[14] The span of multiply connected domains

[14] Menahem Max Schiffer, “The span of multiply connected domains,” in *Duke Mathematical Journal*, Volume **10** (1943), 209–216.

© 1943 Duke University Press. All rights reserved. Reprinted by permission of the publisher. www.dukeupress.edu

THE SPAN OF MULTIPLY CONNECTED DOMAINS

By Menahem Schiffer

1. Let D_n be a domain in the z-plane, containing the point $z = \infty$ and bounded by n continua C_ν ($\nu = 1, \cdots, n$). A function $F(z)$ belongs to the family $\phi(D_n)$, if it is univalent and regular in D_n, the point $z = \infty$ excepted, at which it has the development

$$F(z) = z + \frac{A_2}{z} + \frac{A_3}{z^2} + \cdots . \tag{1}$$

There exists a function $f(z) \subset \phi(D_n)$, mapping D_n on a domain bounded by slits parallel to the real axis. It can be characterized by the following extremal property [5]:

Among all functions $F(z) \subset \phi(D_n)$, $f(z)$ yields the maximal value of $\Re\{A_2\}$.

For the function $g(z) \subset \phi(D_n)$, mapping D_n on a domain bounded by slits parallel to the imaginary axis, an analogous result holds, namely:

Among all functions $F(z) \subset \phi(D_n)$, $g(z)$ yields the minimal value of $\Re\{A_2\}$. The functions

$$f(z) = z + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots, \qquad g(z) = z + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots \tag{2}$$

will be called henceforth the *slit functions* of D_n.

The number

$$S(D_n) = \Re\{a_2 - b_2\} \tag{3}$$

gives the breadth of the interval in which $\Re\{A_2\}$ varies for all functions of $\phi(D_n)$. $S(D_n)$ is a functional of D_n and will be called *the span of* D_n. The aim of this paper is to discuss some of its properties and to connect this number with other characteristics of the domain.

In this chapter some elementary theorems on $S(D_n)$ will be recalled. Let us remark first:

I. *The span is a non-increasing function of the domain.*

This theorem is obvious; for suppose $D_n \subset D'_m$, then $\phi(D'_m) \subset \phi(D_n)$; hence, the interval of variation of $\Re\{A_2\}$ is not smaller for D_n than for D'_m. This proves the assertion.

II. *The span of all domains which can be mapped on each other with the aid of univalent and normalized functions* $p(z) = z + p_1 + p_2/z + \cdots$ *is the same.*

Indeed, take $z = z(\zeta) = \zeta + \pi_1 + \pi_2/\zeta + \cdots$ as the univalent function mapping Δ_n in the ζ-plane on D_n in the z-plane. $F(z) \subset \phi(D_n)$ only if $F[z(\zeta)] -$

Received October 8, 1942.

$\pi_1 \subset \phi(\Delta_n)$. To the second coefficient A_2 of each function $F(z)$ corresponds the coefficient $A_2 + \pi_2$ of $F[z(\zeta)] - \pi_1$. Hence:

IIa. *A conformal and normalized mapping of D_n involves a translation in the space of the coefficients A_2.*

Therefore the span of these coefficients is not changed by this mapping, as was to be proved.

III. *If the domain D_n^+ is obtained from D_n by a homothetic transformation with the factor d, then*

$$S(D_n^+) = d^2 S(D_n). \tag{4}$$

For the slit function $f^+(z^+)$ of D_n^+ is obtained from the slit function $f(z)$ for D_n by the identity

$$f^+(z^+) = df\left(\frac{z^+}{d}\right) = z^+ + \frac{a_2 d^2}{z^+} + \cdots ; \tag{5}$$

and in the same way we have

$$g^+(z^+) = dg\left(\frac{z^+}{d}\right) = z^+ + \frac{b_2 d^2}{z^+} + \cdots . \tag{5'}$$

Hence (4) follows at once.

The unit circle $U : | z | > 1$ has the slit functions

$$f(z) = z + \frac{1}{z}, \qquad g(z) = z - \frac{1}{z}.$$

Hence $S(U) = 2$. A circle C with radius d, $| z | > d$, has according to (4) the span $S(C) = 2d^2$ and an arbitrary simply connected domain with the mapping radius d also has by Theorem II the span $2d^2$.

IV. *The function*

$$H_\varphi(z) = e^{-i\varphi}[f(z) \cos \varphi + ig(z) \sin \varphi] \tag{6}$$

belongs for each real value of φ to $\phi(D_n)$ and maps D_n on a domain bounded by slits parallel to the vector $e^{-i\varphi}$.

Indeed, the function $h_\varphi(z) = e^{i\varphi}H_\varphi(z)$ has a simple pole at $z = \infty$ and is regular elsewhere in D_n; on the boundaries C_ν it is bounded. The integral

$$\frac{1}{2\pi i} \oint_{\Sigma C_\nu} \frac{h'_\varphi(z)}{h_\varphi(z) - \alpha}\, dz = N(\alpha) - 1 \tag{7}$$

($N(\alpha)$ denoting the number of points in D_n, where $h_\varphi(z) = \alpha$) has the value zero at infinity. Because it is an integer, it can change only when α reaches a value ω, assumed by $h_\varphi(z)$ on C_ν. On the other hand, by definition of the slit functions, $h_\varphi(z)$ has a constant imaginary part k_ν on each boundary continuum C_ν of D_n. Hence, the integral (7) changes its value only if $\mathfrak{I}\{\alpha\}$ passes the value k_ν. Now each finite value in the α-plane can obviously be connected with infinity by a path avoiding all the lines $\mathfrak{I}\{\alpha\} = k_\nu$. If, therefore, we run from infinity along this path to the point α considered, the value of integral (7) remains always equal to zero. Hence $N(\alpha) = 1$. Thus, $h_\varphi(z)$ is univalent in D_n and has on each C_ν a constant imaginary part. Hence, $H_\varphi(z)$ has just the properties asserted in Theorem IV.

The functions $H_\varphi(z)$ can also be characterized by an extremal property: *Among all functions $F(z) \subset \phi(D_n)$, $H_\varphi(z)$ yields the maximal value of $\mathfrak{R}\{e^{2i\varphi}A_2\}$.* The second coefficient of $H_\varphi(z)$ is

$$(8) \qquad A_2(\varphi) = (a_2 \cos\varphi + ib_2 \sin\varphi)e^{-i\varphi} = \tfrac{1}{2}(a_2 + b_2) + \tfrac{1}{2}(a_2 - b_2)e^{-2i\varphi}.$$

Thus, the locus of all the extremal coefficients $A_2(\varphi)$ is a circumference about $\frac{1}{2}(a_2 + b_2)$ with radius $\frac{1}{2} \mid a_2 - b_2 \mid$. Since $\mathfrak{R}\{a_2\} = \mathfrak{R}\{A_2(0)\}$ and $\mathfrak{R}\{b_2\} = \mathfrak{R}\{A_2(\frac{1}{2}\pi)\}$ are extremal values for $\mathfrak{R}\{A_2(\varphi)\}$, a_2 and b_2 must obviously have equal imaginary parts, whence

$$(9) \qquad a_2 - b_2 = \mathfrak{R}\{a_2 - b_2\} = S(D_n).$$

Combining (8) with the extremal property of $H_\varphi(z)$ yields the result that outside the circumference C there are no possible values for the coefficients A_2 of $F(z) \subset \phi(D_n)$. To show that each point interior to this circumference is a possible coefficient A_2, let us suppose (without loss of generality in view of IIa) that $b_2 = 0$, that is, D_n is a domain with slits parallel to the imaginary axis. Then the coefficients $A_2(\varphi)$ cover a circumference with radius $\frac{1}{2}a_2 = \frac{1}{2}S(D_n)$, touching the imaginary axis at the origin. By enlarging D_n (preserving its connectivity), $S(D_n)$ will decrease by Theorem I, and it will do so continuously if D_n changes continuously [1]. Since all the domains so obtained remain slit domains, their corresponding circumferences touch the imaginary axis at the origin. If D_n finally becomes the z-plane n times punctured, $S(D_n) = 0$. Therefore, the intermediate circumferences fill all the interior of the original circumference and each point inside this latter is a coefficient of a function $F(z) \subset \phi(D_n)$. Summarizing we get:

V. *The coefficients A_2 of all functions $F(z) \subset \phi(D_n)$ cover exactly a circle with diameter $S(D_n)$* [3].

2. Henceforth we shall suppose that at least one of the boundary continua C_ν of D_n does not reduce to a point. The domain D_n' is called *conformly equivalent* to D_n, if D_n is mapped on D_n' by a function of the family $\phi(D_n)$. Denote the (generalized) Green's function of D_n with the logarithmic pole at infinity by $G(z)$ and let

$$\lim_{z=\infty} (G(z) - \log \mid z \mid) = \lambda.$$

Then $e^{-\lambda} = d(D_n)$, the *transfinite diameter* (a concept due to Fekete, who defined it without conformal mapping [2]) of D_n, and we derive easily from our definition

$$d(D'_n) = d(D_n). \tag{10}$$

Further let $F(D_n)$ be the inner measure of the complementary set $D_n^{\odot}$ of D_n. We have the remarkable inequality [4]

$$F(D_n) \leq \pi d(D_n)^2, \tag{11}$$

which shows that $F(D_n)$ has a common upper bound for all domains which are conformally equivalent to a given domain D_n. There arises therefore the problem:

Given a domain D_n, to find the maximum of $F(D'_n)$ for all domains D'_n, conformally equivalent to D_n.

The existence of a domain D'_n for which $F(D'_n)$ attains its maximal value is a consequence of the compactness of the family $\phi(D_n)$.

Suppose D_n to be a domain with $F(D_n) \geq F(D'_n)$ for all D'_n conformally equivalent to D_n. Then D_n surely has exterior points. We subdivide the complement $D_n^{\odot}$ into an "areal" part A, consisting of all points exterior to the closure of D_n and of the limit points of this exterior, and into the remaining "linear" part L (if such there be). If $z_0 \subset A$, then the function

$$z^* = z + \frac{a\rho^2}{z - z_0}, \qquad \rho > 0, \qquad |a| = 1 \tag{12}$$

belongs to $\phi(D_n)$ for an arbitrary sign of a, if ρ is smaller than the distance of z_0 from D_n. It therefore maps D_n on a conformally equivalent domain D_n^*, and we shall now compute $F(D_n^*)$. For this purpose, we describe a circle K with radius $\rho^{\frac{1}{2}}$ around z_0, choosing ρ so small that this circle also lies entirely in A, and find

$$F(D_n^*) = \int_{A-K} \left| 1 - \frac{a\rho^2}{(z - z_0)^2} \right|^2 d\tau + F(K^*), \tag{13}$$

where $F(K^*)$ denotes the area of the map of K. This latter is an ellipse with the semi-axes $\rho^{\frac{1}{2}} + \rho^{3/2}$, $\rho^{\frac{1}{2}} - \rho^{3/2}$ and area $\pi(\rho - \rho^3)$, while the area K is $\pi\rho$. Thus we have

$$F(D_n^*) = F(D_n) - 2\Re\left\{a\rho^2 \int_{A-K} \frac{d\tau}{(z - z_0)^2}\right\} + O(\rho^3) \tag{14}$$

with $|O(\rho^3)| < C\rho^3$. Denote

$$\lim_{\rho=0} \int_{A-K} \frac{d\tau}{(z - z_0)^2} = \int_A \frac{d\tau}{(z - z_0)^2} \tag{15}$$

and get from (14), in virtue of the extremal property of $D_n : F(D_n^*) \leq F(D_n)$, by passing to the limit $\rho = 0$ the condition

$$\Re\left\{a \int_A \frac{d\tau}{(z - z_0)^2}\right\} \geq 0. \tag{16}$$

This being true for each sign of a, we get the equation

$$\int_A \frac{d\tau}{(z - z_0)^2} = 0, \tag{17}$$

valid for each $z_0 \subset A$.

To deal with (17) which is a rather unusual type of functional equation for D_n, we introduce the function

$$U(x) = \int_A \log \frac{1}{| z - x |} d\tau, \qquad d\tau = d\tau_z, \qquad z = u + iv, \qquad x = \xi + i\eta, \tag{18}$$

representing the potential of the surface distribution with density 1 on A. $U(x)$ is continuous all over the x-plane and so are its first partial derivatives U_ξ and U_η. For the second derivatives

$$\begin{aligned} U_{\xi\xi} &= -\int_A \frac{d\tau}{| z - x |^2} + 2 \int_A \frac{(\xi - u)^2}{| z - x |^4} d\tau; \\ U_{\xi\eta} &= 2 \int_A \frac{(\xi - u)(\eta - v)}{| z - x |^4} d\tau; \\ U_{\eta\eta} &= -\int_A \frac{d\tau}{| z - x |^2} + 2 \int_A \frac{(\eta - v)^2}{| z - x |^4} d\tau \end{aligned} \tag{19}$$

we have the Laplace-Poisson equation

$$\Delta U = -2\pi \tag{20}$$

in A and from (17)

$$U_{\xi\xi} = U_{\eta\eta}, \qquad U_{\xi\eta} = 0 \tag{21}$$

in A. From (20) and (21) we find for $x = \xi + i\eta$ interior to each component of A

$$U(x) = -\frac{\pi}{2} (\xi^2 + \eta^2) + \alpha\xi + \beta\eta + \gamma, \tag{22}$$

with real constants α, β, γ; these constants, however, may be different for different components of A. The function

$$V(x) = \frac{1}{\pi} (U_\xi - iU_\eta) = \frac{1}{\pi} \int_A \frac{d\tau}{z - x} \tag{23}$$

has in the l-th component A_l of A the value $-x^* + c_l$ and is analytic outside A. From its continuity in the whole x-plane, we conclude that the functions

$$\mathfrak{F}(x) = x - V(x), \qquad \mathfrak{G}(x) = x + V(x) \tag{24}$$

are analytic in the exterior of A and approach the values $2\mathfrak{R}\{x_0\} + c_l$, $2i\mathfrak{J}\{x_0\} + c_l$, as x converges to a point x_0 on the boundary of A_l, and have a simple pole at $x = \infty$. Hence, $\mathfrak{F}(x)$ and $\mathfrak{G}(x)$ are the slit functions for the exterior of A. From (24) we have

$$\frac{\mathfrak{F}'(x) - \mathfrak{G}'(x)}{\mathfrak{F}'(x) + \mathfrak{G}'(x)} = -V'(x) = -\frac{1}{\pi}\int_A \frac{d\tau}{(z - x)^2}. \tag{25}$$

Now the left side member of (25) is invariant with respect to normalized conformal mapping of the exterior of A; we can, therefore, compute its value on the boundary of A, by mapping first the exterior of A on the exterior of analytic curves $\zeta = \zeta_l(t)$ and taking the value of the corresponding function

$$Q(\zeta) = \frac{\phi'(\zeta) - \Gamma'(\zeta)}{\phi'(\zeta) + \Gamma'(\zeta)}$$

at the corresponding point. But, by definition of the slit functions, $\phi'(\zeta)d\zeta/dt$ is real and $\Gamma'(\zeta)d\zeta/dt$ is purely imaginary on the analytic boundary of the domain considered; thus, the quotient $Q(\zeta)$ has the modulus 1, and consequently we have

$$\left| \frac{\mathfrak{F}'(x) - \mathfrak{G}'(x)}{\mathfrak{F}'(x) + \mathfrak{G}'(x)} \right| = 1 \qquad \text{on the boundary of } A; \tag{26}$$

hence, by the principle of the maximum, we get

$$\left| \int_A \frac{d\tau}{(z - x)^2} \right| \leq \pi \qquad \text{for all } x \text{ exterior to } A. \tag{26'}$$

We now turn to the linear part L of $D_n^{\odot}$. Let z_0 be a point of L; we take a continuum C_ρ of L containing z_0, such that its exterior can be mapped univalently on the domain $|z^* - z_0| > \rho$ by a mapping function

$$z^* = z + k\rho + \frac{a\rho^2}{(z - z_0)} + \frac{b\rho^3}{(z - z_0)^2} + \frac{c\rho^4}{(z - z_0)^3} + \frac{d\rho^5}{(z - z_0)^4} + \cdots . \tag{27}$$

It is known [6] that $|a| \leq 1$, $|b| \leq 4^3$, $|c| \leq 4^4$, $|d| \leq 4^5$, $\cdots$. We choose now another number $0 < \epsilon < 1$ and form the function

$$z^{**} = z^* + \frac{\epsilon e^{i\varphi}\rho^2}{z^* - z_0} \tag{28}$$

which maps the domain $|z^* - z_0| > \rho$ univalently on the exterior of an ellipse E with semi-axes $\rho(1 + \epsilon)$, $\rho(1 - \epsilon)$ and area $\pi\rho^2(1 - \epsilon^2)$. Comparing (27) and (28) yields

$$z^{**} = z + k\rho + \frac{(a + \epsilon e^{i\varphi})\rho^2}{z - z_0} + O(\rho^3); \tag{29}$$

this function maps D_n on a domain D_n^{**} with the area (see (14))

$$(30)\quad F(D_n^{**}) = \pi\rho^2(1-\epsilon^2) + F(D_n) - 2\Re\left\{(a+\epsilon e^{i\varphi})\rho^2 \int_A \frac{d\tau}{(z-z_0)^2}\right\} + O(\rho^3).$$

The first term on the right side comes from the area of the ellipse E. In view of the extremal property of D_n, we have $F(D_n^{**}) \leq F(D_n)$; hence, from $\rho \to 0$, we obtain

$$(31)\quad \pi(1-\epsilon^2) \leq 2\Re\left\{(a+\epsilon e^{i\varphi}) \int_A \frac{d\tau}{(z-z_0)^2}\right\} \leq 2\,|\,a+\epsilon e^{i\varphi}\,|\cdot\left|\int_A \frac{d\tau}{(z-z_0)^2}\right|.$$

We now choose φ in such a way that $\operatorname{sgn}(-a) = e^{i\varphi}$; since then $|\,a+\epsilon e^{i\varphi}\,| \leq 1-\epsilon$, we get

$$(32)\quad \pi(1+\epsilon) \leq 2\left|\int_A \frac{d\tau}{(z-z_0)^2}\right| \qquad (0<\epsilon<1).$$

Hence

$$(32')\quad \pi \leq \left|\int_A \frac{d\tau}{(z-z_0)^2}\right| \qquad (z_0 \subset L).$$

But this combined with (26′) (applicable to all points of L) yields

$$\left|\int_A \frac{d\tau}{(z-z_0)^2}\right| = \pi$$

on L and consequently (by the principle of the maximum) everywhere outside A, including the point at infinity. But at this point the integral considered vanishes. Thus the assumption of the existence of L leads to a contradiction; D_n° coincides therefore with A, and $\mathfrak{F}(x)$ and $\mathfrak{G}(x)$ are identical with $f(x)$ and $g(x)$.

We summarize:

The extremal domain D_n has a complement consisting of n domains; its slit functions satisfy the equations

$$(33)\quad f'(x) + g'(x) = 2; \qquad f'(x) - g'(x) = -\frac{2}{\pi}\int_{D_n^\circ} \frac{d\tau}{(z-x)^2}.$$

Comparing the coefficients of x^{-2} on both sides of the last equality, we find

$$(34)\quad S(D_n) = \frac{2}{\pi} F(D_n).$$

Now $S(D_n)$ is, by Theorem II, the same for all conformally equivalent domains; $F(D_n)$, on the other hand, furnishes the maximal value within this family of domains. Hence, we proved for an arbitrary domain D_n

$$(35)\quad F(D_n) \leq \frac{\pi}{2} S(D_n).$$

Thus we have obtained for the span a new definition:

The span of a domain is $2/\pi$ times the maximal area, the complement of which is conformally equivalent to this domain.

By integrating the first equality (33) we get $\frac{1}{2}[f(x) + g(x)] = x$ in the case of an *extremal* domain D_n. If D_n is an arbitrary domain of the z-plane, let $x = x(z)$ be the function of $\phi(D_n)$ mapping it on an extremal domain in the x-plane with the slit functions $f(x)$ and $g(x)$. Then we have $\frac{1}{2}[f(x(z)) + g(x(z))] = x(z)$, but $f(x(z))$ and $g(x(z))$ are the slit functions of D_n. Hence:

In each domain D_n, the arithmetic mean of the slit functions belongs to $\phi(D_n)$ and renders maximum the area of the complements of all maps, obtained by functions of $\phi(D_n)$.

From this theorem it can easily be shown *that an extremal domain is always bounded by convex curves.*

Finally we apply the inequality (11) to an *extremal* domain D_n; using (34), we get in this particular case,

$$S(D_n) \leq 2d(D_n)^2. \tag{36}$$

Since both members of the inequality are invariants for conformal normalized mapping, the inequality holds for *each* domain D_n. Equality occurs in (36) only if for the corresponding extremal domain $F(D_n) = \pi d(D_n)^2$ holds. But this equality is known to be valid only in the case of a circle; hence, we have equality in (36) only in the case of simply connected domains, as already pointed out in §1. Thus we get for $S(D_n)$ the double estimate

$$\frac{2}{\pi} F(D_n) \leq S(D_n) \leq 2d(D_n)^2. \tag{37}$$

Bibliography

1. C. Carathéodory, *Untersuchungen über die konformen Abbildungen, von festen und veränderlichen Gebieten,* Mathematische Annalen, vol. 72(1912), pp. 107–144.
2. M. Fekete, *Ueber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten,* Mathematische Zeitschrift, vol. 17(1923), pp. 228–249.
3. H. Grötzsch, *Ueber das Parallelschlitztheorem der konformen Abbildung schlichter Bereiche,* Berichte Leipzig, vol. 84(1932), pp. 15–36.
4. G. Pólya, *Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete,* II, Sitzungsberichte Akad. Berlin, 1928, pp. 280–282.
5. R. de Possel, *Zum Parallelschlitztheorem unendlich-vielfach zusammenhängender Gebiete,* Göttinger Nachrichten, 1931, pp. 199–202.
6. M. Schiffer, *A method of variation within the family of simple functions,* Proceedings of the London Mathematical Society, (2), vol. 44(1938), pp. 432–449.

Hebrew University, Jerusalem.

Commentary on

[14] *The span of multiply connected domains*, Duke Math. J. **10** (1943), 209–216.

In this seminal paper, Schiffer introduced an important conformal invariant for multiply connected plane domains, the span. The definition is in terms of conformal mappings onto canonical slit domains, and there is a complementary characterization (almost literally), in terms of an extremal problem for area under a conformal mapping. While the foundational results on conformal mapping of multiply connected domains concern the existence and uniqueness of mappings onto various canonical domains, Schiffer's paper is an early example of *using* the mappings as a tool in geometric function theory. Moreover, his study of the extremal problem is clearly influenced by his development of variational methods.

Let D be a finitely connected domain containing ∞ in its interior and consider univalent functions F in D with

$$F(z) = z + \frac{A}{z} + \frac{B}{z^2} + \cdots \qquad (1)$$

near ∞. Let $p(z) = z + a/z + \cdots$ and $q(z) = z + b/z + \cdots$ be the unique conformal mappings of D onto a domain bounded by horizontal slits and vertical slits, respectively. Then

$$\operatorname{Re} b \leq \operatorname{Re} A \leq \operatorname{Re} a.$$

Thus $\operatorname{Re} A$ can vary in an interval of length $\operatorname{Re}\{a-b\}$, and Schiffer calls this length the *span* of D. It is the same for any domain that is the image of D under a mapping of the form (1), so in that sense it is a conformal invariant of D. Schiffer proves, among other basic results, that the coefficients A in (1) cover a disk with diameter equal to the span of D.

Now consider conformal mappings of D by functions as in (1) and ask: How large can the area of the complement be? The answer is $\pi/2$ times the span of D. Also, $\Phi(z) = (1/2)(p(z)+q(z))$ is the conformal mapping of D onto the extremal domain, and $\Phi(D)$ is bounded by convex curves. Schiffer's proof is variational, in the course of which he derives some striking identities for the slit mappings of the extremal domain; see [K] for later work on these. Apparently unbeknownst to Schiffer at the time, the convexity property of Φ had already been discovered by Grunsky in his dissertation [G]. But convexity of the boundary curves does not guarantee univalence, a key property that was shown by Schiffer as part of his analysis. This point is also discussed in [34] as an independent observation and without variational methods.

An alternate approach to the area problem, once one knows that Φ is univalent and turns out to be the extremal mapping, uses an expression for Φ in terms of kernel functions and harmonic measures, no less striking, and may be found in [34]. Between [14] and [34], Schiffer revisited the area problem in [22] as an application of orthonormal families to conformal mappings, then quite new. The correct upper bound for the omitted area emerges easily, but the analysis of Φ as the extremal mapping is troublesome, and the full range of representations via kernel functions was not yet realized. Nevertheless, this was a new take on such problems and was influential in subsequent papers. For example, similar area problems were considered by Garabedian and Schiffer in [26], no doubt motivated by [14] and the ideas in [22]. See [N] as well for a compact exposition. Kühnau [Ku] introduced a notion of the span, together with an associated area problem, using quasiconformal mappings onto (inclined) parallel slit domains. He assumed that the complex dilatation of the mapping is identically zero near ∞, thus allowing for a local expansion analogous to (1).

In [AB], analytic and geometric characterizations of the span were reconsidered by Ahlfors and Beurling as examples of their general approach to defining conformal invariants and associated null-sets. Briefly, they define (relative) conformal invariants by forming $M_{\mathfrak{F}}(z_0, D) = \sup_{\mathfrak{F}} |f'(z_0)|$, where z_0 is a fixed point in D and f varies in a class $\mathfrak{F}(D)$ of analytic functions in D that is invariant under conformal mappings of D.[1]

[1] In [AB] most results are given for z_0 a finite point in D, unlike Schiffer's original normalization, but this is not essential (nor was it essential for Schiffer to use $z_0 = \infty$).

If $M_{\mathfrak{F}}(z_0, D) = 0$ (which generally implies that it vanishes identically), then the complement of D is a null-set for the class $\mathfrak{F}(D)$.

For an analytic approach to the span, consider the class $\mathfrak{D}(D)$ of analytic functions with a fixed bound on the Dirichlet integral, specifically

$$\iint_D |f'(z)|^2 \, dxdy \leq \pi.$$

Then $M_{\mathfrak{D}}(z_0, D)^2 = (1/2)\mathrm{span}(D)$, and the extremal is given in terms of the slit mappings as $(p-q)/\sqrt{2\,\mathrm{span}(D)}$. For a geometric characterization, the authors use omitted area to *define* a class of competing functions. Namely, let $\mathfrak{S}\mathfrak{E}(D)$ be the set of univalent functions in D such that $1/(f(z) - f(z_0))$ omits a set of area at least π. Then one has again

$$M_{\mathfrak{S}\mathfrak{E}}(z_0, D)^2 = (1/2)\mathrm{span}(D),$$

and the extremal is $(p+q)/\sqrt{2\,\mathrm{span}(D)}$; the paper includes a proof that $p+q$ is univalent.

Since $M_{\mathfrak{D}}(z_0, D) = M_{\mathfrak{S}\mathfrak{E}}(z_0, D)$, these results serve to describe the identical classes of null-sets for $\mathfrak{D}(D)$ and $\mathfrak{S}\mathfrak{E}(D)$. This direction of research demonstrates the lasting influence of the span in studying small boundaries and removable point sets for classes of analytic functions on plane domains, with similar applications to Riemann surfaces. The latter is tied up with the classification problem for Riemann surfaces, for example, assessing the size of the boundary for the purpose of supporting nonconstant analytic or harmonic functions with a finite Dirichlet integral, necessary for existence theorems. Schiffer himself contributed to this in [85], 22 years after [14]. For a technical discussion see [AS, RS, SN, SO]. For an informal and personal account, see [A].

References

[A] Lars V. Ahlfors, *Riemann surfaces and small point sets*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 49–57.

[AB] Lars Ahlfors and Arne Beurling, *Conformal invariants and function theoretic null-sets*, Acta Math. **83** (1950), 101–129.

[AS] L. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, 1960.

[G] Helmut Grunsky, *Neue Abschätzungen zur konformen Abbildung ein- und mehrfach zusammenhängender Bereiche*, Schriften Math. Sem. Inst. Angew. Math. Univ. Berlin **1** (1932), 95–140.

[K] Yukio Kusunoki, *A new proof of Schiffer's identities on planar Riemann surfaces*, Proc. Japan Acad. Ser. A Math. Sci. **60** (1984), 345–348

[Ku] Reiner Kühnau, *Die Spanne von Gebieten bei quasikonformen Abbildungen*, Arch. Rational Mech. Anal. **65** (1977), 299–303.

[N] Zeev Nehari, *Conformal Mapping*, McGraw-Hill, 1952; Dover edition, 1975.

[RS] Burton Rodin and Leo Sario, *Principal Functions*, D. Van Nostrand Co., 1968

[SN] L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer-Verlag, 1970.

[SO] L. Sario and K. Oikawa, *Capacity Functions*, Springer-Verlag, 1969.

Brad Osgood