10

The Theory of Voting

Voting and elections are part of everyday life. We vote for our representatives in government, and those representatives vote on legislation. We vote for our favorites in television talent quests; committees vote to decide who to hire. Sports leagues have a structure like that of elections (instead of a vote deciding that one team is better than another, we use the result of a match, but the structure is very similar).

When we look more closely at electoral methods, we see that the mathematical structure is very important. We shall see that two different systems, each of which sounds perfectly fair, can give different results. Sometimes groups of voters can manipulate the system to favor their candidates.

Not every election will have a result; it is always possible that the voters like two candidates equally well, and there is a tie. But we shall also see some methods that sound very reasonable, but do not result in the election of a candidate even when there are no ties.

10.1 Simple Elections

Simple Elections

The simplest form of election is where there are a number of candidates for office, and one is to be elected. There is a well defined set of voters—the *electorate*—and each voter casts one vote.

Suppose two people are running for an office. After each person makes his or her one vote, the person who gets more than half the votes wins. This is called the *majority* or *absolute majority* method. Ties are possible—ties are possible in any electoral system—but apart from this the absolute majority method always produces a result.

W.D. Wallis, *A Beginner's Guide to Discrete Mathematics*,
DOI 10.1007/978-0-8176-8286-6_10,

If there are three or more candidates, the majority method is not so good; there may quite easily be no winner. Several schemes have been devised that allow a candidate with an absolute majority to be elected, and try to find a good approximation when there is no "absolute" winner. These are called *majoritarian* or *plurality* systems.

The first generalization is the *plurality* or *simple majority* method; it is often called "first-past-the-post" voting. Each voter makes one vote, and the person who receives the most votes wins. For example, if there were three candidates, A, B, and C and 70 voters, the absolute majority method requires 36 votes for a winner. If A received 30 votes and B and C each got 20, there will be no winner under that method. Under plurality, A would be elected.

The problem with the plurality method is that the winner might be very unpopular with a majority of voters. In our example, suppose all the supporters of B and C thought that both these candidates were better qualified than A. Then the plurality method results in the election of the candidate that the majority thought was the worst possible choice. This problem is magnified if there are more candidates; even if there are only four or five candidates, people often think the plurality method elects the wrong person.

Sequential Voting

One technique used to avoid the problems of the plurality method is sequential voting. In this scheme a vote is taken. As a consequence a new set of candidates is selected; then a new vote is taken. The aim is to reduce the set of candidates to a manageable size—often to size two. The original election is called a *primary.*

For example, when the city of Carbondale, Illinois votes for Mayor, there is a primary election for all the Mayoral candidates. This is run like an ordinary election. Later there is another election; the candidates in the final election are the two candidates who received the most votes in the primary. This second (*runoff*) election is decided by the majority method. A similar method is used in electing the presidents of France and the Ukraine and in a number of other situations.

We shall refer to this as the *runoff method* or *plurality runoff method.* The two top candidates are decided by plurality vote; all other candidates are eliminated. Then a majority vote is taken.

In the real world there is usually a delay after the primary and more campaigning takes place. For simplicity's sake we shall ignore this. We assume that every voter has an order of preference between the candidates that remains fixed throughout the voting process. We define the *preference profile* of an election to be the set of all the voters' preference lists. This can conveniently be written in a table. For example, say there are three candidates, A, B, C and suppose

5 voters like A best, then B, then C;
7 voters like B best, then A, then C;
4 voters like A best, then C, then B;
3 voters like C best, then B, then A;
no voters like B, then C, then A;
no voters like C, then B, then A.

We can represent this as

5	7	4	3
A	B	A	C
B	A	C	B
C	C	B	A

(We can also write

5	7	4	3	0	0
A	B	A	C	B	C
B	A	C	B	C	A
C	C	B	A	A	B

but we shall usually omit zero columns.)

In a simple election, each voter votes for the candidate he or she likes best. In the example, A would receive nine votes—five from those with preference list ABC, and four from those with list ACB. In general, a candidate receives the votes of those who put that candidate first in the preference list.

Sample Problem 10.1. *Suppose the preference profile of an election is*

5	7	4	3
A	B	A	C
B	A	C	B
C	C	B	A

What is the result of the election in the following cases?

(i) *The majority method is used.*

(ii) *The plurality method is used.*

(iii) *The runoff method is used.*

Solution. A receives nine votes, B receives seven, C receives three. So (i) there is no majority winner (as there are 19 voters, 10 votes would be needed), and (ii) A is the plurality winner. In a primary election, A and B are selected to contest the runoff. For the runoff, C is deleted, so the preference profile is

5	7	4	3
A	*B*	*A*	*B*
B	*A*	*B*	*A*

or (combining columns with the same preference list)

9	10
A	*B*
B	*A*

So *B* wins the runoff.

Practice Exercise. Repeat this question for an election with preference profile

7	5	8	3	4
A	*A*	*B*	*B*	*C*
B	*C*	*A*	*C*	*B*
C	*B*	*C*	*A*	*A*

The Hare Method

The *Hare method* or *alternative vote* system was invented by the English lawyer Sir Thomas Hare in 1859. It is most useful in its more general form, for situations where several representatives are to be elected at once (see Section 10.2). In this section we look at the simpler version.

The Hare method *requires* each voter to provide a preference list at the election. This list is called a *ballot*. The candidate with the fewest first place votes is eliminated. Then the votes are tabulated again as if there were one fewer candidate, and again the one with the fewest first place votes in this new election is eliminated. When only two remain, the winner is decided by a majority vote.

Sample Problem 10.2. *Suppose there are four candidates for a position, and 24 voters whose preference profile is as follows:*

5	7	4	3	3	2
A	*B*	*A*	*C*	*D*	*D*
C	*D*	*D*	*D*	*C*	*C*
D	*A*	*C*	*B*	*A*	*B*
B	*C*	*B*	*A*	*B*	*A*

Who would win using the following electoral systems?

(i) *The plurality method.*

(ii) *The runoff method.*

(iii) *The Hare method.*

Solution. (i) The votes for A, B, C, and D are 9, 7, 3, and 5, respectively, so A would win under plurality voting.

(ii) Under the runoff method there is a tie. A and B are retained and the new preference profile is as follows:

5	7	4	3	3	2
A	B	A	B	A	B
B	A	B	A	B	A

giving 12 votes to each candidate.

(iii) In the Hare method we first eliminate C, obtaining

5	7	4	3	3	2
A	B	A	D	D	D
B	D	D	B	A	B
D	A	B	A	B	A

Now we eliminate B:

5	7	4	3	3	2
A	D	A	D	D	D
D	A	D	A	A	A

So D wins 15–9.

Practice Exercise. Repeat the above question for the initial preference profile

6	7	7	7	2	7	5	2
A	B	A	C	D	D	B	D
C	D	D	D	C	C	D	B
D	A	C	A	A	B	C	C
B	C	B	B	B	A	A	A

The Condorcet Method

A multiple use of runoff elections was discussed by Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet, an eighteenth Century French mathematician and political theorist. (Similar ideas were proposed by Ramon Llull as long ago as 1299.)

Suppose we simultaneously conduct all the "runoff" elections among our candidates. For example, in the election discussed in Sample Problem 10.2, there are six runoffs: A versus B, A versus C, A versus D, B versus C, B versus D, and C versus D. If any one candidate wins all his/her runoffs, then surely you would consider that person a winner. Such a candidate is a *Condorcet winner.*

In Sample Problem 10.1, we find:

B beats A 10–9,
A beats C 16–3,
B beats C 12–7,

so B is a Condorcet winner. Similarly, in Sample Problem 10.2, D is a Condorcet winner. But even in the simple example

3	2	2
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*

A beats B 5–2, B beats C 5–2, and C beats A 4–3, so there is no Condorcet winner.

In elections with several candidates, it is very common to have no Condorcet winner, even when there are no ties. This is a serious fault in the Condorcet method.

Sample Problem 10.3. *Consider the election with preference profile*

5	4	3
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*

Who would win under the Hare method? Is there a Condorcet winner?

Solution. The votes for A, B, C, and D are 5, 4, and 3, respectively. Under the Hare method C is eliminated. The new preference profile is

5	4	3
A	*B*	*A*
B	*A*	*B*

So A wins 8–4. Looking at all three runoffs, we see that A beats B 8–4, B beats C 9–3, and C beats A 7–5, so there is no Condorcet winner.

Point Methods

Pointscore methods have often been used in sporting contests. For example, they are commonly used in track meets and in motor racing. When the Olympic Games are being held, many newspapers publish informal medal tallies to rank the performance of the competing nations—the usual method is to allocate three points for a gold medal, two for a silver, one for a bronze, and then add.

In general, a fixed number of points are given for first, second, and so on. The points are totalled and the candidate with the most points wins. If there are n competitors, a common scheme is to allocate n points to first, $n-1$ to second, ..., or equivalently $n-1$ to first, $n-2$ to second, ... This case, where the points go in uniform steps, is called a *Borda count.*

One often see scales like 5, 3, 2, 1, where the winner gets a bonus, or 3, 2, 1, 0, 0, ... (that is, all below a certain point are equal). Sometimes more complicated schemes are used; for example, in the Indy Racing League, the following system has been used:

1st gets	20	5th gets	10	9th gets	4
2nd gets	16	6th gets	8	10th gets	3
3rd gets	14	7th gets	6	11th gets	2
4th gets	12	8th gets	5	12th gets	1

Fastest qualifier gets one point.
Leader of most laps gets one point.

Pointscore methods are occasionally employed for elections, most often for small examples such as the selection of the best applicant for a job.

Sometimes the result depends on the point scheme chosen.

Sample Problem 10.4. *What is the result of an election with preference table*

5	7	4	3
A	B	A	C
C	C	C	B
B	A	B	A

if a 3, 2, 1 *count is used? What is the result if a* 4, 2, 1 *count is used?*

Solution. With a 3, 2, 1 count the totals are $A: 37$, $B: 36$, $C: 41$, so C wins. With a 4, 2, 1 count the totals are $A: 46$, $B: 43$, $C: 44$, and A wins.

Exercises 10.1

1. Eighty-six electors vote between candidates A, B, and C. Their votes are 25 for A, 48 for B, and 13 for C. What is the result under the majority method? What is the result under the plurality method?
2. Twenty-eight electors vote between candidates A, B, and C. Their votes are 5 for A, 14 for B, and 9 for C. What is the result under the majority method? What is the result under the plurality method?

3. Seventy-five electors vote between candidates A, B, and C. Twenty-five vote for A and 28 vote for B. How many votes did C receive? What is the result under the majority method? What is the result under the plurality method?

4. Fifty-four electors vote between candidates A, B, and C. A receives 24 votes while B and C received equal numbers. How many did they each receive? What is the result under the majority method? What is the result under the plurality method?

5. Suppose 73 voters must decide between candidates A, B, C. There are 14 votes for A. How many votes must B receive to win if the majority method is used? How many will suffice under the plurality method?

6. Suppose 50 voters must decide between candidates X, Y, Z. There are 20 votes for X. How many votes must Y receive to win if the majority method is used? How many will suffice under the plurality method?

7. First-year students in a university vote for Class president. There are three candidates, Smith, Jones, and Brown. The preference table is

25	27	14	22	24	11
S	S	J	J	B	B
J	B	S	B	S	J
B	J	B	S	J	S

(i) How many students voted?

(ii) How many first place votes did each candidate receive?

(iii) Who, if anybody, would win under the plurality method?

(iv) Who, if anybody, would win under the majority method?

8. At the Academy Awards there are three nominees for Best Actor: Arthur Andrews, Bob Brown, and Clive Carter. The preference table is

123	101	124	308	222	305
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

(i) How many actors voted?

(ii) How many first place votes did each candidate receive?

(iii) Who, if anybody, would win under the plurality method?

(iv) Who, if anybody, would win under the majority method?

9. In the election described in Exercise 10.1.7, who would win under the runoff method?

10. In the election described in Exercise 10.1.8, who would win under the runoff method?

11. Twenty-three electors vote for three candidates, resulting in the preference table

8	6	5	4
X	Y	Z	Y
Z	Z	X	X
Y	X	Y	Z

(i) Who, if anybody, would win under the majority method?

(ii) Who, if anybody, would win under the plurality method?

(iii) What would be the result if a Borda count were used?

(iv) Would any of these results change if the five voters who preferred Z were to exchange their second and third preferences?

12. In the election of Exercise 10.1.11, it is decided to use a scoring system where first place gets n points, second gets 2, and third gets 1 where n is some whole number greater than 2. For what ranges of values is X winner? What is the range for Y? For Z?

13. In how many ways can a voter rank three candidates if no ties are allowed? In how many ways can it be done if ties are allowed?

14. In how many ways can a voter rank four candidates if no ties are allowed? In how many ways can it be done if ties are allowed?

15. Suppose there are three candidates in an election. Is there any difference whether the runoff method or the Hare method is used?

16. Verify that, in the election described in Sample Problem 10.2, D is a Condorcet winner.

17. Fifteen students vote on three restaurants for a group dinner. Their choices are Chinese, Italian, and Thai. If their preference table is

6	5	4
C	I	T
I	T	I
I	C	C

(i) What will be their decision under the plurality method?

(ii) What will be their decision using a Borda count?

(iii) Is there a Condorcet winner?

18. Nineteen sports announcers rank the final four in a basketball tournament. The teams are Creighton (C), Drake (D), Evansville (E), and Southern Illinois (S). Their preference table is

9	3	2	5
S	C	E	C
C	S	D	E
E	D	C	D
D	E	S	S

Which team will they choose as the favorite if they use the following methods?

(i) Plurality.

(ii) A Borda count.

(iii) The Hare method.

19. There are four candidates for club president; we denote them A, B, C, D. The 17 members rank them as follows:

7	6	4
C	D	A
B	C	B
A	B	D
D	A	C

The club by-laws specify that the Hare system should be used.

(i) Who is elected?

(ii) Just before the election, candidate C withdraws from the race. Who is elected in that case?

20. Twenty-one electors wish to choose between four candidates A, B, C, D. The preference table is

8	9	4
C	B	A
B	C	B
A	D	C
D	A	D

(i) Who is elected under the plurality method?

(ii) Who is elected under the Hare method?

21. Suppose the electors in Exercise 10.1.20 decide to use a pointcount method.

(i) Who would win under the (4, 3, 2, 1) Borda count?

(ii) Who would win if a (5, 3, 1, 0) count is used?

22. A 13-member committee must select from candidates A, B, C, D. Their preferences are

6	3	4
A	C	C
B	B	D
D	A	B
C	D	A

(i) Who is elected under the plurality method?

(ii) Who is elected under the Hare method?

(iii) Who would win under a Borda count?

23. Given the following preference table, who would win under plurality voting? Who would win in a runoff?

6	3	4	2	1
A	C	C	B	E
E	B	D	A	A
B	E	A	C	B
C	D	E	D	C
D	A	B	E	D

Is there a Condorcet winner?

In Exercises 24 to 26, a preference table is given. Who would win in a Borda count? Is there a Condorcet winner?

24.

6	4	4
A	B	C
B	C	B
C	A	A

25.

3	4	2	5	4
A	A	B	B	C
B	C	A	C	B
C	B	C	A	A

26.

2	3	4	2
D	C	B	A
C	A	D	B
A	D	C	C
B	B	A	D

27. Construct a preference profile for four voters and four candidates such that three voters prefer W to X, three prefer X to Y, three prefer Y to Z, and three prefer Z to W.

10.2 Multiple Elections

The Generalized Hare Method

In this section we discuss cases where several candidates are to be elected simultaneously.

The Hare system was modified by Andrew Inglis Clark who was Attorney General of Tasmania in the late 19th century. This *generalized Hare method*, or *quota method*, is used in many countries, including Ireland, Australia, and Malta.

As an example, suppose 24,000 people are to elect five representatives from a larger number of candidates.

Each person makes a *preferential* vote, listing *all* the candidates in order. (In other words, they record all of their preference profiles, not just the first members.) The first name on the list is called the voter's *first preference*, and so on.

The *quota* is 4000 votes. (This number is chosen because you cannot have six candidates who each get more than 4000 first preferences.) Every candidate who gets more than 4000 votes is elected. If a candidate gets more than 4000 votes, the remainder of those votes go to voters' second choices, divided proportionally. To illustrate this, suppose A gets 5000 votes. Of these, 2500 have B as second choice, 2000 have C and 500 have D. Since this is greater than 4000, A is declared elected. The 1000 surplus votes are divided in the proportion 2500 : 2000 : 500, or 50% : 40% : 10%. That is, B gets 50% of A's preferences because 2500 is 50% of 5000; C gets 40%; and D gets 10%. As we said, A has exceeded the quota by 1000 votes, so we say he has 1000 preferences to be distributed. B gets 50% of these; 500 more votes are added to B's total. In the same way C gets 400 added votes (40%) and D gets 100 votes (10%).

Now we check whether B, C, or D has exceeded the quota. For example, if B previously had 3700 votes, the new total would be 4200, exceeding the quota again, so B is declared elected.

If not enough candidates have been elected, and no one has enough votes, the remaining candidate with the *fewest* votes is eliminated. *All* of that candidate's preferences are distributed (in just the same way as in the regular Hare method: no percentages need to be calculated).

In general, suppose there are V voters and N places to be filled. The quota is

$$\frac{V}{N+1},$$

and a candidate is declared to be elected if his or her number of votes exceeds the quota. Notice that, if only one candidate were to be elected, the quota requirement is that a candidate receive a majority of the votes. (Some systems use the formula $\frac{V}{N+1} + 1$ and say a candidate is elected if the quota is equalled or exceeded. This may require one more vote in the case where $\frac{V}{N+1}$ is not an integer.)

In a real example, the whole list of preferences is kept. The process may require a great deal of data; moreover it may result in complicated numbers, fractions, and so on. Historically this was a serious problem and caused long delays in announcing the results of elections, but it is no longer an issue now that computers are available and voting machines can be adapted to keep all the data.

For example, given a preference profile of the form

12	9	6	...	...
A	*A*	*A*	*B*	...
B	*B*	*C*	...	...
C	*D*	*B*	...	...
D	*C*	*D*	...	...

and a quota of 18, A's surplus is nine. The surplus is divided $7 : 2$ between B and C, but if we divide it $4 : 3 : 2$ between the first three columns later calculations will be easier.

Sample Problem 10.5. *Say there are five candidates for three positions; the preference table is*

6	6	9	6	3	2
A	*A*	*C*	*C*	*E*	*E*
B	*B*	*D*	*D*	*C*	*A*
E	*D*	*E*	*E*	*D*	*B*
D	*E*	*A*	*B*	*A*	*C*
C	*C*	*B*	*A*	*B*	*D*

Who will be elected?

Solution. There are 32 voters so the quota is 8. A gets 12 first place votes and C gets 15, both of which exceed the quota. So A and C are elected.

A has a surplus of 4. The second-place candidate in every case in B, but we observe that the votes are divided in proportion 6:6 between A's two lists, so we give two votes to each of A's lists. C is also at the top of two lists, and has a surplus of 7, divided $9 : 6$. This gives surplus allocations of 4.2 and 2.8. After these allocations, the new table is

2	2	4.2	2.8	3	2
B	*B*	*D*	*D*	*E*	*E*
E	*D*	*E*	*E*	*D*	*B*
D	*E*	*B*	*B*	*B*	*D*

B has four votes, D has 7, and E has 5. No one meets the quota. (Note that the quota does not change.) So B (who had the fewest votes) is eliminated. We now have

2	2	4.2	2.8	3	2
E	*D*	*D*	*D*	*E*	*E*
D	*E*	*E*	*E*	*D*	*D*

D now has 9 votes and *E* has 7, so *D* is elected. In total, *A*, *C*, and *D* are elected.

Practice Exercise. Repeat the above problem for preference profile

5	5	7	8	5	2
A	*A*	*B*	*C*	*D*	*E*
B	*E*	*D*	*D*	*C*	*A*
E	*B*	*E*	*E*	*E*	*B*
D	*D*	*A*	*B*	*A*	*C*
C	*C*	*C*	*A*	*B*	*D*

Approval Voting

Approval voting was first used in cases where the number of people to be elected was not fixed. For example, if a committee wants to co-opt a few more members to help organize a function, they might add all those who they think would make a positive contribution, not a fixed number. In that case a member might vote for every suitable candidate, casting as many votes as she wishes; all the votes are considered equal. The number of votes received by a candidate is called his *approval rating*. Candidates with an approval rating of at least 50% (or 60%, or some other agreed figure) are elected.

Sometimes there is a strict requirement that a certain number be elected, or a minimum or maximum is imposed. Ties are possible, and some sort of runoff procedure may be necessary. The following example illustrates these ideas.

Sample Problem 10.6. *Ten board members vote on eight candidates by approval. The candidates are A, B, C, D, E, F, G, H, and the board members are q, r, s, t, u,v, w, x, y, z. They vote as follows* (× *represents approval*)*:*

	q	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*
A	×	×	×		×		×	×		×
B		×	×	×	×	×	×	×	×	
C			×					×		
D	×		×	×	×	×	×	×		×
E	×	×	×	×	×			×		
F	×	×	×	×	×	×	×		×	×
G	×		×	×			×		×	
H		×		×	×	×				×

(i) *Which candidate is chosen if just one is to be elected?*

(ii) *Which candidate is chosen if the top four are to be elected?*

(iii) *Which candidate is chosen if the top two are to be elected?*

(iv) *Which candidate is chosen if at most four are to be elected and 80% approval is required?*

Solution. Here is a summary of the votes received:

$$A-7,\ B-8,\ C-2,\ D-8,\ E-7,\ F-9,\ G-5,\ H-7.$$

(i) F, who received the most votes.

(ii) A, B, D, F.

(iii) F is elected; there will need to be a runoff election between B and D.

(iv) B, D, and F are elected.

The array in the example is called an *approval table*; we shall always denote approval with a cross in such tables.

Approval voting is particularly useful for situations like the selection of new employees. In those circumstances there is usually a minimum requirement, and further applications may be called if not enough good applicants are available.

The Single Transferable Vote

The *single transferable vote* (or STV) system is a variation of the generalized Hare method that contains some aspects of approval voting. Every elector decides whether or not a candidate is approved, and supplies a preference profile of the approved candidates. In the preference table, some columns will have blank spaces at the bottom.

There are various ways in which an STV system can be implemented. Typically, any vote that lists *no* approved candidates is eliminated before the quota is calculated. Then elections proceed similarly to elections under the generalized Hare method. However, it is possible for a vote to disappear because all the candidates named on it have been eliminated. In the basic system this can mean that no candidate reaches the quota, even though not all vacancies have been filled; the standard procedure is to hold another election for the remaining seats. However, another possibility is to recalculate the quotas at each stage. We shall refer to this as *dynamic* STV.

Sample Problem 10.7. *An election for two positions is held under the single transferable vote system; the preference table is*

5	5	5	5	3	1
A	A	A	A	B	C
B	C			C	B
C	B			A	A

What is the result under the basic STV system? What is the result under dynamic STV?

Solution. There are 24 electors, so the quota is 8. *A* is elected and has 12 surplus votes. The new preference table is

3	3	3	3	3	1
B	*C*			*B*	*C*
C	*B*			*C*	*B*

or equivalently

6	4	6
B	*C*	
C	*B*	

B has six votes and *C* has four; neither is elected and a new election is held for the remaining position.

Under the dynamic system, the new preference table is treated as if it were the preference table for a new election; the "empty" ballots are discarded, and we proceed as though there were 10 votes with preference table

6	4
B	*C*
C	*B*

B is elected.

Exercises 10.2

In Exercises 1 to 4, a preference table is shown for an election where two candidates are to be elected using the generalized Hare method. In each case, what is the quota? What is the outcome of the election?

1.

7	8	7	5
A	*B*	*C*	*D*
B	*D*	*B*	*C*
C	*C*	*A*	*B*
D	*A*	*D*	*A*

2.

3	5	8	8	6
A	*A*	*B*	*C*	*D*
B	*C*	*D*	*B*	*C*
C	*D*	*C*	*A*	*B*
D	*B*	*A*	*D*	*A*

3.

4	6	6	6	2	6
A	*D*	*B*	*E*	*C*	*B*
D	*A*	*D*	*A*	*E*	*A*
E	*E*	*C*	*D*	*D*	*E*
B	*B*	*E*	*C*	*A*	*C*
C	*C*	*A*	*B*	*B*	*D*

4.

6	7	7	7	7	5
A	*B*	*C*	*D*	*C*	*E*
E	*D*	*B*	*E*	*D*	*B*
D	*A*	*D*	*A*	*E*	*A*
B	*E*	*E*	*C*	*A*	*C*
C	*C*	*A*	*B*	*B*	*D*

In Exercises 5 to 10, a preference table is shown for an election where three candidates are to be elected using the generalized Hare method. In each case, what is the quota? What is the outcome of the election?

5.

4	7	9	8	7	5
A	*A*	*B*	*C*	*D*	*D*
B	*B*	*D*	*D*	*C*	*A*
C	*D*	*C*	*A*	*B*	*B*
D	*C*	*A*	*B*	*A*	*C*

6.

24	16	19	21	10	10
A	*B*	*C*	*D*	*E*	*E*
B	*A*	*D*	*C*	*C*	*A*
E	*D*	*E*	*E*	*D*	*B*
D	*E*	*A*	*B*	*B*	*D*
C	*C*	*B*	*A*	*A*	*C*

7.

6	6	6	6	5	3
A	*A*	*C*	*C*	*E*	*E*
B	*D*	*D*	*E*	*C*	*A*
E	*E*	*E*	*D*	*D*	*B*
D	*B*	*A*	*B*	*A*	*C*
C	*C*	*B*	*A*	*B*	*D*

8.

9	8	7	6	4	2
A	*B*	*D*	*D*	*C*	*A*
E	*D*	*E*	*E*	*D*	*B*
D	*A*	*C*	*A*	*E*	*E*
B	*E*	*B*	*C*	*A*	*C*
C	*C*	*A*	*B*	*B*	*D*

9.

6	6	7	5	6	5	5
A	*B*	*B*	*C*	*D*	*E*	*E*
B	*E*	*D*	*E*	*E*	*B*	*B*
D	*A*	*A*	*B*	*A*	*C*	*A*
C	*C*	*E*	*A*	*B*	*D*	*D*
E	*D*	*C*	*D*	*C*	*A*	*C*

10.

8	6	7	7	9	3	8
A	*A*	*B*	*C*	*C*	*D*	*E*
D	*B*	*C*	*D*	*E*	*C*	*A*
E	*D*	*E*	*E*	*D*	*B*	*B*
B	*E*	*A*	*A*	*A*	*E*	*D*
C	*C*	*D*	*B*	*B*	*A*	*C*

11. Eight administrators s, t, u, v, w, x, y, z are voting to fill one position in senior management from candidates A, B, C, D, E, F. They use an approval system; their approval table is

	s	*t*	*u*	*v*	*w*	*x*	*y*	*z*
A	×	×				×		
B	×		×	×	×			×
C		×		×			×	×
D		×		×	×		×	
E			×			×	×	
F		×	×	×			×	×

(i) What is the outcome?

(ii) What is the outcome if at least 80% approval rating is required?

(iii) What is the outcome if at least 60% approval rating is required?

(iv) The administrators are told that they can appoint more than one person if they wish. What is the outcome if at least 60% approval rating is required?

In Exercises 12 to 17, an approval table is shown for selection of at most three candidates. In each case

(i) *What is the outcome if there is no minimum requirement?*

(ii) *How many votes are needed if at least* 66% *approval is required?*

(iii) *What is the outcome if at least* 66% *approval is required?*

12.

	r	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*
A	×	×				×			
B	×		×	×	×	×			×
C		×		×			×	×	×
D		×		×	×	×		×	
E			×	×			×	×	
F	×		×	×	×			×	×
G			×	×			×	×	
H	×		×	×	×			×	×

13.

	p	q	r	s	t	u	v	w	x	y	z
A	×	×		×	×	×		×			
B	×	×	×		×	×	×	×			×
C		×		×		×			×	×	×
D		×		×	×	×		×		×	
E	×	×	×	×	×	×	×		×	×	×
F			×	×				×	×	×	
G			×		×	×			×	×	
H	×		×		×	×	×			×	×

14.

	p	q	r	s	t	u	v	w	x	y	z
A		×	×	×	×	×	×	×		×	
B		×	×		×	×		×			×
C									×	×	×
D		×		×	×	×		×		×	
E	×		×	×					×		×
F				×				×	×		
G					×	×			×		
H	×	×	×	×	×	×	×			×	×

15.

	q	r	s	t	u	v	w	x	y	z
A	×		×		×		×	×		
B	×			×	×	×	×			×
C		×	×		×	×		×	×	×
D		×		×	×	×	×		×	
E	×	×	×	×	×	×		×	×	×
F			×	×			×	×	×	
G			×		×	×		×	×	
H	×		×				×		×	×
I			×		×	×			×	
J	×			×	×	×	×			×

16.

	q	r	s	t	u	v	w	x	y	z
A	×			×	×	×	×			×
B		×		×	×		×	×	×	
C			×		×	×	×		×	
D	×	×	×		×		×	×		
E	×		×	×	×	×		×		×
F			×				×		×	×
G			×	×	×			×	×	
H	×		×		×	×			×	
I	×	×	×		×	×		×		×
J				×	×	×	×			×

17.

	o	p	q	r	s	t	u	v	w	x	y	z
A		×	×			×	×	×	×			×
B		×		×		×	×		×	×	×	
C			×		×	×	×		×		×	
D	×	×		×	×		×		×	×		
E	×		×		×	×		×		×		×
F			×						×		×	×
G	×	×			×		×			×	×	
H	×		×		×	×					×	
I	×		×	×	×		×	×		×		×
J			×			×		×	×			×
K				×	×	×	×	×	×			×
L		×	×	×	×	×	×		×	×	×	

18. Suppose three candidates are to be elected. Voters have the following preference profile (if a candidate appears on a list, one may assume the voter approves of that candidate):

3	5	7	4	8	5
A	B	C	D	G	E
E	D	F	F	D	B
D	F	D	A	E	A
B	E	E	C	F	
		A	B	B	
			G	C	
			E	A	

(i) What is the result under approval voting?

(ii) What is the result under the basic STV system?

(iii) What is the result under dynamic STV?

19. An election is to be held under the STV system and four candidates are to be elected. The preference table is

7	7	4	2	4	6
A	A	B	C	D	E
B	C	D	B	B	C
	D	A	A	A	D
	E		D		A
	B		E		B

(i) What is the result under the basic STV system?

(ii) What is the result under dynamic STV?

20. An election is to be held under the STV system and three candidates are to be elected. The preference table is

7	7	4	2	4	6
A	*B*	*C*	*D*	*B*	*F*
B	*A*	*B*	*B*	*A*	*C*
		D	*A*	*E*	*E*
		A	*C*	*D*	
		E	*F*		

(i) What is the quota?

(ii) What is the result under the basic STV system?

(iii) What is the result under dynamic STV?

10.3 Fair Elections

We have seen several different electoral systems. Even in a small election, the same preferences can give rise to a different results depending on the system used. In this section we shall present an extreme case of how the method chosen might affect the outcome of an election in a realistic situation. This means that an electoral system can be chosen to favor one or another candidate.

There are also many instances in which voters can misrepresent their preferences, manipulating an election to give a certain result. Some instances of this are given below and another example will be discussed in Section 10.4.

Five Candidates, Five Winners

Consider a political party convention at which five different voting schemes are adopted. Assume that there are 110 delegates to this national convention, at which five of the party members, denoted by A, B, C, D, and E, have been nominated as the party's presidential candidate. Each delegate must rank all five candidates according to his or her choice. Although there are $5! =: 5 \times 4 \times 3 \times 2 \times 1 = 120$ possible rankings, many fewer will appear in practice because electors typically split into blocs with similar rankings. Let's assume that our 110 delegates submit only six different preference lists, as indicated in the following preference profile:

	Number of delegates					
	36	24	20	18	8	4
First choice	*A*	*B*	*C*	*D*	*E*	*E*
Second choice	*D*	*E*	*B*	*C*	*B*	*C*
Third choice	*E*	*D*	*E*	*E*	*D*	*D*
Fourth choice	*C*	*C*	*D*	*B*	*C*	*B*
Fifth choice	*B*	*A*	*A*	*A*	*A*	*A*

The 36 delegates who most favor nominee A rank D second, E third, C fourth, and B fifth. Although A has the most first-place votes, he is actually ranked last by the other 74 delegates. The 12 electors who most favor nominee E split into two subgroups of 8 and 4 because they differ between B and C on their second and fourth rankings.

We shall assume that our delegates must stick to these preference schedules throughout the following five voting agendas. That is, we will not allow any delegate to switch preference ordering to vote in a more strategic manner or because of new campaigning.

We report the results when six popular voting methods are used. There are six different results.

1. Majority. As one might expect with five candidates, there is no majority winner.

2. Plurality. If the party were to elect its candidate by a simple plurality, nominee A would win with 36 first-place votes, in spite of the fact that A was favored by less than one-third of the electorate and was ranked dead last by the other 74 delegates.

3. Runoff. On the other hand, if the party decided that a runoff election should be held between the top two contenders (A and B), who together received a majority of the first-place votes in the initial plurality ballot, then candidate B outranks A on 74 of the 110 preference schedules and is declared the winner in the runoff.

4. Hare Method. Suppose the Hare method is used: a sequence of ballots is held and eliminates at each stage the nominee with the fewest first-place votes. The last to survive this process becomes the winning candidate. In our example, E, with only 12 first-place votes, is eliminated in the first round. E can then be deleted from the preference profile and all 110 delegates will vote again on successive votes. On the second ballot, the 12 delegates who most favored E earlier now vote for their second choices, that is, 8 for B and 4 for C; the number of first-place votes for the four remaining nominees is

$$\begin{array}{cccc} A & B & C & D \\ 36, & 32, & 24, & 18. \end{array}$$

Thus, D is eliminated. On the third ballot the 18 first-place votes for D are reassigned to C, their second choice, giving

$$\begin{array}{ccc} A & B & C \\ 36, & 32, & 42. \end{array}$$

Now B is eliminated. On the final round, 74 of the 110 delegates favor C over A, and therefore C wins.

5. Borda count. Given that they have the complete preference schedule for each delegate, the party might choose to use a straight Borda count to pick the winner. They assign five points to each first-place vote, four points for each second, three points for a third, two points for a fourth, and one point for a fifth. The scores are

$$\begin{array}{ll} A: & 254 = (5)(36) + (4)(0) + (3)(0) + (2)(0) + (1)(24 + 20 + 18 + 8 + 4), \\ B: & 312 = (5)(24) + (4)(20 + 8) + (3)(0) + (2)(18 + 4) + (1)(36), \\ C: & 324 = (5)(20) + (4)(18 + 4) + (3)(0) + (2)(36 + 24 + 8) + (1)(0), \\ D: & 382 = (5)(18) + (4)(36) + (3)(24 + 8 + 4) + (2)(20) + (1)(0), \\ E: & 378 = (5)(8 + 4) + (4)(24) + (3)(36 + 20 + 18) + (2)(0) + (1)(0). \end{array}$$

The highest total score of 382 is achieved by D, who then wins. A has the lowest score (254) and B the second lowest (312).

6. Condorcet. With five candidates there is often no Condorcet winner. However, when we make the head-to-head comparisons, we see that E wins out over

- A by a vote of 74 to 36,
- B by a vote of 66 to 44,
- C by a vote of 72 to 38,
- D by a vote of 56 to 54.

So there is a Condorcet winner, namely E.

In summary, our political party has employed five different common voting procedures and has come up with five different winning candidates. We see from this illustration that those with the power to select the voting method may well determine the outcome.

Sequential Pairwise Voting

In *sequential pairwise voting,* several candidates are paired in successive runoff elections. There is an *agenda* (an ordered list of candidates). For example, if the agenda is $A, B, C, D, \ldots$, then the elections proceed as follows:

1. A against B.
2. Winner of AB against C.
3. That winner against D.

$\ldots$

Position in the agenda is very important. To see this, consider a four-candidate election with agenda A, B, C, D, in which all four candidates are equally likely to win. If repeated trials are made then we would expect the following results:

- A wins first runoff in half the cases.
- A wins second runoff in half those cases—a quarter overall.
- A wins the third runoff in half those cases—one-eighth overall.

So A has a 1 in 8 chance of winning. B also has a has 1 in 8 chance. However, C has a 1 in 4 chance, and D has a 1 in 2 chance. In this case being later in the list is very beneficial.

Rather than elections, this model is often used for sporting tournaments (the result of a match is used instead of the result of a runoff election). One often sees playoff rules like

(i) Second and third place getters in preliminary competition play each other ("the playoff").
(ii) The winner of the playoff meets the leader from the preliminaries.

In this case it is reasonable that the preliminary leader should get an advantage. However, when the model is used in voting situations it is very subject to manipulation.

Manipulating the Vote

The term *strategic voting* means voting in a way that does not represent your actual preferences to change the result of the election. We would call the resulting ballot *insincere.*

Suppose your favorite is candidate X. (We will call you an *X supporter.*) Then X would normally appear at the top of your preference list. But sometimes you can achieve X's election by voting for another candidate in first place! This is most common in runoff situations; you can ensure that your candidate does not have to face a difficult opponent. The following example illustrates this.

Sample Problem 10.8. *A runoff election has preference profile*

6	2	7	5	4
A	C	C	B	D
D	B	A	A	A
C	D	D	C	B
B	A	B	D	C

Show that the supporters of C can change the result so that their candidate wins, by the two voters in the second column changing their ballots by demoting *their candidate.*

Solution. Initially the first-place votes are A–6, B–5, C–9, D–4, so the runoff election will be between A and C, and A wins 15–9. The revised profile is

6	2	7	5	4
A	B	C	B	D
D	C	A	A	A
C	D	D	C	B
B	A	B	D	C

the first-place votes are A–6, B–7, C–7, D–4, so the runoff election is between A and C, and C wins 13–11.

Even when you cannot ensure victory for your opponent, you may still be able to obtain a preferable result. For example, suppose you support candidate X; you think candidate Y is acceptable, but hate candidate Z. Even if insincere voting cannot ensure victory for candidate X, you may be able to swing the election to Y rather than Z.

Sample Problem 10.9. *An election with four candidates and seven voters is to be decided by the Hare system. The preference profile is*

2	1	2	1	2
B	B	D	C	C
A	D	C	A	D
D	A	B	B	A
C	C	A	D	B

Show that one of the two voters with B, A, D, C can change the outcome to a more favorable one by insincere voting.

Solution. First consider the result of sincere voting. Initially A is eliminated, having no first-place votes

2	1	2	1	2
B	B	D	C	C
D	D	C	B	D
C	C	B	D	B

Next D is eliminated, leaving

2	1	2	1	2
B	B	C	C	C
C	C	B	B	B

The winner is C.

Now suppose one voter changes his ballot from B, A, D, C to D, A, B, C. The profile is

1	1	1	2	1	2
B	*D*	*B*	*D*	*C*	*C*
A	*A*	*D*	*C*	*A*	*D*
D	*B*	*A*	*B*	*B*	*A*
C	*C*	*C*	*A*	*D*	*B*

Again *A* is first eliminated, leaving

1	1	1	2	1	2
B	*D*	*B*	*D*	*C*	*C*
D	*B*	*D*	*C*	*B*	*D*
C	*C*	*C*	*B*	*D*	*B*

In the next round, *B* is eliminated

1	1	1	2	1	2
D	*D*	*D*	*D*	*C*	*C*
C	*C*	*C*	*C*	*D*	*D*

The winner is *D*. This is a preferable outcome for the voter who switched.

Practice Exercise. Consider a Hare system election with preference profile

8	6	5
P	*Q*	*R*
Q	*R*	*S*
R	*P*	*P*
S	*S*	*Q*

Show that *P* would win this election. Show that if one of the six supporters of *Q* changes her vote, she could ensure that *R* wins, even though a majority of voters still prefer *Q* to *R*.

Of course, you do not always know exactly how the votes will go. Strategic voting is usually based on assumptions about the election.

Amendments

We now consider an important example of manipulation of pairwise sequential voting. Suppose three voters on City Council have to decide whether to add a new sales tax. Initially,

- *A* prefers the tax,
- *B* prefers the tax,

- C prefers no tax,

so a tax will be introduced.

However, let's assume A hates income taxes and will never vote for one. On the other hand, B prefers income tax to sales tax. Suppose C moves an amendment to change the tax to an income tax.

We now have the following outcomes.

Original motion: that a city sales tax of 5% be introduced.

Amendment (moved by C): change "sales tax of 5%" to "income tax of 2%."

(We will assume the 2% income tax will provide the same total as the 5% sales tax.)

In the vote on the amendment, both B and C will vote in favor, with A against, so the amendment is carried. So the motion becomes the following: a city income tax of 2% shall be introduced. In the vote on the new motion, both A and C are against, while B votes in favor; so the motion is lost and there is no tax.

Exercises 10.3

1. Consider the preference profile

3	1	1	1	1	1	1
W	W	X	X	Y	Y	Z
Z	Y	Y	Z	X	X	X
Y	Z	Z	Y	Z	W	Y
X	X	W	W	W	Z	W

What is the result of an election using the following systems?

(i) Majority system.

(ii) Plurality system.

(iii) Borda count.

(iv) Hare system.

(v) Condorcet method.

(vi) Pairwise sequential method with agenda X, Y, Z, W.

(vii) Pairwise sequential method with agenda W, X, Z, Y.

(viii) Pairwise sequential method with agenda Y, Z, W, X.

2. Consider the preference profile

2	1	1	1	2
A	*A*	*B*	*C*	*D*
C	*D*	*D*	*D*	*C*
B	*C*	*C*	*B*	*A*
D	*B*	*A*	*A*	*B*

What is the result of an election using the following systems?

(i) Majority system.

(ii) Plurality system.

(iii) Runoff system.

(iv) Borda count.

(v) Hare system.

(vi) Condorcet method.

3. A committee of 11 members needs to elect one representative from four candidates. They plan to use a Borda count. The preferences are

4	2	2	3
A	*A*	*B*	*B*
C	*C*	*C*	*D*
D	*C*	*A*	*C*
B	*D*	*D*	*A*

(i) Who would win the election if all electors vote sincerely?

(ii) Can the voters in the third column vote insincerely so as to change the result in their favor? If so, how?

(iii) Can the voters in the fourth column vote insincerely so as to change the result in their favor? If so, how?

4. Suppose an election with the following preference profile is decided by the Hare system.

3	6	7	8	6
A	*A*	*B*	*C*	*D*
B	*C*	*D*	*D*	*C*
C	*D*	*A*	*A*	*B*
D	*B*	*C*	*B*	*A*

(i) Who will win the election?

(ii) Show that the seven supporters of B can achieve a preferred result if they exchange B and D on their ballots (that is, they vote as if their preference was D, B, A, C).

5. An election with preferences

3	2	2	2
A	B	C	D
B	D	B	C
C	C	A	B
D	A	D	A

is to be decided by Borda count. Who will win? Show that the supporters of C can make sure their candidate wins by insincerely voting preference (C, A, D, B).

6. Consider a runoff election with preferences

4	4	2	3	4
A	A	B	B	C
B	C	A	C	B
C	B	C	A	A

(i) Who wins the election?

(ii) Show that two supporters of A can change the result in favor of their candidate by changing their preferences from (A, C, B) to (C, A, B).

7. Consider the preferences

24	16	19	21	11	10
A	B	C	D	E	E
B	A	D	C	C	A
E	D	E	E	B	A
D	E	A	B	A	B
C	C	B	A	D	D

(i) Suppose everyone votes sincerely. Under the Hare system, who wins the election?

(ii) Show that, if one of the supporters of D votes insincerely by reversing her first two preferences, she achieves a preferable outcome.

8. An election with preference profile

6	4	4
A	B	C
B	C	B
C	A	A

is decided by Borda count. How many of A's supporters must change their ballots to (A, C, B) to make their candidate the winner?

9. An 18-member committee is to elect one of the four candidates Q, R, S, T. Their preference table is as shown. Which candidate wins under pairwise sequential voting with agenda (S, T, Q, R)?

4	6	5	3
Q	R	S	T
T	S	R	S
S	T	T	R
R	Q	Q	Q

10. Consider a pairwise sequential election with preferences

5	3	2	1	8
X	X	Y	Z	T
Y	T	Z	Y	Y
T	Y	X	X	X
Z	Z	T	T	Z

(i) Who will win under agenda X, Y, Z, T?

(ii) Who will win under agenda Y, T, Z, X?

11. A pairwise sequential election has preference profile

2	8	11	7
A	C	B	A
B	B	A	C
C	A	C	B

(i) Who will win under agenda A, B, C?

(ii) Who will win under agenda B, C, A?

(iii) Who will win under agenda C, A, B?

12. A pairwise sequential election has preference profile

40	32	12	18
P	Q	R	S
Q	R	S	R
R	P	P	Q
S	S	Q	P

(i) Who will win under agenda P, Q, R, S?

(ii) Who will win under agenda S, R, Q, P?

(iii) Who will win under agenda S, P, R, Q?

(iv) Who will win under agenda S, Q, P, R?

13. A club with 46 members wishes to elect a president. The post is currently held by Y. The 46 members have preference profile

16	12	10	8
X	Y	Z	Y
Z	Z	X	X
Y	X	Y	Z

(i) The following electoral system is used: a plurality vote is held between the candidates other than the President; there is then a majority election between the winner of that election and the current President. Show that X will win the election.

(ii) The eight voters whose votes form the right-hand column of the table decide to change their votes in an attempt to make Y the winner. Can they do this?

14. Suppose X is a Condorcet winner of an election. Can there be an agenda such that X does not win under pairwise sequential voting?

15. Three committee members must vote to fill one position. There are four candidates, and the preferences are

A	B	C
B	C	D
C	D	A
D	A	B

They decide to use pairwise sequential voting. For each candidate, find an agenda such that the selected candidate will win.

16. An election is to be held under plurality voting. The preferences profile is

1	3	2
A	C	B
C	B	A
B	A	C

Can the supporter of A (first column) achieve a preferable result by insincere voting?

10.4 Properties of Electoral Systems

In this section we shall discuss various properties of electoral systems, most of which are related to the accuracy or fairness of the systems. For simplicity we shall restrict our attention to elections with a single winner, but most of the ideas can be extended to multiple winners, such as the approval and generalized Hare methods.

It will be convenient to define an *instance* of an electoral system to mean an election that uses the system on a particular preference profile.

Polls

Often an election is preceded by an informal count, or poll. For example, even in 2007 there were several polls for the 2008 Presidential election.

If a candidate does badly in polls, his or her supporters may change their votes. For example, consider an election that uses plurality voting. There are 3900 voters and three candidates, A, B, C. The voters' actual preferences are

1800	1700	400
A	B	C
C	C	B
B	A	A

If the election were held immediately, A would win.

However, suppose a poll is held. As usual, the poll uses the same system (plurality voting). Say there are 35 voters in the poll and their preferences are proportional to the overall preferences. The result will show A first (18 votes), with B a close second (17 votes), and C a distant third (with only 4 votes).

When the poll results are reported, the 400 voters who favor C may well reason, "C cannot win the election, so it would be preferable to elect the better of the other two." They all prefer B to A, so they would vote for B. Suppose 300 of the voters decided to change their votes. The vote will be A–1800, B–2000, C–100, and B will win the election.

But observe that C was a Condorcet winner under the original preferences

B beats A 2200–1700,
C beats B 2200–1700,
C beats A 2100–1800.

The type of behavior exhibited by C's followers is frequently observed after polls and leads to the following.

Poll Assumption

- Voters whose favorite is not one of the two top candidates in a poll will adjust their preferences to vote for one of the top two.

For this reason, those who do badly in polls may drop out before the election. In the example, C might easily have dropped out. The Poll Assumption applies primarily to the last poll taken before an election, but candidates frequently drop after earlier polls because they do not expect their showing to improve later.

If the poll assumption holds,

- A Condorcet winner will always lose if s/he is not one of the top two candidates in the poll.
- On the other hand, If the poll assumption holds, a Condorcet winner will always win if s/he *is* one of the top two candidates in the poll.

We say an election satisfies the *poll fairness criterion* if there is no case where the result is changed by a candidate who is not one of the top two drops out before the final vote; a system satisfies the poll fairness criterion if every instance satisfies the condition.

The preceding discussion indicates that the candidate who would eventually win an election may very well withdraw for what seemed like good reason. Even when the candidate does not withdraw, there is no guarantee that an electoral system will behave as we expect. The only completely satisfactory method of voting is majority rule, and even that is not guaranteed to give a result when there are more than two candidates. No matter what system is chosen, there will always be some preference profiles for which the method is inappropriate; either it produces no result (even when there are no ties) or it has a property that is clearly undesirable. The rest of this chapter is devoted to illustrating this fact.

The Condorcet Winner Condition

The *Condorcet winner condition* (or CWC) says

"If the following can happen:

- the election has a winner;
- the election has a Condorcet winner;
- the two are not the same;

then the method of election is unfair."

This seems a reasonable thing to say, and you might well require that, if a preference profile has a Condorcet winner, your electoral system must not elect a different candidate. However, many systems do not follow this principle. For example, consider the 3900 voter election, given earlier in this section. The preferences were

1800	1700	400
A	B	C
C	C	B
B	A	A

we observed that A would win the election under the plurality method, but C is a Condorcet winner. Furthermore, if the runoff method were used, B would receive all 400 preferences from C, and would win.

An election in which there is a Condorcet winner is said to satisfy the *Condorcet criterion* if the Condorcet winner wins the election, or (trivially) if there is no winner or there is no Condorcet winner. An electoral system satisfies the criterion if every instance of it satisfies the criterion. So we have observed that neither plurality nor runoff satisfy this criterion.

Independence of Irrelevant Alternatives

Independence of Irrelevant Alternatives (or IIA) says,

> Suppose candidate A wins. Suppose the voters then change votes, and now B wins. There must be at least one voter who previously preferred A to B and now prefers B to A.

This seems reasonable. But consider the following example.

Sample Problem 10.10. *A Borda count election has preference table*

3	2	2	1
A	B	B	C
B	C	A	A
C	A	C	B

What is the result? What is the effect if the voters in the left-hand column change their preference to A, C, B?

Solution. Using a 2–1–0 count, A receives 9 points, B receives 11, and C receives 4, so B wins. After the votes are changed, the scores are A–9, B–8, and C–7, and A wins.

In the example, no change was made in the relative positions of A and B, but A won the election rather than B. So Borda count does not always satisfy IIA.

If a voting system satisfies both CWC and IIA, then it might not produce a winner. To see this we consider an example. We do not need to know what electoral system is used, we only need to know that it satisfies both CWC and IIA.

Consider an election with three voters that produces the following preference table:

4	3	2
A	B	C
B	C	A
C	A	B

This election has no Condorcet winner. In head-to-head contests, A beats B 6–3, B beats C 7–2, and C beats A 5–4.

Now suppose the second group of voters all change their votes by putting C above B—the relative positions of A and C are unchanged. The preferences become

4	3	2
A	C	C
B	B	A
C	A	B

In this case C is a Condorcet winner (C beats B 5–4, and the other results are unchanged). By CWC, if the election has a winner, it must be C. So A cannot win this new election. Therefore, IIA tells us, A could not have won the original election.

However, suppose we instead considered interchanging A and C in the third group's preferences. A is now a Condorcet winner (beating C 6–3), and the same reasoning as before shows that B did not win the original.

And if we consider interchanging A and B in the first voter's preferences, we can show in the same way that C did not win the original.

We have shown that, if a voting system satisfies both CWC and IIA, it cannot produce a winner, even for the simple preference profile we have given.

Monotonicity

Suppose an election is held and A wins. The only change made in the preference lists is to move A *higher* (nearer to first preference) in some lists. Surely A will win the new election? If this is true we say the election is *monotone*, or say it satisfies the *monotonicity criterion.* A system is called monotone if every instance is monotone.

Not all systems are monotone. In particular, the runoff system and Hare system are not. As an example, consider a runoff election with preference profile

6	1	7	5	2
B	A	A	C	C
A	C	B	B	A
C	B	C	A	B

If the preferences are followed, B will be eliminated, and A defeats C easily, 14–7. However, suppose the supporters of A convince the two electors whose ballots read C, A, B to move A to the top of their ballots. Then C is eliminated, and in the runoff B beats A, 11–10.

Pareto Condition

Vitello Pareto (1848–1923), a French-Italian economist and sociologist, proposed the following: if everybody prefers candidate A to candidate B, then B should not be elected. This *Pareto condition* is not satisfied by sequential pairwise voting. Here is an example. Suppose three voters have preferences

A	C	D
B	D	A
C	A	B
D	B	C

If the agenda D, A, C, B is used, then D beats A 2–1, C beats D 2–1, and B beats C 2–1, so B is elected. But every voter prefers A to B.

Sample Problem 10.11. *Suppose a sequential pairwise election does not satisfy the Pareto condition. Show that there are at least four candidates.*

Solution. Suppose there were an example with three candidates, A, B, and C. Suppose every voter prefers A to C, but C wins the election. The last comparison made must involve C (in order for C to win) and B (C could not beat A). If B is the last member in the agenda, then the first comparison is between A and C, and C would be eliminated. So the agenda must be A, B, C (or, equivalently, B, A, C); in the first comparison, B beats A, and in the second, C beats B.

Suppose the preference profile is

x	y	z
A	A	B
B	C	A
C	B	C

for some nonnegative integers x, y, z. (No other columns are possible because A cannot be below C.) Since B beats A, $z > x + y$. But C beats B, so $y > x + z$. So $y > 2x + y$, and x must be negative—a contradiction.

Arrow's Impossibility Theorem

A *dictator* is a person who completely controls a country or an organization. By analogy, we define a dictator in a voting system to be a voter with similar power: the dictator's first preference will always be elected. A voting system that has a dictator is called a *dictatorship*.

If there is a dictator, there is no point in having elections or collecting preference lists. However, a dictatorship obviously has some of the properties that we would think desirable. It always produces a result. It is monotone. It satisfies the Pareto condition. And it satisfies the independence of irrelevant alternatives.

Surprisingly, the American economist Kenneth Arrow (b. 1927) proved that a dictatorship is the only electoral system for three or more candidates that has all these properties.

Theorem 62 (Arrow's Impossibility Theorem). *For three or more candidates there is no electoral system that always produces a winner* (*rexcept in the case of ties*), *satisfies the Pareto condition, satisfies the independence of irrelevant alternatives, and is not a dictatorship.*

(Notice that we do not even need to add monotonicity.)

Arrow was awarded a Nobel Prize for Economics in 1972.

Exercises 10.4

1. A plurality-vote election has preference table

8	6	4	2
X	Y	Z	T
T	T	T	Z
Y	X	Y	Y
Z	Z	X	X

(i) Is there a Condorcet winner? If so, who?

(ii) Suppose a poll is held and those who place third and fourth drop out. Who will win the final election?

(iii) Suppose a poll is held but only the candidate who placed fourth drops out. Who will win the final election?

2. Fifteen club members are voting to elect a president, using a Borda count. The preference profile is

6	5	4
A	B	C
C	D	B
B	A	A
D	C	E
E	E	D

A poll is conducted; both C and D realize that they are unlikely to win.

(i) Who would win this election?

(ii) Who would win if C drops out before the election?

(iii) Who would win if D drops out before the election?

3. A hiring committee uses the Hare system to select a new foreman. The preference profile is

5	4	3
A	B	C
D	A	D
B	D	B
C	C	A

(i) Who would win this election?

(ii) Who would win if C drops out before the election?

(iii) Who would win if A drops out before the election?

4. Devise a preference profile for three candidates A, B, C with 13 voters, in which C wins under plurality but both A and B would beat C 8–5 in runoff elections.

5. Consider a plurality vote with preference profile

5	2	4
A	B	C
C	C	B
B	A	D
D	D	A

(i) Who would win this election?

(ii) Is there a Condorcet winner? If so, who?

(iii) Does this election satisfy the Condorcet criterion?

(iv) Suppose a poll is conducted, and C decides to drop out before the election. Who would win?

(v) Does this election satisfy the Condorcet criterion?

(vi) Does this election satisfy the poll fairness criterion?

(vii) Does this election satisfy the monotonicity criterion?

6. A runoff election has preference profile

13	9	5	11
X	Y	Y	Z
Y	Z	X	X
Z	X	Z	Y

(i) Who would win this election?

(ii) Is there a Condorcet winner?

(iii) Does this election satisfy the Condorcet criterion?

(iv) Does this election satisfy the monotonicity criterion?

7. Consider the preference profile

4	3	2
A	*B*	*C*
B	*C*	*B*
C	*A*	*A*

(i) Who would win this election under the plurality method?

(ii) By manipulating the third column, show that this election does not satisfy independence of irrelevant alternatives.

8. Consider the preference profile

4	3	2
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*

(i) Who would win this election under the runoff method?

(ii) By manipulating the second column, show that this election does not satisfy independence of irrelevant alternatives.

9. Suppose the preference profile

7	5	4	2
A	*B*	*C*	*D*
D	*A*	*B*	*C*
C	*C*	*D*	*A*
B	*D*	*A*	*B*

is used in a Hare system election.

(i) Show that B would win the election. What is the result of the runoff?

(ii) Suppose the voters in the final column move A above C. Show that A now *loses*. What is the result?

10. A Hare election uses the preference profile

7	6	5	3
A	*B*	*C*	*D*
B	*A*	*B*	*C*
C	*C*	*A*	*B*
D	*D*	*D*	*A*

Show that A wins the election. Consider the result if the voter who preferred D moved A up to first preference. Does this prove that this election is not monotone?

11. Consider the preference profile, in which every voter prefers A to B.

200	200	200	150
A	C	D	A
B	A	C	D
D	B	A	C
C	D	B	B

Find an agenda such that B would win a pairwise sequential election.

12. Consider a plurality election in which one candidate has an absolute majority (first-place in more than half the preference lists). Show that this election satisfies the Condorcet and monotonicity conditions.

13. Does the Condorcet system satisfy the Condorcet criterion?

14. Prove that the Condorcet system is monotone.

15. Show that a dictatorship satisfies the Pareto condition.

16. Prove that the Borda system is monotone.