7

Input Design for Identification in Connected Systems

Input design is of essential importance in system identification to provide sufficient probing capabilities to guarantee the convergence of parameter estimators to their true values; namely, the estimators are consistent. Input conditions for consistent estimation depend on sensor characteristics, system configurations, noise locations and distributions, and identification algorithms. The previous chapters consider only the basic formulation in which the input u_k can be directly designed. This chapter covers input design in more general system configurations.

The system configurations illustrated in Figure 2.2 represent typical scenarios in which identification experiments must be performed. They introduce challenges in input design, signal measurements, and interaction with control tasks. In these configurations, the input u to the plant P, which is either FIR or rational with n_0 parameters, may be measured with noise corruption but cannot be directly selected. Only the external input r can be designed. In these configurations, u_k is the output of a possibly unknown stable system with input r_k. This chapter resolves several issues that are critical for applying the methods of this book to system identification in filtering and closed-loop systems.

Section 7.1 establishes conditions under which a periodic and full-rank signal will retain these features after passing through a stable system, even when the system is unknown. As a result, the design of external probing signals or dithers can be easily accomplished. Section 7.2 details such a design, especially for the typical case of tracking control. In general, input noises, including input measurement noises and actuator noises, will affect signal rank and introduce identification bias. Sections 7.3 and 7.4 are

L.Y. Wang et al., *System Identification with Quantized Observations*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4956-2_7,
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devoted to developing input design principles and modified algorithms to recover signal richness and gain convergence.

7.1 Invariance of Input Periodicity and Rank in Open- and Closed-Loop Configurations

A condition for u_k to provide sufficient probing capability for the convergence of parameter estimates is that u is n_0-periodic and full rank. In this chapter, such conditions will be called "sufficient richness" conditions, to avoid confusion with the typical input "persistent excitation" conditions. Here we would like to establish relationships between periodicity and rank properties of the external signal r and those of u.

Let H be a linear time-invariant and stable system with impulse response $\{h_k\}$. Suppose that $u = Hr$, or in the time domain

$$u_k = \sum_{l=0}^{\infty} h_l r_{k-l}. \tag{7.1}$$

Suppose that the discrete Fourier transform (DFT) of H is

$$H(e^{i\omega}) = \sum_{l=0}^{\infty} h_l e^{-i\omega l}.$$

Theorem 7.1. *Suppose that r is n_0-periodic and full rank. Then u is also n-periodic and full rank if and only if $H(e^{i\omega}) \neq 0$, for $\omega = \omega_k := (2\pi k/n_0), k = 1, \ldots, n_0$.*

Proof. Since r is n_0-periodic and full rank, by Corollary 2.3, the frequency samples of r are nonzero, $R_k = \sum_{l=1}^{n_0} r_l e^{-i\omega_k l} \neq 0$, $k = 1, \ldots, n_0$. Since r is n_0-periodic, and H is LTI and stable, u is also n_0-periodic after a short transient. Furthermore, the frequency samples U_k of u are related to R_k by

$$\begin{aligned} U_k &= \sum_{l=1}^{n_0} u_l e^{-i\omega_k l} = \sum_{l=1}^{n_0} \sum_{t=0}^{\infty} h_t r_{l-t} e^{-i\omega_k l} \\ &= \sum_{t=0}^{\infty} h_t e^{-i\omega_k t} \sum_{l=1}^{n_0} r_{l-t} e^{-i\omega_k (l-t)} = H(e^{i\omega_k}) R_k. \end{aligned}$$

Here, the cyclic property of the DFT is applied:

$$R_k = \sum_{l=1}^{n_0} r_l e^{-i\omega_k l} = \sum_{l=1}^{n_0} r_{l-t} e^{-i\omega_k (l-t)}.$$

By Corollary 2.3, u is full rank if and only if $U_k \neq 0$, $k = 1, \ldots, n_0$. However, by hypothesis, $R_k \neq 0$, $k = 1, \ldots, n_0$. As a result, $U_k \neq 0$ if and only if $H(e^{i\omega_k}) \neq 0$, $k = 1, \ldots, n_0$. □

Example 7.2. The necessity of the condition of Theorem 7.1 can be verified by examining the following second-order system: $u_k = r_k + r_{k-1}$. When r is a 2-periodic signal and full rank, u_k is a constant and hence is not rank 2. This is due to the fact that $H(e^{i\omega}) = 1 + e^{i\omega}$ and for $\omega = \omega_1 = 2\pi/2 = \pi$, $H(e^{i\omega_1}) = 0$.

Remark 7.3. Theorem 7.1 claims that for any system H not having annihilating zeros at n_0 points $e^{i\omega_k}$, $\omega_k = (2\pi k/n_0), k = 1, \ldots, n_0$, on the unit circle, sufficient richness capability of the signal r is always preserved after passing through H. In particular, for the feedback configuration in Figure 2.2, we have the following result indicating that input richness properties are invariant under a feedback mapping.

(A7.1) Consider the feedback configuration in (b) of Figure 2.2. Assume that for $\omega_k = (2\pi k/n_0), k = 1, \ldots, n_0$, $K(e^{i\omega})$ does not have zeros at ω_k; and $P(e^{i\omega})$ and $F(e^{i\omega})$ do not have singularities (such as poles) at ω_k.

Corollary 7.4. *Under Assumption* (A7.1), $M = K/(1 + PKF)$ *does not have annihilating zeros at* $\omega_k = 2\pi k/n_0, k = 1, \ldots, n_0$. *As a result, if* r *is* n_0*-periodic and full rank, so is* u.

Proof. From

$$M(e^{i\omega}) = \frac{K(e^{i\omega})}{1 + P(e^{i\omega})K(e^{i\omega})F(e^{i\omega})},$$

it is clear that the zeros of M are either the zeros of K or the singularities (such as poles) of P or F. By assumption (A7.1), $K(e^{i\omega_k}) \neq 0$, and ω_k is not a singularity point of $P(e^{i\omega})$ or $F(e^{i\omega})$. Hence, $M(e^{i\omega_k}) \neq 0$, $k = 1, \ldots, n_0$. Now by Theorem 7.1, u is n_0-periodic and full rank whenever r is n_0-periodic and full rank. □

7.2 Periodic Dithers

Consider the tracking configuration in Figure 7.1. When the desired output is r_0, usually $r = r_0$ is the set point. However, a constant $r_0 \neq 0$ is 1-periodic. It is only good for the identification of a gain system (namely, $n_0 = 1$). This is an indication that the goals of control and identification are usually not consistent.

To enhance probing capability, one may add a small n_0-periodic dither ϖ_k to r_0, leading to $r_k = \varpi_k + r_0$. Since

$$u_k = M r_k = M \varpi_k + M r_0 = v_k + \mu,$$

where v_k is an n_0-periodic signal and $\mu = M r_0$ a constant, we need to establish rank conditions on u_k.

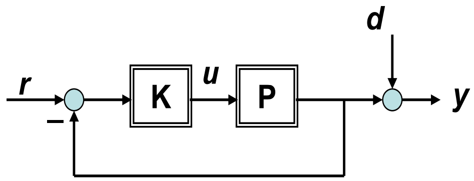

FIGURE 7.1. Tracking configuration

More generally, consider an input signal u: $u_k = v_k + e_k$, which is a perturbation of v. Suppose that v_k is n_0-periodic and full rank. We would like to establish conditions under which u_k is also n_0-periodic and full rank.

(A7.2) Both v_k and e_k are n_0-periodic.

Under Assumption (A7.2), the Toeplitz matrices for v, e, and u, denoted by Φ_v, Φ_e, and Φ_u, respectively, are circulant matrices. Let their corresponding frequency samples be

$$\begin{aligned}
\Gamma^u &= \mathcal{F}[u] = \{\gamma_k^u, k = 1, \ldots, n_0\}, \\
\Gamma^v &= \mathcal{F}[v] = \{\gamma_k^v, k = 1, \ldots, n_0\}, \\
\Gamma^e &= \mathcal{F}[e] = \{\gamma_k^e, k = 1, \ldots, n_0\}.
\end{aligned}$$

Theorem 7.5. *Under Assumption* (A7.2), *u is full rank if and only if $\gamma_k^v + \gamma_k^e \neq 0$, $k = 1, \ldots, n_0$.*

Proof. This follows immediately from $\gamma_k^u = \gamma_k^v + \gamma_k^e$ and the fact that Φ_u is full rank if and only if its frequency samples do not contain 0. □

We now consider the special case when $e_k \equiv \mu$, which is a typical case in tracking problems as shown above.

Corollary 7.6. *Suppose v_k is n_0-periodic and full rank and $e_k = \mu$. Then u_k is n_0-periodic. Let $\eta = \frac{1}{n_0} \sum_{j=1}^{n_0} v_j$. u is full rank if and only if $\mu \neq -\eta$.*

Proof. Since v_k is full rank, by Corollary 2.3 we have $\gamma_k^v \neq 0$, $k = 1, \ldots, n_0$. In particular,

$$\gamma_n^v = \sum_{j=1}^{n_0} v_j = n_0 \eta.$$

Moreover, the frequency samples of $e_k \equiv \mu$ are

$$\gamma_k^e = 0, \; k = 1, \ldots, n_0 - 1, \; \text{and } \gamma_{n_0}^e = n_0 \mu.$$

Consequently, by Theorem 7.5, u_k is full rank if and only if

$$\gamma_{n_0}^v + \gamma_{n_0}^e \neq 0.$$

That is, $n_0\eta + n_0\mu \neq 0$, or $\mu \neq -\eta$, as claimed. □

Corollary 7.6 may be verified directly by matrix manipulations. Toeplitz matrices Φ_u, Φ_v, and Φ_e for u, v, and e, respectively, are

$$\begin{aligned}\Phi_u &= \Phi_v + \Phi_e \\ &\sim \begin{bmatrix} n_0\eta + n_0\mu & 0 & \dots & 0 \\ v_1 + \mu & v_{n_0} - v_1 & \ddots & v_2 - v_1 \\ \vdots & \ddots & \ddots & \vdots \\ v_{n_0-1} + \mu & v_{n_0-2} - v_{n_0-1} & \dots & v_{n_0} - v_{n_0-1} \end{bmatrix},\end{aligned}$$

by adding the second to nth rows to the first row, followed by subtracting the first column from the second to nth columns. The last matrix is full rank since $\eta + \mu \neq 0$ and the lower right $(n_0 - 1) \times (n_0 - 1)$ submatrix, which is obtained by performing elementary operations from Φ_v, is full rank.

7.3 Sufficient Richness Conditions under Input Noise

Under the system configurations in Figure 2.2, $u = Mr$ is generated from r by a possibly unknown system M. In the previous sections, u is assumed to be accurately measured. When u is further corrupted by noise, it can no longer be exactly measured. Furthermore, the actual values of u cannot be directly derived from r since M is unknown. Sufficient richness conditions and identification algorithms under this scenario will be explored in this section.[7.1]

We will consider two cases of input noises shown in Figure 7.2:

1. Input measurement noise ε_k: When u is measured by a regular sensor, the measured values are related to u by $w_k = u_k + \varepsilon_k$, where ε_k is the measurement noise.

2. Actuator noise e_k: In this case, the actual input to the plant is $u_k = v_k + e_k$, where $v_k = Mr$, and e_k is the actuator noise.

As a result, the measured input is $w_k = v_k + e_k + \varepsilon_k$ and identification of the plant must be performed from the observation data on w_k and $s_k = \mathcal{S}(y_k)$.

[7.1] Input noises cause errors in the regressors, leading to a case of errors-in-variables. It is well known in statistical analysis that errors-in-variables will introduce estimation bias. The discussions here involve further complications since the input u cannot be directly designed and measurements are binary valued.

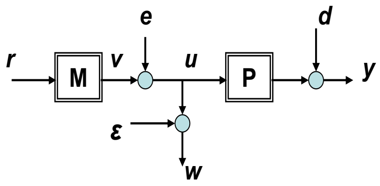

FIGURE 7.2. Input noise configuration

(A7.3) $\{v_k\}$ is n_0-periodic and full rank. $\{e_k\}$ and $\{\varepsilon_k\}$ are sequences of i.i.d. random variables with zero mean and finite variances such that $\{e_k\}$ and $\{\varepsilon_k\}$ are independent.

Denote the $n_0 \times n_0$ Toeplitz matrices for w and v by

$$\Phi_l^w = \begin{bmatrix} w_{ln_0} & w_{ln_0-1} & \cdots & w_{ln_0-n_0+1} \\ w_{ln_0+1} & w_{ln_0} & \ddots & w_{ln_0-n_0+2} \\ \vdots & \ddots & \ddots & \vdots \\ w_{ln_0+n_0-1} & w_{ln_0+n_0-2} & \cdots & w_{ln_0} \end{bmatrix},$$

$$\Phi^v = \begin{bmatrix} v_{n_0} & v_{n_0-1} & \cdots & v_1 \\ v_1 & v_{n_0} & \ddots & v_2 \\ \vdots & \ddots & \ddots & \vdots \\ v_{n_0-1} & v_{n_0-2} & \cdots & v_{n_0} \end{bmatrix}.$$

Although Φ_l^w is not circulant and varies with l, the limit of their averages is a full-rank circulant matrix.

Lemma 7.7. *Under Assumption* (A7.3), $\sum_{l=1}^N \Phi_l^w/N \to \Phi^v$ *w.p.1 as* $N \to \infty$.

Proof. This follows directly from the strong law of large numbers, applied to each element of the matrices. □

We consider first the case of measurement noise only. Actuator noises will be discussed in the next section. In this case, $e_k = 0$, for all k. Hence, $u_k = v_k$, $w_k = u_k + \varepsilon_k$, and $\Phi^u = \Phi^v := \Phi$. Due to measurement noise, the actual u_k is unknown. As a result, Φ is unknown and cannot be used directly in identification algorithms. However, by Lemma 7.7 it can be estimated asymptotically by averaging. The following algorithm utilizes this idea to estimate θ.

We shall use the FIR model for discussion here,

$$y_k = \phi_k'\theta + d_k. \tag{7.2}$$

Assume that y_k is measured by a binary-valued sensor of threshold C. We use the following notation for elementwise vector functions. For the distribution function $F(\cdot)$ and a vector $x = [x_1, \ldots, x_{n_0}]' \in \mathbb{R}^{n_0}$, we define

$$\begin{aligned} F(x) &= [F(x_1), \ldots, F(x_{n_0})]' \in \mathbb{R}^{n_0}, \\ G(x) &= [F^{-1}(x_1), \ldots, F^{-1}(x_{n_0})]' \in \mathbb{R}^{n_0}. \end{aligned} \tag{7.3}$$

Similarly, for

$$\alpha = [\alpha_1, \ldots, \alpha_{m_0}]' \text{ and } c = [c_1, \ldots, c_{m_0}]' \in \mathbb{R}^{n_0},$$

write

$$\mathbf{I}_{\{\alpha \le c\}} = [I_{\{\alpha_1 \le c_1\}}, \ldots, I_{\{\alpha_{m_0} \le c_{m_0}\}}]'.$$

We use $\mathbb{1}_\ell$ and $\mathbf{0}_\ell \in \mathbb{R}^\ell$ to denote column vectors of dimension ℓ with all components being 1 and 0, respectively.

Recall that if Φ were known, a consistent estimator of θ would be

$$\theta_N = \Phi^{-1}(C\mathbb{1} - G(\xi_N)).$$

Define $s_l = [s_{ln_0}, \ldots, s_{ln_0+n_0-1}]'$. This estimator is no longer causal since it employs the unknown Φ in computing θ_N. In other words, one needs the future information on the sequence $\{w_k\}$ in computing θ_N. The following algorithm replaces the future information Φ by a sample average.

Let

$$\Phi_N = \frac{1}{N}\sum_{l=1}^{N} \Phi_l^w.$$

When Φ_N is nonsingular, define

$$\theta_N = \Phi_N^{-1}(C\mathbb{1} - G(\xi_N)).$$

This estimator can be recursively defined as follows.

1. Initial conditions: $\xi_1 = s_1$, $\Phi_1 = \Phi_1^w$ is generated from initial data on w, $\theta_1 = 0$.

2. Recursion: Suppose that at N, ξ_N, Φ_N, and θ_N have been obtained. Then at $N+1$, we update

$$\begin{aligned} \xi_{N+1} &= \xi_N - \frac{1}{N+1}\xi_N + \frac{1}{N+1}s_{N+1}, \\ \Phi_{N+1} &= \Phi_N - \frac{1}{N+1}\Phi_N + \frac{1}{N+1}\Phi_{N+1}^w, \\ \theta_{N+1} &= \begin{cases} \Phi_{N+1}^{-1}(C\mathbb{1} - G(\xi_{N+1})), \text{ if } \Phi_{N+1} \text{ is nonsingular}, \\ \theta_N, \text{ if } \Phi_{N+1} \text{ is singular}. \end{cases} \end{aligned} \tag{7.4}$$

Theorem 7.8. *Under Assumption* (A7.3), $\theta_N \to \theta$ *w.p.1 as* $N \to \infty$.

Proof. Since the true input to the plant is u, $\xi_N \to \xi = F(C\mathbb{1} - \Phi\theta)$ w.p.1. Then $\theta_N - \theta = \Phi_N^{-1}(G(\xi) - G(\xi_N)) + (\Phi_N^{-1} - \Phi^{-1})(C\mathbb{1} - G(\xi))$. By the strong law of large numbers, the convergence $\theta_N - \theta \to 0$ w.p.1 follows from $\Phi_N \to \Phi$ and $\xi_N \to \xi$ w.p.1, the continuity of F^{-1}, and the invertibility of Φ. □

7.4 Actuator Noise

Unlike measurement noise ε_k that affects measured input values but does not enter the plant, actuator noise e_k affects the output y_k of the plant. Consider the case $u_k = v_k + e_k$ and $w_k = u_k$. To understand the impact of e_k, we express the regressor in (7.2) by ϕ_k^u or ϕ_k^v, depending on which signal is used in the regressor. Under Assumption (A7.3), v is n_0-periodic and full rank, but u is not periodic. However, by Lemma 7.7,

$$\frac{1}{N}\sum_{l=1}^{N} \Phi_l^u \to \Phi^v \quad \text{w.p.1 as} \quad N \to \infty.$$

Since $u_k = v_k + e_k$, we have

$$\begin{aligned} y_k &= (\phi_k^u)'\theta + d_k = (\phi_k^v)'\theta + (\phi_k^e)'\theta + d_k \\ &= (\phi_k^v)'\theta + z_k. \end{aligned}$$

Observe that the equivalent noise z_k is

$$\begin{aligned} z_k &= (\phi_k^e)'\theta + d_k \\ &= a_0 e_k + \cdots + a_{n_0-1} e_{k-n_0+1} + d_k. \end{aligned}$$

Under Assumption (A7.3), although $\{z_k\}$ may not be independent, it is strictly stationary. Recall that $\{z_k\}$ is strictly stationary if for any positive integer ν, points $t_1, \ldots, t_\nu \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+$, the joint distribution of $\{z_{t_1}, \ldots, z_{t_\nu}\}$ is the same as that of $\{z_{t_1+l}, \ldots, z_{t_\nu+l}\}$ (i.e., its finite-dimensional distributions are translation invariant; see [47, p. 443]). Denote the distribution function by $F_z(x;\theta)$. A moment of reflection reveals that the sequence is $(n_0 - 1)$-dependent. A precise definition of $(n_0 - 1)$-dependence can be found in [8, p. 167, Example 1]. Since an $(n_0 - 1)$-dependent sequence belongs to the class of ϕ-mixing signals, whose remote past and distant future are asymptotically independent, the sequence is strongly ergodic [47, p. 488]. That is, a strong law of large numbers still holds.

Following (7.4), define

$$\xi_N = \frac{1}{N}\sum_{j=1}^{N} s_j.$$

Let θ_N be the solution to

$$\xi_N = F_z(C\mathbb{1} - \Phi\theta_N; \theta_N). \tag{7.5}$$

For any ϑ, define the Jacobian matrix

$$J(\vartheta) = \frac{\partial F_z(C\mathbb{1} - \Phi\vartheta; \vartheta)}{\partial\vartheta}.$$

A sufficient condition for invertibility of the function in (7.5) is that $J(\theta_N)$ is full rank. In this case, by denoting the inverse of $\xi = F_z(C\mathbb{1} - \Phi\vartheta; \vartheta)$ as $\vartheta = H(\xi)$, the estimate θ_N in (7.5) may be symbolically written as $\theta_N = H(\xi_N)$.

Proposition 7.9. *If $H(\cdot)$ exists and is continuous, then $\theta_N \to \theta$ w.p.1 as $N \to \infty$.*

Proof. By the strong law of large numbers, $\xi_N \to \xi = F_z(C\mathbb{1} - \Phi\theta; \theta)$ w.p.1. Since $H(\cdot)$ exists and is continuous, $\theta_N = H(\xi_N) \to H(\xi) = \theta$ w.p.1. □

For a given ϑ, denote the inverse of $F_z(x; \vartheta)$ (with respect to x) by

$$G_z(x; \vartheta) = F_z^{-1}(x; \vartheta). \tag{7.6}$$

Computationally, it is observed that for a given ξ, the implicit function $\xi = F_z(C\mathbb{1} - \Phi\vartheta; \vartheta)$ of ϑ may be expressed as a fixed-point equation $\vartheta = \Phi^{-1}(C\mathbb{1} - G_z(\xi; \vartheta))$.

Next, a special case will be considered. Suppose that $\{e_k\}$ is a sequence of i.i.d. normal random variables with zero mean and variance σ_e^2, and $\{d_k\}$ is a sequence of i.i.d. normal random variables with zero mean and variance σ_d^2. Then,

$$z = a_0 e_k + \cdots + a_{n_0-1} e_{k-n_0+1} + d_k$$

is also normally distributed and has zero mean and variance

$$\sigma_z^2(\theta) = (a_0^2 + \cdots + a_{n_0-1}^2)\sigma_e^2 + \sigma_d^2 = \sigma_e^2\|\theta\|^2 + \sigma_d^2.$$

Let $F_0(x)$ be the normal distribution function of zero mean and variance 1. Then $F_z(x; \vartheta) = F_0(x/\sigma_z(\vartheta))$. It follows that

$$F_z(C\mathbb{1} - \Phi\vartheta; \vartheta) = F_0\left(\frac{C\mathbb{1} - \Phi\vartheta}{\sigma_z(\vartheta)}\right),$$

and the Jacobian matrix is

$$\begin{aligned} J(\vartheta) &= \frac{dF_z(C\mathbb{1} - \Phi\vartheta; \vartheta)}{d\vartheta} \\ &= -\frac{1}{\sigma_z}\frac{dF_0}{dx}\left[\Phi\left(I_{n_0} - \frac{\sigma_e^2\vartheta\vartheta'}{\sigma_z^2}\right) + \frac{\sigma_e^2 C\mathbb{1}\vartheta'}{\sigma_z^2}\right], \end{aligned}$$

where

$$x = \frac{C\mathbb{1} - \Phi\vartheta}{\sigma_z(\vartheta)}.$$

Since

$$\frac{dF_0}{dx} = \mathrm{diag}(f_z(C - \phi_1'\vartheta), \ldots, f_z(C - \phi_{n_0}'\vartheta))$$

is full rank, where f_z is the density function of F_z, the Jacobian matrix $J(\vartheta)$ is full rank if and only if

$$\Phi(I_{n_0} - \sigma_e^2\vartheta\vartheta'/\sigma_z^2) + \sigma_e^2 C\mathbb{1}\vartheta'/\sigma_z^2$$

is full rank.

Remark 7.10. It is easily verified that if A is an n_0-dimensional square matrix with $\|A\| < 1$, then $I_{n_0} + A$ is invertible, where I_{n_0} denotes the $n_0 \times n_0$ identity matrix. Moreover, if A is an n_0-dimensional invertible matrix and $\|B\| < \|A^{-1}\|^{-1}$, then $A + B$ is invertible.

Theorem 7.11. *If*

$$\|\Phi^{-1}\| < \frac{2\sigma_d^3}{C\sigma_e\sqrt{n_0}(\sigma_e^2\|\theta\|^2 + \sigma_d^2)}, \tag{7.7}$$

then $\theta_N = H(\xi_N) \to \theta$ w.p.1 as $N \to \infty$.

Proof. Noting that

$$\left\|\frac{\sigma_e^2\theta\theta'}{\sigma_z^2}\right\| = \left\|\frac{\sigma_e^2\theta\theta'}{\sigma_e^2\theta'\theta + \sigma_d^2}\right\| = \frac{\sigma_e^2\theta'\theta}{\sigma_e^2\theta'\theta + \sigma_d^2} < 1,$$

by Remark 7.10, $I_{n_0} - \sigma_e^2\theta\theta'/\sigma_z^2$ is full rank. Since

$$\left\|\frac{\sigma_e^2 C\mathbb{1}\theta'}{\sigma_z^2}\right\| \le \frac{\sigma_e^2\|C\mathbb{1}\|\|\theta\|}{\sigma_e^2\theta'\theta + \sigma_d^2} \le \frac{\sigma_e^2 C\sqrt{n}\|\theta\|}{2\sigma_e\sigma_d\|\theta\|} = \frac{\sigma_e C\sqrt{n}}{2\sigma_d},$$

we have

$$\begin{aligned} &\left\|\frac{\sigma_e^2 C\mathbb{1}\theta'}{\sigma_z^2}\left(I_{n_0} - \frac{\sigma_e^2\theta\theta'}{\sigma_z^2}\right)^{-1}\right\| \\ &= \left\|\frac{\sigma_e^2 C\mathbb{1}\theta'}{\sigma_z^2}\sum_{i=0}^{\infty}\left(\frac{\sigma_e^2\theta\theta'}{\sigma_z^2}\right)^i\right\| < \|\Phi^{-1}\|^{-1}. \end{aligned}$$

By Remark 7.10,

$$\Phi + \frac{\sigma_e^2 C \mathbb{1}\theta'}{\sigma_z^2}\left(I_{n_0} - \frac{\sigma_e^2 \theta\theta'}{\sigma_z^2}\right)^{-1}$$

is invertible. Then,

$$\Phi\left(I_{n_0} - \frac{\sigma_e^2 \theta\theta'}{\sigma_z^2}\right) + \frac{\sigma_e^2 C \mathbb{1}\theta'}{\sigma_z^2}$$

is invertible. So, $J(\theta)$ is invertible. Hence, Proposition 7.9 confirms that $\theta_N \to \theta$ w.p.1. □

Remark 7.12. Equation (7.7) can be used to design input signals. Indeed, suppose that the prior information on the unknown parameters is that $\|\theta\| \le \beta$. By using β^2 in place of $\|\theta\|^2$, one can design an input such that Φ satisfies (7.7). Consequently, consistency of the estimates will be guaranteed for any $\theta \in \{\vartheta : \|\vartheta\| \le \beta\}$.

Example 7.13. Suppose the true system is $y_k = 0.9u_k + 1.1u_{k-1} + d_k$. Hence, the true parameters are $\theta = [0.9, 1.1]'$ and $\|\theta\|^2 = 1.93$. Assume that the prior information on θ is that $\|\theta\|^2 \le 2$. The output measurement noise d_k is i.i.d. normally distributed with zero mean and variance $\sigma_d^2 = 4$. The input signal $u_k = v_k + e_k$, where v_k is 2-periodic with its one-period values $v_1 = 3, v_2 = 15$, and e_k is a sequence of i.i.d., normally distributed noise of zero mean and variance $\sigma_e^2 = 1$. By direct calculation, $\|\Phi^{-1}\| = 0.083$. For $C = 20$, and the prior information $\|\theta\|^2 \le 2$, the right-hand side of (7.7) is 0.094. Hence, the input satisfies condition (7.7). In fact, under this input, (7.7) is satisfied for all $\theta \in \{\vartheta : \|\vartheta\|^2 \le 2\}$.

An identification algorithm is devised for this example. At each step N, ξ_N is calculated from (7.4). Then the estimate θ_N is derived by solving (7.5). The inverse function of normal distribution is calculated by the Matlab function *norminv*. The simulation in Figure 7.3 illustrates the convergence of parameter estimates. The relative estimation error $\|\theta_N - \theta\|/\|\theta\|$ is used to evaluate the accuracy and convergence of the estimates. Figure 7.3 shows the parameter convergence of this algorithm.

7.5 Notes

This chapter presents conditions on input ensembles that provide sufficiently rich probing power for convergence of parameter estimates. It is an extension of the basic input design covered in the previous chapters and is based on [109]. We summarize here several reasons that periodic signals are of essential importance for quantized identification.

The classical control theory of Bode and Nyquist characterizes systems by using periodic input signals (frequency responses). They are relatively easy

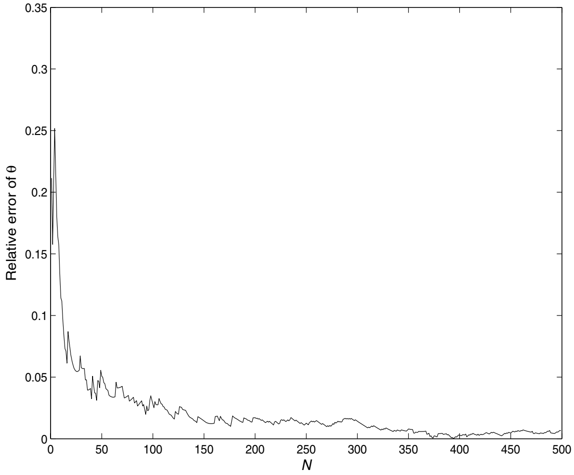

FIGURE 7.3. Relative errors of parameter estimates

to apply and there are many special devices for obtaining system frequency responses. Periodic inputs are especially useful for quantized identification for several technical reasons:

(1) Periodic inputs are uniformly bounded. In contrast, to identify stochastic systems, a typical method uses Gaussian-distributed signals that are unbounded and more difficult to apply in practical systems. The truncation of unbounded signals due to input saturation may cause bias in system identification.

(2) Essential features for a periodic signal to be rich for identification are certain rank conditions, rather than the magnitudes of the signals. As a result, one may use small probing inputs for identification with the benefit of contained perturbation to system operations.

(3) Periods and ranks of periodic signals are shift invariant. As such, they are natural choices for achieving "persistent identification" for time-varying systems [97, 103].

(4) As established in this chapter, periods and ranks of periodic signals are invariant after passing through a linear-time-invariant system (with some mild conditions). Consequently, an externally applied periodic signal can be easily designed for identification of a plant in a closed-loop setting [103].

(5) As shown in the previous chapters, under periodic inputs, the identification of a system with multiple parameters under quantized sensors can often be reduced to a number of greatly simplified identification problems for gains.

(6) Under periodic inputs, our algorithms have been shown to be asymptotically optimal, since they achieve the CR lower bounds asymptotically.