CHAPTER 1

Symplectic Geometry

1.1 Symplectic Manifolds

Let X be a C^∞ manifold in the $\mathbb{R}$-case, or a smooth holomorphic or algebraic variety in the $\mathbb{C}$-case. Let $\mathcal{O}(X)$ denote the algebra of C^∞ (resp. holomorphic, algebraic) functions on X and call it the algebra of *regular* functions on X. We write TX and T^*X for the tangent and cotangent bundles on X respectively, and T_xX, resp. T_x^*X, for the fiber of TX, resp. T^*X, at a point $x \in X$.

Definition 1.1.1. A symplectic structure on X is a non-degenerate regular (i.e., C^∞, resp. holomorphic, algebraic) 2-form ω such that $d\omega = 0$.

Example 1.1.2. Let $X = \mathbb{C}^{2n}$ with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$. Then

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$$

is a symplectic structure.

There are two essential differences between symplectic and Riemannian geometries. First, the Riemannian geometry is "rigid" in the sense that two Riemannian manifolds chosen at random are most likely to be locally non-isometric. On the contrary, *any* two symplectic manifolds are locally isometric in the sense that the symplectic 2-form on *any* symplectic manifold always takes the canonical form of Example 1.1.2 in appropriate local coordinates, due to Darboux's theorem [GS1]. Second, in symplectic geometry the symplectic structure is usually intrinsically associated with the manifold under consideration, while in Riemannian geometry, usually there is no a priori given preferred metric on the manifold under consideration. Here are a few most fundamental examples of such symplectic structures.

Example 1.1.3. Let M be any manifold. Then the cotangent bundle $T^*M = X$ has a canonical symplectic structure.

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CONSTRUCTION. Assume, for concreteness, that the ground field is $\mathbb{C}$. We will construct a 1-form λ on T^*M and set $\omega = d\lambda$. Then the condition $d\omega = 0$ is automatically satisfied.

To construct λ, choose $x \in M$ and $\alpha \in T_x^*M$, a covector in the fiber over x. Let $\pi : T^*M \to M$ be the standard projection and $\pi_* : T_\alpha(T^*M) \to T_xM$ the tangent map. Let ξ be a tangent vector to T^*M at α. Then define $\lambda(\xi)$ to be the image of ξ under the following composition

$$\xi \mapsto \pi_*\xi \mapsto < \alpha, \pi_*\xi > \in \mathbb{C}.$$

Here $<,>$ is the natural pairing $T_x^*M \times T_xM \to \mathbb{C}$.

It is instructive to describe the above in coordinates. Let $q_1, \dots, q_n$ be local coordinates on M, either a C^∞ or a holomorphic manifold, and $p_1, \dots, p_n$ the additional dual coordinates in T^*M. These give a chart in T^*M, and in this chart we write $T^*M \ni \alpha = (q_1(\alpha), \dots, q_n(\alpha), p_1(\alpha), \dots, p_n(\alpha))$. A tangent vector $\xi \in T_\alpha(T^*M)$ has the form $\xi = \sum b_i \frac{\partial}{\partial p_i} + \sum c_i \frac{\partial}{\partial q_i}$ for some $b_i, c_i \in \mathbb{C}$. Thus, for $x = \pi(\alpha) = (q_1(\alpha), \dots, q_n(\alpha))$, the tangent map $\pi_* : T_\alpha(T^*M) \to T_xM$ is given by

$$\xi = \sum b_i \frac{\partial}{\partial p_i} + \sum c_i \frac{\partial}{\partial q_i} \quad \mapsto \quad \pi_*(\xi) = \sum c_i \frac{\partial}{\partial q_i}.$$

We see that $\lambda(\xi) = < \alpha, \pi_*(\xi) > = \sum p_i(\alpha) c_i$. Therefore in our coordinates we find

$$\lambda = \sum p_i dq_i \quad \text{and} \quad d\lambda = \sum dp_i \wedge dq_i$$

Thus, $d\lambda$ is locally the 2-form from example 1.1.2, hence, non-degenerate.

Let G be a Lie group. Throughout this book we let $\mathfrak{g}$ denote the Lie algebra of G, viewed as the tangent space T_eG of G at the identity. The action of G on itself by conjugation $G \ni g : h \mapsto g \cdot h \cdot g^{-1}$ naturally induces a G-action on T_eG, the *adjoint* action on $\mathfrak{g}$. For example, if $G = GL_n(\mathbb{C})$ then $\mathfrak{g} = M_n(\mathbb{C})$ is the matrix algebra and, for $g \in G$ and $x \in M_n(\mathbb{C})$ the adjoint action is again given by conjugation: $g : x \mapsto g \cdot x \cdot g^{-1}$. We adopt the same notation in general. That is, for any Lie group G, we let gxg^{-1} denote (by some abuse of notation) the result of the adjoint action of $g \in G$ on $x \in \mathfrak{g}$. Thus, in the general case, the symbol gxg^{-1} stands for a single object and not a product of 3 factors. Recall further that differentiating the adjoint action at $g = e$ one obtains a $\mathfrak{g}$-action ad on $\mathfrak{g}$ given by the Lie bracket $\operatorname{ad} x : y \mapsto [x, y]$.

Example 1.1.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. The adjoint G-action on $\mathfrak{g}$ gives rise to the transposed *coadjoint* G-action on $\mathfrak{g}^*$, to be denoted by Ad^*. Differentiating the latter at $g = e$, we obtain a $\mathfrak{g}$-action, ad^*, on $\mathfrak{g}^*$.

Proposition 1.1.5. *Any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ has a natural symplectic structure.*

This symplectic structure sometimes called the Kirillov-Kostant-Souriau symplectic structure is at the heart of the orbit method in representation theory (cf., [AuKo],[Ki],[Ko2],[Sou]).

Proof. Pick up a point $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$. We must produce a skew symmetric form on $T_\alpha\mathbb{O}$, the tangent space at the point α. We have a natural isomorphism $\mathbb{O} \simeq G/G^\alpha$, where $G^\alpha =$ the isotropy group of α. Therefore

$$T_\alpha\mathbb{O} = T_\alpha(G/G^\alpha) = \mathfrak{g}/\mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \operatorname{Lie} G^\alpha$. We want to define a skew symmetric 2-form on $T_\alpha\mathbb{O} = \mathfrak{g}/\mathfrak{g}^\alpha$. We define first a skew symmetric form

$$\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \qquad \omega_\alpha : (x, y) \mapsto \alpha([x, y]).$$

To show that the form ω_α descends to $\mathfrak{g}/\mathfrak{g}^\alpha$ we will show that if $x \in \mathfrak{g}^\alpha$ then $\alpha([x,y]) = \omega_\alpha(x,y) \equiv 0$ for all $y \in \mathfrak{g}$. To that end, let us examine more closely the definition of $\mathfrak{g}^\alpha$. We have $g \in G^\alpha \Leftrightarrow \operatorname{Ad}^*g(\alpha) = \alpha$. Therefore, differentiating at $g = e$, for $x \in \mathfrak{g}$, we obtain $\operatorname{Ad}^*x(\alpha) = 0 \Leftrightarrow x \in \mathfrak{g}^\alpha$. Now let $y \in \mathfrak{g}$. Then, for $x \in \mathfrak{g}$, one has $\operatorname{ad} x(y) = [x, y]$, hence $\operatorname{ad}^*x(\alpha)$ is a linear function on $\mathfrak{g}$ given by

$$\operatorname{ad}^*x(\alpha) : y \mapsto \alpha([x, y]).$$

Therefore, since $\alpha([x, y]) = \omega_\alpha(x, y)$, we obtain

$$\omega_\alpha(x, y) = 0 \quad \forall y \in \mathfrak{g} \iff x \in \mathfrak{g}^\alpha.$$

Thus, ω_α descends to $\mathfrak{g}/\mathfrak{g}^\alpha$. The assignment $\alpha \mapsto \omega_\alpha$ clearly gives a regular 2-form, ω, on $\mathbb{O}$.

Claim 1.1.6. $d\omega = 0$.

To prove the claim, recall the following well-known Cartan formula for the exterior differential. Given any vector fields ξ_1, ξ_2, ξ_3 on $\mathbb{O}$, one has

$$(1.1.7) \quad (d\omega)(\xi_1, \xi_2, \xi_3) = \xi_1 \cdot \omega(\xi_2, \xi_3) + \xi_3 \cdot \omega(\xi_1, \xi_2) + \xi_2 \cdot \omega(\xi_3, \xi_1)$$
$$- \Big(\omega([\xi_1, \xi_2], \xi_3) + \omega([\xi_3, \xi_1], \xi_2) + \omega([\xi_2, \xi_3], \xi_1)\Big).$$

Any element $x \in \mathfrak{g}$ gives rise, via the infinitesimal $\mathfrak{g}$-action on $\mathbb{O}$, to ξ_x, a vector field on $\mathbb{O}$. Observe that vector fields of the form ξ_x , $x \in \mathfrak{g}$ span the tangent space at each point of $\mathbb{O}$. Hence, to show that $d\omega = 0$ it is enough to show that, for any $x, y, z \in \mathfrak{g}$, we have $(d\omega)(\xi_x, \xi_y, \xi_z) = 0$.

Observe that, for $y, z, w \in \mathfrak{g}$, the following formulas hold:

$$\omega(\xi_y, \xi_z)(\alpha) = \alpha([y, z]), \quad \text{and} \quad (\xi_x w)(\alpha) = \alpha([x, w])$$

Applying this and the Jacobi identity to the first and second line of the right hand side of (1.1.7) yields that each of these two lines vanishes separately. ■

1.2 Poisson Algebras

Let A be a commutative, associative unital $\mathbb{C}$-algebra with multiplication $\cdot : A \times A \to A$.

Definition 1.2.1. A commutative algebra $(A, \cdot)$ endowed with an additional $\mathbb{C}$-bilinear anti-symmetric bracket $\{\ ,\ \}: A \times A \to A$ is called a *Poisson algebra* if the following conditions hold

(1) A is a Lie algebra with respect to $\{\ ,\ \}$;
(2) Leibniz rule: $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$, $\forall a, b, c \in A$.

The Lie bracket $\{\ ,\ \}$ will be called a *Poisson bracket* on A. We say that $\{\ ,\ \}$ gives a Poisson structure on the commutative algebra $(A, \cdot)$.

We are going to construct a natural Poisson algebra associated with any symplectic manifold. This is the most typical way Poisson algebras arise in geometry.

Let (M, ω) be a symplectic manifold. The non-degenerate 2-form ω gives a canonical isomorphism $TM \simeq T^*M$. Define a unique $\mathbb{C}$-linear map $\mathcal{O}(M) \to \{\textit{Vector fields on } M\}$, denoted $f \mapsto \xi_f$, by the requirement

$$\omega(\cdot, \xi_f) = df, \text{ that is } -df = i_{\xi_f}\omega$$

where i_ξ stands for the contraction with respect to ξ:

$$i_\xi : \{n\text{-forms}\} \to \{(n-1)\text{-forms}\}.$$

Observe that for any vector field η and any function f, by definition of ξ_f we have

$$\omega(\xi_f, \eta) = -\eta f \tag{1.2.2}$$

We define a bracket on $\mathcal{O}(M)$ by any of the following equivalent expressions

$$\{f, g\} = \omega(\xi_f, \xi_g) = -\xi_g f = \xi_f g. \tag{1.2.3}$$

Let L_ξ be the Lie derivative with respect to ξ (see, e.g. [Spiv]). The Lie derivative is related to the contraction operation via the following Cartan *homotopy formula* to be frequently used in the future: $L_\xi \alpha = i_\xi d\alpha + di_\xi \alpha$.

Definition 1.2.4. A vector field ξ is called *symplectic* if it preserves the symplectic form, *i.e.* $L_\xi \omega = 0$.

Lemma 1.2.5. *For any $f \in \mathcal{O}(M)$, one has $L_{\xi_f}\omega = 0$, i.e. ξ_f is symplectic.*

Proof. Observe that: (1) ω is closed and (2) $di_{\xi_f}\omega = -d(df) = 0$. We obtain:

$$L_{\xi_f}\omega = i_{\xi_f}d\omega + d(i_{\xi_f}\omega) = 0 + 0 = 0. \quad \blacksquare$$

We are going to show that $\{\ ,\ \}$ together with pointwise multiplication of functions gives $\mathcal{O}(M)$ a Poisson algebra structure. First we prove

Proposition 1.2.6. *The assignment: $f \to \xi_f$ intertwines the bracket on $\mathcal{O}(M)$ with the commutator, i.e., we have a bracket preserving map*

$$(\mathcal{O}(M), \{\ ,\ \}) \to (\textit{Symplectic Vector Fields on } M\ , [\ ,\ ]).$$

Proof. We have to show that $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. In general, for the Lie derivative, one has an identity (where $\cdot$ stands for the action of a vector field on a function)

$$\xi \cdot \omega(\xi_1, \xi_2) = L_\xi(\omega(\xi_1, \xi_2)) = (L_\xi\omega)(\xi_1, \xi_2) + \omega(L_\xi\xi_1, \xi_2) + \omega(\xi_1, L_\xi\xi_2)$$

for any vector fields ξ, ξ_1, ξ_2 on M. Therefore if $L_\xi\omega = 0$ we have the equality

$$\xi \cdot \omega(\xi_1, \xi_2) = \omega([\xi, \xi_1], \xi_2) + \omega(\xi_1, [\xi, \xi_2])$$

Then, for any vector field η, we get by Lemma 1.2.5

$$\xi_f \cdot \omega(\xi_g, \eta) = \omega([\xi_f, \xi_g], \eta) + \omega(\xi_g, [\xi_f, \eta]).$$

Using 1.2.2, the LHS can be rewritten as $-\xi_f\eta g$, and the second term on the RHS as $-[\xi_f, \eta]g = -\xi_f\eta g + \eta\xi_f g$. Thus we obtain

$$-\xi_f\eta g = \omega([\xi_f, \xi_g], \eta) - \xi_f\eta g + \eta\xi_f g.$$

Canceling terms on the left and on the right and using $\xi_f g = -\{f, g\}$ we find $\omega([\xi_f, \xi_g], \eta) = -\eta\{f, g\}$. The latter equality holds for all vector fields η if and only if $[\xi_f, \xi_g] = \xi_{\{f,g\}}$, and the proposition follows. $\blacksquare$

Theorem 1.2.7. *The algebra $\mathcal{O}(M)$ of regular functions (with pointwise multiplication) on a symplectic manifold M together with $\{\ ,\ \}$ is a Poisson algebra.*

Proof. We first prove the Jacobi identity. By Proposition 1.2.6 we have

$$[\xi_f, \xi_g]h = \xi_{\{f,g\}}h = \{\{f, g\}, h\}, \tag{1.2.8}$$

and

$$[\xi_f, \xi_g]h = \xi_f\xi_g h - \xi_g\xi_f h = \{f, \{g, h\}\} - \{g, \{f, h\}\}. \tag{1.2.9}$$

Now subtracting (1.2.8) from (1.2.9) yields the desired result.

Proving the Leibniz rule is straightforward, since differentiation along any vector field, hence the map: $g \mapsto \xi_f g$, is a derivation of the algebra $\mathcal{O}(M)$. ∎

1.3 Poisson Structures arising from Noncommutative Algebras

Let B be an associative filtered (non-commutative) algebra with unit. In other words there is an increasing filtration by $\mathbb{C}$-vector spaces

$$\mathbb{C} \subset B_0 \subset B_1 \subset \ldots, \qquad \bigcup_{i=0}^{\infty} B_i = B,$$

such that $B_i \cdot B_j \subset B_{i+j} \quad \forall i, j \geq 0$.

Set $A = \operatorname{gr} B = \oplus_i (B_i/B_{i-1})$. The multiplication in B gives rise to a well defined product

$$B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j}/B_{i+j-1},$$

making $A = \operatorname{gr} B$ an associative algebra.

Definition 1.3.1. Call B *almost commutative* if $\operatorname{gr} B$ is commutative with respect to the above product.

Proposition 1.3.2. *If B is almost commutative then* $\operatorname{gr} B$ *has a natural Poisson structure.*

Proof. First we define a bilinear map

$$\{\ ,\ \} : B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

as follows: Let $a_1 \in B_i/B_{i-1}$ and $a_2 \in B_j/B_{j-1}$ and let b_1 (resp. b_2) be a representative of a_1 in B_i (resp. a_2 in B_j). Set

$$\{a_1, a_2\} = b_1 b_2 - b_2 b_1 \pmod{B_{i+j-2}}.$$

Note that $b_1 b_2 - b_2 b_1 \in B_{i+j-1}$ by the almost commutativity of B. Therefore $\{a_1, a_2\}$ is a well-defined element of B_{i+j-1}/B_{i+j-2}. Furthermore, one verifies that this element in B_{i+j-1}/B_{i+j-2} does not depend on the choices of representatives b_1 and b_2.

To prove the axioms for a Poisson algebra, define for any $b_1, b_2 \in B$, $\{b_1, b_2\} = b_1 b_2 - b_2 b_1$. Axioms (2) and (3) of Definition 1.2.1 are satisfied for B with its usual algebra multiplication and $\{\ ,\ \}$, although B is not commutative, so it is not a Poisson algebra. Now, moving $\{\ ,\ \}$ from B to $\operatorname{gr} B$ does not affect the axioms. The proposition follows. ∎

Here are some examples.

Example 1.3.3. (cf. [Di]) Let B be the associative $\mathbb{C}$-algebra with generators

$$p_1, \dots, p_n, q_1, \dots, q_n,$$

and relations

$$[p_i, p_j] = 0 = [q_i, q_j] \quad \text{and} \quad [p_i, q_j] = \delta_{ij} \text{ (Kronecker delta)}.$$

Note that B is filtered but not graded, since the above relations are not homogeneous (the relation $[p_i, q_j] = \delta_{ij}$ is not degree preserving: $[p_i, q_j]$ is of degree 2 and δ_{ij} is degree 0). One has a concrete realization of B given as follows. Let

$$\text{Diff} = \left\{ \sum a_{\mathbf{k}}(x) \frac{\partial^{k_1+\dots+k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \quad a_{\mathbf{k}}(x) \in \mathbb{C}[x_1, \dots, x_n], \ \mathbf{k} = (k_1, \dots, k_n) \right\}$$

be the algebra of the polynomial differential operators on $\mathbb{C}^n$. Define an assignment

$$p_i \leftrightarrow \frac{\partial}{\partial x_i}, \qquad q_i \leftrightarrow x_i.$$

This assignment preserves the relations above, hence, extends to an algebra isomorphism $B \leftrightarrow \text{Diff}$.

We will now give another construction of the same algebra in a coordinate free way. Let (V, ω) be a symplectic vector space, and c a dummy central variable. By the well-known theorem about the canonical form of a skew-symmetric bilinear form, we may find a basis $p_1, \dots, p_n, q_1, \dots, q_n$ of V such that

$$\omega(p_i, p_j) = 0 = \omega(q_i, q_j), \qquad \omega(p_i, q_j) = \delta_{ij}.$$

Form the algebra $TV \otimes \mathbb{C}[c]$ where TV is the tensor algebra of V. Endow both $\mathbb{C}[c]$ and TV with their standard gradings by assigning c and every element $v \in V$ grade degree 1, and put the natural total grading on the tensor product $TV \otimes \mathbb{C}[c]$. Set

$$\tilde{B} = TV \otimes \mathbb{C}[c] / (v_1 \otimes v_2 - v_2 \otimes v_1 - c \cdot \omega(v_1, v_2)).$$

The ideal of relations that we quotient out is not graded, since $v_1 \otimes v_2$ has grade degree 2, while $c \cdot \omega(v_1, v_2)$ has grade degree 1. Therefore the algebra $\tilde{B}$ is not graded. It inherits however a natural increasing filtration, $F_\bullet$. By definition, its k-th term, F_k, is spanned by all monomials of degree $\leq k$ in the generators, written in any order. Moreover, we have

$$\operatorname{gr}_F \tilde{B} = S(V)[c] = \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n, c]$$

where $S(V)$ is the symmetric algebra on V. Since RHS is a commutative algebra, we see that $\tilde{B}$ is almost commutative.

Since $\tilde{B}$ is almost commutative, Proposition 1.3.2 says that $\operatorname{gr}_F \tilde{B}$ has a canonical Poisson structure. An explicit computation yields the following formula for the Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) \cdot c. \tag{1.3.4}$$

Note that if f is a homogeneous element of deree r and g is a homogeneous element of degree s then the RHS has correct degree

$$(\deg f - 1) + (\deg g - 1) + 1 = \deg f + \deg g - 1.$$

To prove formula (1.3.4) we use the following general argument, to be exploited many times later on. We observe first that both sides of (1.3.4) satisfy the Leibniz rule (LHS by Proposition 1.3.2, and RHS as a first order differential operator in both f and g). Hence to show that the above formula yields the Poisson bracket given in Proposition 1.3.2, it is enough to check the equality LHS = RHS only on the generators $p_1, \dots, p_n, q_1, \dots, q_n$. This, however, is trivial and is left to the reader.

One can get a Poisson bracket on the symmetric algebra SV itself by specializing the central variable c to a concrete complex number. For example, taking the quotient of $\tilde{B}$ modulo the relation $c = 1$ we see that (cf. beginning of Example 1.3.3) $B \simeq \tilde{B}/(c-1)$ and

$$\operatorname{gr} B \simeq \operatorname{gr} \tilde{B}/(c-1) \simeq SV.$$

The Poisson bracket on $\operatorname{gr} \tilde{B}$ induces the Poisson bracket on $SV = \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$, given by formula (1.3.4) specialized at $c = 1$.

Further, we may identify SV with the algebra $\mathbb{C}[V^*]$ of polynomial functions on V^*, the dual space. Also, the non-degenerate 2-form ω on V yields a vector space isomorphism $V \simeq V^*$. Transferring the 2-form ω to V^* via this isomorphism makes V^* a symplectic manifold. The base elements $p_1, \dots, p_n, q_1, \dots, q_n \in V$ become canonical linear coordinates on V^*. In these coordinates, the symplectic 2-form on V^* takes the standard form of Example 1.1.2. Thus, we arrive at the following important

OBSERVATION. The Poisson bracket $\{\ ,\ \}$ on $\operatorname{gr} B$ given by formula (1.3.4), specialized at $c = 1$, is the one coming from the symplectic structure on V^*.

The reader is suggested to return to this point after Proposition 1.3.18.

Note next that if $f, g \in SV$ are homogeneous elements of degree 2, then it is clear from (1.3.4) that $\deg\{f, g\}$ is a homogeneous element of degree $\deg f + \deg g - 2 = 2+2-2 = 2$. Therefore the Poisson bracket $\{\ ,\ \}$ makes the space S^2V of degree 2 homogeneous elements a Lie algebra.

Lemma 1.3.5. *The elements of degree 2 form a Lie algebra isomorphic canonically to* $\mathfrak{sp}_{2n} = \mathfrak{sp}(V)$*, the symplectic Lie algebra.*

Proof. Observe that if f, g are homogeneous of degrees 2 and 1 respectively, then $\{f, g\}$ is again homogeneous and we have

$$\deg\{f, g\} = \deg f + \deg g - 2 = 2 + 1 - 2 = 1.$$

This implies that the Lie algebra S^2V acts, via the Poisson bracket, on the vector space V of degree 1 homogeneous elements. Observe also that, for $f, g \in V$, one has $\{f, g\} = \omega(f, g)$. Hence, for homogeneous f, g, h with $\deg h = 2$ and $\deg f = \deg g = 1$, the Jacobi identity for $\{\ ,\ \}$ yields

$$\omega(\{h, f\}, g) + \omega(f, \{h, g\}) = \{h, \omega(f, g)\} = 0\,.$$

This equation shows that the S^2V-action on V is compatible with the symplectic structure on V. We therefore get a Lie algebra morphism

$$S^2V \tilde{\to} \mathfrak{sp}(V).$$

We leave to the reader to check that both sides have the same dimension. We claim further that the morphism above is injective. Indeed, if $f \in S^2V$ commutes with any element of V then it commutes with the whole algebra SV, due to the Leibniz rule. It is clear however from (1.3.4) that the Poisson algebra SV has no center with respect to the Lie bracket. Thus, the above map is an isomorphism. ∎

Example 1.3.6. Let $\mathcal{D}(X)$ be the algebra of of regular (in the corresponding category) differential operators on a manifold X. In general, the notion of a regular differential operator requires the use of sheaf theory. We consider here the following three special cases where the sheaf theoretic language can be avoided, at least in the definitions. Thus we assume that X is

- a C^∞-manifold in the $\mathbb{R}$-case, or
- an open subset in $\mathbb{C}^d$ in the holomorphic case, or
- a smooth complex affine algebraic variety.

In each of these cases we write $\mathcal{T}(X)$ for the vector space of regular (in the corresponding category) vector fields on X, and define $\mathcal{D}(X)$ to be the subalgebra of $\mathrm{End}_{\mathbb{C}}\mathcal{O}(X)$ generated by $\mathcal{O}(X)$ and $\mathcal{T}(X)$, where $\mathcal{O}(X)$ acts on itself via multiplication, and vector fields act via derivations. By definition, the algebra $\mathcal{D}(X)$ comes equipped with an increasing filtration

filtration $\mathcal{O}(X) = \mathcal{D}_0(X) \subset \mathcal{D}_1(X) \subset \mathcal{D}_2(X) \subset \dots$, where $\mathcal{D}_1(X) = \mathcal{O}(X) + \mathcal{T}(X)$ and, for any $n \geq 1$, we put $\mathcal{D}_n(X) = \mathcal{D}_1(X) \cdot \ldots \cdot \mathcal{D}_1(X)$ (n factors). This clearly makes $\mathcal{D}(X)$ a filtered algebra. Elements of $\mathcal{D}_n(X)$ are called differential operators of order n.

Let X be an open subset of $\mathbb{C}^d$, and $x = (x_1, \dots, x_d)$ be some coordinates on $\mathbb{C}^d$. In these coordinates an element $u \in \mathcal{D}(X)$ can be written uniquely as a finite sum

$$(1.3.7) \quad u = \sum_{n_1,\dots,n_d \geq 0} u_{n_1,\dots,n_d}(x)\partial_1^{n_1}\dots\partial_d^{n_d} \quad , \quad u_{n_1,\dots,n_d} \in \mathcal{O}(X),$$

where ∂_i stands for $\frac{\partial}{\partial x_i}$. It is clear that $u \in \mathcal{D}_n(X)$ if and only if the coefficients $u_{n_1,\dots,n_d}$ vanish whenever $\sum_i n_i > n$. If X is a C^∞-manifold then an element $u \in \mathcal{D}(X)$ has the form (1.3.7) in any local chart. Moreover, using partition of unity (this is the instance where sheaf theory implicitly enters), one can prove the following. Let $u : \mathcal{O}(X) \to \mathcal{O}(X)$ be an operator such that in any local chart it restricts (on functions supported there) to an operator of the form (1.3.7), where the summation goes over $n_1 + \dots + n_d \leq n$. Then u is a regular differential operator on X of order n, i.e., $u \in \mathcal{D}_n(X)$.

In the algebraic case, no local coordinates are available so that formula (1.3.7) does not make sense. This obstacle can be (partially) overcome as follows. For any point $x \in X$, one may find a Zariski open affine subset $U \subset X$ such that the tangent bundle on U is trivial, i.e., $\mathcal{T}(U)$ is a free $\mathcal{O}(U)$-module. To construct U one proves first that regular vector fields on an affine variety span the tangent space at each point of the variety. Choose a collection $\{\partial_i, i = 1, 2, \dots, d\}$ of (not necessarily commuting) regular vector fields on X whose values at the point x form a base of the tangent space T_xX. Let U be the affine subset of X consisting of the points where the fields ∂_i are linearly independent. It is then clear that these vector fields form a free basis of $\mathcal{T}(U)$ regarded as a $\mathcal{O}(U)$-module. One can prove that any regular differential operator on U can be written uniquely in the form (1.3.7), where $\partial_1^{n_1}\dots\partial_d^{n_d}$ now stands for the product of the first order differential operators ∂_i written in this particular order.

To any differential operator u of order n on X, one can associate its principal symbol, $\sigma_n(u)$, a regular function (in the corresponding category) on T^*X which is a degree n homogeneous polynomial along each fiber of T^*X. Consider the holomorphic case first. Let X be an open subset of $\mathbb{C}^d$ and $x_1, \dots, x_d, p_1, \dots, p_d$ be the canonical coordinates on T^*X. Then, for $u \in \mathcal{D}_n(X)$ written in the form (1.3.7), the principal symbol is given by (see e.g. [Bj])

$$(1.3.8) \qquad \sigma_n(u) = \sum_{n_1+\dots+n_d=n} u_{n_1,\dots,n_d}(x) \cdot p_1^{n_1} \cdots p_d^{n_d} \in \mathcal{O}(T^*X).$$

A similar formula applies in the C^∞-case in local coordinates.

The principal symbol of a first order differential operator has an especially simple meaning. By definition, any such operator is of the form $u = \xi + f$ where ξ is a regular vector field and f is a regular function (this presentation is canonical, because we have $f = u(1)$). Since $\sigma_1(f) = 0$, it follows that $\sigma_1(u) = \sigma_1(\xi)$. Further, the principal symbol $\sigma_1(\xi)$ is nothing but the linear function on T^*X obtained by contracting covectors with ξ, i.e., given by the assignment $T^*_x X \ni \alpha \mapsto < \alpha, \xi_x >$, where ξ_x is the value of ξ at $x \in X$. Thus, we have defined $\sigma_1(\xi)$ in an intrinsic coordinate free way. Note that, for a coordinate vector field ∂_i in the canonical coordinates we have $\sigma_1(\partial_i) = p_i$.

We can now show that, for a differential operator u of any order n on a C^∞-manifold X, there is a well-defined regular function $\sigma_n(u)$ on T^*X which restricts to the previously defined one (1.3.8) in any local chart. To see this, write u as a linear combination of operators of the type $\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_r\,,\ r \leq n\,,$ where ξ_j are regular vector fields on X. Now fix some local chart. One verifies easily that in this chart, the corresponding symbol σ_n takes a linear combination of such operators into the corresponding linear combination of symbols; furthermore we have $\sigma_n(\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_r) = \sigma_1(\xi_1) \cdot \sigma_1(\xi_2) \cdot \ldots \cdot \sigma_1(\xi_r)$ if $r = n$ and zero otherwise. Thus we get a coordinate free expression for the principal symbol. Hence, $\sigma_n(u)$ is a well-defined regular function on T^*X. Note that while this expression shows the invariance of the principal symbol, it cannot be taken as a definition, since the presentation of a differential operator as a linear combination of operators of the type $\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_r$ is by no means unique.

We now define the principal symbol in the algebraic case. For a first order differential operator, use the above given intrinsic definition in the C^∞-case. Let u be a regular differential operator of order $n \geq 1$ on an affine algebraic variety X. We may find a finite covering of X by Zariski open affine subsets U such that the tangent bundle on U is trivial. As we have explained (two paragraphs above), on U the operator u can be written in the form (1.3.7), which depends of course on the choice of a basis $\{\partial_i\,,\ i = 1, 2, \ldots, d\}$ of $\mathcal{T}(U)$ regarded as a $\mathcal{O}(U)$-module. Using this basis, we define $\sigma_n(u)$ by formula (1.3.8), where p_i is now understood as $\sigma_1(\frac{\partial}{\partial x_i})$. The argument of the preceding paragraph shows that these "local" constructions on the subsets U give rise to a global regular function on T^*X, and that this function is independent of the choices involved.

Observe further that for any differential operator u of order $< n$ we have $\sigma_n(u) = 0$. We therefore get a well-defined morphism given by the principal symbol:

$$\text{(1.3.9)} \quad \sigma_n : \mathcal{D}_n(X)/\mathcal{D}_{n-1}(X) \longrightarrow \begin{array}{c}\textit{Homogeneous polynomial}\\ \textit{functions on } T^*X\\ \textit{of degree } n\end{array}$$

One can prove that for each of the three types of the variety X we are considering here, the above morphism is an isomorphism. This is immediate "locally" from formulas (1.3.7) and (1.3.8). The corresponding global result requires some extra work in "patching local results together" using sheaf theory, see e.g. [Bo5]. The idea is that the local result yields a short exact sequence of *sheaves*

$$0 \to \mathcal{D}_{n-1} \to \mathcal{D}_n \to \mathcal{O}_n \to 0\,,$$

where $\mathcal{D}_i$ is the sheaf of order i differential operators on X, and $\mathcal{O}_n$ is the sheaf (on X) formed by homogeneous polynomial functions on T^*X of degree n. Proving that (1.3.9) is an isomorphism amounts to showing that the short exact sequence of sheaves induces a short exact sequence of the corresponding vector spaces of global sections. In the C^∞-case this can be established using the partition of unity, and in the case of an affine algebraic variety, this can be deduced, cf. [Bj], from Theorem 2.2.7(ii).

Summing up isomorphisms (1.3.9) over all $n \geq 0$ we obtain an algebra isomorphism (cf., [Bj])

$$\operatorname{gr}\ \mathcal{D}(X) \xrightarrow{\ \sim\ } \bigoplus_{n\geq 0} \begin{array}{c}\textit{Polynomial functions}\\ \textit{on } T^*X \textit{ of degree } n\end{array} = \mathcal{O}_{pol}(T^*X).$$

Here $\mathcal{O}_{pol}(T^*X)$ is the algebra of regular functions on T^*X polynomial along the fibers (in the algebraic case we have $\mathcal{O}_{pol}(T^*X) = \mathcal{O}(T^*X)$).

Let ξ, η are regular vector fields on X viewed as first order differential operators. Then $[\xi, \eta]$ is again a first order differential operator corresponding to the vector field given by the Lie bracket of ξ and η, viewed as vector fields. Thus, in $\mathcal{D}(X)$ we have $[\mathcal{T}(X), \mathcal{T}(X)] \subset \mathcal{T}(X)$ and also $[\mathcal{T}(X), \mathcal{O}(X)] \subset \mathcal{O}(X)$. Since the algebra $\mathcal{D}(X)$ is generated by $\mathcal{D}_1(X)$, it follows by the Leibniz rule that $[\mathcal{D}_i(X), \mathcal{D}_j(X)] \subset \mathcal{D}_{i+j-1}(X)$, for any $i, j \geq 0$. Therefore $\operatorname{gr}\mathcal{D}(X)$ is commutative so that $\mathcal{D}(X)$ is an almost commutative algebra. Thus, Proposition 1.3.2 yields a canonical Poisson structure on $\operatorname{gr}\ \mathcal{D}(X) = \mathcal{O}_{pol}(T^*X)$.

On the other hand, T^*X is a symplectic manifold and therefore $\mathcal{O}(T^*X)$ has a Poisson algebra structure arising from its symplectic structure. It turns out that these two structures are the same.

Theorem 1.3.10. *(cf.,* [GS1],[AM]*) The Poisson structure on* $\mathcal{O}_{pol}(T^*X)$ *given by Proposition* 1.3.2 *is the same as the one arising from the symplectic structure on* T^*X*.*

Proof. One can prove that, under our assumptions on X, the algebra $\mathcal{O}_{pol}(T^*X)$ is generated by the subalgebra $\mathcal{O}(X) \subset \mathcal{O}_{pol}(T^*X)$ formed by the pullbacks of functions on X (these are constant along the fibers of $T^*X \to X$) and by the space of functions that are linear along the fibers, i.e., symbols of vector fields on X. As has been explained, checking that the two Poisson brackets in question are the same amounts, due to the Leibniz rule, to checking this on generators. We leave the more simple case involving $\mathcal{O}(X)$ to the reader. For the vector fields, the claim is equivalent to saying that if ξ, η are regular vector fields on X viewed as first order differential operators, then the commutator of these differential operators corresponds to the vector field given by the Lie bracket of ξ and η, viewed as vector fields. This latter result which follows from definitions has been already used in proving that $\mathcal{D}(X)$ is an almost commutative algebra.

Remark. Note that we may avoid any appeal to the fact that $\mathcal{O}_{pol}(T^*X)$ is generated by $\mathcal{O}(X)$ and by the symbols of regular vector fields, using a covering of X by appropriate open subsets U for which the analogous fact is obvious (e.g., in the algebraic case this is obvious if the tangent bundle on U is trivial). Since each of the Poisson brackets has a local definition, it suffices to check that the two brackets are the same when restricted to each T^*U. To prove the latter, the argument of the previous paragraph applies.

It is instructive to check the equality of the two Poisson brackets of the theorem by an explicit computation in local coordinates (assuming X is either an open domain in $\mathbb{C}^n$ or a C^∞-manifold). Given two vector fields, u, v, in coordinates we get

$$u = \sum u_i(x)\frac{\partial}{\partial x_i} \qquad v = \sum v_j(x)\frac{\partial}{\partial x_j}.$$

Writing $\sigma = \sigma_1$ for the symbol of first order differential operators, we have

$$\sigma(u) = \sum u_i(x)p_i \qquad \sigma(v) = \sum v_j(x)p_j.$$

We compute

$$[u, v] = \sum_{i,j}(u_i\frac{\partial v_j}{\partial x_i}\frac{\partial}{\partial x_j} - v_j\frac{\partial u_i}{\partial x_j}\frac{\partial}{\partial x_i})$$

so that

$$\sigma([u, v]) = \sum(u_i\frac{\partial v_j}{\partial x_i}p_j - v_j\frac{\partial u_i}{\partial x_j}p_i).$$

By formula (1.3.4) (with $c = 1$) of Example 1.3.3 we obtain

$$\{\sigma(u), \sigma(v)\} = \sum_k \left(\frac{\partial\sigma(u)}{\partial p_k}\frac{\partial\sigma(v)}{\partial x_k} - \frac{\partial\sigma(v)}{\partial p_k}\frac{\partial\sigma(u)}{\partial x_k}\right)$$
$$= \sum_{i,j}\left(u_i \frac{\partial v_j}{\partial x_i} p_j - v_j \frac{\partial u_i}{\partial x_j} p_i\right) = \sigma([u, v]). \quad \blacksquare$$

To a vector field u on X one associates canonically a vector field $\tilde{u}$ on T^*X as follows: u gives rise to an infinitesimal diffeomorphism of X which naturally induces an infinitesimal diffeomorphism of T^*X, that is it gives rise to the vector field $\tilde{u}$.

$u =$ *Vector field on* X

$\Vert$

Infinitesimal diffeomorphism of X

$\downarrow$

Infinitesimal diffeomorphism of T^*X

$\Vert$

$\tilde{u} =$ *vector field on* T^*X

We sketch here a more explicit construction of the vector field $\tilde{u}$, assuming for concretness that we are in the algebraic setup. Observe that the infinitesimal diffeomorphism of X corresponding to the vector field u acts on $\mathcal{T}(X)$ and on $\mathcal{O}(X)$ via the Lie derivative, see [Ster]. The Lie derivative gives a map

(1.3.11)
$$\tilde{u} : \mathcal{T}(X) + \mathcal{O}(X) \to \mathcal{T}(X) + \mathcal{O}(X) \ , \ \xi + f \mapsto [u, \xi] + u(f) \ ,$$

where $\xi \in \mathcal{T}(X)$ and $f \in \mathcal{O}(X)$.

We proceed now in two steps. Assume first that $\mathcal{T}(X)$ is a free $\mathcal{O}(X)$-module. Then $\mathcal{O}(T^*X) = S\mathcal{T}(X)$, the symmetric algebra on $\mathcal{T}(X)$ over $\mathcal{O}(X)$. Note that giving a vector field $\tilde{u}$ on T^*X is equivalent to giving a derivation of the algebra $\mathcal{O}(T^*X)$. But one can verify easily that the assignment (1.3.11) extends uniquely to a derivation of the symmetric algebra $S\mathcal{T}(X)$. This completes the first step.

In general, a regular vector field on T^*X is a global section of the sheaf of regular vector fields on T^*X. Therefore, to construct $\tilde{u}$ as a global section, we may cover X by appropriate open subsets U, as we have done before, and construct $\tilde{u}$ on each T^*U separately. The first step yields such a construction of the vector $\tilde{u}$ on T^*U. The naturality of the construction insures that the vector fields we obtain in this way for different U's agree with each other.

For any $x \in X$ and any covector $\alpha \in T^*_xX$, we have by the definition of $\tilde{u}$

$$\pi_*(\tilde{u}_\alpha) = u_x \quad \text{where} \quad \pi : T^*X \to X. \tag{1.3.12}$$

Claim 1.3.13. For any vector field u on X, $\tilde{u}$ is a symplectic vector field on T^*X.

Proof. Recall that $\omega = d\lambda$ is the symplectic 2-form on T^*X. The form λ, being constructed in a canonical way, is invariant under all automorphisms of T^*X arising from automorphisms of X. Infinitesimally, this means that $L_{\tilde{u}}\lambda = 0$. It follows that $L_{\tilde{u}}\omega = L_{\tilde{u}}d\lambda = dL_{\tilde{u}}\lambda = 0$ so that $\tilde{u}$ is a symplectic vector field. ∎

Observe next that to any function h on T^*X, one can associate the symplectic vector field ξ_h on T^*X. This applies, in particular, to the function $h_u = \sigma_1(u)$, the linear function on T^*X attached to the vector field u on X. The following result clarifies the relationship between the objects $u, \tilde{u}$ and h_u introduced above.

Lemma 1.3.14. *We have* $\tilde{u} = \xi_{h_u}$ *and, moreover* $h_u = \lambda(\tilde{u})$.

Proof. We have

$$0 = L_{\tilde{u}}\lambda = i_{\tilde{u}}d\lambda + di_{\tilde{u}}\lambda = i_{\tilde{u}}\omega + d(i_{\tilde{u}}\lambda). \tag{1.3.15}$$

Set $h = i_{\tilde{u}}\lambda$ so that $d(i_{\tilde{u}}\lambda) = dh$. Then $\omega(\cdot, \tilde{u}) = d(i_{\tilde{u}}\lambda) = dh$. We want to show that $h_u = h = \lambda(\tilde{u})$. Recall the definition of λ: let ϕ be a tangent vector at a point $\alpha \in T^*X$. Then $\lambda(\phi) = \alpha(\pi_*\phi)$ where $\pi_* : T(T^*X) \to TX$ is the tangent map to the projection $\pi : T^*X \to X$, whence,

$$h(\alpha) = (\lambda(\tilde{u}))(\alpha) = \alpha(\pi_*(\tilde{u})) = \alpha(u) = h_u(\alpha),$$

and the lemma follows. ∎

Second Proof of Theorem 1.3.10. We must prove $\{h_u, h_v\} = h_{[u,v]}$. We already know, by Lemma 1.3.14, that $\xi_{h_u} = \tilde{u}$ and $h_u = \lambda(\tilde{u})$. Observe further that $\widetilde{[u,v]} = [\tilde{u}, \tilde{v}]$. Hence, we get $\{h_u, h_v\} = \xi_{h_u}h_v = \tilde{u}(\lambda(\tilde{v}))$. But

$$\tilde{u}\,(\lambda(\tilde{v})) = L_{\tilde{u}}\,(\lambda(\tilde{v})) = (L_{\tilde{u}}\lambda)(\tilde{v}) + \lambda(L_{\tilde{u}}\tilde{v}) = \lambda([\tilde{u}, \tilde{v}]) \quad (\text{because } L_{\tilde{u}}\lambda = 0)$$

Hence we find $\tilde{u}\lambda(\tilde{v}) = \lambda(\widetilde{[u,v]}) = h_{[u,v]}$. This proves the claim. ∎

Remark. All the above holds in the C^∞-setup provided we take $\mathcal{D}(X)$ to be the algebra of differential operators with C^∞-coefficients and $\mathcal{O}(T^*X)$ to be the algebra of C^∞-functions on T^*X which are *polynomial* along the fibers. An argument involving partition of unity may then be used every time the assumption that X is affine *is* exploited in the algebraic setup above.

Example 1.3.16. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Let $\mathcal{U}\mathfrak{g}$ be its enveloping algebra, that is the quotient of the tensor algebra $T\mathfrak{g}$ modulo the ideal generated by expressions $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in \mathfrak{g}$. The algebra $\mathcal{U}\mathfrak{g}$ has a canonical filtration

$$\mathbb{C} = \mathcal{U}_0\mathfrak{g} \subset \mathcal{U}_1\mathfrak{g} \subset \cdots \quad \text{such that } \mathcal{U}_i\mathfrak{g} \cdot \mathcal{U}_j\mathfrak{g} \subset \mathcal{U}_{i+j}\mathfrak{g}.$$

Here $\mathcal{U}_j\mathfrak{g}$ is the $\mathbb{C}$-linear span of all monomials of degree $\leq j$ formed by elements of $\mathfrak{g}$, i.e., the image of $\mathbb{C} \oplus \mathfrak{g} \oplus T^2\mathfrak{g} \oplus \cdots \oplus T^j\mathfrak{g}$ under the canonical projection $T\mathfrak{g} \twoheadrightarrow \mathcal{U}\mathfrak{g}$. For the proof of the following well-known result, the reader is referred to [Di].

Theorem 1.3.17. *(Poincaré-Birkhoff-Witt) There are canonical graded algebra isomorphisms:*

$$\mathrm{gr}\,\mathcal{U}\mathfrak{g} \simeq S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*] .$$

Thus $\mathcal{U}\mathfrak{g}$ is almost commutative. Hence by Proposition 1.3.2, there is a canonical Poisson bracket $\{\ ,\ \}$ on $\mathbb{C}[\mathfrak{g}^*]$. We will now describe this bracket explicitly.

Let $e_1, \ldots, e_n$ be a base of $\mathfrak{g}$, and $c_{ij}^k \in \mathbb{C}$ the structure constants defined by $[e_i, e_j] = \sum_k c_{ij}^k e_k$. Observe that any element of $\mathfrak{g}$ may be viewed, via the canonical isomorphism $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$ as a linear function on $\mathfrak{g}^*$. In particular let $x_1, \ldots, x_n$ be the coordinate functions on $\mathfrak{g}^*$ corresponding to the base $e_1, \ldots, e_n$.

Proposition 1.3.18. *One has the following two expressions for the Poisson bracket $\{f, g\}$ of $f, g \in \mathbb{C}[\mathfrak{g}^*]$:*

$$\{f,g\} = \sum c_{ij}^k \cdot x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \quad , \quad \{f,g\} : \alpha \mapsto \langle \alpha, [d_\alpha f, d_\alpha g] \rangle \, , \ \alpha \in \mathfrak{g}^*,$$

where $d_\alpha f \in (\mathfrak{g}^*)^* = \mathfrak{g}$ denotes the differential of f at a point α, and $[\,,]$ denotes the Lie bracket on $\mathfrak{g}$.

Proof. Observe first that the polynomial algebra, $\mathbb{C}[\mathfrak{g}^*]$ is generated by linear functions. Observe further, that both the LHS and RHS of either formula clearly satisfies the Leibniz rule. Thus, by our standard argument, we have only to show that the formulas hold for linear functions on $\mathfrak{g}^*$. Such functions may be identified naturally with elements of $\mathfrak{g} = (\mathfrak{g}^*)^*$. For $f = x, g = x \in \mathfrak{g}$, by construction of the Poisson structure (cf. 1.3.18) we have

(1.3.19)

$$\{x, y\} = [x, y] \quad \text{in particular} \quad \{e_i, e_j\} = [e_i, e_j] = \sum_k c_{ij}^k e_k. \quad \blacksquare$$

Remark 1.3.20. Observe that for homogeneous polynomials $f, g \in \mathbb{C}[\mathfrak{g}^*]$, the RHS of Proposition 1.3.18 is a homogeneous polynomial of degree $\deg f + \deg g - 1$, in accordance with the degree of the LHS.

We now reinterpret the Poisson algebra of Example 1.3.3 in our present Lie algebra setup. Thus, given a symplectic vector space (V, ω), set $\mathfrak{g} = V \oplus \mathbb{C}$ and write c for a base vector in the second direct summand. One verifies easily that the following bracket makes $\mathfrak{g}$ a Lie algebra

$$[x \oplus \mu \cdot c, y \oplus \lambda \cdot c] = 0 \oplus \mu \cdot \lambda \cdot \omega(x, y), \quad \forall x, y \in V, \ \mu, \lambda \in \mathbb{C}.$$

The Lie algebra $\mathfrak{g}$ is called the *Heisenberg* algebra. By our general construction we get a Poisson structure on $S\mathfrak{g}$. But we have

$$S(\mathfrak{g}) \simeq S(V \oplus \mathbb{C}) \simeq S(V) \otimes \mathbb{C}[c].$$

The rightmost term here is nothing but the algebra $\operatorname{gr} \tilde{B}$ considered in Example 1.3.3. In fact we have a natural algebra isomorphism $\mathcal{U}\mathfrak{g} = \tilde{B}$. Thus, the Poisson bracket of Example 1.3.3 is essentially the Poisson bracket on the symmetric algebra of the Heisenberg Lie algebra, and formula 1.3.4 is nothing but a special case of the first formula of Proposition 1.3.18.

We recall next that $\mathfrak{g}^*$ is a union of coadjoint orbits, and that each coadjoint orbit $\mathbb{O}$ has a canonical symplectic structure.

Proposition 1.3.21. *(cf.,* [Ki]*) For any regular functions $f, g \in \mathbb{C}[\mathfrak{g}^*]$, and any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ we have*

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{symplectic}.$$

The bracket on the right hand side comes from the symplectic structure on $\mathbb{O}$ while the bracket on the left hand side comes from the Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$ restricted to $\mathbb{O}$.

Proof. Again, by our standard argument, we have only to show that the two brackets are the same for linear functions on $\mathfrak{g}^*$. By formula 1.3.19, $\{x, y\} = [x, y]$ is also a linear function on $\mathfrak{g}^*$. Now, take $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$. We calculate

$$[x, y](\alpha) = \alpha([x, y]) = (\operatorname{ad} x(y))|_{\alpha} = (\xi_x \cdot y)|_{\alpha} = \{x|_{\mathbb{O}}, y|_{\mathbb{O}}\}_{symplectic}. \quad \blacksquare$$

Let (V, ω) be a symplectic vector space. Given a vector subspace $W \subset V$ let $W^{\perp_\omega} \subset V$ denote the annihilator of W with respect to ω, to be distinguished from $W^{\perp}$, the annihilator in V^*.

Definition 1.3.22. A linear subspace $W \subset V$ is called

(1) *Isotropic* if $\omega|_W \equiv 0$, equivalently $W \subset W^{\perp_\omega}$;
(2) *Coisotropic* if $W^{\perp_\omega}$ is isotropic, equivalently, $W^{\perp_\omega} \subset W$;

(3) *lagrangian* if W is both isotropic and coisotropic, i.e., $W = W^{\perp\omega}$.

Example 1.3.23. Let $V = \mathbb{C}^{2n}$ and $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ a basis and let the 2-form ω be given by (cf. Example 1.1.2):

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j), \qquad \omega(e_i, f_j) = \delta_{ij} = -\omega(f_j, e_i).$$

Then we have, for any $k \leq n$,

(1) $W = \langle e_1, \dots, e_k \rangle$ *is isotropic,*
(2) $W^{\perp\omega} = \langle e_1, \dots, e_n, f_{k+1}, \dots, f_n \rangle$ *is coisotropic,*
(3) $\langle e_1, \dots, e_n \rangle$ and $\langle f_1, \dots, f_n \rangle$ *are lagrangian.*

One can show that, in general, a lagrangian subspace of V is always of dimension $1/2 \cdot \dim V$; an isotropic subspace is of dimension less than or equal to $1/2 \cdot \dim V$; a coisotropic subspace is of dimension greater than or equal to $1/2 \cdot \dim V$. These are easy exercises in linear algebra and are left to the reader.

We now extend the "linear setup" above to the nonlinear case. Let M be a symplectic manifold.

Definition 1.3.24. A (possibly singular) subvariety Z of M is called an isotropic (resp. coisotropic, lagrangian) subvariety of M, if for any smooth point $z \in Z$, $T_z Z$ is an isotropic (resp. coisotropic, lagrangian) subspace of $T_z M$.

Example 1.3.25. Let X be any manifold, and $M = T^*X$ its cotangent bundle with canonical 2-form ω. Let $f \in \mathcal{O}(X)$. Then df, the image of a section $X \to T^*X$ given by the differential of f, is a lagrangian subvariety of T^*X. This is clear if $\dim X = 1$; for a proof of the general case, see e.g. [GS1].

Assume from now on that X is a manifold and let T^*X be its cotangent bundle. Given a submanifold $Y \subset X$, define $T^*_Y X$, the conormal bundle of Y, to be the set of all covectors over Y which annihilate the subbundle $TY \subset (T^*X)_{|Y}$. We note that $T^*_Y X$ is a vector bundle over Y, and we have a natural diagram

$$(T^*X)_{|Y} \supset T^*_Y X \twoheadrightarrow Y.$$

Proposition 1.3.26. *The total space of the bundle $T^*_Y X$ is a lagrangian submanifold of T^*X stable under dilations along the fibers of T^*X.*

Proof. To see this we observe first that $T^*_Y X$ has the correct dimension, i.e.,

$$\dim T^*_Y X = 1/2 \cdot \dim T^*X.$$

This follows by observing that if X were a vector space then $T^*X = X \oplus X^*$, in which case we have $T^*_Y X = Y \oplus Y^\perp$, and $\dim Y + \dim Y^\perp = \dim X$, so that $\dim T^*_Y X = \dim X = 1/2 \cdot \dim T^*X$. The general case follows similarly since any manifold is locally isomorphic to a vector space.

Thus, to show that $T^*_Y X$ is lagrangian, it is enough to show that $T^*_Y X$ is isotropic, that is $\omega|_{T^*_Y X} = 0$. It is enough to show that $\lambda|_{T^*_Y X} = 0$ where λ is the canonical 1-form such that $d\lambda = \omega$. But the latter follows from the definition of λ and of $T^*_Y X$. ∎

A subvariety of T^*X stable under dilations along the fibers will be referred to as a *cone subvariety* of T^*X. Let Eu be the Euler vector field generating the $\mathbb{C}^*$-action along the fibers of T^*X. First we note that $i_{Eu}\omega = \lambda$(=standard 1-form). This is easy to verify in local coordinates: if $q_1, \ldots, q_n$ are local coordinates on X, and $p_1, \ldots, p_n$ are the dual "cotangent" coordinates, then we find

$$\lambda = \sum p_i dq_i, \quad Eu = \sum p_i \frac{\partial}{\partial p_i}, \quad \text{and } \omega = \sum dp_i \wedge dq_i.$$

We now give a useful characterization of lagrangian cone-subvarieties in a cotangent bundle. It is due to Kashiwara, though we could not find an appropriate reference in the literature.

Lemma 1.3.27. *Let X be a smooth algebraic variety. Assume $\Lambda \subset T^*X$ is a closed irreducible (possibly singular) algebraic $\mathbb{C}^*$-stable lagrangian subvariety. Write Y for the smooth part of $\pi(\Lambda)$, where $\pi : T^*X \to X$ is the projection. Then $\Lambda = \overline{T^*_Y X}$.*

Proof. It is clear by construction of Y that $\Lambda \subset \pi^{-1}(\overline{Y})$. Since Λ is $\mathbb{C}^*$-stable, Eu is tangent to Λ^{reg}, the regular locus of Λ. Further, Λ being lagrangian, for any vector ξ tangent to Λ^{reg}, we have

$$0 = \omega(Eu, \xi) = \lambda(\xi), \qquad \forall \xi \in T\Lambda^{reg}$$

and therefore $\lambda|_\Lambda \equiv 0$. Fix $\alpha \in \Lambda^{reg}$ such that $y = \pi(\alpha) \in Y$. This implies, by the definition of the 1-form λ, that the covector α vanishes on the image of the map

$$\pi_* : T_\alpha \Lambda \to T_y Y.$$

Furthermore, the Bertini-Sard lemma implies that there exists a Zariski open dense subset $\Lambda^{generic} \subset \Lambda^{reg}$ such that this map is surjective at any point $\alpha \in \Lambda^{generic}$. Hence $\alpha(T_y Y) = 0$, whence $\alpha \in T^*_Y X$. This yields an inclusion

$$\Lambda^{generic} \subset T^*_Y X.$$

Both sets here are irreducible varieties (for Λ is irreducible) of the same dimension. Therefore they have the same closure. Hence, we have $\Lambda = \overline{\Lambda^{generic}} = \overline{T^*_Y X}$. ∎

APPLICATION. Let V be a finite dimensional vector space, $\mathbb{P}(V)$ the corresponding projective space, and $\mathbb{P}(V^*)$ the projectivization of the dual. Let $G \subset \mathrm{PGL}(V)$ be an algebraic subgroup of the group of projective transformations of $\mathbb{P}(V)$.

Theorem 1.3.28. [Pi] *Assume that G has finitely many orbits on $\mathbb{P}(V)$. There is a natural bijection between the G-orbits on $\mathbb{P}(V)$ and the G-orbits on $\mathbb{P}(V^*)$.*

Proof. Let $\tilde{G}$ be the inverse image of G under the projection $\mathrm{GL}(V) \to \mathrm{PGL}(V)$. Thus $\tilde{G}$ is a subgroup of $\mathrm{GL}(V)$ containing the scalars, that is to say, containing the matrices consisting of a scalar times the identity. It suffices to set up a bijection between $\tilde{G}$-orbits in V and V^*.

We have canonical isomorphisms $T^*V = V \times V^* = T^*(V^*)$; let p_V and p_{V^*} denote the 1st and 2nd projections of $V \times V^*$ respectively. Observe that $V \times V^*$ is a $\mathbb{C}^* \times \mathbb{C}^*$-variety, the first copy of $\mathbb{C}^*$ acting on V and the second on V^* by scalar multiplication.

Any $\tilde{G}$-orbit $\mathbb{O} \subset V^*$ is a cone, hence $T^*_{\mathbb{O}}(V^*)$ is a $\mathbb{C}^* \times \mathbb{C}^*$-stable subvariety of $V \times V^*$. Let $\tilde{\mathbb{O}}$ denote the closure in V of the set $p_V(T^*_{\mathbb{O}}(V^*))$. We claim:

(a) $\tilde{\mathbb{O}}$ is the closure of a single $\tilde{G}$-orbit $\mathbb{O}^\vee \subset V$.
(b) The orbit $\mathbb{O}$ can be recovered from the orbit $\mathbb{O}^\vee$.

To prove part (a), recall that the number of $\tilde{G}$-orbits in V is finite, by assumption. We have the following simple result

Lemma 1.3.29. *Let G be a connected algebraic group acting on an algebraic variety X. Then any irreducible G-stable algebraic subvariety of X is the closure of a G-orbit.*

Proof. Let Y be this G-stable subvariety, let $\mathbb{O}$ be an orbit of maximal dimension contained in Y. Since $\mathbb{O}$ cannot be contained in the closure of any other orbit $\mathbb{O}' \subset Y$, and there are only finitely many orbits in Y, we conclude that $\mathbb{O}$ is an open subset of Y (in the Zariski topology). It follows that $\overline{\mathbb{O}}$, the closure of $\mathbb{O}$, is an irreducible component of Y. Since Y is itself irreducible we get $Y = \overline{\mathbb{O}}$. ∎

The lemma implies that $\tilde{\mathbb{O}}$ (notation of the claim before the lemma), being an irreducible $\tilde{G}$-stable subvariety of V, is the closure of a single orbit. This proves claim (a). To prove (b), view $\overline{T^*_{\mathbb{O}}(V^*)}$ as an irreducible $\mathbb{C}^*$-stable lagrangian subvariety of T^*V. By Lemma 1.3.27, we have $\overline{T^*_{\mathbb{O}}(V^*)} = \overline{T^*_Y V}$, where Y is the smooth locus of the image of $\overline{T^*_{\mathbb{O}}(V^*)}$ under the projection

$p_V : V \times V^* \to V$. This image is nothing but $\tilde{\mathbb{O}}$. Hence $Y = \mathbb{O}^\vee$, and we obtain

$$\overline{T^*_{\mathbb{O}}(V^*)} = \overline{T^*_{\mathbb{O}^\vee}(V)}$$

Observe now that this equation is symmetric with respect to $\mathbb{O}$ and $\mathbb{O}^\vee$. Applying Lemma 1.3.27 once again, we find similarly that $\mathbb{O}$ is the smooth locus of the image of $\overline{T^*_{\mathbb{O}^\vee}(V)}$ under the projection $p_{V^*} : V \times V^* \to V^*$. Thus, the assignment $\mathbb{O} \mapsto \mathbb{O}^\vee$ is the bijection we are seeking. ∎

Proof of the following result requires a bit of algebraic geometry and will be sketched in 1.5 below.

Proposition 1.3.30. *Let M be a smooth algebraic symplectic variety and Z a possibly singular isotropic (reduced) algebraic subvariety of M. Then any subvariety of Z is isotropic again.*

The proposition is obvious if Z is a submanifold of M. The point is that the claim holds for a subvariety contained in the singular locus of Z.

1.4 The Moment Map

Let (M, ω) be a symplectic manifold. We have the following exact sequence first considered by Kostant [Ko4]:

$$0 \longrightarrow \begin{matrix}\textit{constant}\\ \textit{functions}\end{matrix} \longrightarrow \mathcal{O}(M) \xrightarrow{\ \partial\ } \begin{matrix}\textit{Symplectic}\\ \textit{vector fields on } M\end{matrix}$$

where the map ∂ sends a function f to the vector field ξ_f. Note that this map need not be surjective. Indeed, the Cartan homotopy formula shows that *a vector field ξ is symplectic (i.e. $L_\xi \omega = 0$) if and only if the 1-form $i_\xi \omega$ is closed.* Notice that if $\xi = \xi_f$ for some $f \in \mathcal{O}(X)$, then $i_{\xi_f}\omega = -df$ is an *exact* form. This way one obtains an isomorphism

$$\mathrm{Coker}(\partial) \simeq \{\textit{closed 1-forms}\}/\{\textit{exact 1-forms}\}.$$

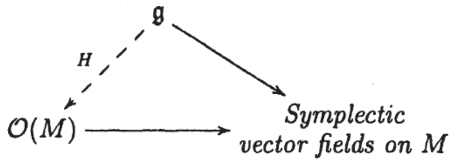
Thus, in the C^∞-setup, for instance, we get $\mathrm{Coker}(\partial) \simeq H^1(M)$, the first de Rham cohomology of M. Thus, in the real case, we get a 4-term exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}(M) \longrightarrow \begin{matrix}\textit{Symplectic}\\ \textit{vector fields}\\ \textit{on } M\end{matrix} \longrightarrow H^1(M, \mathbb{R}) \longrightarrow 0.$$

Suppose that a Lie group G acts on M, preserving the symplectic form, that is $\omega(x, y) = \omega(gx, gy)$ for all $x, y \in T_m M$, $m \in M$ and $g \in G$. The infinitesimal G-action gives a Lie algebra homomorphism

$$\mathfrak{g} := \operatorname{Lie} G \longrightarrow \begin{array}{c}\textit{Symplectic}\\ \textit{vector fields on } M\end{array}$$

Definition 1.4.1. ([Ko4]) A symplectic G-action is said to be *Hamiltonian* if a Lie algebra homomorphism $H : \mathfrak{g} \to \mathcal{O}(M)$, $x \mapsto H_x$ is given, making the following diagram of Lie algebra maps commute:



In other words a symplectic G-action is Hamiltonian if the Lie algebra homomorphism from $\mathfrak{g}$ to symplectic vector fields lifts to $\mathcal{O}(M)$. In case of a Hamiltonian G-action, we assume the Lie algebra lifting $\mathfrak{g} \to \mathcal{O}(M)$ to be fixed once and for all. This map $H : \mathfrak{g} \ni x \mapsto H_x \in \mathcal{O}(M)$ is called the Hamiltonian. We may view H as a function on the cartesian product $M \times \mathfrak{g}$, i.e., as a function in 2 variables. Define the moment map $\mu : M \to \mathfrak{g}^*$ by assigning to $m \in M$ the linear function $\mu(m) : \mathfrak{g} \to \mathbb{C}$, $x \mapsto H_x(m)$, so that $\mu(m)(x) = H_x(m)$.

Lemma 1.4.2. [Ko4] *(i) For any $x \in \mathfrak{g}$ we have $H_x = \mu^* x$, where $\mu^* x$ denotes the pull-back to M of a linear function on $\mathfrak{g}^*$.*

(ii) The map

$$\mu^* : \mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(M)$$

induced by $\mu : M \to \mathfrak{g}^*$ commutes with the Poisson structure.

(iii) If the group G is connected then the moment map μ is G-equivariant (relative to the coadjoint action on $\mathfrak{g}^*$).

Proof. Claim of part (i) is essentially the definition of the moment map. Indeed, we have to show that the following two functions on M are equal: $m \mapsto H_x(m)$ and $m \mapsto \langle \mu(m), x\rangle$. But by definition we have $\langle \mu(m), x\rangle = \mu(m)(x) = H_x(m)$, and the claim follows.

To prove the second claim, it suffices by our usual argument, to verify the assertion on linear functions, that is, elements of $\mathfrak{g}$. For $x, y \in \mathfrak{g}$ we want to check that

$$\{\mu^* x, \mu^* y\} = \mu^*[x, y] = \mu^*\{x, y\}.$$

The first equality here holds since $x \mapsto H_x$ is a Lie algebra homomorphism; the second follows from the definition of the Poisson bracket $\{x, y\}$.

To prove the last claim, write ξ_x for the vector field on M corresponding to the infinitesimal action of $x \in \mathfrak{g}$ on M. Pickup $m \in M$, let $\lambda = \mu(m)$, and let $\mu_* : T_m M \to \mathfrak{g}^*$ denote the differential of the moment map at the point m. The "infinitesimal" Lie algebra version of the G-equivariance of the moment map reads

$$(1.4.3) \qquad \mu_*(\xi_x) = \mathrm{ad}^* x(\lambda), \quad \forall m \in M\,,\, x \in \mathfrak{g}\,.$$

To prove this equation holds, it suffices to check that any linear function on $\mathfrak{g}^*$ takes the same value on both LHS and RHS. For the LHS and any $y \in \mathfrak{g}$, viewed as a linear function on $\mathfrak{g}^*$, we have

$$\langle y, \mu_*(\xi_x)\rangle = \xi_x(\mu^* y) = \{H_x, \mu^* y\} = \{\mu^* x, \mu^* y\}\,,$$

where the last equality is due to part (i). For the RHS of (1.4.3) we find using part (ii):

$$\langle y, \mathrm{ad}^* x(\lambda)\rangle = \lambda([x,y]) = (\mu^*[x,y])(m) = \{\mu^* x, \mu^* y\}(m)\,.$$

This proves (1.4.3), hence, shows that μ is "infinitesimally" G-equivariant. It remains to observe that, for a *connected* Lie group, "infinitesimal" G-equivariance implies G-equivariance. ∎

Example 1.4.4. Let $M = \mathbb{C}^2$ with coordinates (p,q), and $\omega = dp \wedge dq$. Set

$$(1.4.5) \qquad G = \mathrm{SL}_2(\mathbb{C}) \quad , \quad \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a,b,c \in \mathbb{C} \right\}.$$

Then G acts on $M = \mathbb{C}^2$ in a natural way. The induced $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$-action is given by the following symplectic vector fields on $\mathbb{C}^2$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto q\frac{\partial}{\partial p} = \xi_{q^2/2}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto p\frac{\partial}{\partial q} = \xi_{-p^2/2},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} = \xi_{pq}.$$

This action is clearly Hamiltonian with the Hamiltonian functions

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto q^2/2, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto -p^2/2, \quad , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto pq,$$

To compute the moment map $\mu : M \to \mathfrak{sl}_2(\mathbb{C})^*$ explicitly, we identify $\mathfrak{sl}_2(\mathbb{C})^*$ and $\mathfrak{sl}_2(\mathbb{C})$ via the non-degenerate bilinear form: $(A,B) \mapsto \mathrm{Tr}(A \cdot B)$. Then the above formulas yield

$$\mu : (p,q) \mapsto \frac{1}{2}\begin{pmatrix} pq & q^2 \\ -p^2 & -pq \end{pmatrix}.$$

Notice that this matrix has zero determinant, and hence is nilpotent. Therefore μ maps $\mathbb{C}^2$ into the set of nilpotent matrices. This map yields a

2-fold covering of the nilpotent cone in $\mathfrak{sl}_2(\mathbb{C})$ ramified at the origin, which illustrates the phenomena to be studied in more detail in Chapter 3.

The above example is a special case of the following result (cf., e.g. [GS2]).

Proposition 1.4.6. *Let (V, ω) be a symplectic vector space. Then the natural action on V of the symplectic group $Sp(V)$ is Hamiltonian with quadratic Hamiltonian functions given by*

$$H_A(v) = 1/2\, \omega(A \cdot v, v), \qquad A \in \mathfrak{sp}(V)\,,\ v \in V.$$

Proof. Let $A \in \mathfrak{sp}(V)$. Set $H = 1/2\,\omega(A \cdot v, v)$ and let $d_v H$ denote the differential of the function H at a point $v \in V$. We have to check that, for any vector $w \in V$, the following equation holds: $d_v H(w) = \omega(A \cdot v, w)$. We calculate the differential of the quadratic function $H = H_A : v \mapsto 1/2\,\omega(A \cdot v, v)$ at $v \in V$. One finds

$$dH_v(w) = 1/2\,\omega(A \cdot v, w) + 1/2\,\omega(A \cdot w, v) = \omega(A \cdot v, w),$$

(the last equality is due to the skew-symmetry of A). This proves the claim. Thus, it remains only to show that the assignment $A \mapsto H_A$ is a Lie algebra homomorphism. But this amounts essentially to Lemma 1.3.5 which says that the Poisson bracket on the space of quadratic polynomials on V corresponds to the Lie algebra bracket on $\mathfrak{sp}(V)$. ∎

Example 1.4.7. Let $M = T^*X$ and let G act on X. We have Lie algebra homomorphisms

$$\mathfrak{g} \longrightarrow \begin{matrix} \textit{Vector fields} \\ \textit{on } X \end{matrix} \longrightarrow \begin{matrix} \textit{Vector fields} \\ \textit{on } T^*X \end{matrix}\ , \qquad x \mapsto u_x \mapsto \tilde{u}_x$$

The G-action on T^*X arising in this way is clearly symplectic, since any diffeomorphism of X induces a symplectic diffeomorphism of T^*X. Moreover, Lemma 1.3.14 implies

Proposition 1.4.8. *For any G-manifold X, the G action on T^*X is always Hamiltonian with Hamiltonian*

$$x \mapsto H_x = \lambda(\tilde{u}_x) \in \mathcal{O}(T^*X),$$

where λ is the canonical 1-form on T^*X.

Let X be a G-manifold. The Lie algebra homomorphism $\mathfrak{g} \to \{$*vector fields on* $X\}$ given by the "infinitesimal action" can be uniquely extended, by the universal property of the enveloping algebra $\mathcal{U}\mathfrak{g}$, to an associative algebra homomorphism

$$a : \mathcal{U}\mathfrak{g} \to \mathcal{D}(X) = \textit{regular differential operators on } X.$$

Recall that taking differential operators of order $\leq i$, $i = 0, 1, 2, \ldots$ gives a natural increasing filtration on the algebra $\mathcal{D}(X)$ of differential operators. Similarly, there is the standard increasing filtration $\mathbb{C} = \mathcal{U}_0\mathfrak{g} \subset \mathcal{U}_1\mathfrak{g} \subset \cdots$ on the enveloping algebra, where $\mathcal{U}_i\mathfrak{g}$ is the finite-dimensional subspace spanned by all the monomials $x \cdot y \cdot \ldots \cdot z$, $x, y, \ldots, z \in \mathfrak{g}$ of length $\leq i$. Now the map $a : \mathcal{U}\mathfrak{g} \to \mathcal{D}(X)$ is clearly filtration preserving. The leftmost column of the diagram below corresponds to the associated graded map.

$$\begin{array}{ccccc} \mathrm{gr}\,\mathcal{U}\mathfrak{g} & = & S\mathfrak{g} & = & \mathbb{C}[\mathfrak{g}^*] \\ {\scriptstyle \mathrm{gr}\,a}\downarrow & & & & \downarrow{\scriptstyle \mu^*} \\ \mathrm{gr}\,\mathcal{D}(X) & & = & & \mathcal{O}(T^*X) \end{array}$$

Using the identifications provided by the horizontal isomorphisms in the diagram, we get an algebra homomorphism $\mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(T^*X)$ depicted in the rightmost vertical column. This map turns out to be induced by the moment map: $T^*X \to \mathfrak{g}^*$ of the underlying varieties. Thus, the map $\mathcal{U}\mathfrak{g} \to \mathcal{D}(X)$ may be thought of as a "quantization" of the moment map above.

We need an explicit description of the moment map in a special case. Let G be a Lie group and $P \subset G$ a Lie subgroup. Let $\mathfrak{p} = \mathrm{Lie}\,P$ and write $\mathfrak{p}^\perp$ for the annihilator of vector subspace $\mathfrak{p}$ in $\mathfrak{g}^*$. By Claim 1.4.8 the left G-action on G/P induces a Hamiltonian G-action on $T^*(G/P)$. The latter gives rise to the moment map

$$\mu : T^*(G/P) \longrightarrow \mathfrak{g}^* .$$

We would like to calculate μ explicitly. We first describe the cotangent bundle to G/P.

Lemma 1.4.9. *There is a natural G-equivariant isomorphism*

$$T^*(G/P) \simeq G \times_P \mathfrak{p}^\perp,$$

where P acts on $\mathfrak{p}^\perp$ by the coadjoint action.

Proof. Let $e = 1 \cdot P/P \in G/P$ be the base point. We have $T_e(G/P) = \mathfrak{g}/\mathfrak{p}$ and $T_e^*(G/P) = (\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp \subset \mathfrak{g}^*$. It follows that, for any $g \in G$

$$T^*_{g\cdot e}(G/P) = g\mathfrak{p}^\perp g^{-1}.$$

This shows that the vector bundles $T^*(G/P)$ and $G \times_P \mathfrak{p}^\perp$ have the same fibers at each point of G/P, hence are equal as sets. To prove that they are isomorphic as manifolds, one can refine the argument as follows.

Consider the trivial bundle $\mathfrak{g}_{G/P} = G/P \times \mathfrak{g}$ on G/P with fiber $\mathfrak{g}$. The infinitesimal $\mathfrak{g}$-action on G/P gives rise to a vector bundle morphism

$\mathfrak{g}_{G/P} \longrightarrow T(G/P)$. It is clear that the kernel of this morphism is the subbundle $E \subset \mathfrak{g}_{G/P}$ whose fiber at a point $x \in G/P$ is the isotropy Lie algebra $\mathfrak{p}_x \subset \mathfrak{g}$ at x. This gives an isomorphism $T(G/P) \simeq \mathfrak{g}_{G/P}/E$. Further, the description of the fibers of E gives an isomorphism $E \simeq G \times_P (\mathfrak{g}/\mathfrak{p})$. Hence, $T(G/P) \simeq G \times_P (\mathfrak{g}/\mathfrak{p})$, and the result follows by taking the dual on each side. ∎

Observe next that there are two "types" of tangent vectors to $T^*(G/P)$. First there are "vertical" vectors, i.e., vectors which are tangent to the fibers of the projection $T^*(G/P) \to G/P$; since these fibers are themselves vector spaces, we may identify these vertical tangent vectors with elements of the fibers of $T^*(G/P)$. Second, there are tangent vectors of the form ξ_x, $x \in \mathfrak{g}$. Note that $g\mathfrak{p}g^{-1}$ is the Lie algebra of the isotropy group of the point $g \cdot e \in G/P$. Hence, for any $x \in g\mathfrak{p}g^{-1}$ the vector ξ_x is tangent to the fiber $T^*_{g \cdot e}(G/P)$ of $T^*(G/P)$, hence, is a vertical vector.

Proposition 1.4.10. *Under the isomorphism $T^*(G/P) \simeq G \times_P \mathfrak{p}^\perp$ the moment map μ is given explicitly by*

$$(g, \alpha) \mapsto g\alpha g^{-1}, \quad g \in G,\, \alpha \in \mathfrak{p}^\perp.$$

Note that this map is well-defined on $G \times_P \mathfrak{p}^\perp$, a quotient of $G \times \mathfrak{p}^\perp$.

Proof. The moment map sends (g, α) to the linear function $\mu(g, \alpha) : \mathfrak{g} \to \mathbb{C}$ given by $x \mapsto H_x(g, \alpha)$, $x \in \mathfrak{g}$, where H_x is the Hamiltonian for x. By Lemma 1.3.14 we have $H_x = \lambda(\tilde{x})$. The differential of the projection $\pi : T^*(G/P) \to G/P$ takes $\tilde{x}$ to x. Hence, we find

$$\lambda(\tilde{x})(g, \alpha) = gpg^{-1}(\pi_* \tilde{x}) = gpg^{-1}(x).$$

Thus, $\mu(g, \alpha)(x) = gpg^{-1}(x)$ as was shown. ∎

It is often useful in concrete computations to also have an explicit description of the canonical symplectic form, ω, on $T^*(G/P)$. This is provided by

Proposition 1.4.11. *The canonical symplectic form ω is given by the formulas:*

(a) $\omega(\alpha_1, \alpha_2) = 0$, *for any vertical vectors $\alpha_1, \alpha_2 \in T^*_g(G/P)$.*
(b) $\omega(\xi_x, \xi_y)|_\alpha = \alpha(g[x, y]g^{-1})$ *for $x, y \in \mathfrak{g}$, $\alpha \in T^*_{g \cdot e}(G/P)$, a covector.*
(c) $\omega(\beta, \xi_x)|_\alpha = \beta(gxg^{-1})$ *for any vertical $\beta \in T^*_{g \cdot e}(G/P)$ viewed as a tangent vector to $T^*(G/P)$ at $\alpha \in T^*_{g \cdot e}(G/P)$.*

Proof. Given a 1-form α on $X = G/P$, let $\tilde{\alpha}$ denote the vertical vector field on T^*X whose restriction to any fiber of $T^*_x X$ is the constant vector field α_x, the value of α at x.

Proving (a) amounts to showing that $\omega(\tilde{\alpha}_1, \tilde{\alpha}_2) = 0$ for any 1-forms α_1 and α_2. Now the canonical 1-form λ on T^*X vanishes on any vertical vector field. Hence $\lambda(\tilde{\alpha}_1) = \lambda(\tilde{\alpha}_2) = 0$. Furthermore, the field $[\tilde{\alpha}_1, \tilde{\alpha}_2]$ is also vertical. Hence, $\lambda([\tilde{\alpha}_1, \tilde{\alpha}_2]) = 0$ and part (a) follows from

$$\omega(\tilde{\alpha}_1, \tilde{\alpha}_2) = d\lambda(\tilde{\alpha}_1, \tilde{\alpha}_2) = \tilde{\alpha}_1 \cdot \lambda(\tilde{\alpha}_2) - \tilde{\alpha}_2 \cdot \lambda(\tilde{\alpha}_1) - \lambda([\tilde{\alpha}_1, \tilde{\alpha}_2]) = 0 + 0 + 0.$$

The left hand side of the equality in (b) can be rewritten as $\{\mu^* x, \mu^* y\}(\alpha)$ and the right hand side as $\mu^*([x, y])(\alpha)$. The claim now follows from Lemma 1.4.2.

To prove (c) observe first that $[\xi_x, \tilde{\beta}] = [\tilde{x}, \tilde{\beta}] = L_{\tilde{x}}\tilde{\beta} = \widetilde{(L_x\beta)}$ (the first equality is due to Lemma 1.3.14). Then we obtain

$$\omega(\tilde{\beta}, \xi_x) = (d\lambda)(\tilde{\beta}, \xi_x) = \tilde{\beta} \cdot \lambda(\xi_x) - \xi_x \cdot \lambda(\tilde{\beta}) - \lambda(\widetilde{(L_x\beta)}) = \tilde{\beta} \cdot \lambda(\xi_x),$$

for λ vanishes on vertical vector fields $\tilde{\beta}$ and $\widetilde{(L_x\beta)}$. To compute $\tilde{\beta} \cdot \lambda(\xi_x)$ note that the restriction of $\lambda(\xi_x)$ to a fiber $T^*_x X$ is a linear function: $\alpha \mapsto \lambda(\xi_x)(\alpha) = \lambda(\tilde{x})(\alpha) = \alpha(x)$. Hence the derivative of that function in the direction of the constant vector field β_x is the constant function $\beta_x(\xi_x)$. ∎

The following elementary result will be frequently used in the future.

Lemma 1.4.12. *Let P be an algebraic group with Lie algebra $\mathfrak{p}$. Let V be a finite-dimensional representation of P and $E \subset V$ a P-stable linear subspace. Then*

(i) If P is connected, then for $v \in V$, the following conditions are equivalent:

(1) *The affine linear subspace $v + E \subset V$ is P-stable;*
(2) *We have $\mathfrak{p} \cdot v \subset E$, that is the image of $\mathfrak{p}$ under the induced "infinitesimal" Lie algebra action-map $\mathfrak{p} \to V$, $x \mapsto x \cdot v$ is contained in E.*

(ii) Moreover, if the linear map $\mathfrak{p} \to E$, $x \mapsto x \cdot v$ is surjective, then $P \cdot v$, the P-orbit of v is a Zariski open dense subset of $v + E$.

Proof. If condition (1) holds, then we have $P \cdot v \subset v + E$. Differentiating this condition at the identity of the group P yields $\mathfrak{p} \cdot v \subset E$, hence, condition (2). Conversely, assume condition (2) holds. The tangent space to $v + E$ at any point $u \in v + E$ clearly gets identified with E. Given $x \in \mathfrak{p}$, let ξ_x be the vector field on V arising from the action of x on V. Then, for any $u \in v + E$, we find

(1.4.13)
$$\xi_x(u) = x \cdot u \in x(v + E) = x \cdot v + x \cdot E \subset \mathfrak{p} \cdot v + x \cdot E \subset E,$$

since $\mathfrak{p} \cdot v \subset E$ by (2) and $\mathfrak{p} \cdot E \subset E$ by the P-invariance of E. Formula (1.4.13) shows that the vector field ξ_x is tangent to the subspace $(v + E)$

at any of its points. Hence, this subspace is stable under the action of a small neighborhood of the identity in P. Since P is connected, it follows that $v+E$ is P-stable.

To prove (ii) consider a morphism of algebraic varieties $f : P \to v+E$ given by $p \mapsto p \cdot v$ (which is well-defined due to part (i)). The image, $f(P)$, is connected and is known to be a locally closed subset of $v+E$ in the Zariski topology. Observe that the differential of the map f at the identity is the map $\mathfrak{p} \mapsto V$ given by the linear $\mathfrak{p}$-action on v as in (2). If the differential is surjective, then $f(P)$ contains an open neighborhood (in the usual topology) of v in $v+E$ by the implicit function theorem. Hence, $f(P)$ cannot be contained in any proper closed algebraic subvariety of $v+E$. Hence $f(P)$ is an irreducible Zariski open subset of $v+E$. Since $v+E$ is itself irreducible, it follows that $f(P)$ is dense in $v+E$. ∎

We conclude this section with the following generalization of Proposition 1.4.11.

Proposition 1.4.14. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, P a closed connected subgroup of G with Lie algebra $\mathfrak{p}$. Let λ be a linear function on $\mathfrak{g}$ such that $\lambda|_{[\mathfrak{p},\mathfrak{p}]} = 0$. Then*

(1) The affine linear subspace $\lambda + \mathfrak{p}^{\perp} \subset \mathfrak{g}^*$ is stable under the coadjoint P-action of $\mathfrak{g}^*$.

(2) The space $G \times_P (\lambda + \mathfrak{p}^{\perp})$ has the natural G-invariant symplectic structure, ω, it is given by formulas, cf. (1.4.11)

(1) $\omega(\alpha_1, \alpha_2) = 0$ *if α_1, α_2 are vertical, i.e., tangent to the fibers of the projection $\pi : G \times_P (\lambda + \mathfrak{p}^{\perp}) \to G/P$.*
(2) $\omega(\xi_x, \xi_y)|_\alpha = \alpha(g[x,y]g^{-1})$ *for any point $(g, \alpha) \in G \times_P (\lambda + \mathfrak{p}^{\perp})$ and any $x \in \mathfrak{g}$.*
(3) $\omega(\beta, \xi_x) = \beta(gxg^{-1})$ *for any β tangent to $gP \times_P (\lambda + \mathfrak{p}^{\perp})$. In particular, the fibers of the projection π are lagrangian affine subspaces.*

Proof. Part (1) follows from Lemma 1.4.12. Proof of part (2) is similar to the proof of Proposition 1.4.11 and is left to the reader. ∎

The first projection $\pi : G \times_P (\lambda + \mathfrak{p}^{\perp}) \to G/P$ clearly has a natural structure of an affine fibration, i.e., a locally trivial fibration with canonical affine linear space strucure on every fiber (put differently, the structure group of the fibration is reduced from the whole group of diffeomorphisms of $\lambda + \mathfrak{p}^{\perp}$ to the subgroup of *affine* automorphisms). It is clear also that we have

$$\dim(\lambda + \mathfrak{p}^{\perp}) = \dim(\mathfrak{p}^{\perp}) = \dim \mathfrak{g}/\mathfrak{p} = \dim G/P.$$

We see that the fiber dimension equals half the dimension of the total space of the fibration. Furthermore, looking at formulas of Proposition

1.4.14, one finds that the symplectic 2-form ω vanishes on each fiber. Thus, all fibers are lagrangian submanifolds. For this reason one calls $\pi : G \times_P (\lambda + \mathfrak{p}^\perp) \to G/P$ an affine lagrangian fibration. Motivated by comparison with Proposition 1.4.11, the space $G \times_P (\lambda + \mathfrak{p}^\perp)$ should be thought of as a "twisted cotangent bundle" on G/P.

We mention the following interesting result about general lagrangian fibrations. If M is a symplectic manifold and $p : M \to B$ a smooth fibration with lagrangian fibers, then it is shown in [AG] that every fiber of the fibration has a natural affine linear structure, i.e., has a canonical infinitesimal transitive free action of the additive group of a vector space. It follows that any lagrangian fibration with connected and simply connected fibers is isomorphic (as lagrangian fibration) to an open subset of an appropriate twisted cotangent bundle.

1.5 Coisotropic Subvarieties

Let (M, ω) be a symplectic manifold with Poisson bracket $\{\ ,\ \}$ on $\mathcal{O}(M)$. Recall that a subvariety $\Sigma \subset M$ is called coisotropic if the tangent space at any smooth point $m \in \Sigma$ is a coisotropic subspace of the whole tangent space, i.e.,

$$T_m\Sigma \supseteq T_m\Sigma^{\perp_\omega},$$

where $\perp_\omega$ stands for the annihilator in T_mM with respect to the symplectic form. Let $\mathcal{J}_\Sigma \subset \mathcal{O}(M)$ be the defining ideal of Σ.

Proposition 1.5.1. *(cf.,* [Bj], [GS1]*) The subvariety Σ is coisotropic if and only if $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$, that is, if and only if $\mathcal{J}_\Sigma$ is a Lie subalgebra (not necessarily a Lie ideal) in $\mathcal{O}(M)$.*

Proof. Suppose $\{\mathcal{J}_\Sigma, \mathcal{J}_\Sigma\} \subset \mathcal{J}_\Sigma$. This occurs if and only if the following implication holds

$$f, g \in \mathcal{J}_\Sigma \Rightarrow \omega(\xi_f, \xi_g)(m) \equiv 0, \qquad \forall m \in \Sigma^{reg}. \tag{1.5.2}$$

Let $f \in \mathcal{J}_\Sigma$. Write $W = T_m\Sigma$ and $V = T_mM$ for the tangent spaces at a smooth point $m \in \Sigma$. The differential df clearly vanishes on $W = T_m\Sigma$, hence $df \in W^\perp$ where $W^\perp \subset V^*$. Therefore $\xi_f \in W^{\perp_\omega} \subset V$. Furthermore, the vectors of the form ξ_f, $f \in \mathcal{J}_\Sigma$, span $W^{\perp_\omega}$. This combined with (1.5.2) implies

$$\omega(W^{\perp_\omega}, W^{\perp_\omega}) \equiv 0.$$

But this occurs if and only if $W^{\perp_\omega}$ is isotropic which occurs if and only if W is coisotropic. This proves the "if" part of the proposition. The argument can be reversed to complete the proof. ∎

Let $\Sigma \subset M$ be a smooth coisotropic subvariety and $m \in \Sigma$. The restriction of the symplectic form ω to $T_m\Sigma$ is a degenerate 2-form, and one checks easily that

$$\mathrm{Rad}(\omega|_{T_m\Sigma}) = (T_m\Sigma)^{\perp_\omega} \subset T_m\Sigma.$$

Thus the radicals of the form ω at each fiber of the tangent bundle assembled together form the vector subbundle $(T\Sigma)^{\perp_\omega} \subset T\Sigma$ of the tangent bundle $T\Sigma$. We claim that this subbundle is *integrable*, i.e., we have

Proposition 1.5.3. *There exists a foliation on Σ such that, for any $m \in \Sigma$, the space $(T_m\Sigma)^{\perp_\omega}$, the fiber of the subbundle given above is equal to the tangent space at m to the leaf of the foliation.*

The foliation arising in this way is called the *0-foliation* on the coisotropic subvariety Σ. Its existence is guaranteed by the following general criterion:

Theorem 1.5.4. *(Frobenius Integrability Theorem) Let $E \subset T\Sigma$ be a vector subbundle of the tangent bundle on a manifold Σ. Then E is integrable if and only if sections of E form a Lie subalgebra, i.e., for any sections ξ, η of E, viewed as vector fields on Σ, we have $[\xi, \eta] \in E$.*

In fact it suffices, for integrability, to check this only for all pairs within a family of sections of E that span the fibers of E at every point $m \in \Sigma$, and not necessarily for all pairs (ξ, η).

Proof of Proposition 1.5.3. Observe that

(1.5.5) $\quad f|_\Sigma \equiv const \Leftrightarrow (\xi_f)|_\Sigma$ belongs to the subbundle $(T_m\Sigma)^{\perp_\omega}$.

Clearly the family of vector fields $\{\xi_f \,,\, f|_\Sigma \equiv const\}$ spans the space $(T_m\Sigma)^{\perp_\omega}$ for any $m \in \Sigma$. Hence, proving integrability amounts to showing that $f|_\Sigma \equiv constant$ and $g|_\Sigma \equiv constant$ implies $[\xi_f, \xi_g] \in (T\Sigma)^{\perp_\omega}$. But this follows from formula (1.5.5) and the equality $[\xi_f, \xi_g] = \xi_{\{f,g\}}$. ∎

Example 1.5.6. Let M be symplectic and let $f \in \mathcal{O}(M)$. Let Σ be the zero variety of f. Suppose that df does not vanish on Σ, so that Σ is a smooth coisotropic codimension 1 subvariety. Then the 0-foliation ("null"-foliation) on Σ is generated by the vector field ξ_f.

The rest of this section is devoted to the proof of the theorem below. This theorem will play an important role in our study of the Springer resolutions in Chapter 3.

Theorem 1.5.7. *Let A be a solvable algebraic group with a Hamiltonian action on a symplectic algebraic variety M. Let $\mathfrak{a} = \mathrm{Lie}\, A$ and let μ be the moment map*

$$\mu : M \longrightarrow \mathfrak{a}^* .$$

Then for any coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$ the set $\mu^{-1}(\mathbb{O})$ is either empty or is a coisotropic subvariety of M.

In the theorem, $\mu^{-1}(\mathbb{O})$, stands for the set-theoretic preimage, which, in the algebro-geometric language, means the *reduced* scheme associated to the scheme-theoretic inverse image, cf. Remark 1.5.8 below and also §2.2.

Remark 1.5.8. For any Lie algebra $\mathfrak{a}$, the defining ideal $\mathcal{J}_{\mathbb{O}} \subset \mathbb{C}[\mathfrak{a}^*]$ of a coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$ is stable under the natural Poisson structure because

$$\{f, g\}_{|\mathbb{O}} = \{f_{|\mathbb{O}}, g_{|\mathbb{O}}\}_{symplectic} = 0$$

if f, g vanish on $\mathbb{O}$ (the first equality follows from Proposition 1.3.21). It follows that the ideal $\mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}} \subset \mathcal{O}(M)$ is stable under the Poisson bracket on M, due to Lemma 1.4.2. The above theorem is equivalent to saying that in the solvable case the *radical*, (see §2.2), $\sqrt{\mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}}}$ is stable with respect to $\{\ ,\ \}$.

Remark 1.5.9. Assume that the orbit $\mathbb{O}$ consists of regular values of the moment map μ, i.e., the differential $d\mu$ is surjective at every point of the inverse image of $\mathbb{O}$. Then $\mathcal{J}_{\mu^{-1}(\mathbb{O})} = \sqrt{\mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}}} = \mathcal{O}(M) \cdot \mu^* \mathcal{J}_{\mathbb{O}}$ and the theorem is well-known (see e.g. [GS2]) and holds without any solvability assumption.

First, we prove some general results that will be used in the proof of the theorem. Let (M, ω) be a symplectic manifold with a Hamiltonian action of a Lie group A. Set $\mathfrak{a} = \mathrm{Lie}\, A$ and let $\mu : M \to \mathfrak{a}^*$ be the moment map. Let $P \in \mathbb{C}[\mathfrak{a}^*]$ and write $\widetilde{P} = \mu^* P$. Then

Lemma 1.5.10. *For a point $m \in M$ let $\alpha = \mu(m) \in \mathfrak{a}^*$. Then*

$$\xi_{\widetilde{P}}(m) = dP(\alpha).$$

Here $(dP)(\alpha) \in \mathfrak{a}$ because it is a linear function on $\mathfrak{a}^*$.

Proof. We will assume first that P is a linear function so that $P = dP = a \in \mathfrak{a}$. Then the statement of the lemma is true, since by definition of the moment map we have

$$\widetilde{P} = \mu^* a \Rightarrow d(\mu^* a) = \omega(\cdot, a).$$

This implies $a = \xi_{\mu^* a} = \xi_{\widetilde{P}}$.

Now we prove the lemma for arbitrary functions. Note that locally

$$P = P(\alpha) + dP(\alpha) + \text{ “higher order terms”}$$

where $P(\alpha)$ is constant and $dP(\alpha)$ is linear. The lemma is trivial for constant functions (these give rise to zero vector fields) so we are done because the higher order terms do not come into play, since both sides of the equality are completely determined by first derivatives of P and $\widetilde{P}$.

Lemma 1.5.11. *Let (V,ω) be a symplectic vector space. A vector subspace $\Sigma \subset V$ is coisotropic if and only if it contains a lagrangian subspace $\Lambda \subset \Sigma$.*

Proof. (i) If $\Sigma \supset \Lambda$, then Λ is lagrangian implies $\Sigma \supset \Sigma^{\perp_\omega}$ because

$$\Sigma \supset \Lambda = \Lambda^{\perp_\omega} \supset \Sigma^{\perp_\omega}.$$

Therefore Σ is coisotropic.

(ii) Assume that Σ is coisotropic. Then $\Sigma \supset \Sigma^{\perp_\omega}$ and $\Sigma/\Sigma^{\perp_\omega}$ is again a symplectic vector space. Choose any lagrangian subspace $\overline{\Lambda} \subset \Sigma/\Sigma^{\perp_\omega}$ and let Λ be the pre-image of $\overline{\Lambda}$ in Σ with respect to $\Sigma \to \Sigma/\Sigma^{\perp_\omega}$. Then $\Lambda \supset \Sigma^{\perp_\omega}$ which implies $\Sigma \supset \Lambda^{\perp_\omega}$. Taking into account that $\overline{\Lambda}^{\perp_\omega} = \overline{\Lambda}$, we obtain $\Lambda = \Lambda^{\perp_\omega}$, and Λ is lagrangian.

Lemma 1.5.12. *Let $N \subset M$ be an irreducible subvariety in the smooth algebraic variety M, and $f \in \mathcal{O}(N)$, a nonconstant regular function. For any $c \in \mathbb{C}$ define the hypersurface $D_c = f^{-1}(c)$ as the set-theoretic (reduced) preimage of c and assume D_0 is nonempty. Then there is a Zariski-open dense subset $D_0^{gen} \subset D_0$ with the following properties:*

D_0^{gen} is contained in the smooth locus of D_0 and for any point $x \in D_0^{gen}$, there exists a sequence of complex numbers $c_1, c_2, \ldots \to 0$ and a sequence of points $x_i \in D_{c_i}$, $i = 1, 2, \ldots$ such that

(a) *$x_i \to x$ (in the ordinary Hausdorff topology), and x_i is a smooth point of the divisor D_{c_i};*
(b) *$T_{x_i}D_{c_i} \to T_xD_0$, where the convergence (in the ordinary topology) takes place in the Grassmannian of $(\dim N - 1)$-planes in TM;*
(c) *The values $c_1, c_2, \ldots$ of the function f are generic in the sense that they can be chosen in the complement to any finite subset of $\mathbb{C}$.*

Proof. It suffices to prove the lemma locally so that we assume N and M are affine and D_0 is irreducible. Let N^{sing} and D_0^{sing} be the singular loci of N and D_0 respectively. There are two cases:

(i) First assume that $D_0 \not\subset N^{sing}$. Then set $D_0^{gen} = D_0 \setminus (N^{sing} \cup D_0^{sing})$, a Zariski-open dense subset of D_0. Let $x \in D_0^{gen}$. Since x is a smooth point of N and the claim of the lemma is local with respect to the ordinary

Hausdorff topology, we may regard N as a holomorphic complex manifold and choose a local chart on N with coordinates $(t_1, \dots, t_n)$ such that $x = 0$. Moreover, since D_0 is locally a codimension 1 smooth subvariety of N, we may assume without loss of generality that $D_0 = \{t_1 = 0\}$. Since D_0 is the zero set of f, the function f viewed as a holomorphic function in the local coordinates t must be of the form $f(t) = t_1^k \cdot g(t)$, where g is a holomorphic function such that $g(0) \neq 0$. Hence, locally one can define a holomorphic function $t \mapsto g(t)^{\frac{1}{n}}$, a branch of the k-th root of g. It is easy to see from the implicit function theorem that the functions $(\tau, t_2, \dots, t_n)$, where $\tau = g(t)^{\frac{1}{n}} \cdot t_1$, form a local chart on N again. In this new chart we have $f(t) = \tau^n$, so that the level sets of f are disjoint unions of hyperplanes $\tau = const$. Thus, the claim of the lemma is clear in this case.

(ii) Assume now that $D_0 \subset N^{sing}$. Since normal varieties are smooth in codimension one [Ha] we can find a Zariski open subset $U \subset N$ such that (a) its normalization $\tilde{U}$ is smooth; and (b) $U \cap D_0$ is dense in D_0.

Remark. The non-expert in algebraic geometry may feel uneasy about using such results as codimension 1 smoothness of normal varieties. Here is another argument which is, hopefully, more convincing intuitively. Let $R(D_0)$ be the field of all rational functions on D_0, let $I \subset \mathcal{O}(N)$ be the defining ideal of the divisor D_0, and $S = \mathcal{O}(N) \setminus I$. Let $\mathcal{O}(N)_S$ denote the localization with respect to the multiplicative set S. Thus, $\mathcal{O}(N)_S$ is a local ring with maximal ideal I_S, the localization of I. We have $\mathcal{O}(N)_S/I_S = R(D_0)$ so that the field $R(D_0)$ may be thought of as the "coordinate ring of the generic point of D_0" and the ring $\mathcal{O}(N)_S$ as the "coordinate ring of a small neighborhood" of that generic point in N. Let $\tilde{R}(D_0)$ be the normalization of $R(D_0)$, cf., e.g., [Ha]. The local ring $R(D_0)$ is 1-dimensional, hence, its normalization, $\tilde{R}(D_0)$ is a regular local ring (it is an elementary fact, see [Ha], that a normal curve is always smooth). In geometric terms this translates into the existence of a Zariski open subset $U \subset N$ with the above specified properties (a)–(b).

We now complete the proof of the lemma. Shrinking U if necessary, one may assume $U \cap D_0$ to be smooth as well. Set $D_0^{gen} = U \cap D_0$, let $\nu : \tilde{U} \to U$ be the normalization map and $\tilde{f} = \nu^* f$ the pullback of the function f. Since $\tilde{U}$ is smooth, the first part of the proof applies to the function $\tilde{f}$. Hence, the lemma holds for $\tilde{f}$. We may now transfer the information from $\tilde{U}$ to U because each component of D_0 is the image of an irreducible component of $\tilde{f}^{-1}(0) = \nu^{-1}(D_0)$, and because the map $\tilde{U} \to M$ is smooth when restricted to $\nu^{-1}(D_0^{gen})$. The lemma follows. ∎

Let A be a solvable Lie group, with Lie algebra $\mathfrak{a}$. Choose a codimension 1 normal subgroup $A_1 \subset A$ and let $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$ be the inclusion of Lie algebras. If we let μ_1 be the corresponding moment map for $\mathfrak{a}_1$ then we have the following commutative diagram (left triangle)

where p is the natural projection induced by the inclusion $\mathfrak{a}_1 \hookrightarrow \mathfrak{a}$. We are interested in the special case where $M = \mathbb{O}$ is a coadjoint orbit in $\mathfrak{a}^*$. In this case the map $\mu : M \to \mathfrak{a}^*$ becomes the tautological inclusion and the above diagram reduces to the right triangle above.

Claim 1.5.13. (cf. [Di]) There are only 2 alternatives.

(1) $\dim p(\mathbb{O}) = \dim \mathbb{O}$. *In this case* $p(\mathbb{O})$ *is a single* A_1*-orbit.*
(2) $\dim p(\mathbb{O}) < \dim \mathbb{O}$. *In this case the dimension of any* A_1*-orbit in* $p(\mathbb{O})$ *equals* $\dim \mathbb{O} - 2$.

Proof. There is a natural A-action on $\mathfrak{a}_1$, hence $\mathfrak{a}_1^*$, since A_1 is normal in A.

Observe that $p(\mathbb{O})$ is an A-stable subvariety of $\mathfrak{a}_1^*$ which implies that $p(\mathbb{O})$ (being the image of an A-orbit) is an A-orbit.

Let $o \in p(\mathbb{O})$. Then $\dim(\mathfrak{a}_1 \cdot o) \geq \dim \mathfrak{a} \cdot o - 1$, since $\dim \mathfrak{a}_1 = \dim \mathfrak{a} - 1$. Hence, $\dim A_1 \cdot o \geq (\dim A \cdot o) - 1 = \dim p(\mathbb{O})$ because $p(\mathbb{O})$ is a single A-orbit. Moreover, all A_1-orbits in $p(\mathbb{O})$ are symplectic manifolds, hence have even dimensions; similarly $\dim \mathbb{O}$ is even. It follows that in alternative (1), A_1-orbits in $p(\mathbb{O})$ cannot have dimension equal to $\dim p(\mathbb{O}) - 1 = \dim \mathbb{O} - 1$, hence they are of dimension equal to $\dim p\,(\mathbb{O})$, hence, $p(\mathbb{O})$ is a single A_1-orbit. Similarly, if $\dim p(\mathbb{O}) = \dim \mathbb{O} - 1$ then A_1-orbits in $p(\mathbb{O})$ cannot have odd dimension $\dim p(\mathbb{O})$, hence, are all of dimension $\dim p(\mathbb{O}) - 1 = \dim \mathbb{O} - 2$. ■

Proof of Theorem 1.5.7. We proceed by induction on $\dim A$. Choose a codimension 1 normal subgroup $A_1 \subset A$.

Assume we have alternative (2) above. In this case $\mathbb{O}$ is an open part of $p^{-1}(p\ (\mathbb{O}))$ so it suffices to prove that $\mu^{-1}(p^{-1}p(\mathbb{O})) = {\mu_1}^{-1}(p(\mathbb{O}))$ is coisotropic. But $p(\mathbb{O})$ is a union of A_1-coadjoint orbits. This implies ${\mu_1}^{-1}(p(\mathbb{O}))$ is coisotropic by induction.

Now assume alternative (1) of claim 1.5.13. By induction $\mu^{-1}(p^{-1}p(\mathbb{O})) = N$ is coisotropic (as a union of coisotropic subvarieties, the preimages of coadjoint orbits in $\mathfrak{a}_1^*$).

This is all right because increasing the dimension of a coisotropic subvariety keeps it coisotropic. Now $\mathbb{O}$ has codimension 1 in $p^{-1}p(\mathbb{O})$. We may argue locally. Let P be a local equation of $\mathbb{O}$, i.e., a function on $p^{-1}p(\mathbb{O})$, such that $P \not\equiv 0$, $P_{|\mathbb{O}} = 0$. This implies

$$\mu^{-1}(\mathbb{O}) = N \cap \{\mu^* P = 0\}.$$

Write f for $\mu^* P$. Since we work locally assume that N is irreducible and f does not identically vanish on N. Put

$$\Sigma_c = N \cap \{f = c\}, \quad c \in \mathbb{C}. \quad \blacksquare$$

Lemma 1.5.14. *For generic $c \in \mathbb{C}$ we have Σ_c is coisotropic.*

Proof. From now on we will write $\perp$ for the annihilator with respect to ω dropping the subscript ω for short. We want to show $(T_m\Sigma_c)^\perp$ is isotropic (where m is a smooth point of Σ). We know that $T_m N^\perp$ is isotropic. Now $\dim \Sigma_c = \dim N - 1$ which implies that $\dim T_m{\Sigma_c}^\perp = \dim T_m N^\perp + 1$. We have

$$T_m\Sigma_c = \{df_{|T_m N} = 0\}.$$

Therefore $T_m{\Sigma_c}^\perp = T_m N^\perp + \mathbb{C}\xi_f$. The space $T_m N^\perp$ is isotropic by induction. Further, the vector field ξ_f is tangent to N by Lemma 1.5.10. Hence we find

$$\begin{aligned} \omega(T_m{\Sigma_c}^\perp, T_m{\Sigma_c}^\perp) &= \omega(T_m N^\perp + \mathbb{C}\xi_f\,, T_m N^\perp + \mathbb{C}\xi_f) \\ &= \omega(T_m N^\perp, T_m N^\perp) + \omega(\mathbb{C}\xi_f, \mathbb{C}\xi_f) + \omega(T_m N^\perp, \mathbb{C}\xi_f) \\ &= 0 + 0 + \omega(T_m N^\perp, \mathbb{C}\xi_f) = 0 + 0 + 0\,. \end{aligned}$$

Thus, we see that $T_m{\Sigma_c}^\perp$ is isotropic, and Lemma 1.5.14 follows. $\blacksquare$

We wish to show that Σ_0 is also coisotropic. By Lemma 1.5.12 choose a sequence $x_i \to x \in \Sigma_0$ such that the lemma holds. Then

$$T_{x_i}\Sigma_{c_i} \to T_{x_0}\Sigma_0$$

in the Grassmannian of $\dim N - 1$ subspaces of TM. By 1.5.11 and 1.5.14 there exist lagrangian subspaces $\Lambda_i \subset T_{x_i}\Sigma_{c_i}$. Choose a subsequence i_k such that $\Lambda_{i_k} \to \Lambda \subset T_{x_0}\Sigma_0$. This is possible because Grassmannians are compact. Then Λ is isotropic since all Λ_{i_k} are lagrangian. Therefore $\dim \Lambda = \dim \Lambda_i$ implies Λ is lagrangian. Now we apply Lemma 1.5.11 again to complete the result. This completes the proof of Theorem 1.5.7.

Example 1.5.15. We now give an example where Theorem 1.5.7 fails because the group A is not solvable.

Let $M = \mathbb{C}^2$, $\omega = dp \wedge dq$ and $A = \mathrm{SL}_2(\mathbb{C})$. The standard $\mathrm{SL}_2(\mathbb{C})$-action on $\mathbb{C}^2$ is Hamiltonian, see Example 1.4.4, and the corresponding moment map has been computed to be

$$\mu : M \to (\mathfrak{sl}_2(\mathbb{C}))^* \simeq \mathbb{C}^3 \quad , \quad (p, q) \mapsto (q^2/2, -p^2/2, pq).$$

The origin in $\mathfrak{sl}_2(\mathbb{C})^*$ constitutes a coadjoint orbit. But $\mu^{-1}(0,0,0) = (0,0)$ is not coisotropic. Thus the solvability condition in Theorem 1.5.7 is really necessary.

Proof 1.5.16 of Proposition 1.3.30. We must show that if Z is isotropic then so is any (reduced) subvariety $N \subset Z$. This is obvious if $\dim N = \dim Z$. Assume $\dim N = \dim Z - 1$. Our claim being local in Z, we may assume without loss of generality that there is a non-constant regular function $f \in \mathcal{O}(Z)$ such that $N = f^{-1}(0)$. Hence we are in a position to apply Lemma 1.5.12.

Let $x \in N$. We must show that T_xN is an isotropic vector subspace in T_xM. Lemma 1.5.12 implies that there exists a sequence $\{x_i\,,\, i = 1, 2, \ldots\}$ of regular points of Z and a sequence of vector spaces $W_i \subset T_{x_i}Z\,,\, i = 1, 2, \ldots$, such that $x_i \to x$ and moreover $W_i \to T_xN$ in an appropriate Grassmannian. Each of the spaces W_i is isotropic since Z is an isotropic subvariety. It follows by continuity that T_xN is also isotropic.

Assume finally that $\dim N < \dim Z - 1$. Then we may find, shrinking N if necessary, a codimension one subvariety $Z' \subset Z$ that contains N. It follows from the argument above that Z' is isotropic. We now complete the proof by induction on the codimension of N in Z using that $\operatorname{codim}_{Z'} N < \operatorname{codim}_Z N$. ∎

We end this section with one more result involving coisotropic subvarieties, the so-called integrability of characteristics theorem ([Ga], [GQS], [Ma], [SKK]). Although not directly related to the subject of this book, this theorem has important applications in representation theory (cf. [Bj], [Gi2], [Jo2]) and is somewhat reminiscent of Theorem 1.5.7.

Let A be a filtered ring such that $\operatorname{gr} A$ is a commutative ring. Let I be a left ideal in A. Form $\operatorname{gr} I \subset \operatorname{gr} A$. This is an ideal, and moreover it is stable under the Poisson bracket (see 1.3.2), i.e.,

$$\{\operatorname{gr} I, \operatorname{gr} I\} \subset \operatorname{gr} I$$

because $x, y \in I$ implies $xy - yx \in I$ (this is the case even though I is only a left ideal.)

Integrability of Characteristics Theorem 1.5.17. *Now assume that* $\operatorname{gr} A$ *is a commutative Noetherian ring. Then* $\sqrt{\operatorname{gr} I}$*, the radical of* $\operatorname{gr} I$*, is stable under the Poisson bracket* $\{\ ,\ \}$.

Remark 1.5.18. If $\operatorname{gr} A = \mathcal{O}(M)$ then the theorem amounts to the claim that the zero variety of $\operatorname{gr} I$ is a coisotropic subvariety.

1.6 Lagrangian Families

In this section we introduce the notion of a coisotropic cone subvariety which is of independent interest, and explain its relation to families of lagrangian subvarieties in a symplectic manifold.

Definition 1.6.1. A *symplectic cone variety* is a symplectic manifold (M, ω) with a vector field ξ on M such that $L_\xi \omega = \omega$.

Remark 1.6.2. It is important not to confuse the vector field ξ in definition 1.6.1 with a symplectic vector field, which would satisfy the condition $L_\xi \omega = 0$.

Example 1.6.3. Let $T^*X = M$ and $\xi = Eu$. Then we have already verified that this is a symplectic cone variety.

Lemma 1.6.4. *The symplectic form on any symplectic cone variety is exact, that is to say $\omega = d\lambda_M$. More precisely, if we set $\lambda_M = i_\xi \omega$ then $\omega = d\lambda_M$.*

Proof. We calculate

$$d\lambda_M = di_\xi \omega = L_\xi \omega - i_\xi d\omega = L_\xi \omega$$

where the second equality is the Cartan homotopy formula and the last is due to $d\omega = 0$. ∎

Let (M, ω) be a symplectic cone variety. A subvariety $\Lambda \subset M$ is called a cone subvariety if ξ is tangent to Λ at any smooth point of Λ.

Let $\Lambda_x, x \in X$ be a family of lagrangian cone subvarieties of M parametrized by a space X. Observe that giving such a family is the same thing as giving the subset

$$\Sigma = \{(m, x) \in M \times X \mid m \in \Lambda_x,\ x \in X\}.$$

with the property that the fibers of the projection $\Sigma \to X$ are lagrangian cone subvarieties of M.

Example 1.6.5. Let $M = T^*X$. Set $\{\Lambda_x = T^*_x X, \quad x \in X\}$. Then $\Sigma \simeq T^*X$.

Theorem 1.6.6. *(Resolution of lagrangian families) Suppose that M is a symplectic cone variety, X is a manifold, and $\Sigma \subset M \times X$ is a submanifold such that the projections $p_M : \Sigma \to M$ and $p_X : \Sigma \to X$ to the first and second factors are smooth fibrations with surjective differentials. Assume moreover, that the fibers, $\Lambda_x \subset M$, of the projection $\Sigma \to X$ are lagrangian cone subvarieties. Then there exists an immersion (i.e., a map with injective differential) $i : \Sigma \hookrightarrow T^*X$ making Σ an immersed coisotropic subvariety of T^*X. Moreover,*

(a) *The following diagram commutes*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\;i\;} & T^*X \\ & {\scriptstyle p_X}\searrow & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

(b) $p_M^* \lambda_M = i^* \lambda_{T^*X}$, *where* λ_M *and* λ_{T^*X} *are the canonical 1-forms on* M *and* T^*X, *respectively.*

(c) *the 0-foliation on* Σ *coincides with the fibration* $\Sigma \to M$.

Proof. First we construct a map $i : \Sigma \to T^*X$ as follows. Let $\phi \in \Sigma$ and $x = p_X(\phi) \in X$. The tangent map $(p_X)_* : T_\phi \Sigma \to T_x X$ is surjective by assumption. Hence, given $\eta \in T_x X$, one can choose a tangent vector $\tilde{\eta} \in T_\phi \Sigma$ such that $(p_X)_*(\tilde{\eta}) = \eta$. We claim that for any fixed η, the value of the 1-form $p_M^* \lambda_M$ on $\tilde{\eta}$ does not depend on the choice of $\tilde{\eta}$. To prove this, let $\tilde{\tilde{\eta}}$ be another vector such that $(p_X)_*(\tilde{\tilde{\eta}}) = \eta$. Then, $\tilde{\tilde{\eta}} - \tilde{\eta} = v$ is a vector tangent to the fiber of the projection $\Sigma \to X$ over x. This fiber can be identified naturally via p_M with the lagrangian subvariety $\Lambda_x \subset M$ so that we have

$$p_M^*(\lambda_M)(v) = \lambda_M((p_M)_* v) = (i_\xi \omega_M)((p_M)_* v) = \omega_M(\xi, (p_M)_* v) = 0,$$

for Λ_x is a lagrangian cone subvariety. Hence, $(p_M^* \lambda_M)(\tilde{\eta}) = (p_M^* \lambda_M)(\tilde{\tilde{\eta}})$ and the claim follows.

Thus, the map $\eta \mapsto (p_M^* \lambda_M)(\tilde{\eta})$ gives rise to a well-defined linear function on $T_x X$, i.e., to an element $i(\phi) \in T_x^* X$. The assignment $\phi \mapsto i(\phi)$ thus defined gives a map $i : \Sigma \to T^*X$. Furthermore, it follows by the construction of i that $p_M^* \lambda_M = i^*(\lambda_{T^*X})$ and that the diagram of part (a) of the proposition commutes.

We now show that the map i is an immersion. We have a commutative diagram of linear maps of tangent spaces induced by the diagram in (a):

$$\begin{array}{ccc} T_\phi \Sigma & \xrightarrow{\;i_*\;} & T_{i(\phi)}(T^*X) \\ & {\scriptstyle (p_X)_*}\searrow & \downarrow{\scriptstyle \pi_*} \\ & & T_x X \end{array}$$

Let $v \in T_\phi \Sigma$ be a nonzero vector such that $i_*(v) = 0$. Then by the above diagram we have $(p_X)_*(v) = 0$, hence v is tangent to the fiber Λ_x of the projection $p_X : \Sigma \to X$. Hence $(p_M)_*(v) \neq 0$. Since the symplectic 2-form ω_M on M is non-degenerate, one can find a vector $u \in T_m M$, where $m = p_M(\phi)$, such that $\omega_M((p_M)_* v,\, u) = 0$. The map $(p_M)_* : T_\phi \Sigma \to T_m M$ is surjective, by the hypothesis of the theorem, so that there exists $\tilde{u} \in T_\phi \Sigma$ such that $(p_M)_*(\tilde{u}) = u$. Furthermore, by part (b) we have $p_M^*(\omega_M) =$

$i^*\omega_{T^*X}$, whence we obtain

$$0 \neq \omega_M((p_M)_*v, u) = p_M^*\omega_M(v, \tilde{u}) = i^*\omega_{T^*X}(v, \tilde{u}) = \omega(i_*v, i_*\tilde{u}).$$

It follows that $i_*v \neq 0$, a contradiction.

To complete the proof of the theorem, it suffices to show that $i_*(T_\phi\Sigma)$ is a coisotropic subspace of $T_{i(\phi)}(T^*X)$, for any $\phi \in \Sigma$. To that end, put $\dim M = 2n$, an even integer. Then $\dim \Lambda_x = n$, for any $x \in X$, since Λ_x is lagrangian. Hence

$$\dim \Sigma = n + \dim X, \tag{1.6.7}$$

since $\Sigma \to X$ is a fibration with fiber Λ_x.

Let $W \subset T_\phi\Sigma$ be the kernel of the projection $(p_M)_* : T_\phi\Sigma \to T_mM$, the tangent space to the fiber over $m(:= p_M(\phi))$ of the fibration $p_M : \Sigma \to M$. Clearly, the space W is the radical of the 2-form $p_M^*\omega_M$. Using the equality $p_M^*\omega_M = i^*\omega_{T^*X}$ we see that the spaces i_*W and $i_*T_\phi\Sigma$ are orthogonal with respect to the symplectic form on T^*X, i.e.,

$$i_*(T_\phi\Sigma) \subset (i_*W)^\perp \tag{1.6.8}$$

where $\perp$ stands for $\perp_\omega$, for short. On the other hand, from (1.6.7) one obtains

$$\dim W = \dim p_M^{-1}(m) = \dim \Sigma - \dim M = (n + \dim X) - 2n = \dim X - n.$$

The map i_* being injective, we get $\dim (i_*W) = \dim W - \dim X - n$. It follows that

$$\dim (i_*W)^\perp = \dim T^*X - \dim (i_*W) = \dim X + n.$$

Using (1.6.7) one obtains

$$\dim (i_*W)^\perp = \dim \Sigma = \dim T_\phi\Sigma = \dim i_*(T_\phi\Sigma). \tag{1.6.9}$$

Formulas (1.6.8) and (1.6.9) yield $(i_*W)^\perp = i_*T_\phi\Sigma$, hence $(i_*T_\phi\Sigma)^\perp \subset (i_*W)^\perp = i_*(T_\phi\Sigma)$ and the coisotropicness follows. Finally, we have $(i_*W) = ((i_*W)^\perp)^\perp = (i_*(T_\phi\Sigma))^\perp$ and part (c) follows. ∎

Remark 1.6.10. For any $x \in X$ we have $\Lambda_x = i^{-1}(T_x^*X)$ which explains the name of the proposition.