1

Introduction

1.1 So what *is* a differential form?

A differential form is simply this: an integrand. In other words, it is a thing which can be integrated over some (often complicated) domain. For example, consider the following integral: $\int_0^1 x^2 dx$. This notation indicates that we are integrating x^2 over the interval $[0,1]$. In this case, $x^2 dx$ is a differential form. If you have had no exposure to this subject, this may make you a little uncomfortable. After all, in calculus we are taught that x^2 is the integrand. The symbol "dx" is only there to delineate when the integrand has ended and what variable we are integrating with respect to. However, as an object in itself, we are not taught any meaning for "dx." Is it a function? Is it an operator on functions? Some professors call it an "infinitesimal" quantity. This is very tempting. After all, $\int_0^1 x^2 dx$ is defined to be the limit, as $n \to \infty$, of $\sum_{i=1}^{n} x_i^2 \Delta x$, where $\{x_i\}$ are n evenly spaced points in the interval $[0,1]$ and $\Delta x = 1/n$. When we take the limit, the symbol "$\sum$" becomes "$\int$," and the symbol "Δx" becomes "dx." This implies that $dx = \lim_{\Delta x \to 0} \Delta x$, which is absurd. $\lim_{\Delta x \to 0} \Delta x = 0$!! We are not trying to make the argument that the symbol "dx" should be eliminated. It does have meaning. This is one of the many mysteries that this book will reveal.

One word of caution here: Not all integrands are differential forms. In fact, in Section 4.8 we will see how to calculate arc length and surface area. These calculations involve integrands which are not differential forms. Differential forms are simply natural objects to integrate and also the first that one should study. As we will see, this is much like beginning the study of all functions by understanding linear functions. The naive student may at first object to this, since linear functions are a very restrictive class. On the other hand, eventually we learn that any differentiable function (a much more general class) can be

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locally approximated by a linear function. Hence, in some sense, the linear functions are the most important ones. In the same way, one can make the argument that differential forms are the most important integrands.

1.2 Generalizing the integral

Let's begin by studying a simple example and trying to figure out how and what to integrate. The function $f(x, y) = y^2$ maps $\mathbb{R}^2$ to $\mathbb{R}$. Let M denote the top half of the circle of radius 1, centered at the origin. Let's restrict the function f to the domain, M, and try to integrate it. Here we encounter our first problem: The given description of M is not particularly useful. If M were something more complicated, it would have been much harder to describe it in words as we have just done. A parameterization is far easier to communicate and far easier to use to determine which points of $\mathbb{R}^2$ are elements of M and which are not. However, there are lots of parameterizations of M. Here are two which we will use:

$\phi_1(a) = (a, \sqrt{1 - a^2})$, where $-1 \leq a \leq 1$, and

$\phi_2(t) = (\cos t, \sin t)$, where $0 \leq t \leq \pi$.

Here is the trick: Integrating f over M is difficult. It may not even be clear what this means. However, perhaps we can use ϕ_1 to translate this problem into an integral over the interval $[-1, 1]$. After all, an integral is a big sum. If we add up all the numbers $f(x, y)$ for all the points, (x, y), of M, shouldn't we get the same thing as if we added up all the numbers $f(\phi_1(a))$ for all the points, a, of $[-1, 1]$ (see Figure 1.1)?

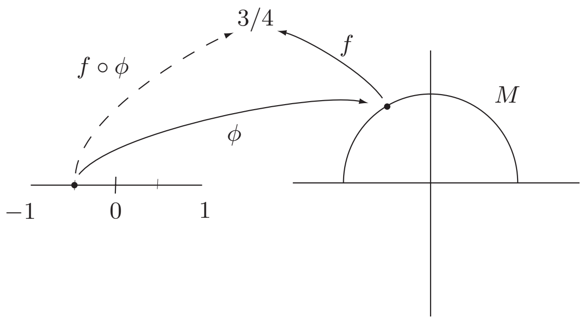

Fig. 1.1. Shouldn't the integral of f over M be the same as the integral of $f \circ \phi$ over $[-1, 1]$?

Let's try it. $\phi_1(a) = (a, \sqrt{1 - a^2})$, so $f(\phi_1(a)) = 1 - a^2$. Hence, we are saying that the integral of f over M should be the same as $\int\limits_{-1}^{1} (1 - a^2)\ da$. Using a little calculus, we can determine that this evaluates to $4/3$.

Let's try this again, this time using ϕ_2. Using the same argument, the integral of f over M should be the same as $\int_0^\pi f(\phi_2(t))\ dt = \int_0^\pi \sin^2 t\ dt = \pi/2$.

Hold on! The problem was stated *before* any parameterizations were chosen. Shouldn't the answer be independent of which one was picked? It would not be a very meaningful problem if two people could get different correct answers, depending on how they went about solving it. Something strange is going on!

In order to understand what happened, we must first review the definition of the Riemann Integral. In the usual definition of the integral, the first step is to divide the interval up into n evenly spaced subintervals. Thus, $\int_a^b f(x)\ dx$ is defined to be the limit, as $n \to \infty$, of $\sum_{i=1}^{n} f(x_i)\Delta x$, where $\{x_i\}$ are n evenly spaced points in the interval $[a, b]$ and $\Delta x = (b - a)/n$. Hpowever, what if the points $\{x_i\}$ are not evenly spaced? We can still write down a reasonable sum: $\sum_{i=1}^{n} f(x_i)\Delta x_i$, where now $\Delta x_i = x_{i+1} - x_i$. In order to make the integral well defined, we can no longer take the limit as $n \to \infty$. Instead, we must let $\max\{\Delta x_i\} \to 0$. It is a basic result of analysis that if this limit converges, then it does not matter how we picked the points $\{x_i\}$; the limit will converge to the same number. It is this number that we define to be the value of $\int_a^b f(x)\ dx$.

1.3 What went wrong?

We are now ready to figure out what happened in Section 1.2. Obviously, $\int_{-1}^{1} f(\phi_1(a))\ da$ was not what we wanted. Let's not give up on our general approach just yet; it would still be great if we could use ϕ_1 to find *some* function that we can integrate on $[-1, 1]$ that will give us the same answer as the integral of f over M. For now, let's call this mystery function "$F(a)$."

Let's look at the Riemann Sum that we get for $\int_{-1}^{1} F(a)\ da$ when we divide the interval up into n pieces, each of width Δa: $\sum_{i=1}^{n} F(a_i)\Delta a$. Examine Figure 1.2 to see what happens to the points a_i under the function, ϕ_1. Notice that the points $\{\phi_1(a_i)\}$ are not evenly spaced along M. To use these points to estimate the integral of f over M, we would have to use the approach from the previous section. A Riemann Sum for f over M would be $\sum_{i=1}^{n} f(\phi_1(a_i))l_i$, where the l_i represent the arc length, along M, between $\phi_1(a_i)$ and $\phi_1(a_{i+1})$.

This is a bit problematic, however, since arc length is generally hard to calculate. Instead, we can approximate l_i by substituting in the length of the

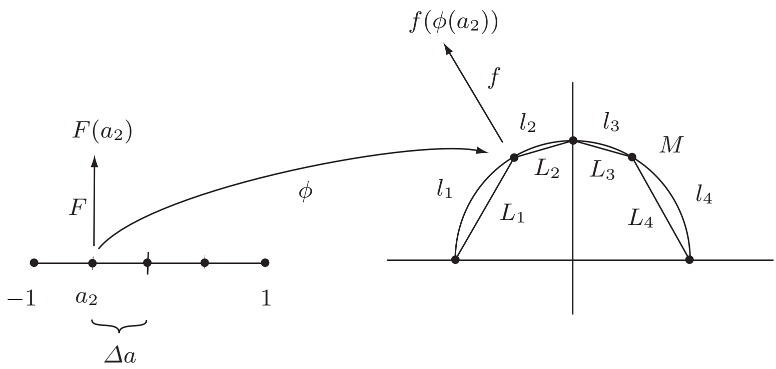

Fig. 1.2. We want $\sum_{i=1}^{n} F(a_i)\Delta a = \sum_{i=1}^{n} f(\phi_1(a_i))L_i$.

line segment which connects $\phi_1(a_i)$ to $\phi_1(a_{i+1})$, which we will denote as L_i. Note that this approximation gets better and better as we let $n \to \infty$. Hence, when we take the limit, it does not matter if we use l_i or L_i.

So our goal is to find a function, $F(a)$, on the interval $[-1, 1]$ so that

$$\sum_{i=1}^{n} F(a_i)\Delta a = \sum_{i=1}^{n} f(\phi_1(a_i))L_i.$$

Of course this equality will hold if $F(a_i)\Delta a = f(\phi_1(a_i))L_i$. Solving, we get $F(a_i) = \frac{f(\phi_1(a_i))L_i}{\Delta a}$.

What happens to this function as $\Delta a \to 0$? First, note that $L_i = |\phi_1(a_{i+1}) - \phi_1(a_i)|$. Hence,

$$\begin{aligned}
\lim_{\Delta a \to 0} F(a_i) &= \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))L_i}{\Delta a} \\
&= \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))|\phi_1(a_{i+1}) - \phi_1(a_i)|}{\Delta a} \\
&= f(\phi_1(a_i)) \lim_{\Delta a \to 0} \frac{|\phi_1(a_{i+1}) - \phi_1(a_i)|}{\Delta a} \\
&= f(\phi_1(a_i)) \left| \lim_{\Delta a \to 0} \frac{\phi_1(a_{i+1}) - \phi_1(a_i)}{\Delta a} \right|.
\end{aligned}$$

However, $\lim_{\Delta a \to 0} \frac{\phi_1(a_{i+1}) - \phi_1(a_i)}{\Delta a}$ is precisely the definition of the derivative of ϕ_1 at a_i, $\frac{d\phi_1}{da}(a_i)$. Hence, we have $\lim_{\Delta a \to 0} F(a_i) = f(\phi_1(a_i)) \left| \frac{d\phi_1}{da}(a_i) \right|$. Finally, this means that the integral we want to compute is $\int_{-1}^{1} f(\phi_1(a)) \left| \frac{d\phi_1}{da} \right| da$.

1.1. Check that $\int_{-1}^{1} f(\phi_1(a))|\frac{d\phi_1}{da}|\ da = \int_{0}^{\pi} f(\phi_2(t))|\frac{d\phi_2}{dt}|\ dt$, using the function, f, defined in Section 1.2.

Recall that $\frac{d\phi_1}{da}$ is a vector, based at the point $\phi(a)$, tangent to M. If we think of a as a time parameter, then the length of $\frac{d\phi_1}{da}$ tells us how fast $\phi_1(a)$ is moving along M. How can we generalize the integral, $\int_{-1}^{1} f(\phi_1(a))|\frac{d\phi_1}{da}|\ da$? Note that the bars $|\cdot|$ denote a function that "eats" vectors and "spits out" real numbers. So we can generalize the integral by looking at other such functions. In other words, a more general integral would be $\int_{-1}^{1} f(\phi_1(a))\omega(\frac{d\phi_1}{da})\ da$, where f is a function of points and ω is a function of vectors.

It is not the purpose of the present work to undertake a study of integrating with respect to all possible functions, ω. However, as with the study of functions of real variables, a natural place to start is with *linear* functions. This is the study of differential forms. A differential form is precisely a linear function which eats vectors, spits out numbers and is used in integration. The strength of differential forms lies in the fact that their integrals do not depend on a choice of parameterization.

1.4 What about surfaces?

Let's repeat the previous discussion (faster this time), bumping everything up a dimension. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = z^2$. Let M be the top half of the sphere of radius 1, centered at the origin. We can parameterize M by the function ϕ, where $\phi(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1 - r^2}), 0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Again, our goal is not to figure out how to actually integrate f over M but to use ϕ to set up an equivalent integral over the rectangle $R = [0, 1] \times [0, 2\pi]$. See Figure 1.3 for an illustration of how we do this.

Let $\{x_{i,j}\}$ be a lattice of evenly spaced points in R. Let $\Delta r = x_{i+1,j} - x_{i,j}$, and $\Delta\theta = x_{i,j+1} - x_{i,j}$. By definition, the integral over R of a function, $F(x)$, is equal to $\lim_{\Delta r, \Delta\theta \to 0} \sum F(x_{i,j})\Delta r\Delta\theta$.

To use the mesh of points, $\phi(x_{i,j})$, in M to set up a Riemann Sum, we write down the following sum: $\sum f(\phi(x_{i,j}))\text{Area}(L_{i,j})$, where $L_{i,j}$ is the rectangle spanned by the vectors $\phi(x_{i+1,j}) - \phi(x_{i,j})$ and $\phi(x_{i,j+1}) - \phi(x_{i,j})$. If we want our Riemann Sum over R to equal this sum, then we end up with $F(x_{i,j}) = \frac{f(\phi(x_{i,j}))\text{Area}(L_{i,j})}{\Delta r\Delta\theta}$.

We now leave it as an exercise to show that as Δr and $\Delta\theta$ get small, $\frac{\text{Area}(L_{i,j})}{\Delta r\Delta\theta}$ converges to the area of the parallelogram spanned by the vectors $\frac{\partial\phi}{\partial r}(x_{i,j})$ and $\frac{\partial\phi}{\partial\theta}(x_{i,j})$. The upshot of all this is that the integral we want to evaluate is the following:

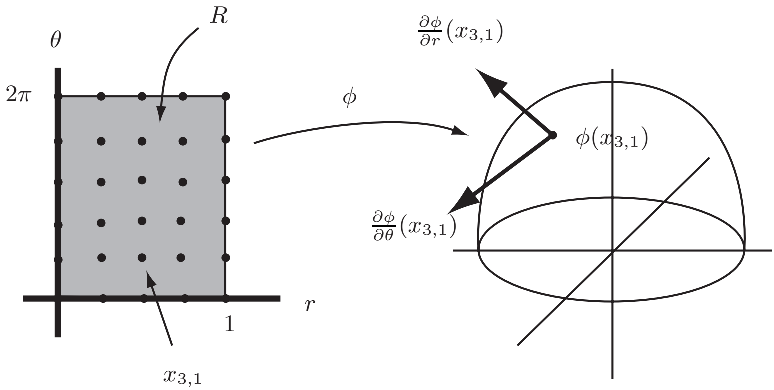

Fig. 1.3. Setting up the Riemann Sum for the integral of z^2 over the top half of the sphere of radius 1.

$$\int_R f(\phi(r,\theta))\text{Area}\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}\right) dr\ d\theta.$$

The point of all this is not the specific integral that we have arrived at, but the *form* of the integral. We integrate $f \circ \phi$ (as in the previous section), times a function which takes *two* vectors and returns a real number. Once again, we can generalize this by using other such functions:

$$\int_R f(\phi(r,\theta))\omega\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}\right) dr\ d\theta.$$

In particular, if we examine linear functions for ω, we arrive at a differential form. The moral is that if we want to perform an integral over a region parameterized by $\mathbb{R}$, as in the previous section, then we need to multiply by a function which takes a vector and returns a number. If we want to integrate over something parameterized by $\mathbb{R}^2$, then we need to multiply by a function which takes *two* vectors and returns a number. In general, an n-form is a linear function which takes n vectors and returns a real number. One integrates n-forms over regions that can be parameterized by $\mathbb{R}^n$. Their strength is that the value of such an integral does not depend on the choice of parameterization.