Chapter 7

Smooth and Heavy Viable Solutions

Introduction

Let us still consider the problem of regulating a control system

$$(i) \quad \text{for almost all } t \geq 0, \ x'(t) = f(x(t), u(t)) \ \text{ where } \ u(t) \in U(x(t))$$

where $U : K \rightsquigarrow Z$ associates with each state x the set $U(x)$ of feasible controls (in general state-dependent) and $f : \text{Graph}(U) \mapsto X$ describes the dynamics of the system.

For simplicity, we take for viability subset the domain $K := \text{Dom}(U)$ of U[1]. We have seen in the preceding chapter that viable controls (which provide viable solutions $x(t) \in K := \text{Dom}(U)$) are the ones obeying the regulation law

$$\forall\, t \geq 0, \ \ u(t) \in R_K(t) \ \ (\text{or} \ \ (x(t), u(t)) \in \text{Graph}(R_K))$$

where

$$\forall\, x \in K, \ \ R_K(x) = \{ u \in U(x) \mid f(x,u) \in T_K(x) \}$$

In this chapter, we are looking for a system of differential equations or of differential inclusions governing *the evolution of both viable states and controls*, so that we can look for

[1] or we replace U by its restriction to K. It is closed whenever $U : X \rightsquigarrow Z$ is upper semicontinuous.

— *heavy solutions*, which are evolutions where the controls evolve with minimal velocity

— *punctuated equilibria*, i.e., evolutions in which the control $\bar{u}$ remains constant whereas the state may evolve in the associated *viability cell*, which is the viability domain of $x \mapsto f(x, \bar{u})$,

— *regulation by ramp controls*, i.e., evolutions in which the open-control is linear, and more generally, polynomial open-loop controls

— and other related ideas.

The idea which allows us to achieve these aims is quite simple: *we differentiate the regulation law.*

This is possible whenever we know how to differentiate set-valued maps. Hence the first section is devoted to the definition and the elementary properties of the contingent derivative[2] $DF(x,y)$ of a set-valued map $F : X \rightsquigarrow Y$ at a point (x,y) of its graph: By definition, its graph is the contingent cone to the graph of F at (x,y). We refer to Chapter 5 of SET-VALUED ANALYSIS for further information on the differential calculus of set-valued maps.

In the second section, we differentiate the regulation law and deduce that

$$(ii) \quad \text{for almost all } t \geq 0, \quad u'(t) \in DR_K(x(t), u(t))(f(x(t), u(t)))$$

whenever the viable control $u(\cdot)$ is absolutely continuous,

This is the second half of the system of differential inclusions we are looking for.

Observe that this new differential inclusion has a meaning whenever the state-control pair $(x(\cdot), u(\cdot))$ remains viable in the graph of R_K.

Fortunately, by the very definition of the contingent derivative, the graph of R_K is a viability domain of the new system (i), (ii).

Unfortunately, as soon as viability constraints involve inequalities, there is no hope for the graph of the contingent cone, and thus, for the graph of the regulation map, to be closed, so that, the Viability Theorem cannot apply.

[2]We set $Df(x) := Df(x, f(x))$ whenever f is single-valued. When f is Fréchet differentiable at x, then $Df(x)(v) = f'(x)v$ is reduced to the usual directional derivative.

However, if the contingent derivative of U obeys a growth condition:

$$(\mathcal{G})\ \forall\,(x,u)\in \mathrm{Graph}(U),\quad \inf_{v\in DU(x,u)(f(x,u))}\|v\|\leq c(\|u\|+\|x\|+1)$$

then there exists an absolutely continuous solution $(x(\cdot),u(\cdot))$ of (i) verifying

$$(iii)\ \text{ for almost all }\ t\ \geq\ 0,\quad \|u'(t)\|\ \leq\ c(\|u(t)\|+\|x(t)\|+1)$$

So, *a strategy to overcome the above difficulty is to introduce*[3] *the a priori growth condition (iii)* and to look for *graphs of closed set-valued maps R* contained in $\mathrm{Graph}(U)$ which are viable under this system of differential inclusions. We already illustrated that in the simple economic example of Section 6.2.

Such set-valued maps R are solutions to the *partial differential inclusion*

$$\forall\,x\in K,\ \ 0\in DR(x,u)(f(x,u))-c(\|x\|+\|u\|+1)B$$

satisfying the constraint

$$\forall\,(x,u)\in \mathrm{Graph}(R),\ \ R(x)\ \subset\ U(x)$$

Since we shall show that such closed set-valued maps R are all contained in the regulation map R_K, we call them *subregulation maps* associated with the system $i), iii)$. In particular, there exists a largest subregulation map denoted R^c.

In particular, any *single-valued* $r\,:\,K\,\mapsto\,Z$ with closed graph which is a solution to the *partial differential inclusion*

$$\forall\,x\in K,\ \ 0\in Dr(x)(f(x,r(x)))-c(\|x\|+\|r(x)\|+1)B$$

satisfying the constraint

$$\forall\,x\in K,\ \ r(x)\ \in\ U(x)$$

provides feedback controls regulating smooth solutions to the control system.

[3]even if growth conditions on the contingent derivative of U are absent.

The set-valued and single-valued solutions to these partial differential inclusions are studied in Section 6 of Chapter 8.

Let us consider such a subregulation map R. Theorem 4.1.2 implies that *whenever the initial state* x_0 *is chosen in* $\mathrm{Dom}(R)$) *and the initial control* u_0 *in* $R(x_0)$*, there exists a solution to the system of differential inclusions* $i), iii)$ *viable in* $\mathrm{Graph}(R)$. The regulation law for the viable state-controls becomes

$$(iv) \quad u'(t) \in DR(x(t), u(t))(f(x(t), u(t))) \cap c(\|x(t)\| + \|u(t)\| + 1)B$$

We call it the *metaregulation law* associated with the subregulation map R.

This is how we can obtain *smooth viable state-control solutions* to our control problem by solving the system of differential inclusions $i), v)$.

Actually, the graphs of all such regulation maps are contained in the *viability kernel* of $\mathrm{Graph}(U)$ for the system of differential inclusions $i), iii)$. This viability kernel is then the graph of the largest subregulation map $R^c \subset U$.

We shall construct explicitly in the third section such a regulation map in the case of the simplest economic model we can think of.

To the extent where second order differential equations and inclusions are first-order systems in disguise, we devote section 7.4 to viability problems for second order differential inclusions. The situation is not as simple as in the first order case, because the viability constraint $x(t) \in K$ becomes $x'(t) \in T_K(x(t))$, or again, $(x(t), x'(t)) \in \mathrm{Graph}(T_K)$. It no longer defines closed (or even, locally compact) viability domains. So, here again, we shall overcome this type of difficulty by using the concept of viability kernel.

We can naturally follow the same route to obtain smoother open-loop controls by setting bounds on the m-th derivatives: for almost all $t \geq 0$,

$$(v) \quad \|u^{(m)}(t)\| \;\leq\; c(\|u^{(m-1)}(t)\| + \ldots + \|u(t)\| + \|x(t)\| + 1)$$

This is the topic of the fifth section.

We devote the sixth section to the particular case when $c = 0$. We observe that equation (iii) then yields constant controls u_0 and thus solutions $x(\cdot)$ to the problem $x'(t) = f(x(t), u_0)$ which are viable in the closed subset $U^{-1}(u_0)$ (whenever this subset is not empty.) If this is the case, we shall say that u_0 is a *punctuated equilibrium* and that $(R^0)^{-1}(u_0)$ is the associated *viability cell, the closed subset of states regulated by the constant control* u_0.

In the general case of smooth systems of order m, the 0-growth condition yields open-loop controls which are polynomial of degree m. In particular, for $m = 1$, first-degree polynomials open-loop controls are known under the more descriptive label of *ramp controls*.

The seventh section is devoted to selection procedures of *dynamical closed loops*, and, among them, of heavy viable solutions.

Instead of looking for closed loop control selections of the regulation map R_K as we did in Chapter 6, we now look for selections $g(\cdot,\cdot)$ of the metaregulation map

$$(x,u) \rightsquigarrow DR(x,u)(f(x,u)) \cap c(\|x\| + \|u\| + 1)B$$

called *dynamical closed-loops.*

Naturally, under adequate assumptions, Michael's Theorem implies the existence of a continuous dynamical closed loop. But under the same assumptions, we can take as dynamical closed-loop the minimal selection $g^\circ(\cdot,\cdot)$ defined by $\|g^\circ(x,u)\| = \min_{v \in DR(x,u)(f(x,u))} \|v\|$, which, in general, is not continuous.

However, we shall prove that this minimal dynamical feedback still yields smooth viable control-state solutions to the system of differential equations

$$x'(t) = f(x(t),u(t)) \quad \& \quad u'(t) = g^\circ(x(t),u(t))$$

called *heavy viable solutions*, (heavy in the sense of heavy trends.) They are the ones for which *the control evolves with minimal velocity*. In the case of the usual differential inclusion $x' \in F(x)$, where the controls are the velocities, they are the solutions with minimal acceleration (or maximal inertia.)

Heavy viable solutions obey the *inertia principle*: *"keep the controls constant as long as they provide viable solutions"*.

Indeed, if zero belongs to $DR(x(t_1),u(t_1))(f(x(t_1),u(t_1)))$, then the control will remain equal to $u(t_1)$ as long as for $t \geq t_1$, a solution $x(\cdot)$ to the differential equation $x'(t) = f(x(t),u(t_1))$ satisfies the condition $0 \in DR(x(t_1),u(t_1))(f(x(t_1),u(t_1)))$.

If at some time t_f, $u(t_f)$ is a punctuated equilibrium, then the solution enters the viability cell associated to this control and may remain in this viability cell forever[4] and the control will remain equal to this punctuated equilibrium.

The concept of a heavy viable solution will be extended to the m-th order, where we look for controls whose m-th derivative evolves as slowly as possible. They obey an *m-th order inertia principle: keep an m-degree polynomial open-loop control as long as the solution it regulates is viable.*

7.1 Contingent Derivatives

By coming back to the original point of view proposed by Fermat, we are able to geometrically define the derivatives of set-valued maps from the choice of tangent cones to the graphs, even though they yield very strange limits of differential quotients.

Definition 7.1.1 *Let $F : X \rightsquigarrow Y$ be a set-valued map from a normed space X to another normed space Y and $y \in F(x)$.*

The contingent derivative *$DF(x,y)$ of F at $(x,y) \in \text{Graph}(G)$ is the set-valued map from X to Y defined by*

$$\text{Graph}(DF(x,y)) \ := \ T_{\text{Graph}(F)}(x,y)$$

When $F := f$ is single-valued, we set $Df(x) := Df(x, f(x))$ and $Cf(x) := Cf(x, f(x))$.

We shall say that F is sleek *at $(x,y) \in \text{Graph}(F)$ if and only if the map*

$$(x',y') \in \text{Graph}(F) \rightsquigarrow \text{Graph}(DF(x',y'))$$

is lower semicontinuous at (x,y) (i.e., if the graph of F is sleek at (x,y).) The set-valued map F is sleek *if it is sleek at every point of its graph.*

Naturally, *when the map is sleek at (x,y), the contingent derivative $DF(x,y)$ is a closed convex process.*

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[4]as long as the viability domain does not change for external reasons which are not taken into account here.

We can easily compute the derivative of the inverse of a set-valued map F (or even of a noninjective single-valued map): *The contingent derivative of the inverse of a set-valued map F is the inverse of the contingent derivative*:

$$D(F^{-1})(y,x) \quad = \quad DF(x,y)^{-1}$$

If K is a subset of X and f is a single-valued map which is Fréchet differentiable around a point $x \in K$, then *the contingent derivative of the restriction of f to K is the restriction of the derivative to the contingent cone*:

$$D(f|_K)(x) = D(f|_K)(x, f(x)) = f'(x)|_{T_K(x)}$$

These contingent derivatives can be characterized by adequate limits of differential quotients[5]:

Proposition 7.1.2 *Let $(x,y) \in \mathrm{Graph}(F)$ belong to the graph of a set-valued map $F : X \leadsto Y$ from a normed space X to a normed space Y. Then*

$$\begin{cases} v \in DF(x,y)(u) \ \text{ if and only if} \\ \mathrm{liminf}_{h\to 0+, u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \end{cases}$$

If $x \in \mathrm{Int}(\mathrm{Dom}(F))$ and F is Lipschitz around x, then

$$v \in DF(x,y)(u) \ \text{ if and only if } \ \liminf_{h\to 0+} d\left(v, \frac{F(x+hu)-y}{h}\right) = 0$$

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[5]We can reformulate Proposition 7.1.2 by saying that *the contingent derivative $DF(x,y)$ is the graphical upper limit (See Definition 3.6.3) of the differential quotients*

$$u \leadsto \nabla_h F(x,y)(u) \ := \ \frac{F(x+hu)-y}{h}$$

Indeed, we know that the contingent cone

$$T_{\mathrm{Graph}(F)}(x,y) \ = \ \mathrm{Limsup}_{h\to 0+} \frac{\mathrm{Graph}(F)-(x,y)}{h}$$

is the upper limit of the differential quotients $\frac{\mathrm{Graph}(F)-(x,y)}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathrm{Graph}(DF(x,y)) := T_{\mathrm{Graph}(F)}(x,y) \ \& \ \mathrm{Graph}(\nabla_h F(x,y)) = \frac{\mathrm{Graph}(F)-(x,y)}{h}$$

to conclude.

If moreover the dimension of Y is finite, then

$$\mathrm{Dom}(DF(x,y)) = X \text{ and } DF(x,y) \text{ is Lipschitz}$$

Proof — The first two statements being obvious, let us check the last one. Let u belong to X and l denote the Lipschitz constant of F on a neighborhood of x. Then, for all $h > 0$ small enough and $y \in F(x)$,

$$y \in F(x) \subset F(x+hu) + lh\|u\|B$$

Hence there exists $y_h \in F(x+hu)$ such that $v_h := (y_h - y)/h$ belongs to $l\|u\|B$, which is compact. Therefore the sequence v_h has a cluster point v, which belongs to $DF(x,y)(u)$. □

Remark — Lower Semicontinuously Differentiable Maps
The lower semicontinuity of the set-valued map

$$(x,y,u) \in \mathrm{Graph}(F) \times X \rightsquigarrow DF(x,y)(u)$$

at some point (x_0, y_0, u_0) is often needed. Observe that it implies that F is sleek at (x_0, y_0). The converse needs further assumptions. We derive for instance from Theorem 2.5.7 the following criterion:

Proposition 7.1.3 *Assume that X and Y are Banach spaces and that F is sleek on some neighborhood $\mathcal{U}$ of $(x_0, y_0) \in \mathrm{Graph}(F)$. If the boundedness property*

$$\forall\, u \in X, \quad \sup_{(x,y)\in\mathcal{U}\cap\mathrm{Graph}(F)} \ \inf_{v\in DF(x,y)(u)} \|v\| < +\infty$$

holds true, then the set-valued map

$$(x,y,u) \in \mathrm{Graph}(F) \times X \rightsquigarrow DF(x,y)(u)$$

is lower semicontinuous on $(\mathcal{U} \cap \mathrm{Graph}(F)) \times X$

7.2 Smooth Viable Solutions

7.2.1 Regularity Theorem

Let us consider a finite dimensional vector space Z and a control system (U, f) defined by a set-valued map $U : X \rightsquigarrow Z$ and a single-valued map $f : \mathrm{Graph}(U) \mapsto X$, where X is regarded as the state space, Z the control space, f as a description of the dynamics and U as the a priori feedback. The evolution of a state-control solution $(x(\cdot), u(\cdot))$ viable in Graph(U) is governed by

$$x'(t) \;=\; f(x(t), u(t)), \;\; u(t) \;\in\; U(x(t)) \tag{7.1}$$

We shall look for viable solutions in $K := \text{Dom}(U)$ which are smooth in the following sense:

Definition 7.2.1 (Smooth State-Control) *We say that the pair $(x(\cdot), u(\cdot))$ is* smooth *if both $x(\cdot)$ and $u(\cdot)$ are absolutely continuous and* m-smooth *if both $x(\cdot)$ and $u^{(m-1)}(\cdot)$ are absolutely continuous.*

It is said to be φ-smooth *(respectively φ-smooth of m-th order) if in addition for almost all $t \geq 0$, $\|u'(t)\| \leq \varphi(x(t), u(t))$ (respectively $\|u^{(m)}(t)\| \leq \varphi(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t))$), where $\varphi : X \times Z \mapsto \mathbf{R}_+$ (respectively $\varphi : X \times Z^m \mapsto \mathbf{R}_+$) is a given function.*

We obtain smooth viable solutions by setting a bound to the growth to the evolution of controls, as we did in the simple economic example of Section 6.2.

For that purpose, we associate to this control system and to any nonnegative continuous function $u \to \varphi(x, u)$ with linear growth[6] the system of differential inclusions

$$\begin{cases} i) & x'(t) \;=\; f(x(t), u(t)) \\ \\ ii) & u'(t) \;\in\; \varphi(x(t), u(t))B \end{cases} \tag{7.2}$$

Observe that any solution $(x(\cdot), u(\cdot))$ to (7.2) viable in Graph(U) is a φ-smooth solution to the control system (7.1).

We thus deduce from the Viability Theorem applied to the system (7.2) on the graph of U the following Regularity Theorem:

Theorem 7.2.2 *Assume that U is closed and f, φ are continuous with linear growth. Then the following two statements are equivalent:*

a) — For any initial state $x_0 \in \text{Dom}(U)$ and control $u_0 \in U(x_0)$, there exists a φ-smooth state-control solution $(x(\cdot), u(\cdot))$ to the control system (7.1) starting at (x_0, u_0).

[6] which can be a constant ρ, or the function $(x, u) \to c\|u\|$, or the function $(x, u) \to c(\|u\| + \|x\| + 1)$. One could also take other dynamics $u' \in \Phi(x, u)$ where Φ is a Marchaud map.

b) — *The set-valued map U satisfies*

$$\forall\, (x,u) \in \text{Graph}(U),\;\; 0 \;\in\; DU(x,u)(f(x,u)) - \varphi(x,u)B \qquad (7.3)$$

Proof — The conclusion of the theorem amounts to saying that the closed subset Graph(U) enjoys the viability property. By Viability Theorem 3.3.5, which we can apply because $(x,u) \rightsquigarrow \{f(x,u)\} \times \varphi(x,u)B$ is a Marchaud map, this is the case if and only if it is a viability domain, i.e., if and only if

$$\forall\, (x,u) \in \text{Graph}(U), T_{\text{Graph}(U)}(x,u) \cap (\{f(x,u)\} \times \varphi(x,u)B) \neq \emptyset$$

By the very definition of the contingent derivative of U, this is the necessary and sufficient condition of the theorem. □

We know that whenever the right-hand side of an ordinary differential equation is differentiable, its solutions are twice differentiable. The extension of this property to the case of differential inclusions is just a consequence of the above theorem when we take $f(x,u) = u$:

Corollary 7.2.3 *Let $F : X \rightsquigarrow X$ be a closed set-valued map such that*

$$\forall\, x \in \text{Dom}(F),\; \forall\, v \in F(x),\; 0 \;\in\; DF(x,v)(v) - \varphi(x,v)B$$

where $(x,u) \rightarrow \varphi(x,u)$ is a nonnegative continuous function with linear growth.

Then, for any $x_0 \in \text{Dom}(F)$ and $v_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to the differential inclusion

$$x'(t) \;\in\; F(x(t)),\;\; x(0) \;=\; x_0 \;\&\; x'(0) \;=\; v_0$$

which belongs to the Sobolev space $W^{2,1}(0,\infty;X;e^{-bt}dt)$ (both $x(\cdot)$ and $x'(\cdot)$ are absolutely continuous.)

Remark — Naturally, we can consider other evolution laws of open-loop controls associated with the control system (U,f) which provide smooth open-loop controls yielding viable solutions.

First, we can introduce an observation space Y, replace the initial control space Z by another finite dimensional space Z_1, an observation map $\beta : X \mapsto Y$ and relate the new controls $v \in Z_1$ and the observation y to the former controls $u \in Z$ by a single-valued map of the form

$$u = \alpha(\beta(x), v)$$

where

$$\alpha : Y \times Z_1 \mapsto Z$$

We then define a new control system (g, V) defined by

$$\begin{cases} i) & g(x,v) := f(x, \alpha(\beta(x), v)) \\ ii) & V(x) := \{v \in Z_1 \mid \alpha(\beta(x), v) \in U(x)\} \end{cases}$$

Therefore the new control system governed by

$$\begin{cases} i) & x'(t) = g(x(t), v(t)) \\ ii) & v(t) \in V(x(t)) \end{cases} \tag{7.4}$$

provides the same dynamics of the state although through another parametrization.

This being done, we can propose any evolution law of the open-loop controls as long as they are compatible with the constraints $v(t) \in V(x(t))$ (or $u(t) \in U(x(t))$.)

For instance, if $A \in \mathcal{L}(Z_1, Z_1)$ and $\Phi : X \times Z_1 \rightsquigarrow Z_1$ and $\varphi : X \times Z_1 \rightsquigarrow Z_1$ is a Marchaud map, we can replace system (7.2) by the system of differential inclusions

$$\begin{cases} i) & x'(t) = g(x(t), v(t)) \\ ii) & v'(t) \in Av(t) + \Phi(x(t), v(t)) \end{cases} \tag{7.5}$$

(With an adequate choice of A, we are able to study the evolution of m time differentiable open-loop controls in next section.)

Then the Regularity Theorem becomes:

Theorem 7.2.4 *Assume that U is closed and sleek, that f, φ are continuous with linear growth, that the maps α and β are continuously differentiable with linear growth and that*

$$\forall\, (x,v) \in \mathrm{Graph}(V),\;\; \alpha'_v(\beta(x),v) \text{ is surjective}$$

Then the following two statements are equivalent:

a) — *For any initial state $x_0 \in \mathrm{Dom}(V)$ and control $v_0 \in V(x_0)$, there exists a solution $(x(\cdot),v(\cdot))$ to the control system (7.5) starting at (x_0,v_0) (so that $x(\cdot)$ is still a solution to the control system (7.1)).*

b) — *The set-valued map V satisfies: for every $(x,v) \in \mathrm{Graph}(V)$,*

$$Av \in -\Phi(x,v)+$$
$$\alpha'_v(\beta(x),v)^{-1}\left[DU(x,\alpha(\beta(x),v))(g(x,v)) - \alpha'_y(\beta(x),v)\beta'(x)g(x,v)\right]$$

Proof — By the Viability Theorem 3.3.5, we have to check that the graph of V is a viability domain for the set-valued map

$$(x,v) \rightsquigarrow \{g(x,v)\} \times (Av + \Phi(x,v))$$

Since the graph of V is the inverse image of the graph of U under the differentiable map $h : X \times Z_1 \mapsto X \in Z$ defined by

$$h(x,v) \;=\; (x,\alpha(\beta(x),v))$$

we can derive a formula to compute its contingent cone whenever U is sleek and the following transversality condition holds true:

$$\mathrm{Im}(h'(x,v)) - T_{\mathrm{Graph}(U)}(h(x,v)) \;=\; X \times Z$$

But the surjectivity of $\alpha'_v(\beta(x),v)$ implies obviously the surjectivity of $h'(x,v)$, so that this condition is satisfied. Hence, the contingent derivative of V is given by the formula

$$\begin{cases} DV(x,v)(x') \;=\; \alpha'_v(\beta(x),v)^{-1}\,[\\ DU(x,\alpha(\beta(x),v))(x') - \alpha'_y(\beta(x),v)\beta'(x)x'] \end{cases}$$

Therefore, we observe that the second statement of the theorem states that the graph of V is a viability domain. □

7.2.2 Subregulation and Metaregulation Maps

The assumption of the above theorem is too strong, since it requires that property (7.3) is satisfied for all controls u of $U(x)$ (so that we have a solution for every initial control chosen in $U(x_0)$.) This means that, setting

$$R_K(x) := \{u \in U(x) \mid f(x,u) \in T_K(x)\}$$

we are in the situation where $R_K = U$.

We may very well be content with the existence of a smooth solution for only some initial control in a subset $R(x_0)$ of $U(x_0)$.

So, we can relax the problem by looking for closed set-valued feedback maps R contained in U in which we can find the initial state-controls yielding smooth viable solutions to the control system.

The Viability Theorem implies the following

Theorem 7.2.5 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \text{ is continuous and has linear growth} \end{cases} \tag{7.6}$$

Let $(x,u) \to \varphi(x,u)$ be a nonnegative continuous function with linear growth and $R : Z \rightsquigarrow X$ a closed set-valued map contained in U. Then the two following conditions are equivalent:

a) — R regulates φ-smooth viable solutions in the sense that for any initial state $x_0 \in \text{Dom}(R)$ and any initial control $u_0 \in R(x_0)$, there exists a φ-smooth state-control solution $(x(\cdot), u(\cdot))$ to the control system (7.1) starting at (x_0, u_0) and viable in the graph of R.

b) — R is a solution to the partial differential inclusion

$$\forall\ (x,u) \in \text{Graph}(R),\ \ 0 \in DR(x,u)(f(x,u)) - \varphi(x,u)B \tag{7.7}$$

satisfying the constraint: $\forall\ x \in K,\ \ R(x) \subset U(x)$.

In this case, such a map R is contained in the regulation map R_K, and is thus called a φ-subregulation map of U *or simply a* subregulation map . *The* metaregulation law *regulating the evolution of state-control solutions viable in the graph of R takes the form of the system of differential inclusions*

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) \in G_R(x(t), u(t)) \end{cases} \tag{7.8}$$

where the set-valued map G_R defined by

$$G_R(x,u) \;:=\; DR(x,u)(f(x,u)) \cap \varphi(x,u)B$$

is called the metaregulation map *associated with R.*

Furthermore, there exists a largest φ-subregulation map denoted R^φ contained in U.

Proof — Indeed, to say that R is a regulation map regulating φ-smooth solutions amounts to saying that its graph is viable under the system (7.2).

In this case, we deduce that for any $(x_0, u_0) \in \text{Graph}(R)$, there exists a solution $(x(\cdot), u(\cdot))$ viable in the graph of U, so that $x(\cdot)$ is in particular viable in K. Since $x'(t) = f(x(t), u(t))$ is absolutely continuous, we infer that $f(x_0, u_0)$ is contingent to K at x_0, i.e., that u_0 belongs to $R_K(x_0)$.

The regulation map for the system (7.2) associates with any $(x, u) \in \text{Graph}(R)$ the set of pairs $(x', u') \in \{f(x,u)\} \times \varphi(x,u)B$ such that (x', u') belongs to the contingent cone to the graph of R at (x, u), i.e., such that

$$u' \;\in\; DR(x,u)(f(x,u)) \cap \varphi(x,u)B \;=:\; C_R(x,u)$$

The graph of R^φ is the viability kernel of $\text{Graph}(U)$ for the system of differential inclusions (7.2). □

Proposition 7.2.6 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & U \text{ maps a neighborhood of every point to a compact subset} \\ ii) & \text{Graph}(U) \textit{ is upper semicontinuous with compact values} \\ iii) & f \textit{ is continuous and has linear growth} \end{cases}$$

Then the domain of every subregulation map is closed.

Proof — Let $x_n \in \text{Dom}(R)$ be a sequence converging to x_0 and let u_n belong to $R(x_n) \subset U(x_n)$. By assumption, the sequence u_n remains in a compact subset, so that a subsequence (again denoted by) u_n converges to some $u \in U(x)$. Since R is a subregulation map, there exist solutions $(x_n(\cdot), u_n(\cdot))$ to the system (7.2) of differential inclusions viable in the graph of R. Theorem 3.5.2 implies that a subsequence (again denoted by) $(x_n(\cdot), u_n(\cdot))$ converges to a solution $(x(\cdot), u(\cdot))$ starting at (x, u). Hence $u \in R(x)$ and thus, $x \in \mathrm{Dom}(R)$. □

We can be particularly interested in *single-valued regulation maps* $r : K \mapsto Z$, which are *closed-loop (feedback) controls regulating* φ-*smooth viable solutions*:

Proposition 7.2.7 *A closed single-valued continuous map r is a feedback control regulating φ-smooth viable solutions to the control problem if and only if r is a single-valued solution to the* inclusion

$$\forall\, x \in K, \;\; 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraint

$$\forall\, x \in K, \;\; r(x) \;\in\; U(x)$$

Then for any $x_0 \in K$, there exists a solution to the differential equation $x'(t) = f(x(t), r(x(t)))$ starting at x_0 such that

$$\forall\, t \geq 0, \; u(t) \;:=\; r(x(t)) \;\in\; U(x(t))$$

and

$$\text{for almost all } t \geq 0, \;\; \|u'(t)\| \leq \varphi(x(t), r(x(t)))$$

Remark — The study of set-valued and single-valued solutions to partial differential inclusion (7.7) will be carried over in Chapter 8 in the framework of the more general "tracking property". □

Remark — We observe that any φ-subregulation map remains a ψ-subregulation map for $\psi \geq \varphi$ and in particular, that the largest subregulation maps R^{φ} are increasing with φ. □

Example: Equality Constraints

Consider the case when $h : X \mapsto Y$ is a twice continuously differentiable map and when the viability domain is $K := h^{-1}(0)$.

Since $T_K(x) = \ker h'(x)$ when $h'(x)$ is surjective, we deduce that the regulation map is equal to

$$R_K(x) = \{u \in U(x) \mid h'(x)f(x,u) = 0\}$$

Proposition 7.2.8 *Assume that $h'(x) \in \mathcal{L}(X,Y)$ is surjective whenever $h(x) = 0$, that the graph of U is sleek and that for any $y \in Y$ and $v \in X$, the subsets*

$$DU(x,u)(v) \cap (h'(x)f'_u(x,u))^{-1}(y - h''(x)(f(x,u),v) - h'(x)f'_x(x,u)v)$$

are not empty. Then the contingent derivative $DR_K(x,u)(v)$ of the regulation map is equal to

$$DU(x,u)(v) \cap -(h'(x)f'_u(x,u))^{-1}(h''(x)(f(x,u),v) - h'(x)f'_x(x,u)v)$$

when $h'(x)v = 0$ and $DR_K(x,v) = \emptyset$ if not. In particular, if $U(x) \equiv Z$, then it is sufficient to assume that $h'(x)f'_u(x,u)$ is surjective and we have in this case

$$DR_K(x,u)(v) = -(h'(x)f'_u(x,u))^{-1}(h''(x)(f(x,u),v) - h'(x)f'_x(x,u)v)$$

when $h'(x)v = 0$ and $DR_K(x,v) = \emptyset$ if not.

Proof — The graph of R_K can be written as the subset of pairs $(x,u) \in \text{Graph}(U)$ such that $C(x,u) := (h(x), h'(x)f(x,u)) = 0$. Since the graph of U is closed and sleek, we know that the transversality condition

$$C'(x,u)T_{\text{Graph}(U)}(x,u) = C'(x,u)\text{Graph}(DU(x,u)) = Y \times Y$$

implies that the contingent cone to the graph of U is the set of elements $(v,w) \in \text{Graph}(DU(x,u))$ such that

$$\begin{cases} C'(x,u)(v,w) = \\ (h'(x)v, h'(x)f'_u(x,u)w + h'(x)f'_x(x,u)v + h''(x)(f(x,u),v)) = 0 \end{cases}$$

But the surjectivity of $h'(x)$ and the nonemptiness of the intersection imply this transversality condition. □

Therefore, the right-hand side of the metaregulation rule is equal to

$$\begin{cases} -(h'(x)f'_u(x,u))^{-1}(h''(x)(f(x,u),f(x,u)) - h'(x)f'_x(x,u)f(x,u)) \\ \cap DU(x,u)(f(x,u)) \cap \varphi(x,u)B \end{cases}$$

Example: Inequality Constraints

Consider the case when

$$K \ := \ \{x \in X \ : \ \forall\, i = 1, \ldots, p, \ \ g_i(x) \ \geq \ 0\}$$

is defined by inequality constraints (for simplicity, we do not include equality constraints.)

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of *active constraints* and we assume once and for all that for every $x \in K$,

$$\exists\, v_0 \in C_L(x) \ \text{ such that } \forall\, i \in I(x), \ < g_i'(x), v_0 >> 0$$

so that, by Theorem 5.1.10,

$$R_K(x) \ := \ \{u \in U(x) \mid \forall\, i \in I(x), \langle g_i'(x), f(x,u)\rangle \geq 0\}$$

We set $g(x) := (g_1(x), \ldots, g_p(x))$.

We have seen that the graph of the set-valued map $x \rightsquigarrow R_K(x)$ is not necessarily closed. However, we can find explicit subregulation maps by using Theorem 5.1.11. We thus introduce the set-valued map $R_K^\diamond : X \rightsquigarrow Z$ defined by

$$R_K^\diamond(x) \ := \ \{u \in U(x) \mid g(x) + g'(x) f(x,u) \geq 0\} \ \subset \ R_K(x)$$

We can regulate solutions viable in K by smooth open-loop controls by looking for solutions to the system of differential inclusions (7.2) which are viable in the graph of $R_K^\diamond$.

We thus need to compute the derivative of $R_K^\diamond$ in order to characterize the associated metaregulation map:

Proposition 7.2.9 *Assume that the stronger viability condition*[7]

$$\forall\, x \in K, \ \ R_K^\diamond(x) \ \neq \ \emptyset$$

[7] which holds true whenever K is a viability domain for the control system and

$$\forall\, x \in K, \ \exists\, u \in U(x) \ \text{ such that } \|f(x,u)\| \ \leq \ \gamma_K(x)$$

where the function γ_K is defined by (5.1) in Section 5.1. See Theorem 5.1.11.

is satisfied. We set

$$I(x,u) := \{i = 1,\ldots,p \mid g_i(x) + \langle g_i'(x), f(x,u)\rangle = 0\}$$

Assume that U *is sleek and closed and that for every* $(x,u) \in \mathrm{Graph}(R_K^\diamond)$, *there exists* $u_0' \in DU(x,u)(x_0')$ *satisfying*

$$\forall\, i \in I(x,u),\ \langle g_i'(x), x_0' + f_x'(x,u)x_0' + f_u'(x,u)u_0'\rangle + g_i''(x)(f(x,u), x_0') \geq 0$$

Then the contingent derivative $DR_K^\diamond(x,u)(v)$ *of the subregulation map* $R_K^\diamond$ *is defined by:* $u' \in DR_K^\diamond(x,u)(x')$ *if and only if* $u' \in DU(x,u)(x')$ *and*

$$\forall\, i \in I(x,u),\ \langle g_i'(x), x' + f_x'(x,u)x' + f_u'(x,u)u'\rangle + g_i''(x)(f(x,u), x') \geq 0$$

If $U(x) \equiv Z$, *then it is sufficient to assume that* $g'(x)f_u'(x,u)$ *is surjective. We then have in this particular case*

$$\left\{ \begin{array}{l} DR_K^\diamond(x,u)(x') \ := \{u' \in Z \mid \forall\, i \in I(x,u), \\ \langle g_i'(x), f_u'(x,u)u'\rangle \ \geq \ -\langle g_i'(x), x' + f_x'(x,u)x'\rangle - g_i''(x)(f(x,u), x')\} \end{array} \right.$$

Proof — By Theorem 5.1.10 applied to $L := \mathrm{Graph}(U)$ and to the constraints defined by $\tilde{g}_i(x,u) := g_i(x) + \langle g_i'(x), f(x,u)\rangle$, we deduce that $u' \in DR_K^\diamond(x,u)(x')$ if and only if $u' \in DU(x,u)(x')$ and

$$\forall\, i \in I(x,u),\ \langle g_i'(x), x' + f_x'(x,u)x' + f_u'(x,u)u'\rangle + g_i''(x)(f(x,u), x') \geq 0 \quad \square$$

We then deduce from the above Proposition and the Regularity Theorem the following consequence:

Proposition 7.2.10 *We posit the assumptions of Proposition 7.2.9. If for any* $(x,u) \in \mathrm{Graph}(R_K^\diamond)$, *there exists* u' *such that* $\|u'\| \leq \varphi(x,u)$, *then for any initial state* x_0 *and any* $u_0 \in R_K^\diamond(x_0)$, *there exists a solution* $(x(\cdot), u(\cdot))$ *to the control system (7.2) such that* $x(\cdot)$ *is viable in the set* K *defined by inequality constraints. The metaregulation law can then be written*

$$\left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t), u(t)) \\ ii) & u'(t) & \in & G(x(t), u(t)) \end{array} \right. \tag{7.9}$$

where the metaregulation map G associated to $R_K^\diamond$

$$G(x,u) \;:=\; DR_K^\diamond(x,u)(f(x,u)) \cap \varphi(x,u)B$$

defined by:
$w \in G(x,u)$ *if and only if* $w \in DU(x,u)(f(x,u)) \cap \varphi(x,u)B$ *and*

$$\begin{cases} \forall\, i \in I(x,u), \;\; \langle g_i'(x), f_u'(x,u)u'\rangle \\ \geq \; -\langle g_i'(x), f(x,u) + f_x'(x,u)f(x,u)\rangle - g_i''(x)(f(x,u), f(x,u)) \end{cases}$$

Naturally, the graph of the metaregulation map G is not necessarily closed. However, we can still use Theorem 5.1.11 to obtain a "submetaregulation map" of this system of differential inclusions. We introduce the set-valued map $G^\diamond$ defined by: $u' \in G^\diamond(x,u)$ if and only if $\|u'\| \leq \varphi(x,u)$ and

$$\begin{cases} \forall\, i = 1,\ldots,p, \;\; \langle g_i'(x), f_u'(x,u)u'\rangle \\ \geq \; -g_i(x) - \langle g_i'(x), 2f(x,u) + f_x'(x,u)f(x,u)\rangle - g_i''(x)(f(x,u), f(x,u)) \end{cases}$$

Hence the system of differential inclusions

$$\begin{cases} i) \quad x'(t) \;=\; f(x(t),u(t)) \\ \\ ii) \quad u'(t) \;\in\; G^\diamond(x(t),u(t)) \cap \varphi(x(t),u(t))B \end{cases} \tag{7.10}$$

regulates φ-smooth solutions which are viable in K.

7.3 Second Order Differential Inclusions

Viability problems for second order differential inclusions also require the use of viability kernels.

Let us consider a set-valued map $F : X \times X \leadsto X$ and the second order differential inclusion

$$\text{for almost all } t \geq 0, \;\; x''(t) \in F(x(t), x'(t)) \tag{7.11}$$

If we are looking for differentiable solutions $x(\cdot)$ which are viable in K, we know that $\forall\, t \geq 0, \;\; x'(t) \in T_K(x(t))$, i.e., $(x(t), x'(t)) \in \text{Graph}(T_K)$. So the viability condition $x(t) \in K$ involves the underlying viability condition $x'(t) \in T_K(x(t))$. Hence, a necessary condition for having viable solutions is that the closure of the graph of T_K is contained in the domain of F.

As usual, we regard the second order differential inclusion as the system of first order differential inclusions

$$\begin{cases} i) & \text{for almost all } t \geq 0, \quad x'(t) = u(t) \\ ii) & \text{and} \quad u'(t) \in F(x(t), u(t)) \end{cases}$$

and the viability condition $x(t) \in K$ as the first order viability constraint

$$\forall\, t \geq 0, \quad (x(t), x'(t)) \in \text{Graph}(T_K)$$

So, by the very definition of contingent derivatives, the necessary condition of viability can be expressed in the form

$$\forall\, (x, u) \in \text{Graph}(T_K), \quad F(x, u) \cap DT_K(x, u)(u) \neq \emptyset \qquad (7.12)$$

Viability Theorem 3.3.5 implies the following result:

Proposition 7.3.1 *Assume that the graph of the contingent cone $T_K(\cdot)$ is closed and contained in the domain of a Marchaud map F.*

Then the necessary and sufficient condition for the second order differential inclusion (7.11) to have viable solutions starting from any initial state $x_0 \in K$ and any initial velocity $u_0 \in T_K(x_0)$ is that condition (7.12) *is satisfied.*

This condition is satisfied whenever K is a smooth subset of the form $h^{-1}(0)$:

Corollary 7.3.2 *Let $h : X \mapsto Y$ be a twice continuously differentiable map such that $h'(x) \in \mathcal{L}(X, Y)$ is surjective whenever $h(x) = 0$ and $K := h^{-1}(0)$. Then differential inclusion (7.11) has a viable solution starting from any initial state $x_0 \in K$ and any initial velocity u_0 satisfying $h'(x_0)u_0 = 0$ if and only if*

$$\forall\, x \in K, \ \forall\, u \in \ker h'(x), \quad -h'(x)F(x, u) \cap h''(x)(u, u) \neq \emptyset$$

Proof — We already know that $T_K(x) = \ker h'(x)$ because $h'(x)$ is surjective, so that the transversality condition is satisfied.

Since the graph of T_K can be described by the equation $B(x,u) = 0$ where

$$B(x,u) \ := \ (h(x), h'(x)v)$$

Its derivative $B'(x,u) \in \mathcal{L}(X \cap X, X \cap X)$ is equal to

$$B'(x,u)(v,w) = (h'(x)v, h''(x)(u,v) + h'(x)w)$$

and is surjective thanks to the surjectivity of $h'(x)$. Therefore, the contingent cone to the graph of the set-valued map $T_K(\cdot)$ is the subset of elements (v,w) such that $B'(x,u)(v,w) = 0$, i.e., the subset of elements $v \in T_K(x)$ and $w \in -h'(x)^{-1}h''(x)(u,v)$. In other words,

$$DT_K(x,u)(v) = \begin{cases} -h'(x)^{-1}h''(x)(u,v) & \text{if } v \in T_K(x) \\ \emptyset & \text{if } v \notin T_K(x) \end{cases}$$

Hence tangential condition (7.12) is equivalent to the condition of the corollary. □

Unfortunately, the graph of the contingent cone is not closed, nor even locally compact, as soon as the viability constraints involve inequality constraints. In this case, this condition is no longer sufficient, as the following example shows.

Example Take $X := \mathbf{R}$ and $K := \mathbf{R}_+$ and the differential inclusion $x''(t) = x(t)+1$. We see easily that the tangential condition (7.12) is satisfied. However, there is no solution to this second order differential equation starting from $(0,0)$. □

If the graph of $T_K(\cdot)$ is not closed, we can look for explicit closed set-valued maps contained in $T_K(\cdot)$, such as the maps $T_K^c(\cdot)$ (see Definition 4.4.1), or the maps $T_K^\diamond(\cdot)$ introduced by N. Maderner in the case of inequality constraints (see Theorem 5.1.11).

In the general case, we can regard the viability kernel of its closure as the graph of a closed set-valued map (possibly empty) R. Theorem 4.1.2 implies the following consequence:

Theorem 7.3.3 *Assume that $F : X \times X \leadsto X$ is a Marchaud map. Let K be a subset such that $\overline{\mathrm{Graph}(T_K)} \subset \mathrm{Dom}(F)$.*

Then there exists a largest closed set-valued map $R : X \rightsquigarrow X$ such that second order differential inclusion (7.11) has a viable solution for any initial state $x_0 \in \mathrm{Dom}(R)$ and initial velocity $u_0 \in R(x_0)$.

If we are not interested by global properties, but are satisfied with local properties, we can look for locally compact viability domains of $u \times F(x,u)$ comprised between the graph of R (the largest closed viability domain) and the graph of T_K (a viability domain which may not be locally compact), because Viability Theorem 3.3.2 requires only local compactness for having local viable solutions.

This happens whenever the graph of the interior of the contingent cone $\mathrm{Int}(T_K)$ is open (this is the case when the interior of a closed convex subset K is not empty, for instance.) Then, by taking initial velocities $u_0 \in \mathrm{Int}(T_K(x_0))$, we deduce from Theorem 3.3.2 the existence of a viable solution $x(\cdot)$ on some $[0,T]$.

In the nonconvex case, one can take initial velocities u_0 in the Dubovitsky-Miliutin cone $D_K(x_0)$ (see Definition 4.3.1.)

7.4 Metaregulation Map of High Order

The above results can naturally be extended to the regulation of control systems by smooth controls of order $m > 1$.

We introduce a set-valued map $U_m : X \times Z^{m-2} \rightsquigarrow Z$ satisfying

$$\text{if } \exists\, u_0, \ldots, u_{m-1} \mid u_{m-1} \in U_m(x, u_0, \ldots, u_{m-2}), \text{ then } u_0 \in U(x)$$

We can take for instance $\mathrm{Graph}(U_m) := \mathrm{Graph}(U) \times Z^{m-2}$, but we shall propose later other choices of closed maps U_m.

Let us consider a nonnegative continuous function

$$(x, u_0, \ldots, u_{m-1}) \in \mathrm{Graph}(U_m) \to \varphi(x, u_0, \ldots, u_{m-1}) \in \mathbf{R}_+$$

with linear growth.

We obtain smooth viable solutions of order m by setting a bound to the m-th derivative of the control. For that purpose, we associate with this control system and φ the system of differential inclusions

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u^{(m)}(t) \in \varphi(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t))B \end{cases} \tag{7.13}$$

Let us consider a closed set-valued map $R_m : X \times Z^{m-1} \rightsquigarrow Z$. We also regard the graph of R_m as the graph of the set-valued map $N_m : Z^m \rightsquigarrow X$ defined by

$$x \in N_m(u_0, \ldots, u_{m-1}) \text{ if and only if } u_{m-1} \in R_m(x, u_0, \ldots, u_{m-2})$$

and $K_m := \operatorname{Im}(N_m)$ its image.

Theorem 7.4.1 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & \operatorname{Graph}(U_m) \text{ is closed} \\ ii) & f \text{ is continuous and has linear growth} \end{cases}$$

Let us consider a closed set-valued map $R_m : X \times Z^{m-2} \rightsquigarrow Z$ contained in U_m. Then the two following conditions are equivalent:

a) — R_m regulates φ-smooth viable solutions of order m in the sense that for any initial $(x_0, u_0, u_1, \ldots, u_{m-1}) \in \operatorname{Graph}(R_m)$, there exists a solution $x(\cdot) \in W^{1,1}(0, \infty; X, e^{bt})$ and a control $u(\cdot) \in W^{m,1}(0, \infty; Z, e^{bt})$ to the control system (7.1) satisfying the initial conditions

$$x(0) = x_0, \; u(0) = u_0, \; u'(0) = u_1 \; \ldots, \; u^{(m-1)}(0) = u_{m-1}$$

the growth condition

$$\left\| u^{(m)}(t) \right\| \leq \varphi(x(t), \ldots, u^{(m-1)}(t))$$

and the constraints[8]

$$\forall\, t \geq 0, \; x(t) \in N_m(u(t), u'(t), \ldots, u^{(m-2)}(t))$$

b) — R_m is a solution to the partial differential inclusion[9]

$$\begin{cases} \forall\, (x, u_0, \ldots, u_{m-1}) \in \operatorname{Graph}(R_m), \\ 0 \in DR_m(x, u_0, \ldots, u_{m-1})(f(x, u_0), u_1, \ldots, u_{m-1}) \\ -\varphi(x, u_0, \ldots, u_{m-1})B \end{cases}$$

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[8] which can also be written in the form

$$\forall\, t \geq 0, \; u^{(m-1)}(t) \in R_m(x(t), u(t), u'(t), \ldots, u^{(m-2)}(t))$$

[9] or N_m is a solution to the partial differential inclusion

$$0 \in DN_m(u_0, \ldots, u_{m-1})(u_1, \ldots, u_{m-1}, \varphi(x, \ldots, u_{m-1})) - f(x, u_0)$$

satisfying the constraint: $R_m(x, u_0, \ldots, u_{m-2}) \subset U_m(x, u_0, \ldots, u_{m-2})$.

In this case, such a map R_m *is called a* φ-growth subregulation map of order m of U *or simply a* subregulation map of order m .

The metaregulation law of order m *regulating the evolution of state-control solutions viable in the graph of* R *takes the form of the system of differential inclusions*

$$\begin{cases} i) \quad x'(t) = f(x(t), u(t)) \\ ii) \quad u^{(m)}(t) \in G_{R_m}(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t)) \end{cases} \tag{7.14}$$

where the metaregulation map G_{R_m} of order m *is defined by*

$$\begin{cases} G_{R_m}(x, u_0, \ldots, u_{m-1}) := \\ DR_m(x, u_0, \ldots, u_{m-1})(f(x, u_0), \ldots, u_{m-1}) \cap \varphi(x, u_0, \ldots, u_{m-1})B \end{cases}$$

There exists a largest φ*-growth subregulation map denoted* R_m^{φ} *contained in* U_m.

Proof — We introduce the differential inclusion

$$\begin{cases} x'(t) = f(x(t), u_0(t)) \\ u_0'(t) = u_1(t) \\ \ldots \\ u_{m-2}'(t) = u_{m-1}(t) \\ u_{m-1}'(t) \in \varphi(x(t), u_0(t), \ldots, u_{m-1}(t))B \end{cases} \tag{7.15}$$

where the state space is $X \times Z^m$ and the set of constraints is $\mathrm{Graph}(U_m) \subset X \times Z^m$.

To say that R_m is a subregulation map regulating smooth solutions of order m amounts to saying that its closed graph is viable under the above system (7.15).

The metaregulation map of order m, which is the regulation map of the system (7.15) yielding viable solutions in the graph of R_m, is the set of velocities

$$(f(x, u_0), u_1, \ldots, u_{m-1}, u')$$

where $u' \in \varphi(x, u_0, \ldots, u_{m-1})B$ which are contingent to the graph of R_m at $(x, u_0, \ldots, u_m)$, i.e., which satisfy

$$u' \in DR_m(x, u_0, u_1, \ldots, u_{m-1})(f(x, u_0), u_1, \ldots, u_{m-1})$$

The graph of the largest subregulation map R_m^φ of order m is the viability kernel of Graph(U_m) for this system of differential inclusions. □

7.5 Punctuated Equilibria, Ramp Controls and Polynomial Open-Loop Controls

The case when the growth φ is equal to 0 is particularly interesting, because the inverse N^0 of the 0-growth regulation map R^0 determines the areas $N^0(u)$ regulated by constant control u.

One could call $N^0(u)$ the *viability cell or niche* of u. A control u is called a *punctuated equilibrium* if and only if its viability cell is not empty. Naturally, *when the viability cell of a punctuated equilibrium is reduced to a point, this point is an equilibrium.*

So, punctuated equilibria are constant controls which regulate the control systems (in its viability cell):

Proposition 7.5.1 *The viability cell of a control u is the viability kernel of $U^{-1}(u)$ for the differential equation $x'(t) = f(x(t), u)$ parametrized by the constant control u.*

Proof — Indeed, viability cells describe the regions of Dom(U) which are controlled by the constant control u because for any initial state x_0 given in $N^0(u)$, there exists a viable solution $x(\cdot)$ to the differential inclusion

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) = 0 \end{cases}$$

starting at (x_0, u), i.e., of the differential equation $x'(t) = f(x(t), u)$ which is *viable in the viability cell* $N^0(u)$ because $u \in R^0(x(t))$ for every $t \geq 0$. □

One can ask more generally whether linear open-loop controls $u(t) := u_0 + tu_1$ can regulate viable solutions to the control systems, and what are the largest areas of the viability domain which can be regulated by linear controls. Such controls are called *ramp controls* .

The advantage is that in such areas, finding the ramp controls amounts to looking only for two elements u_0 and u_1 in the finite dimensional space Z^2 rather than a general function $u(\cdot)$ in an infinite-dimensional space $W^{1,1}(0, \infty; Z, e^{bt})$.

Pursuing this point of view, the problem arises of *regulating viable solutions to a control system by polynomial open-loop controls of degree m.* For $m = 0$, we find the punctuated equilibria, for $m = 1$ the ramp controls, and so on.

We consider the graph $\mathrm{Graph}(R_m^0)$ of the largest m-smooth 0-growth regulation map of the system (7.15) and we denote by $K_m^0 := \mathrm{Im}(N_m^0)$.

Proposition 7.5.2 *We posit the assumptions of Theorem 7.4.1. Then $K_m^0 \subset K$ is the largest subset of initial states from which there exist viable solutions regulated by m-degree polynomial open-loop controls.*

Controlling the system from $x_0 \in K_m^0$ amounts to choosing initial controls $(u_0, u_1, \ldots, u_m) \in (N_m^0)^{-1}(x_0) \subset Z^{m+1}$. In this case, there exists a viable solution $x(\cdot)$ to the control system

$$x'(t) \;=\; f\left(x(t), u_0 + u_1 t + \ldots + u_{m-1}\frac{t^{m-1}}{(m-1)!}\right)$$

satisfying

$$x(0) \;=\; x_0, \;\; u(0) \;=\; u_0, \;\; u'(0) \;=\; u_1, \;\; \ldots, \;\; u^{(m-1)}(0) \;=\; u_{m-1}$$

and the regulation law written in the form

$$\forall\, t \geq 0, \;\; x(t) \;\in\; N_m^0\left(\sum_{j=0}^{m-1} u_j\frac{t^j}{j!}, \ldots, \sum_{j=0}^{m-k-1} u_{j+k}\frac{t^j}{j!}, \ldots, u_{m-1}\right)$$

We naturally obtain

$$K^0 \;:=\; K_1^0 \;\subset\; K_1^0 \;\subset\; \ldots \;\subset K_m^0 \;\subset\; \ldots \;\; K := \mathrm{Dom}(U)$$

and, for $k \leq m$,

$$N_k^0(u_0, \ldots, u_{k-1}) = N_m^0(u_0, \ldots, u_{k-1}, 0, \ldots, 0)$$

Remark — In the case of the general evolution of open-loop controls, the regulation maps are solutions to the partial differential inclusion

$$\forall (x,v) \in \text{Graph}(V), \quad Av \in DR(x,v)(g(x,v)) - \Phi(x,v)$$

subject to the constraint

$$\forall x \in X, \quad R(x) \subset V(x)$$

In particular, for $\Phi = 0$, we obtain the subset of initial states x_0 from which there exist viable solutions to the control system

$$x'(t) = f(x(t), e^{At}v_0)$$

regulated by open-loop controls

$$v(t) = e^{At}v_0$$

which are solutions to the system of differential equations

$$v'(t) = Av(t), \quad v(0) = v_0$$

7.6 Heavy Viable Solutions

7.6.1 Dynamical Closed Loops

Let us consider a control system (U, f), a regulation map $R \subset U$ which is a solution to the partial differential inclusion (7.7) and the metaregulation map

$$(x,u) \rightsquigarrow G_R(x,u) := DR(x,u)(f(x,u)) \cap \varphi(x,u)B$$

regulating smooth state-control solutions viable in the graph of R through the system (7.8) of differential inclusions.

The question arises as to whether we can construct selection procedures of the control component of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 7.6.1 (Dynamical Closed Loops) *Let R be a φ-growth subregulation map of U. We shall say that a selection g of the contingent derivative of the metaregulation map G_R associated with R mapping every $(x,u) \in \text{Graph}(R)$ to*

$$g(x,u) \in G_R(x,u) := DR(x,u)(f(x,u)) \cap \varphi(x,u)B \tag{7.16}$$

is a dynamical closed loop *of R.*

The system of differential equations

$$\begin{cases} i) & x'(t) = f(x(t),u(t)) \\ ii) & u'(t) = g(x(t),u(t)) \end{cases} \tag{7.17}$$

is called the associated closed loop differential system.

Clearly every solution to (7.17) is also a solution to (7.8). Therefore, a dynamical closed loop being given, solutions to the system of ordinary differential equations (7.17) (if any) are smooth state-control solutions of the initial control problem (7.1).

Such solutions do exist when g is continuous (and if such is the case, they will be continuously differentiable.) But they also may exist when g is no longer continuous, as we saw when we built closed loop controls in Chapter 6. This is the case for instance when $g(x,u)$ is the element of minimal norm in $G_R(x,u)$.

In both cases, we need to assume that the metaregulation map G_R associated with R is lower semicontinuous with closed convex images. By Proposition 7.1.3, it will be sufficient to assume that:

$$\begin{cases} i) & R \text{ is sleek} \\ ii) & \sup_{(x,u)\in\text{Graph}(R)} \|DR(x,u)\| < +\infty \end{cases} \tag{7.18}$$

Indeed, assumptions (7.18)i) and ii) imply that the set-valued map $(x,u,v) \rightsquigarrow DR(x,u,v)$ is lower semicontinuous. Since φ is continuous, we infer from Proposition 6.3.2 that the metaregulation map G_R is also lower semicontinuous.

We thus begin by deducing from Michael's Theorem 6.5.7 the existence of continuously differentiable viable state-control solutions.

Theorem 7.6.2 *Assume that* U *is closed and that* f, φ *are continuous and have linear growth. Let* $R(\cdot) \subset U(\cdot)$ *be a* φ*-growth subregulation map satisfying assumption (7.18). Then there exists a continuous dynamical closed loop* g *associated with* R. *The associated closed loop differential system (7.17) regulates continuously differentiable state-control solutions to (7.1) defined on* $[0, \infty[$.

7.6.2 Heavy Viable Solutions

Since we do not know constructive ways to build continuous dynamical closed loops, we shall investigate whether some explicit dynamical closed loop provides closed loop differential systems which do possess solutions.

The simplest example of dynamical closed loop control is the minimal selection of the metaregulation map G_R, which in this case is equal to the map g_R° associating with each state-control pair (x, u) the element $g_R^\circ(x, u)$ of minimal norm of $DR(x, u)(f(x, u))$ because for all (x, u), $\|g_R^\circ(x, u)\| \leq \varphi(x, u)$ whenever $G_R(x, u) \neq \emptyset$.

Definition 7.6.3 (Heavy Viable Solutions) *Denote by* $g_R^\circ(x, u)$ *the element of minimal norm of* $DR(x, u)(f(x, u))$. *We shall say that the solutions to the associated closed loop differential system*

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) = g_R^\circ(x(t), u(t)) \end{cases}$$

are heavy viable solutions *to the control system* (U, f) *associated with* R.

Theorem 7.6.4 (Heavy Viable Solutions) *Let us assume that* U *is closed and that* f, φ *are continuous and have linear growth. Let* $R(\cdot) \subset U(\cdot)$ *be a* φ*-growth subregulation map satisfying assumption (7.18). Then for any initial state-control pair* (x_0, u_0) *in Graph(R), there exists a heavy viable solution to the control system (7.1).*

Remark — *Any heavy viable solution* $(x(\cdot), u(\cdot))$ *to the control system (7.1) satisfies the inertia principle*: Indeed, we observe that if for some t_1, the solution enters the subset $C_R(u(t_1))$ where we set

$$C_R(u) := \{x \in K \mid 0 \in DR(x, u)(f(x, u))\}$$

the control $u(t)$ remains equal to $u(t_1)$ as long as $x(t)$ remains in $C_R(u(t_1))$. Since such a subset is not necessarily a viability domain, the solution may leave it.

If for some $t_f > 0$, $u(t_f)$ is a punctuated equilibrium, then $u(t) = u_{t_f}$ for all $t \geq t_f$ and thus, *$x(t)$ remains in the viability cell $N_1^0(u(t_f))$ for all $t \geq t_f$.* □

The reason why this theorem holds true is that the minimal selection is obtained through the selection procedure defined by

$$S^{\circ}_{G_R}(x,u) \; := \; \|g^{\circ}_R(x,u)\| \, B \tag{7.19}$$

It is this fact which matters. So, Theorem 7.6.4 can be extended to any selection procedure of the metaregulation map $G_R(x,u)$ defined in Chapter 6 (See Definition 6.5.2).

Theorem 7.6.5 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \text{ is continuous and has linear growth} \end{cases} \tag{7.20}$$

Let $(x,u) \rightarrow \varphi(x,u)$ be a nonnegative continuous function with linear growth and $R : Z \rightsquigarrow X$ a closed set-valued map contained in U.

Let S_{G_R} : Graph(R) $\rightsquigarrow X$ be a selection procedure with convex values of the metaregulation map G_R. Then, for any initial state $(x_0, u_0) \in$ Graph(R), there exists a state-control solution to the associated closed loop system

$$x' = f(x,u), \quad u' \in G_R(x,u) \cap S_{G_R}(x,u) \tag{7.21}$$

defined on $[0, \infty[$ and starting at (x_0, u_0). In particular, if for any $(x,u) \in$ Graph(R), the intersection

$$G_R(x,u) \cap S_{G_R}(x,u) \; = \; \{s\,(G_R(x,u))\}$$

is a singleton, then there exists a state-control solution defined on $[0, \infty[$ and starting at (x_0, u_0) to the associated closed loop system

$$x'(t) \; = \; f(x(t), u(t)), \;\; u'(t) \; = \; s\,(G_R(x(t), u(t)))$$

Proof — Consider the system of differential inclusions

$$x' = f(x,u), \quad u' \in S_{G_R}(x,u) \cap \varphi(x,u)B \tag{7.22}$$

subject to the constraints

$$\forall\, t \geq 0, \quad (x(t),u(t)) \in \text{Graph}(R)$$

Since the selection procedure S_{G_R} has a closed graph and convex values, the right-hand side is an upper semicontinuous set-valued map with nonempty compact convex images and with linear growth. On the other hand Graph(R) is a viability domain of the map $\{f(x,u)\}\times(S_{G_R}(x,u) \times \varphi(x,u)B)$. Therefore, the Viability Theorem can be applied. For any initial state-control $(x_0,u_0) \in$ Graph(R), there exists a solution $(x(\cdot),u(\cdot))$ to (7.22) which is viable in Graph(R). Consequently, for almost all $t \geq 0$, the pair $(x'(t),u'(t))$ belongs to the contingent cone to the graph of R at $(x(t),u(t))$, which is the graph of the contingent derivative $DR(x(t),u(t))$. In other words, for almost all $t \geq 0$, $u'(t) \in DR(x(t),u(t))(f(x(t),u(t)))$. Since $\|u'(t)\| \leq \varphi(x(t),u(t))$, we deduce that $u'(t) \in G_R(x(t),u(t))$ for almost all $t \geq 0$. Hence, the state-control pair $(x(\cdot),u(\cdot))$ is a solution to (7.21). □

Proof of Theorem 7.6.4 — By the Maximum Theorem 2.1.6 the map $(x,u) \mapsto \|g_R^\circ(x,u)\|$ is upper semicontinuous. It has a linear growth on Graph(R). Thus the set-valued map $(x,u) \rightsquigarrow \|g_R^\circ(x,u)\|\, B$ is a selection procedure satisfying the assumptions of Theorem 7.6.5. □

Since we know many examples of selection procedures, it is possible to multiply examples of dynamical closed-loops as we did for usual closed loops. We shall see some examples in the framework of differential games in Chapter 14.

7.6.3 Heavy Viable Solutions under Equality Constraints

Consider the case when $h : X \mapsto Y$ is a twice continuously differentiable map, when the viability domain is $K := h^{-1}(0)$ and when there are no constraints on the controls ($U(x) = Z$ for all $x \in K$). We derive from Proposition 7.2.8 the following explicit formulas for the dynamical closed loop yielding heavy solutions.

Proposition 7.6.6 *We posit assumptions of Theorem 7.2.8. Assume further that $U(x) \equiv Z$, that the regulation map*

$$R(x) := \{u \in Z \mid h'(x)f(x,u) = 0\}$$

has nonempty values, that $h(x)$ is surjective whenever $x \in K$ and that $h'(x)f'_u(x,u) \in \mathcal{L}(Z,Y)$ is surjective whenever $u \in R(x)$.

Then there exist heavy solutions viable in K, which are the solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -f'_u(x,u)^\star h'(x)^\star p(x,u) \text{ where} \\ & p(x,u) := (h'(x)f'_u(x,u)f'_u(x,u)^\star h'(x)^\star)^{-1} h'(x)f'_x(x,u)f(x,u) \end{cases}$$

Proof — The element $g(x,u) \in G(x,u)$ of minimal norm, being a solution to the quadratic minimization problem with equality constraints

$$h'(x)f'_u(x,u)w = -h'(x)f'_x(x,u)f(x,u) - h''(x)(f(x,u), f(x,u))$$

is equal to

$$\begin{array}{l} g(x,u) = -f'_u(x,u)^\star h'(x)^\star (h'(x)f'_u(x,u)f'_u(x,u)^\star h'(x)^\star)^{-1} \\ (h'(x)f'_x(x,u)f(x,u) + h''(x)(f(x,u), f(x,u))) \end{array}$$

because the linear operator $B := h'(x)f'_u(x,u) \in \mathcal{L}(Z,Y)$ is surjective.

Example: Heavy solutions viable in affine spaces. Consider the case when $K := \{x \in X \mid Lx = y\}$ where $L \in \mathcal{L}(X,Y)$ is surjective.

Let us assume that

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \text{ such that } Lf(x,u) = 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ Lf'_u(x,u) \text{ is surjective} \end{cases}$$

Then, for any initial state $x_0 \in K$ and initial velocity u_0 satisfying $Lf(x_0,u_0) = 0$, there exists a heavy viable solution given by the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -f'_u(x,u)^\star L^\star (Lf'_u(x,u)f'_u(x,u)^\star L^\star)^{-1} Lf'_x(x,u)f(x,u) \end{cases}$$

When $Y := \mathbf{R}$ and $K := \{x \in X \mid \ < p,x >= y\}$ is an hyperplane, the above assumption becomes

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \text{ such that } < p, f(x,u) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ f'_u(x,u)^\star p \neq 0 \end{cases}$$

and heavy viable solutions are solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -\frac{<p, f'_x(x,u), f(x,u)>}{\|f'_u(x,u)^\star p\|^2} f'_u(x,u)^\star p \end{cases}$$

Example: Heavy solutions viable in the sphere.

Let $L \in \mathcal{L}(X,X)$ be a symmetric positive-definite linear operator, with which we associate the viability subset

$$K := \{x \in X \mid < Lx, x >= 1\}$$

We assume that

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \text{ such that } < Lx, f(x,u) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ \ f'_u(x,u)^\star Lx \neq 0 \end{cases}$$

Then there exist heavy viable solutions in the sphere, which are solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -\frac{f'_u(x,u)^\star Lx}{\|f'_u(x,u)^\star Lx\|^2} p(x,u) \text{ where} \\ & p(x,u) :=< Lf(x,u), f(x,u) > + < Lx, f'_x(x,u) f(x,u) > \end{cases}$$

7.6.4 Heavy Viable Solutions of High Order

We shall extend the concept of heavy viable solutions to higher order.

For simplicity, we explain what happens for the first order, in the case when we want to satisfy both the inertia principle and a first-order inertia principle: keep a ramp control as long as it regulates a viable smooth solution.

We begin with the control system (U, f), we set $U_1 := U$ and $\varphi_1 := \varphi$, we choose a φ_1-growth subregulation map $R_1(\cdot) := R(\cdot) \subset U(\cdot)$ and we denote by

$$G_1(x,u) := DR_1(x,u)(f(x,u)) \cap \varphi_1(x,u)B$$

the metaregulation rule associated with R_1.

Since we know that the evolution of heavy viable solutions is governed by the differential equation

$$u_0'(t) = u_1(t) = g_1^\circ(x(t), u_0(t))$$

where g_1° is the minimal selection of G_1, the instinctive idea which comes to mind is to take for set-valued map U_1 the (single-valued) map g_1°. Unfortunately, its graph is not closed.

Since the minimal selection g_1° is obtained through the selection procedure defined by (7.19), another idea is to use any selection procedure S_{G_1} of the set-valued map G_1 and in particular the one defined by (7.19):

$$S_{G_1}^\circ(x, u) := \|g_1^\circ(x, u)\| \, B$$

We then define U_2 by

$$\text{Graph}(U_2) := (\text{Graph}(R_1) \times Z) \cap \text{Graph}(S_{G_1})$$

and we introduce a continuous function $\varphi_2 : \text{Graph}(U_2) \mapsto \mathbf{R}_+$ with linear growth.

The graph of U_2 is closed. This choice being made, we associate a φ_2-growth subregulation map $R_2 \subset U_2$ (for instance, the viability kernel of the graph of U_2.) We know that the evolution of the second derivative of the control is governed by the metaregulation law

$$u''(t) \in G_2(x(t), u(t), u'(t))$$

where we denote by

$$G_2(x, u_0, u_1) := DR_2(x, u_0, u_1)(f(x, u_0), u_1) \cap \varphi_2(x, u_0, u_1)B$$

the metaregulation map associated with R_2. We propose to govern the evolution of the second derivative of the control by selections of the map G_2, and in particular, by its selection of minimal norm g_2°, which then yields a second-order heavy viable solution.

Theorem 7.6.7 (Second-Order Heavy Viable Solutions) *Let us assume that U_1 is closed and that f, φ_1, φ_2 are continuous and have linear growth, that conditions (7.18) and*

$$\begin{cases} i) & \text{the subregulation map } R_2 \text{ is sleek} \\ ii) & \sup_{(x,u_0,u_1)\in\text{Graph}(R_2)} \|DR_2(x, u_0, u_1)\| < +\infty \end{cases} \tag{7.23}$$

hold true. Then for any initial data $u_1 \in R_2(x_0, u_0)$, there exists a second-order heavy viable solution to the control system (7.1), i.e., a solution to the system

$$\begin{cases} x'(t) &= f(x(t), u(t)) \\ u'(t) &= g_1^\circ(x(t), u(t)) \\ u''(t) &= g_2^\circ(x(t), u(t), u'(t)) \end{cases}$$

Remark — *Any second-order heavy viable solution satisfies the first-order inertia principle.*

For explaining why , let us introduce the subsets

$$\begin{cases} C_k(u_0, \ldots, u_{k-1}) \\ := \{x \in K | 0 \in DR_k(x, u_0, \ldots, u_{k-1})(f(x, u_0), \ldots, u_{k-1})\} \end{cases}$$

for $k = 1, 2$.

If for some t_0, the solution enters the subset $C_1(u(t_0))$, then the open-loop control $u(t)$ becomes constant as long as $x(t)$ remains in $C_1(u(t_0))$.

If for some t_1, the solution enters the subset $C_2(u(t_1), u'(t_1))$, then the open-loop control $u(t)$ becomes a ramp control as long as $x(t)$ remains in $C_2(u(t), u'(t))$. In this case, it is regulated by

$$x(t) \in N_2^0\left(u_{t_1} + (t - t_1)u'(t_1), u'(t_1)\right)$$

Since such a subset is not necessarily a viability domain, the solution may leave it.

If for some $t_r > 0$, the solution $x(t)$ enters the subset K_2^0, then it will be regulated by a ramp control, until some time[10] $t_f \in [t_r, \infty[$ where $x(t_f) \in K_1^0$. Then $u_{t_f} \in R_1(x(t_f))$ is a punctuated equilibrium, and $u(t) = u_{t_f}$ for all $t \geq t_f$, so that $x(t)$ *remains in the viability cell* $N_1^0(u(t_f))$ *for all* $t \geq t_f$. □

Naturally, as for heavy viable solutions, this theorem follows from:

[10] which may never be reached

Theorem 7.6.8 *Let us assume that the control system (7.1) and the functions φ_1, φ_2 satisfy*

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \ \& \ \varphi_i \text{ are continuous and have linear growth } (i = 1,2) \end{cases}$$

Let S_{G_1} : Graph$(R_1) \rightsquigarrow X$ be a selection procedure of the metaregulation map G_1, U_2 be defined by

$$\text{Graph}(U_2) \ := \ (\text{Graph}(R_1) \times Z) \cap \text{Graph}(S_{G_1})$$

$R_2 \subset U_2$ be a subregulation map and S_{G_2} : Graph$(R_2) \rightsquigarrow Z$ be a selection procedure of the metaregulation map G_2 with convex values. Set

$$\begin{cases} S(G_1)(x,u) \ := \ G_1(x,u) \cap S_{G_1}(x,u) \\ S(G_2)(x,u,u') \ := \ G_2(x,u,u') \cap S_{G_2}(x,u,u') \end{cases}$$

Then, for any initial state $(x_0, u_0, u_1) \in$ Graph(R_2), there exists a solution to the system

$$\begin{cases} x'(t) \ = \ f(x(t), u(t)) \\ u'(t) \ \in \ S(G_1)(x(t), u(t)) \\ u''(t) \ \in \ S(G_2)(x(t), u(t), u'(t)) \end{cases} \tag{7.24}$$

defined on $[0, \infty[$ and starting at (x_0, u_0, u_1).

In particular, if for any $(x, u, u') \in$ Graph(R_2), the intersections

$$S(G_1)(x,u) \ \& \ S(G_2)(x,u,u')$$

are singleta $\{s(G_1)(x,u)\}$ and $\{s(G_2)(x,u,u')\}$, then there exists a state-control solution defined on $[0, \infty[$ and starting at (x_0, u_0) to the associated closed loop system

$$\begin{cases} x'(t) \ = \ f(x(t), u(t)) \\ u'(t) \ = \ s(G_1)(x(t), u(t)) \\ u''(t) \ = \ s(G_2)(x(t), u(t), u'(t)) \end{cases}$$

Proof — We consider the system

$$\begin{cases} i) & x'(t) = f(x(t), u_0(t)) \\ ii) & u_0'(t) = u_1(t) \\ iii) & u_1'(t) \in S_{G_2}(x(t), u_0(t), u_1(t)) \cap \varphi_2(x(t), u_0(t), u_1(t))B \end{cases} \tag{7.25}$$

Since the selection procedure S_{G_2} has a closed graph and convex values, the right-hand side of this system of differential inclusions is a Marchaud map.

The closed subset $\text{Graph}(R_2)$ is a viability domain. Indeed, we know that there exists an element w in the selection $S(G_2)(x, u_0, u_1)$. Since $w \in G_2(x, u_0, u_1)) \subset DR_2(x, u_0, u_1)(f(x, u_0), u_1))$, we infer that

$$(f(x, u_0), u_1, w) \in \text{Graph}(DR_2(x, u_0, u_1)) := T_{\text{Graph}(R_2)}(x, u_0, u_1)$$

Hence $(f(x, u_0), u_1, w)$ is a velocity which is contingent to the graph of R_2.

Therefore the Viability Theorem implies the existence of a solution $(x(\cdot), u_0(\cdot), u_1(\cdot))$ to the system of differential inclusions (7.25) viable in the graph of R_2. This implies that for almost all $t \geq 0$, setting $u(\cdot) := u_0(\cdot)$,

$$u''(t) = u_1'(t) \in DR_2(x(t), u(t), u'(t))(f(x(t), u(t)), u'(t))$$

This, together with (7.25)iii), implies that for almost all $t \geq 0$,

$$u''(t) \in G_2(x(t), u(t), u'(t)) \cap S_{G_2}(x(t), u(t), u'(t))$$

Furthermore, since $\text{Graph}(R_2)$ is contained in $\text{Graph}(R_1) \times Z$, we deduce that

$$\forall\, t \geq 0, \;\; u(t) := u_0(t) \in R_1(x(t))$$

so that

$$\forall\, t \geq 0, \;\; u'(t) \in DR_1(x(t), u(t))(f(x(t)), u(t)) \subset G_1(x(t), u(t))$$

On the other hand, by the very choice of U_2, we know that

$$\forall\, t \geq 0,\;\; u'(t) \;:=\; u_1(t) \;\in\; R_2(x(t), u(t)) \;\subset\; S_{G_1}(x(t), u(t))$$

Hence we have proved the existence of a solution to the second-order system of partial differential inclusions (7.24) with a right-hand side which is not a Marchaud map. □

Naturally, we can extend this theorem up to the order m, by recursively choosing the map U_m by formula

$$\text{Graph}(U_m) \;:=\; (\text{Graph}(R_{m-1}) \times Z) \cap \text{Graph}(S_{G_{m-1}})$$

and by taking a subregulation map $R_m \subset U_m$ (for instance, the map whose graph is a viability kernel for the system (7.15).)

In the case of the minimal selection, we take as selection procedure

$$S^{\circ}_{G_m}(x, u_0, \ldots, u_{m-1}) \;:=\; \|g^{\circ}_m(x, u_0, \ldots, u_{m-1})\|\, B$$

where actually, $g^{\circ}_m(x, u_0, \ldots, u_{m-1})$ is the element of minimal norm of

$$DR_m(x, u_0, \ldots, u_{m-1})(f(x, u_0), \ldots, u_{m-1})$$

Theorem 7.6.9 (m-th Order Heavy Viable Solutions) *Assume that U is closed and that f, φ_k are continuous and have linear growth for $0 \leq k \leq m$. We assume further that for $0 \leq k \leq m$,*

$$\begin{cases} i) & \text{the subregulation map } R_k \subset U_k \text{ is sleek} \\ ii) & \sup_{(x,u_0,\ldots,u_{k-1})\in\text{Graph}(R_k)} \|DR_k(x, u_0, \ldots, u_{k-1})\| < +\infty \end{cases}$$

Then for any initial data $u_{m-1} \in R_m(x_0, u_0, \ldots, u_{m-2})$, there exists an m-th order heavy viable solution to the control system

$$\begin{cases} x'(t) \;=\; f(x(t), u(t)) \\ u'(t) \;=\; g^{\circ}_0(x(t), u(t)) \\ \cdots \\ u^{(m)}(t) \;=\; g^{\circ}_m(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t)) \end{cases}$$

It obeys an *m-th order inertia principle: keep an m-degree polynomial open-loop control as long as the solution it regulates is viable.*

This theorem follows from the more general

Theorem 7.6.10 *Let us assume that the control system (7.1) and the functions* φ_k *satisfy for* $0 \leq k \leq m$

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \ \& \ \varphi_k \text{ are continuous and have linear growth} \end{cases}$$

Let $S_{G_k} : \text{Graph}(R_k) \rightsquigarrow X$ *be selection procedures with convex values of the set-valued maps* G_k. *Set*

$$S(G_k)(x, u_0, \ldots, u_{k-1}) := S_{G_k}(x, u_0, \ldots, u_{k-1}) \cap G_k(x, u_0, \ldots, u_{k-1})$$

Then, for any initial state $(x_0, u_0, u_1, \ldots, u_{m-1}) \in \text{Graph}(R_m)$, *there exists a solution to the system*

$$\begin{cases} x'(t) & = & f(x(t), u(t)) \\ u'(t) & \in & S(G_1)(x(t), u(t)) \\ \ldots & & \\ u^{(m)}(t) & \in & S(G_m)(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t)) \end{cases}$$

defined on $[0, \infty[$ *and starting at* $(x_0, u_0, u_1, \ldots u_{m-1})$.