2

Prerequisites

2.1 Multivariable calculus

We denote by $\mathbb{R}^n$ the set of points with n real coordinates. In this text we will often represent functions abstractly by saying how many numbers go into the function and how many come out. So, if we write $f : \mathbb{R}^n \to \mathbb{R}^m$, we mean f is a function whose input is a point with n coordinates and whose output is a point with m coordinates.

2.1. Sketch the graphs of the following functions $f : \mathbb{R}^2 \to \mathbb{R}^1$:

1. $z = 2x - 3y$.
2. $z = x^2 + y^2$.
3. $z = xy$.
4. $z = \sqrt{x^2 + y^2}$.
5. $z = \frac{1}{\sqrt{x^2+y^2}}$.
6. $z = \sqrt{x^2 + y^2 + 1}$.
7. $z = \sqrt{x^2 + y^2 - 1}$.
8. $z = \cos(x + y)$.
9. $z = \cos(xy)$.
10. $z = \cos(x^2 + y^2)$.
11. $z = e^{-(x^2+y^2)}$.

2.2. Find functions whose graphs are the following:

1. A plane through the origin at 45° to both the x- and y-axes.
2. The top half of a sphere of radius 2.
3. The top half of a torus centered around the z-axis (i.e., the tube of radius 1, say, centered around a circle of radius 2 in the xy-plane).
4. The top half of the cylinder of radius 1 which is centered around the line where the plane $y = x$ meets the plane $z = 0$.

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You may find it helpful to check your answers to the above exercises with a computer graphing program.

The volume under the graph of a multivariable function is given by a multiple integral, as in the next example.

Example 1. To find the volume under the graph of $f(x, y) = xy^2$ and above the rectangle R with vertices at $(0, 0)$, $(2, 0)$, $(0, 3)$ and $(2, 3)$ we compute

$$\begin{aligned} \int_R xy^2 \ dx\ dy &= \int_0^3 \int_0^2 xy^2 \ dx\ dy \\ &= \int_0^3 \left[\frac{1}{2} x^2 y^2 \Big|_{x=0}^{2} \right] dy \\ &= \int_0^3 2y^2 \ dy \\ &= 18. \end{aligned}$$

2.3. Let R be the rectangle in the xy-plane with vertices at $(1, 0)$, $(2, 0)$, $(1, 3)$ and $(2, 3)$. Integrate the following functions over R:

1. x^2y^2.
2. 1.
3. $x^2 + y^2$.
4. $\sqrt{x + \frac{2}{3}y}$.

As in the one-variable case, derivates of multivariable functions give the slope of a tangent line. The relevant question here is "*Which* tangent line?" The answer to this is another question: "Which derivative?" For example, suppose we slice the graph of $f : \mathbb{R}^2 \to \mathbb{R}^1$ with the plane parallel to the yz-plane, through the point $(x_0, y_0, 0)$. Then we get a curve which represents some function of y. We can then ask "What is the slope of the tangent line to this curve when $y = y_0$?" The answer to this question is precisely the definition of $\frac{\partial f}{\partial y}(x_0, y_0)$ (see Figure 2.1). Similarly, slicing the graph with a plane parallel to the xz-plane gives rise to the definition of $\frac{\partial f}{\partial x}(x_0, y_0)$.

2.4. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

1. x^2y^3.
2. $\sin(x^2y^3)$.
3. $x \sin(xy)$.

When you take a partial derivative, you get another function of x and y. You can then do it again to find the *second partials*. These are denoted by

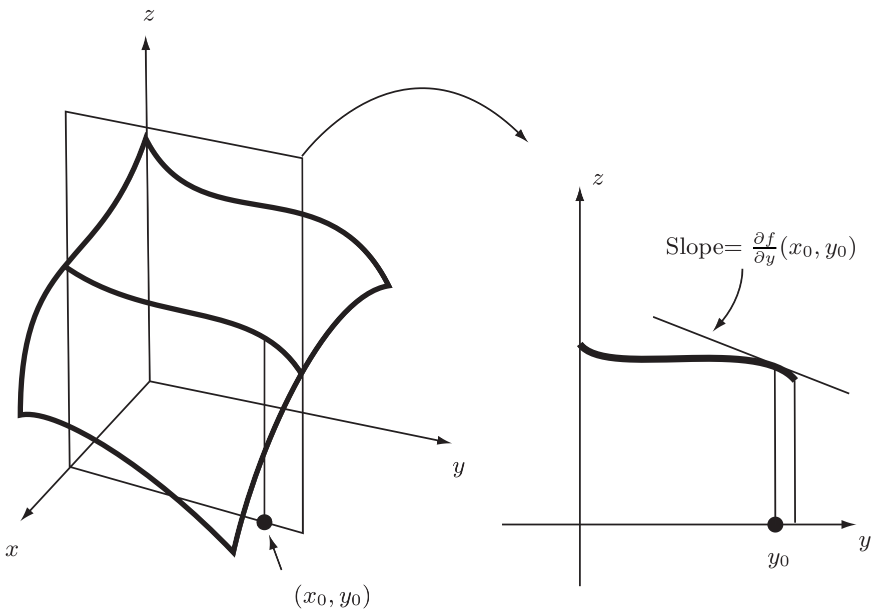

Fig. 2.1. The partial derivative with respect to y.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

2.5. Find all second partials for each of the functions in the previous exercise.

Recall that amazingly the "mixed" partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are always equal. This is not a coincidence! Somehow the mixed partials measure the "twisting" of the graph, and this is the same from every direction.

2.6. Let

$$D(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

If, for some point (x_0, y_0), you know $D(x_0, y_0) > 0$, then show that the signs of $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$ and $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ are the same.

2.2 Gradients

Let's look back to Figure 2.1. What if we sliced the graph of $f(x, y)$ with some vertical plane through the point (x_0, y_0) that was *not* parallel to the xz-plane or yz-plane, as in Figure 2.2? How could we compute the slope?

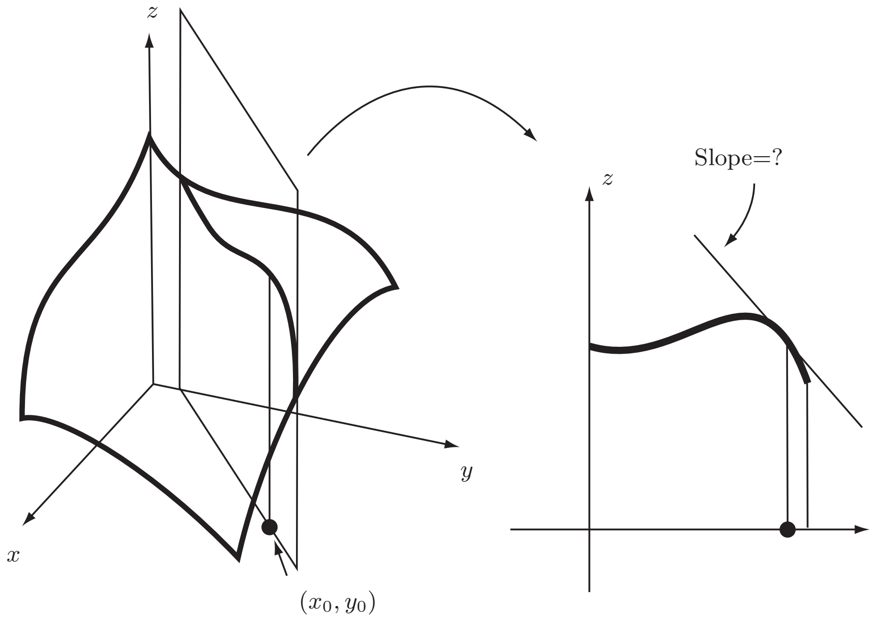

Fig. 2.2. A directional derivative.

To answer this, visualize the set of *all* lines tangent to the graph of $f(x, y)$ at the point (x_0, y_0). This is a tangent *plane*.

The equation for a plane through the origin in $\mathbb{R}^3$ is of the form $z = m_x x + m_y y$. Notice that the intersection of such a plane with the xz-plane is the graph of $z = m_x x$. Hence, m_x is the slope of this line of intersection. Similarly, the quantity m_y is the slope of the line which is the intersection with the yz-plane.

To get a plane through the point $(x_0, y_0, f(x_0, y_0))$, we can translate the origin to this point by replacing x with $x - x_0$, y with $y - y_0$ and z with $z - f(x_0, y_0)$:

$$z - f(x_0, y_0) = m_x(x - x_0) + m_y(y - y_0).$$

Since we want this to actually be a tangent plane, it follows that m_x must be equal to $\frac{\partial f}{\partial x}$ and m_y must be equal to $\frac{\partial f}{\partial y}$. Hence, the equation of the tangent plane T is given by

$$T(x,y) = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + f,$$

where $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and f are all evaluated at the point (x_0, y_0).

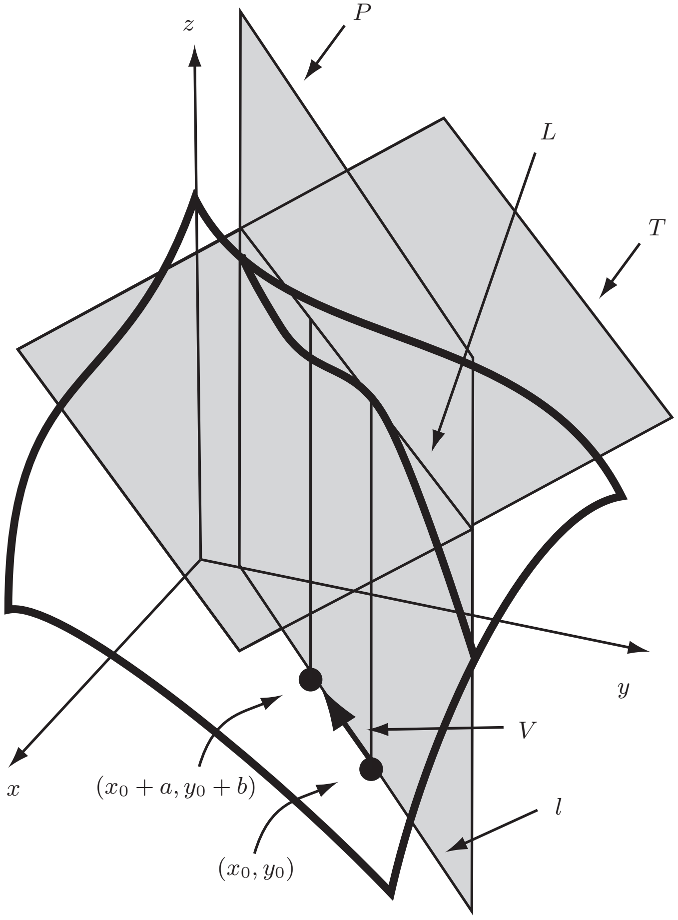

Fig. 2.3. Computing the slope of the tangent line L.

Now, suppose P is the vertical plane through the point (x_0, y_0) depicted in Figure 2.3. Let l denote the line where P intersects the xy-plane. The tangent line L to the graph of f, which lies above l, is also the line contained in T, which lies above l. To figure out the slope of L, we will simply compute "rise over run."

Suppose l contains the vector $V = \langle a, b \rangle$, where $|V| = 1$. Then two points on l, a distance of 1 apart, are (x_0, y_0) and $(x_0 + a, y_0 + b)$. Thus, the "run" will be equal to 1. The "rise" is the difference between $T(x_0, y_0)$ and $T(x_0 + a, y_0 + b)$, which we compute as follows:

$$\begin{aligned}
&T(x_0 + a, y_0 + b) - T(x_0, y_0) \\
&\qquad = \left[\frac{\partial f}{\partial x}(x_0 + a - x_0) + \frac{\partial f}{\partial y}(y_0 + b - y_0) + f\right] \\
&\qquad\quad - \left[\frac{\partial f}{\partial x}(x_0 - x_0) + \frac{\partial f}{\partial y}(y_0 - y_0) + f\right] \\
&\qquad = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}.
\end{aligned}$$

Since the slope of L is "rise" over "run" and the "run" equals 1, we conclude that the slope of L is $a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are evaluated at the point (x_0, y_0).

2.7. Suppose $f(x, y) = x^2y^3$. Compute the slope of the line tangent to $f(x, y)$, at the point $(2, 1)$, in the direction $\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$.

The quantity $a\frac{\partial f}{\partial x}(x_0, y_0) + b\frac{\partial f}{\partial y}(x_0, y_0)$ is defined to be the *directional derivative* of f, at the point (x_0, y_0), in the direction V. We will adopt the notation $\nabla_V f(x_0, y_0)$ for this quantity.

Let $f(x, y) = xy^2$. Let's compute the directional derivative of f, at the point $(2, 3)$, in the direction $V = \langle 1, 5 \rangle$. We compute

$$\begin{aligned}
\nabla_V f(2, 3) &= 1\frac{\partial f}{\partial x}(2, 3) + 5\frac{\partial f}{\partial y}(2, 3) \\
&= 1 \cdot 3^2 + 5 \cdot 2 \cdot 2 \cdot 3 \\
&= 69.
\end{aligned}$$

Is 69 the slope of the tangent line to some curve that we get when we intersect the graph of xy^2 with some plane? What this number represents is the rate of change of f, as we walk along the line l of Figure 2.3 *with speed* $|V|$. To find the desired slope, we would have to walk with speed 1. Hence, the directional derivative only represents a slope when $|V| = 1$.

2.8. Let $f(x, y) = xy + x - 2y + 4$. Find the slope of the tangent line to the graph of $f(x, y)$, in the direction of the vector $\langle 1, 2 \rangle$, at the point $(0, 1)$.

To proceed further, we write the definition of $\nabla_V f$ as a dot product:

$$\nabla_{\langle a,b \rangle} f(x_0, y_0) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle.$$

The vector $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ is called the *gradient* of f and is denoted ∇f. Using this notation, we obtain the following formula:

$$\nabla_V f(x_0, y_0) = \nabla f(x_0, y_0) \cdot V.$$

Note that this dot product is greatest when V points in the same direction as ∇f. This fact leads us to the geometric significance of the gradient vector. Think of $f(x, y)$ as a function which represents the altitude in some mountain range, given a location in longitude x and latitude y. Now, if all you know is f and your location x and y, and you want to figure out which way "uphill" is, all you have to do is point yourself in the direction of ∇f.

What if you wanted to know what the slope was in the direction of steepest ascent? You would have to compute the directional derivative, using a vector of length 1 which points in the same direction as ∇f. Such a vector is easy to find: $U = \frac{\nabla f}{|\nabla f|}$. Now we compute this slope:

$$\begin{aligned}
\nabla_U f &= \nabla f \cdot U \\
&= \nabla f \cdot \frac{\nabla f}{|\nabla f|} \\
&= \frac{1}{|\nabla f|}(\nabla f \cdot \nabla f) \\
&= \frac{1}{|\nabla f|}|\nabla f|^2 \\
&= |\nabla f|.
\end{aligned}$$

Hence, the magnitude of the gradient vector represents the largest slope of a tangent line through a particular point.

2.9. Let $f(x, y) = \sin(xy^2)$. Calculate the directional derivative of $f(x, y)$ at the point $\left(\frac{\pi}{4}, 1\right)$, in the direction of $\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$.

2.10. Let $f(x, y) = xy^2$.

1. Compute ∇f.
2. Use your answer to the previous question to compute $\nabla_{\langle 1,5\rangle} f(2, 3)$.
3. Find a vector of length 1 that points in the direction of steepest ascent, at the point $(2, 3)$.
4. What is the largest slope of a tangent line to the graph of f when $(x, y) = (2, 3)$?

2.11. Let $f(x, y)$ be the following function:

$$f(x, y) = \frac{1}{2}x^2 + 3xy.$$

Find the largest slope of any tangent line to the graph of $f(x, y)$ at the point $(1, 1)$.

2.12. Suppose (x_0, y_0) is a point where ∇f is nonzero and let $n = f(x_0, y_0)$. Show that the vector $\nabla f(x_0, y_0)$ is perpendicular to the set of points (x, y) such that $f(x, y) = n$ (i.e., a level curve).

2.3 Polar, cylindrical and spherical coordinates

The two most common ways of specifying the location of a point in $\mathbb{R}^2$ are *rectangular* and *polar* coordinates. Rectangular coordinates on $\mathbb{R}^2$ will always be denoted in this text as (x, y) and polar coordinates by (r, θ). As is standard, r is the distance to the origin and θ is the angle a ray makes with the (positive) x-axis. Some basic trigonometry gives the relationships between these quantities:

$$\begin{array}{r|l} x = r\cos\theta & r = \sqrt{x^2 + y^2}, \\ y = r\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right). \end{array}$$

In $\mathbb{R}^3$ we will mostly use three different coordinate systems: *rectangular* (x, y, z), *cylindrical* (r, θ, z), and *spherical* (ρ, θ, ϕ).

Cylindrical coordinates describe the location of a point by using polar coordinates (r, θ) to describe the projection onto the xy-plane and the quantity z to describe the height off of the xy-plane (see Figure 2.4). It follows that the relationships among r, θ, x and y are the same as for polar coordinates.

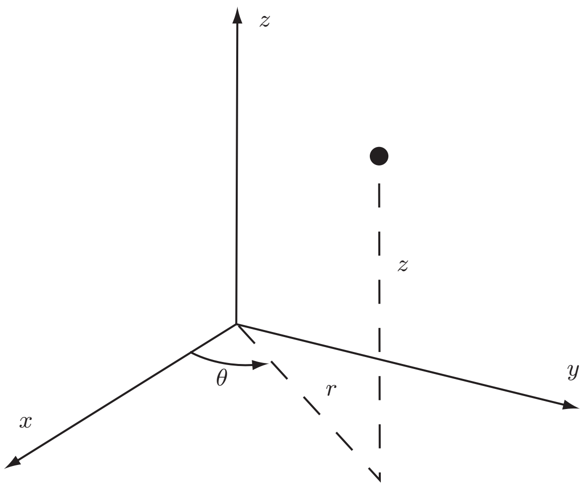

Fig. 2.4. Cylindrical coordinates.

In spherical coordinates, the quantity ρ specifies the distance from the origin, the angle θ is the same as from cylindrical coordinates and the angle ϕ is what a ray to the origin makes with the z-axis (see Figure 2.5). A little basic trigonometry yields the relationships

$x = \rho \sin\phi \cos\theta$	$\rho = \sqrt{x^2 + y^2 + z^2}$,
$y = \rho \sin\phi \sin\theta$	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$,
$z = \rho \cos\phi$	$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$.

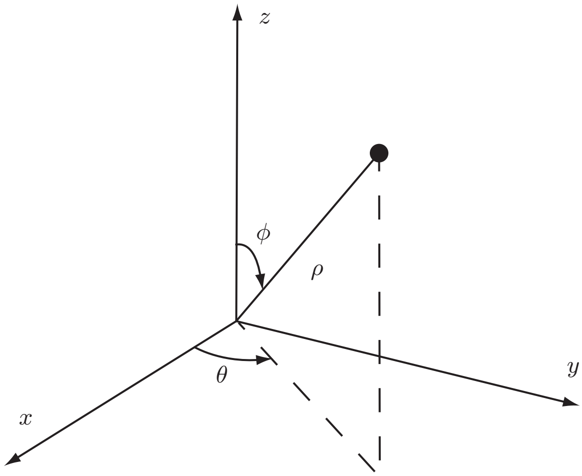

Fig. 2.5. Spherical coordinates.

2.13. Find all of the relationships between the quantities r, θ and z from cylindrical coordinates and the quantities ρ, θ and ϕ from spherical coordinates.

Each coordinate system is useful for describing different graphs, as can be seen in the following examples.

Example 2. A cylinder of radius 1, centered on the z-axis, can be described by equations in each coordinate system as follows:

- Rectangular: $x^2 + y^2 = 1$.
- Cylindrical: $r = 1$.
- Spherical: $\rho \sin\phi = 1$.

Example 3. A sphere of radius 1 is described by the following equations:

- Rectangular: $x^2 + y^2 + z^2 = 1$.
- Cylindrical: $r^2 + z^2 = 1$.
- Spherical: $\rho = 1$.

2.14. Sketch the shape described by the following equations:

1. $\theta = \frac{\pi}{4}$.
2. $z = r^2$.
3. $\rho = \phi$.
4. $\rho = \cos\phi$.
5. $r = \cos\theta$.
6. $z = \sqrt{r^2 - 1}$.
7. $z = \sqrt{r^2 + 1}$.
8. $r = \theta$.

2.15. Find rectangular, cylindrical and spherical equations that describe the following shapes:

1. A right, circular cone centered on the z-axis, with vertex at the origin.
2. The xz-plane.
3. The xy-plane.
4. A plane that is at an angle of $\frac{\pi}{4}$ with both the x- and y-axes.
5. The surface found by revolving the graph of $z = x^3$ (where $x \geq 0$) around the z-axis.

2.16. Let S be the surface which is the graph of $z = \sqrt{r^2 + 1}$ (in cylindrical coordinates).

1. Describe and/or sketch S.
2. Write an equation for S in rectangular coordinates.

2.4 Parameterized curves

Given a curve C in $\mathbb{R}^n$, a *parameterization* for C is a (one-to-one, onto, differentiable) function of the form $\phi : \mathbb{R}^1 \to \mathbb{R}^n$ whose image is C.

Example 4. The function $\phi(t) = (\cos t, \sin t)$, where $0 \leq t < 2\pi$, is a parameterization for the circle of radius 1. Another parameterization for the same circle is $\psi(t) = (\cos 2t, \sin 2t)$, where $0 \leq t < \pi$. The difference between these two parameterizations is that as t increases, the image of $\psi(t)$ moves twice as fast around the circle as the image of $\phi(t)$.

2.17. A function of the form $\phi(t) = (at + c, bt + d)$ is a parameterization of a line.

1. What is the slope of the line parameterized by ϕ?
2. How does this line compare to the one parameterized by $\psi(t) = (at, bt)$?

2.18. Draw the curves given by the following parameterizations:

1. (t, t^2), where $0 \leq t \leq 1$.
2. (t^2, t^3), where $0 \leq t \leq 1$.
3. $(2\cos t, 3\sin t)$, where $0 \leq t \leq 2\pi$.

4. $(\cos 2t, \sin 3t)$, where $0 \le t \le 2\pi$.
5. $(t\cos t, t\sin t)$, where $0 \le t \le 2\pi$.

Given a curve, it can be very difficult to find a parameterization. There are many ways of approaching the problem, but nothing which always works. Here are a few hints:

1. If C is the graph of a function $y = f(x)$, then $\phi(t) = (t, f(t))$ is a parameterization of C. Notice that the y-coordinate of every point in the image of this parameterization is obtained from the x-coordinate by applying the function f.
2. If one has a polar equation for a curve like $r = f(\theta)$, then, since $x = r\cos\theta$ and $y = r\sin\theta$, we get a parameterization of the form $\phi(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$.

Example 5. The top half of a circle of radius 1 is the graph of $y = \sqrt{1-x^2}$. Hence, a parameterization for this is $(t, \sqrt{1-t^2})$, where $-1 \le t \le 1$. This figure is also the graph of the polar equation $r = 1, 0 \le \theta \le \pi$, hence the parameterization $(\cos t, \sin t)$, where $0 \le t \le \pi$.

2.19. Sketch and find parameterizations for the curves described by the following:

1. The graph of the polar equation $r = \cos\theta$.
2. The graph of $y = \sin x$.
3. The set of points such that $x = \sin y$.

2.20. Find a parameterization for the line segment which connects the point $(1, 1)$ to the point $(2, 5)$.

The derivative of a parameterization $\phi(t) = (f(t), g(t))$ is defined to be the vector

$$\phi'(t) = \frac{d\phi}{dt} = \frac{d}{dt}(f(t), g(t)) = \langle f'(t), g'(t)\rangle.$$

This vector has important geometric significance. The slope of a line containing this vector when $t = t_0$ is the same as the slope of the line tangent to the curve at the point $\phi(t_0)$. The magnitude (length) of this vector gives one a concept of the *speed* of the point $\phi(t)$ as t is increases through t_0. For convenience, one often draws the vector $\phi'(t_0)$ based at the point $\phi(t_0)$ (see Figure 2.6).

2.21. Let $\phi(t) = (\cos t, \sin t)$ (where $0 \le t \le \pi$) and $\psi(t) = (t, \sqrt{1-t^2})$ (where $-1 \le t \le 1$) be parameterizations of the top half of the unit circle. Sketch the vectors $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt}$ at the points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(0, 1)$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

2.22. Let C be the set of points in $\mathbb{R}^2$ that satisfy the equation $x = y^2$.

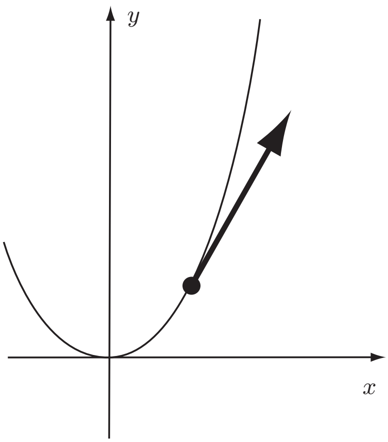

Fig. 2.6. The derivative of the parameterization $\phi(t) = (t, t^2)$ is the vector $\langle 1, 2t\rangle$. When $t = 1$, this is the vector $\langle 1, 2\rangle$, which we picture based at the point $\phi(1) = (1, 1)$.

1. Find a parameterization for C.
2. Find a tangent vector to C at the point $(4, 2)$.

We now move on to parameterized curves in $\mathbb{R}^3$. We begin with an example.

Example 6. The function $\phi(t) = (\cos t, \sin t, t)$ parameterizes a curve that spirals upward around a cylinder of radius 1.

2.23. Describe the difference between the curves with the following parameterizations:

1. $(\cos t^2, \sin t^2, t^2)$.
2. $(\cos t, \sin t, t^2)$.
3. $(t \cos t, t \sin t, t)$.
4. $(\cos \frac{1}{t}, \sin \frac{1}{t}, t)$.

2.24. Describe the lines given by the following parameterizations:

1. $(t, 0, 0)$.
2. $(0, 0, t)$.
3. $(0, t, t)$.
4. (t, t, t).

In $\mathbb{R}^2$ we saw that a parameterization for a curve could be found if you first use polar coordinates to describe it, then solve for r or θ, and then translate to rectangular coordinates. To find a parameterization of a curve in $\mathbb{R}^3$, an effective strategy is to find some way to “eliminate” *two* coordinates (in some coordinate system) and then translate into rectangular coordinates.

By “eliminating” a coordinate we mean either expressing it as some constant or expressing it as a function of the third, unknown coordinate.

Example 7. We demonstrate two ways to parameterize one of the lines that is at the intersection of the cone $z^2 = x^2 + y^2$ and the plane $y = 2x$. The coordinate y is already expressed as a function of x. To express z as a function of x, we substitute $2x$ for y in the first equation. This gives us $z^2 = x^2+(2x)^2 = 5x^2$, or $z = \sqrt{5}x$ (the negative root would give us the other intersection line). Hence, we get the parameterization

$$\phi(x) = (x, 2x, \sqrt{5}x).$$

Another way to describe this line is with spherical coordinates. Note that for every point on the line, $\phi = \frac{\pi}{4}$ (from the first equation) and $\theta = \tan^{-1} 2$ (because $\tan\theta = \frac{y}{x} = 2$, from the second equation). Converting to rectangular coordinates then gives us

$$\phi(\rho) = \left(\rho \sin\frac{\pi}{4}\cos(\tan^{-1} 2), \rho\sin\frac{\pi}{4}\sin(\tan^{-1} 2), \rho\cos\frac{\pi}{4}\right),$$

which simplifies to

$$\psi(\rho) = \left(\frac{\sqrt{10}\rho}{10}, \frac{\sqrt{10}\rho}{5}, \frac{\sqrt{2}\rho}{2}\right).$$

Note that dividing the first parameterization by $\sqrt{10}$ and simplifying yields the second parameterization.

2.25. Find a parameterization for the curve that is at the intersection of the plane $x + y = 1$ and the cone $z^2 = x^2 + y^2$.

2.26. Find two parameterizations for the circle that is at the intersection of the cylinder $x^2 + y^2 = 4$ and the paraboloid $z = x^2 + y^2$.

2.27. Parameterize the curve that lies on a sphere of radius 1 such that $\theta = \phi$.

2.5 Parameterized surfaces in $\mathbb{R}^3$

A parameterization for a surface S in $\mathbb{R}^3$ is a (one-to-one, onto, differentiable) function from some subset of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is S.

Example 8. The function $\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, where (u, v) lies inside a disk of radius 1, is a parameterization for the top half of the unit sphere.

One of the best ways to parameterize a surface is to find an equation *in some coordinate system* which can be used to eliminate one unknown coordinate. Then translate back to rectangular coordinates.

Example 9. An equation for the top half of the sphere in cylindrical coordinates is $r^2 + z^2 = 1$. Solving for z then gives us $z = \sqrt{1 - r^2}$. Translating to rectangular coordinates, we have

$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$z = \sqrt{1 - r^2}.$$

Hence, a parameterization is given by the function

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{1 - r^2}),$$

where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

Example 10. The equation $\rho = \phi$ describes some surface in spherical coordinates. Translating to rectangular coordinates then gives us

$$x = \rho\sin\rho\cos\theta,$$

$$y = \rho\sin\rho\sin\theta,$$

$$z = \rho\cos\rho.$$

Hence, a parameterization for this surface is given by

$$\phi(\rho, \theta) = (\rho\sin\rho\cos\theta, \rho\sin\rho\sin\theta, \rho\cos\rho).$$

2.28. Find parameterizations of the surfaces described by the equations in Problem 2.14.

2.29. Find a parameterization for the graph of an equation of the form $z = f(x, y)$.

2.30. Parameterize the portion of the graph of $z = 4 - x^2 - y^2$ that lies above the set of points in the xy-plane that are inside

1. the rectangle with vertices at $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$.
2. the circle of radius 1, centered on the origin.

2.31. Use the rectangular, cylindrical and spherical equations found in Problem 2.15 to parameterize the surfaces described there.

2.32. Use spherical coordinates to find a parameterization for the portion of the sphere of radius 2, centered at the origin, which lies below the graph of $z = r$ and above the xy-plane.

2.33. Sketch the surfaces given by the following parameterizations:

1. $\psi(\theta, \phi) = (\phi\sin\phi\cos\theta, \phi\sin\phi\sin\theta, \phi\cos\phi)$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$.

2. $\phi(r,\theta) = (r\cos\theta, r\sin\theta, \cos r)$, $0 \le r \le 2\pi$, $0 \le \theta \le 2\pi$.

Just as we could differentiate parameterizations of curves in $\mathbb{R}^2$, we can also differentiate parameterizations of surfaces in $\mathbb{R}^3$. In general, such a parameterization for a surface S can be written as

$$\phi(u,v) = (f(u,v), g(u,v), h(u,v)).$$

Thus, there are two variables we can differentiate with respect to: u and v. Each of these gives a vector which is tangent to the parameterized surface:

$$\frac{\partial \phi}{\partial u} = \left\langle \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right\rangle,$$

$$\frac{\partial \phi}{\partial v} = \left\langle \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right\rangle.$$

The vectors $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ determine a plane which is tangent to the surface S at the point $\phi(u,v)$.

2.34. Suppose some surface is described by the parameterization

$$\phi(u,v) = (2u, 3v, u^2 + v^2).$$

Find two (nonparallel) vectors which are tangent to this surface at the point $(4,3,5)$.

2.35. Consider the parameterization

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{r}),$$

where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

1. Sketch the surface parameterized by ϕ.
2. Find two tangent vectors to this surface at the point $(0,1,1)$

2.6 Parameterized regions in $\mathbb{R}^2$ and $\mathbb{R}^3$

A parameterization of a region R in $\mathbb{R}^n$ is a (differentiable, one-to-one) function $f : \mathbb{R}^n \to \mathbb{R}^n$ whose image is R. A change of coordinates, for example, can give a parameterization of a region.

Example 11. A parameterization for the disk of radius 1 (i.e., the set of points in $\mathbb{R}^2$ which are at a distance of at most 1 from the origin) is given using polar coordinates:

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta),\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Note that the interesting part of this parameterization is the specification of the domain. Without this, we are just giving the translation from polar to rectangular coordinates. It is the restrictions $0 \le r \le 1$, $0 \le \theta \le 2\pi$ that give us just the points inside the disk of radius 1.

2.36. Let B be the ball of radius 1 in $\mathbb{R}^3$ (i.e., the set of points satisfying $x^2 + y^2 + z^2 \le 1$).

1. Use spherical coordinates to find a parameterization for B.
2. Find a parameterization for the intersection of B with the first octant.
3. Find a parameterization for the intersection of B with the octant where x, y, and z are all negative.

2.37. The "solid cylinder" of height 1 and radius r in $\mathbb{R}^3$ is the set of points inside the cylinder $x^2 + y^2 = r^2$ and between the planes $z = 0$ and $z = 1$.

1. Use cylindrical coordinates to find a parameterization for the solid cylinder of height 1 and radius 1.
2. Find a parameterization for the region that is inside the solid cylinder of height 1 and radius 2 and outside the cylinder of radius 1.

Example 12. A common type of region to integrate over is one that is bounded by the graphs of two functions. Suppose R is the region in $\mathbb{R}^2$ above the graph of $y = g_1(x)$, below the graph of $y = g_2(x)$ and between the lines $x = a$ and $x = b$. A parameterization for R (check this!) is given by

$$\phi(x,t) = (x, tg_2(x) + (1-t)g_1(x)), \; a \le x \le b, \; 0 \le t \le 1.$$

2.38. Let R be the region between the (polar) graphs of $r = f_1(\theta)$ and $r = f_2(\theta)$, where $a \le \theta \le b$. Find a parameterization for R.

2.39. Find a parameterization for the region in $\mathbb{R}^2$ bounded by the ellipse whose x-intercepts are 3 and -3 and y-intercepts are 2 and -2. (Hint: Start with the parameterization given in Example 11.)

2.40.

1. Sketch the region in $\mathbb{R}^2$ parameterized by
$$\phi(r,\theta) = (2r\cos\theta, r\sin\theta),$$
where $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$.
2. Sketch the region in $\mathbb{R}^3$ parameterized by
$$\psi(\rho,\theta,\phi) = (2\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi),$$
where $0 \le \rho \le 1$, $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \pi$.

2.41. Consider the following parameterization:

$$\psi(r,\theta) = (2r\cos\theta, r\sin\theta + 1),$$

where $0 \le r \le 1$ and $0 \le \theta \le \pi$.

1. Sketch the region of $\mathbb{R}^2$ parameterized by ψ.
2. Find the vectors $\frac{\partial\psi}{\partial r}$ and $\frac{\partial\psi}{\partial\theta}$
3. Find the area of the parallelogram spanned by $\frac{\partial\psi}{\partial r}$ and $\frac{\partial\psi}{\partial\theta}$.