Chapter 5

Invariance Theorems for Differential Inclusions

Introduction

We devote this chapter to subsets *invariant* under a set-valued map, to invariance domains, kernels and envelopes, and to some of their properties.

Since the invariance property of a subset K involves the behavior of F outside of K, we need to extend the contingent cone to a subset K to the whole domain of F: we define for that purpose the concept of *external contingent cone* to K at any element $x \in X$.

Also, to proceed further, we need some regularity property of the subset, a kind of "$\mathcal{C}^1$-regularity", which here takes the following form: *the set-valued map* $K \ni x \rightsquigarrow T_K(x)$ *is lower semicontinuous.* Since this property will be used quite often, we give it a name: *sleekness*. We shall check that *the contingent cones to sleek subsets are convex. Convex subsets as well as smooth manifolds are sleek.*

Since we have seen the crucial role played by these contingent cones in viability theorems, we take this opportunity to study them further and to mention their calculus summarized in Table 5.2 for the convenience of the reader. Details are provided in chapter 4 of SET-VALUED ANALYSIS.

The second section is devoted to criteria for a subset to be *invariant under a set-valued map*. These criteria involve the concepts of external contingent cone introduced in the first section.

In the third section, we shall derive from Filippov's Theorem[1] the characterization of closed subsets K locally invariant under a Lipschitz set-valued map F as closed invariance domains.

We define in the fourth section the *invariance kernel* of a closed subset K, which is the largest closed subset of K invariant under F. We prove its existence when the solution map of the differential inclusion is lower semicontinuous. We also introduce the *invariance envelopes*, which are the smallest closed subsets containing K invariant under F, and relate them to the backward invariance kernel of the complement of K.

We study the stability of sequences of closed subsets invariant by set-valued maps F_n and invariance kernels, by showing for instance that the lower limit of invariance kernels of closed subsets K_n is contained in the invariance kernel of the lower limit.

We devote the fifth section to the study of semipermeability and viability properties of the boundaries of the viability and invariance kernels of a closed subset. We apply these results to define the defeat and victory domains of an open target and show that the boundary of the victory domain is a semipermeable barrier.

We illustrate in the sixth section the notions and results obtained so far with the example of *linear differential inclusions* $x' \in F(x)$, where the right-hand side F is a closed convex process. We mention in particular that in the case of linear differential inclusions, a closed convex cone is an invariance domain if and only if its polar cone is a viability domain of the transpose. In this sense, one can say that the concepts of viability and invariance are dual.

5.1 External Contingent Cones

5.1.1 External Contingent Cones

We begin by introducing the following notation:

$$D_{\uparrow} d_K(x)(v) \ := \ \liminf_{h \to 0+} (d_K(x + hv) - d_K(x))/h$$

[1]that we shall not prove here. We refer to Hélène Frankowska's monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS or to DIFFERENTIAL INCLUSIONS for an exposition of the fundamental Filippov's Theorem and its numerous applications.

which will be justified later[2]. We observe that when $x \in K$, a direction v is contingent to K at x if and only if $D_{\uparrow}d_K(x)(v) \leq 0$.

In order to study invariance properties of a subset K which involve the behavior of the set-valued map F outside of K, we need to extend our definition of the contingent cone to points outside of K:

Definition 5.1.1 *Let K be a subset of a finite dimensional vector-space X and x belong to X. We extend the notion of contingent cone to the one of* external contingent cone *to K at points outside of K in the following way:*

$$T_K(x) := \{v \mid D_{\uparrow}d_K(x)(v) \leq 0\}$$

We point out an easy but important relation between the external contingent cone at a point and the contingent cone at its projection:

Lemma 5.1.2 *Let K be a closed subset of a finite dimensional vector-space and $\Pi_K(y)$ be the set of projections of y onto K, i.e., the subset of $z \in K$ such that $\|y - z\| = d_K(y)$. Then the following inequalities:*

$$D_{\uparrow}d_K(y)(v) \quad \leq \quad d(v, T_K(\Pi_K(y)))$$

hold true. Therefore,

$$T_K(\Pi_K(y)) \subset T_K(y)$$

Proof — Choose $z \in \Pi_K(y)$ and $w \in T_K(z)$. Then

$$\begin{cases} \dfrac{d_K(y+hv) - d_K(y)}{h} \leq \dfrac{\|y-z\| + d_K(z+hv) - d_K(y)}{h} \\[2ex] = \dfrac{d_K(z+hv)}{h} \leq \dfrac{d_K(z+hw)}{h} + \|v - w\| \end{cases}$$

Since z belongs to K and $w \in T_K(z)$, the above inequality implies that

$$D_{\uparrow}d_K(y)(v) \quad \leq \quad d(v, T_K(z)) \;\square$$

[2]this is the contingent epiderivative of the distance functions d_K. (See Definition 9.1.2 of Chapter 9.)

5.1.2 Sleek Subsets

We define now the tangent cone $C_K(x)$ introduced in 1975 by F. H. Clarke.

Definition 5.1.3 *Let $K \subset X$ be a subset of a normed space X and $x \in \overline{K}$ belong to the closure of K. We define the* (Clarke) tangent cone *(or* circatangent cone*)* $C_K(x)$ *by*

$$C_K(x) := \{v \mid \lim_{h \to 0+, K \ni x' \to x} d_K(x' + hv)/h = 0\}$$

We see at once that $C_K(x) \subset T_K(x)$ and that if x belongs to Int(K), then $C_K(x) = X$.

It is very convenient to observe that when x belongs to $\overline{K}$,

$$\begin{cases} v \in C_K(x) \text{ if and only if } \forall\ h_n \to 0+,\ \forall\ K \ni x_n \to x, \\ \exists\ v_n \to v \text{ such that } \forall\ n,\ x_n + h_n v_n \in K \end{cases}$$

The charm of the tangent cone C_K at x is that *it is always convex*[3]. Unfortunately, *the price to pay for enjoying this convexity property of the Clarke tangent cones is that they may often be reduced to the trivial cone* $\{0\}$.

However, we shall show that the Clarke tangent cone and the contingent cone do coincide at those points x where the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous:

Definition 5.1.4 (Sleek Subsets) *We shall say that a subset K of X is* sleek *at $x \in K$ if the set-valued map*

$$K \ni x' \rightsquigarrow T_K(x') \text{ is lower semicontinuous at } x$$

and that it is sleek *if and only if it is sleek at every point of K.*

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[3]Let v_1 and v_2 belong to $C_K(x)$. To prove that $v_1 + v_2$ belongs to this cone, let us choose any sequence $h_n > 0$ converging to 0 and any sequence of elements $x_n \in K$ converging to x. There exists a sequence of elements v_{1n} converging to v_1 such that the elements $x_{1n} := x_n + h_n v_{1n}$ do belong to K for all n. But since x_{1n} does also converge to x in K, there exists a sequence of elements v_{2n} converging to v_2 such that

$$\forall\ n,\ x_{1n} + h_n v_{2n} = x_n + h_n(v_{1n} + v_{2n}) \in K$$

This implies that $v_1 + v_2$ belongs to $C_K(x)$ because the sequence of elements $v_{1n} + v_{2n}$ converges to $v_1 + v_2$.

Theorem 5.1.5 *Let K be a closed subset of a finite dimensional vector-space X. Consider a set-valued map $F : K \rightsquigarrow X$ satisfying*

$$\begin{cases} i) & F \text{ is lower semicontinuous at } x \\ ii) & \exists\, \delta > 0 \quad \text{such that } \forall\, z \in B_K(x,\delta),\ \ F(z) \subset T_K(z) \end{cases}$$

Then $F(x) \subset C_K(x)$.

In particular, if K is sleek at $x \in K$, then $T_K(x) = C_K(x)$ is a closed convex cone.

Proof — Let us take $x \in K$ and $v \in F(x)$, assumed to be different from 0. Since F is lower semicontinuous at x, Corollary 2.1.7 implies that we can associate with any $\varepsilon > 0$ a number $\eta \in]0,\delta[$ such that $d(v,F(z)) \leq d(v,F(x)) + \varepsilon = \varepsilon$ for any $z \in B_K(x,\eta)$ (because $d(v,F(x)) = 0$). Therefore, for any $y \in B(x,\eta/4)$ and $\tau \leq \eta/4\|v\|$, the inequality

$$\forall\, z \in \Pi_K(y+\tau v),\ \ \|z-x\| \leq 2\|y+\tau v - x\| \leq 2\|x-y\| + 2\tau\|v\| \leq \eta$$

implies that

$$\begin{cases} d(v, T_K(\Pi_K(y+\tau v))) & \leq \ d(v, F(\Pi_K(y+\tau v))) \\ & \leq \ d(v,F(x)) + \varepsilon \ = \ \varepsilon \end{cases}$$

We set $g(\tau) := d_K(y+\tau v)$. By Lemma 5.1.2, we obtain

$$\begin{cases} \liminf_{h\to 0+} \left(g(\tau+h) - g(\tau)\right)/h \ = \ D_\uparrow d_K(y+\tau v)(v) \\ \leq \ d(v, T_K(\Pi_K(y+\tau v))) \ \leq \ \varepsilon \end{cases}$$

The function g being Lipschitz, it is almost everywhere differentiable, so that $g'(t) \leq \varepsilon$ for almost all t small enough. Integrating this inequality from 0 to h, we obtain

$$d_K(y+hv) \ = \ g(h) \ = \ g(h) - g(0) \ \leq \ \varepsilon h$$

for any $y \in B(x,\eta/4)$ and $\tau \leq \eta/4\|v\|$. This shows that v belongs to $C_K(x)$.

By taking $F(x) = T_K(x)$, we deduce that $T_K(x) \subset C_K(x)$ whenever K is sleek at $x \in K$, and thus, that they coincide. □

5.1.3 Tangent Cones to Convex Sets

For convex subsets K, the Clarke tangent cone and the contingent cone coincide with the closed cone spanned by $K - x$:

Proposition 5.1.6 (Tangent Cones to Convex Sets) *We denote by*

$$S_K(x) \;:=\; \bigcup_{h>0} \frac{K-x}{h}$$

the cone spanned by $K - x$. If K is convex, the contingent cone $T_K(x)$ to K at x is convex and

$$C_K(x) \;=\; T_K(x) \;=\; \overline{S_K(x)}$$

The subnormal cone is equal to

$$N_K(x) \;:=\; S_K(x)^- \;=\; \{\, p \in X^\star \mid \max_{y\in K} < p, y >=< p, x > \}$$

Furthermore, the normal cones $N_K(x)$ to a convex subset K are contained in the barrier cone of K: for every $x \in K$, $N_K(x) \subset b(K)$.

Remark — We shall denote by $T_K(x)$ the common value of these cones, and call it the *tangent cone* to the convex subset K at x. The subnormal cone coincides with the normal cone of K at x of convex analysis. □

Actually, closed convex subsets are sleek:

Theorem 5.1.7 *Any closed convex subset of a finite dimensional vector-space X is sleek.*

We refer to Theorem 4.2.2 of SET-VALUED ANALYSIS for the proof of this Theorem. □

It may be useful to recall the characterization of the interior of the tangent cone to a convex subset.

Proposition 5.1.8 (Interior of a Tangent Cone) *Assume that the interior of $K \subset X$ is not empty. Then*

$$\forall\, x \in K, \quad \mathrm{Int}(T_K(x)) \;=\; \bigcup_{h>0} \left(\frac{\mathrm{Int}(K) - x}{h} \right)$$

Furthermore, the graph of the set-valued map $K \ni x \rightsquigarrow \mathrm{Int}(T_K(x))$ is open.

For the convenience of the reader, we list in the Table 5.1 some useful formulas of the calculus of tangent cones to convex subsets (see Section 4.1. of APPLIED NONLINEAR ANALYSIS, in which the subsets K, K_i, L, M, ... are assumed to be convex.)

We shall need the following characterization of the normal cone to a convex cone:

Lemma 5.1.9 *Let $K \subset X$ be a convex cone of a normed space X and $x \in K$. Then*

$$p \in N_K(x) \iff x \in K,\ p \in K^- \ \&\ < p, x > = 0 \iff x \in N_{K^-}(p)$$

where $N_{K^-}(p) := \{x \in K \mid \forall\, q \in K^-,\ \ < q - p, x > \leq 0\}$.

Proof — To say that $p \in N_K(x)$ means that $< p, x >= \sigma_K(p)$, which is equal to 0 if and only if $p \in K^-$, and the first statement of the lemma follows. □

5.1.4 Calculus of Contingent Cones

We summarize in Table 5.2 the calculus of contingent cones. Formulas (1) to (4) are straightforward. The other properties are valid when K is sleek, and are a consequence of the Constrained Inverse Function Theorem, which we do not prove in this book[4].

See also Quincampoix's Theorem 4.3.3 and the remark following it for another set of sufficient conditions.

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[4]We refer to Chapter 4 of SET-VALUED ANALYSIS for the proofs of these formulas and more detailed results.

We mention also that transversality condition of formula (5) implies the constraint qualification assumption $0 \in \mathrm{Int}(f(L) - M)$ and that the stronger transversality condition

$$\exists\, c > 0 \mid \forall\, x \in K,\ \ B_Y \subset f'(x)(T_L(x) \cap cB_X) - T_M(Ax)$$

implies that if L and M are sleek and f is continuously differentiable, then K is also sleek.

Table 5.1: **Properties of Tangent Cones to Convex Sets.**

(1)	▷	If $x \in K \subset L \subset X$, then $T_K(x) \subset T_L(x)$ & $N_L(x) \subset N_K(x)$
(3)	▷	If $x_i \in K_i \subset X_i, \quad (i = 1, \cdots, n)$, then $T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n T_{K_i}(x_i)$ $N_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n N_{K_i}(x_i)$
(4)a)	▷	If $A \in \mathcal{L}(X, Y)$ and $x \in K \subset X$, then $T_{A(K)}(Ax) = \overline{A(T_K(x))}$ $N_{A(K)}(Ax) = A^{\star^{-1}} N_K(x)$
(4)b)	▷	If $K_1,\ K_2 \subset X$, $x_i \in K_i$, $i = 1, 2$, then $T_{K_1+K_2}(x_1 + x_2) = \overline{T_{K_1}(x_1) + T_{K_2}(x_2)}$ $N_{K_1+K_2}(x_1 + x_2) = N_{K_1}(x_1) \cap N_{K_2}(x_2)$ In particular, if $x_1 \in K$ and x_2 belongs to a closed subspace P of X, then $T_{K+P}(x_1 + x_2) = \overline{T_{K_1}(x_1) + P}$ $N_{K+P}(x_1 + x_2) = N_K(x_1) \cap P^{\perp}$
(5)	▷	If $L \subset X$ and $M \subset Y$ are *closed convex* subsets and $A \in \mathcal{L}(X, Y)$ satisfies the *constraint qualification assumption* $0 \in \mathrm{Int}(M - A(L))$, then, for every $x \in L \cap A^{-1}(M)$, $T_{L \cap A^{-1}(M)} = T_L(x) \cap A^{-1} T_M(Ax)$ $N_{L \cap A^{-1}(M)} = N_L(x) + A^{\star} N_M(Ax)$
(5)a)	▷	If $M \subset Y$ is *closed convex* and if $A \in \mathcal{L}(X, Y)$ satisfies $0 \in \mathrm{Int}(\mathrm{Im}(A) - M)$, then, for any $x \in A^{-1}(M)$, $T_{A^{-1}(M)}(x) = A^{-1} T_M(Ax)$ $N_{A^{-1}(M)}(x) = A^{\star} N_M(Ax)$
(5)b)	▷	If $K_1,\ K_2 \subset X$ are closed convex and satisfy $0 \in \mathrm{Int}(K_1 - K_2)$, then, for any $x \in K_1 \cap K_2$ $T_{K_1 \cap K_2}(x) = T_{K_1}(x) \cap T_{K_2}(x)$ $N_{K_1 \cap K_2}(x) = N_{K_1}(x) + N_{K_2}(x)$
(5)c)	▷	If $K_i \subset X, \quad (i = 1, \ldots, n)$, are closed and convex, $x \in \bigcap_{i=1}^n K_i$ and if there exists $\gamma > 0$ satisfying $\forall x_i$ such that $\|x_i\| \leq \gamma,\ \bigcap_{i=1}^n (K_i - x_i) \neq \emptyset$, then $T_{\bigcap_{i=1}^n K_i}(x) = \bigcap_{i=1}^n T_{K_i}(x)$ $N_{\bigcap_{i=1}^n K_i}(x) = \sum_{i=1}^n N_{K_i}(x)$

Table 5.2: **Properties of Contingent Cones.**

(1)	▷	If $K \subset L$ and $x \in \overline{K}$, then $T_K(x) \subset T_L(x)$
(2)	▷	If $K_i \subset X$, $(i = 1, \cdots, n)$ and $x \in \overline{\bigcup_i K_i}$, then $T_{\cup_{i=1}^n K_i}(x) = \cup_{i \in I(x)} T_{K_i}(x)$ where $I(x) := \{i \mid x \in \overline{K_i}\}$
(3)	▷	If $K_i \subset X_i$, $(i = 1, \cdots, n)$ and $x_i \in \overline{K_i}$, then $T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) \subset \prod_{i=1}^n T_{K_i}(x_i)$
(4)	▷	If $g \in \mathcal{C}^1(X, Y)$, if $K \subset X$, $x \in \overline{K}$ and $M \subset Y$, then $g'(x)(T_K(x)) \subset T_{g(K)}(g(x))$ $T_{g^{-1}(M)}(x) \subset g'(x)^{-1} T_M(g(x))$
(5)	▷	If $L \subset X$ and $M \subset Y$ are *closed sleek* subsets, $f \in \mathcal{C}^1(X, Y)$ is a continuously differentiable map and $x \in L \cap f^{-1}(M)$ satisfies the *transversality condition* $f'(x)T_L(x) - T_M(f(x)) = Y$, then $T_{L \cap f^{-1}(M)}(x) = T_L(x) \cap f'(x)^{-1} T_M(f(x))$
(5)a)	▷	If $M \subset Y$ is a *closed sleek* subset, $f \in \mathcal{C}^1(X, Y)$ is a continuously differentiable map and $x \in f^{-1}(M)$ satisfies $\mathrm{Im}(f'(x)) - T_M(f(x)) = Y$, then $T_{f^{-1}(M)}(x) = f'(x)^{-1} T_M(f(x))$
(5)b)	▷	If K_1 and K_2 are *closed sleek* subsets contained in X and $x \in K_1 \cap K_2$ satisfies $T_{K_1}(x) - T_{K_2}(x) = X$, then $T_{K_1 \cap K_2}(x) = T_{K_1}(x) \cap T_{K_2}(x)$
(5)c)	▷	If $K_i \subset X$, $(i = 1, \ldots, n)$, are *closed sleek* and $x \in \bigcap_i K_i$ satisfies $\forall v_i = 1, \ldots, n, \ \bigcap_{i=1}^n (T_{K_i}(x) - v_i) \neq \emptyset$ then, $T_{\bigcap_{i=1}^n K_i}(x) = \bigcap_{i=1}^n T_{K_i}(x)$

5.1.5 Inequality Constraints

We also state the following example of the contingent cone to a set defined by equality and inequality constraints[5]:

Theorem 5.1.10 *Let us consider a closed subset L of a finite dimensional vector-space X and two continuously differentiable maps $g := (g_1, \ldots, g_p) : X \mapsto \mathbf{R}^p$ and $h := (h_1, \ldots, h_q) : X \mapsto \mathbf{R}^q$ defined on an open neighborhood of L.*

Let K be the subset of L defined by the constraints

$$K := \{x \in L \mid g_i(x) \geq 0,\ i = 1, \ldots, p,\ \ \&\ h_j(x) = 0,\ j = 1, \ldots, q\}$$

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of active constraints.

We posit the following transversality condition at a given $x \in K$:

$$\begin{cases} i) & h'(x)C_L(x) = \mathbf{R}^q \\ ii) & \exists\, v_0 \in C_L(x) \quad \text{such that } h'(x)v_0 = 0 \\ & \text{and } \forall\, i \in I(x),\ \ < g_i'(x), v_0 >> 0 \end{cases}$$

Then u belongs to the contingent cone to K at x if and only if u belongs to the contingent cone to L at x and satisfies the constraints

$$\forall\, i \in I(x),\ \ < g_i'(x), u > \geq\ 0\ \ \&\ \ \forall\, j = 1, \ldots, q,\ \ h_j'(x)u = 0$$

Unfortunately, the graph of $T_K(\cdot)$ is not necessarily closed. However, there exists a closed set-valued map $T_K^{\diamond}(\cdot)$ contained in $T_K(\cdot)$ introduced by N. Maderner. Set

$$\gamma_K(x) := \min_{i \notin I(x)} \frac{g_i(x)}{\|g_i'(x)\|} \in \,]0, +\infty] \tag{5.1}$$

We observe that γ_K *is upper semicontinuous whenever the functions g_i are continuously differentiable*. Indeed, let $x_n \in K$ converge to x_0 and $a_n \leq \gamma_K(x_n)$ converge to a_0. Since $g_i(x_0) > 0$ whenever $i \notin I(x_0)$, we infer that $i \notin I(x_n)$ for n large enough. Hence inequalities

[5] See Proposition 4.3.6 of SET-VALUED ANALYSIS

$a_n\|g_i'(x_n)\| \leq g_i(x_n)$ hold true for any $i \notin I(x_0)$ and imply at the limit that $a_0 \leq \gamma_K(x_0)$.

The growth of the function γ_K *is linear whenever we assume that there exists a constant* $c > 0$ *such that*

$$\forall\, i = 1, \ldots, p, \;\; \|g_i'(x)\| \;\geq\; c\,\frac{g_i(x)}{\|x\|+1}$$

Theorem 5.1.11 (Maderner) *We posit the assumptions of Theorem 5.1.10. Then the set-valued map* $T_K^{\diamond}(\cdot) : K \rightsquigarrow X$ *defined by:*
$u \in T_K^{\diamond}(x)$ *if and only if* $u \in T_L(x)$ *and*

$$\begin{cases} \forall\, i = 1, \ldots, p, & g_i(x) + < g_i'(x), u > \;\geq\; 0 \\ \forall\, j = 1, \ldots, q, & h_j'(x)u \;=\; 0 \end{cases}$$

is contained in the contingent cone $T_K(x)$ *and satisfy*

$$T_K(x) \cap \gamma_K(x)B \;\subset\; T_K^{\diamond}(x)$$

Its graph is closed whenever the graph of $x \rightsquigarrow T_L(x)$ *is closed.*

Proof — Let u belong to $T_K^{\diamond}(x)$. Then, when $i \in I(x)$, we see that $\langle g_i'(x), u\rangle = g_i(x) + \langle g_i'(x), u\rangle \geq 0$, so that $u \in T_K(x)$.

Conversely, let us choose u in $T_K(x)$ satisfying $\|u\| \leq \gamma_K(x)$. Then either $i \in I(x)$ and $g_i(x) + \langle g_i'(x), u\rangle = \langle g_i'(x), u\rangle \geq 0$ or $g_i(x) > 0$ so that

$$i \notin I(x) \;\;\&\;\; g_i(x) + \langle g_i'(x), u\rangle \;\geq\; g_i(x) \;-\; \|g_i'(x)\|\|u\| \;\geq\; 0$$

because $\|u\| \leq \gamma_K(x) \leq g_i(x)/\|g_i'(x)\|$. Thus, in both cases, $g_i(x) + \langle g_i'(x), u\rangle \geq 0$, so that u belongs to $T_K^{\diamond}(x)$. □

5.2 Invariance Domains

Let us consider the differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t)) \tag{5.2}$$

We recall the definition of invariant subsets K under a set-valued map F: *A subset* K *is said to be* (locally) invariant under F *(or* *enjoys the* invariance property*) if for any initial state* x_0 *of* K*,* all *solutions to the differential inclusion (5.2) starting at* x_0 *are* viable (on some interval $[0,T]$).

We emphasize again that the concept of invariance *depends upon the behavior of* F *on the domain of* F *outside of* K. But we can tackle this issue since we have extended the concept of contingent cone to K at points outside of K (Definition 5.1.1). This enables us to provide an *Invariance Criterion* (by contrast with the Strict Invariance Theorem 4.3.6).

Theorem 5.2.1 *Let* K *be a subset of the domain of a nontrivial set-valued map* F*. If* F *is locally bounded and if*

$$\forall\, x \in \mathrm{Dom}(F),\quad F(x) \;\subset\; T_K(x)$$

then K *is invariant under* F*.*

Proof — Let $x(\cdot) \in \mathcal{S}(x_0)$ be any solution to the differential inclusion (5.2) defined on some interval $[0,T]$. Let us set $g(t) := d_K(x(t))$, which is absolutely continuous. Let t be a point where both $x'(t)$ and $g'(t)$ exist. Then there exists $\varepsilon(h)$ converging to 0 with h such that $x(t+h) = x(t) + hx'(t) + h\varepsilon(h)$ and

$$\begin{cases} g'(t) & = \lim_{h\to 0+} \frac{d_K(x(t)+hx'(t)+h\varepsilon(h))-d_K(x(t))}{h} \\ & = D_\uparrow d_K(x(t))(x'(t)) \end{cases}$$

Since $x'(t) \in F(x(t)) \subset T_K(x(t))$ almost everywhere, we infer that $g'(t) \le 0$ for almost all t. Therefore $x(\cdot)$ is viable whenever the initial state x_0 is in K. If not, there would exist $t > 0$ such that $x(t) \notin K$. But we derive a contradiction since:

$$0 < d_K(x(t)) = d_K(x(t)) - d_K(x(0)) = g(t) - g(0) = \int_0^t g'(\tau)d\tau \le 0 \quad \square$$

We are tempted to call an invariance domain of F a subset $K \subset \mathrm{Dom}(F)$ satisfying the condition $F(x) \subset T_K(x)$ for all $x \in \mathrm{Dom}(F)$. But actually, we shall study the stronger property where the above condition holds true only for $x \in K$.

Definition 5.2.2 (Invariance Domain) *Let* $F : X \rightsquigarrow X$ *be a nontrivial set-valued map. We shall say that a subset* $K \subset \mathrm{Dom}(F)$ *is an* invariance domain *of* F *if*

$$\forall x \in K, \quad F(x) \subset T_K(x)$$

Since the contingent cone to a singleton is reduced to 0, we observe that a singleton $\{\overline{x}\}$ is an invariance domain if and only if $\overline{x}$ is a "stopping point" of F, i.e., a solution to the inclusion

$$F(\overline{x}) \;=\; \{\, 0 \,\}$$

(No velocity can take such a stopping point away.)

Corollary 5.2.3 *Let* K *be a subset of the domain of a nontrivial set-valued map* F*. Assume that* F *satisfies*

$$\forall\, x \in \mathrm{Dom}(F), \;\; F(x) \;\subset\; F(\Pi_K(x))$$

If K *is an invariance domain, then it is invariant under* F*.*

Proof — It follows from Theorem 5.2.1, since $F(x) \subset F(\Pi_K(x)) \subset T_K(\Pi_K(x)) \subset T_K(x)$ thanks to Lemma 5.1.2. □

For instance, when K is a closed convex set, we can extend a set-valued map $F : K \rightsquigarrow X$ to a set-valued map $\widetilde{F} : X \rightsquigarrow X$ by setting

$$\forall\, x \in X, \;\; \widetilde{F}(x) \;:=\; F(\pi_K(x))$$

Corollary 5.2.4 *Let* K *be a closed convex subset and* $F : K \rightsquigarrow X$ *be a set-valued map satisfying*

$$\forall\, x \in K, \;\; F(x) \;\subset\; T_K(x)$$

Then K *is invariant under the extension* $\widetilde{F}$ *of* F*.*

Corollary 5.2.5 *Let* K *be a closed subset of the domain of a nontrivial set-valued map* F*. If*

$$\forall\, x \in \mathrm{Dom}(F), \;\; \forall\, v \in F(x), \;\; \forall\, y \in \Pi_K(x), \;\; < x - y, v > \;\leq\; 0$$

then K *is invariant under* F*.*

Proof — It follows trivially from Corollary 5.2.3 and Proposition 3.2.3. □

We can regard the next result as a *structural stability* property:

Proposition 5.2.6 *Let us assume that K is convex with nonempty interior. Assume that the graph of F is compact and that*

$$\forall\, x \in K, \;\; F(x) \subset \mathrm{Int}(T_K(x))$$

Then there exists a neighborhood $\mathcal{U}$ of the graph of F such that the above condition is verified for all set-valued maps G whose graph is contained in $\mathcal{U}$.

Proof — Since the graph of F is compact and contained in the graph of $K \ni x \rightsquigarrow \mathrm{Int}\,(T_K(x))$ which is open by Proposition 5.1.8, the latter is such a neighborhood $\mathcal{U}$. □

5.3 Invariance Theorem

5.3.1 Filippov's Theorem

In order to characterize the local invariance property of a closed subset K, i.e., to prove that K is an invariance domain of F, we need to know that given any $x \in K$ and $v \in F(x)$, there exists a solution $x(\cdot)$ to differential inclusion (5.2) such that $x(0) = x$ and $x'(0) = v$.

This is the case when the right-hand side F is Lipschitz in a neighborhood of K, thanks to the Filippov Theorem[6]. Actually, Filippov's Theorem is much more than a mere existence theorem. It also provides an estimate of the distance between a function $y(\cdot)$ and the set $\mathcal{S}_F(x_0)$ of solutions starting at some initial state x_0.

Theorem 5.3.1 (Filippov) *Assume that $F : X \rightsquigarrow X$ is λ-Lipschitz with closed values on the interior of its domain. Let $y(\cdot)$ be a given*

[6]We do not provide the proof of the Filippov Theorem, but refer the reader to Corollary 2.4.1, p.121 of DIFFERENTIAL INCLUSIONS or to Hélène Frankowska's CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS.

absolutely continuous function such that $t \to d(y'(t), F(y(t)))$ *is integrable(for the measure* $e^{-\lambda s}ds$*). We associate with a fixed* x_0 *the function* η *defined by*

$$\eta(t) \;=\; e^{\lambda t}\left(\|x_0 - y(0)\| \;+\; \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right)$$

Let $T > 0$ *be finite or infinite chosen such that the tube*

$$\{y(t) + \eta(t)B\}_{t\in[0,T[}$$

is contained in the interior of the domain of F.

Then there exists a solution $x(\cdot)$ *to differential inclusion (5.2) such that, for all* $t \in [0, T[$,

$$\|x(t) - y(t)\| \leq e^{\lambda t}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \tag{5.3}$$

and for almost all $t \in [0, T[$,

$$\begin{cases} \|x'(t) \;-\; y'(t)\| \;\leq\; d(y'(t), F(y(t))) \\ \\ +\lambda e^{\lambda t}\left(\|x_0 - y(0)\| \;+\; \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \end{cases}$$

Proof — Filippov's Theorem yields an estimate on any finite interval $[0, T]$ such that the tube $\{y(t) + \eta(t)B\}_{t\in[0,T]}$ is contained in the interior of the domain of F.

Actually, we can extend it to the interval $[0, +\infty[$ if the tube

$$\{y(t) + \eta(t)B\}_{t\in[0,+\infty[}$$

is contained in the interior of the domain of F. Indeed, there exists a solution $x(\cdot)$ to differential inclusion (5.2) defined on $[0, T]$ starting at x_0 satisfying estimate (5.3) and in particular

$$\|x(T) - y(T)\| \;\leq\; e^{\lambda T}\left(\|x_0 - y(0)\| \;+\; \int_0^T d(y'(s), F(y(s)))e^{-\lambda s}ds\right)$$

There also exists a solution $z(\cdot)$ to differential inclusion (5.2) starting at $x(T)$ estimating the function $t \mapsto y(t + T)$ and satisfying

$$\begin{cases} \|z(t) - y(t+T)\| \\ \leq\; e^{\lambda t}\left(\|x(T) - y(T)\| \;+\; \int_0^t d(y'(s+T), F(y(s+T)))e^{-\lambda s}ds\right) \end{cases}$$

Hence we can extend $x(\cdot)$ on the interval $[0, 2T]$ by concatenating it with the function $t \mapsto x(t) := z(t-T)$ on the interval $[T, 2T]$ and we observe that the above estimates yield (5.3) for $t \in [0, 2T]$. We reiterate this process as long as the tube $\{y(t) + \eta(t)B\}_{t\in[0,nT]}$ is contained in the interior of the domain of F. □

It implies the existence of a solution:

Corollary 5.3.2 *Assume that F is Lipschitz on the interior of its domain. Then, for any $x_0 \in \mathrm{Int}(\mathrm{Dom}(F))$ and $v_0 \in F(x_0)$, there exist $T > 0$ and a solution $x(\cdot)$ to differential inclusion (5.2) defined on $[0, T]$ and satisfying $x(0) = x_0$ and $x'(0) = v_0$.*

Proof — We apply Filippov's Theorem with $y(t) := x_0 + tv_0$ and $x_0 := y(0)$. Then $d(y'(t), F(y(t))) \leq \lambda t \|v_0\|$ and

$$\eta(t) \;\leq\; e^{\lambda t}\int_0^t \lambda\tau\|v_0\|e^{-\lambda\tau}d\tau \;\leq\; \frac{\|v_0\|}{\lambda}\left(e^{\lambda t} - 1 - \lambda t\right)$$

Filippov's Theorem implies the existence of a solution $x(\cdot)$ to differential inclusion (5.2) starting at x_0 and satisfying

$$\|x(t) - x_0 - tv_0\| \;\leq\; \frac{\|v_0\|}{\lambda}\left(e^{\lambda t} - 1 - \lambda t\right)$$

Dividing by $t > 0$ and letting t converge to $0+$, we infer $x'(0) = v_0$.
□

It also implies the Lipschitz dependence of the solution map on the initial condition:

Corollary 5.3.3 *Let $y(\cdot) \in \mathcal{S}_F(y_0)$ and assume that F, $y(\cdot)$ satisfy the assumptions of Filippov's Theorem 5.3.1. Then*

$$d\left(y(t),\; \mathcal{S}_F(x_0)(t)\right) \;\leq\; \|x_0 - y_0\|\, e^{\lambda t}$$

so that the solution map $\mathcal{S}_F$ is lower semicontinuous.

Remark — Observe that if we set

$$\delta(t) \;:=\; d\left(\mathrm{Exit}_F(K, t), \partial K\right)$$

Filippov's Theorem 5.3.1 implies that for all $0 < T_0 < T$,

$$\forall\, x \in \mathrm{Exit}_F(K, T),\;\; B\left(x, \frac{\delta(T - T_0)}{e^{\lambda T_0}}\right) \;\subset\; \mathrm{Exit}_F(K, T_0) \;\square$$

5.3.2 Characterization of Local Invariance

We are ready to prove the characterization of invariant domains under a Lipschitz map:

Theorem 5.3.4 *Let us assume that F is Lipschitz on the interior of its domain and has compact values. Then a closed subset $K \subset$ $\mathrm{Int}(\mathrm{Dom}(F))$ is locally invariant under F if and only if K is an invariance domain.*

Proof — Let us assume that K is an invariance domain and let $x(\cdot)$ be any solution to differential inclusion (5.2) starting at x_0 and defined on some interval $[0,T]$. Let us set $g(t) := d_K(x(t))$, which is absolutely continuous on $[0,T]$.

Let t be a point such that both $x'(t)$ and $g'(t)$ exist and $x'(t)$ belongs to $F(x(t))$. Then there exists $\varepsilon(h)$ converging to 0 with h such that

$$x(t+h) \;=\; x(t) + hx'(t) + h\varepsilon(h)$$

and

$$\begin{cases} g'(t) \;=\; \lim_{h\to 0+}(d_K(x(t)+hx'(t)+h\varepsilon(h)) - d_K(x(t)))/h \\ \;=\; D_{\uparrow}d_K(x(t))(x'(t)) \end{cases}$$

Lemma 5.1.2 implies that

$$D_{\uparrow}d_K(x)(x'(t)) \;\leq\; d(x'(t), T_K(\Pi_K(x(t))))$$

Let us denote by $\lambda > 0$ the Lipschitz constant of F and choose any y in $\Pi_K(x(t))$. We deduce that:

$$d(x'(t), T_K(\Pi_K(x(t)))) \leq d(x'(t), T_K(y)) \leq d(x'(t), F(y))$$
(since K is an invariance domain)

$$\leq d(x'(t), F(x(t))) + \lambda\|y - x(t)\| \quad \text{(since } F \text{ is Lipschitz)}$$

$$=\; 0 + \lambda d_K(x(t)) \;=\; \lambda g(t)$$

Then g is a solution to

$$\text{for almost all } t \in [0,T],\;\; g'(t) \;\leq\; \lambda g(t) \;\;\&\;\; g(0) = 0$$

We deduce that $g(t) = 0$ for all $t \in [0, T]$, and therefore, that $x(t)$ is viable in K on $[0, T]$.

— Let us assume that K is locally invariant under F. Let $x_0 \in K$. We have to prove that any $u_0 \in F(x_0)$ is contingent to K at x_0. Corollary 5.3.2 implies that for all x_0 and $u_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to differential inclusion (5.2) satisfying $x(0) = x_0$ and $x'(0) = u_0$. Since K is locally invariant under F, it is viable on some interval $[0, T]$. We thus infer that u_0 belongs to $T_K(x_0)$. Hence $F(x_0)$ is contained in $T_K(x_0)$. □

5.3.3 Graphical Lower Limits of Solution Maps

Let us recall the concepts of *lower limits* of subsets and of *graphical lower limit* of set-valued maps.

Let K_n be a sequence of subsets of a metric space X. We say that

$$K^\flat \ := \ \mathrm{Liminf}_{n\to\infty} K_n \ := \ \{y \in X \mid \lim_{n\to\infty} d(y, K_n) = 0\}$$

is its *lower limit.* In other words, it is the closed subset of limits of sequences of elements $x_n \in K_n$.

We shall say that the set-valued map $\mathrm{Lim}^\flat{}_{n\to\infty} F_n$ from X to X defined by

$$\mathrm{Graph}(\mathrm{Lim}^\flat{}_{n\to\infty} F_n) \ := \ \mathrm{Liminf}_{n\to\infty} \mathrm{Graph}(F_n)$$

is the *graphical lower limit of the set-valued maps* F_n. For simplicity, we set $F^\flat := \mathrm{Lim}^\flat{}_{n\to\infty} F_n$.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L, M) \ := \ \sup_{y\in L} \inf_{z\in M} d(y, z) \ = \ \sup_{y\in L} d(y, M)$$

their *semi-Hausdorff distance*[7], and recall that $\Delta(L, M) = 0$ if and only if $L \subset M$. If Φ and Ψ are two set-valued maps, we set

$$\Delta(\Phi, \Psi)_\infty \ = \ \sup_{x\in X} \Delta(\Phi(x), \Psi(x))$$

[7]The Hausdorff distance between L and M is equal to $\max(\Delta(L, M), \Delta(M, L))$.

Filippov's Theorem provides an example of a situation where the solution map $\mathcal{S}_F$ is the graphical lower limit of a sequence of solution maps $\mathcal{S}_{F_n}$.

Theorem 5.3.5 *Let $F_n : X \rightsquigarrow X$ and $F : X \rightsquigarrow X$ be λ-Lipschitz set-valued maps with closed images and uniform linear growth: there exists $c > 0$ such that*

$$\forall\, n \geq 0, \ \ \forall\, x \in X, \ \ \|F_n(x)\| \ \leq \ c(\|x\| + 1)$$

Then

$$\Delta\left(\mathcal{S}_F(x_0), \mathcal{S}_{F_n}(x_{0n})\right)_\infty \ \leq \ e^{\lambda t}\|x_0 - x_{0n}\| + \frac{e^{\lambda t} - 1}{\lambda}\Delta(F, F_n)_\infty$$

and

$$\Delta\left(\mathcal{S}_F, \mathcal{S}_{F_n}\right)_\infty \ \leq \ \frac{e^{\lambda t} - 1}{\lambda}\Delta(F, F_n)_\infty$$

Consequently, if[8] $\lim_{n\to\infty} \Delta(F, F_n)_\infty = 0$*, then*

$$\mathcal{S}_F \ \subset \ \mathrm{Lim}^\flat{}_{n\to\infty}\left(\mathcal{S}_{F_n}\right)$$

Proof — Let us consider any solution $x(\cdot) \in \mathcal{S}_F(x_0)$ to differential inclusion (5.2). Therefore,

$$d(x'(t), F_n(x(t))) \ \leq \ \Delta(F(x(t)), F_n(x(t))) \ \leq \ \Delta(F, F_n)_\infty$$

By Filippov Theorem 5.3.1 applied to the map F_n, there exists a solution $x_n(\cdot) \in \mathcal{S}_{F_n}(x_{0n})$ such that

$$\begin{cases} \|x_n(t) - x(t)\| \ \leq \ e^{\lambda t}\|x_0 - x_{0n}\| + \displaystyle\int_0^t e^{\lambda(t-s)}\Delta(F, F_n)_\infty ds \\ \\ = \ e^{\lambda t}\|x_0 - x_{0n}\| + \Delta(F, F_n)_\infty \dfrac{e^{\lambda t} - 1}{\lambda} \end{cases}$$

Then for any $t \geq 0$, $x(t)$ is the limit of $x_n(t)$, so that our claim is proved. □

[8]This implies that F is contained in the graphical lower limit $F^\flat$ of the set-valued maps F_n.

Remark — We can obtain other estimates. Set

$$\Delta(\Phi,\Psi)_1 \;=\; \sup_{x\in X}\frac{\Delta(\Phi(x),\Psi(x))}{\|x\|+1}$$

Let $F_n : X \rightsquigarrow X$ and $F : X \rightsquigarrow X$ be λ-Lipschitz set-valued maps with closed images and uniform linear growth. *Then, for any $\lambda > c$,*

$$\Delta\left(\mathcal{S}_F,\mathcal{S}_{F_n}\right)_1 \;\leq\; \frac{e^{\lambda t}-e^{ct}}{\lambda - c}\Delta(F,F_n)_1$$

so that $\Delta\left(\mathcal{S}_F,\mathcal{S}_{F_n}\right)_1$ converges to 0 and thus

$$\mathcal{S}_F \;\subset\; \mathrm{Lim}^{\flat}{}_{n\to\infty}\left(\mathcal{S}_{F_n}\right)$$

when $\lim_{n\to\infty}\Delta(F,F_n)_1 \;=\; 0.$

Indeed, consider any solution $x(\cdot)\in\mathcal{S}_F(x_0)$ to differential inclusion (5.2). Since

$$\begin{cases} d(x'(t),F_n(x(t))) \leq \Delta(F(x(t)),F_n(x(t))) \leq \Delta(F,F_n)_1(\|x(t)\|+1) \\ \leq\; \Delta(F,F_n)_1(\|x_0\|+1)e^{ct} \end{cases}$$

Filippov Theorem 5.3.1 applied to the map F_n implies that there exists a solution $x_n(\cdot)\in\mathcal{S}_{F_n}(x_0)$ such that

$$\begin{cases} \|x_n(t)-x(t)\| \;\leq\; e^{\lambda t}\int_0^t \Delta(F,F_n)_1(\|x_0\|+1)e^{-(\lambda-c)s}ds \\ =\; \Delta(F,F_n)_1(\|x_0\|+1)\frac{e^{\lambda t}-e^{ct}}{\lambda-c} \quad \square \end{cases}$$

5.3.4 Accessibility Map

We recall that the *reachable map* R_F is defined by

$$R_F(t)x \;:=\; (\mathcal{S}_F(x))(t)$$

(See Definition 3.5.4.)

Definition 5.3.6 *We shall denote by $\mathcal{R}_F : X \rightsquigarrow X$ the map defined by*

$$\mathcal{R}_F(x) \;:=\; \bigcup_{T\geq 0} R_F(T)x$$

and call it the accessibility map.

Proposition 5.3.7 *Assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty closed values. Then $\mathcal{R}_F$ maps open subsets onto open subsets. If K is a closed subset satisfying $K = \overline{\mathrm{Int}(K)}$, then*

$$\overline{\mathrm{Int}\,(\mathcal{R}_F(K))} \;=\; \overline{\mathcal{R}_F\,(\mathrm{Int}(K))}$$

Proof — Let Ω be an open subset and fix any $y \in \mathcal{R}_F(x)$ where $x \in \Omega$: By definition, there exist $T > 0$ and a solution $x(\cdot)$ on $[0, T]$ to the differential inclusion (5.2) starting at x such that $x(T) = y$. Let $y(\cdot)$ be a solution to the backward inclusion $y' \in -F(y)$ starting at x and consider the solution $\widetilde{y}(\cdot) \in \mathcal{S}_{-F}(y)$ to the reverse differential inclusion defined by

$$\widetilde{y}(s) \;:=\; \begin{cases} x(T-s) & \text{if } 0 \le s \le T \\ y(s-T) & \text{if } T \le s < \infty \end{cases}$$

Since $-F$ is Lipschitz, Filippov's Theorem 5.3.1 implies that there exists a neighborhood $\mathcal{N}(y)$ of y such that, for every $z \in \mathcal{N}(y)$, one can find a solution $z(\cdot) \in \mathcal{S}_{-F}(z)$ satisfying $z(T) \in \Omega$. This means that z can be reached from Ω in finite time.

We thus deduce that $\mathcal{R}_F(\mathrm{Int}(K))$ is contained in $\mathrm{Int}(\mathcal{R}_F(K))$, so that the inclusion

$$\overline{\mathcal{R}_F(\mathrm{Int}(K))} \;\subset\; \overline{\mathrm{Int}\,(\mathcal{R}_F(K))}$$

holds true. It remains to prove the converse inclusion when we assume that $K = \overline{\mathrm{Int}(K)}$. We shall actually prove that any $y \in \mathcal{R}_F(K)$ belongs to $\overline{\mathcal{R}_F(\mathrm{Int}(K))}$. We know that there exist $x \in K$, $T > 0$ and a solution $x(\cdot)$ to differential inclusion (5.2) defined on $[0, T]$ starting at x such that $x(T) = y$. Take any $\varepsilon > 0$. Since $x \in \overline{\mathrm{Int}(K)}$, Filippov's Theorem 5.3.1 implies that there exists $\delta > 0$ such that for any $z \in B(x, \delta) \cap \mathrm{Int}(K)$, one can obtain a solution $z(\cdot)$ to differential inclusion (5.2) starting at z and satisfying $z(T) \in B(y, \varepsilon)$. Hence y can be approximated by elements $z(T) \in \mathcal{R}_F(\mathrm{Int}(K))$. □

5.3.5 Proof of Convergence of the Fast Viability Kernel Algorithm

Proposition 5.3.8 *Assume that F is both Marchaud and λ-Lipschitz. Let x belong to the outward area $K_{\Rightarrow}$ and set*

$$\delta_K(x) \ := \ d(F(x), T_K(x))/2 \ > \ 0$$

We denote by $\theta_K(x) > 0$ the largest positive number θ such that

$$\forall\, h \in]0, \theta], \ \ d\left(x + h\left(F(x) + \delta_K(x)B\right), K\right) \ > \ 0$$

(which does exist). Let us fix $r > 0$ and set

$$\begin{cases} C_x \ := \ F(B(x,r)), \ \ T := \min\{r/\|C_x\|, 1/\lambda\} \\ t_K(x) \ := \ \min\{\delta_K(x)/2\lambda\|C_x\|, \theta_K(x), T\} \\ \varepsilon_K(x) := \delta_K(x)t_K(x)/2e^{\lambda t_K(x)} \end{cases}$$

Then $\varepsilon_K(x)$, which depends only upon x and K and does not involve $\mathrm{Viab}_F(K)$*, satisfies*

$$\overset{\circ}{B}\,(x, \varepsilon_K(x)) \cap \mathrm{Viab}_F(K) \ = \ \emptyset$$

Proof — The compactness of $F(x) + \delta_K(x)B$ and the very definition of the contingent cone imply that there exists a positive $\theta > 0$ such that

$$\forall\, h \in]0, \theta], \ \ d\left(x + h\left(F(x) + \delta_K(x)B\right), K\right) \ > \ 0$$

(See the proof of Proposition 4.3.5.) Therefore $\theta_K(x) > 0$ is positive and we observe that

$$\forall\, h \in]0, \theta_K(x)], \ \ d\left(x + h\left(F(x) + \frac{\delta_K(x)}{2}B\right), K\right) \ > \ \frac{\delta_K(x)h}{2}$$

Let us consider any solution $x(\cdot) \in \mathcal{S}(x)$ starting at x. Since $F(y) \subset C_x$ when y ranges over the ball $B(x,r)$, we first infer that

$$\forall\, t \leq r/\|C_x\|, \ \ \|x(t) - x\| \ \leq \ \int_0^t \|F(x(\tau))\|d\tau \ \leq \ \|C_x\|t$$

Since F is λ-Lipschitz, we deduce that

$$\begin{cases} x(t) - x \in \int_0^t F(x(\tau)d\tau \subset \int_0^t (F(x) + \lambda\|x(\tau) - x\|B)d\tau \\ \subset t\left(F(x) + \frac{\delta_K(x)}{2}B\right) \end{cases}$$

whenever $t < t_K(x) := \min\{\delta_K(x)/2\lambda\|C_x\|, \theta_K(x), T\}$. Consequently, for every positive $t < t_K(x)$,

$$d(x(t), K) \geq d\left(x + t\left(F(x) + \frac{\delta_K(x)}{2}B\right), K\right) \geq \frac{\delta_K(x)}{2}t$$

Furthermore, by the Filippov Theorem 5.3.1, we know that for any $y(\cdot) \in \mathcal{S}(y)$, there exists a solution $x(\cdot) \in \mathcal{S}(x)$ such that

$$\|x(t) - y(t)\| \leq e^{\lambda t}\|x - y\|$$

We set $\varepsilon_K(x) := \delta_K(x)t_K(x)/2e^{\lambda t_K(x)}$. This implies that for any $y \in \overset{\circ}{B}(x, \varepsilon_K(x))$,

$$\begin{cases} d(y(t_K(x)), K) \geq d(x(t_K(x)), K) - \|x(t_K(x)) - y(t_K(x))\| \\ \geq \delta_K(x)t_K(x)/2 - e^{\lambda t_K(x)}\|x - y\| > 0 \end{cases}$$

This means that such initial states y do not belong to the viability kernel of K, because all solutions leave K in finite time. □

We shall need the following result.

Lemma 5.3.9 *Let P be a convex closed cone with compact sole*[9] *and M be a compact subset of X. Then there exists $y \in M$ such that:*

$$(y + P) \cap M = \{y\}$$

---

[9]Let P be a closed convex cone. We recall that the following conditions are equivalent:

$$\begin{cases} i) & \text{P is spanned by a convex compact set disjoint from } 0 \\ ii) & \text{the interior of the polar cone } P^+ \text{ is not empty} \\ iii) & S := \{x \in P | < p_0, x >= 1\} \text{ where } p_0 \in \mathrm{Int}(P^+) \text{ spans } P; \end{cases}$$

The compact convex subset S is called the *sole*, and such closed convex cones are called *cones with compact sole.*

Proof — The proof follows from Zorn's lemma. Let us define the following preorder relation on M:

$$a \leq b \iff b \in a+P$$

which is actually an order since P has a compact sole. We next prove that every subset L of M *which is totally ordered* has a majorant.

Clearly, for any $a \in L$, $(a+P) \cap M \neq \emptyset$. Since these sets are nonempty and compact and since $(b+P) \cap M \subset (a+P) \cap M$ whenever $a \leq b$, we deduce that:

$$\bigcap_{a \in L} ((a+P) \cap M) \neq \emptyset$$

Let b belong to $\bigcap_{a \in L}(a+P) \cap M$. Obviously, b is larger than any element of L for the order $\leq$. According to Zorn's lemma, there exists a maximal element $y \in M$: Namely, if $z \in M$ is different from y , then, $y \notin z+P$. Hence, $(y+P) \cap M = \{y\}$. □

Proof of Theorem 4.4.6 — By Lemma 4.4.5, we already know that

$$\mathrm{Viab}_F(K) = \mathrm{Viab}_F(\widetilde{K}_\infty) \subset \widetilde{K}_\infty$$

Assume that $\widetilde{K}_\infty$ is not a viability domain: there would exist $x \in \widetilde{K}_{\infty \Rightarrow}$. Set

$$\delta_\infty := \delta_{\widetilde{K}_\infty}(x) := \frac{d\left(F(x), T_{\widetilde{K}_\infty}(x)\right)}{2}$$

and

$$\theta_\infty := \theta_{\widetilde{K}_\infty}(x) > 0$$

We shall derive a contradiction by constructing a sequence of elements $x_n \in \widetilde{K}_{n \Rightarrow}$ converging to x such that $\varepsilon_{\widetilde{K}_n}(x_n)$ is bounded below by some $\varepsilon_\infty > 0$ that we shall define: In this case, we would have $\|x_n - x\| \geq \varepsilon_{\widetilde{K}_n}(x_n) \geq \varepsilon_\infty$ because

$$x \in \widetilde{K}_{n+1} \subset \widetilde{K}_n \backslash \overset{\circ}{B}(x_n, \varepsilon_{\widetilde{K}_n}(x_n))$$

by the very definition of the algorithm and thus, the contradiction ensues. We thus have to define this positive lower bound ε_∞.

Since the convex compact set $F(x) + \delta_\infty B$ does not contain 0, the cone P spanned by this set has a compact sole. Set

$$M_n := \widetilde{K}_n \cap (x + [0, \theta_\infty](F(x) + \delta_\infty B))$$

We can assert, thanks to Lemma 5.3.9, that:

$$\exists\, x_n \in M_n \text{ such that } (x_n + P) \cap M_n \;=\; \{x_n\}$$

On the other hand, by the very definition of $\widetilde{K}_\infty$ and the choice of x, the sequence x_n converges to x. Hence for all n large enough,

$$x_n \in\; x + \left[0, \frac{\theta_\infty}{2}\right] (F(x) + \delta_\infty B)$$

Thus,

$$\widetilde{K}_n \cap \left(x_n + \left[0, \tfrac{\theta_\infty}{2}\right] (F(x) + \delta_\infty B)\right) \;\subset$$

$$\widetilde{K}_n \cap (x + [0, \theta_\infty] \,(F(x) + \delta_\infty B)) \cap (x_n + P) = (x_n + P) \cap M_n = \{x_n\}$$

Since F is Lipschitz, we have for n large enough, $F(x_n) \subset F(x) + \delta_\infty B/2$, so that for any $t < \theta_\infty/2$,

$$d\left(x_n + t\,(F(x_n) + \delta_\infty B/2)\,, \widetilde{K}_n\right) \;\geq\; d\left(x_n + t\,(F(x) + \delta_\infty B)\,, \widetilde{K}_n\right) \;>\; 0$$

Thus $d(F(x_n), T_{\widetilde{K}_n}(x_n)) \geq \delta_\infty/2$, i.e., $\delta_{\widetilde{K}_n}(x_n) \geq \delta_\infty/4$. By Proposition 5.3.8, we deduce that $\theta_{\widetilde{K}_n}(x_n) \geq \theta_\infty/2$ and thus, setting

$$t_\infty \;:=\; \min\left\{\frac{\theta_\infty}{2}, \frac{\delta_\infty}{2\lambda\|C_x\|}, T\right\}$$

that $t_{\widetilde{K}_n}(x_n) \geq t_\infty/2$. Since $t \mapsto t/e^{\lambda t}$ is increasing for $0 \leq t \leq 1/\lambda$, we infer that

$$\varepsilon_\infty \;:=\; \frac{\delta_\infty t_\infty}{16 e^{\lambda t_\infty/2}} \;\leq\; \frac{\delta_{\widetilde{K}_n}(x_n) t_{\widetilde{K}_n}(x_n)}{2e^{\lambda t_{\widetilde{K}_n}(x_n)}} \;:=\; \varepsilon_{\widetilde{K}_n}(x_n)$$

We have thus constructed a lower bound ε_∞ of the radii $\varepsilon_{\widetilde{K}_n}(x_n)$ for n large enough which implies the contradiction we claimed at the beginning of the proof. □

5.4 Invariance Kernels

We now introduce the concepts of invariance kernel and envelope:

Definition 5.4.1 (Invariance Kernels and Envelopes) *Let K be a subset of the domain of a set-valued map $F : X \rightsquigarrow X$. The largest closed subset of K invariant under F, which we denote by* $\mathrm{Inv}_F(K)$ *or* $\mathrm{Inv}(K)$*, is called the* invariance kernel *of K. We shall say that the smallest closed subset invariant under F containing K is the* invariance envelope $\mathrm{Env}_F(K)$ *of K.*

Since the intersection of closed subsets invariant under F is still a closed subset invariant under F, the invariance envelope of a closed subset does exist.

5.4.1 Existence of the Invariance Kernel

We now prove the existence of the invariance kernel of a closed subset (possibly empty).

Recall that $\mathcal{S}_F$ denotes the solution map associating with any x_0 the set of solutions to differential inclusion $x' \in F(x)$ starting at x_0 and that it is lower semicontinuous when F is Lipschitz with closed values (see Corollary 5.3.3.) We shall set

$$\Omega := \mathrm{Dom}\,(\mathcal{S}_F)$$

Naturally, invariant subsets are necessarily contained in Ω. We supply the space $\mathcal{C}(0,\infty;X)$ with the topology of pointwise convergence.

Theorem 5.4.2 *Let us assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from Ω to $\mathcal{C}(0,\infty;X)$. Then, for any closed subset $K \subset \Omega$, there exists an invariance kernel (possibly empty) of K. It is the subset of initial points such that all solutions starting from them are viable in K.*

Proof — Let us denote by $\mathcal{K} \subset \mathcal{C}(0,+\infty;X)$ the subset of continuous functions $x(\cdot)$ which are viable in K and by Inv(K) the subset of initial state $x \in K$ such that $\mathcal{S}_F(x) \subset \mathcal{K}$, possibly empty.

Since the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ supplied with the topology of pointwise convergence and since $\mathcal{K}$ is closed, we deduce that Inv(K) is also a closed subset of K (See Proposition 1.4.4 of SET-VALUED ANALYSIS.)

It obviously contains any closed subset of K invariant under F.

It remains to be shown that it is also invariant under F. For that purpose, let us take $x \in$ Inv(F) and show that any solution $x(\cdot) \in \mathcal{S}_F(x)$ is viable in Inv(K) (by checking that for any $T > 0$, $x(T) \in$ Inv(K)). Let $y(\cdot)$ belongs to $\mathcal{S}_F(x(T))$. Hence the function $z(\cdot)$ defined by

$$z(t) := \begin{cases} x(t) & \text{if } t \in [0, T] \\ y(t-T) & \text{if } t \in [T, \infty[\end{cases}$$

is a solution to the differential inclusion (5.2) starting at x at time 0, and thus, is viable in K by the very definition of Inv(K). Hence for all $t \geq 0$, $y(t) = z(t+T)$ belongs to K, so that we have proved that $\mathcal{S}_F(x(T)) \subset \mathcal{K}$, i.e., $x(T) \in$ Inv(K). □

Remark — It is clear that

$$\text{Inv}(K_1 \cap K_2) = \text{Inv}(K_1) \cap \text{Inv}(K_2)$$

and more generally, that the invariance kernel of any intersection of closed subsets K_i $(i \in I)$ is the intersection of the invariance kernels of the K_i. □

5.4.2 Complement of the Invariance Kernel

Proposition 5.4.3 *Assume that $K \subset \Omega :=$ Dom$(\mathcal{S}_F)$ is compact with nonempty interior, that $F(K)$ is bounded and that its invariance kernel $\text{Inv}_F(K)$ is contained in the interior of K. Then the complement $\Omega \backslash \text{Inv}_F(K)$ of the invariance kernel is viable under F.*

Proof — Since we assume that the invariance kernel is compact, there exists $\eta > 0$ such that $\text{Inv}_F(K) + 2\eta B \subset K$.

We observe that property

$$\forall\, x(\cdot) \in \mathcal{S}_F(x),\ \exists\, t \leq \tau_K(x(\cdot)) \quad \text{such that} \quad x(t) \in \text{Inv}_F(K)$$

implies that x belongs to the invariance kernel of K.

Therefore, if $x_0 \in K \backslash \mathrm{Inv}_F(K)$, there exists a solution $x_1(\cdot) \in \mathcal{S}_F(x_0)$ such that $x_1(t) \notin \mathrm{Inv}_F(K)$ for every $t \in [0, \tau_K(x_1(\cdot))]$.

If $\tau_K(x_1(\cdot)) = +\infty$, we deduce that $x_1(\cdot)$ is viable in $K \backslash \mathrm{Inv}_F(K)$. If not, we set $t_1 := \tau_K(x_1(\cdot))$ and $x_1 := x_1(t_1) \in \partial K$.

Let $x_2(\cdot) \in \mathcal{S}_F(x_1)$ and define $\rho(x_2(\cdot)) := \inf\{t \geq 0 \mid x_2(t) \in \mathrm{Inv}_F(K) + \eta B\}$. Then either $\rho(x_2(\cdot)) = +\infty$ and the solution obtained in concatenating $x_1(\cdot)$ and $x_2(\cdot)$ is viable in $\Omega \backslash \mathrm{Inv}_F(K)$, or $t_2 := \rho(x_2(\cdot))$ is finite and $x_2 := x_2(t_2) \in \partial(\mathrm{Inv}_F(K) + \eta B)$.

We also check that $t_2 - t_1 \geq \eta / \|F(K)\|$ because $\|x_2 - x_1\| \leq (t_2 - t_1)\|F(K)\|$ and $\|x_2 - x_1\| \geq \eta$.

Now we iterate this procedure to construct a solution $x(\cdot)$ which is viable in $X \backslash \mathrm{Inv}_F(K)$. □

Let us point out this easy but useful remark:

Proposition 5.4.4 *If the boundary ∂K of a closed subset $K \subset \mathrm{Dom}(\mathcal{S}_F)$ is invariant under F, so is K.*

Proof — Indeed, take x_0 in the interior of K and any solution $x(\cdot) \in \mathcal{S}_F(x_0)$. If it is not viable in K, there would exist a finite exit time $T := \inf\{s \geq 0 \mid x(s) \notin K\}$, at which $x(T) \in \partial K$. Since the boundary is invariant, any solution starting at $x(T)$ remains in ∂K. This is the case of the solution $y(\cdot)$ defined by $y(t) := x(t + T)$, so that $x(t) \in \partial K$ for every $t \geq T$. This contradicts the assumption that $x(\cdot)$ is not viable in K. □

5.4.3 Stability of Invariance Domains

Let us consider now a sequence of closed subsets K_n invariant under a set-valued map F. *Is their lower limit still invariant under F?*

Proposition 5.4.5 *Let us assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from Ω to $\mathcal{C}(0, \infty; X)$. Then the lower limit of closed subsets $K_n \subset \Omega$ invariant under F is also invariant under F.*

In particular, the lower limit of the invariance kernels of a sequence of closed subsets $K_n \subset \Omega$ contains the invariance kernel of the lower limit of the sequence K_n:

$$\mathrm{Liminf}_{n\to\infty}\left(\mathrm{Inv}(K_n)\right) \supset \mathrm{Inv}\left(\mathrm{Liminf}_{n\to\infty} K_n\right)$$

Proof — Let the initial set $x_0 := \lim_{n\to\infty} x_{0n}$ belong to the lower limit $K^\flat$ of the sequence K_n and $x(\cdot) \in \mathcal{S}_F(x_0)$ be any solution to differential inclusion (5.2). Since the solution map is lower semicontinuous, there exist solutions $x_n(\cdot) \in \mathcal{S}_F(x_{0n})$ converging pointwise to $x(\cdot)$. The subsets K_n being invariant under F, we conclude that for any $t \geq 0$, $x_n(t) \in K_n$. This implies that $x(t) \in K^\flat$ for every $t \geq 0$. Hence $K^\flat$ is invariant under F. □

More generally, we can prove that the lower limit $K^\flat$ of a sequence of closed subsets K_n invariant under set-valued maps F_n are invariant under some set-valued map F.

Theorem 5.4.6 (Stability) *Let us consider set-valued maps $F_n : X \rightsquigarrow X$ and $F : X \rightsquigarrow X$ such that the solution map $\mathcal{S}_F$ is contained in the graphical lower limit of the solution maps $\mathcal{S}_{F_n}$. Then if the closed subsets $K_n \subset \mathrm{Dom}(\mathcal{S}_{F_n})$ are invariant under the set-valued maps F_n, their lower limit $K^\flat$ is invariant under F.*

In particular, the lower limit of the invariance kernels of closed subsets K_n for the set-valued maps F_n contains the invariance kernel of the lower limit $K^\flat$ for F:

$$\mathrm{Liminf}_{n\to\infty}\left(\mathrm{Inv}_{F_n}(K_n)\right) \supset \mathrm{Inv}_F\left(\mathrm{Liminf}_{n\to\infty} K_n\right)$$

5.4.4 Global Exit and Hitting Functions

When the solution map $\mathcal{S}_F$ is lower semicontinuous, we can deduce from Proposition 4.2.2 and the Maximum Theorem 2.1.6 that the function $\theta_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\theta_K^\sharp(x) := \sup_{x(\cdot)\in\mathcal{S}_F(x)} \theta_K(x(\cdot))$$

is lower semicontinuous and that the function $\tau_K^\flat : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_K^\flat(x) := \inf_{x(\cdot)\in\mathcal{S}_F(x)} \tau_K(x(\cdot))$$

is upper semicontinuous.

Therefore the graphs of the "tubes" associating with $t \in [0, +\infty[$ the subsets

$$\begin{cases} \left\{x \in K \mid \theta_K^\sharp(x) \leq T\right\} \\ \\ \left\{x \in K \mid \tau_K^\flat(x) \geq T\right\} \end{cases} \tag{5.4}$$

are closed.

The first subset is the subset of initial states $x \in K$ such that the boundary ∂K is reached before T by all solutions $x(\cdot)$ to the differential inclusion (5.2) starting at x.

The second subset is the subset of initial states $x \in K$ such that all solutions $x(\cdot)$ to the differential inclusion (5.2) starting at x remain in K for all $t \in [0, T]$.

We then observe that the invariance kernel is equal to

$$\mathrm{Inv}_F(K) = \bigcap_{T \geq 0} \left\{x \in K \mid \tau_K^\flat(x) \geq T\right\}$$

5.4.5 Invariance Envelopes

One can relate invariance envelopes with the accessibility map:

Proposition 5.4.7 *Assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty closed values. Then the invariance envelope and the accessibility map are related by*

$$\mathrm{Env}_F(K) = \overline{\mathcal{R}_F(K)}$$

Proof — The subset $\mathcal{R}_F(K)$ is obviously contained in any closed invariant subset M containing K and in particular, in the invariance envelope of K.

Conversely, it is enough to prove that $\overline{\mathcal{R}_F(K)}$ is invariant. If not, there would exist $x_0 \in \overline{\mathcal{R}_F(K)}$, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ and $T > 0$ such that $x(T)$ does not belong to $\overline{\mathcal{R}_F(K)}$. Let $\varepsilon > 0$ be such that

$$B(x(T), \varepsilon) \cap \overline{\mathcal{R}_F(K)} = \emptyset$$

By Filippov's Theorem 5.3.1, there exists $\delta > 0$ such that for every $y_0 \in B(x_0, \delta)$, one can find a solution $y(\cdot) \in \mathcal{S}_F(y_0)$ starting from y_0 such that

$$y(T) \in B(x(T), \varepsilon) \subset X \setminus \overline{\mathcal{R}_F(K)}$$

Since x_0 belongs to the closure of $\mathcal{R}_F(K)$, one can choose such an initial state y_0 in $\mathcal{R}_F(K)$, so that there exists $z_0 \in K$, a solution $z(\cdot) \in \mathcal{S}_F(z_0)$ and $T_0 > 0$ satisfying $z(T_0) = y_0$. We then introduce the concatenation $\widetilde{y}(\cdot)$ defined by

$$\widetilde{y}(s) := \begin{cases} z(s) & \text{if } 0 \le s \le T_0 \\ y(s - T_0) & \text{if } T_0 \le s < \infty \end{cases}$$

Therefore $\widetilde{y}(\cdot) \in \mathcal{S}_F(z_0)$ is a solution starting from K such that $\widetilde{y}(T + T_0) = y(T)$, so that $y(T)$ belongs to $\mathcal{R}_F(K)$, a contradiction. □

Proposition 5.4.8 *Assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty closed values and that $K = \overline{\text{Int}(K)}$. Then*

$$\text{Env}_F(K) = \overline{X \setminus \text{Inv}_{-F}(\widehat{K})} \text{ where } \widehat{K} := \overline{X \setminus K}$$

Proof — Since these two sets contain K, it is enough to prove the equality for the elements outside of K.

Let x_0 be outside of both K and $\text{Inv}_{-F}(\widehat{K})$. We infer that there exists a solution $x(\cdot) \in \mathcal{S}_{-F}(x_0)$ and $T > 0$ such that $x(T) \in X \setminus \widehat{K} = \text{Int}(K)$. Let us associate with a solution $y(\cdot) \in \mathcal{S}_F(x_0)$ the solution $\widetilde{y}(\cdot) \in \mathcal{S}_F(x(T))$ defined by

$$\widetilde{y}(s) := \begin{cases} x(T - s) & \text{if } 0 \le s \le T \\ y(s - T) & \text{if } T \le s < \infty \end{cases}$$

which thus satisfies $\widetilde{y}(T) = x_0 \in \mathcal{R}_F(\text{Int}(K))$. Proposition 5.4.7 implies that the latter subset is contained in $\text{Env}_F(K)$.

Conversely, let y belong to $\text{Int}(\mathcal{R}_F(K)) \setminus K$. Since the interior of $\mathcal{R}_F(K)$ is equal to $\mathcal{R}_F(\text{Int}(K))$ by Proposition 5.3.7, there exist $x_0 \in \text{Int}(K)$, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ and $T > 0$ such that $y = x(T) \in X \setminus K$. We then associate with a solution $y(\cdot) \in \mathcal{S}_F(x_0)$ the solution $\widetilde{y}(\cdot) \in \mathcal{S}_F(y)$ defined by

$$\widetilde{y}(s) := \begin{cases} x(T - s) & \text{if } 0 \le s \le T \\ y(s - T) & \text{if } T \le s < \infty \end{cases}$$

which thus satisfies $\widetilde{y}(T) = x_0 \in \text{Int}(K)$. Hence such a solution is not viable in $\widehat{K} = X\backslash\text{Int}(K)$ and thus, $y = x(T)$ does not belong to the invariance kernel of $\widehat{K}$, so that we have proved that

$$\text{Int}(\mathcal{R}_F(K)) \subset \overline{X \setminus \text{Inv}_{-F}(\widehat{K})}$$

We conclude, thanks to Proposition 5.4.7. □

5.5 Boundaries of Viability and Invariance Kernels

5.5.1 Semipermeability of the Boundary of the Viability Kernel

We shall prove in this section that if the solution map is lower semicontinuous, then every viable solution starting on the boundary of the exit tube (respectively the viability kernel) remains on it.

Theorem 5.5.1 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$.*

Then, if

$$x \in \partial(\text{Exit}_F(K,T)) \cap \text{Limsup}_{t\to T-}\,(\text{Exit}_F(K,t)\backslash\text{Exit}_F(K,T))$$

every solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0,T]$ remains on the boundary of the exit tube:

$$\forall\, t \in [0,T],\ \ x(t)\ \in\ \partial\,(\text{Exit}_F(K,T-t))$$

Proof — Let $x(\cdot) \in \mathcal{S}_F(x)$ be a solution viable in K on $[0,T]$, which exists by assumption, and which thus satisfies

$$\forall\, t \in [0,T],\ \ x(t) \in \text{Exit}_F(K,T-t)$$

Also by assumption, there exists a sequence of $T_n < T$ converging to T and a sequence of elements $x_n \in \text{Exit}_F(K,T_n)\backslash\text{Exit}_F(K,T)$ converging to x.

Since the solution map is assumed to be lower semicontinuous, there exist solutions $x_n(\cdot)$ to the differential inclusion (5.2) starting at x_n defined on $[0,T]$ converging pointwise to $x(\cdot)$. On the other hand, by Proposition 4.2.8, we know that for any $t \in [0,T_n]$,

$$x_n(t) \in \mathrm{Exit}_F(K,T_n-t)\backslash\mathrm{Exit}_F(K,T-t)$$

Consequently, by passing to the limit, we obtain for all $t \in [0,T]$,

$$x(t) \in \mathrm{Exit}_F(K,T-t) \cap \overline{X\backslash\mathrm{Exit}_F(K,T-t)} = \partial\left(\mathrm{Exit}_F(K,T-t)\right)$$

i.e., the solution remains in the boundary of the exit tube. □

By using Proposition 4.2.9 instead of Proposition 4.2.8 in the proof of Theorem 5.5.1, we obtain the following statement:

Theorem 5.5.2 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ and that* $\mathrm{Exit}_F(K,T)$ *is contained in the interior of K. Then, if $x \in \partial(\mathrm{Exit}_F(K,T))$, every solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0,T]$ remains on the boundary of the exit tube:*

$$\forall\, t \in [0,T],\;\; x(t) \in \partial\left(\mathrm{Exit}_F(K,T-t)\right)$$

For $T = +\infty$, we obtain the following consequence:

Theorem 5.5.3 (Quincampoix) *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ and that the viability kernel of K is contained in the interior of K. Then the viability kernel is* semipermeable *in the sense that if $x \in \partial(\mathrm{Viab}_F(K))$, every solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K remains in the boundary of the viability kernel.*

In other words, this means that every solution starting from the boundary of the viability kernel can either remain in the boundary or leave the viability kernel, or equivalently, that no solution starting from outside of the viability kernel can cross its boundary: such solutions can only remain on the boundary of the viability kernel, or leave it.

5.5.2 Viability of the Boundary of the Invariance Kernel

In a symmetric way, we can prove that the boundary of the invariance kernel is viable:

Theorem 5.5.4 (Quincampoix) *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a compact subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0, \infty; X)$ and that the invariance kernel of K is contained in the interior of K. Then, the boundary $\partial(\text{Inv}_F(K))$ is viable under F.*

Proof — Let x_0 belong to $\partial(\text{Inv}_F(K))$ and consider a sequence of elements $x_n \in K \backslash \text{Inv}_F(K)$ converging to x_0.

By Proposition 5.4.3, we know that $X \backslash \text{Inv}_F(K)$ is viable under F: there exist solutions $x_n(\cdot)$ to differential inclusion (5.2) starting at x_n which are viable in $X \backslash \text{Inv}_F(K)$.

Since F is a Marchaud map, we infer from Theorem 3.5.2 that a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x(\cdot) \in \mathcal{S}_F(x_0)$ which is viable in the closure of the complement of $\text{Inv}_F(K)$. Theorem 5.4.2 implies that this solution is also viable in the invariance kernel of K, and thus, that it is viable in the boundary of $\partial(\text{Inv}_F(K))$. □

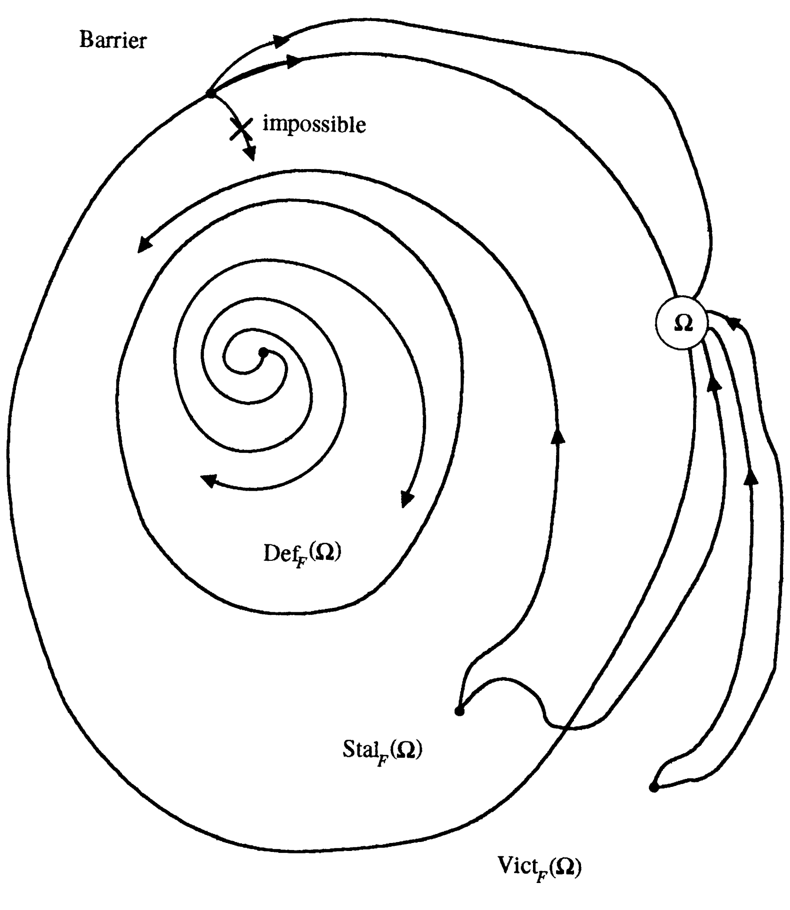
5.6 Defeat and Victory domains of a Target and its Barrier

We can apply the above theorems to the complement of an open target Ω. Let us introduce the following notations:

$$\begin{cases} i) & \text{Defeat}_F(\Omega) := \text{Inv}_F(X \backslash \Omega) \\ ii) & \text{Stal}_F(\Omega) := \text{Viab}_F(X \backslash \Omega) \setminus \text{Inv}_F(X \backslash \Omega) \\ iii) & \text{Vict}_F(\Omega) := X \backslash \text{Viab}_F(X \backslash \Omega) \end{cases}$$

Theorem 5.6.1 (Quincampoix) *Let $F : X \rightsquigarrow X$ be a Marchaud and Lipschitz map. Consider an open target $\Omega \subset X$. Then*

Figure 5.1: Victory and Defeat Domains



The target Ω, the defeat domain, the barrier and the victory domains are symbolically represented.

1. $\text{Defeat}_F(\Omega)$ *is the* defeat domain*:* $\forall\, x_0 \in \text{Defeat}_F(\Omega)$, *every solution starting from* x_0 *never reaches the target* Ω

2. $\text{Vict}_F(\Omega)$ *is the* victory domain: $\forall\, x_0 \in \text{Vict}_F(\Omega)$, *every solution reaches the target* Ω *in finite time*

3. $\text{Stal}_F(\Omega)$ *is the* stalemate domain: $\forall\, x_0 \in \text{Stal}_F(\Omega)$

   - *there exists one solution which never reaches the target* Ω
   - *there exists one solution hitting the target* Ω

4. $\partial\,(\text{Viab}_F(X\backslash\Omega))$ *is the* barrier: $\forall\, x_0 \in \partial\,(\text{Viab}_F(X\backslash\Omega))$, *there exists a solution which is viable in the barrier as long as it does not reach the target* Ω, *and no solution enters the interior of* $\text{Stal}_F(\Omega)$

5. $\partial\,(\text{Defeat}_F(\Omega))$ *is viable under* F

We can also introduce

$$\text{Vict}_F(\Omega, T) \;:=\; X\backslash\text{Exit}_F(X\backslash\Omega, T)$$

which is the open subset

$$\text{Vict}_F(\Omega, T) \;=\; \left\{x \quad \text{such that} \;\; \tau^\sharp_{X\backslash\Omega}(x) < T\right\}$$

of initial states from which all solutions reach the target Ω before T.

We deduce that the victory domain is equal to:

$$\text{Vict}_F(\Omega) \;=\; \bigcup_{T>0} \text{Vict}_F(\Omega, T)$$

The subset

$$\begin{cases} \text{vict}_F(\Omega, T) \;:=\; \text{Hit}_F(X\backslash\Omega, T) \\[2ex] = \left\{x \notin \Omega \mid \exists\, x(\cdot) \in \mathcal{S}_F(x), \;\; \exists\, t \in [0, T] \;\; \text{such that} \;\; x(t) \in \overline{\Omega}\right\} \end{cases}$$

is the set of initial states such that at least one solution to the differential inclusion reaches the closure of Ω at some $t \leq T$.

For compact targets C, we obtain the following characterization:

Proposition 5.6.2 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $C \subset X$ be a nonempty compact subset. The set*

$$\{x \notin C \mid \exists\, x(\cdot) \in \mathcal{S}_F(x),\ \ \exists\, t \in [0,T] \quad \text{such that } x(t) \in C\}$$

of initial states such that at least one solution to the differential inclusion reaches the target C at some $t \leq T$ is equal to

$$\bigcup_{\eta>0} \bigcap_{0<\varepsilon\leq\eta} \mathrm{vict}_F(\overset{\circ}{B}(C,\varepsilon),T)$$

Proof — Let us choose $x \notin C$. Then we know that there exists $\eta > 0$ such that for any $\varepsilon \leq \eta$, there exists at least one solution $x_\varepsilon(\cdot) \in \mathcal{S}_F(x)$ to the differential inclusion reaching the ball $B(C,\varepsilon)$ at some $t_\varepsilon \leq T$, thanks to the above remark with $\Omega := \overset{\circ}{B}(C,\varepsilon)$. Since $\mathcal{S}_F(x)$ is compact in $\mathcal{C}(0,\infty;X)$ supplied with the compact convergence topology, subsequences (again denoted by) $x_\varepsilon(\cdot)$ and t_ε converge to $x(\cdot) \in \mathcal{S}_F(x)$ and $t \in [0,T]$ respectively, so that the limit $x(t)$ of $x_\varepsilon(t_\varepsilon) \in B(C,\varepsilon)$ belongs to the closed target C. □

5.7 Linear Differential Inclusions

5.7.1 Viability Cones

Let us consider the case when the right-hand side of the differential inclusion is a closed convex process. Since closed convex processes are set-valued analogues of continuous linear operators, it is legitimate to call such differential inclusions *linear differential inclusions*.

The domain of a closed convex process being a convex cone, it is quite natural to restrict the class of viability domains of closed convex processes to closed convex cones.

Theorem 5.7.1 (Linear Differential Inclusions) *Let X be a finite dimensional vector-space, $F : X \rightsquigarrow X$ be a closed convex process and $K \subset X$ be a closed convex cone. We posit the following assumptions:*

$$\begin{cases} i) & \forall\, x \in K,\ \ R(x) \ := \ F(x) \cap (\overline{K + \mathbf{R}x}) \ \neq \emptyset \\ ii) & \text{the norm (see Definition 2.5.3) of } \|R\| \text{ is finite} \end{cases}$$

Then, for any initial state $x_0 \in K$, there exists a solution $x(\cdot)$ to the linear differential inclusion

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t)) \tag{5.5}$$

starting at x_0 and viable in the cone K.

Proof — It is a direct consequence of the Second Viability Theorem 3.3.6 and formula

$$T_K(x) = \overline{K + \mathbf{R}x}$$

since, by the very definition of the norm of R, we have:

$$\forall\, x \in K, \quad d(0, F(x) \cap T_K(x)) \leq \|R\|\|x\| \quad \square$$

Hence it remains to prove the following

Lemma 5.7.2 *Let $K \subset X$ be a convex cone of a normed space X and $x \in K$. Then*[10] *$T_K(x) = \overline{K + \mathbf{R}x}$.*

The proof is left as an exercise (see also Lemma 4.2.5 of SET-VALUED ANALYSIS.) □

Example — Let $A \in \mathcal{L}(X, X)$ be a linear operator and $P \subset X$ and $Q \subset X$ be closed convex cones. Then the set-valued map F defined by

$$F(x) := Ax + Q \text{ if } x \in P \ \& \ \emptyset \text{ if not} \tag{5.6}$$

is a closed convex process. We then deduce a useful corollary for linear control systems with inequality constraints on both the state and the control variables:

Corollary 5.7.3 *Let X be a finite dimensional vector-space, $A \in \mathcal{L}(X, X)$ be a linear operator and $P \subset X$ and $Q \subset Y$ be closed convex cones. If*

$$\begin{cases} i) & \forall\, x \in P, \ Ax \in \overline{P + \mathbf{R}x} - Q \\ ii) & \exists\, c > 0 \ \text{ such that } \inf_{u \in \left(\overline{P + \mathbf{R}x}\right) \cap (Q + Ax)} \|u\| \leq c\|x\| \end{cases}$$

then, for any initial state $x_0 \in P$, there exists a solution to the differential equation $x'(t) = Ax(t) + u(t)$, where $u(t) \in Q$, which is viable in the closed convex cone P. □

[10] If we assume that $K^- + \{x\}^- = X^\star$ and that X is reflexive, then $T_K(x) = K - \mathbf{R}_+ x$ thanks to Closed Range Theorem 2.3.4.

5.7.2 Projection on the sphere

We shall "project" the solutions $x(\cdot)$ onto the unit sphere Σ. We shall show that the evolutions of these projections are governed by a differential inclusion the right-hand side of which is the "projection" of the linear differential inclusion onto the tangent space to this sphere defined in the following way: we associate with any $y \in \Sigma$ the orthogonal projector $\pi(y)$ onto the tangent space $T_\Sigma(y)$ to Σ at y defined by

$$\pi(y)z \;:=\; z - < y, z > y$$

We observe the following property:

Lemma 5.7.4 *If K is a convex cone of a finite dimensional vector-space X, then, for any $y \in K \cap \Sigma$, $\pi(y)T_K(y) \subset T_{K\cap\Sigma}(y)$.*

Proof — Let $z \in T_K(y)$. We already know that $\pi(y)z$ belongs to $T_\Sigma(y)$. It belongs to $T_K(y)$ because

$$\pi(y)z \;=\; z - < y, z > y \in \overline{K + \mathbf{R}y} + \mathbf{R}y \;\subset\; \overline{K + \mathbf{R}y}$$

Then it belongs to the intersection of $T_\Sigma(y) = \{y\}^\perp$ and $T_K(y)$. It is equal to $T_{K\cap\Sigma}(y)$ (see Table 5.2), because the transversality condition $T_\Sigma(y) - T_K(y) = X$ is satisfied since we can write

$$\forall\, z \in X, \;\; z \;=\; \pi(y)z + < y, z > y \;\; \square$$

We now associate with a closed convex process $F : X \rightsquigarrow X$ its "projection" defined by

$$H(y) \;:=\; \pi(y)(F(y) \cap \|R\|B)$$

It is obviously a set-valued map with closed convex images contained in the ball $\|R\|B$, which is compact.

We deduce from the above lemma that *if a closed convex cone K is a viability domain of F, then $K \cap \Sigma$ is a viability domain of its projection H.* This implies the following consequence:

Proposition 5.7.5 *We posit the assumptions of Theorem 5.7.1.*

Then $x(\cdot)$ is a never vanishing viable solution to linear differential inclusion (5.5) if and only if $y(\cdot) := x(\cdot)/\|x(\cdot)\|$ is a solution to the projected differential inclusion

$$\text{for almost all } t \geq 0, \;\; y'(t) = \pi(y(t))z(t) \;\; \text{where} \;\; z(t) \in F(y(t))$$

viable in $K \cap \Sigma$ and we can write:

$$x(t) = y(t)\|x_0\|e^{\int_0^t < y(\tau), z(\tau) > d\tau}$$

Proof — The proof follows easily from the relation

$$y(t) = \frac{x(t)}{\|x(t)\|} \ \& \ z(t) = \frac{x'(t)}{\|x(t)\|}$$

and the property

$$\frac{d}{dt}\|x(t)\| = \|x(t)\| < y(t), z(t) > \ \square$$

Remark — Let us introduce the constants

$$\lambda_- := \inf_{y \in \Sigma \cap K, v \in F(y) \cap \|R\|B} < v, y > \ \& \ \lambda_+ := \sup_{y \in \Sigma \cap K, v \in F(y) \cap \|R\|B} < v, y >$$

We deduce that the solutions $x(\cdot)$ obey the estimates

$$\|x_0\|e^{\lambda_- t} \le \|x(t)\| \le \|x_0\|e^{\lambda_+ t} \ \square$$

We deduce that if $\lambda_+ < 0$, then the origin is an *attractor* and that if $\lambda_- > 0$, the origin is a *source* of the system.

5.7.3 Projection on a compact sole

It may be advantageous to project a linear differential inclusion on the sole of a cone instead of the sphere, if one needs convexity, for instance. In particular, this allows us to prove that a closed convex process F does have an eigenvector in cones with compact soles.

We associate with the closed convex cone K and an element $p_0 \in \mathrm{Int}(K^+)$ the "compact sole"

$$S := \{x \in K \mid < p_0, x > = 1\}$$

We associate with any element $y \in S$ the projector $\varpi(y)$ onto the orthogonal hyperplane to p_0, defined by

$$\forall z \in X, \ \varpi(y)z := z - < p_0, z > y$$

We then remark that:

Lemma 5.7.6 *If K is a convex cone with compact sole of a finite dimensional vector-space X, then, for any $y \in S$, $\varpi(y)T_K(y) \subset T_S(y)$.*

Proof — The tangent cone to the sole S of K is equal to

$$T_S(x) = \{v \in T_K(x) \mid < p_0, v >= 0\} \quad (5.7)$$

since S can be written in the form $K \cap p_0^{-1}(1)$. Indeed, the constraint qualification assumption $0 \in \text{Int}(p_0(K) - 1)$ is satisfied because $p_0(K)$ is a cone of $\mathbf{R}$ containing 1. We then deduce from Table 5.1 that $T_S(x) = T_K(x) \cap p_0^{-1} T_{\{1\}}(1)$, i.e., formula (5.7).

We now check that

$$\forall\, y \in S, \;\; \varpi(y)T_K(y) \;\subset\; T_S(y) \tag{5.8}$$

Indeed, Lemma 5.7.2 implies that if $u \in T_K(y)$, then

$$\varpi(y)u := u - < p_0, u > y \;\in\; \overline{(K + \mathbf{R}y)} + \mathbf{R}y \subset \overline{(K + \mathbf{R}y)} = T_K(y)$$

(because K is a closed convex cone) and

$$< p_0, \varpi(y)u > = < p_0, u > - < p_0, u >< p_0, y > = 0$$

(because $< p_0, y > = 1$). We deduce that $\varpi(y)u$ belongs to $T_S(y)$ thanks to (5.7). □

Let us project the closed convex process F to the set-valued map G defined on the compact sole S by

$$G(y) \;:=\; \varpi(y)(F(y) \cap \|R\|B)$$

which is naturally a set-valued map with closed convex images contained in the ball $\|R\|B$, which is compact. Since its graph is closed, we deduce that G is upper semicontinuous from S to X. By (5.8), S is a viability domain of the set-valued map G since the cone K is a viability domain of the closed convex process F, so that:

Proposition 5.7.7 *We posit the assumptions of Theorem 5.7.1. Therefore $x(\cdot)$ is a never vanishing viable solution to linear differential inclusion (5.5) if and only if*

$$y(\cdot) \;:=\; x(\cdot)/ < p_0, x(\cdot) >$$

is a solution to the projected differential inclusion

$$\text{for almost all } t \geq 0, \;\; y'(t) = \varpi(y(t))z(t) \;\; \text{where} \;\; z(t) \in F(y(t))$$

viable in the sole S and we can write:

$$x(t) \;=\; y(t) < p_0, x_0 > e^{\int_0^t < y(\tau), z(\tau) > d\tau}$$

Proof — The solutions $x(\cdot)$ and $y(\cdot)$ are related by

$$y(t) \;=\; \frac{x(t)}{< p_0, x(t) >} \;\;\&\;\; z(t) \;=\; \frac{x'(t)}{< p_0, x(t) >} \quad \square$$

Since the compact sole is a compact viability domain of the projection G, the Equilibrium Theorem 3.7.6 implies the existence of eigenvectors:

Theorem 5.7.8 (Eigenvector of a Closed Convex Process) *Let X be a finite dimensional vector-space and $F : X \leadsto X$ be a closed convex process. Assume that a closed convex cone $K \subset X$ enjoys the following properties.*

$$\begin{cases} i) & K \text{ has a compact sole} \\ ii) & K \text{ is a viability domain of } F \\ iii) & \text{the norm } \|R\| \text{ is finite} \end{cases}$$

Then there exists a nonzero eigenvector $\bar{x} \in K$ *of the closed convex process F associated with an* eigenvalue $\bar{\lambda}$, *i.e., a solution to the problem*

$$\bar{x} \in K, \ \bar{x} \neq 0, \ \bar{\lambda} \in \mathbf{R} \ \& \ \bar{\lambda}\bar{x} \in F(\bar{x}) \tag{5.9}$$

The eigenvalue $\bar{\lambda}$ is therefore nonnegative whenever $F(K) \subset K$.

Proof — Indeed, there exists an equilibrium $\bar{x} \in S$ of G, i.e., a solution to $0 \in G(\bar{x})$, in other words, a solution to

$$\bar{x} \in S, \ 0 = \varpi(y)(\bar{x})\bar{y} = \bar{y} - \langle p_0, \bar{y} \rangle \bar{x} \text{ where } \bar{y} \in F(\bar{x})$$

By setting $\bar{\lambda} := \langle p_0, \bar{y} \rangle$, we see that the pair $(\bar{\lambda}, \bar{x})$ is a solution to inclusion (5.9). □

5.7.4 Duality between Viability and Invariance

Let us consider the case when the right-hand side of the differential inclusion is a closed convex process F whose domain is the whole space.

Then we know that F is Lipschitz and that its transpose $F^\star$ is upper semicontinuous with compact images on its domain $F(0)^+$.

Theorem 5.7.9 (Polar of a Viability Domain) *Let X be a finite dimensional vector-space, $F : X \leadsto X$ be a strict closed convex process and K be a closed convex cone. Then K is an invariance domain of F if and only if K^+ is a viability domain of its transpose:*

$$\begin{cases} i) & \forall\, x \in K, \quad F(x) \subset T_K(x) \\ & \Updownarrow \\ ii) & \forall\, q \in K^+, \quad F^\star(q) \cap T_{K^+}(q) \neq \emptyset \end{cases}$$

where $K^+ := -K^- = \{p \in X^\star \mid \forall\, x \in K, \ \langle p, x \rangle \geq 0\}$.

We refer to Section 4.2 (Theorem 4.2.6) of SET-VALUED ANALYSIS for the proof of this Theorem. □