CHAPTER 7

Hecke Algebras and K-Theory

7.1 Affine Weyl Groups and Hecke Algebras

From now on fix a complex torus T. Let $P = \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ denote the weight lattice and $P^\vee = \mathrm{Hom}_{alg}(\mathbb{C}^*, T)$ the coweight lattice. Both P and $P^\vee$ are free abelian groups of rank $\dim_{\mathbb{C}} T$. There is a natural duality pairing

$$\langle\ ,\ \rangle : P \times P^\vee \to \mathbb{Z}$$

defined as follows. Let $\alpha \in P$ and $a \in P^\vee$. Then the composition $\mathbb{C}^* \xrightarrow{a} T \xrightarrow{\alpha} \mathbb{C}^*$ is an algebraic group homomorphism $\mathbb{C}^* \to \mathbb{C}^*$, hence is of the form $z \mapsto z^n$, for a certain $n = n(\alpha, a) \in \mathbb{Z}$. We set $\langle \alpha, a\rangle := n(\alpha, a)$. The pairing is *perfect*, i.e., induces natural group isomorphisms

$$P^\vee = Hom(P, \mathbb{Z}), \quad \text{and} \quad P = \mathrm{Hom}\,(P^\vee, \mathbb{Z}).$$

Let $R \subset P$ be a reduced (not necessarily finite) root system as defined, e.g., in 3.1.22. There is a slight difference with 3.1.22, since now we are working with lattices instead of vector spaces. This makes axiom 3.1.22(3) superfluous. Thus it is assumed only that, in addition to the above data, a subset $R^\vee \subset P^\vee$, called the dual root system, and a specified bijection $R \leftrightarrow R^\vee$, $\alpha \leftrightarrow \check\alpha$ are given such that the following three properties hold.

(1) $\langle \alpha, \check\alpha\rangle = 2$ for any $\alpha \in R$;
(2) For any $\alpha \in R$ the transformation $s_\alpha : P \to P$ (resp. $s_{\check\alpha} : P^\vee \to P^\vee$) given by the formula $s_\alpha(x) = x - \langle x, \check\alpha\rangle \cdot \alpha$ (resp. $s_{\check\alpha}(y) = y - \langle \alpha, y\rangle \cdot \check\alpha$) preserves the subset $R \subset P$ (resp. $R^\vee \subset P^\vee$).
(3) If $\alpha \in R$ then $c \cdot \alpha \in R$ if and only if $c = \pm 1$.

Throughout this chapter we view W, the Weyl group of the root system R, as a Coxeter group with respect to the set of simple reflections corresponding to a fixed choice of simple roots $S \subset R$. Later on, R will be the root system of a complex semisimple group G. In this case T and W will

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become the "abstract" maximal torus and the "abstract" Weyl group of G introduced in Section 3.1.22. Abusing the notation we will write W instead of $\mathbb{W}$ in this chapter.

Let $w \in W$. A factorization $w = s_1 \cdot \ldots \cdot s_r$, $s_i \in S$, is said to be a *reduced expression* if it has a minimal possible number of factors. Although the reduced expression of a given element is by no means unique, all reduced expressions have the same number of factors, to be denoted $\ell(w)$. We put $\ell(1) = 0$, where $1 \in W$ is the identity element. The function $\ell : w \mapsto \ell(w)$ on W is called the *length function.*

We can now introduce

Definition 7.1.1. The Hecke algebra of the Coxeter group (W, S) is a $\mathbb{Z}[q, q^{-1}]$-algebra H_W with generators T_s, $s \in S$ subject to the following defining relations, cf. (3.1.24)–(3.1.25):

(i) $T_{s_\alpha} \cdot T_{s_\beta} \cdot T_{s_\alpha} \ldots = T_{s_\beta} \cdot T_{s_\alpha} \cdot T_{s_\beta} \ldots,$ $\quad m(\alpha, \beta)$ factors;

(ii) $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0$.

These relations specialize at $q = 1$ to the relations (3.1.24)–(3.1.25) in the group algebra of the Weyl group. Thus, one may think of H_W as a q-analogue of $\mathbb{Z}[W]$. The definition above will not be used in this book, since it is typically rather difficult to verify the braid relation 7.1.1(i) in practice. Instead, we will use the following result, proved e.g. in [Bour, Chap. IV, sec. 2, Ex 34], that provides a very convenient "characterization" of the Hecke algebra. This 'characterization' is most useful in applications, so that we will sometimes refer to it as a "definition."

Proposition 7.1.2. *The Hecke algebra $H = H_W$ associated to W, has a free $\mathbb{Z}[q, q^{-1}]$-basis $\{T_w \mid w \in W\}$ such that the following multiplication rules hold:*

(a) $(T_s + 1)(T_s - q) = 0$ *if $s \in S$ is a simple reflection.*

(b) $T_y \cdot T_w = T_{yw}$ *if* $\ell(y) + \ell(w) = \ell(yw)$.

Note that (b) implies $T_w = T_{s_1} \cdot \ldots \cdot T_{s_k}$ if $w = s_1 \cdot \ldots \cdot s_k$ is a reduced expression for w. It then follows that the rules above completely determine the ring structure in H_W. Thus, any algebra satisfying the properties of the proposition is isomorphic to the Hecke algebra H_W.

We next define the affine Weyl group associated to the quadruple $(P, P^\vee, R, R^\vee)$ to be the semidirect product $W_{aff} = W \ltimes P$ where the group W acts naturally on the lattice P by group automorphisms.

Remark 7.1.3. The group W_{aff} as defined above is not a Coxeter group. Classically, the affine Weyl group has been defined as the semidirect product $W \ltimes Q$, where Q is the subgroup in P generated by the set R, the *root* lattice. The group $W \ltimes Q$ is indeed the Coxeter group associated to an affine root system. This group is a normal subgroup of finite index in W_{aff}, since Q is clearly a W-stable subgroup of finite index in P.

Definition 7.1.4. Call a root system (P, R) *simply connected* if the coroots $R^\vee$ generate the lattice $P^\vee$.

The following claim, whose proof is left to the reader, may be taken as an alternative definition of a simply connected root system (P, R).

Claim 7.1.5. If (P, R) is a simply connected root system with simple roots $\{\alpha_1, \ldots, \alpha_n\}$, then there exists a basis $e_1, \ldots, e_n$ of the lattice P such that $\langle e_i, \check{\alpha}_j\rangle = \delta_{ij}$.

Remark 7.1.6. Let G be a connected linear algebraic group over $\mathbb{C}$. Let $T \subset G$ be a maximal torus; let $P = \mathrm{Hom}_{alg}(T, \mathbb{C}^*)$, and let $R \subset P$ be the roots of (G, T). Then if G is simply connected in the sense of Lie groups then (P, R) is simply connected in the sense of root systems.

For the rest of this chapter we will assume that all the semisimple groups and all the root systems under consideration are *simply connected.*

Recall that the group algebra $\mathbb{Z}[P]$ is isomorphic to $R(T)$, the representation ring of the torus T. We write e^λ for the element of $\mathbb{Z}[P] = R(T)$ corresponding to a weight $\lambda \in P$.

The natural group embeddings $P \hookrightarrow W_{aff}$ and $W \hookrightarrow W_{aff}$ induce the corresponding group algebra embeddings

$$R(T) \hookrightarrow \mathbb{Z}[W_{aff}] \quad \text{and} \quad \mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}], \tag{7.1.7}$$

and the multiplication map in $\mathbb{Z}[W_{aff}]$ gives rise to a $\mathbb{Z}$-module (but not algebra) isomorphism

$$\mathbb{Z}[W_{aff}] \simeq R(T) \otimes_{\mathbb{Z}} \mathbb{Z}[W]. \tag{7.1.8}$$

We now define the affine Hecke algebra associated to the simply connected root system (R, P). The algebra presented below was introduced by J. Bernstein (unpublished; relation 7.1.9(d) first appeared in [Lu3]), and is isomorphic to the so-called Iwahori-Hecke algebra of a split p-adic group with connected center, see Introduction. The latter was discovered by Iwahori and Matsumoto, see [IM].

Definition 7.1.9. The affine Hecke algebra $\mathbf{H}$ is a free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{e^\lambda \cdot T_w \mid w \in W, \lambda \in P\}$, such that

(a) The $\{T_w\}$ span a subalgebra of $\mathbf{H}$ isomorphic to H_W.
(b) The $\{e^\lambda\}$ span a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathbf{H}$ isomorphic to $R(T)[q, q^{-1}]$.
(c) For $s = s_\alpha \in S$ with $\langle \lambda, \check{\alpha}_s\rangle = 0$ we have $T_s e^\lambda = e^\lambda T_s$.
(d) For $s = s_\alpha \in S$ with $\langle \lambda, \check{\alpha}_s\rangle = 1$ we have $T_s e^{s(\lambda)} T_s = q \cdot e^\lambda$.

Conditions (c) and (d) together are equivalent to the following more general formula which will be useful in some later calculations.

Lemma 7.1.10. *Let α be a simple root and s_α the corresponding simple reflection. Then for $\lambda \in P$,*

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}}. \tag{7.1.11}$$

Proof. Note that if (7.1.11) holds for λ and λ' then it clearly holds for $n\lambda + \lambda'$, $n \in \mathbb{Z}$. Therefore it is enough to prove the equality for any set of generators of P. But claim 7.1.5 yields a set of generators $e_1, \ldots, e_n$ ($n =$ rank R) such that, e.g., $\langle e_1, \check{\alpha}\rangle = 1$, and $\langle e_i, \check{\alpha}\rangle = 0, i > 1$. Thus it is enough to prove (7.1.11) for elements $\lambda \in P$ such that $\langle \lambda, \check{\alpha}\rangle = 0$ and $\langle \lambda, \check{\alpha}\rangle = 1$.

Now if $\langle \lambda, \check{\alpha}\rangle = 0$, then Definition 7.1.9(c) implies $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$. Thus

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = T_{s_\alpha} e^\lambda - e^\lambda T_{s_\alpha} = 0.$$

But then the RHS of (7.1.11) is equal to

$$-(q-1)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}} = 0,$$

proving the lemma for this case. Suppose now that $\langle \lambda, \alpha\rangle = 1$. Then Definition 7.1.9(d) says

$$T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = q \cdot e^\lambda. \tag{7.1.12}$$

From 7.1.2(a) we immediately compute that $T_{s_\alpha}^{-1} = q^{-1} \cdot T_{s_\alpha} + (q^{-1} - 1)$. Thus, $T_{s_\alpha} e^{s_\alpha(\lambda)} = q \cdot e^\lambda \cdot T_{s_\alpha}^{-1} = e^\lambda T_{s_\alpha} + (1-q)e^\lambda$. Rewriting we have

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q)e^\lambda.$$

Evaluating the RHS of (7.1.11) in the case $\langle \lambda, \check{\alpha}\rangle = 1$ we see

$$(1-q)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}} = (1-q)\frac{e^\lambda - e^{\lambda-\alpha}}{1-e^{-\alpha}} = (1-q)e^\lambda,$$

and the lemma is proved. ∎

By properties (a) and (b) in Definition 7.1.9, there are canonical algebra embeddings

$$R(T)[q, q^{-1}] \hookrightarrow \mathbf{H} \quad \text{and} \quad H_W \hookrightarrow \mathbf{H}. \tag{7.1.13}$$

Furthermore, multiplication in $\mathbf{H}$ gives rise to a $\mathbb{Z}[q, q^{-1}]$-module (but not algebra) isomorphism

$$\mathbf{H} \simeq R(T)[q, q^{-1}] \otimes_{\mathbb{Z}[q,q^{-1}]} H_W$$

which is a q-analogue of (7.1.8).

It is rather important to know the center of $\mathbf{H}$. To that end, view $R(T)^W \subset R(T)$, the subring of W-invariants, as a subring of $\mathbb{Z}[W_{aff}]$ by means of the embedding (7.1.7). One verifies easily that $R(T)^W$ is in fact the center of the algebra $\mathbb{Z}[W_{aff}]$. The following q-analogue of this result is due to J. Bernstein (see [Lu5]).

Proposition 7.1.14. *The algebra* $R(T)^W[q,q^{-1}]$, *identified with a subset of* $\mathbf{H}$ *by means of (7.1.13), is the center of the algebra* $\mathbf{H}$.

Proof. For $\lambda \in P$ let $W \cdot \lambda$ be the W-orbit of λ. Then let

$$z(e^\lambda) = \sum_{\lambda' \in W\cdot\lambda} e^{\lambda'} \tag{7.1.15}$$

be the corresponding W-invariant element in $R(T)$. We will prove that the $z(e^\lambda)$'s belong to Z, the center of $\mathbf{H}$. We then show that Z is a free $\mathbb{Z}[q,q^{-1}]$-module with base $\{z(e^\lambda)\}$.

A direct calculation using (7.1.11) yields

$$T_{s_\alpha}(e^{s_\alpha(\lambda)} + e^\lambda) = (e^{s_\alpha(\lambda)} + e^\lambda)T_{s_\alpha}, \quad \lambda \in P, \alpha \in R. \tag{7.1.16}$$

Set $z_1(e^\lambda) = \sum_{w\in W} e^{w(\lambda)}$, and fix $s_\alpha \in S$, a simple reflection. Let $W' \subset W$ be the set of $w' \in W$ such that $\ell(s_\alpha w') = \ell(s_\alpha)+\ell(w')$. Write $W'' = W \setminus W'$. Using the bijection

$$W' \leftrightarrow W\setminus W' \quad , \quad w' \leftrightarrow s_\alpha w' ,$$

we may write the sum $\sum_{w\in W} e^{w(\lambda)}$ in the form

$$z_1(e^\lambda) = \sum_{w'\in W'} \left(e^{w'(\lambda)} + e^{s_\alpha w'(\lambda)}\right).$$

Thus by (7.1.16) we get $z_1(e^\lambda)T_{s_\alpha} = T_{s_\alpha} z_1(e^\lambda)$, for any $\alpha \in R, \lambda \in P$, and therefore by 7.1.2(b) $z_1(e^\lambda)T_w = T_w z_1(e^\lambda)$ for each $w \in W$.

To prove that the element (7.1.15) is central, observe that $z_1(e^\lambda) = k \cdot z(e^\lambda)$ where k is the order of the stabilizer of λ in W. Since the algebra $\mathbf{H}$ has no $\mathbb{Z}$-torsion it follows that

$$0 = z_1(e^\lambda)T_w - T_w z_1(e^\lambda) \quad \Rightarrow \quad 0 = z(e^\lambda)T_w - T_w z(e^\lambda).$$

Furthermore, each $z(e^\lambda)$ clearly commutes with each e^μ, $\mu \in P$. Since the T_w and the e^μ generate $\mathbf{H}$, we deduce that $z(e^\lambda) \in Z$.

We next use a specialization argument due to Lusztig [Lu4] to show that the $z(e^\lambda)$'s form a $\mathbb{Z}[q,q^{-1}]$-basis of Z. Mapping $q \to 1$ defines a specialization homomorphism of $\mathbb{Z}[q,q^{-1}]$-modules

$$\text{sp}\colon \mathbf{H} \to \mathbb{Z}[W_{aff}]. \tag{7.1.17}$$

This map is surjective and its kernel is the two-sided ideal in $\mathbf{H}$ generated by $\mathfrak{m} := (1-q)$, the principal ideal in $\mathbb{Z}[q,q^{-1}]$ with generator $1-q$.

Write R for the $\mathbb{Z}[q,q^{-1}]$-span of the $z(e^\lambda)$'s. Clearly R is a subring in Z isomorphic to $R(T\times\mathbb{C}^*)^W$. To show that $R=Z$ recall that the center of $\mathbb{Z}[W_{aff}]$ is known to be isomorphic to $R(T)^W$. Therefore specializing $q\to 1$ defines a ring homomorphism $\mathrm{sp}\colon Z\twoheadrightarrow R(T)^W$. Observe further that if $a\in\mathbf{H}$ is such that $(1-q)\cdot a\in Z$ then $a\in Z$. This implies that $\mathrm{Ker}[Z\overset{\mathrm{sp}}{\twoheadrightarrow}R(T)^W]=(1-q)Z$, and one gets an exact sequence of R-modules

$$0\to\mathfrak{m}Z\to Z\overset{\mathrm{sp}}{\to}R(T)^W\to 0. \tag{7.1.18}$$

Let $R_\mathfrak{m}$ be the local ring obtained by localizing R at the maximal ideal $\mathfrak{m}$. Since the localization is an exact functor the short exact sequence above yields an isomorphism of $R_\mathfrak{m}$-modules $Z_\mathfrak{m}/\mathfrak{m}Z_\mathfrak{m}\simeq R(T)^W$. On the other hand, obviously we have an isomorphism $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}\simeq R(T)^W$. Thus

$$R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}\simeq Z_\mathfrak{m}/\mathfrak{m}Z_\mathfrak{m}. \tag{7.1.19}$$

Observe further that $R\simeq R(T\times\mathbb{C}^*)^W$ is a polynomial ring (see [Bour]) and hence noetherian. By the Pittie-Steinberg Theorem 6.1.2 we know that $\mathbf{H}$ is a finitely generated R-module. Hence Z is finitely generated over R as a submodule of the finitely generated noetherian R-module $\mathbf{H}$. It follows that $Z_\mathfrak{m}$ is finitely generated over $R_\mathfrak{m}$ and the Nakayama Lemma applied to isomorphism (7.1.19) yields $R_\mathfrak{m}=Z_\mathfrak{m}$.

We see that every element $z\in Z$ can be written as a finite linear combination of $z(e^\lambda)$'s with coefficients possibly in the localization $\mathbb{Z}[q,q^{-1}]_\mathfrak{m}$. Since $z\in\mathbf{H}$ the coefficients must be in fact in $\mathbb{Z}[q,q^{-1}]$. Finally, the $z(e^\lambda)$'s are clearly independent. This completes the proof. ■

7.2 Main Theorems

We return now to our basic geometric setup, so that G is a complex semisimple *simply connected* group with Lie algebra $\mathfrak{g}$, $\mathcal{B}$ is the flag variety of G, $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$, and $\mu:\tilde{\mathcal{N}}\to\mathcal{N}$ is the Springer resolution and $\tilde{\mathcal{N}}\simeq T^*\mathcal{B}$.

Let $Z_\Delta\subset\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}$ be the diagonal copy of $\tilde{\mathcal{N}}$. Recall that the variety Z_Δ gets identified, under the isomorphism

$$\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}\simeq T^*(\mathcal{B}\times\mathcal{B})$$

(cf. the sign convention as in section 3.3.3), with $T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B})$, the conormal bundle to the diagonal $\mathcal{B}_\Delta\subset\mathcal{B}\times\mathcal{B}$. This yields the following canonical isomorphisms of $R(G)$-algebras

$$K^G(Z_\Delta)=K^G\left(T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B})\right)\simeq K^G(\mathcal{B}_\Delta)\simeq R(T), \tag{7.2.1}$$

where the second isomorphism is the Thom isomorphism 5.4.17, the last one is the canonical isomorphism (6.1.6), and algebra structures are given by the tensor product in K-theory, see (5.2.12). Here and below T and W stand for the 'abstract' maximal torus and the abstract Weyl group associated to G.

Further, let $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \subset T^*(\mathcal{B} \times \mathcal{B})$ be the Steinberg variety. Clearly $Z_\Delta \subset Z$. Furthermore, we have $Z_\Delta \circ Z_\Delta = Z_\Delta$, and $Z \circ Z = Z$. Hence, the K-groups $K^G(Z_\Delta)$ and $K^G(Z)$ acquire natural associative algebra structures by means of convolution, and $K^A(Z_\Delta)$ is a subalgebra in $K^G(Z)$. Note that by Lemma 5.2.25 the convolution product on $K^A(Z_\Delta)$ coincides with the ring structure introduced in the preceding paragraph, by means of tensor product in K-theory. Thus, the leftmost term of (7.2.1) may be viewed as a convolution algebra. Moreover, the natural map $K^A(Z_\Delta) \to K^A(Z)$ is injective, see Corollaries 6.2.6 - 6.2.7.

The first main result of this chapter is the following equivariant K-theoretic counterpart to Theorem 3.4.1.

Theorem 7.2.2. *There is a natural algebra isomorphism $K^G(Z) \simeq \mathbb{Z}[W_{aff}]$ making the following diagram commute*

$$\begin{array}{ccc} K^G(Z_\Delta) & \hookrightarrow & K^G(Z) \\ {\scriptstyle (7.2.1)}\downarrow\wr & & \wr\downarrow \\ R(T) & \xhookrightarrow{(7.1.7)} & \mathbb{Z}[W_{aff}] \end{array}$$

We now introduce an extra variable q. To that end note that any irreducible representation of the group $\mathbb{C}^*$ is an integral tensor power of the tautological representation $q : \mathbb{C}^* \to \mathbb{C}^*$ given by the identity map. Therefore we have the natural ring isomorphisms

$$(7.2.3) \qquad R(\mathbb{C}^*) \simeq \mathbb{Z}[q, q^{-1}] \quad \text{and} \quad R(T \times \mathbb{C}^*) \simeq R(T)[q, q^{-1}] .$$

For the rest of this book we put $A := G \times \mathbb{C}^*$. In §6.2 we have defined natural A-actions on $T^*\mathcal{B}$, $T^*\mathcal{B} \times T^*\mathcal{B}$, Z, and $Z_\Delta = T^*_{\mathcal{B}_\Delta}(\mathcal{B} \times \mathcal{B})$. There is the following "A-counterpart" of isomorphism (7.2.1):

$$(7.2.4) \qquad K^A(Z_\Delta) \simeq K^A(T^*_{\mathcal{B}_\Delta}(\mathcal{B} \times \mathcal{B})) \simeq K^A(\mathcal{B}_\Delta) \simeq R(T)[q, q^{-1}] .$$

Here is the second main result of this chapter which is the q-analogue of Theorem 7.2.2 above.

Theorem 7.2.5. *There is a natural algebra isomorphism* $K^A(Z) \simeq \mathbf{H}$

making the following diagram commute

(7.2.6)
$$\begin{array}{ccc} K^A(Z_\Delta) & \hookrightarrow & K^A(Z) \\ {\scriptstyle (7.2.4)}\downarrow\wr & & \downarrow\wr \\ R(T)[q,q^{-1}] & \xhookrightarrow{(7.1.13)} & \mathbf{H} \end{array}$$

Remark 7.2.7. Note that Theorem 7.2.2 follows from Theorem 7.2.5 by setting the parameter q equal to 1. We will, however, give a separate proof of 7.2.2 along the lines of the proof of the analogous theorem for the finite Weyl group. Though the two theorems look very similar Theorem 7.2.2 is far more elementary than Theorem 7.2.5.

We now discuss the role of the center of $\mathbf{H}$ from the geometric point of view. Observe first that

(7.2.8)
$$R(A) = R(G \times \mathbb{C}^*) = R(G) \otimes_{\mathbb{Z}} R(\mathbb{C}^*) = R(G)[q,q^{-1}] \overset{6.1.4}{\simeq} R(T)^W[q,q^{-1}].$$

Next, recall that for any A-variety X there is a natural homomorphism

$$R(A) = K^A(\mathrm{pt}) \xrightarrow{p^*} K^A(X)$$

induced by the constant map $p : X \to \mathrm{pt}$. The image of $R(A)$ is formed by trivial vector bundles on X with possibly non-trivial A-actions. Tensoring with those vector bundles makes $K^A(X)$ an $R(A)$-module.

For the Steinberg variety Z, the $R(A)$-module $K^A(Z)$ has its own algebra structure by means of convolution. From the convolution point of view, tensoring with a representation $E \in R(A)$ amounts to taking convolution with the sheaf $E \otimes_{\mathbb{C}} \mathcal{O}_{Z_\Delta}$ supported on the diagonal Z_Δ. Thus, the diagram of Theorem 7.2.5 can be supplemented by the following natural commutative diagram.

(7.2.9)
$$\begin{array}{ccc} R(A) & \hookrightarrow & K^A(Z_\Delta) \\ \wr\downarrow{\scriptstyle (7.2.8)} & & {\scriptstyle (7.2.4)}\downarrow\wr \\ R(T)^W[q,q^{-1}] & \hookrightarrow & R(T)[q,q^{-1}] \end{array}$$

Observe that, for any $\mathcal{F}, \mathcal{F}' \in K^A(Z)$ and $E \in R(A)$, one has

(7.2.10)
$$E \otimes (\mathcal{F} * \mathcal{F}') = (E \otimes \mathcal{F}) * \mathcal{F}' = \mathcal{F} * (E \otimes \mathcal{F}').$$

Equation (7.2.10) shows that the natural homomorphism $R(A) \to K^A(Z)$ given by the composition of the top rows of the diagrams (7.2.6) and (7.2.9) maps $R(A)$ into the center of the algebra $K^A(Z)$. Looking now at the bottom rows of the diagrams and using Theorem 7.2.5, we see that the image of the composition

$$R(T)^W[q,q^{-1}] \hookrightarrow R(T)[q,q^{-1}] \overset{(7.1.13)}{\hookrightarrow} \mathbf{H}$$

belongs to the center of the algebra **H**, which is nothing but Proposition 7.1.14. Thus, Theorem 7.2.5 gives a geometric proof of an essential part of Proposition 7.1.14, and conversely the proposition says that the whole center of the algebra $K^A(Z)$ is given by the representation ring $R(A)$.

We also mention the following result, which indicates somewhat why we require G to be simply connected.

Lemma 7.2.11. *The algebra* **H** *is a free* $R(T)^W[q,q^{-1}]$*-module of rank* $(\#W)^2$.

Proof. By Theorem 6.1.2 which applies since G is simply connected, $R(T)$ is a free $R(T)^W$-module of rank $\#W$. Let $r_w, w \in W$ be a basis for that module. Then the elements $\{r_w \cdot T_y \mid w, y \in W\}$ form a free basis of **H** viewed as a $R(T)^W[q,q^{-1}]$-module. ∎

Let $\mathcal{I}^{R(T)}$ denote the ideal in $R(T)$ generated by all W-invariant functions vanishing at $1 \in T$. The results announced in this section may be summarized in the following commutative diagram of algebra homomorphisms:

(7.2.12)

$$\begin{array}{ccccccc}
K^{G\times\mathbb{C}^*}(Z) & \xrightarrow[\mathbb{C}^*-action]{forgetting} & K^G(Z) & \xrightarrow[G-action]{forgetting} & K(Z) & \xrightarrow[cycle]{support} & H(Z)\\
\Vert\ 7.2.5 & & \Vert\ 7.2.2 & & \Vert & & \Vert\ 3.4.1\\
\mathbf{H} & \xrightarrow{q\mapsto 1} & \mathbb{Z}[W_{aff}] & \xrightarrow{proj} & \mathbb{Z}[W]\ltimes(R(T)/\mathcal{I}^{R(T)}) & \xrightarrow{T\to 1} & \mathbb{Z}[W]
\end{array}$$

where $H(Z)$ stands for the top Borel-Moore homology group of the Steinberg variety and the square on the right commutes due to the bivariant Riemann-Roch Theorem 5.11.11.

7.2.13. QUANTIZED W-ACTION AND DEMAZURE-LUSZTIG FORMULAS. Recall that the restriction to the Steinberg variety $Z \subset T^*(\mathcal{B}\times\mathcal{B})$ of either of the two projections $T^*(\mathcal{B}\times\mathcal{B}) \to T^*(\mathcal{B})$ is proper. Thus, the general procedure of section 5.4.22 yields a $K^A(Z)$-module structure on $K^A(T^*\mathcal{B})$. Recall that we have $K^A(T^*\mathcal{B}) \simeq K^A(\mathcal{B}) \simeq R(T)[q,q^{-1}]$ by (7.2.4), and $K^A(Z) \simeq \mathbf{H}$ by Theorem 7.2.5. Hence, there is a natural action of the algebra **H** on the polynomial ring $R(T)[q,q^{-1}]$ arising from the convolution-action. This action can be written out explicitly. Its restriction to the finite Hecke algebra $H_W \subset \mathbf{H}$ is especially interesting and is given by the so called "Demazure-Lusztig" operators, which we now describe.

We begin with the special case $q = 1$. Thus we forget about the $\mathbb{C}^*$-action and replace the group $A = G \times \mathbb{C}^*$ by G everywhere. By Theorem 7.2.2 we have an isomorphism $\mathbb{C}[W_{aff}] \simeq K^G(Z)$. Hence, the natural embedding $W \hookrightarrow W_{aff}$ makes $\mathbb{Z}[W]$ a subalgebra of $K^G(Z)$ and hence makes $K^G(T^*\mathcal{B})$ a W-module.

Proposition 7.2.14. *The W-action on $K^G(T^*\mathcal{B})$ by convolution with $K^G(Z)$ gets identified, by means of the canonical isomorphism $K^G(T^*\mathcal{B}) \simeq R(T)$ of* (6.1.6), *with the W-action on $R(T)$ induced by the standard W-action on T.*

Corollary 7.2.15. *The composition*

$$\mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}] \xrightarrow[\sim]{7.2.2} K^G(Z) \xrightarrow{G \to 1} K(Z) \xrightarrow[cycle]{support} H(Z) \xrightarrow[\sim]{3.4.1} \mathbb{Z}[W],$$

is the identity map.

The situation is more complicated if $q \neq 1$. Recall that the finite Hecke algebra H_W is generated, as an algebra, by the elements T_{s_α}, one for each reflection with respect to a simple root $\alpha \in R$.

Theorem 7.2.16. *The T_{s_α}-action on $K^A(T^*\mathcal{B})$ arising from convolution with $K^A(Z)$ gets transported, by means of the canonical isomorphism $K^A(T^*\mathcal{B}) \simeq R(T)[q, q^{-1}]$, to the following map $\widehat{T}_{s_\alpha} \in \mathrm{End}_{\mathbb{Z}[q,q^{-1}]} R(T)[q, q^{-1}]$:*

$$\widehat{T}_{s_\alpha} : e^\lambda \mapsto \frac{e^\lambda - e^{s_\alpha(\lambda)}}{e^\alpha - 1} - q\frac{e^\lambda - e^{s_\alpha(\lambda)+\alpha}}{e^\alpha - 1}. \tag{7.2.17}$$

This formula was discovered by Lusztig in [Lu6], and it was the starting point of the K-theoretic approach to Hecke algebras. Observe that if $q = 1$ the RHS of (7.2.17) reduces to $e^{s_\alpha(\lambda)}$ in accordance with Proposition 7.2.14. The expression $\frac{e^\lambda - e^{s_\alpha(\lambda)}}{e^\alpha - 1}$ in the first term of (7.2.17) was introduced much earlier by Demazure [Dem] in his study of the K-theory of the flag variety (cf. [BGG] for a similar formula).

7.3 Case q = 1: Deformation Argument

In this section we prove Theorem 7.2.2 following the lines of the proof of Theorem 3.4.1, which is based on a deformation argument. Recall the notation of the proof of Theorem 3.4.1 and a basic diagram

$$\begin{array}{ccccc} \tilde{\mathcal{N}} & \hookrightarrow & \tilde{\mathfrak{g}} & & \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & \searrow{\scriptstyle \nu} & \\ \mathcal{N} & \hookrightarrow & \mathfrak{g} & & \mathfrak{h} \end{array}$$

For any $(h, \mathfrak{b}) \in \tilde{\mathfrak{g}}$, where $h \in \mathfrak{g}^{sr}$ (= semisimple regular elements) and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n} \in \mathcal{B}$, set $\tilde{\mathfrak{g}}^h = \nu^{-1}(h) \subset \tilde{\mathfrak{g}}$. Then there is a natural projection

$$\pi : \tilde{\mathfrak{g}}^h \simeq G \times_B (h + \mathfrak{n}) \to G/B = \mathcal{B} \tag{7.3.1}$$

making $\tilde{\mathfrak{g}}^h$ a G-equivariant affine bundle over $\mathcal{B}$ with fiber $h + \mathfrak{n}$.

Recall next that the map $\mu : \tilde{\mathfrak{g}}^{sr} \to \mathfrak{g}^{sr}$ is a Galois covering with Galois group W, the abstract Weyl group. For each $w \in W$ the action of w gives an isomorphism $\tilde{\mathfrak{g}}^h \xrightarrow{\sim} \tilde{\mathfrak{g}}^{w(h)}$ and we let

$$\Lambda_w^h \subset \tilde{\mathfrak{g}}^h \times \tilde{\mathfrak{g}}^{w(h)}$$

denote the graph of that action. The composition $\Lambda_w^h \xrightarrow{\mathrm{pr}_1} \tilde{\mathfrak{g}}^h \xrightarrow{\pi} \mathcal{B}$ gives rise to the following isomorphisms of K-groups

$$R(T) \xrightarrow{\sim} K^G(\mathcal{B}) \xrightarrow{(\pi \circ \mathrm{pr}_1)^*} K^G(\Lambda_w^h), \tag{7.3.2}$$

where the first map is the canonical isomorphism assigning to $\lambda \in P$ the line bundle L_λ on $\mathcal{B}$, and the second map is the Thom isomorphism, since pr_1 maps Λ_w^h isomorphically to $\tilde{\mathfrak{g}}^h$ and $\tilde{\mathfrak{g}}^h \to \mathcal{B}$ is an affine fibration. We set $\tilde{L}_\lambda = (\pi \circ \mathrm{pr}_1)^* L_\lambda$, a G-equivariant line bundle on Λ_w^h.

Using the tensor product decomposition (7.1.8) we may write a direct sum decomposition

$$\mathbb{Z}[W_{aff}] = \bigoplus_{w \in W} R(T) \cdot w,$$

where w on the right is viewed as an element of $\mathbb{Z}[W] \subset \mathbb{Z}[W_{aff}]$. Given $h \in \mathfrak{g}^{sr}$ as above and $w \in W$, define an isomorphism of $R(T)$-modules

$$\Theta^h : R(T) \cdot w \xrightarrow{\sim} K^G(\Lambda_w^h), \qquad e^\lambda \cdot w \mapsto \tilde{L}_\lambda. \tag{7.3.3}$$

Lemma 7.3.4. *The following diagram, whose first row is induced by multiplication in $\mathbb{Z}[W_{aff}]$ and the second row is induced by convolution, commutes for any $w, y \in W$:*

$$\begin{array}{ccc} R(T) \cdot w \otimes R(T) \cdot y & \longrightarrow & R(T) \cdot wy \\ \downarrow{\scriptstyle \Theta^h \otimes \Theta^h} & & \downarrow{\scriptstyle \Theta^h} \\ K^G(\Lambda_w^h) \otimes K^G(\Lambda_y^{w(h)}) & \longrightarrow & K^G(\Lambda_{wy}^h). \end{array}$$

Proof. Note that, for any $R_1, R_2 \in R(T)$, the map of the top row of the diagram sends $(R_1 \cdot w) \otimes (R_2 \cdot y)$ to $(R_1\, R_2^w) \cdot wy$, where R_2^w stands for the action of w on R_2. The result follows now by a straightforward computation using the definition of convolution and the fact that $\Lambda_w^h \circ \Lambda_y^{w(h)} = \Lambda_{wy}^h$ and that all the intersections involved in this composition are transverse. ∎

With this lemma in hand, to prove 7.2.2 we can just copy the proof of Theorem 3.4.1 step by step. First fix a regular semisimple element h, introduce the line $\mathfrak{l} = \mathbb{C} \cdot h$, and set $\mathfrak{l}^* = \mathbb{C}^* \cdot h$. Then we have a locally trivial fibration $\tilde{\mathfrak{g}}^{\mathfrak{l}} = \nu^{-1}(\mathfrak{l}) \to \mathfrak{l}$. Letting in the previous construction the element h vary within $\mathfrak{l}^*$, we obtain in particular, for each $w \in W$, the graph variety $\Lambda_w^{\mathfrak{l}^*}$ fibered over $\mathfrak{l}^*$, and an $R(T)$-linear homomorphism $\Theta^{\mathfrak{l}^*} : R(T) \cdot w \to K^G(\Lambda_w^{\mathfrak{l}^*})$ satisfying an analogue of Lemma 7.3.4.

We now extend $\Lambda_w^{\mathfrak{l}^*}$ to a closed subvariety $\Lambda_w^{\mathfrak{l}}$ as in the proof of Theorem 3.4.1 and observe that

$$\Lambda_w^{\mathfrak{l}} \cap (\nu^{-1}(0) \times \nu^{-1}(0)) = Z = \text{Steinberg variety}.$$

Hence, there is a well defined specialization map in equivariant K-theory

$$\lim_{h \to 0} : K^G(\Lambda_w^{\mathfrak{l}^*}) \to K^G(Z).$$

Form the composite

$$\Theta^{0 \cdot h} : R(T) \cdot w \xrightarrow{\Theta^{\mathfrak{l}^*}} K^G(\Lambda_w^{\mathfrak{l}^*}) \xrightarrow{\lim_{h \to 0}} K^G(Z). \tag{7.3.5}$$

Proof of the following analogue of Lemma 3.4.11 will be postponed until after the end of the proof of the theorem.

Claim 7.3.6. The above map $\Theta^{0 \cdot h}$ does not depend on the choice of h.

In view of the claim we drop the superscript $0 \cdot h$ from the notation and write Θ instead of $\Theta^{0 \cdot h}$. Assembling the homomorphisms (7.3.5) for all $w \in W$ we get a map

$$\Theta : \mathbb{Z}[W_{aff}] = \bigoplus_{w \in W} R(T) \cdot w \to K^G(Z). \tag{7.3.7}$$

Lemma 7.3.4 combined with the fact that the specialization homomorphism $\lim_{h \to 0}$ commutes with convolution (see 5.3.9) implies that the map (7.3.7) is an algebra homomorphism.

It remains to show that that map (7.3.7) is bijective. We need some preparations.

We enumerate G-diagonal orbits on $\mathcal{B} \times \mathcal{B}$ in some order $pt = Y_1, Y_2, \ldots, Y_m$, so that $\dim Y_1 \geq \dim Y_2 \geq \cdots \geq \dim Y_m$, cf. proof of Theorem 6.2.4. This way we get a total linear order on the set W by declaring $y \leq w$ if Y_y goes *after* Y_w in our enumeration. It is clear that, for $i = 1, 2, \ldots, m$, the sets $\sqcup_{j \geq i} Y_j$ are closed in $\mathcal{B} \times \mathcal{B}$. Thus, our *total* linear order extends the Bruhat order on W, a *partial* order given by the closure relation of the G-diagonal orbits.

7.3.8. CONVENTION. For the rest of this chapter we will fix such a total linear order on W and use the notation "$\leq$" for this order, and *not* for the

Bruhat order. Thus an expression such as $y \leq w,\ y, w \in W$ will always be with respect to this total order.

We now define certain filtrations analogous to those used in the proof of Theorem 6.2.4. For each $w \in W$ set

$$\mathbb{Z}_{\leq w}[W_{aff}] = \bigoplus_{y \leq w} R(T) \cdot y \quad \text{and} \quad \mathbb{Z}_{<w}[W_{aff}] = \bigoplus_{y<w} R(T) \cdot y\,.$$

The submodules $\mathbb{Z}_{\leq w}[W_{aff}]$ form a filtration on $\mathbb{Z}[W_{aff}]$ by the totally ordered set W and there is an obvious isomorphism

$$\operatorname{gr}_w \mathbb{Z}[W_{aff}] := \mathbb{Z}_{\leq w}[W_{aff}]/\mathbb{Z}_{<w}[W_{aff}] \simeq R(T) \cdot w.$$

We also filter the Steinberg variety Z by G-stable closed subvarieties

$$Z_{\leq w} := \sqcup_{y \leq w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B})\,, \quad \text{and put} \quad Z_{<w} := \sqcup_{y<w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B})\,.$$

Clearly $y \leq w$ implies $Z_{\leq y} \subset Z_{\leq w}$, and we have $Z_{\leq w} \setminus Z_{<w} = T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$. The arguments of section 6.2, based on the Cellular Fibration Lemma, yield the following result.

Lemma 7.3.9. *(1) For any $y < w$, the homomorphism $K^G(Z_{\leq y}) \to K^G(Z_{\leq w})$ induced by the inclusion $Z_{\leq y} \hookrightarrow Z_{\leq w}$ is injective; in particular, all the maps $K^G(Z_{\leq y}) \to K^G(Z)$ are injective.*

(2) For any $w \in W$, we have a short exact sequence $K^G(Z_{<w}) \hookrightarrow K^G(Z_{\leq w}) \twoheadrightarrow K^G(T^*_{T_w}(\mathcal{B} \times \mathcal{B}))$, which gives a natural isomorphism

$$K^G(Z_{\leq w})/K^G(Z_{<w}) \xrightarrow{\sim} K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))\,. \tag{7.3.10}$$

By part (1) of the lemma we may view the groups $K^G(Z_{\leq y}), y \in W$, as subgroups of $K^G(Z)$. The subgroups form a filtration of $K^G(Z)$ indexed by the totally ordered set W. The associated graded group, $\operatorname{gr} K^G(Z)$, is described by part (2) of the lemma; that is

$$\operatorname{gr}_w K^G(Z) \simeq K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B})).$$

Lemma 7.3.11. *The morphism Θ in (7.3.7) is filtration preserving.*

Proof. Recall the natural projection $\pi^2 : \Lambda^h_w \hookrightarrow \mathfrak{g}^h \times \mathfrak{g}^{w(h)} \xrightarrow{\pi \times \pi} \mathcal{B} \times \mathcal{B}$. It was shown in 3.4.4 that $\pi^2(\Lambda^h_w) \subset Y_w$. This inclusion holds for any h, in particular for all $h \in \mathfrak{l}^*$. It follows that $\pi^2(\Lambda^{\mathfrak{l}^*}_w) \subset Y_w$. Hence $\pi^2(\overline{\Lambda^{\mathfrak{l}^*}_w}) \subset \overline{Y_w}$. Therefore the specialization at $h = 0$ gives a morphism

$$\lim_{h \to 0} : K^G(\Lambda^{\mathfrak{l}^*}_w) \to K^G\left(Z \cap (\pi^2)^{-1}(\bar{Y}_w)\right)\,,$$

where $\pi^2 : T^*\mathcal{B} \times T^*\mathcal{B} \to \mathcal{B} \times \mathcal{B}$ is the projection. But it is immediate from definitions that $Z \cap (\pi^2)^{-1}(\bar{Y}_w) = Z_{\leq w}$, and the lemma follows. ∎

By Lemma 7.3.11 the map (7.3.7) induces the associated graded map

$$\text{gr } \Theta : \text{gr } \mathbb{Z}[W_{aff}] \to \text{gr } K^G(Z). \tag{7.3.12}$$

To describe the morphism gr Θ explicitly we let $\pi_w : T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \to Y_w$ denote the bundle projection and let $\text{pr}_1 : Y_w \hookrightarrow \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ denote the map induced by the first projection.

Lemma 7.3.13. *For any $w \in W$ the composition morphism*

$$R(T) \cdot w \simeq \text{gr}_w \mathbb{Z}[W_{aff}] \xrightarrow{\text{gr } \Theta} \text{gr}_w K^G(Z) \simeq K^G(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))$$

is given by assignment $e^\lambda \cdot w \mapsto \pi_w^* \circ \text{pr}_1^* L_\lambda$, $\lambda \in P$.

Proof. Given $(h, \mathfrak{b}) \in \tilde{\mathfrak{g}}^{sr}$, we have the following commutative diagram, cf., (3.4.3).

$$\begin{array}{ccccc} \Lambda^h_w & \hookrightarrow & \tilde{\mathfrak{g}}^h \times \mathfrak{g}^{w(h)} & \xrightarrow{\text{pr}_1} & \tilde{\mathfrak{g}}^h \\ \downarrow{\scriptstyle \pi^2} & & \downarrow{\scriptstyle \pi^2} & & \downarrow{\scriptstyle \pi} \\ Y_w & \hookrightarrow & \mathcal{B} \times \mathcal{B} & \xrightarrow{\text{pr}_1} & \mathcal{B} \end{array} \tag{7.3.14}$$

In more concrete terms, this diagram is isomorphic, by means of Lemma 3.4.3 and the definition of $\tilde{\mathfrak{g}}^h$, to the following natural diagram, where $\mathfrak{n}^w$ and B^w stand for w-conjugates of $\mathfrak{n}$ and B respectively:

$$\begin{array}{ccc} G \times_{B \cap w(B)} (h + \mathfrak{n} \cap \mathfrak{n}^w) & \xrightarrow{\text{pr}_1} & G \times_B (h + \mathfrak{n}) \\ \downarrow{\scriptstyle \pi^2} & & \downarrow{\scriptstyle \pi} \\ G/(B \cap B^w) & \xrightarrow{\text{pr}_1} & G/B \end{array} \tag{7.3.15}$$

For $\lambda \in P$, we have by (7.3.3) and the commutativity of (7.3.14):

$$\Theta^h(e^\lambda \cdot w) = \text{pr}_1^* \cdot \pi^* L_\lambda = (\pi^2)^* \, \text{pr}_1^* L_\lambda. \tag{7.3.16}$$

Replace now h by the line $\mathbb{1}$ through h everywhere in (7.3.15). Observe that for $h = 0$ the morphism π^2 on the LHS of (7.3.15) gets identified with the projection $\pi_w : G \times_{B \cap w(B)} (\mathfrak{n} \cap \mathfrak{n}^w) \simeq T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \to Y_w$. Hence taking the specialization at $h = 0$ in formula (7.3.16) we find

$$[\lim_{h \to 0} \Theta^h(e^\lambda \cdot w)]|_{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})} = [\lim_{h \to 0} (\pi^2)^* \, \text{pr}_1^* L_\lambda]|_{T^*_{Y_w}(\mathcal{B} \times \mathcal{B})} = \pi_w^* \, \text{pr}_1^* L_\lambda.$$

The lemma follows. ∎

We see that the map of Lemma 7.3.13 equals the one given by formula (7.3.2). Since the latter is an isomorphism, it follows that gr Θ is an isomorphism. Hence, by Proposition 2.3.20(ii), Θ is itself an isomorphism, and the theorem is proved. ∎

It remains to prove Claim 7.3.6. We can not apply an argument of the type used in the proof of the analogous result 3.4.11 in Borel-Moore homology, since the argument there was not completely algebraic: in the course of that proof we connected two lines $\mathbf{l}$ and $\mathbf{l}'$ by a path built out of $\mathbb{R}$-linear segments. Instead, we will now use an algebraic homotopy construction, which is an algebraic adaptation of the construction used in the proof of Lemma 2.6.35 on the specialization in Borel-Moore homology. Note that the specialization in Borel-Moore homology was defined for a smooth base of *arbitrary* dimension, while the specialization in the algebraic K-theory was only defined for a 1-dimensional base. The argument below shows that, in some favorable situations, one can in effect define specialization in K-theory over a higher dimensional base.

Though our proof works in greater generality, we will not attempt to formulate it in the most general form, and will stick to the framework of Claim 7.3.6 that we intend to prove.

Thus we fix two linearly independent regular elements $h_1, h_2 \in \mathfrak{H}$. We must show that the two maps (7.3.5) corresponding respectively to h_1 and h_2 are equal. To that end, consider the *complex* path

$$\tau \mapsto \gamma(\tau) = (1-\tau)\cdot h_1 + \tau \cdot h_2\,, \quad \tau \in \mathbb{C}\,.$$

We have, $\gamma(0) = h_1$ and $\gamma(1) = h_2$. Observe further that the path γ intersects the root hyperplanes in $\mathfrak{H}$ at finitely many points. Thus, there is a finite set $S \subset \mathbb{C}$ such that, for all $\tau \in \mathbb{C}\setminus S$, the element $\gamma(\tau) \in \mathfrak{H}$ is regular. We put $\mathbb{C}_S := \mathbb{C}\setminus S$ for convenience. We would like to emphasize at this point that a very essential special feature of the situation we are dealing with is that the points h_1 and h_2 in the base of the specialization are connected by a *straight* line, more precisely, by a set of the form $\mathbb{C}_S$, as opposed to an arbitrary complex curve.

Following the pattern of the proof of Lemma 2.6.35 we consider the map

$$\Phi : \mathbb{C}\times\mathbb{C}_S \to \mathfrak{H}\ ,\ \ (t,\tau) \mapsto t\cdot\gamma(\tau) = t(1-\tau)\cdot h_1 + t\,\tau\cdot h_2\,.$$

Clearly, the image of Φ is a Zariski open subset in the 2-dimensional complex vector space $\mathbf{h} \subset \mathfrak{H}$ spanned by h_1 and h_2. Note also that $\Phi(0,\mathbb{C}_S) = \{0\}$. Now, fix $w \in W$, and define a variety $\Lambda_w^{\mathbf{h}}$ by

$$\Lambda_w^{\mathbf{h}} \;=\; (\tilde{\mathfrak{g}}^{w(\mathbf{h})} \times_{\mathbf{h}_w} \tilde{\mathfrak{g}}^{\mathbf{h}}) \cap (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})\,,$$

which is obtained from formula (3.4.10) by replacing everywhere the line $\mathbf{l}$ by the 2-dimensional vector space $\mathbf{h}$. Following the strategy of the corresponding argument in the Borel-Moore homology case, we define a variety

$\mathcal{X}$ by means of the cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \Lambda^{\mathbf{k}}_{w} \\ \big\downarrow & & \big\downarrow{\scriptstyle \nu\times\nu} \\ \mathbb{C}\times\mathbb{C}_S & \xrightarrow[\Phi]{} & \mathbf{k} \end{array} \tag{7.3.17}$$

where $\nu : \tilde{\mathfrak{g}} \to \mathfrak{H}$ is the standard map $(x, \mathfrak{b}) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$. It is convenient at this stage to formalize the properties of the map $\mathcal{X} \to \mathbb{C} \times \mathbb{C}_S$ resulting from this construction as follows.

7.3.18. ALGEBRAIC HOMOTOPY ARGUMENT. Let C be a smooth complex curve with base point o, and let $\mathcal{X}$ be a G-variety over $C \times \mathbb{C}_S$ (with G acting trivially on both C and $\mathbb{C}_S$). We write $\pi : \mathcal{X} \to C$ and $p : \mathcal{X} \to \mathbb{C}_S$ for the compositions of the map $\mathcal{X} \to C\times\mathbb{C}_S$ with the projections of $C\times\mathbb{C}_S$ to the factors $\mathbb{C}$ and $\mathbb{C}_S$, respectively. Set $\mathcal{X}^o = \pi^{-1}(o)$, our usual notation for the special fiber. We assume the following holds:

(P1) The projection $p : \mathcal{X} \to \mathbb{C}_S$ is flat;

(P2) The projection $p : \mathcal{X}^o \to \mathbb{C}_S$ is split, that is, there is a G-variety X and a G-equivariant isomorphism $\mathcal{X}^o \simeq X\times\mathbb{C}_S$ making the projection $\mathcal{X}^o \to \mathbb{C}_S$ into the second projection $X \times \mathbb{C}_S \to \mathbb{C}_S$.

In the case we are interested in, we have $C = \mathbb{C}$, $o = 0$, and $\mathcal{X}$ is the space defined by diagram (7.3.17). Property (P1) is then clear. Property (P2) holds because the map Φ maps $\{0\} \times \mathbb{C}_S$ to $\{0\}$, and the fiber of $\Lambda^{\mathbf{k}}_{w}$ over $0 \in \mathfrak{H}$ is Z, the Steinberg variety. We thus put $X := Z$ in our case.

In the general case, the embedding $\mathcal{X}^o \hookrightarrow \mathcal{X}$ gives by property (P2) a natural diagram

$$\begin{array}{ccccc} X\times\mathbb{C}_S = \mathcal{X}^o & \overset{\epsilon}{\hookrightarrow} & \mathcal{X} & \overset{j}{\hookleftarrow} & \mathcal{X}^* = \mathcal{X} \setminus \mathcal{X}^o \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ \{0\}\times\mathbb{C}_S & \hookrightarrow & \mathbb{C}\times\mathbb{C}_S & \hookleftarrow & (\mathbb{C}\setminus\{0\})\times\mathbb{C}_S \end{array} \tag{7.3.19}$$

Also, for any $\tau \in \mathbb{C}_S$, put $\mathcal{X}_\tau = p^{-1}(\tau)$ and $\mathcal{X}^*_\tau = p^{-1}(\tau) \cap \mathcal{X}^*$, and let $i_\tau : \mathcal{X}^*_\tau \hookrightarrow \mathcal{X}^*$ denote the embedding. The embedding induces a well defined pullback morphism $i^*_\tau : K^G(\mathcal{X}^*) \to K^G(\mathcal{X}^*_\tau)$, since the map $\mathcal{X} \to \mathbb{C}_S$ is flat. Note finally that, for each $\tau \in \mathbb{C}_S$, the projection $\pi : \mathcal{X}_\tau \to C$ has the special fiber over o isomorphic to $X\times\{\tau\} = X$, due to property (P2). Thus, there is a specialization map $\lim_{t\to 0} : K^G(\mathcal{X}^*_\tau) \to K^G(X)$.

The key result, that clearly implies Claim 7.3.6, is

Proposition 7.3.20. *(Homotopy Invariance of Specialization) If the conditions (P1)-(P2) hold, then the following composite map is independent of*

$\tau \in \mathbb{C}_S$:

$$K^G(\mathcal{X}^*) \xrightarrow{i_\tau^*} K^G(\mathcal{X}_\tau^*) \overset{\lim_{t\to 0}}{\longrightarrow} K^G(X).$$

The proof of the proposition will be based on three general lemmas.

Let (C, o) be a smooth curve and $f : Y \to C$ an arbitrary G-variety over C (with trivial G-action on C). Let $\varepsilon : Y^\circ \hookrightarrow Y$ be the embedding of the special fiber over o. Although the variety Y is not assumed to be smooth, the restriction functor $\varepsilon^* : K^G(Y) \to K^G(Y^\circ)$ can still be defined as it was implicitly done in the course of the definition of the specialization in K-theory, see 5.3. Namely, choose a local coordinate t on C such that $t(o) = 0$. We may view t as a function on Y by means of pullback. Then, $Y^\circ = t^{-1}(0)$, and for any sheaf $\mathcal{F} \in K^G(Y)$, we put

$$\text{(7.3.21)}\quad \varepsilon^*[\mathcal{F}] := \mathcal{T}or^0_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y/t\cdot\mathcal{O}_Y) - \mathcal{T}or^1_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y/t\cdot\mathcal{O}_Y)$$
$$= \mathrm{Coker}(\mathcal{F} \xrightarrow{t\cdot} \mathcal{F}) - \mathrm{Ker}(\mathcal{F} \xrightarrow{t\cdot} \mathcal{F}) \in K^G(Y^\circ).$$

Lemma 7.3.22. *The composite map $\varepsilon^*\ \varepsilon_* : K^G(Y^\circ) \to K^G(Y^\circ)$ is the zero-map.*

Proof. In view of the above definition of ε^* by means of equation (7.3.21), the two term complex $\mathcal{O}_Y \xrightarrow{t\cdot} \mathcal{O}_Y$ given by multiplication by t plays here the role of the Koszul resolution of the sheaf $\varepsilon_*\mathcal{O}_{Y^\circ}$. Thus, for any $\mathcal{F} \in K^G(Y^\circ)$, we deduce (either directly or as in Lemma 5.4.9) $\varepsilon^*\varepsilon_*\mathcal{F} = \lambda(N) \otimes \mathcal{F}$, where N is the "conormal bundle" to Y°. We put quotation marks because Y° is singular in general, and the bundle N is by definition given by the pullback by means of f of the cotangent space T_o^*C. Since C is a curve, T_o^*C is a 1-dimensional vector space, and N is the trivial 1-dimensional vector bundle on Y°. Hence $\lambda(N) = [\Lambda^0 N] - [\Lambda^1 N] = [N] - [N] = 0$, and the lemma follows. ∎

The second lemma is a variation of Lemma 5.3.6.

Lemma 7.3.23. *In the above setting, let $\mathcal{G}$ be a G-equivariant coherent sheaf on $Y^* = Y \setminus Y^\circ$, and $\overline{\mathcal{G}}$ its G-equivariant coherent extension to Y, that is, $\overline{\mathcal{G}}|_{Y^*} = \mathcal{G}$. Then in $K^G(Y^\circ)$ we have $\lim_{t\to 0}[\mathcal{G}] = \varepsilon^*\overline{\mathcal{G}}$.*

Proof. If the sheaf $\overline{\mathcal{G}}$ is a *lattice*, i.e., is t-torsion free, then the equation of the lemma is just the definition of the specialization. The point is that we do not assume $\overline{\mathcal{G}}$ to be t-torsion free. In this general case we argue as follows (cf. proof of Lemma 5.3.6).

Let $\mathcal{G}_\infty$ be the maximal subsheaf of $\overline{\mathcal{G}}$ supported on Y° and $\tilde{\mathcal{G}} = \overline{\mathcal{G}}/\mathcal{G}_\infty$. Then $\lim_{t\to 0}[\overline{\mathcal{G}}] = \varepsilon^*[\tilde{\mathcal{G}}]$ since $\tilde{\mathcal{G}}$ is a lattice for $\mathcal{G}$. On the other hand, in $K^G(Y)$ we have $[\overline{\mathcal{G}}] = [\mathcal{G}_\infty] + [\tilde{\mathcal{G}}]$. It follows that $\varepsilon^*[\overline{\mathcal{G}}] = \varepsilon^*[\mathcal{G}_\infty] + \varepsilon^*[\tilde{\mathcal{G}}]$, since ε^* is a homomorphism of K-groups. Thus, it suffices to prove that

$\varepsilon^*[\mathcal{G}_\infty] = 0$. But any sheaf supported on Y° is represented in $K^G(Y)$ by a class of the form $\varepsilon_*\mathcal{F}$ for some $\mathcal{F} \in K^G(Y^\circ)$. Hence, $\mathcal{G}_\infty = \varepsilon_*\mathcal{F}$, and Lemma 7.3.21 implies $\varepsilon^*\mathcal{G}_\infty = \varepsilon^*\varepsilon_*\mathcal{F} = 0$. ∎

Lemma 7.3.24. *For any G-variety X and any finite set $S \subset \mathbb{C}$, the projection $p : X \times \mathbb{C}_S \to X$ induces an isomorphism $p^* : K^G(X) \xrightarrow{\sim} K^G(X \times \mathbb{C}_S)$.*

Proof. The obvious diagram $X \times S \hookrightarrow X \times \mathbb{C} \hookleftarrow X \times \mathbb{C}_S$ gives rise to the standard exact sequence of K-groups, see 5.2.14:

$$K^G(X \times S) \to K^G(X \times \mathbb{C}) \to K^G(X \times \mathbb{C}_S) \to 0. \tag{7.3.25}$$

We claim that the first map in (7.3.25) vanishes, and hence, the second map is an isomorphism.

To prove the claim, we may assume without loss of generality, that the finite set S consists of a single point and that this point is $0 \in \mathbb{C}$, the origin. Let $\varepsilon : X \times \{0\} \hookrightarrow X \times \mathbb{C}$ denote the corresponding embedding. It suffices to show that the composite map $\varepsilon^*\varepsilon_* : K^G(X \times \{0\}) \to K^G(X \times \mathbb{C}) \to K^G(X \times \{0\})$ vanishes, since the second map ε^* is the Thom isomorphism. But $\varepsilon^*\varepsilon_* = 0$ by Lemma 7.3.21, applied to the projection $X \times \mathbb{C} \to \mathbb{C}$, and the claim follows.

To complete the proof of the lemma, we observe that the pullback morphism $p^* : K^G(X) \to K^G(X \times \mathbb{C}_S)$ may be factored as the composition $K^G(X) \to K^G(X \times \mathbb{C}) \to K^G(X \times \mathbb{C}_S)$, where the first map is the Thom isomorphism and the second map is the restriction map in (7.3.25), which is also an isomorphism, due to the claim. ∎

7.3.26. *Proof of Proposition* 7.3.20. Fix $\tau \in \mathbb{C}_S$ and consider the following commutative diagram of embeddings

$$\begin{array}{ccccc} X & \xhookrightarrow{\varepsilon_\tau} & \mathcal{X}_\tau & \hookleftarrow & \mathcal{X}^*_\tau \\ {\scriptstyle \tilde{i}_\tau}\big\downarrow & & {\scriptstyle \bar{i}_\tau}\big\downarrow & & \big\downarrow{\scriptstyle i_\tau} \\ X \times \mathbb{C}_S & \xhookrightarrow{\varepsilon} & \mathcal{X} & \hookleftarrow & \mathcal{X}^* \end{array} \tag{7.3.27}$$

Let $\mathcal{F}$ be a G-equivariant coherent sheaf on $\mathcal{X}^*$ and $\overline{\mathcal{F}}$ its G-equivariant coherent extension to $\mathcal{X}$. Then the restriction $\bar{i}_\tau^*\overline{\mathcal{F}}$ (notation $\bar{i}_\tau$ and $\tilde{i}_\tau$ is clear from (7.3.27)) is a well-defined coherent sheaf on $\mathcal{X}_\tau$, due to condition (P1). Hence, by Lemma 7.3.23 we obtain

$$\lim_{t\to 0}[\mathcal{F}] = \epsilon^*\overline{\mathcal{F}} \quad , \quad \lim_{t\to 0}[i_\tau^*\mathcal{F}] = \epsilon_\tau^*[\overline{i}_\tau^*\mathcal{F}].$$

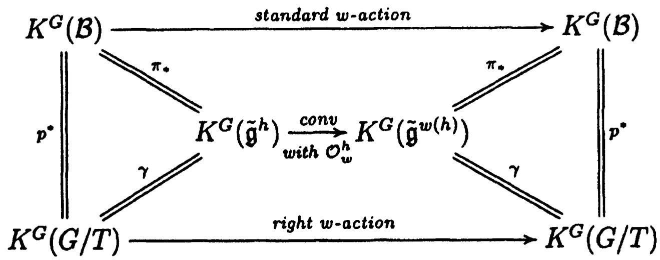
Set $\mathcal{G} = \lim_{t\to 0}[\mathcal{F}] \in K^G(X \times \mathbb{C}_S)$. Using the commutativity of diagram (7.3.27) and functoriality of restriction, one therefore finds

$$\lim_{t\to 0}[i_\tau^*\mathcal{F}] = \epsilon_\tau^*\,\overline{i}_\tau^*\overline{\mathcal{F}} = \overline{i}_\tau^*\,\epsilon^*\overline{\mathcal{F}} = \overline{i}_\tau^*\mathcal{G}.$$

We see that the only thing that has to be shown in order to prove the Proposition is that the class $\overline{i}_\tau^*\mathcal{G} \in K^G(X)$ is independent of the choice of τ. But using Lemma 7.3.24 we can write $\mathcal{G} = p^*\mathcal{G}'$, for some class $\mathcal{G}' \in K^G(X)$ and $p : X \times \mathbb{C}_S \to X$. Hence, for any τ, we have $\overline{i}_\tau^*\mathcal{G} = \overline{i}_\tau^* p^*\mathcal{G}' = \mathcal{G}'$. This proves the proposition, hence completes the proof of Claim 7.3.6. ∎

Proof of Proposition 7.2.14. We adapt to the K-theoretic setup the argument used in the proof of the corresponding result 3.6.17 in homology. In the notation of the proof of Theorem 7.2.2 write $\mathcal{O}_w^h \in K^G(\Lambda_w^h)$ for the class of the structure sheaf of the smooth variety Λ_w^h. We have the following commutative diagram, which is an analogue of (3.6.22).

(7.3.28)



Here the rectangle along the perimeter commutes by the definition of the standard W-action on $K^G(\mathcal{B})$, see (6.4.15). The isomorphism γ is induced by composition of the isomorphisms $\tilde{\mathfrak{g}}^h \overset{\mu}{\xrightarrow{\sim}} \mathrm{Ad}G\cdot h$ and $\mathrm{Ad}G\cdot h \simeq G/T$, so that the triangles $p\circ\gamma = \pi$ commute. Also, the trapezoid at the bottom of (7.3.28) commutes since $\Lambda_w^h = \mathrm{Graph}(w\text{-}action)$. Hence, it follows from the diagram that the upper "inverted trapezoid" in diagram (7.3.28) commutes.

Now replace h by the line $\mathfrak{l} = \mathbb{C}\cdot h$, and set $\mathcal{O}_w = \lim_{h\to 0}\mathcal{O}_w^{\mathfrak{l}^\bullet}$. By the proof of Theorem 7.2.2 the class $\mathcal{O}_w \in K^G(Z)$ is the image of the element $1\cdot w \in \mathbb{Z}[W]$ under the composition

$$\mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}] \overset{7.2.2}{\xrightarrow{\sim}} K^G(Z).$$

Taking the specialization of the upper inverted trapezoid in (7.3.28) as $h \to 0$, and using the fact that specialization commutes with convolution (see 5.3.9), we obtain the following commutative diagram

$$\begin{array}{ccc} K^G(\mathcal{B}) & \xrightarrow{\text{standard } w\text{-action}} & K^G(\mathcal{B}) \\ \pi^* \Big\| & & \Big\| \pi^* \\ K^G(T^*\mathcal{B}) & \xrightarrow[\text{with } \mathcal{O}_w]{\text{convolution}} & K^G(T^*\mathcal{B}) \end{array}$$

Thus, convolution with $\mathcal{O}_w$ transferred to $K^G(\mathcal{B})$ by means of the Thom isomorphism π^*, is the same thing as the right w-action on $K^G(G/T)$ transferred by means of p^*. This proves the proposition. ∎

Proof of Corollary 7.2.15. We retain the notation of the previous proof. By construction, the composition

$$\phi : \mathbb{Z}[W] \hookrightarrow \mathbb{Z}[W_{aff}] \overset{7.2.2}{\longrightarrow} K^G(Z) \to K(Z)$$

takes $w \in W$ to $\mathcal{O}_w$. Since assigning the support cycle to a sheaf intertwines the specialization map in K-theory with the one in homology, we compute

$$[\operatorname{supp} \mathcal{O}_w] = \operatorname{supp}(\lim_{\hbar \to 0} \mathcal{O}_w^{1^*}) \overset{5.9.17}{=} \lim_{\hbar \to 0}[\operatorname{supp} \mathcal{O}_w^{1^*}] = \lim_{\hbar \to 0}[\Lambda_w^{1^*}] = [\Lambda_w].$$

This agrees with the deformation proof of Theorem 3.4.1 and the result follows. ∎

7.3.29. Some Compatibilities for W-action. We complete this section by analyzing compatibility of the various natural isomorphisms introduced earlier in the book. These results are a bit technical and may be omitted by the reader without much trouble.

A linear map $f : V_1 \to V_2$ of two W-modules is said to be *sign-commuting* with the W-actions if we have

$$f(w \cdot v_1) = (-1)^{\ell(w)} \cdot w \cdot f(v_1), \qquad \forall v_1 \in V_1, w \in W.$$

First recall the Poincaré duality isomorphism 2.6(4),

$$H_i(\mathcal{B}) \simeq H^{2n-i}(\mathcal{B}), \qquad 2n = \dim_{\mathbb{R}} \mathcal{B}.$$

This isomorphism *does not* commute with the W-actions on homology and cohomlogy. The reason is that the W-action does not preserve the orientation (= fundamental) class of $\mathcal{B}$, hence does not commute with the intersection pairing (see the warning directly before 2.6.19). Specifically, the Weyl group acts on $H_{2n}(\mathcal{B})$, hence on the fundamental class of $\mathcal{B}$, by the sign representation $w \mapsto (-1)^{\ell(w)}$. It follows that the Poincaré duality isomorphism sign-commutes with the W-action. For example, for $i = 2n$ we have

$$H^0(\mathcal{B}) \simeq \text{sign} \otimes H_{2n}(\mathcal{B}) = \text{sign} \otimes \text{sign} = \textit{trivial representation}.$$

On the contrary, the Weyl group acts on G/T (on the right) by holomorphic transformations, hence, preserves the orientation (arising from the complex structure). It follows that the intersection pairing on G/T *commutes* with the W-action. Thus the Poincaré duality isomorphism

$$H_i(G/T) \simeq H^{2n-i}(G/T) \tag{7.3.30}$$

commutes with the natural W-action.

We turn next to the case of the cotangent bundle $\pi : T^*\mathcal{B} \to \mathcal{B}$ with zero-section $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$. In the following lemma, the W-action related to $\mathcal{B}$ is always understood to be the "standard" one (see 6.4.15) and the W-action related to $T^*\mathcal{B}$ is always understood to be the one arising by convolution.

Lemma 7.3.31. *Let $n = \dim_{\mathbb{C}}\mathcal{B}$. Then*

(i) The Thom isomorphism $\pi^*: H_i(\mathcal{B}) \xrightarrow{\sim} H_{i+2n}(T^*\mathcal{B})$ sign-commutes with the W-actions; hence, the same holds for the inverse, $i^*: H_{i+2n}(T^*\mathcal{B}) \xrightarrow{\sim} H_i(\mathcal{B})$, see 2.6.43.

(ii) The Thom isomorphism $\pi^* : K^G(T^*\mathcal{B}) \to K^G(\mathcal{B})$ commutes with the W-actions; hence the same holds for the inverse $i^* : K^G(T^*\mathcal{B}) \to K^G(\mathcal{B})$. The same statement also holds for non-equivariant K-groups.

*(iii) The homological Chern character map $K(T^*\mathcal{B}) \to H_*(T^*\mathcal{B})$ (*5.8.1*), commutes with the W-actions, while the the homological Chern character map $K(\mathcal{B}) \to H_*(\mathcal{B})$ sign-commutes with the W-actions.*

(iv) The co-homological Chern character map $K(\mathcal{B}) \to H^*(\mathcal{B})$ commutes with the Weyl group actions.

Proof of Lemma 7.3.31. (i) It suffices to prove the statement for i^*. This is a restriction with support map which is constructed by definition, by means of intersecting in $T^*\mathcal{B}$ with $[\mathcal{B}]$, the fundamental class of the zero-section. Thus, proving the claim for i^* amounts to proving two facts: (1) convolution with $H(Z)$ acts on $[\mathcal{B}]$ as the sign representation of W; and (2) convolution with $H(Z)$ acts on $[T^*\mathcal{B}]$ as the trivial representation of W. The first fact follows from Claim 3.6.17 and the remark above it. The second fact follows from the equation $[T^*\mathcal{B}] = \lim_{t\to 0}[\tilde{\mathfrak{g}}^{t\cdot h}]$, since the fundamental class of $\tilde{\mathfrak{g}}^h \simeq G/T$ is preserved by the convolution (cf. the discussion leading to formula (7.3.30)).

(ii) Let L_λ be the G-equivariant line bundle on $\mathcal{B}$ associated with a character $\lambda \in X^*(T)$ and let π^*L_λ be its pullback to $T^*\mathcal{B}$. The deformation argument in the proof of Theorem 7.2.2 shows that, for any $w \in W$, convolution with $\mathcal{O}_w^h =$ *structure sheaf of* Λ_w^h, takes π^*L_λ to $\pi^*L_{w(\lambda)}$. Since the standard w-action on $K^G(\mathcal{B})$ takes L_λ to $L_{w(\lambda)}$ as well, the map π^* commutes with the W-actions.

(iv) is clear from the construction of the W-action on $H^*(\mathcal{B})$, cf. 6.4.15.

(iii) Observe first that for a smooth variety, the homological Chern character is obtained from the cohomological Chern character by means of Poincaré duality. By the discussion preceding the Lemma we know that the Poincaré duality isomorphism for $\mathcal{B}$ sign-commutes with W. The cohomological Chern character for $\mathcal{B}$ commutes with W by (iv). Further, the cohomological Chern character always commutes with restriction. But restriction to the zero-section in homology sign-commutes with W, by (i) and the restriction map in K-theory commutes with W by (ii). This proves that the homological Chern character map on $T^*\mathcal{B}$ commutes with W. ∎

So far, we have always used the "standard" W-action as long as the variety $\mathcal{B}$ was concerned. There is, however, a convolution action of W on $H_*(\mathcal{B})$ and on $K(\mathcal{B})$ arising from convolution with $H(Z)$, resp. $K(Z)$, due to the set-theoretic equation $Z \circ \mathcal{B} = \mathcal{B}$. The two actions, the "standard" action and the "convolution" action, turn out to be equal on the homology of $\mathcal{B}$ due to Claim 3.6.17. This is not the case in K-theory, as is shown by the result below.

Let $e^\rho \in R(\mathbb{T})$ denote the element corresponding to the half-sum of positive roots (recall that there is a preferred choice of "geometric" positive roots, see 6.1.9, for the "abstract" maximal torus).

Lemma 7.3.32. *The W-action on $K^G(\mathcal{B})$ arising from convolution with $K^G(Z)$ is expressed in terms of the "standard" W-action on $K^G(\mathcal{B})$ by*

$$w : R \mapsto (-1)^{\ell(w)} e^{-\rho} \cdot w(e^\rho R), \qquad R \in R(\mathbb{T}).$$

Proof. The zero section $i \colon \mathcal{B} \hookrightarrow T^*\mathcal{B}$ induces the following two diagrams in K-theory

(7.3.33)

$$\begin{array}{ccc} K^G(T^*\mathcal{B}) & \xrightarrow[\text{with } K^G(Z)]{\text{conv}} & K^G(T^*\mathcal{B}) \\ {\scriptstyle i_*}\uparrow & & \uparrow{\scriptstyle i_*} \\ K^G(\mathcal{B}) & \xrightarrow[\text{with } K^G(Z)]{\text{conv}} & K^G(\mathcal{B}) \end{array} \qquad \begin{array}{ccc} K^G(T^*\mathcal{B}) & \xrightarrow[\text{with } K^G(Z)]{\text{conv}} & K^G(T^*\mathcal{B}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ K^G(\mathcal{B}) & \xrightarrow[w\text{-action}]{\text{standard}} & K^G(\mathcal{B}) \end{array}$$

The diagram on the left commutes due to Lemma 5.2.23 (put $M_3 = pt$, $\widetilde{Y} = \mathcal{B}$, $Y = T^*\mathcal{B}$ in the discussion after the lemma). The diagram on the right commutes due to Lemma 7.3.31(ii). Hence for any $R \in R(\mathbb{T})$ and $w \in W$,

$$i^* i_*(\text{convolution action of } w \text{ on } R) = w(i^* i_* R).$$

By Lemma 5.4.9 the map $i^* i_*$ is given by the tensor product with the class $\lambda(T^*\mathcal{B})$. The cotangent bundle on $\mathcal{B}$ is isomorphic to $G \times_B \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of the Lie algebra of a Borel subgroup B. Recall that the weights of the torus action on $\mathfrak{n}$ are the *negative* roots relative to the "geometric" choice of positive roots. Hence we find

$$\lambda(T^*\mathcal{B}) = \prod_{\alpha>0}(1-e^{-\alpha}) \overset{6.1.10}{=} e^{-\rho}\cdot\prod_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2}) = e^{-\rho}\cdot\Delta.$$

We get

$$e^{-\rho}\cdot\Delta\cdot(\textit{convolution with } w \textit{ on } R) = w(e^{-\rho}\cdot\Delta\cdot R) = (-1)^{\ell(w)}\cdot\Delta\cdot w(e^{-\rho}\cdot R),$$

since Δ is a skew-symmetric element. The ring $R(\mathbb{T})$ being an integral domain, it follows that

$$\textit{convolution action of } w \textit{ on } R = (-1)^{\ell(w)}\cdot e^{\rho}\cdot w(e^{-\rho}\,R).$$

This completes the proof. ∎

7.4 Hilbert Polynomials and Orbital Varieties

In this section we digress from the main theme of this chapter, as set out in §7.2. Our aim here is to prove an important result (Theorem 7.4.1 below) relating harmonic polynomials on the Cartan subalgebra to some equivariant Hilbert polynomials, see §6.6, and to Springer representations of the Weyl group.

Let $\mathbb{O}\subset\mathcal{N}$ be a nilpotent orbit. Throughout this section we fix a Borel subalgebra $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}\subset\mathfrak{g}$, where $\mathfrak{h}$ is a fixed Cartan subalgebra and $\mathfrak{n}$ is the nilradical of $\mathfrak{b}$. Write Σ for the closure of $\mu^{-1}(\mathbb{O})\subset\tilde{\mathcal{N}}$, the orbital variety associated to $\mathbb{O}$.

Recall next that at the end of section 6.5 we have introduced a map $\epsilon: H(\Sigma)\to\mathcal{H}^d$. Let $\mathcal{H}(\mathbb{O})$ denote the image of ϵ, a linear subspace of $\mathcal{H}^d$ with a distinguished basis formed by the images of the fundamental classes of the irreducible components of Σ. The latter are, by (6.5.10), in bijective correspondence with the irreducible components of the variety $\mathbb{O}\cap\mathfrak{n}$. Given such a component Λ, write Σ_Λ for the irreducible component of the orbital variety Σ associated to Λ by means of (6.5.9), and let $\epsilon(\Sigma_\Lambda)\in\mathcal{H}(\mathbb{O})$ be the corresponding harmonic polynomial. We shall now proceed to an alternative direct construction of those distinguished polynomials.

Let $T\subset B$ be the maximal torus and the Borel subgroup with Lie algebras $\mathfrak{h}\subset\mathfrak{b}$, respectively. Clearly, $\bar{\Lambda}$, the closure of Λ, is a B-stable, hence a T-stable, closed subvariety of $\mathfrak{n}$ which can be equivalently defined as an irreducible component of $\overline{\mathbb{O}\cap\mathfrak{n}}$. Let $P_\Lambda\in\mathbb{C}[\mathfrak{h}]$ denote the T-equivariant Hilbert polynomial (see §6.6) of the subvariety $\bar{\Lambda}\subset\mathfrak{n}$ and write $Comp(\mathbb{O}\cap\mathfrak{n})$ for the set of irreducible components of $\overline{\mathbb{O}\cap\mathfrak{n}}$.

Fix a point $x\in\mathbb{O}\cap\mathfrak{n}$ and put $d=\dim_{\mathbb{C}}\mathcal{B}_x$. The following result plays an important role in representation theory of semisimple Lie algebras.

Theorem 7.4.1. *([BBM],[Jo3],[Ve]) The equivariant Hilbert polynomials*

$$\{P_\Lambda \ , \ \Lambda \in Comp(\mathbb{O} \cap \mathfrak{n})\}$$

are homogeneous W-harmonic polynomials on $\mathfrak{h}$ of degree d. Moreover, for any Λ, the polynomial P_Λ is proportional to $\epsilon(\Sigma_\Lambda)$. In particular, the Hilbert polynomials form the distinguished basis of the vector space $\mathcal{H}(\mathbb{O})$.

Let $\mathbb{C}^d[\mathfrak{h}]$ denote the vector space of degree d homogeneous polynomials on $\mathfrak{h}$, and $\mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ the ideal in $\mathbb{C}[\mathfrak{h}]$ generated by W-invariant polynomials without constant term. We have a natural projection

$$\text{(7.4.2)} \qquad \text{proj} \ : \ \mathbb{C}[\mathfrak{h}] \to \mathbb{C}^d[\mathfrak{h}]/(\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}^{\mathbb{C}[\mathfrak{h}]}).$$

We will deduce Theorem 7.4.1 from the following two results.

Proposition 7.4.3. $\text{proj}(P_\Lambda) = \text{proj}\Big(\epsilon(\Sigma_\Lambda)\Big)$, *for any $\Lambda \in Comp(\mathbb{O} \cap \mathfrak{n})$.*

Proposition 7.4.4. *The equivariant Hilbert polynomials $\{P_\Lambda, \ \Lambda \in Comp(\mathbb{O} \cap \mathfrak{n})\}$ span a W-stable subspace in $\mathbb{C}^d[\mathfrak{h}]$.*

To begin the proof, we first reformulate the results about equivariant Hilbert polynomials proved in section 6.6 in a slightly different way.

Recall that to any T-equivariant coherent sheaf $\mathcal{M}$ on $\mathfrak{n}$ (here $\mathfrak{n}$ may be replaced by any finite dimensional vector space with a contracting T-action) we have associated its formal character, $\text{ch}_T(\mathcal{M})$, which has the form, see Proposition 6.6.6

$$\text{ch}_T(\mathcal{M}) = \frac{\chi_{\mathcal{M}}}{\prod_{\alpha \in \text{Sp}\, \mathfrak{n}}(1 - e^\alpha)}, \qquad \text{where } \chi_{\mathcal{M}} \in R(T).$$

We may view $\chi_{\mathcal{M}}$ as a function on T and pull it back to $\mathfrak{h}$ by means of the exponential map. Taking the Taylor expansion at $0 \in \mathfrak{h}$ we get, by additivity of formal characters, a well-defined group homomorphism:

$$\text{(7.4.5)} \qquad \chi : K^T(\mathfrak{n}) \to \mathbb{C}[[\mathfrak{h}]] \quad , \quad \mathcal{M} \mapsto \exp^*(\chi_{\mathcal{M}}).$$

This map should be rather denoted "$\exp^* \circ \chi$", but abusing the notation, we will write χ for short, thinking of the function $\chi_{\mathcal{M}}$ in terms of its Taylor expansion on $\mathfrak{h}$.

Let $I \subset \mathbb{C}[[\mathfrak{h}]]$ denote the augmentation ideal consisting of the formal power series without constant term (not to be confused with $\mathcal{I}^{\mathbb{C}[[\mathfrak{h}]]}$, the ideal generated by the augmentation ideal of the subalgebra $\mathbb{C}[[\mathfrak{h}]]^W$). On $\mathbb{C}[[\mathfrak{h}]]$ introduce the I-adic filtration $\mathbb{C}[[\mathfrak{h}]] = I^0 \supset I^1 \supset I^2 \supset \ldots$ where I^k, the k-th power of I, is the ideal of the power series vanishing at $0 \in \mathfrak{h}$ up to order $\geq k$. Put $n = \dim \mathfrak{n}$. Then, Theorem 6.6.12 clearly implies the following claim.

Claim 7.4.6. The map χ takes $\Gamma_q K^T(\mathfrak{n})$ to I^{n-q}, for any $q \geq 0$.

By the crucial dimension equality 3.3.6 we have $\dim(\mathbb{O} \cap \mathfrak{n}) = 1/2 \dim \mathbb{O} = \dim \mathfrak{n} - d$, where $d = \dim_{\mathbb{C}} \mathcal{B}_x$ is the dimension of the Springer fiber. Therefore the map χ restricts by Claim 7.4.6 to a homomorphism

$$\chi : K^T(\overline{\mathbb{O} \cap \mathfrak{n}}) \to I^d. \tag{7.4.7}$$

Next, recall our basic diagram

$$\begin{array}{ccc} & T^*\mathcal{B} & \\ {}^{\mu}\swarrow & & \searrow^{\pi} \\ \mathcal{N} & & \mathcal{B} \end{array}$$

where $\mu : T^*\mathcal{B} \to \mathcal{N}$ is the Springer resolution and π is the cotangent bundle projection. Let $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ denote the zero-section. Let $\Sigma =$ *the closure of* $\mu^{-1}(\mathbb{O}) \subset T^*\mathcal{B}$, be our orbital variety. The projection $\pi : \Sigma \to \mathcal{B}$ makes Σ a G-equivariant fibration over $\mathcal{B}$ with fiber $\overline{\mathbb{O} \cap \mathfrak{n}}$ so that we get the commutative diagram

$$\begin{array}{ccccc} \Sigma \simeq G \times_B \overline{\mathbb{O} \cap \mathfrak{n}} & \overset{j}{\hookrightarrow} & T^*\mathcal{B} \simeq G \times_B \mathfrak{n} & \overset{i}{\hookleftarrow} & \mathcal{B} \\ \uparrow & & \uparrow & & \uparrow {\scriptstyle \mathfrak{b} \mapsto \mathrm{pt}} \\ \overline{\mathbb{O} \cap \mathfrak{n}} & \overset{j}{\hookrightarrow} & \mathfrak{n} & \overset{0 \leftarrow \mathrm{pt}}{\hookleftarrow} & \mathrm{pt} \end{array} \tag{7.4.8}$$

The second row of the diagram is obtained from the first by restriction to $T^*_{\mathfrak{b}}\mathcal{B} = \mathfrak{n}$, the fiber over the base point $\mathfrak{b} =$ *our fixed Borel subalgebra.* This gives the induced commutative diagram of K-groups where "res" is the isomorphism induced by restriction:

(7.4.9)

$$\begin{array}{ccccccccc} K^G(\Sigma) & \xrightarrow{j_*} & K^G(T^*\mathcal{B}) & \xrightarrow{i^*} & K^G(\mathcal{B}) & \overset{6.1.11}{=\!=} & R(T) & \xrightarrow{\exp^*} & \mathbb{C}[[\mathfrak{h}]] \\ \| \,{\scriptstyle \mathrm{res}} & & \| \,{\scriptstyle \mathrm{res}} & & \| \,{\scriptstyle \mathrm{res}} & & \| & & \| \\ K^T(\overline{\mathbb{O} \cap \mathfrak{n}}) & \xrightarrow{j_*} & K^T(\mathfrak{n}) & \xrightarrow{i^*} & K^T(\mathrm{pt}) & \overset{5.2.1}{=\!=} & R(T) & \xrightarrow{\exp^*} & \mathbb{C}[[\mathfrak{h}]] \end{array}$$

Let Ψ_{top}, resp. Ψ_{bot}, denote the composition of maps in the top, resp. bottom, row of diagram (7.4.9). The main point we need about this diagram in order to prove Proposition 7.4.3 is the following result.

Lemma 7.4.10. *(i)* $\Psi_{bot} = \chi$, *see* (7.4.5)*;*
(ii) The image of Ψ_{bot} *is contained in* I^d.

Proof. Part (i) is due to claim 6.6.8, and part (ii) is immediate from (i) and (7.4.7). ∎

A few general remarks are in order. Write $\mathcal{I} = \mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ for short. An obvious isomorphism $I^k/I^{k+1} \simeq \mathbb{C}^k[\mathfrak{h}]$ yields

$$(7.4.11) \qquad I^k/(I^{k+1} + I^k \cap \mathcal{I}) \simeq \mathbb{C}^k[\mathfrak{h}]/\mathbb{C}^k[\mathfrak{h}] \cap \mathcal{I} \simeq H^{2k}(\mathcal{B}),$$

where the last isomorphism is the Borel isomorphism β, see (6.4.13). Recall further that there is an increasing Γ-filtration on K-groups, the filtration by dimension of support, defined in 5.9. By equation (6.4.25) the cohomological Chern character maps $\Gamma_{n-k}K(\mathcal{B})$ to $\oplus_{p\geq k} H^{2p}(\mathcal{B})$. Note that the latter space corresponds to $I^k/I^k \cap \mathcal{I}$ under the Borel isomorphism.

Proof of Proposition 7.4.3. By definition, the equivariant Hilbert polynomial of a sheaf $\mathcal{M}$ is given by the first non-vanishing term in the Taylor expansion of the function $\exp^*(\chi_{\mathcal{M}})$. In the case we are interested in, the polynomials P_Λ are obtained this way from the structure sheaves of the components of $\overline{\mathbb{O} \cap \mathfrak{n}}$. Thus, in view of Lemma 7.4.10(i) and isomorphisms (7.4.11), the Hilbert polynomials P_Λ are the images of the corresponding classes $[\mathcal{O}_\Lambda]$ under the composition

(7.4.12)
$$K^T(\overline{\mathbb{O} \cap \mathfrak{n}}) \to I^d \to I^d/I^{d+1} \to I^d/(I^{d+1} + I^d \cap \mathcal{I}), \quad d = \dim_{\mathbb{C}} \mathcal{B}_x,$$

where the first arrow is the map Ψ_{bot} and all the others are natural projections.

By commutativity of diagram (7.4.9), we may replace the map Ψ_{bot} in (7.4.12) by Ψ_{top}. To study the latter, recall that $\dim \Sigma = 2n - d$ by 6.5.12. We see that the isomorphism $res : K^G(\Sigma) \to K^T(\overline{\mathbb{O} \cap \mathfrak{n}})$ from (7.4.9) composed with all the maps in (7.4.12) is equal to the composition of maps along the top row and then along the right vertical arrows of the following diagram.

(7.4.13)
$$\begin{array}{ccccccc}
K^G(\Sigma) & \xrightarrow{j_*} & \Gamma_{2n-d}K(T^*\mathcal{B}) & \xrightarrow{i^*} & \Gamma_{n-d}K(\mathcal{B}) & \xrightarrow{\mathrm{ch}^* \circ \beta} & I^d/I^d \cap \mathcal{I} \\
\downarrow{\scriptstyle \mathrm{ch}_*} & & \downarrow{\scriptstyle \mathrm{ch}_*} & & \downarrow{\scriptstyle \mathrm{ch}_*} & & \Vert{\scriptstyle \beta} \\
H_*(\Sigma) & \xrightarrow{j_*} & \bigoplus_{p\leq 2n-d} H_{2p}(T^*\mathcal{B}) & \xrightarrow{i^*} & \bigoplus_{p\leq 2n-d} H_{2p}(\mathcal{B}) & \overset{\mathbb{D}}{=\!=} & \bigoplus_{p\geq d} H^{2p}(\mathcal{B}) \\
\downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle \mathrm{proj}} \\
H(\Sigma) & \xrightarrow{j_*} & H_{2(2n-d)}(T^*\mathcal{B}) & \xrightarrow{i^*} & H_{2(n-d)}(\mathcal{B}) & \overset{\mathbb{D}}{=\!=} & H^{2d}(\mathcal{B})
\end{array}$$

In this diagram, $\mathbb{D}$ stands for Poincaré duality and $H(\Sigma)$ stands for the top dimensional Borel-Moore homology group of Σ, as usual. The compositions $ch_* \circ$ proj of the vertical maps in each of the three left columns are given by the support cycle map (Lemma 5.9.4). Clearly, the above diagram commutes. Thus, two paths from the top left corner to the bottom right corner, the first all the way down along left vertical arrows and then right, and the second along the top row and then along the right vertical arrows, coincide. Combining this last observation with Lemma 7.4.10(i), and with the commutativity of diagram (7.4.9), we deduce commutativity of the left square in the following diagram, where i^* and j_* are as in (7.4.13):

(7.4.14)

$$\begin{array}{ccccccc}
K^G(\Sigma) & \xlongequal{\text{res}} & K^T(\overline{\mathbb{O}\cap\mathfrak{n}}) & \xrightarrow{\ \chi\ } & I^d & \xrightarrow{\ p\ } & \mathbb{C}^d[\mathfrak{h}] \\
{\scriptstyle supp}\downarrow & & & & \text{proj}\downarrow & & {\scriptstyle proj}\downarrow \\
H(\Sigma) & \xrightarrow{\epsilon=j_*\circ i^*\circ\mathbb{D}} & H^{2d}(\mathcal{B}) & = & I^d/(I^{d+1}+I^d\cap\mathcal{I}) & = & \mathbb{C}^d[\mathfrak{h}]/\mathbb{C}^d[\mathfrak{h}]\cap\mathcal{I}
\end{array}$$

The map $p : I^d \to \mathbb{C}^d[\mathfrak{h}] = I^d/I^{d+1}$ in (7.4.14) is the projection, so that the right square of (7.4.14) trivially commutes. Thus, the whole diagram commutes.

Observe further that if Σ_Λ denotes the irreducible component of Σ corresponding to $\Lambda \in Comp(\mathbb{O}\cap\mathfrak{n})$ then, for the map *res* in diagram (7.4.14), we have

$$\text{res}(\mathcal{O}_{\Sigma_\Lambda}) = \mathcal{O}_\Lambda \quad \text{and} \quad \text{supp}\,(\mathcal{O}_{\Sigma_\Lambda}) = [\Sigma_\Lambda].$$

By definition, for any irreducible component of Λ of $\overline{\mathbb{O}\cap\mathfrak{n}}$, the composition $p\circ\chi$ in the top row of (7.4.14) sends $\mathcal{O}_\Lambda$ to the equivariant Hilbert polynomial P_Λ. It follows by commutativity of the diagram that the composition in the bottom row of (7.4.14) sends the fundamental class $[\Sigma_\Lambda]$ to $\text{proj}(P_\Lambda)$. Observe finally that the map $\epsilon = j_*\circ i^*\circ\mathbb{D}$ in the diagram, composed with the isomorphism $H^d(\mathcal{B}) \simeq \mathcal{H}^d$ is exactly the map assigning to an irreducible component of Σ a distinguished harmonic polynomial. This proves Proposition 7.4.3. ∎

Proof of Proposition 7.4.4. Let Φ_{top}, resp. Φ_{bot}, be the composition of the map Ψ_{top}, resp. Ψ_{bot}, in the top (resp. bottom) row of diagram (7.4.9) followed by the projection $I^d \to I^d/I^{d+1} = \mathbb{C}^d[\mathfrak{h}]$ (thus, Φ_{bot} equals the composition of all the maps in (7.4.12) but the last one). We must show that the subspace spanned by the polynomials

$$\{\Phi_{bot}(\mathcal{O}_\Lambda) \in \mathbb{C}^d[\mathfrak{h}] \mid \Lambda \in Comp(\mathbb{O}\cap\mathfrak{n})\}$$

is W-stable. To that end, observe first that this subspace equals the whole image of the map Φ_{bot}, due to part (ii) of Theorem 6.6.12. By commutativity of diagram (7.4.9) this is the same as $\text{Image}(\Phi_{top})$.

To show that the image of Φ_{top} is W-stable, consider the Steinberg variety $Z \subset T^*\mathcal{B} \times T^*\mathcal{B}$, and recall that

$$Z \circ T^*\mathcal{B} = T^*\mathcal{B}, \qquad Z \circ \Sigma = \Sigma.$$

Hence, convolution in K-theory makes $K^G(T^*\mathcal{B})$ and $K^G(\Sigma)$ into $K^G(Z)$-modules, in particular, into W-modules. Furthermore, the map j_*: $K^G(\Sigma) \to K^G(T^*\mathcal{B})$ commutes with the $K^G(Z)$-action, by Lemma 5.2.23. This shows that the first map in the top row of (7.4.9) is W-equivariant. The second map is W-equivariant, by Lemma 7.3.31(ii). The third map is W-equivariant by definition of the map, see 6.1.11. That last map, exp*, is clearly W-equivariant. Thus, the composition Φ_{top} commutes with the W-action, and Image(Φ_{top}) is a W-stable subspace. ∎

Proof of Theorem 7.4.1. We have already shown in the course of the proof of Proposition 7.4.4 that all of the maps in the top row of diagram (7.4.9) commute with the W-actions. Therefore, the map $\Phi_{top} : K^G(\Sigma) \to \mathbb{C}^d[\mathfrak{h}]$ is a W-map. The map Φ_{top} also equals the composition of maps in the top row of diagram (7.4.14). The map $\text{res} \circ \chi$ in that diagram is completely determined–due to Theorem 6.6.12(ii)–by its value on the structure sheaves of the irreducible components of Σ. The latter project isomorphically onto the basis of $H(\Sigma)$ under the support cycle map $K^G(\Sigma) \to H(\Sigma)$. It follows that the map Φ_{top} descends to a well-defined W-equivariant linear map $\overline{\Phi} : H(\Sigma) \to \mathbb{C}^d[\mathfrak{h}]$.

Next, we use the W-stable direct sum decomposition $\mathbb{C}[\mathfrak{h}] = \mathcal{H} \oplus \mathcal{I}$ (see section 6.4) and write $\overline{\Phi}$ as the sum of two W-maps

$$\overline{\Phi} = \Phi_{\mathcal{H}} + \Phi_{\mathcal{I}}, \quad \Phi_{\mathcal{H}} : H(\Sigma) \to \mathcal{H}^d, \quad \Phi_{\mathcal{I}} : H(\Sigma) \to \mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}.$$

To analyze the map $\Phi_{\mathcal{H}}$ observe that, by definition, $\text{proj} \circ \Phi_{\mathcal{I}} = 0$, where $\text{proj} : \mathbb{C}^d[\mathfrak{h}] \twoheadrightarrow \mathbb{C}^d[\mathfrak{h}]/\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}$. Hence, $\text{proj} \circ \Phi_{top} = \text{proj} \circ \overline{\Phi} = \text{proj} \circ \Phi_{\mathcal{H}}$. Therefore, Proposition 7.4.3 yields $\text{proj} \circ \Phi_{\mathcal{H}} = \text{proj} \circ \epsilon$. But the projection proj restricted to the subspace $\mathcal{H}^d$ becomes an isomorphism. Thus we get $\Phi_{\mathcal{H}} = \epsilon$.

To study the map $\Phi_{\mathcal{I}}$ we compose it with the isomorphism ϕ: $H(\mathcal{B}_x)^{C(x)} \xrightarrow{\sim} H(\Sigma)$ of Proposition 6.5.13, where $C(x)$ stands for the component group of the centralizer of x in G. This way we get a map $\Phi'_{\mathcal{I}} = \phi \circ \Phi_{\mathcal{I}} : H(\mathcal{B}_x)^{C(x)} \to \mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$. Since the map $\overline{\Phi}$ is W-equivariant, it follows that $\Phi'_{\mathcal{I}}$ is W-equivariant. Therefore, its image is a W-stable subspace in $\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}^{\mathbb{C}[\mathfrak{h}]}$ isomorphic to the irreducible representation $H(\mathcal{B}_x)^{C(x)}$, corresponding to the trivial representation of $C(x)$. But the graded space factorization $\mathbb{C}[\mathfrak{h}] \simeq \mathbb{C}[\mathfrak{h}]^W \otimes \mathcal{H}$ shows that the simple W-modules appearing in the decomposition of $\mathbb{C}^d[\mathfrak{h}] \cap \mathcal{I}$ are only those that occur in $\mathcal{H}^i$, for some $i < d$ with non-zero multiplicity. By Corollary 6.5.3, the representation $H(\mathcal{B}_x)^{C(x)}$ never occurs in $\mathcal{H}^i$ for $i < d$. Hence the map $\Phi'_{\mathcal{I}}$ vanishes. Thus, $\Phi_{\mathcal{I}} = 0$, and we obtain $\overline{\Phi} = \Phi_{\mathcal{H}} = \epsilon$. The theorem follows. ∎

7.5 The Hecke Algebra for SL_2

Before proving Theorem 7.2.5 in the general case, we consider in more detail the special case $G = SL_2(\mathbb{C})$. Let T be the standard maximal torus, the group of (2×2)-diagonal matrices with determinant 1, and B the Borel subgroup of (2×2)-upper triangular matrices with determinant 1. The group $\mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ is isomorphic to $\mathbb{Z}$ with the generator chosen to be the homomorphism

$$\omega : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to t^{-1}, \tag{7.5.1}$$

the dominant fundamental weight of $SL_2(\mathbb{C})$ with respect to the geometric choice of positive roots, see 6.1.9. We write the group $\mathrm{Hom}_{alg}(T, \mathbb{C}^*)$ additively, so that any element of the group is of the form

$$\lambda = n\omega : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to t^{-n} \quad , \quad n \in \mathbb{Z}.$$

Let $X = e^{\omega}$ denote the element of the group algebra $\mathbb{Z}[P] = R(T)$ corresponding to ω. Thus $R(T) = \mathbb{Z}[X, X^{-1}]$.

The group $SL_2(\mathbb{C})$ acts transitively on $\mathbb{C}^2 \setminus \{0\}$, by linear transformations. The isotropy group of the vector $\binom{1}{0}$ is the subgroup U of upper triangular unipotent matrices in $SL_2(\mathbb{C})$. The subgroup B stabilizes the line spanned by $\binom{1}{0}$. Thus, there are natural $SL_2(\mathbb{C})$-equivariant isomorphisms:

$$G/U \simeq \mathbb{C}^2 \setminus \{0\} \quad , \quad \mathcal{B} = G/B = \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1 . \tag{7.5.2}$$

Observe further that there is a T-action on G/U *on the right.* An element $t = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in T$ acts by the assignment $t : g \cdot U \mapsto g \cdot t \cdot U$. This action is well defined, since the torus T normalizes U. The right T-action clearly commutes with the standard left G-action.

Lemma 7.5.3. *For any $t \in \mathbb{C}^*$, the right $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$-action on G/U corresponds by means of isomorphism* (7.5.2) *to the standard t-action on $\mathbb{C}^2 \setminus \{0\}$ by dilations, i.e., the action $t : (z_1, z_2) \mapsto (t \cdot z_1, t \cdot z_2)$.*

Proof. Note that G-actions on G/U and $\mathbb{C}^2 \setminus \{0\}$ are transitive and both T-actions commute with the G-action. Hence it suffices to show that the two T-actions correspond on a single vector, say the base vector $\binom{1}{0}$. But in that case we find

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \quad \blacksquare$$

Recall that $\mathcal{O}(n)$ denotes the line bundle (invertible sheaf) on $\mathbb{P}^1$ whose germs of sections are regular homogeneous functions of degree n on open

$\mathbb{C}^*$-stable subsets of $\mathbb{C}^2 \setminus \{0\}$. On the other hand, to any weight $\lambda : T \to \mathbb{C}^*$ we have associated in section 6.1.11 a canonical line bundle L_λ on $\mathcal{B} = \mathbb{P}^1$.

Lemma 7.5.4. *For any weight $n \in \mathbb{Z}$, there is a natural G-equivariant isomorphism $L_{n\omega} \simeq \mathcal{O}(n)$.*

Proof. A germ of a section of $L_{n\omega}$ may be viewed, see (6.1.12), as the germ of function f on G/U such that, for any $t \in T$ and $g \in G$, we have:

$$f(g \cdot t \cdot U) = (n\omega)(t)^{-1} \cdot f(g \cdot U) = t^n \cdot f(g \cdot U).$$

This equation translates, by means of the isomorphism (7.5.2) and Lemma 7.5.3, to the condition that f is homogeneous of degree n on $\mathbb{C}^2 \setminus \{0\}$. ∎

We now discuss the Weyl Character Formula 6.1.17 for $G = SL_2(\mathbb{C})$. First note that in this case we have $W = \textit{Weyl group} = \mathbb{Z}/2$. The constant map $p : \mathbb{P}^1 \to \text{pt}$ induces a morphism of equivariant K-groups: $p_* : K^G(\mathbb{P}^1) \to K^G(\text{pt})$. The generator of the Weyl group $\mathbb{Z}/2$ acts on $R(T) = \mathbb{Z}[X, X^{-1}]$ as the involution $X \leftrightarrow X^{-1}$, and we identify $K^G(\text{pt})$ with the subring stable under this involution. Then in our special case Corollary 6.1.17 reads

Lemma 7.5.5. *For any integer $n \in \mathbb{Z}$ we have*

$$p_*\mathcal{O}(n) = \frac{X^{n+1} - X^{-(n+1)}}{X - X^{-1}} \in R(T)^W.$$

Proof. Although the result is a special case of Corollary 6.1.17 it is instructive, we believe, to give here a direct argument. Since $\dim_{\mathbb{C}} \mathbb{P}^1 = 1$, for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$, we have $H^i(\mathbb{P}^1, \mathcal{F}) = 0$, for all $i > 1$, see [Ha]. Therefore, for $\mathcal{F} = \mathcal{O}(n)$ we find

$$p_*\mathcal{O}(n) = H^0(\mathbb{P}^1, \mathcal{O}(n)) - H^1(\mathbb{P}^1, \mathcal{O}(n)) \in R(T),$$

and we have only to compute the two cohomology groups on the right.

The space $H^0(\mathbb{P}^1, \mathcal{O}(n))$ consists by definition of homogeneous algebraic regular functions on $\mathbb{C}^2 \setminus \{0\}$ of degree n. Observe that there are no non-zero regular homogeneous functions on $\mathbb{C}^2 \setminus \{0\}$ of degree $n < 0$. To see this, view such a function f as a holomorphic function. Then, since the origin in $\mathbb{C}^2$ is a codimension 2 subset, the function f can be extended to a holomorphic function on the whole of $\mathbb{C}^2$, due the the removable singularity theorem. But clearly, a homogeneous function of negative degree cannot be regular at the origin. Thus, we get

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = 0 \quad \text{if } n < 0. \tag{7.5.6}$$

The second thing that we use is Serre duality, cf. [GH]:

$$H^1(\mathbb{P}^1, \mathcal{O}(n))^* \simeq H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1} \otimes \mathcal{O}(-n)) = H^0(\mathbb{P}^1, \mathcal{O}(-n-2)). \tag{7.5.7}$$

Formulas (7.5.6) and (7.5.7) combined together show that the group $H^1(\mathbb{P}^1, \mathcal{O}(n))$ vanishes for all $n \geq 0$. Furthermore, the class $H^0(\mathbb{P}^1, \mathcal{O}(n)) - H^1(\mathbb{P}^1, \mathcal{O}(n))$ only changes sign under substitution $n \mapsto -n-2$, due to Serre duality (7.5.7). If, in particular, $n = -1$ then both cohomology groups vanish. Thus everything is reduced to computing the class $H^0(\mathbb{P}^1, \mathcal{O}(n)) \in K^G(\mathrm{pt})$ for $n \geq 0$.

Any regular homogeneous function on $\mathbb{C}^2 \setminus \{0\}$ of non-negative degree is a polynomial (see [Ha] or note that it is an entire holomorphic function on $\mathbb{C}^2$ of polynomial growth). Hence, for $n \geq 0$, the space $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is the vector space of homogeneous polynomials of degree n in two variables z_1, z_2. The monomials $z_1^i \cdot z_2^{n-i}$, $i = 0, 1, \dots, n$, form a weight basis of this $(n+1)$-dimensional vector space. The torus T acts on z_1 and z_2 by means of the weights $-\omega$ and $+\omega$ respectively. Hence a monomial $z_1^i \cdot z_2^{n-i}$ has weight $-i\omega + (n-i)\omega = (n-2i)\omega$. Therefore the class of $H^0(\mathbb{P}^1, \mathcal{O}(n))$ in $R(T) = \mathbb{Z}[X, X^{-1}]$ is represented by the element

$$X^n + X^{n-2} + \ldots + X^{2-n} + X^{-n} = \frac{X^{-(n+1)} - X^{n+1}}{X^{-1} - X}. \quad \blacksquare$$

From now on set $\mathbb{P} = \mathbb{P}^1$. There are two G-orbits of the diagonal action on $\mathbb{P} \times \mathbb{P}$. The first one is $\mathbb{P}_\Delta$, the diagonal copy of $\mathbb{P}$ in $\mathbb{P} \times \mathbb{P}$, the closed orbit, and the second one is $Y = (\mathbb{P} \times \mathbb{P}) \setminus \mathbb{P}_\Delta$, a Zariski open subset in $\mathbb{P} \times \mathbb{P}$. Thus, the Steinberg variety Z consists of two components $Z_\Delta = T^*_{\mathbb{P}_\Delta}(\mathbb{P} \times \mathbb{P})$, and $Z_Y = T^*_{\bar{Y}}(\mathbb{P} \times \mathbb{P}) = T^*_{\mathbb{P} \times \mathbb{P}}(\mathbb{P} \times \mathbb{P}) =$ *zero-section of* $T^*(\mathbb{P} \times \mathbb{P})$. Thus the projection $\pi_Y : Z_Y \to \mathbb{P} \times \mathbb{P}$ is an isomorphism.

Let $\Omega^1_{\mathbb{P} \times \mathbb{P}/\mathbb{P}}$ be the sheaf of relative 1-forms along the projection to the first factor $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$. Put $\mathcal{Q} = \pi_Y^* \Omega^1_{\mathbb{P} \times \mathbb{P}/\mathbb{P}}$, a sheaf on $Z_Y \subset T^*(\mathbb{P} \times \mathbb{P})$. Further, for an integer n we set $\mathcal{O}_n = \pi_\Delta^* \mathcal{O}(n)$ where $\pi_\Delta : Z_\Delta \to \mathbb{P}_\Delta$ is the natural projection.

Recall now that the affine Hecke algebra $\mathbf{H}$ for $SL_2(\mathbb{C})$ is an associative $\mathbb{C}[q, q^{-1}]$-algebra on 3 generators, T, X and X^{-1} subject to the relations

$$(7.5.8) \qquad (T+1)(T-q) = 0, \qquad X \cdot X^{-1} = X^{-1} \cdot X = 1,$$

$$(7.5.9) \qquad \text{and} \qquad T \cdot X^{-1} - X \cdot T = (1-q)X$$

where T is the unique generator of the subalgebra H_W, the finite Hecke algebra.

Write $c = -(T+1) \in \mathbf{H}$. Note that the set $\{c, X, X^{-1}\}$ also generates $\mathbf{H}$, and the relations (7.5.8) and (7.5.9) can be written

$$(7.5.10) \qquad c^2 = -(q+1)c, \qquad cX^{-1} - Xc = qX - X^{-1}.$$

Our aim is to construct an algebra isomorphism

$$\mathbf{H} \xrightarrow{\sim} K^{G\times\mathbb{C}^*}(Z), \tag{7.5.11}$$

where the algebra structure on the right hand side is given by convolution, see 5.2.20.

As a first step towards constructing the isomorphism (7.5.11) we define a map

$$\Theta : \{c, X, X^{-1}\} \to K^{G\times\mathbb{C}^*}(Z)$$

by the following assignment

$$X \mapsto [\mathcal{O}_{-1}]\,, \quad X^{-1} \mapsto [\mathcal{O}_1]\,, \quad c \mapsto [q\mathcal{Q}],$$

where $q \in R(\mathbb{C}^*)$ as in 7.2.3.

Theorem 7.5.12. *The map Θ can be extended to an algebra homomorphism $\Theta : \mathbf{H} \to K^{G\times\mathbb{C}^*}(Z)$, that is, the following relations (cf. (7.5.10)) hold in the algebra $K^{G\times\mathbb{C}^*}(Z)$:*

$$(q\mathcal{Q}) * (q\mathcal{Q}) = -(q+1)q\mathcal{Q}\,, \quad \textit{and} \tag{7.5.13}$$

$$(q\mathcal{Q}) * \mathcal{O}_1 - \mathcal{O}_{-1} * (q\mathcal{Q}) = q\mathcal{O}_{-1} - \mathcal{O}_1, \quad \mathcal{O}_1 * \mathcal{O}_{-1} = \mathcal{O}_0. \tag{7.5.14}$$

Proof of Theorem 7.5.12. If $\mathrm{pr}_2 : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ is the second projection, then by definition we get

$$\mathcal{Q} = \pi_Y^* \Omega_{\mathbb{P}\times\mathbb{P}/\mathbb{P}} = \Omega^1_{\mathbb{P}\times\mathbb{P}/\mathbb{P}} = \mathrm{pr}_2^*\Omega^1_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}}\,.$$

It will be useful for us, in order to perform convolution computations, to know the class of $\mathcal{Q}$ in the K-group of $\mathbb{P} \times T^*\mathbb{P}$, where $\mathcal{Q}$ is viewed as a sheaf supported on the subset $\mathbb{P} \times T^*_{\mathbb{P}}\mathbb{P} \simeq \mathbb{P} \times \mathbb{P}$. To that end, write

$$\mathbb{P} \underset{\pi}{\overset{i}{\rightleftarrows}} T^*\mathbb{P}$$

for the zero section and vector bundle projection respectively. We have the Koszul complex

$$0 \to \mathcal{O}_{T^*\mathbb{P}} \to \pi^*\Omega^1_{\mathbb{P}} \to i_*\Omega^1_{\mathbb{P}} \to 0, \tag{7.5.15}$$

given by viewing $\Omega^1_{\mathbb{P}}$ as a sheaf, sitting on the zero section of $T^*\mathbb{P}$. Tensoring with $\mathcal{O}_{\mathbb{P}}$ we obtain the resolution

$$0 \to \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}} \to \mathcal{Q} \to 0. \tag{7.5.16}$$

The differential δ in the Koszul complex is a linear function along the fibers, hence is not a morphism of $\mathbb{C}^*$-equivariant sheaves. We may restore $\mathbb{C}^*$-equivariance of the above complex by tensoring $\mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}}$ with the character $q^{-1} \in R(\mathbb{C}^*)$, normalized as in (7.2.3). This way, we obtain the following equality in $K^{G\times\mathbb{C}^*}(\mathbb{P}\times T^*\mathbb{P})$

$$(7.5.17) \qquad qQ = q\cdot(\mathcal{O}_{\mathbb{P}}\boxtimes\pi^*\Omega^1_{\mathbb{P}}) - \mathcal{O}_{\mathbb{P}}\boxtimes\mathcal{O}_{T^*\mathbb{P}}.$$

We can now compute the convolution $qQ * qQ$ in K-theory, using (5.2.29) and the above equation as follows:

$$\begin{aligned} qQ * qQ &= (q\cdot\mathcal{O}_{\mathbb{P}}\boxtimes\pi^*\Omega^1_{\mathbb{P}} - \mathcal{O}_{\mathbb{P}}\boxtimes\mathcal{O}_{T^*\mathbb{P}}) * (q\mathcal{O}_{\mathbb{P}}\boxtimes\Omega^1_{\mathbb{P}}) \\ &= q\cdot\langle\pi^*\Omega^1_{\mathbb{P}},\mathcal{O}_{\mathbb{P}}\rangle\mathcal{O}_{\mathbb{P}}\boxtimes\pi^*\Omega^1_{\mathbb{P}} - \langle\mathcal{O}_{T^*\mathbb{P}},\mathcal{O}_{\mathbb{P}}\rangle\cdot q\mathcal{O}_{\mathbb{P}}\boxtimes\Omega^1_{\mathbb{P}} \\ &= q\cdot(p_*\Omega^1_{\mathbb{P}})\cdot(qQ) - (p_*\mathcal{O}_{\mathbb{P}})\cdot qQ \\ &= q\cdot(p_*\mathcal{O}_{\mathbb{P}}(-2))\cdot(qQ) - (p_*\mathcal{O}_{\mathbb{P}})\cdot qQ = -(q+1)q\ Q, \end{aligned}$$

where $p:\mathbb{P}\to\{pt\}$ is the projection and in the last equality we used Lemmas 7.5.4 and 7.5.5 to find $p_*\mathcal{O}_{\mathbb{P}}(-2) = -1$ and $p_*\mathcal{O}_{\mathbb{P}} = 1$. This proves 7.5.13.

To verify (7.5.14), we first transport this equation from $K^{G\times\mathbb{C}^*}(Z)$ to a more easily computable K-group. To that end, consider the maps

$$Z \overset{\bar\pi}{\hookrightarrow} \mathbb{P}\times T^*\mathbb{P} \overset{\bar\imath}{\hookleftarrow} \mathbb{P}\times\mathbb{P},$$

where $\bar\pi$ is the restriction to Z of the natural projection $T^*\mathbb{P}\times T^*\mathbb{P} \xrightarrow{\pi\times\mathrm{id}} \mathbb{P}\times T^*\mathbb{P}$ and $\bar\imath = \mathrm{id}\times(\textit{zero-section})$. We now apply Corollary 5.4.34 to the vector bundle $E = T^*\mathcal{B}$ and $M = \mathcal{B}$. It is implicit in the statement of that Corollary that the map

$$\Phi := \bar\imath^*\bar\pi_* : K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P}),$$

is an algebra homomorphism. Moreover, it will be shown in the next section (see proof of 7.6.7) that this homomorphism is injective. Thus to verify (7.5.14) it suffices to prove the following equality in $K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P})$:

$$(7.5.18) \quad \Phi(qQ) * \Phi(\mathcal{O}_1) - \Phi(\mathcal{O}_{-1}) * \Phi(qQ) = q\Phi(\mathcal{O}_{-1}) - \Phi(\mathcal{O}_1).$$

This equation is much easier to handle than the original equation (7.5.14) because we know by the Künneth formula that

$$K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P}) = K^{G\times\mathbb{C}^*}(\mathbb{P})\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(\mathbb{P}),$$

so that we have only to present explicitly each side of (7.5.18) as an element of $K^{G\times\mathbb{C}^*}(\mathbb{P})\otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(\mathbb{P})$.

To that end, write $\mathcal{O}_\Delta(n)$ for the direct image of the sheaf $\mathcal{O}(n)$ under the diagonal embedding $\Delta:\mathbb{P}\to\mathbb{P}\times\mathbb{P}$. Using 7.5.17 one verifies immediately that we have

$$\bar\imath^*\bar\pi_*(qQ) = q\mathcal{O}\boxtimes\mathcal{O}(-2) - \mathcal{O}\boxtimes\mathcal{O} \quad , \quad \bar\imath^*\bar\pi_*(\mathcal{O}_n) = \mathcal{O}_\Delta(n)\ ,\ \forall n\in\mathbb{Z}.$$

Recall the general fact that, for a sheaf $\mathcal{L}_\Delta$ supported on the diagonal $\mathbb{P}_\Delta \subset \mathbb{P} \times \mathbb{P}$ and $\mathcal{F} \in K^{G\times\mathbb{C}^*}(\mathbb{P} \times \mathbb{P})$, we have the general equalities, see Corollary 5.2.25, $\mathcal{F} * \mathcal{L}_\Delta = \mathrm{pr}_2^*\mathcal{L}_\Delta \otimes \mathcal{F}$ and $\mathcal{L}_\Delta * \mathcal{F} = \mathrm{pr}_1^*\mathcal{L}_\Delta \otimes \mathcal{F}$, where $\mathrm{pr}_i : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ stands for the projection to the i-th factor. Thus we find

$$\text{(7.5.19)}\quad \text{LHS of (7.5.18)} = \bar{i}^*\bar{\pi}_*(q\mathcal{Q}) * \mathcal{O}_\Delta(1) - \mathcal{O}_\Delta(-1) * \bar{i}^*\bar{\pi}_*(q\mathcal{Q}) = \\ = q\mathcal{O} \boxtimes \mathcal{O}(-1) - \mathcal{O} \boxtimes \mathcal{O}(1) - q\mathcal{O}(-1) \boxtimes \mathcal{O}(-2) + \mathcal{O}(-1) \boxtimes \mathcal{O}.$$

To compute RHS of (7.5.18) resolve the sheaves $\mathcal{O}_\Delta(\pm 1)$ by locally free sheaves on $\mathbb{P} \times \mathbb{P}$. For this we use Beilinson's resolution (5.7.4), which in the special case of $\mathbb{P} = \mathbb{P}^1$ yields

$$\text{(7.5.20)}\qquad 0 \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

To get a resolution of $\mathcal{O}_\Delta(\pm 1)$ we tensor the above exact sequence by $\mathcal{O}(\pm 1)$ on the right hand side and note that $\Omega^1 = \mathcal{O}(-2)$. Thus we obtain exact sequences:

$$0 \to \mathcal{O}(-1) \boxtimes \mathcal{O}(-2) \to \mathcal{O} \boxtimes \mathcal{O}(-1) \to \mathcal{O}_\Delta(-1) \to 0$$
$$0 \to \mathcal{O}(-1) \boxtimes \mathcal{O} \to \mathcal{O} \boxtimes \mathcal{O}(1) \to \mathcal{O}_\Delta(1) \to 0.$$

This yields the following equality in $K^{G\times\mathbb{C}^*}(\mathbb{P} \times \mathbb{P})$:

$$\text{(7.5.21)}\quad q\mathcal{O}_\Delta(-1) - \mathcal{O}_\Delta(1) \\ = q\mathcal{O} \boxtimes \mathcal{O}(-1) - q\mathcal{O}(-1) \boxtimes \mathcal{O}(-2) - \mathcal{O} \boxtimes \mathcal{O}(1) + \mathcal{O}(-1) \boxtimes \mathcal{O}.$$

Comparing the RHS of (7.5.19) with the RHS of (7.5.21) we see that (7.5.18) is indeed an equality in $K^{G\times\mathbb{C}^*}(\mathbb{P} \times \mathbb{P})$.

Proving the second equation in (7.5.14) is trivial and is left to the reader. This completes the proof of the theorem. ∎

We now prove

Theorem 7.5.22. *The algebra homomorphism $\Theta : \mathbf{H} \to K^{G\times\mathbb{C}^*}(Z)$ is an isomorphism.*

Proof. Write $\mathbf{H}_0 \subset \mathbf{H}$ for the subalgebra of $\mathbf{H}$ generated by X and X^{-1}. Then by construction we see $\Theta(\mathbf{H}_0) \subset K^{G\times\mathbb{C}^*}(Z_\Delta)$. Furthermore, it is easy to see that the map Θ is nothing but the composition of the following natural isomorphisms, see (7.2.4)

$$\mathbf{H}_0 \xrightarrow{\sim} R(T)[q, q^{-1}] \simeq K^{G\times\mathbb{C}^*}(\mathbb{P}) \simeq K^{G\times\mathbb{C}^*}(Z_\Delta)$$

where the first one sends $e^\lambda \mapsto e^{-\lambda}$. Hence Θ maps $\mathbf{H}_0$ isomorphically onto $K^{G\times\mathbb{C}^*}(Z_\Delta)$.

Observe next that the Cellular Fibration Lemma 5.5 applied to Z yields the short exact sequence of $R(G \times \mathbb{C}^*)$-modules:

$$(7.5.23)\ 0 \to K^{G\times\mathbb{C}^*}(Z_\Delta) \to K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})) \to 0.$$

By the Thom isomorphism Theorem 5.4.17 we have the isomorphisms

$$K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})) \simeq K^{G\times\mathbb{C}^*}(Y) \simeq K^{G\times\mathbb{C}^*}(\mathbb{P}).$$

By 5.2.16 and 5.2.18 respectively we have $K^{G\times\mathbb{C}^*}(\mathbb{P}) \simeq R(B\times\mathbb{C}^*) \simeq R(T\times\mathbb{C}^*)$, so that it is clear that $K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P}))$ is a free $R(T\times\mathbb{C}^*)$-module with generator $[\mathcal{O}_{T^*_Y(\mathbb{P}\times\mathbb{P})}]$. Thus, from (7.5.23) we deduce an isomorphism

$$K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta) \simeq R(T\times\mathbb{C}^*).$$

This shows that the induced map

$$(7.5.24)\ \Theta : \mathbf{H}/\mathbf{H}_0 \to K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta) \simeq K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})),$$

sends T to $u\cdot[\mathcal{O}_{T^*_Y(\mathbb{P}\times\mathbb{P})}]$, where u is an invertible element of $R(T\times\mathbb{C}^*)$. Hence, (7.5.24) is an isomorphism (of free $R(T)$-modules of rank 1). Since $\mathbf{H}_0 \simeq K^{G\times\mathbb{C}^*}(Z_\Delta)$ we deduce, by Proposition 2.3.20(ii), that the map Θ is itself an isomorphism. ∎

7.6 Proof of the Main Theorem

This section is entirely devoted to proving Theorem 7.2.5, i.e., to constructing an algebra isomorphism $\Theta : \mathbf{H} \xrightarrow{\sim} K^A(Z)$, where $A = G\times\mathbb{C}^*$, and Z is the Steinberg variety, see 3.3. As in the $G = SL_2(\mathbb{C})$-case, worked out in the previous section, we begin with defining the map Θ on generators. Let S be the set of simple reflections in W, the Weyl group. Observe that the $\mathbb{Z}[q,q^{-1}]$-algebra $\mathbf{H}$ is generated by definition by the following set

$$\mathcal{S} = \{e^\lambda \mid \lambda\in P\} \cup \{T_s \mid s\in S\} \subset \mathbf{H}.$$

We construct a map $\Theta : \mathcal{S} \to K^A(Z)$ as follows. The assignment $e^\lambda \mapsto \Theta(e^\lambda)$ is given, up to sign, by isomorphism (7.2.4). In more detail, to any $\lambda\in P$ we have associated a canonical G-equivariant line bundle L_λ on $\mathcal{B}$. Identify $\mathcal{B}$ with the diagonal $\mathcal{B}_\Delta \subset \mathcal{B}\times\mathcal{B}$. Let $\pi_\Delta : Z_\Delta \simeq T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B}) \to \mathcal{B}_\Delta$ be the natural projection (cf. 7.1). Set $\mathcal{O}_\lambda = \pi_\Delta^* L_\lambda$. Thus $\mathcal{O}_\lambda$ is a line bundle on Z_Δ which comes equipped with a natural $G\times\mathbb{C}^*$-equivariant structure. Thus, we may view $\mathcal{O}_\lambda$ as an $G\times\mathbb{C}^*$-equivariant sheaf on Z supported on $Z_\Delta \subset Z$.

Next, for each $s\in S$, let $Y_s \subset \mathcal{B}\times\mathcal{B}$ be the corresponding G-orbit. We observe that the closure, $\bar{Y}_s = Y_s \sqcup \mathcal{B}_\Delta$, is a smooth variety, fibered over $\mathcal{B}$ by means of the first projection $\bar{Y}_s \hookrightarrow \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ with 1-dimensional fibers isomorphic to the projective line $\mathbb{P}^1$. Denote by $\Omega^1_{\bar{Y}_s/\mathcal{B}}$ the sheaf on $\bar{Y}_s$ of relative 1-forms with respect to the first projection. Further, the conormal bundle $\pi_s : T^*_{\bar{Y}_s}(\mathcal{B} \times \mathcal{B}) \to \bar{Y}_s$ is a smooth irreducible component of Z. Set $\mathcal{Q}_s = \pi_s^* \Omega^1_{\bar{Y}_s/\mathcal{B}}$. The sheaf $\mathcal{Q}_s$ comes equipped with a natural $G \times \mathbb{C}^*$-equivariant structure. We now define the map $\Theta : \mathcal{S} \to K^A(Z)$ by the following assignment (which agrees with the definition of Θ given in the previous section in the special case $G = SL_2(\mathbb{C})$):

$$(7.6.1) \qquad e^\lambda \mapsto [\mathcal{O}_{-\lambda}], \quad T_s \mapsto -([q\mathcal{Q}_s] + [\mathcal{O}_0]), \quad (\lambda \in P, s \in S).$$

Our first task is to show that (7.6.1) can be extended to an algebra homomorphism $\mathbf{H} \to K^A(Z)$, i.e., that the above defined elements $\Theta(u) \in K^A(Z)$, $u \in \mathcal{S}$, satisfy all the relations that hold for the u's in $\mathbf{H}$. It is rather difficult to verify the relations among the $\Theta(u)$'s directly, as we did in the SL_2-case, so we will adopt the following strategy. We will construct a $\mathbb{C}$-vector space M and we will define an action on M of both the algebra $\mathbf{H}$ and the algebra $K^A(Z)$. These actions make M an $\mathbf{H}$-module as well as an $K^A(Z)$-module, i.e., give rise to algebra homomorphisms

$$\rho_1 : \mathbf{H} \to \mathrm{End}_{\mathbb{C}} M \quad , \quad \rho_2 : K^A(Z) \to \mathrm{End}_{\mathbb{C}} M .$$

We first show that M is a faithful module with respect to each of the two actions, that is, the above homomorphisms are both injective. We then prove, by a direct computation, that for any $u \in \mathcal{S}$, the u-action on M and the $\Theta(u)$-action on M are given by the same operator, in other words that $\rho_1(u) = \rho_2(\Theta(u))$. This clearly implies, due to the faithfulness of the two actions, that the elements $\Theta(u) \in K^A(Z)$, $u \in \mathcal{S}$, satisfy all the relations that hold for the u's themselves.

To construct the vector space M we need an

ALGEBRAIC DIGRESSION. Set $\mathbf{e} = \sum_{w \in W} T_w \in H_W \subset \mathbf{H}$.

Lemma 7.6.2. *(1) The assignment $T_w \mapsto q^{\ell(w)}$ extends by $\mathbb{Z}[q, q^{-1}]$-linearity to an algebra homomorphism*

$$\epsilon : H_W \to \mathbb{Z}[q, q^{-1}].$$

(2) For any $a \in H_W$ we have the equation $a \cdot \mathbf{e} = \epsilon(a)\mathbf{e} = \mathbf{e} \cdot a$.
(3) $\mathbf{H} \cdot \mathbf{e}$ is a free $R(T)[q, q^{-1}]$-module with generator $\mathbf{e}$.

Proof. It is immediate to check that the assigment $T_s \mapsto q$, $s \in S$ is compatible with relations 7.1.1(i)-(ii), hence extends to an algebra homomorphism $H_W \to \mathbb{Z}[q, q^{-1}]$. Then, for any element $w \in W$, given a reduced expression $w = s_1 \cdot \ldots \cdot s_k$ we have $T_w = T_{s_1} \cdot \ldots \cdot T_{s_k}$, and hence the homomorphism takes T_w to $q^k = q^{\ell(w)}$. This is exactly the formula of part (1).

To prove (2), fix $s \in S$ and observe that we have a decomposition $W = W' \sqcup W''$, where

$$W' = \{w \in W \mid \ell(w) = \ell(s) + \ell(sw)\},$$
$$W'' = \{v \in W \mid \ell(sv) = \ell(s) + \ell(v)\}.$$

Thus for $w \in W'$, we can write $w = sw'$ where $w' = sw$, and $\ell(w) = \ell(s) + \ell(w')$. Hence $T_w = T_s T_{w'}$. Using the relation $T_s^2 = (q-1)T_s + q$, see 7.1.2(a), we rewrite this as $T_s T_w = (q-1)T_w + qT_{w'}$. Likewise, for $v \in W''$, we have $T_s T_v = T_{sv}$. Note that $y \in W' \Leftrightarrow sy \in W''$. Now we calculate

$$\begin{aligned} T_s\mathbf{e} = T_s \cdot \sum_{w \in W} T_w &= \sum_{w \in W'} T_s T_w + \sum_{v \in W''} T_s T_v \\ &= \sum_{\substack{w \in W' \\ w = sw'}} ((q-1)T_w + qT_{w'}) + \sum_{v \in W''} T_{sv} \\ &= \sum_{w \in W'} qT_w + \sum_{w' \in W''} qT_{w'} = q \cdot \mathbf{e}. \end{aligned}$$

This gives the first equality in (2). The second equality in (2) is proved similarly.

To prove part (3), recall that the elements $\{e^\lambda T_w,\ \lambda \in P,\ w \in W\}$ form a $\mathbb{Z}[q,q^{-1}]$-basis for $\mathbf{H}$. It follows that the elements $\{e^\lambda \mathbf{e},\ \lambda \in P\}$, are $\mathbb{Z}[q,q^{-1}]$-linearly independent. On the other hand, it is immediate from part (2) that these elements span the $\mathbb{Z}[q,q^{-1}]$-module $\mathbf{H} \cdot \mathbf{e}$. ∎

Let $\mathbf{H} \cdot \mathbf{e} \subset \mathbf{H}$ be the left ideal in $\mathbf{H}$ generated by the element $\mathbf{e}$. Thus, $\mathbf{H} \cdot \mathbf{e}$ has a natural left $\mathbf{H}$-module structure. Furthermore, Lemma 7.6.2(1)-(2) implies that the map $u \otimes 1 \mapsto u \cdot \mathbf{e}$ gives rise to a well-defined homomorphism $\mathrm{Ind}_{H_W}^{\mathbf{H}} \epsilon \to \mathbf{H} \cdot \mathbf{e}$, where $\mathrm{Ind}_{H_W}^{\mathbf{H}} \epsilon = \mathbf{H} \otimes_{H_W} \epsilon$ is the induced module. Part (3) of the same lemma shows that this homomorphism is bijectitive. Thus we have an $\mathbf{H}$-module isomorphism

$$\mathbf{H} \cdot \mathbf{e} \simeq \mathrm{Ind}_{H_W}^{\mathbf{H}} \epsilon. \tag{7.6.3}$$

The space $\mathbf{H} \cdot \mathbf{e}$ is the vector space M we have mentioned earlier while sketching the strategy of the proof. The $\mathbf{H}$-module structure on $\mathbf{H} \cdot \mathbf{e}$ clearly gives rise to an algebra homomorphism

$$\rho_{\mathbf{H}} : \mathbf{H} \to \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathbf{H} \cdot \mathbf{e}). \tag{7.6.4}$$

The next step is to construct a $K^A(Z)$-action on the same vector space. Recall that the geometric meaning of the variable q was explained in 7.2.3, so throughout we keep the convention that $q \in R(\mathbb{C}^*)$ is as in 7.2.3. The crucial role in relating the algebraic construction above to geometry is played by a $\mathbb{Z}[q,q^{-1}]$-module isomorphism given by the composition

$$(7.6.5) \qquad K^A(T^*\mathcal{B}) \overset{\mathrm{Th}}{\underset{\sim}{\to}} K^A(\mathcal{B}) \overset{\alpha}{\underset{\sim}{\to}} R(T)[q,q^{-1}] \overset{\beta}{\underset{\sim}{\to}} \mathbf{H}\cdot\mathbf{e},$$

where the map Th is the Thom isomorphism, the map α is the canonical isomorphism, cf. (6.1.6),

$$K^A(\mathcal{B}) \simeq K^{G\times\mathbb{C}^*}(G/B) \simeq K^{B\times\mathbb{C}^*}(\mathrm{pt}) \simeq R(T\times\mathbb{C}^*) \simeq R(T)[q,q^{-1}],$$

and the map β is given by the assignment $e^\lambda \mapsto e^{-\lambda}\cdot\mathbf{e}$, $\lambda\in P$, which is an isomorphism due to Lemma 7.6.2(3).

Further, in the setup of section 5.4.22 put $M_1 = M_2 = \mathcal{B}$ and $E_1 = E_2 = T^*\mathcal{B}$. Observe that the natural projection $\mathrm{id}\times\pi : T^*\mathcal{B}\times T^*\mathcal{B} \to (T^*\mathcal{B})\times\mathcal{B}$ becomes a closed embedding when restricted to the Steinberg variety (this is obvious if Z is viewed as the variety of triples, see the second formula at the beginning of §3.3). Hence the assumption 5.4.24 holds for the Steinberg variety Z. The construction of that section yields a $K^A(Z)$-module structure on $K^A(T^*\mathcal{B})$, that is, an algebra homomorphism

$$(7.6.6) \qquad \rho_{T^*\mathcal{B}} : K^A(Z) \to \mathrm{End}_{R(A)}(K^A(T^*\mathcal{B})).$$

We now make the following claims whose proofs will be delayed.

Claim 7.6.7. The homomorphism $\rho_{T^*\mathcal{B}}$ in (7.6.6) is injective, i.e., $K^A(T^*\mathcal{B})$ is a faithful $K^A(Z)$-module.

Now, isomorphism (7.6.5) induces an algebra isomorphism

$$\mathrm{End}_{\mathbb{Z}[q,q^{-1}]}K^A(T^*\mathcal{B}) \overset{\Phi}{\underset{\sim}{\to}} \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}\mathbf{H}\cdot\mathbf{e}$$

and we have

Claim 7.6.8. The following diagram (with the exception of the dashed arrow) commutes:

$$\begin{array}{ccccc}
\mathcal{S} & \longrightarrow & \mathbf{H} & \xrightarrow{\rho_{\mathbf{H}}} & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathbf{H}\cdot\mathbf{e}) \\
 & \searrow^{\Theta} & \downarrow & & \wr\downarrow\Phi \\
 & & K^A(Z) & \xrightarrow{\rho_{T^*\mathcal{B}}} & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}K^A(T^*\mathcal{B})
\end{array}$$

From these claims we obtain the following result.

Proposition 7.6.9. *The map Θ in the diagram can be uniquely extended to an algebra homomorphism $\mathbf{H}\to K^A(Z)$ that makes the above diagram (including the dashed arrow) commute.*

Proof of the Proposition. Let $T(\mathcal{S})$ be the free associative $\mathbb{Z}[q,q^{-1}]$-algebra generated by $\mathcal{S}$, that is the tensor algebra on the free $\mathbb{Z}[q,q^{-1}]$-module with base $\mathcal{S}$. The universal property of free algebras ensures that, for any $\mathbb{Z}[q,q^{-1}]$-algebra B and any map $\mathcal{S} \to B$, there exists a unique algebra homomorphism $T(\mathcal{S}) \to B$ extending that map. In particular, there is an algebra homomorphism $\widehat{\Theta} : T(\mathcal{S}) \to K^A(Z)$ that extends the map (7.6.1) and a homomorphism $\tau : T(\mathcal{S}) \to \mathbf{H}$ that extends the tautological embedding $\mathcal{S} \hookrightarrow \mathbf{H}$. The homomorphism τ is surjective, since the set $\mathcal{S}$ generates $\mathbf{H}$. Hence, proving the existence of the dashed arrow in the diagram amounts to showing that $\widehat{\Theta}$ vanishes on $\mathrm{Ker}\,(T(\mathcal{S}) \to \mathbf{H})$. To that end, assume a is in the kernel of $T(\mathcal{S}) \to \mathbf{H}$. Then $\tau(a) = 0$, hence, $\Phi \circ \rho_{\mathbf{H}} \circ \tau(a) = 0$. By Claim 7.6.8 we obtain $\rho_{T^*\mathcal{B}} \circ \widehat{\Theta}(a) = 0$. Now, the injectivity of $\rho_{T^*\mathcal{B}}$, ensured by Claim 7.6.7, yields $\widehat{\Theta}(a) = 0$ and the proposition follows. ∎

The proofs of Claims 7.6.7 and 7.6.8 will be postponed until the end of this section. We first prove the following result, which is a more precise version of the main Theorem 7.2.5.

Theorem 7.6.10. *The algebra homomorphism $\Theta : \mathbf{H} \to K^A(Z)$ constructed in Proposition 7.6.9 is a bijection.*

The strategy of proof of Theorem 7.6.10 is quite similar to the argument used in the proof of Theorem 7.2.2. We recall that we have fixed a total linear order on W extending the Bruhat order, see 7.3.8. Write Y_w for the G-diagonal orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w \in W$. We have an A-stable filtration of Z indexed by the elements of W:

$$Z_{\leq w} = \sqcup_{y \leq w} T^*_{Y_y}(\mathcal{B} \times \mathcal{B}).$$

The following analogue of Lemma 7.3.9 follows from the Cellular Fibration Lemma 5.5.

Lemma 7.6.11. *(1) The natural maps $K^A(Z_{\leq w}) \to K^A(Z)$ induced by the embeddings $Z_{\leq w} \hookrightarrow Z$ are injective and their images form a filtration on $K^A(Z)$ indexed by the set W;*

(2) For any $w \in W$, the restriction to the open subset $T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \hookrightarrow Z_{\leq w}$ induces an isomorphism

$$K^A(Z_{\leq w})/K^A(Z_{<w}) \simeq K^A(T^*_{Y_w}(\mathcal{B} \times \mathcal{B}))\,.$$

Moreover, the RHS is a free $R(T \times \mathbb{C}^*)$-module with generator $[\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}]$.

Similarly, on $\mathbf{H}$ we introduce a filtration $\mathbf{H}_{\leq w}$, $w \in W$, setting $\mathbf{H}_{\leq w}$ to be the span of the basis elements $\{e^\lambda T_y \mid \lambda \in P\,, y \leq w\}$. Clearly $\mathbf{H}_{\leq y} \subset \mathbf{H}_{\leq w}$ whenever $y \leq w$ and $\mathbf{H}_{\leq w}/\mathbf{H}_{<w}$ is a free left $R(T \times \mathbb{C}^*)$-module with generator T_w.

Proposition 7.6.12. *We have*

(1) *The homomorphism* $\Theta : \mathbf{H} \to K^A(Z)$ *is filtration preserving, i.e., for any* $w \in W$, *we have* $\Theta(\mathbf{H}_{\leq w}) \subset K^A(Z_{\leq w})$; *Moreover,*
(2) *for any* $w \in W$ *the induced map*

$$\Theta : \mathbf{H}_{\leq w}/\mathbf{H}_{<w} \to K^A(Z_{\leq w})/K^A(Z_{<w}) \simeq K^A(T^*_{Y_w}(\mathcal{B}\times\mathcal{B}))$$

takes T_w *to* $c_w \cdot [\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}]$, *where* c_w *is an invertible element of* $R(T \times \mathbb{C}^*)$.

We note at this point that part (2) of Proposition 7.6.12 implies that the associated graded map $\operatorname{gr}\Theta\colon \mathbf{H} \to \operatorname{gr} K^A(Z)$, corresponding to the above defined filtrations, is an isomorphism of $R(T \times \mathbb{C}^*)$-modules. Hence Proposition 7.6.12, combined with Proposition 2.3.20(ii), yields Theorem 7.6.10.

To prove Proposition 7.6.12 we first make a few general remarks concerning composition of sets.

Let M be a manifold, and $Y \subset M \times M$ a subset. There are two maps $Y \to M$ by means of the two projections $M \times M \xrightarrow{p_i} M$, $i = 1,2$. Given two subsets Y_{12} and Y_{23} of $M \times M$ one may form a fiber product

$$Y_{12} \times_M Y_{23} = \{(y_{12}, y_{23}) \in Y_{12} \times Y_{23} \mid p_2(y_{12}) = p_1(y_{23})\}.$$

Explicitly, writing $y_{12} = (m_1, m_2)$ and $y_{23} = (m_2', m_3)$, we have

$$Y_{12} \times_M Y_{23} = \{(m_1, m_2, m_2', m_3) \mid m_2 = m_2'\}.$$

Let $p_{ij} : M \times M \times M \to M \times M$ denote the projection along the factor not named. Then the map $(m_1, m_2, m_2', m_3) \mapsto (m_1, m_2, m_3)$ gives a natural isomorphism

$$(7.6.13) \qquad Y_{12} \times_M Y_{23} \simeq p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \subset M \times M \times M.$$

On the other hand we have defined, see (2.7.6), a subset $Y_{12} \circ Y_{23} \subset M \times M$. By definition, $Y_{12} \circ Y_{23}$ is the image of the projection p_{13} : $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to M \times M$. Using (7.6.13) we may view this projection as a map $Y_{12} \times_M Y_{23} \to Y_{12} \circ Y_{23}$. More generally, given several subsets $Y_{12}, Y_{23}, \ldots, Y_{k-1,k} \subset M \times M$, we have a natural surjective map

$$(7.6.14) \qquad Y_{12} \times_M Y_{23} \times_M \ldots \times_M Y_{k-1,k} \twoheadrightarrow Y_{12} \circ Y_{23} \circ \ldots \circ Y_{k-1,k}.$$

We now turn to the case $M = \mathcal{B}$, the flag manifold. Fix an element $w \in W$ and a reduced expression, cf. 7.1, $w = s_1 \cdot \ldots \cdot s_r$, $s_i \in S$. Recall that each of the varieties $\bar{Y}_{s_i}$ is smooth and the fibers of $\bar{Y}_{s_i} \to \mathcal{B}$, with respect to either of the projections $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, are isomorphic to $\mathbb{P}^1$. It follows that $\bar{Y}_{s_1} \times_{\mathcal{B}} \bar{Y}_{s_2} \times \cdots \times_{\mathcal{B}} \bar{Y}_{s_r}$ is a smooth compact variety. On the other hand one finds, computing the set-theoretic composition (see 2.7), that

$$\bar{Y}_{s_1} \circ \cdots \circ \bar{Y}_{s_r} = \bar{Y}_w. \tag{7.6.15}$$

Further, one has the following well-known result [Dem].

Proposition 7.6.16. *(Demazure resolution) The natural projection* (7.6.14)

$$p: \bar{Y}_{s_1} \times_{\mathcal{B}} \times \cdots \times_{\mathcal{B}} \bar{Y}_{s_r} \twoheadrightarrow \bar{Y}_{s_r} \circ \cdots \circ \bar{Y}_{s_r} \overset{(7.6.15)}{=} \bar{Y}_w$$

gives a resolution of singularities of $\bar{Y}_w$ (i.e., is birational and proper). Moreover, it induces the isomorphism of Zariski open subsets

$$p: Y_{s_1} \times_{\mathcal{B}} \times \cdots \times_{\mathcal{B}} Y_{s_r} \xrightarrow{\sim} Y_w.$$

We remark that the first claim of the proposition can be easily proved by induction on the Bruhat order.

Proof of Proposition 7.6.12. Fix some w and choose a reduced decomposition $w = s_1 \cdot \ldots \cdot s_r$. Clearly, each $T^*_{\bar{Y}_{s_i}}(\mathcal{B} \times \mathcal{B})$ is a smooth irreducible component of the Steinberg variety Z. Hence, the composition $T^*_{\bar{Y}_{s_1}}(\mathcal{B} \times \mathcal{B}) \circ \cdots \circ T^*_{\bar{Y}_{s_r}}(\mathcal{B} \times \mathcal{B})$ is a closed subvariety of Z. Observe further that the natural projection $T^*(\mathcal{B} \times \mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ commutes with the compositions of subsets in $T^*(\mathcal{B} \times \mathcal{B})$ and $\mathcal{B} \times \mathcal{B}$, respectively. It follows easily that, set-theoretically, we have

$$T^*_{\bar{Y}_{s_1}}(\mathcal{B} \times \mathcal{B}) \circ \cdots \circ T^*_{\bar{Y}_{s_r}}(\mathcal{B} \times \mathcal{B}) = T^*_{Y_w}(\mathcal{B} \times \mathcal{B}) \sqcup \mathcal{V}, \tag{7.6.17}$$

where $\mathcal{V} \subset Z_{<w}$ is a closed subset. In particular, the LHS belongs to $Z_{\leq w}$. Hence, in the notation of (7.6.1), we get $\operatorname{supp}(\mathcal{Q}_{s_1} * \cdots * \mathcal{Q}_{s_r}) \subset Z_{\leq w}$, so that in K-theory one has $[\mathcal{Q}_{s_1}] * \cdots * [\mathcal{Q}_{s_r}] \in K^A(Z_{\leq w})$. Thus, for any $\lambda \in P$, formula (7.6.1) yields

$$\Theta(e^\lambda) * \Theta(T_{s_1}) * \cdots * \Theta(T_{s_r}) \in K^A(Z_{\leq w}).$$

On the other hand, the multiplication rule 7.1.2(b) for the Hecke algebra $\mathbf{H}$ yields $T_{s_1} \cdot \ldots \cdot T_{s_r} = T_w$, since $\ell(s_1) + \cdots + \ell(s_r) = \ell(w)$. The map $\Theta : \mathbf{H} \to K^A(Z)$ being an algebra homomorphism, we obtain $\Theta(e^\lambda T_w) = \Theta(e^\lambda) * \Theta(T_{s_1}) * \cdots * \Theta(T_{s_r}) \in K^A(Z_{\leq w})$, and part (1) of the proposition follows.

To prove part (2) note first that by (7.6.1), we have

$$\Theta(T_{s_i})|_{T^*_{\bar{Y}_{s_i}}(\mathcal{B}\times\mathcal{B})} = \mathcal{Q}_{s_i} \tag{7.6.18}$$

is a line bundle on $T^*_{\bar{Y}_{s_i}}(\mathcal{B} \times \mathcal{B})$.

Next put $w_1 = s_1, w_2 = s_1 \cdot s_2, \ldots, w_r = s_1 \cdot \ldots \cdot s_r = w$. For each $1 \leq j \leq r$, clearly, $w_j = s_1 \cdot \ldots \cdot s_j$ is a reduced expression for w_j.

Proposition 7.6.16 shows that, for any $j = 1, 2, 3, \ldots, r-1$, the varieties $Y_1 = Y_{w_j}$ and $Y_2 = Y_{s_{j+1}}$ satisfy the assumptions of Remark 2.7.27(ii). Write $\mathrm{pr}_{j,j+1} : (T^*\mathcal{B})^{r+1} \to T^*(\mathcal{B} \times \mathcal{B})$ for the natural projection to the $(j, j+1)$-factor, and put $\mathcal{Z}_i = \mathrm{pr}_{i,i+1}^{-1}(T^*_{Y_{s_i}}(\mathcal{B} \times \mathcal{B}))$. The repeated use of Remark 2.7.27(iii) yields an isomorphism

$$\mathcal{Z}_1 \cap \mathcal{Z}_2 \cap \ldots \cap \mathcal{Z}_r \xrightarrow{\sim} T^*_{Y_w}(\mathcal{B} \times \mathcal{B}), \tag{7.6.19}$$

and implies, moreover, that the intersections on the LHS of (7.6.19) are transverse. Let $\tilde{\mathcal{Q}}_{s_i}$ denote the direct image of the sheaf $\mathrm{pr}^*_{i,i+1}\mathcal{Q}_{s_i}$ under the embedding $\mathcal{Z}_i \hookrightarrow (T^*\mathcal{B})^{r+1}$. It now follows from (7.6.18) and the definition of convolution that under isomorphism (7.6.19) we get

$$[\Theta(T_{s_1}) * \Theta(T_{s_2}) * \cdots * \Theta(T_{s_r})]_{|T^*_{Y_w}(\mathcal{B}\times\mathcal{B})} = [\tilde{\mathcal{Q}}_{s_1}] \otimes \ldots \otimes [\tilde{\mathcal{Q}}_{s_r}].$$

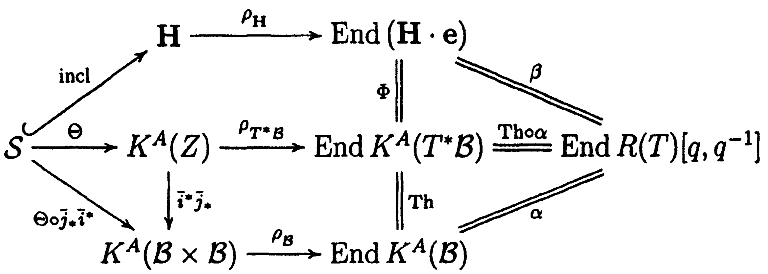
The RHS represents the class of a line bundle on $T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$. This completes the proof of part (2) of the proposition. ∎

The rest of this section is devoted to proving Claims 7.6.7 and 7.6.8. Recall that the projection $T^*(\mathcal{B}\times\mathcal{B}) = T^*\mathcal{B}\times T^*\mathcal{B} \xrightarrow{\mathrm{id}\times\pi} (T^*\mathcal{B})\times\mathcal{B}$ becomes injective when restricted to the Steinberg variety $Z \subset T^*(\mathcal{B} \times \mathcal{B})$. Thus we get the following natural embeddings

$$Z \xhookrightarrow{\bar{j}} (T^*\mathcal{B}) \times \mathcal{B} \xleftarrow{\bar{i}} \mathcal{B} \times \mathcal{B}, \qquad \bar{i} = (\textit{zero section}) \times \mathrm{id}_{\mathcal{B}}$$

We introduce the following expanded version of the diagram of Claim 7.6.8.

(7.6.20)



The maps Th, α and β on the right of the diagram arise from the corresponding isomorphisms (7.6.5). This part of the diagram is just a more detailed definition of the isomorphism Φ in Claim 7.6.8. The rectangle at the bottom of the diagram comes from Lemma 5.4.27. Thus there are three paths in the diagram starting at S on the left and ending up at

$\mathrm{End}_{Z[q,q^{-1}]}\Big(R(T)[q,q^{-1}]\Big)$ on the right. They are given by the compositions

$$\text{(7.6.21)}\quad \Psi_1 = \beta \circ \rho_{\mathrm{H}} \circ \mathrm{incl}, \quad \Psi_2 = \alpha \circ \mathrm{Th} \circ \rho_{T^*\mathcal{B}} \circ \Theta, \quad \Psi_3 = \alpha \circ \rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* \circ \Theta.$$

Using the notation above, Claim 7.6.8 amounts to the equation $\Psi_1 = \Psi_2$. By Corollary 5.4.34 we know that the rectangle at the bottom of diagram (7.6.20) commutes. This yields $\Psi_2 = \Psi_3$. Thus, it suffices to prove that $\Psi_1 = \Psi_3$. The strategy of the proof of this last equation is based on a reduction from $G\times\mathbb{C}^*$-equivariant K-theory to $T\times\mathbb{C}^*$-equivariant K-theory.

Fix a point $\mathfrak{b} \in \mathcal{B}$. Let B be the Borel subgroup corresponding to $\mathfrak{b}$ and $T \subset B$ a maximal torus. Identify $\mathcal{B}$ with G/B (using the choice of B) and view $\mathcal{B} \times \mathcal{B}$ as a G-equivariant fibration over G/B by means of the second projection. Restricting to the fiber $\mathcal{B}$ of this fibration over the base point $1 \in G/B$ gives an isomorphism $K^G(\mathcal{B}\times\mathcal{B}) \simeq K^B(\mathcal{B})$, see 5.2.16. Composing it with the reduction isomorphism $K^B(\mathcal{B}) \simeq K^T(\mathcal{B})$, see 5.2.18, one obtains an isomorphism

$$\text{(7.6.22)}\qquad res : K^G(\mathcal{B}\times\mathcal{B}) \xrightarrow{\sim} K^T(\mathcal{B}).$$

Further, let $\mathcal{B} = \sqcup_{w\in W}\mathcal{B}_w$ be the Bruhat cell stratification by B-orbits. Let $Z_\mathfrak{b} := \sqcup_{w\in W} T^*_{\mathcal{B}_w}\mathcal{B} \subset T^*\mathcal{B}$ be the fiber over $\{\mathfrak{b}\}$ of the composition $Z \overset{\bar{j}}{\hookrightarrow} (T^*\mathcal{B})\times\mathcal{B} \xrightarrow{\mathrm{pr}_2} \mathcal{B}$.

We have the following commutative diagram

(7.6.23)

$$\begin{array}{ccccccccc}
Z & \overset{\bar{j}}{\hookrightarrow} & (T^*\mathcal{B})\times\mathcal{B} & \overset{\bar{i}}{\hookleftarrow} & \mathcal{B}\times\mathcal{B} & \xrightarrow{\mathrm{pr}_2} & \mathcal{B} & = & G/B \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Z_\mathfrak{b}\times\{\mathfrak{b}\} & \overset{j}{\hookrightarrow} & (T^*\mathcal{B})\times\{\mathfrak{b}\} & \overset{i}{\hookleftarrow} & \mathcal{B}\times\{\mathfrak{b}\} & \xrightarrow{\mathrm{pr}_2} & \{\mathfrak{b}\} & = & 1\cdot B/B
\end{array}$$

All the varieties in the top row of the diagram are fibered naturally over $\mathcal{B}$ by means of the second projection pr_2, and the corresponding varieties in the second row are obtained as fibers of those fibrations over the base point $\mathfrak{b} \in \mathcal{B}$. The fibrations being $G\times\mathbb{C}^*$-equivariant, the above diagram induces, as in (7.6.22) (by the induction property 5.2.16 and the identification $\mathcal{B} = G/B$ given by the base point $\mathfrak{b}$), the following commutative diagram of K-groups:

$$\text{(7.6.24)}\quad \begin{array}{ccccc}
K^{G\times\mathbb{C}^*}(Z) & \xrightarrow{\bar{i}^*\circ\bar{j}_*} & K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B}) & = & K^G(\mathcal{B}\times\mathcal{B})[q,q^{-1}] \\
{\scriptstyle res}\Big\| & & {\scriptstyle res}\Big\| & & {\scriptstyle res}\Big\| \\
K^{T\times\mathbb{C}^*}(Z_\mathfrak{b}) & \xrightarrow{i^*\circ j_*} & K^{T\times\mathbb{C}^*}(\mathcal{B}) & = & K^T(\mathcal{B})[q,q^{-1}]
\end{array}$$

In this diagram one writes B instead of T first and then replaces B by a maximal torus $T \subset B$ by the reduction property 5.2.18. Thus the vertical isomorphism res in the middle is essentially the isomorphism (7.6.22).

Proof of the Injectivity Claim 7.6.7. By 5.4.27 we have, see (7.6.20),

$$\alpha \circ \mathrm{Th} \circ \rho_{T^*\mathcal{B}} = \alpha \circ \rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* .$$

Since α is an isomorphism, it follows that

$$\mathrm{Th} \circ \rho_{T^*\mathcal{B}} = \rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* .$$

It is clear that $\rho_{T^*\mathcal{B}}$ is injective if and only if so is $\mathrm{Th} \circ \rho_{T^*\mathcal{B}}$, since Th is the Thom isomorphism. Further, the Künneth theorem for the flag variety 6.1.19(b) implies that $\rho_{\mathcal{B}}$ is injective. Thus, to show that $\rho_{T^*\mathcal{B}}$ is injective it is enough to prove that $\bar{i}^* \bar{j}_*$ is injective. Using commutativity of (7.6.24) we see that proving the injectivity claim reduces to showing injectivity of the composition

$$(7.6.25) \qquad i^* j_* : K^{T \times \mathbb{C}^*}(Z_{\mathfrak{b}}) \xrightarrow{j_*} K^{T \times \mathbb{C}^*}(T^*\mathcal{B}) \xrightarrow{i^*} K^{T \times \mathbb{C}^*}(\mathcal{B}),$$

For this we apply the Localization Theorem in equivariant K-theory. Choose a complex number $z \neq 1$ and set $a = (1, z) \in T \times \mathbb{C}^*$. Write $K^{T \times \mathbb{C}^*}(\bullet)_a$ for the K-groups localized at the maximal ideal in $R(T \times \mathbb{C}^*)$ corresponding to a. The maps (7.6.25) induce the corresponding maps of the localized groups

$$(7.6.26) \quad i^* j_* : K^{T \times \mathbb{C}^*}(Z_{\mathfrak{b}})_a \xrightarrow{j_*} K^{T \times \mathbb{C}^*}(T^*\mathcal{B})_a \xrightarrow{i^*} K^{T \times \mathbb{C}^*}(\mathcal{B})_a .$$

Observe that each K-group in (7.6.25) is a free $R(T \times \mathbb{C}^*)$-module (this follows from the Cellular Fibration Lemma, see 6.2.8), hence any morphism between these modules which is injective under localization is itself injective. Thus, we must only prove that both maps in (7.6.26) are injective. Consider the cartesian square of $G \times \mathbb{C}^*$-equivariant morphisms given by the left diagram below:

$$(7.6.27) \qquad \begin{array}{ccc} Z_{\mathfrak{b}} & \xrightarrow{j} & T^*\mathcal{B} \\ {\scriptstyle i_Z}\uparrow & & \uparrow{\scriptstyle i} \\ Z_{\mathfrak{b}} \cap \mathcal{B} & \hookrightarrow & \mathcal{B} \end{array} \qquad\qquad \begin{array}{ccc} Z_{\mathfrak{b}}^a & = & (T^*\mathcal{B})^a \\ \| {\scriptstyle i} & & \| \\ (Z_{\mathfrak{b}} \cap \mathcal{B})^a & = & \mathcal{B}^a \end{array}$$

where the vertical map $i_Z : Z_{\mathfrak{b}} \cap \mathcal{B} \hookrightarrow Z_{\mathfrak{b}}$ is viewed as being induced by the embedding $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ of the corresponding smooth ambient spaces given by the other vertical arrow of the square. We apply the Localization Theorem for cellular fibrations 5.10.5 in this situation. The theorem says that the composite map in (7.6.26) gets identified, by means of restriction to the a-fixed point sets, to the map $i^* : K^{T \times \mathbb{C}^*}(Z_{\mathfrak{b}}^a)_a \to K^{T \times \mathbb{C}^*}(Z_{\mathfrak{b}}^a \cap \mathcal{B}^a)_a$.

The latter is the restriction with supports corresponding to the fixed-point cartesian square on the right of (7.6.27). But the a-fixed point sets in the four varieties are all the same:

$$Z_b^a = (T^*\mathcal{B})^a = \mathcal{B}^a = Z_b^a \cap \mathcal{B}^a .$$

Thus, the map i^* for the fixed point sets is an isomorphism, and Claim 7.6.7 follows. ∎

We begin proving Claim 7.6.8 with some preparations that will facilitate an explicit computation of the operators $\rho_{T^*\mathcal{B}}(\Theta(u))$, $u \in \mathcal{S}$.

Recall that we have fixed $T \subset B \subset G$. Compose the natural "forgetful" morphism $K^G(\mathcal{B}) \to K^T(\mathcal{B})$ with the duality pairing (5.2.27) to define a morphism "tr" as the composition

$$\mathrm{tr} : K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) \to K^T(\mathcal{B}) \otimes K^T(\mathcal{B}) \xrightarrow{\langle\,,\,\rangle} R(T) .$$

The result below provides a technical tool for computing the convolution action $K^G(\mathcal{B} \times \mathcal{B}) \otimes K^G(\mathcal{B}) \xrightarrow{*} K^G(\mathcal{B})$.

Lemma 7.6.28. *The following diagram, where the isomorphism res is given by* (7.6.22), *commutes*

$$\begin{array}{ccc} K^G(\mathcal{B} \times \mathcal{B}) \otimes K^G(\mathcal{B}) & \xrightarrow{\ *\ } & K^G(\mathcal{B}) \\ {\scriptstyle res\otimes \mathrm{id}}\Big\| & & \Big\|{\scriptstyle (6.1.6)} \\ K^T(\mathcal{B}) \otimes K^G(\mathcal{B}) & \xrightarrow{\ tr\ } & R(T) . \end{array}$$

Proof. Recall the isomorphisms (6.1.19)(a) and (6.1.22)(a):

$$K^G(\mathcal{B} \times \mathcal{B}) \simeq K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B}) \quad \text{and} \quad K^T(\mathcal{B}) \simeq R(T) \otimes_{R(G)} K^G(\mathcal{B})$$

Using these morphisms the diagram of the lemma can be written as

$$\begin{array}{ccc} (K^G(\mathcal{B}) \otimes_{R(G)} K^G(\mathcal{B})) \otimes K^G(\mathcal{B}) & \xrightarrow{\ *\ } & K^G(\mathcal{B}) \\ {\scriptstyle \phi\otimes\mathrm{id}\otimes\mathrm{id}}\Big\downarrow\wr & & {\scriptstyle \phi}\Big\downarrow\wr \\ (R(T) \otimes_{R(G)} K^G(\mathcal{B})) \otimes K^G(\mathcal{B}) & \xrightarrow{\ tr\ } & R(T), \end{array}$$

where ϕ is the canonical isomorphism 6.1.6. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G} \in K^G(\mathcal{B})$. Writing $\boxtimes$ to distinguish external tensor product from the tensor product in K-theory we find:

$$\phi((\mathcal{F}_1 \boxtimes \mathcal{F}_2) * \mathcal{G}) = \phi(\mathcal{F}_1 \cdot \langle \mathcal{F}_2, \mathcal{G} \rangle) = \phi(\mathcal{F}_1) \cdot \langle \mathcal{F}_2, \mathcal{G} \rangle$$

and also

$$\operatorname{tr}\circ(\phi\otimes\mathrm{id}\otimes\mathrm{id})(\mathcal{F}_1\boxtimes\mathcal{F}_2\boxtimes\mathcal{G})$$
$$=\operatorname{tr}(\phi(\mathcal{F}_1)\cdot(\mathcal{F}_2\boxtimes\mathcal{G}))=\phi(\mathcal{F}_1)\cdot\operatorname{tr}(\mathcal{F}_2\otimes\mathcal{G})=\phi(\mathcal{F}_1)\cdot\langle\mathcal{F}_2,\mathcal{G}\rangle\,.$$

Thus, the two expressions are equal. (Note that the steps here are similar to those used in the proof of Lemma 5.2.28.) ∎

Proposition 7.6.29. *For any $\mu\in X^*(T)$ we have $\Psi_3(e^\mu): e^\lambda\to e^{-\mu+\lambda}$, and for any simple root $\alpha\in R$, the operator $\Psi_3(T_{s_\alpha})$ is given by formula (7.2.17).*

Proof. We keep the setup of diagram (7.6.20), choose a simple reflection $s=s_\alpha\in W$, and let $\mathfrak{b}$ be the Borel subalgebra corresponding to the fixed Borel subgroup B. Recall that $\bar{Y}_s=Y_s\sqcup\mathcal{B}_\Delta\subset\mathcal{B}\times\mathcal{B}$ is the closure of the G-diagonal orbit of pairs of Borel subalgebras in relative position s. The second projection $\mathrm{pr}_2:\bar{Y}_s\to\mathcal{B}$ is a G-equivariant fibration with fiber $\mathrm{pr}_2^{-1}\{\mathfrak{b}\}=\bar{\mathcal{B}}_s\simeq\mathbb{P}^1$, where $\bar{\mathcal{B}}_s$ is the set of all Borel subalgebras in relative position $\le s$ with $\mathfrak{b}$. Write $\varepsilon:\bar{\mathcal{B}}_s\hookrightarrow\mathcal{B}$ for the embedding. We have defined in (7.6.1) the sheaf $\mathcal{Q}_s$ on $T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$ and, for any $\lambda\in X^*(T)$, the sheaf $\mathcal{O}_\lambda$ on the diagonal $\mathcal{B}_\Delta\subset\mathcal{B}\times\mathcal{B}$.

We claim first that, in the setup of (7.6.24), the following equations hold in $K^{T\times\mathbb{C}^*}(\mathcal{B})$:

(7.6.30)
$$res\circ\bar{i}^*\bar{j}_*[\mathcal{Q}_s]=\varepsilon_*(q\cdot[\Omega^1_{\bar{\mathcal{B}}_s}]-[\mathcal{O}_{\bar{\mathcal{B}}_s}])\quad,\quad res\circ\bar{i}^*\bar{j}_*[\mathcal{O}_\lambda]=[\mathbb{C}_{\{\mathfrak{b}\},\lambda}]\,,$$

where $\mathbb{C}_{\{\mathfrak{b}\},\lambda}$ is the skyscraper sheaf on $\{\mathfrak{b}\}$ with one dimensional fiber and the T-action given by λ and trivial $\mathbb{C}^*$-action. We begin proving the first equation.

Using the commutativity of diagram (7.6.24) we find

$$res\circ\bar{i}^*\bar{j}_*[\mathcal{Q}_s]=i^*j_*\circ res\;[\mathcal{Q}_s]\,,$$

where $i:\mathcal{B}\hookrightarrow T^*\mathcal{B}$ is the zero section. To compute $res[\mathcal{Q}_s]$, note that the embedding $Z_\mathfrak{b}\times\{\mathfrak{b}\}\hookrightarrow Z$ restricted to $T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$ gives the embedding $T^*_{\bar{\mathcal{B}}_s}\mathcal{B}\times\{\mathfrak{b}\}\hookrightarrow T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})$, since $(Z_\mathfrak{b}\times\{\mathfrak{b}\})\cap T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B})=T^*_{\bar{\mathcal{B}}_s}\mathcal{B}$. Therefore, it is clear that $res[\mathcal{Q}_s]=\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}$, where $\pi_s:T^*_{\bar{\mathcal{B}}_s}\mathcal{B}\to\bar{\mathcal{B}}_s$ is the natural projection. Thus, we are reduced to computing $i^*j_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}=i^*(\bar{\varepsilon}\circ\tilde{j})_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}$, where $\bar{\varepsilon}$ and $\tilde{j}$ are the embeddings defined in the diagrams (so, $j=\bar{\varepsilon}\circ\tilde{j}$)

(7.6.31)
$$\begin{array}{ccc} T^*\mathcal{B}|_{\bar{\mathcal{B}}_s} & \xhookrightarrow{\ \bar{\varepsilon}\ } & T^*\mathcal{B} \\ \big\uparrow{\scriptstyle \bar{i}} & & \big\uparrow{\scriptstyle i} \\ \bar{\mathcal{B}}_s & \xhookrightarrow{\ \varepsilon\ } & \mathcal{B} \end{array}\qquad\qquad \begin{array}{ccc} & \bar{\mathcal{B}}_s & \\ {\scriptstyle i_s}\swarrow & & \searrow{\scriptstyle \tilde{i}} \\ T^*_{\bar{\mathcal{B}}_s}\mathcal{B} & \xhookrightarrow{\ \tilde{j}\ } & T^*\mathcal{B}|_{\bar{\mathcal{B}}_s} \end{array}$$

We apply the base change (case (b) of Proposition 5.3.15) for the cartesian square on the left in (7.6.31) to deduce

$$i^*j_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s} = i^*\bar{\varepsilon}_*\tilde{j}_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_*\tilde{i}^*\tilde{j}_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}\,.$$

Decompose the map $\bar{\mathcal{B}}_s \hookrightarrow T^*\mathcal{B}_{|\bar{\mathcal{B}}_s}$ as $\tilde{i} = \tilde{j} \circ i_s$ where (see triangle on the right of (7.6.31)) $i_s : \bar{\mathcal{B}}_s \hookrightarrow T^*_{\bar{\mathcal{B}}_s}\mathcal{B}$ is the zero section. Thus, we have:

$$i^*j_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_*\tilde{i}^*\tilde{j}_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_*i_s^*\tilde{j}^*\tilde{j}_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s}\,.$$

We first compute $\tilde{j}^*\tilde{j}_*$ using Proposition 5.4.10. We have the canonical short exact sequence of vector bundles on $\bar{\mathcal{B}}_s$

$$0 \to T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \xrightarrow{\tilde{j}} T^*\mathcal{B}_{|\bar{\mathcal{B}}_s} \to T^*\bar{\mathcal{B}}_s \to 0. \tag{7.6.32}$$

The short exact sequence shows that the normal bundle to $T^*_{\bar{\mathcal{B}}_s}\mathcal{B}$ in $T^*\mathcal{B}_{|\bar{\mathcal{B}}_s}$ is isomorphic to the pullback by means of $\pi_s : T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \to \bar{\mathcal{B}}_s$ of the cotangent bundle $T^*\bar{\mathcal{B}}_s$ on $\bar{\mathcal{B}}_s$ Hence, applying Proposition 5.4.10 we find :

$$\varepsilon_*\tilde{i}^*\tilde{j}^*\tilde{j}_*\pi_s^*\Omega^1_{\bar{\mathcal{B}}_s} = \varepsilon_*i_s^*\pi_s^*(\lambda(T\bar{\mathcal{B}}_s)\otimes\Omega^1_{\bar{\mathcal{B}}_s}) = \varepsilon_*(\lambda(T\bar{\mathcal{B}}_s)\otimes\Omega^1_{\bar{\mathcal{B}}_s}).$$

The class $\lambda(T\bar{\mathcal{B}}_s)$ has been, in effect, computed in Section 7.5, since $T\bar{\mathcal{B}}_s$ is a 1-dimensional vector bundle on $\mathbb{P}^1$. We have (see (7.5.15) and (7.5.17)):

$$\lambda(T\bar{\mathcal{B}}_s) = \mathcal{O}_{\bar{\mathcal{B}}_s} - q^{-1}T\bar{\mathcal{B}}_s, \tag{7.6.33}$$

where the factor q^{-1} takes into account that the differential in the Koszul complex is *not* $\mathbb{C}^*$-equivariant, see 7.5.17. Combining all the previous computations together we obtain

$$\begin{aligned} res\circ\tilde{i}^*\tilde{j}_*[q\mathcal{Q}_s] &= i^*j_*\pi_s^*[q\cdot\Omega^1_{\bar{\mathcal{B}}_s}] = \varepsilon_*[q\cdot\lambda(T\bar{\mathcal{B}}_s)\otimes\Omega^1_{\bar{\mathcal{B}}_s}] \\ &= \varepsilon_*[(q\mathcal{O}_{\bar{\mathcal{B}}_s} - T\bar{\mathcal{B}}_s)\otimes\Omega^1_{\bar{\mathcal{B}}_s}] = \varepsilon_*(q[\Omega^1_{\bar{\mathcal{B}}_s}] - [\mathcal{O}_{\bar{\mathcal{B}}_s}])\,. \end{aligned} \tag{7.6.34}$$

This proves the first equation in (7.6.30). The proof of the second equation is much simpler and is left to the reader.

We can now continue the proof of Proposition 7.6.29. By definition, for a simple reflection $s \in W$, we have

$$\Psi_3(T_s) = \alpha\circ\rho_{\mathcal{B}}\circ\tilde{i}^*\tilde{j}_*\circ\Theta(T_s) = \alpha\circ\rho_{\mathcal{B}}\circ\tilde{i}^*\tilde{j}_*(\mathcal{Q}_s)\,.$$

Set $\mathcal{F} = \tilde{i}^*\tilde{j}_*(\mathcal{Q}_s) \in K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B})$. By Lemma 7.6.28 the operator $\rho_{\mathcal{B}}$: $K^{G\times\mathbb{C}^*}(\mathcal{B}) \to K^{G\times\mathbb{C}^*}(\mathcal{B})$ is given by

$$L \mapsto \rho_{\mathcal{B}}(\mathcal{F})(L) = \mathrm{tr}(res(\mathcal{F})\otimes L)\,.$$

Thus, using equation (7.6.30) and putting $L = L_\lambda$ we see that the operator

$\rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* \circ \Theta(T_s)$ is given by

(7.6.35)

$$\begin{aligned}\rho_{\mathcal{B}} \circ \bar{i}^* \bar{j}_* \circ \Theta(T_s) : [L_\lambda] \mapsto\ & \mathrm{tr}\left(\varepsilon_*(-q\Omega_{\bar{\mathcal{B}}_s} + \mathcal{O}_{\mathcal{B}_s} - \mathbb{C}_{\{\mathfrak{b}\}}) \otimes L_\lambda\right) \\ &= -q \cdot [p_*(\Omega_{\bar{\mathcal{B}}_s} \otimes \varepsilon^* L_\lambda)] + p_* \varepsilon^* [L_\lambda] - [L_\lambda],\end{aligned}$$

where $p : \bar{\mathcal{B}}_s \to \{\mathrm{pt}\}$ is a constant map.

To complete the proof of the proposition we must express the class in the second line of (7.6.35) as an element of $R(T)[q, q^{-1}]$. Let P_s be the unique parabolic subgroup of G of type s containing B, and let $R \subset P_s$ be the centralizer of T in P_s. Then R is a reductive subgroup of G, a Levi component of P_s. Let $B_R := R/R \cap B$ be a Borel subgroup in R. Note that

$$\bar{\mathcal{B}}_s \simeq P_s/B \simeq R/B_R$$

is the flag manifold for R. Observe further that T is a maximal torus in R. Therefore, we have $K^{R\times\mathbb{C}^*}(R/B_R) = K^{B_R\times\mathbb{C}^*}(\mathrm{pt}) = R(T)[q, q^{-1}]$. We see that in order to compute the rightmost term in (7.6.35) there will be no loss of information if L_λ is replaced by its restriction to $\bar{\mathcal{B}}_s$, viewed as an element of $K^{R\times\mathbb{C}^*}(R/B_R)$. Thus, we have reduced our computation from the case of the semisimple group G to that of R.

Digression: Semisimple Rank 1 Case. By construction, the group R is a connected reductive group, and R^{der}, the derived group, is a semisimple group of rank 1. We will say that R has *semisimple rank* 1. Such a group can always be written as a semidirect product $R = \bar{R} \cdot H$, where $\bar{R}$ is either $SL_2(\mathbb{C})$ or $PGL_2(\mathbb{C})$, and H is a torus. Therefore, we have

$$\mathrm{Lie}\, R = \mathrm{Lie}\, \bar{R} \oplus \mathrm{Lie}\, T = \mathfrak{sl}_2(\mathbb{C}) \oplus \text{abelian Lie algebra}.$$

Hence the variety $\mathcal{B}_R$ of all Borel subalgebras in $\mathrm{Lie}\, R$ is isomorphic to that for $\mathfrak{sl}_2(\mathbb{C})$. Thus, writing B_R for a Borel subgroup of R, we have an isomorphism $\mathcal{B}_R \simeq R/B_R \simeq \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, where $\mathcal{B}_R$ is, of course, the flag manifold for R. We fix such an isomorphism once and for all.

For concreteness, we choose the Borel subgroup $B_R \subset R$ to be the stabilizer of the line spanned by the vector $\binom{1}{0}$. We also let $T \subset R$ be the maximal torus with eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$. Since the group R is of semisimple rank 1, it has a unique positive root, $\alpha \in X^*(T)$ (with respect to the geometric choice of positive roots). Write $\check{\alpha}$ for the corresponding coroot. The Weyl group, W_R, of R is generated by the reflection $s : \lambda \mapsto \lambda - \langle\lambda, \check{\alpha}\rangle\alpha$.

The following result is a generalization of Lemma 7.5.4: *for any $\lambda \in X^*(T)$ there is a natural R-equivariant isomorphism of line bundles on $\mathbb{P}^1$*

$$L_\lambda \simeq \mathcal{O}(\langle\lambda, \check{\alpha}\rangle). \tag{7.6.36}$$

The proof is very similar to that of Lemma 7.5.4 and is left to the reader.

We would like to use the Weyl character formula 6.1.17. Note that in our case $\rho = \alpha/2$, since we have only one positive root α. Therefore the RHS of the formula in Corollary 6.1.17 reads

$$\frac{e^{\lambda+\rho} - e^{s(\lambda+\rho)}}{e^{\alpha/2} - e^{-\alpha/2}} = \frac{e^{\lambda+\alpha/2} - e^{\lambda+\alpha/2-\langle\lambda+\alpha/2,\check{\alpha}\rangle\alpha}}{e^{\alpha/2} - e^{-\alpha/2}} = e^{\lambda}\frac{e^{\alpha/2} - e^{\alpha/2-\langle\lambda,\check{\alpha}\rangle\alpha-\alpha}}{e^{\alpha/2} - e^{-\alpha/2}}.$$

Since $e^{\alpha/2}/(e^{\alpha/2} - e^{-\alpha/2}) = 1/(1-e^{-\alpha})$, applying Corollary 6.1.17 to the line bundle L_λ on $\mathcal{B}_R$ and a constant map $p : \mathcal{B}_R \to \mathrm{pt}$, we obtain

$$(7.6.37) \qquad p_*L_\lambda = e^{\lambda}\frac{1 - e^{-(\langle\lambda,\check{\alpha}\rangle+1)\alpha}}{1 - e^{-\alpha}} \in K^R(\mathrm{pt}) = R(T)^s.$$

Recall further that $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$. Now using the identification $\tilde{\mathcal{B}}_s = R/B_R$ and $K^{R\times\mathbb{C}^*}(R/B_R) \simeq R(T)[q, q^{-1}]$, we view the class $\varepsilon^*[L_\lambda]$ in the last line of (7.6.35) as an element $e^\lambda \in R(T)[q, q^{-1}]$. Using formula (7.6.37) we see that the last line of equation (7.6.35) takes the form

$$\begin{aligned}
&-q\cdot e^{\lambda-\alpha}\frac{1 - e^{-(\langle\lambda,\check{\alpha}\rangle-1)\alpha}}{1-e^{-\alpha}} + e^{\lambda}\frac{1 - e^{-(\langle\lambda,\check{\alpha}\rangle+1)\alpha}}{1-e^{-\alpha}} - e^{\lambda}\\
&= -q\frac{e^{\lambda-\alpha} - e^{\lambda-\alpha-(\langle\lambda,\check{\alpha}\rangle-1)\alpha}}{1-e^{-\alpha}} + e^{\lambda}\left(\frac{e^{\alpha} - e^{\alpha-(\langle\lambda,\check{\alpha}\rangle+1\alpha}}{e^{\alpha}-1} - \frac{e^{\alpha}-1}{e^{\alpha}-1}\right)\\
&= -q\frac{e^{\lambda} - e^{\lambda-(\langle\lambda,\check{\alpha}\rangle-1)\alpha}}{e^{\alpha}-1} + \frac{e^{\lambda} - e^{\lambda-\langle\lambda,\check{\alpha}\rangle\alpha}}{e^{\alpha}-1}\\
&= \frac{e^{\lambda} - e^{s(\lambda)}}{e^{\alpha}-1} - q\frac{e^{\lambda} - e^{s(\lambda)+\alpha}}{e^{\alpha}-1},
\end{aligned}$$

which is precisely the formula (7.2.17). This completes the proof of Proposition 7.6.29. ∎

Proposition 7.6.38. *For any $\mu \in X^*(T)$ we have $\Psi_1(e^\mu) : e^\lambda \to e^{-\mu+\lambda}$, and for any simple root $\alpha \in R$, the operator $\Psi_1(T_{s_\alpha})$ is given by formula (7.2.17).*

Proof. The claim for $\Psi_1(e^\mu)$ is clear. It remains to compute the action of $\Psi_1(T_s)$ on e^λ which is by definition $\rho_{\mathbf{H}}(T_s)(e^{-\lambda}\mathrm{e})$. First, by Lemma 7.1.10 in $\mathbf{H}$ we have

$$(7.6.39) \qquad T_s\cdot e^{-\lambda} = e^{-s(\lambda)}T_s - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1-e^{-\alpha}}.$$

Thus

$$\rho_{\mathbf{H}}(T_s)(e^{-\lambda}\mathrm{e}) = T_s e^{-\lambda}\mathrm{e} = \left(e^{-s(\lambda)}T_s - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1-e^{-\alpha}}\right)\mathrm{e}.$$

By Lemma 7.6.2(2) we have the equality $T_s\mathrm{e} = q\mathrm{e}$ and therefore the equation above becomes

$$\rho_{\mathbf{H}}(T_s)(e^{-\lambda}\mathbf{e}) = \left(qe^{-s(\lambda)} - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1 - e^{-\alpha}}\right)\mathbf{e}. \tag{7.6.40}$$

To complete the computation we must map the RHS of (7.6.40) into $R(T)[q, q^{-1}]$ by means of the isomorphism $\beta : \mathbf{H}\cdot\mathbf{e} \xrightarrow{\sim} R(T)[q, q^{-1}]$. Applying β to the RHS of (7.6.40) we find

$$\beta\left(qe^{-s(\lambda)} - (q-1)\frac{e^{-s(\lambda)} - e^{-\lambda}}{1 - e^{-\alpha}}\mathbf{e}\right) = qe^{s(\lambda)} - (q-1)\frac{e^{s(\lambda)} - e^{\lambda}}{1 - e^{\alpha}}.$$

Finally we note the equality

$$qe^{s(\lambda)} - (q-1)\frac{e^{s(\lambda)} - e^{\lambda}}{1 - e^{\alpha}} = \frac{e^{\lambda} - e^{s(\lambda)}}{e^{\alpha} - 1} + q\frac{e^{s(\lambda)+\alpha} - e^{\lambda}}{e^{\alpha} - 1}.$$

This completes the proof. ∎

The preceding two propositions show that indeed $\Psi_1 = \Psi_3$ and hence the main theorem is proved.