2_{i,j}|$. We now have

$$f(x_i, y_j) = \frac{\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})}{|V^1_{i,j}| \cdot |V^2_{i,j}|}.$$

Using the bilinearity of ω, this reduces to

$$f(x_i, y_j) = \omega_{\phi(x_i,y_j)}\left(\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|}, \frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}\right).$$

As the distance between adjacent points of our partition tends toward zero,

$$\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|} = \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{|(x_{i+1}, y_j) - (x_i, y_j)|} = \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{|x_{i+1} - x_i|} \to \frac{\partial \phi}{\partial x}(x_i, y_j).$$

Similarly, $\frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}$ converges to $\frac{\partial \phi}{\partial y}(x_i, y_j)$.

Let's summarize what we have so far. We defined $f(x, y)$ so that

$$\begin{aligned}
&\sum_i \sum_j \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j}) \\
&\quad = \sum_i \sum_j f(x_i, y_j)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j}) \\
&\quad = \sum_i \sum_j \omega_{\phi(x_i,y_j)}\left(\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|}, \frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}\right)\ dx \wedge dy(V^1_{i,j}, V^2_{i,j}).
\end{aligned}$$

We have also shown that when we take the limit as the distance between adjacent partition points tends toward zero, this sum converges to the sum

$$\sum_i \sum_j \omega_{\phi(x,y)}\left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y)\right) dx \wedge dy(V^1_{i,j}, V^2_{i,j}).$$

Hence, it must be that

$$\boxed{\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)}\left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y)\right) dx \wedge dy. \qquad (4.1)}$$

At first glance, this seems like a very complicated formula. Let's break it down by examining the integrand on the right. The most important thing to notice is that this is just a differential 2-form on R, even though ω is a 2-form on $\mathbb{R}^3$. For each pair of numbers, (x, y), the function $\omega_{\phi(x,y)}\left(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y)\right)$ just returns some real number. Hence, the entire integrand is of the form $g\ dx \wedge dy$, where $g : R \to \mathbb{R}$.

The only way to *really* convince oneself of the usefulness of this formula is to actually use it.

Example 19. Let M denote the top half of the unit sphere in $\mathbb{R}^3$. Let $\omega = z^2 dx \wedge dy$ be a differential 2-form on $\mathbb{R}^3$. Calculating $\int\limits_M \omega$ directly by setting up a Riemann Sum would be next to impossible. So we employ the parameterization $\phi(r,t) = (r\cos t, r\sin t, \sqrt{1-r^2})$, where $0 \le t \le 2\pi$ and $0 \le r \le 1$.

$$\begin{aligned}
\int\limits_M \omega &= \int\limits_R \omega_{\phi(r,t)}\left(\frac{\partial\phi}{\partial r}(r,t), \frac{\partial\phi}{\partial t}(r,t)\right) dr \wedge dt \\
&= \int\limits_R \omega_{\phi(r,t)}\left(\left\langle \cos t, \sin t, \frac{-r}{\sqrt{1-r^2}}\right\rangle, \langle -r\sin t, r\cos t, 0\rangle\right) dr \wedge dt \\
&= \int\limits_R (1-r^2)\begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} dr \wedge dt \\
&= \int\limits_R (1-r^2)(r)\ dr \wedge dt \\
&= \int\limits_0^{2\pi}\int\limits_0^{1} r - r^3\ dr\ dt = \frac{\pi}{2}.
\end{aligned}$$

Notice that, as promised, the term $\omega_{\phi(r,t)}\left(\frac{\partial\phi}{\partial r}(r,t), \frac{\partial\phi}{\partial t}(r,t)\right)$ in the second integral simplified to a function from R to $\mathbb{R}$: $r - r^3$.

4.3. Integrate the 2-form

$$\omega = \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz$$

over the following surfaces:

1. The top half of the unit sphere using the following parameterizations from cylindrical and spherical coordinates:
   a) $(r, \theta) \to (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$, where $0 \le \theta \le 2\pi$ and $0 \le r \le 1$.
   b) $(\theta, \phi) \to (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{2}$.

2. The surface parameterized by

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta, \cos r), \quad 0 \le r \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi.$$

3. The surface parameterized by

$$\Psi(\theta,\phi) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi), \quad 0 \le \theta \le 2\pi, \quad -\frac{\pi}{4} \le \phi \le \frac{\pi}{4}.$$

4.4. Let S be the surface in $\mathbb{R}^3$ parameterized by

$$\Psi(\theta, z) = (\cos\theta, \sin\theta, z),$$

where $0 \le \theta \le \pi$ and $0 \le z \le 1$. Let $\omega = xyz\ dy \wedge dz$. Calculate $\int\limits_S \omega$.

4.5. Let ω be the differential 2-form on $\mathbb{R}^3$ given by

$$\omega = xyz\ dx \wedge dy + x^2 z\ dy \wedge dz - y\ dx \wedge dz.$$

1. Let P be the portion of the plane $3 = 2x + 3y - z$ in $\mathbb{R}^3$ that lies above the square $\{(x,y)|0 \le x \le 1, 0 \le y \le 1\}$. Calculate $\int\limits_P \omega$.
2. Let M be the portion of the graph of $z = x^2 + y^2$ in $\mathbb{R}^3$ that lies above the rectangle $\{(x,y)|0 \le x \le 1, 0 \le y \le 2\}$. Calculate $\int\limits_M \omega$.

4.6. Let S be the surface given by the parameterization

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{r^2+1}), \quad 0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Let ω be the differential 2-form given by

$$\omega = xz\ dx \wedge dz - yz\ dy \wedge dz.$$

Compute $\int\limits_S \omega$.

4.7. Let S be the surface in $\mathbb{R}^3$ parameterized by

$$\Psi(u,v) = (2u, v, u^2 + v^3), \quad 0 \le u \le 1, \quad 0 \le v \le 2.$$

Integrate the 2-form $(x + 2y)\ dx \wedge dz$ over S.

4.8. Let D be some region in the xy-plane. Let M denote the portion of the graph of $z = g(x,y)$ that lies above D.

1. Let $\omega = f(x,y)\ dx \wedge dy$ be a differential 2-form on $\mathbb{R}^3$. Show that

$$\int\limits_M \omega = \int\limits_D f(x,y)\ dx\ dy.$$

Notice the answer does not depend on the function $g(x,y)$.

2. Now suppose $\omega = f(x,y,z)\ dx \wedge dy$. Show that

$$\int\limits_M \omega = \int\limits_D f(x,y,g(x,y))\ dx\ dy.$$

4.9. Let S be the surface obtained from the graph of $z = f(x) = x^3$, where $0 \le x \le 1$, by rotating around the z-axis. Integrate the 2-form $\omega = y\ dx \wedge dz$ over S. (Hint: Use cylindrical coordinates to parameterize S.)

4.3 Orientations

What would have happened in Example 19 if we had used the parameterization $\phi'(r,t) = (-r\cos t, r\sin t, \sqrt{1-r^2})$ instead? We leave it to the reader to check that we end up with the answer $-\pi/2$ rather than $\pi/2$. This is a problem. We defined $\int\limits_M \omega$ before we started talking about parameterizations. Hence, the value which we calculate for this integral should not depend on our choice of parameterization. So what happened?

To analyze this completely, we need to go back to the definition of $\int\limits_M \omega$ from the previous section. We noted at the time that a choice was made to calculate $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ instead of $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$. Was this choice correct? The answer is a resounding *maybe!* We are actually missing enough information to tell. An *orientation* is precisely some piece of information about M which we can use to make the right choice. This way we can tell a friend what M is, what ω is, and what the orientation on M is, and they are sure to get the same answer. Recall Equation 4.1:

$$\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)}\left(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y)\right) dx \wedge dy.$$

Depending on the specified orientation of M, it may be incorrect to use Equation 4.1. Sometimes we may want to use

$$\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)}\left(-\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y)\right) dx \wedge dy.$$

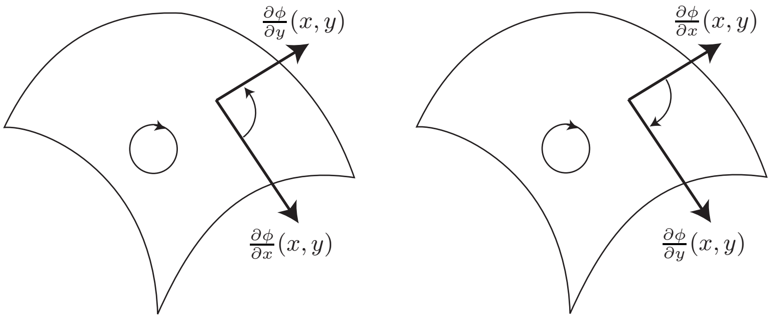
Both ω and $\int$ are linear. This just means the negative sign in the integrand on the right can go all the way outside. Hence, we can write both this equation and Equation 4.1 as

$$\int\limits_M \omega = \pm\int\limits_R \omega_{\phi(x,y)}\left(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y)\right) dx \wedge dy. \tag{4.2}$$

We define an *orientation on M* to be any piece of information that can be used to decide, for each choice of parameterization ϕ, whether to use the "+"

or "$-$" sign in Equation 4.2, so that the integral will always yield the same answer.

We will see several ways to specify an orientation on M. The first will be geometric. It has the advantage that it can be easily visualized, but it has the disadvantage that it is actually much harder to use in calculations. All we do is draw a small circle on M with an arrowhead on it. To use this "oriented circle" to tell if we need the "+" or "$-$" sign in Equation 4.2, we draw the vectors $\frac{\partial \phi}{\partial x}(x,y)$ and $\frac{\partial \phi}{\partial y}(x,y)$ and an arc with an arrow from the first to the second. If the direction of this arrow agrees with the oriented circle, then we use the "+" sign. If they disagree, then we use the "$-$" sign. See Figure 4.5.



Use the "$-$" sign when integrating. Use the "+" sign when integrating.

Fig. 4.5. An orientation on M is given by an oriented circle.

A more algebraic way to specify an orientation is to simply pick a point p of M and choose any 2-form ν on $T_p\mathbb{R}^3$ such that $\nu(V_p^1, V_p^2) \neq 0$ whenever V_p^1 and V_p^2 are vectors tangent to M and V_1 is not a multiple of V_2. Do not confuse this 2-form with the *differential* 2-form, ω, of Equation 4.2. The 2-form ν is only defined at the single tangent space $T_p\mathbb{R}^3$, whereas ω is defined everywhere.

Let us now see how we can use ν to decide whether to use the "+" or "$-$" sign in Equation 4.2. All we must do is calculate $\nu\left(\frac{\partial \phi}{\partial x}(x_p,y_p), \frac{\partial \phi}{\partial y}(x_p,y_p)\right)$, where $\phi(x_p, y_p) = p$. If the result is positive, then we will use the "+" sign to calculate the integral in Equation 4.2. If it is negative, then we use the "$-$" sign. Let's see how this works with an example.

Example 20. Let's revisit Example 19. The problem was to integrate the form $z^2 dx \wedge dy$ over M, the top half of the unit sphere. However, no orientation was ever given for M, so the problem was not very well stated. Let's pick an easy point, p, on M: $(0, \sqrt{2}/2, \sqrt{2}/2)$. The vectors $\langle 1,0,0\rangle_p$ and $\langle 0,1,-1\rangle_p$ in $T_p\mathbb{R}^3$ are both tangent to M. To give an orientation on M, all we do is specify a 2-form ν on $T_p\mathbb{R}^3$ such that $\nu(\langle 1,0,0\rangle, \langle 0,1,-1\rangle) \neq 0$. Let's pick an easy one: $\nu = dx \wedge dy$.

Now let's see what happens when we try to evaluate the integral by using the parameterization $\phi'(r,t) = (-r\cos t, r\sin t, \sqrt{1-r^2})$. First, note that $\phi'(\sqrt{2}/2, \pi/2) = (0, \sqrt{2}/2, \sqrt{2}/2)$ and

$$\left(\frac{\partial\phi'}{\partial r}\left(\frac{\sqrt{2}}{2},\frac{\pi}{2}\right), \frac{\partial\phi'}{\partial t}\left(\frac{\sqrt{2}}{2},\frac{\pi}{2}\right)\right) = \left(\langle 0,1,-1\rangle, \left\langle \frac{\sqrt{2}}{2},0,0\right\rangle\right).$$

Now we check the value of ν when this pair is plugged in:

$$dx\wedge dy\left(\langle 0,1,-1\rangle, \left\langle \frac{\sqrt{2}}{2},0,0\right\rangle\right) = \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 1 & 0 \end{vmatrix} = -\frac{\sqrt{2}}{2}.$$

The sign of this result is "−," so we need to use the negative sign in Equation 4.2 in order to use ϕ' to evaluate the integral of ω over M:

$$\begin{aligned}\int_M \omega &= -\int_R \omega_{\phi'(r,t)}\left(\frac{\partial\phi'}{\partial r}(r,t), \frac{\partial\phi'}{\partial t}(r,t)\right) dr\wedge dt \\ &= -\int_R (1-r^2)\begin{vmatrix} -\cos t & r\sin t \\ \sin t & r\cos t\end{vmatrix}\, dr\ dt = \frac{\pi}{2}.\end{aligned}$$

Very often, the surface that we are going to integrate over is given to us by a parameterization. In this case, there is a very natural choice of orientation. Just use the "+" sign in Equation 4.2! We will call this the orientation of M *induced* by the parameterization. In other words, if you see a problem phrased like "*Calculate the integral of the form ω over the surface M given by parameterization ϕ with the induced orientation,*" then you should just go back to using Equation 4.1 and do not worry about anything else. In fact, this situation is so common that when you are asked to integrate some form over a surface which is given by a parameterization but no orientation is specified, then you should assume the induced orientation is the desired one.

4.10. Let M be the image of the parameterization, $\phi(a,b) = (a, a+b, ab)$, where $0\le a\le 1$ and $0\le b\le 1$. Integrate the form $\omega = 2z\ dx\wedge dy + y\ dy\wedge dz - x\ dx\wedge dz$ over M using the orientation induced by ϕ.

4.11. Let S be the frustrum parameterized by

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta, r), \quad 1\le r\le 2, \quad 0\le\theta\le 2\pi.$$

Integrate the 2-form

$$\omega = z\ dx\wedge dy + \frac{1}{x}dy\wedge dz + \frac{1}{y}dx\wedge dz$$

over S with the orientation induced by ϕ.

There is one subtle technical point here that should be addressed. The novice reader may want to skip this for now. Suppose someone gives you a surface defined by a parameterization and tells you to integrate some 2-form over it using the induced orientation. However, you are clever and you realize that if you change parameterizations, you can make the integral easier. Which orientation do you use? The problem is that the orientation induced by your new parameterization may not be the same as the one induced by the original parameterization.

To fix this, we need to see how we can define a 2-form on some tangent space $T_p\mathbb{R}^3$, where p is a point of M, that yields an orientation of M consistent with the one induced by a parameterization ϕ. This is not so hard. If $dx \wedge dy\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right)$ is positive, then we simply let $\nu = dx \wedge dy$. If it is negative, then we let $\nu = -dx \wedge dy$. In the unlikely event that $dx \wedge dy\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right) = 0$, we can remedy things by either changing the point p or by using $dx \wedge dz$ instead of $dx \wedge dy$. Once we have defined ν, we know how to integrate M using any other parameterization.

4.12. Let ψ be the following parameterization of the sphere of radius 1:

$$\psi(\theta, \phi) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi).$$

Which of the following 2-forms on $T_{(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})}\mathbb{R}^3$ determine the same orientation on the sphere as that induced by ψ?

1. $\alpha = dx \wedge dy + 2dy \wedge dz$.
2. $\beta = dx \wedge dy - 2dy \wedge dz$.
3. $\gamma = dx \wedge dz$.

4.4 Integrating 1-forms on $\mathbb{R}^m$

In the previous sections we saw how to integrate a 2-form over a region in $\mathbb{R}^2$, or over a subset of $\mathbb{R}^3$ parameterized by a region in $\mathbb{R}^2$. The reason that these dimensions were chosen was because there is nothing new that needs to be introduced to move to the general case. In fact, if the reader were to go back and look at what we did, he/she would find that almost nothing would change if we had been talking about n-forms instead.

Before we jump to the general case, we will work one example showing how to integrate a 1-form over a parameterized curve.

Example 21. Let C be the curve in $\mathbb{R}^2$ parameterized by

$$\phi(t) = (t^2, t^3),$$

where $0 \leq t \leq 2$. Let ν be the 1-form $y\ dx + x\ dy$. We calculate $\int\limits_C \nu$.

The first step is to calculate

$$\frac{d\phi}{dt} = \langle 2t, 3t^2 \rangle.$$

So, $dx = 2t$ and $dy = 3t^2$. From the parameterization, we also know $x = t^2$ and $y = t^3$. Hence, since $\nu = y\ dx + x\ dy$, we have

$$\nu_{\phi(t)}\left(\frac{d\phi}{dt}\right) = (t^3)(2t) + (t^2)(3t^2) = 5t^4.$$

Finally, we integrate:

$$\begin{aligned}\int\limits_C \nu &= \int\limits_0^2 \nu_{\phi(t)}\left(\frac{d\phi}{dt}\right) dt \\ &= \int\limits_0^2 5t^4\ dt \\ &= t^5\Big|_0^2 \\ &= 32.\end{aligned}$$

4.13. Let C be the curve in $\mathbb{R}^2$ given by $\psi(t) = (t\cos t, t\sin t)$, where $0 \le t \le 2$. Let $\nu = -y\ dx + x\ dy$. Compute $\int\limits_C \nu$ (with the induced orientation).

4.14. Let C be the curve in $\mathbb{R}^3$ parameterized by $\phi(t) = (t, t^2, 1+t)$, where $0 \le t \le 2$. Integrate the 1-form $\omega = y\ dx + z\ dy + xy\ dz$ over C using the induced orientation.

4.15. Let C be the curve parameterized by the following:

$$\phi(t) = (2\cos t, 2\sin t, t^2), \quad 0 \le t \le 2.$$

Integrate the 1-form $(x^2 + y^2)\ dz$ over C.

4.16. Let C be the subset of the graph of $y = x^2$, where $0 \le x \le 1$. An orientation on C is given by the 1-form dx on $T_{(0,0)}\mathbb{R}^2$. Let ω be the 1-form $-x^4\ dx + xy\ dy$. Integrate ω over C.

4.17. Let M be the line segment in $\mathbb{R}^2$ which connects $(0,0)$ to $(4,6)$. An orientation on M is specified by the 1-form $-dx$ on $T_{(2,3)}\mathbb{R}^2$. Integrate the form $\omega = \sin y\ dx + \cos x\ dy$ over M.

Just as there was for surfaces, for parameterized curves there is also a pictorial way to specify an orientation. All we have to do is place an arrowhead somewhere along the curve and ask whether or not the derivative of the parameterization gives a tangent vector that points in the same direction. We illustrate this in the next example.

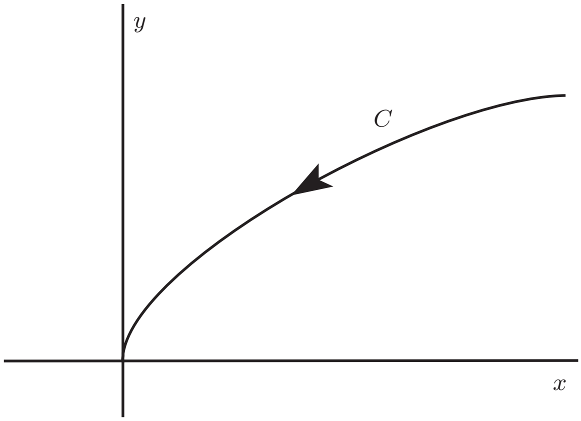


Fig. 4.6. An orientation on C is given by an arrowhead.

Example 22. Let C be the portion of the graph of $x = y^2$, where $0 \leq x \leq 1$, as pictured in Figure 4.6. Notice the arrowhead on C. We integrate the 1-form $\omega = dx + dy$ over C with the indicated orientation.

First, parameterize C as $\phi(t) = (t^2, t)$, where $0 \leq t \leq 1$. Now notice that the derivative of ϕ is

$$\frac{d\phi}{dt} = \langle 2t, 1\rangle.$$

At the point $(0,0)$ this is the vector $\langle 0, 1\rangle$, which points in a direction opposite to that of the arrowhead. This tells us to use a negative sign when we integrate, as follows:

$$\begin{aligned}\int_C \omega &= -\int_0^1 \omega_{(t^2,t)}(\langle 2t, 1\rangle)\\ &= -(2t+1)\big|_0^1\\ &= -2.\end{aligned}$$

4.5 Integrating n-forms on $\mathbb{R}^m$

To proceed to the general case, we need to know what the integral of an n-form over a region of $\mathbb{R}^n$ is. The steps to define such an object are precisely the same as before, and the results are similar. If our coordinates in $\mathbb{R}^n$ are $(x_1, x_2, ..., x_n)$, then an n-form is always given by

$$f(x_1, ..., x_n)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Going through the steps, we find that the definition of $\int_{\mathbb{R}^n} \omega$ is exactly the same as the definition we learned in Chapter 2 for $\int_{\mathbb{R}^n} f\ dx_1 dx_2 \cdots dx_n$.

4.18. Let Ω be the cube in $\mathbb{R}^3$:

$$\{(x, y, z)|\ 0 \le x, y, z \le 1\}.$$

Let γ be the 3-form $x^2 z\ dx \wedge dy \wedge dz$. Calculate $\int_{\Omega} \gamma$.

Moving on to integrals of n-forms over subsets of $\mathbb{R}^m$ parameterized by a region in $\mathbb{R}^n$, we again find nothing surprising. Suppose we denote such a subset by M. Let $\phi : R \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ be a parameterization. Then we find that the following generalization of Equation 4.2 must hold:

$$\int_M \omega = \pm \int_R \omega_{\phi(x_1,\ldots,x_n)} \left(\frac{\partial \phi}{\partial x_1}(x_1, ..., x_n), ..., \frac{\partial \phi}{\partial x_n}(x_1, ..., x_n) \right) dx_1 \wedge \cdots \wedge dx_n. \tag{4.3}$$

To decide whether or not to use the negative sign in this equation, we must specify an orientation. Again, one way to do this is to give an n-form ν on $T_p\mathbb{R}^m$, where p is some point of M. We use the negative sign when the value of

$$\nu \left(\frac{\partial \phi}{\partial x_1}(x_1, ..., x_n), ..., \frac{\partial \phi}{\partial x_n}(x_1, ..., x_n) \right)$$

is negative, where $\phi(x_1, ..., x_n) = p$. If M was originally given by a parameterization and we are instructed to use the induced orientation, then we can ignore the negative sign.

Example 23. Suppose $\phi(a, b, c) = (a + b, a + c, bc, a^2)$, where $0 \le a, b, c \le 1$. Let M be the image of ϕ with the induced orientation. Suppose $\omega = dy \wedge dz \wedge dw - dx \wedge dz \wedge dw - 2y\ dx \wedge dy \wedge dz$. Then

$$\begin{aligned}
\int_M \omega &= \int_R \omega_{\phi(a,b,c)} \left(\frac{\partial \phi}{\partial a}(a, b, c), \frac{\partial \phi}{\partial b}(a, b, c), \frac{\partial \phi}{\partial c}(a, b, c) \right) da \wedge db \wedge dc \\
&= \int_R \omega_{\phi(a,b,c)} \left(\langle 1, 1, 0, 2a\rangle, \langle 1, 0, c, 0\rangle, \langle 0, 1, b, 0\rangle \right) da \wedge db \wedge dc \\
&= \int_R \left| \begin{matrix} 1\ 0\ 1 \\ 0\ c\ b \\ 2a\ 0\ 0 \end{matrix} \right| - \left| \begin{matrix} 1\ 1\ 0 \\ 0\ c\ b \\ 2a\ 0\ 0 \end{matrix} \right| - 2(a + c) \left| \begin{matrix} 1\ 1\ 0 \\ 1\ 0\ 1 \\ 0\ c\ b \end{matrix} \right| da \wedge db \wedge dc \\
&= \int_0^1 \int_0^1 \int_0^1 2bc + 2c^2\ da\ db\ dc = \frac{7}{6}.
\end{aligned}$$

4.6 The change of variables formula

There is a special case of Equation 4.3 which is worth noting. Suppose ϕ is a parameterization that takes some subregion, R, of $\mathbb{R}^n$ into some other subregion, M, of $\mathbb{R}^n$ and ω is an n-form on $\mathbb{R}^n$. At each point, ω is just a volume form, so it can be written as $f(x_1, ..., x_n)\ dx_1 \wedge \cdots \wedge dx_n$. If we let $\bar{x} = (x_1, ..., x_n)$, then Equation 4.3 can be written as

$$\int_M f(\bar{x}) dx_1 \cdots dx_n = \pm \int_R f(\phi(\bar{x})) \left| \frac{\partial \phi}{\partial x_1}(\bar{x}) \cdots \frac{\partial \phi}{\partial x_n}(\bar{x}) \right| dx_1 \cdots dx_n. \quad (4.4)$$

The bars $|\cdot|$ indicate that we take the determinant of the matrix whose column vectors are $\frac{\partial \phi}{\partial x_i}(\bar{x})$.

4.6.1 1-Forms on $\mathbb{R}^1$

When $n = 1$, this is just the reverse of the substitution rule for integration from calculus. We demonstrate this in the following example.

Example 24. Let's integrate the 1-form $\omega = \sqrt{u}\ du$ over the interval $[1, 5]$. This would be easy enough to do directly, but using a parameterization of this interval will be instructive. Let $\phi : [0, 2] \to [1, 5]$ be the parameterization given by $\phi(x) = x^2 + 1$. Then $\frac{d\phi}{dx} = \langle 2x \rangle$. Now we compute:

$$\begin{aligned} \int_1^5 \sqrt{u}\ du = \int_{[1,5]} \omega &= \int_{[0,2]} \omega_{\phi(x)} \left(\frac{d\phi}{dx} \right) dx \\ &= \int_{[0,2]} \omega_{x^2+1} \left(\langle 2x \rangle \right) dx \\ &= \int_{[0,2]} 2x\sqrt{x^2+1}\ dx \\ &= \int_0^2 2x\sqrt{x^2+1}\ dx. \end{aligned}$$

Reading this backwards is doing the integral $\int_0^2 2x\sqrt{x^2+1}\ dx$ by "u-substitution."

Employing a parameterization to integrate a 1-form on $\mathbb{R}^1$ is a common technique to handle certain integrands. This is often called "trigonometric substitution" in a first-year calculus class.

Example 25. Let $\omega = \frac{1}{\sqrt{1-x^2}}dx$ be a 1-form on $\mathbb{R}^1$. We wish to integrate ω over the interval $[0,1]$, with the standard orientation on $\mathbb{R}^1$. To do this, we employ the parameterization $\phi : [0, \frac{\pi}{2}] \to [0,1]$ given by $\phi(\theta) = \sin\theta$. To use ϕ to perform the desired integration, we will need its derivative: $\frac{d\phi}{d\theta} = \langle \cos\theta \rangle$. We may now compute:

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1-x^2}} dx &= \int_{[0,1]} \omega \\
&= \int_{[0,\frac{\pi}{2}]} \omega_{\phi(\theta)} \left(\frac{d\phi}{d\theta} \right) d\theta \\
&= \int_{[0,\frac{\pi}{2}]} \omega_{\sin\theta}(\langle \cos\theta \rangle) d\theta \\
&= \int_{[0,\frac{\pi}{2}]} \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \; d\theta \\
&= \int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

4.6.2 2-Forms on $\mathbb{R}^2$

For other n, Equation 4.4 is the general change of variables formula.

Example 26. We will use the parameterization $\Psi(u,v) = (u, u^2+v^2)$ to evaluate

$$\iint_R (x^2 + y) \; dA,$$

where R is the region of the xy-plane bounded by the parabolas $y = x^2$ and $y = x^2 + 4$ and the lines $x = 0$ and $x = 1$.

The first step is to find out what the limits of integration will be when we change coordinates:

$$\begin{aligned}
y = x^2 &\Rightarrow u^2 + v^2 = u^2 \Rightarrow v = 0, \\
y = x^2 + 4 &\Rightarrow u^2 + v^2 = u^2 + 4 \Rightarrow v = 2, \\
x = 0 &\Rightarrow u = 0, \\
x = 1 &\Rightarrow u = 1.
\end{aligned}$$

Next, we will need the partial derivatives:

$$\frac{\partial \Psi}{\partial u} = \langle 1, 2u \rangle,$$
$$\frac{\partial \Psi}{\partial v} = \langle 0, 2v \rangle.$$

Finally, we can integrate:

$$\begin{aligned}
\iint\limits_{R} (x^2 + y)\ dA &= \int\limits_{R} (x^2 + y)\ dx \wedge dy \\
&= \int\limits_0^2 \int\limits_0^1 u^2 + (u^2 + v^2) \begin{vmatrix} 1 & 0 \\ 2u & 2v \end{vmatrix}\ du\ dv \\
&= \int\limits_0^2 \int\limits_0^1 4vu^2 + 2v^3\ du\ dv \\
&= \int\limits_0^2 \frac{4}{3}v + 2v^3\ dv \\
&= \frac{8}{3} + 8 = \frac{32}{3}.
\end{aligned}$$

Example 27. In our second example, we revisit Fubini's Theorem, which says that the order of integration does not matter in a multiple integral. Recall from Section 4.2 the curious fact that $\int f\ dx\ dy = \int f\ dx \wedge dy$ but $\int f\ dy\ dx \neq \int f\ dy \wedge dx$. We are now prepared to see why this is.

Let's suppose we want to integrate the function $f(x, y)$ over the rectangle R in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b). We know the answer is just $\int\limits_0^b \int\limits_0^a f(x, y)\ dx\ dy$. We also know this integral is equal to $\int\limits_R f\ dx \wedge dy$, where R is given the "standard" orientation (e.g., the one specified by a counter-clockwise oriented circle).

Let's see what happens if we try to compute the integral using the following parameterization:

$$\phi(y, x) = (x, y), \quad 0 \le y \le b, \quad 0 \le x \le a.$$

First, we need the partials of ϕ:

$$\frac{\partial \phi}{\partial y} = \langle 0, 1 \rangle,$$

$$\frac{\partial \phi}{\partial x} = \langle 1, 0 \rangle.$$

Next, we have to deal with the issue of orientation. The pair of vectors we just found — $\langle 0,1\rangle$ and $\langle 1,0\rangle$ — are in an order which does not agree with the orientation of R. So we have to use the negative sign when employing Equation 4.4:

$$\begin{aligned}\int\limits_R f(x,y)\ dx\ dy &= -\int\limits_R f(\phi(y,x))\left|\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\right| dy\ dx \\ &= -\int\limits_R f(x,y)\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix} dy\wedge dx \\ &= -\int\limits_R f(x,y)(-1)\ dy\wedge dx \\ &= \int\limits_R f(x,y)\ dy\ dx.\end{aligned}$$

From the above, we see one of the reasons why Fubini's Theorem is true is because when the order of integration is switched, there are *two* negative signs. So, $\int\limits_R f\ dy\ dx$ actually *does* equal $\int\limits_R f\ dy\wedge dx$, *but only if you remember to switch the orientation of* R*!*

4.19. Let E be the region in $\mathbb{R}^2$ parameterized by $\Psi(u,v) = (u^2+v^2, 2uv)$, where $0 \le u \le 1$ and $0 \le v \le 1$. Evaluate

$$\int\limits_E \frac{1}{\sqrt{x-y}}\ dx\wedge dy.$$

Up until this point, we have only seen how to integrate functions $f(x,y)$ over regions in the plane which are rectangles. Let's now see how we can use parameterizations to integrate over more general regions. Suppose, first, that R is the region of the plane below the graph of $y = g(x)$, above the x-axis, and between the lines $x = a$ and $x = b$.

The region R can be parameterized by

$$\Psi(u,v) = (u, vg(u)),$$

where $a \le u \le b$ and $0 \le v \le 1$. The partials of this parameterization are

$$\frac{\partial\Psi}{\partial u} = \left\langle 1, v\frac{dg(u)}{du}\right\rangle,$$

$$\frac{\partial\Psi}{\partial v} = \langle 0, g(u)\rangle.$$

Hence,

$$dx \wedge dy = \begin{vmatrix} 1 & 0 \\ v\frac{dg(u)}{du} & g(u) \end{vmatrix} = g(u).$$

We conclude with the identity

$$\int_R f(x,y)\ dy\ dx = \int_a^b \int_0^1 f(u, vg(u))g(u)\ dv\ du.$$

4.20. Let R be the region below the graph of $y = x^2$ and between the lines $x = 0$ and $x = 2$. Calculate

$$\int_R xy^2\ dx\ dy.$$

A slight variant is to integrate over a region bounded by the graphs of $y = g_1(x)$ and $y = g_2(x)$ and by the lines $x = a$ and $x = b$, where $g_1(x) < g_2(x)$ for all $x \in [a, b]$. To compute such an integral, we may simply integrate over the region below $g_2(x)$, then integrate over the region below $g_1(x)$, and subtract.

4.21. Let R be the region to the right of the y-axis, to the left of the graph of $x = g(y)$, above the line $y = a$, and below the line $y = b$. Find a formula for $\int_R f(x,y)\ dx\ dy$.

4.22. Let R be the region in the first quadrant of $\mathbb{R}^2$, below the line $y = x$, and bounded by $x^2 + y^2 = 4$. Integrate the 2-form

$$\omega = \left(1 + \frac{y^2}{x^2}\right) dx \wedge dy$$

over R.

4.23. Let R be the region of the xy-plane bounded by the ellipse

$$9x^2 + 4y^2 = 36.$$

Integrate the 2-form $\omega = x^2 dx \wedge dy$ over R (Hint: See Problem 2.39 of Chapter 2.)

4.24. Integrate the 2-form

$$\omega = \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz$$

over the top half of the unit sphere using the following parameterization from rectangular coordinates:

$$(x, y) \to (x, y, \sqrt{1 - x^2 - y^2}),$$

where $\sqrt{x^2 + y^2} \leq 1$. Compare your answer to Problem 4.3.

4.25. Let R be the region of $\mathbb{R}^2$ parameterized by

$$\phi(r,t) = (r\cosh t, r\sinh t), \quad 0 \le r \le 1, \quad -1 \le t \le 1.$$

Integrate the function $f(x,y) = x^2 - y^2$ over R. Hints:

$$\frac{d}{dt}\sinh t = \cosh t, \frac{d}{dt}\cosh t = \sinh t,$$

$$\cosh^2 t - \sinh^2 t = 1.$$

4.6.3 3-Forms on $\mathbb{R}^3$

Example 28. Let $V = \{(r,\theta,z)|1 \le r \le 2, \ 0 \le z \le 1\}$. ($V$ is the region between the cylinders of radii 1 and 2 and between the planes $z = 0$ and $z = 1$.) Let's calculate

$$\int_V z(x^2+y^2)\ dx \wedge dy \wedge dz.$$

The region V is best parameterized using cylindrical coordinates:

$$\Psi(r,\theta,z) = (r\cos\theta, r\sin\theta, z),$$

where $1 \le r \le 2$, $1 \le \theta \le 2\pi$ and $0 \le z \le 1$.

We compute the partials:

$$\begin{aligned}\frac{\partial\Psi}{\partial r} &= \langle\cos\theta, \sin\theta, 0\rangle,\\ \frac{\partial\Psi}{\partial\theta} &= \langle -r\sin\theta, r\cos\theta, 0\rangle,\\ \frac{\partial\Psi}{\partial z} &= \langle 0,0,1\rangle.\end{aligned}$$

Hence,

$$dx \wedge dy \wedge dz = \begin{vmatrix} \cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix} = r.$$

Additionally,

$$z(x^2+y^2) = z(r^2\cos^2\theta + r^2\sin^2\theta) = zr^2.$$

So we have

$$\int_V z(x^2+y^2)\,dx\wedge dy\wedge dz = \int_0^1\int_0^{2\pi}\int_1^2 (zr^2)(r)\;dr\;d\theta\;dz$$
$$= \int_0^1\int_0^{2\pi}\int_1^2 zr^3\;dr\;d\theta\;dz$$
$$= \frac{15}{4}\int_0^1\int_0^{2\pi} z\;d\theta\;dz$$
$$= \frac{15\pi}{2}\int_0^1 z\;dz$$
$$= \frac{15\pi}{4}.$$

4.26. Integrate the 3-form $\omega = x\;dx\wedge dy\wedge dz$ over the region of $\mathbb{R}^3$ in the first octant bounded by the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ and the plane $z=2$.

4.27. Let R be the region in the first octant of $\mathbb{R}^3$ bounded by the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$. Integrate the 3-form $\omega = dx\wedge dy\wedge dz$ over R.

4.28. Let V be the volume *in the first octant*, inside the cylinder of radius 1 and below the plane $z=1$. Integrate the 3-form

$$2\sqrt{1+x^2+y^2}\;dx\wedge dy\wedge dz$$

over V.

4.29. Let V be the region inside the cylinder of radius 1, centered around the z-axis and between the planes $z=0$ and $z=2$. Integrate the function $f(x,y,z)=z$ over V.

4.30. Let V be the volume in $\mathbb{R}^3$ parameterized by

$$\Psi(r,\theta,t) = (r\cos\theta, r\sin\theta, t^2),\quad 0\le r\le 1,\quad 0\le\theta\le 2\pi,\quad 0\le t\le 1.$$

(Note that this is *not* a parameterization of a surface by cylindrical coordinates.) Use the parameterization Ψ to integrate the function $f(x,y,z)=\sqrt{z}$ over V.

4.31. Let ω be the 3-form on $\mathbb{R}^3$ given by

$$\omega = \frac{z}{x^2+y^2}dx\wedge dy\wedge dz.$$

Compute the integral of ω over the region under the graph of $f(x,y)=x^2+y^2$, in the positive octant, and

1. above the square with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$.
2. at most one unit away from the z-axis.

4.7 Summary: How to integrate a differential form

4.7.1 The steps

To compute the integral of a differential n-form ω over a region S, the steps are as follows:

1. Choose a parameterization $\Psi : R \to S$, where R is a subset of $\mathbb{R}^n$ (see Figure 4.7).

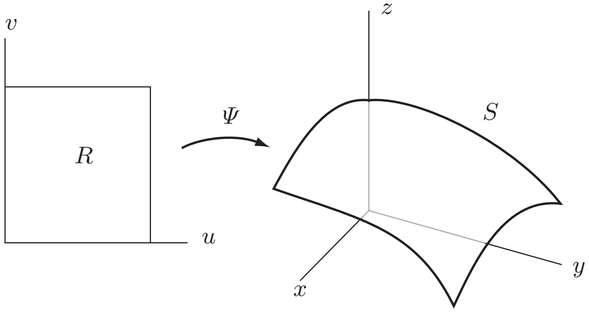


Fig. 4.7.

2. Find all n vectors given by the partial derivatives of Ψ. Geometrically, these are tangent vectors to S which span its tangent space (see Figure 4.8).

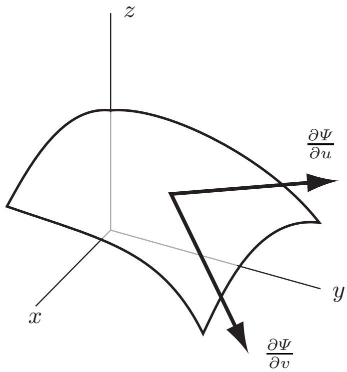


Fig. 4.8.

3. Plug the tangent vectors into ω at the point $\Psi(u_1, u_2, ..., u_n)$.
4. Integrate the resulting function over R.

4.7.2 Integrating 2-forms

The best way to see the above steps in action is to look at the integral of a 2-form over a surface in $\mathbb{R}^3$. In general, such a 2-form is given by

$$\omega = f_1(x,y,z)\ dx \wedge dy + f_2(x,y,z)\ dy \wedge dz + f_3(x,y,z)\ dx \wedge dz.$$

To integrate ω over S, we now follow steps 1–4:

1. Choose a parameterization $\Psi : R \to S$, where R is a subset of $\mathbb{R}^2$:
$$\Psi(u,v) = (g_1(u,v), g_2(u,v), g_3(u,v)).$$
2. Find both vectors given by the partial derivatives of Ψ:
$$\frac{\partial \Psi}{\partial u} = \left\langle \frac{\partial g_1}{\partial u}, \frac{\partial g_2}{\partial u}, \frac{\partial g_3}{\partial u} \right\rangle,$$
$$\frac{\partial \Psi}{\partial v} = \left\langle \frac{\partial g_1}{\partial v}, \frac{\partial g_2}{\partial v}, \frac{\partial g_3}{\partial v} \right\rangle.$$
3. Plug the tangent vectors into ω at the point $\Psi(u,v)$. To do this, x, y and z will come from the coordinates of Ψ; that is, $x = g_1(u,v)$, $y = g_2(u,v)$ and $z = g_3(u,v)$. Terms like $dx \wedge dy$ are determinants of 2×2 matrices, whose entries come from the vectors computed in the previous step. Geometrically, the value of $dx \wedge dy$ is the area of the parallelogram spanned by the vectors $\frac{\partial \Psi}{\partial u}$ and $\frac{\partial \Psi}{\partial v}$ (tangent vectors to S), projected onto the $dxdy$-plane (see Figure 4.9).
The result of all this is:
$$f_1(g_1,g_2,g_3)\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{vmatrix} + f_2(g_1,g_2,g_3)\begin{vmatrix} \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{vmatrix}$$
$$+ f_3(g_1,g_2,g_3)\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{vmatrix}.$$
Note that simplifying this gives a function of u and v.
4. Integrate the resulting function over R. In other words, if $h(u,v)$ is the function you ended up with in the previous step, then compute
$$\int_R h(u,v)\ du\ dv.$$

If R is not a rectangle, you may have to find a parameterization of R whose domain is a rectangle and repeat the above steps to compute this integral.

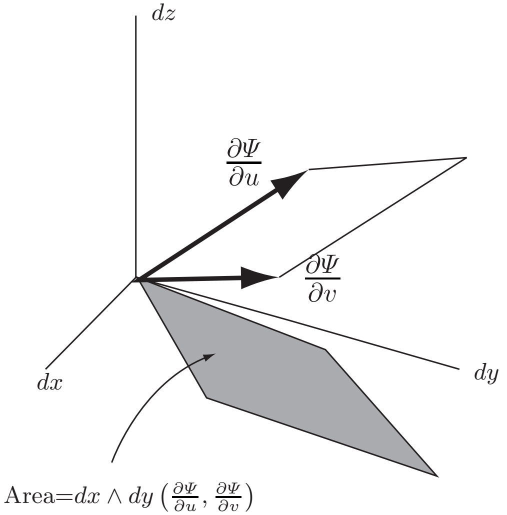


Fig. 4.9. Evaluating $dx \wedge dy$ geometrically.

4.7.3 A sample 2-form

Let $\omega = (x^2 + y^2)\, dx \wedge dy + z\, dy \wedge dz$. Let S denote the subset of the cylinder $x^2 + y^2 = 1$ that lies between the planes $z = 0$ and $z = 1$.

1. Choose a parameterization $\Psi : R \to S$:
$$\Psi(\theta, z) = (\cos\theta, \sin\theta, z),$$
where $R = \{(\theta, z) | 0 \le \theta \le 2\pi, 0 \le z \le 1\}$.
2. Find both vectors given by the partial derivatives of Ψ.
$$\frac{\partial \Psi}{\partial \theta} = \langle -\sin\theta, \cos\theta, 0\rangle$$
$$\frac{\partial \Psi}{\partial z} = \langle 0, 0, 1\rangle.$$
3. Plug the tangent vectors into ω at the point $\Psi(\theta, z)$. We get
$$(\cos^2\theta + \sin^2\theta)\begin{vmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} + z\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix}.$$
This simplifies to the function $z\cos\theta$.
4. Integrate the resulting function over R:
$$\int_0^1 \int_0^{2\pi} z\cos\theta\ d\theta\ dz.$$

Note that the integrand comes from Step 3 and the limits of integration come from Step 1.

4.8 Nonlinear forms (optional)

4.8.1 Surface area

Now that we have developed some proficiency with integrating differential forms, let's see what *else* we can integrate. A basic assumption that we used to come up with the definition of an n-form was the fact that at every point, it is a *linear* function which "eats" n vectors and returns a number. What about the non-linear functions?

Let's go all the way back to Section 1.4. There we decided that the integral of a function f over a surface R in $\mathbb{R}^3$ should look something like

$$\int_R f(\phi(r,\theta))\text{Area}\left[\frac{\partial\phi}{\partial r}(r,\theta), \frac{\partial\phi}{\partial\theta}(r,\theta)\right] dr\ d\theta. \tag{4.5}$$

At the heart of the integrand is the Area function, which takes two vectors and returns the area of the parallelogram that it spans. The 2-form $dx \wedge dy$ does this for two vectors in $T_p\mathbb{R}^2$. In $T_p\mathbb{R}^3$, the right function is the following:

$$\text{Area}(V_p^1, V_p^2) = \sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}.$$

(The reader may recognize this as the magnitude of the cross product between V_p^1 and V_p^2.) This is clearly nonlinear!

Example 29. The area of the parallelogram spanned by $\langle 1,1,0\rangle$ and $\langle 1,2,3\rangle$ can be computed as follows:

$$\begin{aligned}\text{Area}(\langle 1,1,0\rangle, \langle 1,2,3\rangle) &= \sqrt{\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}^2} \\ &= \sqrt{3^2 + 3^2 + 1^2} \\ &= \sqrt{19}.\end{aligned}$$

We will see in Chapter 6 that the thing that makes (linear) differential forms so useful is the *Generalized Stokes Theorem*. We do not have anything like this for nonlinear forms, but that is not to say that they do not have their uses. For example, there is no differential 2-form on $\mathbb{R}^3$ that one can integrate over arbitrary surfaces to find their surface area. For that, we would need to compute the following:

$$\text{Area}(R) = \int_S \sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}.$$

For relatively simple surfaces, this integrand can be evaluated by hand. Integrals such as this play a particularly important role in certain applied problems. For example, if one were to dip a loop of bent wire into a soap film, the resulting surface would be the one of minimal area. Before one can even begin to figure out what surface this is for a given piece of wire, one must be able to know how to compute the area of an arbitrary surface, as above.

Example 30. We compute the surface area of a sphere of radius r in $\mathbb{R}^3$. A parameterization is given by

$$\Phi(\theta, \phi) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi),$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

Now we compute:

$$\begin{aligned}
&\text{Area}\left(\tfrac{\partial\Phi}{\partial\theta}, \tfrac{\partial\Phi}{\partial\phi}\right) \\
&\quad = \text{Area}\left(\langle -r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0\rangle, \langle r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi\rangle\right) \\
&\quad = \sqrt{(-r^2\sin^2\phi\cos\theta)^2 + (r^2\sin^2\phi\sin\theta)^2 + (-r^2\sin\phi\cos\phi)^2} \\
&\quad = r^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} \\
&\quad = r^2\sin\phi.
\end{aligned}$$

Thus, the desired area is given by

$$\begin{aligned}
&\int_S \text{Area}\left(\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\phi}\right) d\theta\ d\phi \\
&\qquad = \int_0^{\pi}\int_0^{2\pi} r^2\sin\phi\ d\theta\ d\phi \\
&\qquad = 4\pi r^2.
\end{aligned}$$

4.32. Compute the surface area of a sphere of radius r in $\mathbb{R}^3$ using the parameterizations

$$\Phi(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, \pm\sqrt{r^2 - \rho^2})$$

for the top and bottom halves, where $0 \leq \rho \leq r$ and $0 \leq \theta \leq 2\pi$.

Let's now go back to Equation 4.5. Classically, this is called a *surface integral*. It might be a little clearer how to compute such an integral if we write it as follows:

$$\int_R f(x, y, z)\ dS = \int_R f(x, y, z)\sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}.$$

4.8.2 Arc length

Lengths are very similar to areas. In calculus you learn that if you have a curve C in the plane, for example, parameterized by the function $\phi(t) = (x(t), y(t))$, where $a \leq t \leq b$, then its arc length is given by

$$\text{Length}(C) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We can write this without making reference to the parameterization by employing a nonlinear 1-form:

$$\text{Length}(C) = \int_C \sqrt{dx^2 + dy^2}.$$

Finally, we can define what is classically called a *line integral* as follows:

$$\oint_C f(x,y)\; ds = \int_C f(x,y)\sqrt{dx^2 + dy^2}.$$