3

Forms

3.1 Coordinates for vectors

Before we begin to discuss functions of vectors, we first need to learn how to specify a vector. Before we can answer that, we must first learn where vectors live. In Figure 3.1 we see a curve, C, and a tangent line to that curve. The line can be thought of as the set of all tangent vectors at the point p. We denote that line as T_pC, the *tangent space* to C at the point p.

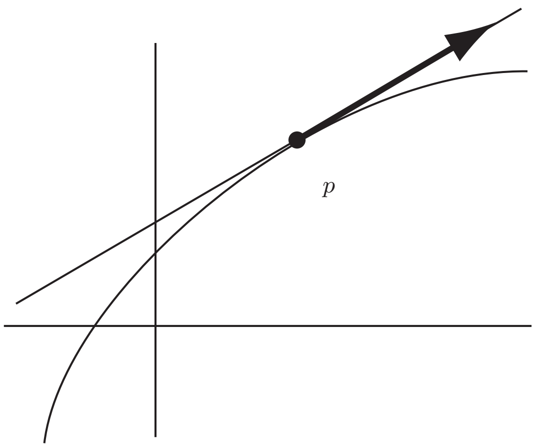

Fig. 3.1. T_pC is the set of all vectors tangents to C at p.

What if C is actually a straight line? Will T_pC be the same line? To answer this, let's instead think about the real number line $L = \mathbb{R}^1$. Suppose p is the point corresponding to the number 2 on L. We would like to understand T_pL,

D. Bachman, *A Geometric Approach to Differential Forms*,
DOI 10.1007/978-0-8176-8304-7_3, © Springer Science+Business Media, LLC 2012

the set of all vectors tangent to L at the point p. For example, where would you draw a vector of length 3? Would you put its base at the origin on L? Of course not. You would put its base at the point p. This is really because the origin for T_pL is different than the origin for L. We are thus thinking about L and T_pL as two different lines, placed right on top of each other.

The key to understanding the difference between L and T_pL is their *coordinate systems*. Let's pause here for a moment to look a little more closely. What are "coordinates" anyway? They are a way of assigning a number (or, more generally, a set of numbers) to a point in space. In other words, coordinates are functions which take points of a space and return (sets of) numbers. When we say that the x-coordinate of p in $\mathbb{R}^2$ is 5, we really mean that we have a function $x : \mathbb{R}^2 \to \mathbb{R}$, such that $x(p) = 5$.

Of course we need two numbers to specify a point in a plane, which means that we have two coordinate functions. Suppose we denote the plane by P and $x : P \to \mathbb{R}$ and $y : P \to \mathbb{R}$ are our coordinate functions. Then, saying that the coordinates of a point, p, are $(2, 3)$ is the same thing as saying that $x(p) = 2$ and $y(p) = 3$. In other words, the coordinates of p are $(x(p), y(p))$.

So what do we use for coordinates in the tangent space? Well, first we need a *basis* for the tangent space of P at p. In other words, we need to pick two vectors which we can use to give the relative positions of all other points. Note that if the coordinates of p are (x, y), then $\frac{d(x+t,y)}{dt} = \langle 1, 0\rangle$ and $\frac{d(x,y+t)}{dt} = \langle 0, 1\rangle$. We have switched to the notation "$\langle \cdot, \cdot\rangle$" to indicate that we are not talking about points of P anymore, but rather vectors in T_pP. We take these two vectors to be a basis for T_pP. In other words, any point of T_pP can be written as $dx\langle 0, 1\rangle + dy\langle 1, 0\rangle$, where $dx, dy \in \mathbb{R}$. Hence, "dx" and "dy" are coordinate functions for T_pP. Saying that the coordinates of a vector V in T_pP are $\langle 2, 3\rangle$, for example, is the same thing as saying that $dx(V) = 2$ and $dy(V) = 3$. In general, we may refer to the coordinates of an arbitrary vector in T_pP as $\langle dx, dy\rangle$, just as we may refer to the coordinates of an arbitrary point in P as (x, y).

It will be helpful in the future to be able to distinguish between the vector $\langle 2, 3\rangle$ in T_pP and the vector $\langle 2, 3\rangle$ in T_qP, where $p \neq q$. We will do this by writing $\langle 2, 3\rangle_p$ for the former and $\langle 2, 3\rangle_q$ for the latter.

Let's pause for a moment to address something that may have been bothering you since your first term of calculus. Let's look at the tangent line to the graph of $y = x^2$ at the point $(1, 1)$. We are no longer thinking of this tangent line as lying in the same plane that the graph does. Rather, it lies in $T_{(1,1)}\mathbb{R}^2$. The horizontal axis for $T_{(1,1)}\mathbb{R}^2$ is the "dx" axis and the vertical axis is the "dy" axis (see Figure 3.2). Hence, we can write the equation of the tangent line as $dy = 2dx$. We can rewrite this as $\frac{dy}{dx} = 2$. Look familiar? This is one explanation for why we use the notation $\frac{dy}{dx}$ in calculus to denote the derivative.

3.1.

1. Draw a vector with $dx = 1$, $dy = 2$ in the tangent space $T_{(1,-1)}\mathbb{R}^2$.

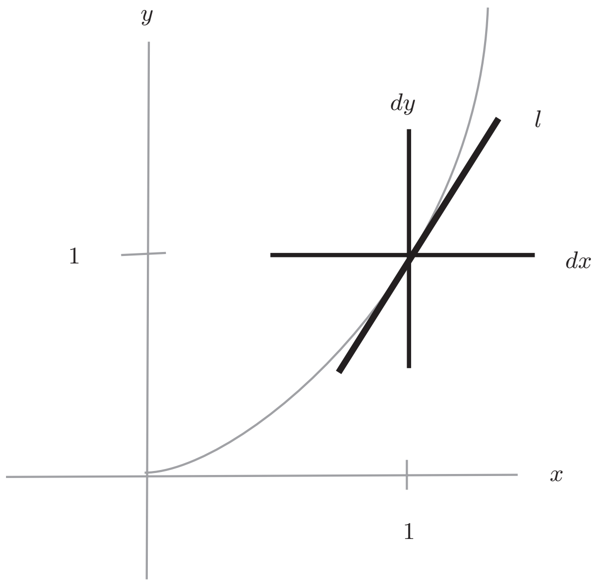

Fig. 3.2. The line l lies in $T_{(1,1)}\mathbb{R}^2$. Its equation is $dy = 2dx$.

2. Draw $\langle -3, 1\rangle_{(0,1)}$.

3.2 1-Forms

Recall from the previous chapter that a 1-form is a linear function which acts on vectors and returns numbers. For the moment let's just look at 1-forms on $T_p\mathbb{R}^2$ for some fixed point, p. Recall that a linear function, ω, is just one whose graph is a plane through the origin. Hence, we want to write down an equation of a plane through the origin in $T_p\mathbb{R}^2 \times \mathbb{R}$, where one axis is labeled dx, another dy and the third ω (see Figure 3.3). This is easy: $\omega = a\ dx + b\ dy$. Hence, to specify a 1-form on $T_p\mathbb{R}^2$ we only need to know two numbers: a and b.

Here is a quick example. Suppose $\omega(\langle dx, dy\rangle) = 2dx + 3dy$; then

$$\omega(\langle -1, 2\rangle) = 2 \cdot -1 + 3 \cdot 2 = 4.$$

The alert reader may see something familiar here: the dot product; that is, $\omega(\langle -1, 2\rangle) = \langle 2, 3\rangle \cdot \langle -1, 2\rangle$. Recall the geometric interpretation of the dot product: You project $\langle -1, 2\rangle$ onto $\langle 2, 3\rangle$ and then multiply by $|\langle 2, 3\rangle| = \sqrt{13}$. In other words:

> *Evaluating a 1-form on a vector is the same as projecting onto some line and then multiplying by some constant.*

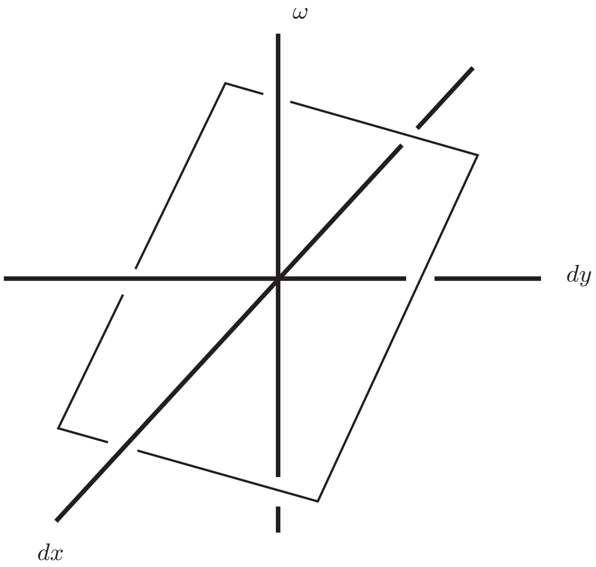

Fig. 3.3. The graph of ω is a plane through the origin.

In fact, we can even interpret the act of multiplying by a constant geometrically. Suppose ω is given by $a\ dx + b\ dy$. Then the value of $\omega(V_1)$ is the length of the projection of V_1 onto the line, l, *where* $\frac{\langle a,b\rangle}{|\langle a,b\rangle|^2}$ *is a basis vector for* l.

This interpretation has a huge advantage ... it is coordinate free. Recall from the previous section that we can think of the plane P as existing independent of our choice of coordinates. We only pick coordinates so that we can communicate to someone else the location of a point. Forms are similar. They are objects that exist independently of our choice of coordinates. This is one key as to why they are so useful outside of mathematics.

There is still another geometric interpretation of 1-forms. Let's first look at the simple example $\omega(\langle dx, dy\rangle) = dx$. This 1-form simply returns the first coordinate of whatever vector you feed into it. This is also a projection; it's the projection of the input vector onto the dx-axis. This immediately gives us a new interpretation of the action of a general 1-form $\omega = a\ dx + b\ dy$.

Evaluating a 1-form on a vector is the same as projecting onto each coordinate axis, scaling each by some constant and adding the results.

Although this interpretation is more cumbersome, it is the one that will generalize better when we get to n-forms.

Let's move on now to 1-forms in n dimensions. If $p \in \mathbb{R}^n$, then we can write p in coordinates as $(x_1, x_2, ..., x_n)$. The coordinates for a vector in $T_p\mathbb{R}^n$ are $\langle dx_1, dx_2, ..., dx_n \rangle$. A 1-form is a linear function, ω, whose graph (in $T_p\mathbb{R}^n \times \mathbb{R}$) is a plane through the origin. Hence, we can write it as $\omega = a_1\ dx_1 + a_2\ dx_2 + \cdots + a_n\ dx_n$. Again, this can be thought of as either projecting onto the vector $\langle a_1, a_2, ..., a_n \rangle$ and then multiplying by $|\langle a_1, a_2, ..., a_n \rangle|$ or as projecting onto each coordinate axis, multiplying by a_i, and then adding.

3.2. Let $\omega(\langle dx, dy \rangle) = -dx + 4dy$.

1. Compute $\omega(\langle 1, 0 \rangle)$, $\omega(\langle 0, 1 \rangle)$ and $\omega(\langle 2, 3 \rangle)$.
2. What line does ω project vectors onto?

3.3. Find a 1-form which computes the length of the projection of a vector onto the indicated line, multiplied by the indicated constant c.

1. The dx-axis, $c = 3$.
2. The dy-axis, $c = \frac{1}{2}$.
3. Find a 1-form that does both of the two preceding operations and adds the result.
4. The line $dy = \frac{3}{4}dx$, $c = 10$.

3.4. If ω is a 1-form show the following:

1. $\omega(V_1 + V_2) = \omega(V_1) + \omega(V_2)$, for any vectors V_1 and V_2.
2. $\omega(cV) = c\omega(V)$, for any vector V and constant c.

3.3 Multiplying 1-forms

In this section we would like to explore a method of multiplying 1-forms. You may think "What is the big deal? If ω and ν are 1-forms, can't we just define $\omega \cdot \nu(V) = \omega(V) \cdot \nu(V)$?" Well, of course we *can*, but then $\omega \cdot \nu$ is not a linear function, so we have left the world of forms.

The trick is to define the product of ω and ν to be a 2-form. So as not to confuse this with the product just mentioned, we will use the symbol "$\wedge$" (pronounced "wedge") to denote multiplication. So how can we possibly define $\omega \wedge \nu$ to be a 2-form? We must define how it acts on a pair of vectors, (V_1, V_2).

Note first that there are four ways to combine all of the ingredients:

$$\omega(V_1), \quad \nu(V_1), \quad \omega(V_2), \quad \nu(V_2).$$

The first two of these are associated with V_1 and the second two with V_2. In other words, ω and ν together give a way of taking each vector and returning a *pair* of numbers. How do we visualize pairs of numbers? In the plane, of course! Let's define a new plane with one axis as the ω-axis and the other as the ν-axis. So, the coordinates of V_1 in this plane are $[\omega(V_1), \nu(V_1)]$ and the coordinates of V_2 are $[\omega(V_2), \nu(V_2)]$. Note that we have switched to the notation "$[\cdot,\cdot]$" to indicate that we are describing points in a new plane. This may seem a little confusing at first. Just keep in mind that when we write something like $(1,2)$, we are describing the location of a point in the xy-plane, whereas $\langle 1,2\rangle$ describes a vector in the $dxdy$-plane and $[1,2]$ is a vector in the $\omega\nu$-plane.

Let's not forget our goal now. We wanted to use ω and ν to take the pair of vectors (V_1, V_2) and return a number. So far, all we have done is to take this pair of vectors and return another pair of vectors. Do we know of a way to take these vectors and get a number? Actually, we know several, but the most useful one turns out to be the area of the parallelogram that the vectors span. This is precisely what we define to be the value of $\omega \wedge \nu(V_1, V_2)$ (see Figure 3.4).

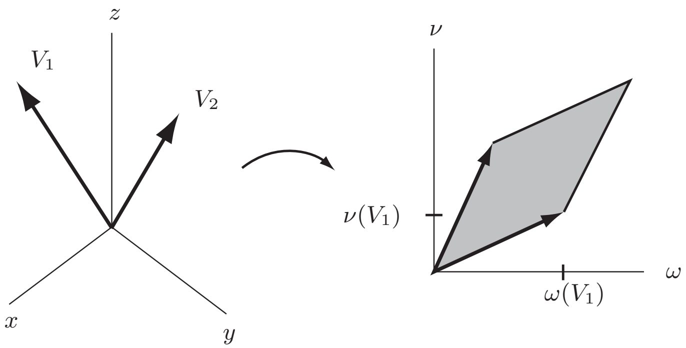

Fig. 3.4. The product of ω and ν.

Example 13. Let $\omega = 2dx - 3dy + dz$ and $\nu = dx + 2dy - dz$ be two 1-forms on $T_p\mathbb{R}^3$ for some fixed $p \in \mathbb{R}^3$. Let's evaluate $\omega \wedge \nu$ on the pair of vectors $(\langle 1,3,1\rangle, \langle 2,-1,3\rangle)$. First, we compute the $[\omega,\nu]$ coordinates of the vector $\langle 1,3,1\rangle$:

$$\begin{aligned}[\omega(\langle 1,3,1\rangle), \nu(\langle 1,3,1\rangle)] &= [2\cdot 1 - 3\cdot 3 + 1\cdot 1, 1\cdot 1 + 2\cdot 3 - 1\cdot 1] \\ &= [-6, 6].\end{aligned}$$

Similarly, we compute $[\omega(\langle 2,-1,3\rangle), \nu(\langle 2,-1,3\rangle)] = [10,-3]$. Finally, the area of the parallelogram spanned by $[-6,6]$ and $[10,-3]$ is

$$\begin{vmatrix} -6 & 6 \\ 10 & -3 \end{vmatrix} = 18 - 60 = -42.$$

Should we have taken the absolute value? Not if we want to define a *linear* operator. The result of $\omega \wedge \nu$ is not just an area, it is a *signed* area; it can either be positive or negative. We will see a geometric interpretation of this soon. For now, we define

$$\omega \wedge \nu(V_1, V_2) = \begin{vmatrix} \omega(V_1) & \nu(V_1) \\ \omega(V_2) & \nu(V_2) \end{vmatrix}.$$

3.5. Let ω and ν be the following 1-forms:

$$\omega(\langle dx, dy \rangle) = 2dx - 3dy,$$

$$\nu(\langle dx, dy \rangle) = dx + dy.$$

1. Let $V_1 = \langle -1, 2 \rangle$ and $V_2 = \langle 1, 1 \rangle$. Compute $\omega(V_1)$, $\nu(V_1)$, $\omega(V_2)$ and $\nu(V_2)$.
2. Use your answers to the previous question to compute $\omega \wedge \nu(V_1, V_2)$.
3. Find a constant c such that $\omega \wedge \nu = c\ dx \wedge dy$.

3.6. $\omega \wedge \nu(V_1, V_2) = -\omega \wedge \nu(V_2, V_1)$ ($\omega \wedge \nu$ is *skew-symmetric*).

3.7. $\omega \wedge \nu(V, V) = 0$. (This follows immediately from the previous exercise. It should also be clear from the geometric interpretation.)

3.8. $\omega \wedge \nu(V_1 + V_2, V_3) = \omega \wedge \nu(V_1, V_3) + \omega \wedge \nu(V_2, V_3)$ and $\omega \wedge \nu(cV_1, V_2) = \omega \wedge \nu(V_1, cV_2) = c\, \omega \wedge \nu(V_1, V_2)$, where c is any real number ($\omega \wedge \nu$ is *bilinear*).

3.9. $\omega \wedge \nu(V_1, V_2) = -\nu \wedge \omega(V_1, V_2)$.

It is interesting to compare Problems 3.6 and 3.9. Problem 3.6 says that the 2-form, $\omega \wedge \nu$, is a skew-symmetric operator on pairs of vectors. Problem 3.9 says that $\wedge$ can be thought of as a skew-symmetric operator on 1-forms.

3.10. $\omega \wedge \omega(V_1, V_2) = 0$.

3.11. $(\omega + \nu) \wedge \psi = \omega \wedge \psi + \nu \wedge \psi$ ($\wedge$ is distributive).

There is another way to interpret the action of $\omega \wedge \nu$ which is much more geometric. First, let $\omega = a\ dx + b\ dy$ be a 1-form on $T_p\mathbb{R}^2$. Then we let $\langle \omega \rangle$ be the vector $\langle a, b \rangle$.

3.12. Let ω and ν be 1-forms on $T_p\mathbb{R}^2$. Show that $\omega \wedge \nu(V_1, V_2)$ is the area of the parallelogram spanned by V_1 and V_2, times the area of the parallelogram spanned by $\langle \omega \rangle$ and $\langle \nu \rangle$.

3.13. Use the previous problem to show that if ω and ν are 1-forms on $\mathbb{R}^2$ such that $\omega \wedge \nu = 0$, then there is a constant c such that $\omega = c\nu$.

There is also a more geometric way to think about $\omega \wedge \nu$ if ω and ν are 1-forms on $T_p\mathbb{R}^3$, although it will take us some time to develop the idea. Suppose $\omega = a\ dx + b\ dy + c\ dz$. Then we will denote the vector $\langle a, b, c\rangle$ as $\langle\omega\rangle$. From the previous section, we know that if V is any vector, then $\omega(V) = \langle\omega\rangle \cdot V$ and that this is just the projection of V onto the line containing $\langle\omega\rangle$, times $|\langle\omega\rangle|$.

Now suppose ν is some other 1-form. Choose a scalar x so that $\langle\nu - x\omega\rangle$ is perpendicular to $\langle\omega\rangle$. Let $\nu_\omega = \nu - x\omega$. Note that $\omega \wedge \nu_\omega = \omega \wedge (\nu - x\omega) = \omega \wedge \nu - x\omega \wedge \omega = \omega \wedge \nu$. Hence, any geometric interpretation we find for the action of $\omega \wedge \nu_\omega$ is also a geometric interpretation of the action of $\omega \wedge \nu$.

Finally, we let $\overline{\omega} = \frac{\omega}{|\langle\omega\rangle|}$ and $\overline{\nu_\omega} = \frac{\nu_\omega}{|\langle\nu_\omega\rangle|}$. Note that these are 1-forms such that $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$ are perpendicular unit vectors. We will now present a geometric interpretation of the action of $\overline{\omega} \wedge \overline{\nu_\omega}$ on a pair of vectors (V_1, V_2).

First, note that since $\langle\overline{\omega}\rangle$ is a unit vector, then $\overline{\omega}(V_1)$ is just the projection of V_1 onto the line containing $\langle\overline{\omega}\rangle$. Similarly, $\overline{\nu_\omega}(V_1)$ is given by projecting V_1 onto the line containing $\langle\overline{\nu_\omega}\rangle$. As $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$ are perpendicular, we can think of the quantity

$$\overline{\omega} \wedge \overline{\nu_\omega}(V_1, V_2) = \begin{vmatrix} \overline{\omega}(V_1) & \overline{\nu_\omega}(V_1) \\ \overline{\omega}(V_2) & \overline{\nu_\omega}(V_2) \end{vmatrix}$$

as the area of parallelogram spanned by V_1 and V_2, projected onto the plane containing the vectors $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$. This is the same plane as the one which contains the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$.

Now observe the following:

$$\overline{\omega} \wedge \overline{\nu_\omega} = \frac{\omega}{|\langle\omega\rangle|} \wedge \frac{\nu_\omega}{|\langle\nu_\omega\rangle|} = \frac{1}{|\langle\omega\rangle||\langle\nu_\omega\rangle|}\omega \wedge \nu_\omega.$$

Hence,

$$\omega \wedge \nu = \omega \wedge \nu_\omega = |\langle\omega\rangle||\langle\nu_\omega\rangle|\overline{\omega} \wedge \overline{\nu_\omega}.$$

Finally, note that since $\langle\omega\rangle$ and $\langle\nu_\omega\rangle$ are perpendicular, the quantity $|\langle\omega\rangle||\langle\nu_\omega\rangle|$ is just the area of the rectangle spanned by these two vectors. Furthermore, the parallelogram spanned by the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$ is obtained from this rectangle by skewing. Hence, they have the same area. We conclude the following:

Evaluating $\omega \wedge \nu$ on the pair of vectors (V_1, V_2) gives the area of parallelogram spanned by V_1 and V_2 projected onto the plane containing the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$, and multiplied by the area of the parallelogram spanned by $\langle\omega\rangle$ and $\langle\nu\rangle$.

CAUTION: While every 1-form can be thought of as projected length, *not* every 2-form can be thought of as projected area. The only 2-forms for which this interpretation is valid are those that are the product of 1-forms. See Problem 3.18.

Let's pause for a moment to look at a particularly simple 2-form on $T_p\mathbb{R}^3$, $dx \wedge dy$. Suppose $V_1 = \langle a_1, a_2, a_3 \rangle$ and $V_2 = \langle b_1, b_2, b_3 \rangle$. Then

$$dx \wedge dy(V_1, V_2) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

This is precisely the (signed) area of the parallelogram spanned by V_1 and V_2 projected onto the $dxdy$-plane.

3.14. Show that for any 1-forms ω and ν on $T\mathbb{R}^3$, there are constants c_1, c_2, and c_3 such that

$$\omega \wedge \nu = c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dy \wedge dz.$$

The preceding comments and this last exercise give the following geometric interpretation of the action of a 2-form on the pair of vectors (V_1, V_2):

Every 2-form projects the parallelogram spanned by V_1 and V_2 onto each of the (2-dimensional) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results.

This interpretation holds in all dimensions. Hence, to specify a 2-form, we need to know as many constants as there are 2-dimensional coordinate planes. For example, to give a 2-form in 4-dimensional Euclidean space we need to specify six numbers:

$$c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dx \wedge dw + c_4 dy \wedge dz + c_5 dy \wedge dw + c_6 dz \wedge dw.$$

The skeptic may argue here. Problem 3.14 only shows that a 2-form which is a product of 1-forms can be thought of as a sum of projected, scaled areas. What about an arbitrary 2-form? Well, to address this, we need to know what an arbitrary 2-form is! Up until now we have not given a complete definition. Henceforth, we will define a 2-form to be a bilinear, skew-symmetric, real-valued function on $T_p\mathbb{R}^n \times T_p\mathbb{R}^n$. That is a mouthful. This just means that it is an operator which eats pairs of vectors, spits out real numbers, and satisfies the conclusions of Problems 3.6 and 3.8. Since these are the only ingredients necessary to do Problem 3.14, our geometric interpretation is valid for all 2-forms.

3.15. If $\omega(\langle dx, dy, dz \rangle) = dx + 5dy - dz$ and $\nu(\langle dx, dy, dz \rangle) = 2dx - dy + dz$, compute

$$\omega \wedge \nu(\langle 1, 2, 3 \rangle, \langle -1, 4, -2 \rangle).$$

3.16. Let $\omega(\langle dx, dy, dz\rangle) = dx + 5dy - dz$ and $\nu(\langle dx, dy, dz\rangle) = 2dx - dy + dz$. Find constants c_1, c_2 and c_3, such that

$$\omega \wedge \nu = c_1 dx \wedge dy + c_2 dy \wedge dz + c_3 dx \wedge dz.$$

3.17. Express each of the following as the product of two 1-forms:

1. $3dx \wedge dy + dy \wedge dx$.
2. $dx \wedge dy + dx \wedge dz$.
3. $3dx \wedge dy + dy \wedge dx + dx \wedge dz$.
4. $dx \wedge dy + 3dz \wedge dy + 4dx \wedge dz$.

3.4 2-Forms on $T_p\mathbb{R}^3$ (optional)

This text is about differential n-forms on $\mathbb{R}^m$. For most of it, we keep $n, m \leq 3$ so that everything we do can be easily visualized. However, very little is special about these dimensions. Everything we do is presented so that it can easily generalize to higher dimensions. In this section and the next we break from this philosophy and present some special results when the dimensions involved are 3 or 4.

3.18. Find a 2-form which is *not* the product of 1-forms.

In doing this exercise, you may guess that, in fact, all 2-forms on $T_p\mathbb{R}^3$ *can* be written as a product of 1-forms. Let's see a proof of this fact that relies heavily on the geometric interpretations we have developed.

Recall the correspondence introduced above between vectors and 1-forms. If $\alpha = a_1 dx + a_2 dy + a_3 dz$, then we let $\langle\alpha\rangle = \langle a_1, a_2, a_3\rangle$. If V is a vector, then we let $\langle V\rangle^{-1}$ be the corresponding 1-form.

We now prove two lemmas.

Lemma 1. *If α and β are 1-forms on $T_p\mathbb{R}^3$ and V is a vector in the plane spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$, then there is a vector, W, in this plane such that $\alpha \wedge \beta = \langle V\rangle^{-1} \wedge \langle W\rangle^{-1}$.*

Proof. The proof of the above lemma relies heavily on the fact that 2-forms which are the product of 1-forms are very flexible. The 2-form $\alpha \wedge \beta$ takes pairs of vectors, projects them onto the plane spanned by the vectors $\langle\alpha\rangle$ and $\langle\beta\rangle$, and computes the area of the resulting parallelogram times the area of the parallelogram spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$. Note that for every nonzero scalar c, the area of the parallelogram spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$ is the same as the area of the parallelogram spanned by $c\langle\alpha\rangle$ and $1/c\langle\beta\rangle$. (This is the same thing as saying that $\alpha \wedge \beta = c\alpha \wedge \frac{1}{c}\beta$.) The important point here is that we can scale one of the 1-forms as much as we want at the expense of the other and get the same 2-form as a product.

Another thing we can do is apply a rotation to the pair of vectors $\langle\alpha\rangle$ and $\langle\beta\rangle$ in the plane which they determine. As the area of the parallelogram spanned by these two vectors is unchanged by rotation, their product still determines the same 2-form. In particular, suppose V is any vector in the plane spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$. Then we can rotate $\langle\alpha\rangle$ and $\langle\beta\rangle$ to $\langle\alpha'\rangle$ and $\langle\beta'\rangle$ so that $c\langle\alpha'\rangle = V$ for some scalar c. We can then replace the pair $(\langle\alpha\rangle, \langle\beta\rangle)$ with the pair $(c\langle\alpha'\rangle, 1/c\langle\beta'\rangle) = (V, 1/c\langle\beta'\rangle)$. To complete the proof, let $W = 1/c\langle\beta'\rangle$.

Lemma 2. *If $\omega_1 = \alpha_1 \wedge \beta_1$ and $\omega_2 = \alpha_2 \wedge \beta_2$ are 2-forms on $T_p\mathbb{R}^3$, then there exist 1-forms, α_3 and β_3, such that $\omega_1 + \omega_2 = \alpha_3 \wedge \beta_3$.*

Proof. Let's examine the sum $\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2$. Our first case is that the plane spanned by the pair $(\langle\alpha_1\rangle, \langle\beta_1\rangle)$ is the same as the plane spanned by the pair $(\langle\alpha_2\rangle, \langle\beta_2\rangle)$. In this case, it must be that $\alpha_1 \wedge \beta_1 = C\alpha_2 \wedge \beta_2$ and, hence, $\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = (1 + C)\alpha_1 \wedge \beta_1$.

If these two planes are not the same, then they intersect in a line. Let V be a vector contained in this line. Then by the preceding lemma, there are 1-forms γ and γ' such that $\alpha_1 \wedge \beta_1 = \langle V\rangle^{-1} \wedge \gamma$ and $\alpha_2 \wedge \beta_2 = \langle V\rangle^{-1} \wedge \gamma'$. Hence,

$$\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = \langle V\rangle^{-1} \wedge \gamma + \langle V\rangle^{-1} \wedge \gamma' = \langle V\rangle^{-1} \wedge (\gamma + \gamma').$$

Now note that *any* 2-form is the sum of products of 1-forms. Hence, this last lemma implies that any 2-form on $T_p\mathbb{R}^3$ is a product of 1-forms. In other words:

> *Every 2-form on $T_p\mathbb{R}^3$ projects pairs of vectors onto some plane and returns the area of the resulting parallelogram, scaled by some constant.*

This fact is precisely why all of classical vector calculus works. We explore this in the next few exercises and further in Section 6.3.

3.19. Use the above geometric interpretation of the action of a 2-form on $T_p\mathbb{R}^3$ to justify the following statement: For every 2-form ω on $T_p\mathbb{R}^3$ there are nonzero vectors V_1 and V_2 such that V_1 is not a multiple of V_2 but $\omega(V_1, V_2) = 0$.

3.20. Does Problem 3.19 generalize to higher dimensions?

3.21. Show that if ω is a 2-form on $T_p\mathbb{R}^3$, then there is a line l in $T_p\mathbb{R}^3$ such that if the plane spanned by V_1 and V_2 contains l, then $\omega(V_1, V_2) = 0$.

Note that the conditions of Problem 3.21 are satisfied when the vectors that are perpendicular to both V_1 and V_2 are also perpendicular to l.

3.22. Show that if all you know about V_1 and V_2 is that they are vectors in $T_p\mathbb{R}^3$ that span a parallelogram of area A, then the value of $\omega(V_1, V_2)$ is maximized when V_1 and V_2 are perpendicular to the line l of Problem 3.21.

Note that the conditions of this exercise are satisfied when the vectors perpendicular to V_1 and V_2 are parallel to l.

3.23. Let N be a vector perpendicular to V_1 and V_2 in $T_p\mathbb{R}^3$ whose length is precisely the area of the parallelogram spanned by these two vectors. Show that there is a vector V_ω in the line l of Problem 3.21 such that the value of $\omega(V_1, V_2)$ is precisely $V_\omega \cdot N$.

Remark. You may have learned that the vector N of the previous exercise is precisely the cross product of V_1 and V_2. Hence, the previous problem implies that if ω is a 2-form on $T_p\mathbb{R}^3$ then there is a vector V_ω such that $\omega(V_1, V_2) = V_\omega \cdot (V_1 \times V_2)$.

3.24. Show that if $\omega = F_x \; dy \wedge dz - F_y \; dx \wedge dz + F_z \; dx \wedge dy$, then $V_\omega = \langle F_x, F_y, F_z \rangle$.

3.5 2-Forms and 3-forms on $T_p\mathbb{R}^4$ (optional)

Many of the techniques of the previous section can be used to prove results about 2- and 3-forms on $T_p\mathbb{R}^4$.

3.25. Show that any 3-form on $T_p\mathbb{R}^4$ can be written as the product of three 1-forms. (Hint: Two 3-dimensional subspaces of $T_p\mathbb{R}^4$ must meet in at least a line.)

We now give away an answer to Problem 3.18. Let $\omega = dx \wedge dy + dz \wedge dw$. Then an easy computation shows that $\omega \wedge \omega = 2dx \wedge dy \wedge dz \wedge dw$. However, if ω were equal to $\alpha \wedge \beta$ for some 1-forms α and β, then $\omega \wedge \omega$ would be zero (why?). This argument shows that, in general, if ω is any 2-form such that $\omega \wedge \omega \neq 0$, then ω cannot be written as the product of 1-forms.

3.26. Let ω be a 2-form on $T_p\mathbb{R}^4$. Show that ω can be written as the sum of exactly two products; that is, $\omega = \alpha \wedge \beta + \delta \wedge \gamma$. (Hint: Given three planes in $T_p\mathbb{R}^4$, there are at least two of them that intersect in more than a point.)

Above, we saw that if ω is a 2-form such that $\omega \wedge \omega \neq 0$, then ω is not the product of 1-forms. We now use the previous exercise to show the converse.

Theorem 1. *If ω is a 2-form on $T_p\mathbb{R}^4$ such that $\omega \wedge \omega = 0$, then ω can be written as the product of two 1-forms.*

Our proof of this again relies heavily on the geometry of the situation. By the previous exercise, $\omega = \alpha \wedge \beta + \delta \wedge \gamma$. A short computation then shows

$$\omega \wedge \omega = 2\alpha \wedge \beta \wedge \delta \wedge \gamma.$$

If this 4-form is the zero 4-form, then it must be the case that the (4-dimensional) volume of the parallelepiped spanned by $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\delta\rangle$ and $\langle\gamma\rangle$ is zero. This, in turn, implies that the plane spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$ meets the plane spanned by $\langle\delta\rangle$ and $\langle\gamma\rangle$ in at least a line (show this!). Call such an intersection line $\mathcal{L}$.

As in the previous section, we can now rotate $\langle\alpha\rangle$ and $\langle\beta\rangle$, in the plane they span, to vectors $\langle\alpha'\rangle$ and $\langle\beta'\rangle$ such that $\langle\alpha'\rangle$ lies in the line $\mathcal{L}$. The 2-form $\alpha' \wedge \beta'$ must equal $\alpha \wedge \beta$ since they determine the same plane and span a parallelogram of the same area. Similarly, we rotate $\langle\delta\rangle$ and $\langle\gamma\rangle$ to vectors $\langle\delta'\rangle$ and $\langle\gamma'\rangle$ such that $\langle\delta'\rangle \subset \mathcal{L}$. It follows that $\delta \wedge \gamma = \delta' \wedge \gamma'$.

Since $\langle\alpha'\rangle$ and $\langle\delta'\rangle$ lie on the same line, there is a constant c such that $c\alpha' = \delta'$. We now put all of this information together:

$$\begin{aligned}
\omega &= \alpha \wedge \beta + \delta \wedge \gamma \\
&= \alpha' \wedge \beta' + \delta' \wedge \gamma' \\
&= (c\alpha') \wedge \left(\frac{1}{c}\beta'\right) + \delta' \wedge \gamma' \\
&= \delta' \wedge \left(\frac{1}{c}\beta'\right) + \delta' \wedge \gamma' \\
&= \delta' \wedge \left(\frac{1}{c}\beta' + \gamma'\right).
\end{aligned}$$

3.6 *n*-Forms

Let's think a little more about our multiplication operator $\wedge$. If it is really going to be anything like multiplication, we should be able to take three 1-forms — ω, ν and ψ — and form the product $\omega \wedge \nu \wedge \psi$. How can we define this? A first guess might be to say that $\omega \wedge \nu \wedge \psi = \omega \wedge (\nu \wedge \psi)$, but $\nu \wedge \psi$ is a 2-form and we have not defined the product of a 2-form and a 1-form. We take a different approach and define $\omega \wedge \nu \wedge \psi$ directly.

This is completely analogous to the previous section. ω, ν and ψ each act on a vector, V, to give three numbers. In other words, they can be thought of as coordinate functions. We say the coordinates of V are $[\omega(V), \nu(V), \psi(V)]$. Hence, if we have three vectors — V_1, V_2 and V_3 — we can compute the $[\omega, \nu, \psi]$ coordinates of each. This gives us three new vectors. The value of $\omega \wedge \nu \wedge \psi(V_1, V_2, V_3)$ is then defined to be the signed volume of the parallelepiped which they span.

There is no reason to stop at 3 dimensions. Suppose $\omega_1, \omega_2, ..., \omega_n$ are 1-forms and $V_1, V_2, ..., V_n$ are vectors. Then we define the value of

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, V_2, ..., V_n)$$

to be the signed (n-dimensional) volume of the parallelepiped spanned by the vectors $[\omega_1(V_i), \omega_2(V_i), ..., \omega_n(V_i)]$. Algebraically,

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, V_2, ..., V_n) = \begin{vmatrix} \omega_1(V_1) & \omega_2(V_1) & \cdots & \omega_n(V_1) \\ \omega_1(V_2) & \omega_2(V_2) & \cdots & \omega_n(V_2) \\ \vdots & \vdots & & \vdots \\ \omega_1(V_n) & \omega_2(V_n) & \cdots & \omega_n(V_n) \end{vmatrix}$$

3.27. Let γ be the 3-form $2dx \wedge dy \wedge dz$. Let

$$V_1 = \langle 1, 2, 1 \rangle, \quad V_2 = \langle 0, 1, 1 \rangle, \quad V_3 = \langle -1, -2, 1 \rangle.$$

Compute $\gamma(V_1, V_2, V_3)$.

3.28. Calculate $\alpha \wedge \beta \wedge \gamma(V_1, V_2, V_3)$, where

$$\alpha = dx + 2dy + dz, \quad \beta = dx - dz, \quad \gamma = -dy + 3dz,$$

$$V_1 = \langle 1, 2, 3 \rangle, \quad V_2 = \langle -1, 1, 1 \rangle, \quad V_3 = \langle 0, 1, 1 \rangle.$$

3.29. Note that, just as in Problem 3.12, if α, β and γ are 1-forms on $T_p\mathbb{R}^3$, then $\alpha \wedge \beta \wedge \gamma(V_1, V_2, V_3)$ is the (signed) volume of the parallelepiped spanned by V_1, V_2 and V_3 times the volume of the parallelepiped spanned by $\langle \alpha \rangle$, $\langle \beta \rangle$ and $\langle \gamma \rangle$. Suppose ω is a 2-form on $T_p\mathbb{R}^3$ and ν is a 1-form on $T_p\mathbb{R}^3$. Show that if $\omega \wedge \nu = 0$, then there is a 1-form γ such that $\omega = \nu \wedge \gamma$. (Hint: Combine the given geometric interpretation of a 3-form which is the product of 1-forms on T_pR^3, with the results of Section 3.4.)

It follows from linear algebra that if we swap any two rows or columns of this matrix, the sign of the result flips. Hence, if the n-tuple $\mathbf{V}' = (V_{i_1}, V_{i_2}, ..., V_{i_n})$ is obtained from $\mathbf{V} = (V_1, V_2, ..., V_n)$ by an even number of exchanges, then the sign of $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(\mathbf{V}')$ will be the same as the sign of $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(\mathbf{V})$. If the number of exchanges is odd, then the sign is opposite. We sum this up by saying that the n-form, $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ is *alternating*.

The wedge product of 1-forms is also *multilinear*, in the following sense:

$$\begin{aligned} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, ..., V_i + V_i', ..., V_n) \\ = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, ..., V_i, ..., V_n) \\ + \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, ..., V_i', ..., V_n) \end{aligned}$$

and

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, ..., cV_i, ..., V_n) = c\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n(V_1, ..., V_i, ..., V_n)$$

for all i and any real number c.

In general, we define an n-form to be any alternating, multilinear real-valued function which acts on n-tuples of vectors.

3.30. Prove the following geometric interpretation (Hint: All of the steps are completely analogous to those in Section 3.3.):

> *An m-form on $T_p\mathbb{R}^n$ can be thought of as a function which takes the parallelepiped spanned by m vectors, projects it onto each of the m-dimensional coordinate planes, computes the resulting areas, multiplies each by some constant, and adds the results.*

3.31. How many numbers do you need to give to specify a 5-form on $T_p\mathbb{R}^{10}$?

We turn now to the simple case of an n-form on $T_p\mathbb{R}^n$. Notice that there is only one n-dimensional coordinate plane in this space — namely the space itself. Such a form, evaluated on an n-tuple of vectors, must therefore give the n-dimensional volume of the parallelepiped which it spans, multiplied by some constant. For this reason, such a form is called a *volume form* (in 2-dimensions, an *area form*).

Example 14. Consider the forms, $\omega = dx + 2dy - dz, \nu = 3dx - dy + dz$ and $\psi = -dx - 3dy + dz$, on $T_p\mathbb{R}^3$. By the above argument, $\omega \wedge \nu \wedge \psi$ must be a volume form. Which volume form is it? One way to tell is to compute its value on a set of vectors which we *know* span a parallelepiped of volume one — namely $\langle 1,0,0\rangle, \langle 0,1,0\rangle$ and $\langle 0,0,1\rangle$. This will tell us how much the form scales volume.

$$\omega \wedge \nu \wedge \psi(\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & -1 & -3 \\ -1 & 1 & 1 \end{vmatrix} = 4.$$

So, $\omega \wedge \nu \wedge \psi$ must be the same as the form $4dx \wedge dy \wedge dz$.

3.32. Let $\omega(\langle dx, dy, dz\rangle) = dx + 5dy - dz$, $\nu(\langle dx, dy, dz\rangle) = 2dx - dy + dz$ and $\gamma(\langle dx, dy, dz) = -dx + dy + 2dz$.

1. If $V_1 = \langle 1,0,2\rangle, V_2 = \langle 1,1,2\rangle$ and $V_3 = \langle 0,2,3\rangle$, compute $\omega \wedge \nu \wedge \gamma(V_1, V_2, V_3)$.
2. Find a constant c such that $\omega \wedge \nu \wedge \gamma = c\ dx \wedge dy \wedge dz$.
3. Let $\alpha = 3dx \wedge dy + 2dy \wedge dz - dx \wedge dz$. Find a constant c such that $\alpha \wedge \gamma = c\ dx \wedge dy \wedge dz$.

3.33. Simplify

$$dx \wedge dy \wedge dz + dx \wedge dz \wedge dy + dy \wedge dz \wedge dx + dy \wedge dx \wedge dy.$$

3.34.

1. Expand and simplify

$$(dx + dy) \wedge (2dx + dz) \wedge dz.$$

2. Plug the following vectors into the above 3-form:

$$V_1 = \langle 1, 1, 1 \rangle, \quad V_2 = \langle 1, 0, 1 \rangle, \quad V_3 = \langle 0, 1, -1 \rangle$$

3.35. Let ω be an n-form and ν an m-form.

1. Show that
$$\omega \wedge \nu = (-1)^{nm} \nu \wedge \omega.$$
2. Use this to show that if n is odd, then $\omega \wedge \omega = 0$.

3.7 Algebraic computation of products

In this section, we break with the spirit of the text briefly. At this point, we have amassed enough algebraic identities that multiplying forms becomes similar to multiplying polynomials. We quickly summarize these identities and work a few examples.

Let ω be an n-form and ν be an m-form. Then we have the following identities:

$$\begin{aligned} \omega \wedge \nu &= (-1)^{nm} \nu \wedge \omega, \\ \omega \wedge \omega &= 0 \text{ if } n \text{ is odd}, \\ \omega \wedge (\nu + \psi) &= \omega \wedge \nu + \omega \wedge \psi, \\ (\nu + \psi) \wedge \omega &= \nu \wedge \omega + \psi \wedge \omega. \end{aligned}$$

Example 15.

$$\begin{aligned} (x\ dx + y\ dy) \wedge (y\ dx + x\ dy) &= \cancel{xy\ dx \wedge dx} + x^2\ dx \wedge dy + y^2\ dy \wedge dx \\ &\quad + \cancel{yx\ dy \wedge dy} \\ &= x^2\ dx \wedge dy + y^2\ dy \wedge dx \\ &= x^2\ dx \wedge dy - y^2\ dx \wedge dy \\ &= (x^2 - y^2)\ dx \wedge dy. \end{aligned}$$

Example 16.

$$\begin{aligned} &(x\ dx + y\ dy) \wedge (xz\ dx \wedge dz + yz\ dy \wedge dz) \\ &= \cancel{x^2 z\ dx \wedge dx \wedge dz} + xyz\ dx \wedge dy \wedge dz \\ &\quad + yxz\ dy \wedge dx \wedge dz + \cancel{y^2 z\ dy \wedge dy \wedge dz} \\ &= xyz\ dx \wedge dy \wedge dz + yxz\ dy \wedge dx \wedge dz \\ &= xyz\ dx \wedge dy \wedge dz - xyz\ dx \wedge dy \wedge dz \\ &= 0. \end{aligned}$$

3.36. Expand and simplify the following:

1. $[(x - y)\ dx + (x + y)\ dy + z\ dz] \wedge [(x - y)\ dx + (x + y)\ dy]$.
2. $(2dx + 3dy) \wedge (dx - dz) \wedge (dx + dy + dz)$.