Chapter 3

Viability Theorems for Differential Inclusions

Introduction

This is the basic chapter of this book, where the main viability theorems for differential inclusions in finite dimensional vector spaces are gathered and proved. (Invariance Theorems are the topic of Chapter 5.)

We must begin by defining the class of functions in which to seek solutions to differential inclusions. An adequate choice is a weighted Sobolev space, made of absolutely continuous functions. The first section is devoted to these spaces and the derivatives in the sense of distributions.

Viability domains K of a set-valued map F are presented and studied in the second section: They are defined by

$$\forall\, x \in K,\ F(x) \cap T_K(x) \neq \emptyset$$

or, equivalently, when K is closed and F is upper semicontinuous with convex compact values, by

$$\forall\, x \in K,\ F(x) \cap \overline{co}\,(T_K(x)) \neq \emptyset$$

or also, by a dual condition involving the polar cone of the contingent cone (called the *subnormal cone*).

Viability Theorems are stated in the third section. They claim that a subset K is viable under F (in the sense that for any initial state x_0, there exists one solution starting at x_0 which is viable in K) if and only if K is a *viability domain* of F.

We consider successively the cases when K is locally compact, open and closed. The proofs are gathered in the fourth section.

We then show in the fifth section that the *solution map* $\mathcal{S}$ associating with any initial state the (possibly empty) subset of solutions to the differential inclusion *is upper semicontinuous.*

We also prove Kurzhanski's Representation Theorem stating that the restriction of a set-valued map to a closed convex subset is a *countable intersection of unconstrained set-valued maps.* In the same way that Lagrange multipliers allows us to replace a constrained optimization problem with unconstrained problems by "adding the constraints to the functional", this representation theorem enables us to represent the set of viable solutions to a differential inclusion as a countable intersection of sets of solutions to unconstrained differential inclusions obtained by "adding the constraints " to the right-hand of the original differential inclusion.

We recall that the *upper limit* of a sequence of subsets K_n is the set of cluster points of sequences of elements $x_n \in K_n$. We then answer in the sixth section a natural *stability question*: does the upper limit of a sequence of viability domains remain a viability domain? We also extend this result to the case when the subsets K_n are viability domains of maps F_n. We define the *upper graphical limit* $F^\sharp$ of a sequence of set-valued maps F_n by saying that the graph of $F^\sharp$ is the upper limit of the graphs of F_n's. We then prove that the *upper limit of viability domains of set-valued maps F_n is a viability domain of the map $\overline{co}(F^\sharp)$*.

We proceed by giving examples of closed viability domains. In the seventh section, we show that the *limit sets* of solutions to a differential inclusion are closed viability domains. In particular, *trajectories of periodic solutions are closed viability domains* and thus, *limits of solutions when $t \to +\infty$, if they exist, are equilibria.* These limit sets are among the most interesting features of a dynamical system. They are naturally subsets of the largest closed viability domain contained in a closed set K, the existence of which is proved in Chapter 4. This set, which we call the *viability kernel* of K, plays such an important role that we devote the whole chapter 4 to some of its properties, which we shall use throughout this book.

This motivates a further study of existence theorems of an equilibrium. We begin by pointing out that an equilibrium does exist if there exists a solution $x(\cdot)$ viable in a compact subset such that a sequence of average velocities

$$v_n := \frac{1}{t_n}\int_0^{t_n} \|x'(\tau)\| d\tau$$

converges to 0.

The question arises as to whether a *closed viability domain K contains an equilibrium.* This is the case when K is compact and the range $F(K)$ is convex.

This is also the case when K is compact and convex. This striking statement, linking viability and nonlinear analysis, is actually equivalent to the Brouwer Fixed Point Theorem. In both cases, one can say that *viability implies stationarity.*

In Section 8, we adapt to the set-valued case an efficient result of D. Saari on the *chaotic behavior* of discrete systems. Assume that the domain of a differential inclusion is covered by a family of compact subsets K_a satisfying an adequate controllability property: Any point can be reached from any subset K_a. Take any arbitrary sequence $K_{a_0}, K_{a_1}, K_{a_2}, \ldots$ of such sets. Then there exist a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ and a sequence of instants t_j such that $x(t_j) \in K_{a_j}$ for all j.

Throughout this chapter, X denotes a finite dimensional vector-space so long as it is not explicitly mentioned that this is not the case.

3.1 Solution Class

We are going to extend Nagumo's Theorem 1.2.1 to the case of differential inclusions $x'(t) \in F(x(t))$. But we have first to agree on what we shall call a solution to such differential inclusions.

In the case of differential equations, there is no ambiguity since the derivative $x'(\cdot)$ of one solution $x(\cdot)$ to a differential equation

$x'(t) = f(t, x(t))$ inherits the properties of the map f and of the function $x(\cdot)$. It is continuous whenever f is continuous and measurable whenever f is continuous with respect to x and measurable with respect to t.

This is no longer the case with differential inclusions. We have to choose a space of functions or distributions in which we shall look for a solution.

We cannot hope to obtain without further restrictions a continuously differentiable, or even a plain differentiable solution. We shall be content to deal only with functions which are almost everywhere differentiable. Namely, we shall look for solutions among *absolutely continuous* functions, as it was proposed by T. Ważewski at the beginning of the sixties.

We denote by $L^1(0,\infty;X,e^{-bt}dt)$ the *weighted Lebesgue space* of (classes of) measurable functions $x(\cdot)$ from $[0,\infty[$ to X satisfying

$$\|x(\cdot)\| := \int_0^\infty e^{-bt}\|x(t)\|dt < +\infty$$

Definition 3.1.1 (Absolutely Continuous Functions) *A continuous function $x : [0,T] \mapsto X$ is said to be* absolutely continuous *if there exists a locally integrable function v such that*

$$\text{for all } t,s \in [0,T], \quad \int_t^s v(\tau)d\tau = x(s) - x(t)$$

In this case,

$$\text{for almost all } t \in [0,T], \quad x'(t) := v(t)$$

and we shall say that $x'(\cdot)$ is the weak derivative *of the function $x(\cdot)$.*

We shall denote by $W^{1,1}(0,\infty;X;e^{-bt}dt)$ (for some $b \geq 0$) the space of absolutely continuous functions defined by

$$\{x(\cdot) \in L^1(0,\infty;X,e^{-bt}dt) \mid x'(\cdot) \in L^1(0,\infty;X,e^{-bt}dt)\}$$

and, when $T < +\infty$, by $W^{1,1}(0,T;X)$ the space

$$\{x(\cdot) \in L^1(0,T;X) \mid x'(\cdot) \in L^1(0,T;X)\}$$

We shall supply them with the topology for which a sequence $x_n(\cdot)$ converges to $x(\cdot)$ if and only if

$$\begin{cases} i) & x_n(\cdot) \text{ converges } \textit{uniformly} \text{ to } x(\cdot) \\ & \text{(on compact intervals if } T = \infty) \\ & \\ ii) & x_n'(\cdot) \text{ converges } \textit{weakly} \text{ to } x'(\cdot) \text{ in } L^1(0,T;X) \\ & \text{(in } L^1(0,\infty;X,e^{-bt}dt) \text{ if } T = +\infty) \end{cases}$$

Remark — The above spaces are *weighted Sobolev spaces*. To define them, it may be best to recall what *distributions* and *derivatives in the sense of distributions* are[1].

We denote by $\mathcal{D}(0,T;X)$ the space of indefinitely differentiable functions from $]0,T[$ to the finite dimensional vector space X *with compact support in* $]0,T[$. The choice of the simplest scalar product

$$< x, y >:= \int_0^T x(t)y(t)dt$$

allows us to identify the space $\mathcal{D}(0,T;X)$ with a subspace of the dual $D^\star(0,T;X)$ of continuous linear functionals on $\mathcal{D}(0,T;X)$, called *distributions* since their discovery by Laurent Schwartz.

For that purpose, we identify a function $x(\cdot)$ with the continuous linear functional

$$y \mapsto \int_0^T x(t)y(t)dt$$

which belongs to the dual of $\mathcal{D}(0,T;X)$.

In other words, the fundamental idea is to regard the usual functions in a novel way: Instead of viewing them as maps from $]0,T[$ to X, we shall also regard them as continuous linear functionals on the infinite dimensional space $\mathcal{D}(0,T;X)$. In particular, *integrable functions* (actually, classes of measurable and integrable functions) are instances of distributions.

This very same scalar product defines the topology of quadratic convergence on $\mathcal{D}(0,T;X)$. Taking the completion of this space for this scalar product, we obtain the celebrated space $L^2(0,T;X)$. Since this scalar product was already used to identify $\mathcal{D}(0,T,X)$ with a subspace of its dual, it will also be used to identify $L^2(0,T,X)$ with its dual thanks to Riesz' Theorem. We thus obtain the inclusions:

$$\mathcal{D}(0,T;X) \subset L^2(0,T;X) = L^2(0,T;X)^\star \subset \mathcal{D}^\star(0,T;X)$$

[1] We refer to the text APPLIED FUNCTIONAL ANALYSIS by the author or any of the many books on distributions for more details.

The first (and most important) consequence of this concept is the possibility of differentiating integrable functions, and more generally, distributions.

Definition 3.1.2 (Distributional Derivative) *If $x(\cdot)$ is a measurable locally integrable function from $]0,T[$ to a finite dimensional vector space X, we shall say that the continuous linear functional $x' \in \mathcal{D}^\star(0,T;X)$ defined on the space $\mathcal{D}(0,T;X)$ by*

$$y(\cdot) \longmapsto -\int_0^T x(t)y'(t)dt$$

is the weak derivative *(or the* distributional derivative*) of $x(\cdot)$.*

A distributional derivative defined in such a way does not need to be a function, even measurable. In any case, *it is a distribution.* The weak derivative of a function of $\mathcal{D}(0,T,X)$ naturally coincides with the usual derivative.

Sobolev spaces are then defined in the following way:

Definition 3.1.3 *Let $a(\cdot)$ be a strictly positive measurable function. We denote by*

$$W^{1,p}(0,T;X;a) \;:=\; \{x \in L^p(0,T;X;a) \mid x' \in L^p(0,T;X;a)\}$$

the weighted Sobolev space *of measurable p^{th}-integrable functions $x(\cdot)$ (for the measure $a(t)dt$) whose derivative $x'(\cdot)$ in the sense of distributions belongs to the space $L^p(0,T;X;a)$.*

If $a \equiv 1$, we set $W^{1,p}(0,T;X) := W^{1,p}(0,T;X;a)$ This is a Sobolev space. *If $p = 2$, we often use the notation*

$$H^1(0,T;X) \;=\; W^{1,2}(0,T;X)$$

They are Banach spaces for the norm:

$$\|x\|_{1,p;a} \;:=\; (\|x\|_{p,a}^p + \|x'\|_{p,a}^p)^{1/p}$$

For our study, we endowed $W^{1,1}(0,\infty;X;e^{-bt}dt)$ with a weaker topology, for reasons which will soon become clear. □

The generalization of the concept of derivative provided by the theory of distributions is not the only one we can conceive. This approach allows us to *keep the linearity properties of the differential operator* $x \mapsto x'$. Actually, one can show that the distribution x' is the limit in the space $\mathcal{D}^\star(0,T;X)$ of the differential quotients

$$\frac{x(\cdot+h)-x(\cdot)}{h}$$

The topology of $\mathcal{D}^\star(0,T;X)$ is so much weaker than the pointwise convergence topology that not only do differential quotients of any function converge, but also differential quotients of distributions. In this distributional sense, functions and distributions are indefinitely differentiable.

The price one pays to obtain this paradisiac situation is that the space of distributions may be too large, and that distributions are no longer functions.

We will propose in Chapter 9 another concept of derivative (contingent epiderivative) for studying Lyapunov functions: They are *lower epilimits of these difference quotients*, as we shall explain later, and are usual functions instead of distributions. But the contingent epiderivative of a function no longer depends linearly on this function.

3.2 Viability Domains

Let X be a finite dimensional vector-space. We describe the (nondeterministic) dynamics of the system by a set-valued map F from the finite dimensional vector-space X to itself.

The contingent cone was introduced by G. Bouligand[2] in the early thirties: When K is a subset of X and x belongs to K, we recall that the *contingent cone* $T_K(x)$ to K at x is the closed cone of elements

[2]who wrote: "... Nous poserons les définitions suivantes:

1. Une demi-droite OT, issue du point d'accumulation O de l'ensemble E, sera dite une *demi-tangente* au point O, à l'ensemble E, si tout cône droit à base circulaire, de sommet O et d'axe OT, contient (si faibles en soient la hauteur et l'angle au sommet) un point de l'ensemble E distinct du point O;
2. L'ensemble de toutes les demi-tangentes à l'ensemble E en un même point d'accumulation sera appelé, moyennant une désignation abrégée conforme à l'étymologie, le *contingent* de l'ensemble E au point O.

Le mot *contingent* a déjà été employé comme adjectif, en matière philosophique, ou comme substentif, au point de vue militaire. L'emploi nouveau que nous en faisons ne peut évidemment créer aucune équivoque."

v such that

$$\liminf_{h \to 0+} \frac{d(x+hv, K)}{h} = 0$$

(see Definition 1.1.3 and Section 5.1 below[3].)

3.2.1 Definition of Viability Domains

There are two ways to extend the concept of viability domain K to set-valued maps. The first one is to require that for any state x, there exists *at least* one velocity $v \in F(x)$ which is *contingent* to K at x. The second demands that *all* velocities $v \in F(x)$ are *contingent* to K at x.

Definition 3.2.1 (Viability Domain) *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map. We shall say that a subset $K \subset \mathrm{Dom}(F)$ is a* viability domain *of F if and only if*

$$\forall\, x \in K, \quad F(x) \cap T_K(x) \neq \emptyset$$

Since the contingent cone to a singleton is obviously reduced to 0, we observe that *a singleton $\{\overline{x}\}$ is a viability domain if and only if $\overline{x}$ is an equilibrium of F*, i.e., a stationary solution to the differential inclusion, which is a solution to the inclusion

$$0 \in F(\overline{x}) \tag{3.1}$$

In other words, the equilibria of a set-valued map provide the first examples of viability domains, actually, the *minimal viability domains.*

Remark — If K is a viability domain of a set-valued map F, the subset

$$D := \bigcap_{x \in K} (T_K(x) - F(x))$$

is the subset of *disturbances* of the system which do not destroy the fact that K is still a viability domain, because K remains a viability domain of any *perturbed set-valued map* $x \rightsquigarrow F(x) + G(x)$ where $x \mapsto G(x)$ maps K into D. □

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[3]By using the concept of *upper limits* of sets introduced in Definition 3.6.1 of Section 3.3.6 below, we observe that the *contingent cone $T_K(x)$ is the upper limit of the differential quotients $\frac{K-x}{h}$ when $h \to 0+$.*

3.2.2 Subnormal Cones

In order to provide a dual characterization of viability domains, we need to introduce the dual concept of a contingent cone: the *subnormal cone*.

Definition 3.2.2 *Let x belong to $K \subset X$. We shall say that the (negative) polar cone*

$$N_K^0(x) := T_K(x)^- = \{p \in X^\star \mid \forall\, v \in T_K(x),\ < p, v > \leq 0\}$$

is the subnormal cone *to K at x.*

We see at once that

$$N_K^0(x) := (\overline{co}(T_K(x)))^-$$

The subnormal cone is equal to the whole space whenever the tangent cone $T_K(x)$ is reduced to 0.

Let us point out the following property:

Proposition 3.2.3 *Let K be a subset of a finite dimensional vector-space and Π_K denote its projector of best approximation. Then*

$$\forall\, y \notin K,\ \forall\, x \in \Pi_K(y),\ y - x \in N_K^0(x)$$

Proof — Let v belong to the contingent cone $T_K(x)$: there exists a sequence $h_n > 0$ converging to 0 and a sequence v_n converging to v such that $x + h_n v_n$ belongs to K for all n. Since $\|y - x\| \leq \|y - x - h_n v_n\|$, we deduce that $< x - y, v > \geq 0$ for all $v \in T_K(x)$. □

3.2.3 Dual Characterization of Viability Domains

We now prove a very important characterization of viability domains:

Theorem 3.2.4 *Assume that the set-valued map $F : K \leadsto X$ is upper semicontinuous with convex compact values. Then the three following properties are equivalent:*

$$\begin{cases} i) & \forall\, x \in K,\ F(x) \cap T_K(x) \neq \emptyset \\ ii) & \forall\, x \in K,\ F(x) \cap \overline{co}\,(T_K(x)) \neq \emptyset \\ iii) & \forall\, x \in K,\ \forall\, p \in N_K^0(x),\ \sigma(F(x), -p) \geq 0 \end{cases} \tag{3.2}$$

Proof — Since property i) implies ii), assume that ii) holds true and fix $x \in K$. Let $u \in F(x)$ and $v \in T_K(x)$ achieve the distance between $F(x)$ and $T_K(x)$:

$$\|u - v\| = \inf_{y \in F(x),\, z \in T_K(x)} \|y - z\|$$

and set $w := \frac{u+v}{2}$. We have to prove that $u = v$. Assume the contrary.

Since v is contingent to K at x, there exist sequences $h_n > 0$ converging to 0 and v_n converging to v such that $x + h_n v_n$ belongs to K for every $n \geq 0$. We also introduce a projection of best approximation

$$x_n \in \Pi_K(x + h_n w) \text{ of } x + h_n w \text{ onto } K \text{ and we set } z_n := \frac{x_n - x}{h_n}$$

so that, by Proposition 3.2.3, we know that

$$w - z_n \in N_K^0(x_n)$$

By assumption ii), there exists an element $y_n \in F(x_n) \cap \overline{co}\,(T_K(x_n))$. Consequently,

$$\langle w - z_n, y_n \rangle \leq 0 \tag{3.3}$$

Since x_n converges to x, the upper semicontinuity of F at x implies that for any $\varepsilon > 0$, there exists N_ε such that for $n \geq N_\varepsilon$, y_n belongs to the neighborhood $F(x) + \varepsilon B$, which is compact. Thus a subsequence (again denoted by) y_n converges to some element $y \in F(x)$.

We shall now prove that z_n converges to v. Indeed, the inequality

$$\begin{cases} \|w - z_n\| = \frac{1}{h_n} \|x + h_n w - x_n\| \\ \leq \frac{1}{h_n} \|x + h_n w - x - h_n v_n\| = \|w - v_n\| \end{cases}$$

implies that the sequence z_n has a cluster point and that every cluster point z of the sequence z_n belongs to $T_K(x)$, because $x + h_n z_n = x_n \in K$ for every $n \geq 0$. Furthermore, every such z satisfies $\|w - z\| \leq \|w - v\|$.

We now observe that *v is the unique best approximation of w by elements of $T_K(x)$.* If not, there would exist $p \in T_K(x)$ satisfying either $\|w-p\| < \|w-v\|$ or $p \neq v$ and $\|w-p\| = \|w-v\| = \|w-u\|$. In the latter case, we have $\langle u-w, w-p\rangle < \|u-w\|\|w-p\|$, since the equality holds true only for $p = v$. Each of these conditions together with the estimates

$$\begin{cases} \|u-p\|^2 &= \|u-w\|^2 + \|w-p\|^2 + 2\langle u-w, w-p\rangle \\ &\leq (\|u-w\| + \|w-p\|)^2 \leq \|u-v\|^2 \end{cases}$$

imply the strict inequality $\|u-p\| < \|u-v\|$, which is impossible since v is the projection of u onto $T_K(x)$. Hence $z = v$.

Consequently, all the cluster points being equal to v, and we conclude that z_n converges to v.

Therefore, we can pass to the limit in inequality (3.3) and obtain, observing that $w - v = (u-v)/2$,

$$\langle u-v, y\rangle = 2\langle w-v, y\rangle \leq 0 \text{ where } y \in F(x) \tag{3.4}$$

Since $F(x)$ is closed and convex and since $u \in F(x)$ is the projection of v onto $F(x)$, we infer that

$$\langle u-v, u-y\rangle \leq 0 \tag{3.5}$$

Finally, $T_K(x)$ being a cone and $v \in T_K(x)$ being the projection of u onto this cone, and in particular onto the half-line $v\mathbf{R}_+$, we deduce that

$$\langle u-v, v\rangle = 0 \tag{3.6}$$

Therefore, properties (3.4), (3.5) and (3.6) imply that

$$\|u-v\|^2 = \langle u-v, -v\rangle + \langle u-v, u-y\rangle + \langle u-v, y\rangle \leq 0$$

and thus, that $u = v$.

The equivalence between ii) and iii) follows from the Separation Theorem. Indeed, by ii), to saying that K is a viability domain amounts to say that for all $x \in K$, 0 belongs to $F(x) - \overline{co}(T_K(x))$, which is closed and convex whenever $F(x)$ is compact. Hence the Separation Theorem implies that this condition is equivalent to the one stated in the Theorem. □

We can deduce right away from Theorem 3.2.4 the following very useful fact:

Proposition 3.2.5 *Let us assume that two set-valued maps F_1 and F_2 are upper semicontinuous with compact convex images. If K is a viability domain of F_1 and F_2, it is still a viability domain of $\lambda_1 F_1 + \lambda_2 F_2$ (where $\lambda_1, \lambda_2 > 0$.)*

3.3 Statement of Viability Theorems

We now consider initial value problems (or Cauchy problems) associated with the differential inclusion

$$\text{for almost all } t \in [0,T], \quad x'(t) \in F(x(t)) \tag{3.7}$$

satisfying the initial condition $x(0) = x_0$.

Definition 3.3.1 (Viability and Invariance Properties) *Let K be a subset of the domain of F. A function $x(\cdot) : I \mapsto X$ is said to be* viable in K on the interval I *if and only if*

$$\forall\, t \in I, \quad x(t) \in K$$

We shall say that K is locally viable under F *(or enjoys the* local viability property for the set-valued map F*) if for any initial state x_0 in K,* there exist *$T > 0$ and a solution on $[0,T]$ to differential inclusion (3.7) starting at x_0 which is* viable *in K. It is said to be* (globally) viable under F *(or to enjoy the* (global) viability property*) if we can take $T = \infty$.*

The subset K is said to be locally invariant (respectively invariant) under F *if for any initial state x_0 of K,* all *solutions to differential inclusion (3.7) are* viable in K on some interval (respectively for all $t \geq 0$)*. We also say that F enjoys the* local invariance *(respectively* invariance*) property.*

Remark — We should emphasize as we did for ordinary differential equations that the concept of invariance *depends upon the behavior of F on its domain outside of K.* □

We would naturally like to characterize closed subsets viable under F as closed viability domains. This is more or less the situation that we shall meet: The main viability theorems hold true for the class of *Marchaud maps, i.e., the nontrivial upper hemicontinuous* *set-valued maps with nonempty compact convex images and with linear growth* (or equivalently, in the case of finite dimensional state spaces, *closed set-valued maps with closed domain, convex values and linear growth*. (See Corollary 2.2.3).)

We observe that the only truly restrictive condition is the *convexity* of the images of these set-valued maps, since the continuity requirements are kept minimal. But we cannot dispense with it, as the following counter example shows.

Example — Let us consider $X := \mathbf{R}$, $K := [-1,+1]$ and the set-valued map $F : K \leadsto \mathbf{R}$ defined by

$$F(x) \quad := \quad \begin{cases} -1 & \text{if} \quad x > 0 \\ \{-1,+1\} & \text{if} \quad x = 0 \\ +1 & \text{if} \quad x < 0 \end{cases}$$

Obviously, no solution to the differential inclusion $x'(t) \in F(x(t))$ can start from 0, since 0 is not an equilibrium of this set-valued map!

We note however that

- The graph of F is closed
- F is bounded
- K is convex and compact
- K is a viability domain of F.

But the value $F(0)$ of F at 0 is not convex. Observe that if we had set $F(0) := [-1,+1]$, then 0 would have been an equilibrium.

This example shows that upper semicontinuity is not strong enough to compensate the lack of convexity. Stronger continuity or differentiability requirements allow us to relax this assumption.

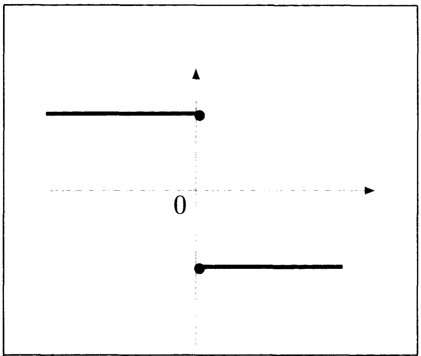
But we shall keep our continuity requirements minimal, and thus, be ready to pay the price of considering systems whose sets of velocities are convex. This is possible thanks to the extension of the Nagumo Theorem 1.2.1. □

Theorem 3.3.2 *Let us assume that*

$$\begin{cases} i) & F : X \leadsto X \text{ is upper semicontinuous} \\ ii) & \text{the images of } F \text{ are convex and compact} \\ iii) & K \text{ is locally compact} \end{cases}$$

Then K is locally viable under F if and only if K is a viability domain of F.

Figure 3.1: Example of a Map without Convex Values



There is no solution starting at 0

Since open subsets of finite dimensional vector spaces are locally compact viability domains of any set-valued map, we obtain the extension of Peano's Theorem 1.2.2 to differential inclusions due to Marchaud, Zaremba[4] and Ważewski[5]:

Theorem 3.3.3 *Let Ω be an open subset of a finite dimensional vector space X and $F : \Omega \rightsquigarrow X$ be a strict upper semicontinuous set-valued map with convex compact images.*

Then, for any $x_0 \in \Omega$, there exists $T > 0$ such that differential inclusion (3.7) has a solution on the interval $[0, T]$ starting from x_0.

[4]who proved independently in the thirties the existence of respectively *contingent* and *paratingent* solutions to differential inclusions (called *champs de demi-cônes* at the time.) The generalization of the concept of derivative to the notion of contingent derivative is due to B. Bouligand, who wrote: "... Nous ferons tout d'abord observer ... que la notion de contingent éclaire celle de différentielle".)

[5]who wrote: "... I learned the results of Zaremba's dissertation before the second world war, since I was a referee of that paper. Then a few years ago I came across with some results on optimal control and I have noticed a close connection between the optimal control problem and the theory of Marchaud-Zaremba." The author learned that this "coming across" happened during a seminar talk of C. Olech on a paper by LaSalle at Ważewski's seminar.

Ważewski proved that one can replace the contingent or paratingent derivatives of functions by derivatives of absolutely continuous functions defined almost everywhere in the definition of a solution to a differential inclusion, that he called *orientor field*.

The interesting case from the viability point of view is the one when the viability subset K is *closed*. In this case, we derive from Theorem 3.3.2 a more precise statement.

Theorem 3.3.4 (Local Viability Theorem) *Consider a* nontrivial upper semicontinuous set-valued map F with compact convex images *from* X *to* X *and a* closed *subset* $K \subset \mathrm{Dom}(F)$.

If K *is a* viability domain, *then for any initial state* $x_0 \in K$, *there exist a positive* T *and a solution on* $[0, T]$ *to differential inclusion (3.7) starting from* x_0, *viable in* K *and satisfying*

$$\begin{cases} \textit{either} & T = \infty \\ \quad\textit{or} & T < \infty \quad \textit{and} \quad \limsup_{t \to T-} \|x(t)\| = \infty \end{cases}$$

Further adequate information — a priori estimates on the growth of F — allow us to exclude the case when $\limsup_{t \to T-} \|x(t)\| = \infty$.

This is the case for instance when F is bounded on K, and, in particular, when K is bounded.

More generally, we can take $T = \infty$ when F enjoys linear growth:

Theorem 3.3.5 (Viability Theorem) *Consider a* Marchaud map $F : X \leadsto X$ *and a* closed *subset* $K \subset \mathrm{Dom}(F)$ *of a finite dimensional vector space* X.

If K *is a* viability domain, *then for any initial state* $x_0 \in K$, *there exists a* viable *solution on* $[0, \infty[$ *to differential inclusion (3.7.) More precisely, if we set*

$$c := \sup_{x \in \mathrm{Dom}(F)} \frac{\|F(x)\|}{\|x\| + 1}$$

then every solution $x(\cdot)$ *starting at* x_0 *satisfies the estimates*

$$\begin{cases} \forall\, t \geq 0, \;\; \|x(t)\| \;\leq\; (\|x_0\| + 1)e^{ct} \\ \text{and} \\ \text{for almost all } t \geq 0, \;\; \|x'(t)\| \;\leq\; c(\|x_0\| + 1)e^{ct} \end{cases}$$

and thus belongs to the space $W^{1,1}(0, \infty; X; e^{-bt}dt)$ *for* $b > c$.

Actually, we shall also use another more convenient formulation of this theorem. We agree for that purpose to set the distance $d(x, \emptyset)$ to the empty set equal to $+\infty$.

Theorem 3.3.6 (Second Viability Theorem) *Let us consider a* Marchaud map $F : X \rightsquigarrow X$ *and a* closed *subset* $K \subset \mathrm{Dom}(F)$ *of a finite dimensional vector space* X. *We assume that there exists a constant* $c > 0$ *such that*

$$\sup_{x \in K} \frac{d(0, F(x) \cap T_K(x))}{\|x\| + 1} \leq c < +\infty \tag{3.8}$$

Then for any initial state $x_0 \in K$, *there exists a* viable *solution on* $[0, \infty[$ *to differential inclusion (3.7) starting from* x_0, *which belongs to the space* $W^{1,1}(0, \infty; X; e^{-bt}dt)$ *for* $b > c$.

One can look right away at the *control version of the viability Theorems* in Section 6.1 in the framework of control systems and a very simple economic example in Section 6.2, in which other concepts such as viability kernels and heavy solutions are illustrated. Viability (and invariance) theorems for *linear differential inclusions* are presented in section 5.6 and can be checked over now.

3.4 Proofs of the Viability Theorems

We gather in this section the proofs of the theorems stated in the preceding one.

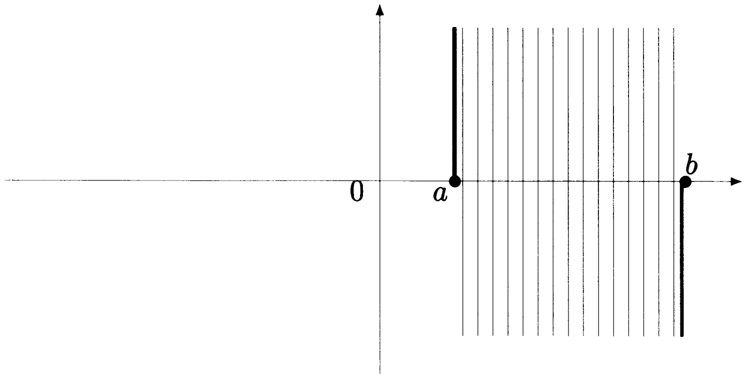
Since viable absolutely continuous functions $x(\cdot) : [0, T] \mapsto K$ satisfy $x'(t) \in T_K(x(t))$ for almost all $t \in [0, T]$, we could be tempted to derive viability theorems from existence theorems of solutions to differential inclusion $x'(t) \in R_K(x(t))$ where we set $R_K(x) := F(x) \cap T_K(x)$. Unfortunately, this is not possible because $T_K(\cdot)$ may be neither upper semicontinuous nor lower semicontinuous[6]. For instance, it is not upper semicontinuous as soon as inequality constraints are involved: take for example $K := [-1, +1]$. *The graph of* $T_K(\cdot)$, *equal to*

$$\{-1\} \times \mathbf{R}_+ \cup] -1, +1[\times \mathbf{R} \cup \{+1\} \times \mathbf{R}_-$$

is not closed, and not even locally compact: See figure 3.2.

[6] See Section 4.1., p. 178 of DIFFERENTIAL INCLUSIONS for an example of subset K such that $T_K(\cdot)$ is neither upper semicontinuous nor lower semicontinuous.

Figure 3.2: The Graph of $T_{[a,b]}(\cdot)$



So we have to devise a specific proof of Theorem 3.3.2, which consists in proving Propositions 3.4.1 and 3.4.2 below.

Proposition 3.4.1 (Necessary Condition) *Let us assume that*

$$\begin{cases} i) & F : X \rightsquigarrow X \text{ is upper hemicontinuous} \\ ii) & \text{the images of } F \text{ are convex and compact} \end{cases}$$

Let us consider a solution $x(\cdot)$ to differential inclusion (3.7) starting at x_0 and satisfying

$$\forall\, T > 0,\ \exists\, t \in]0, T] \text{ such that } x(t) \in K \tag{3.9}$$

(Naturally, viable solutions do satisfy this property.) Then

$$F(x_0) \cap T_K(x_0) \neq \emptyset$$

Proof — By assumption (3.9), there exists a sequence $t_n \to 0+$ such that $x(t_n) \in K$. Since F is upper hemicontinuous at x_0, we can associate with any $p \in X^\star$ and $\varepsilon > 0$ an $\eta_p > 0$ such that

$$\forall\, \tau \in [0, \eta_p],\ < p, x'(\tau) > \leq\ \sigma(F(x(\tau)), p)\ \leq\ \sigma(F(x_0), p) + \varepsilon\|p\|_\star$$

By integrating this inequality from 0 to t_n, setting $v_n := \frac{x(t_n)-x_0}{t_n}$ and dividing by $t_n > 0$, we obtain for n larger than some N_p

$$\forall\, p \in X^\star,\ \ \forall\, n \geq N_p,\ \ < p, v_n > \leq\ \sigma(F(x_0), p) + \varepsilon \|p\|_\star$$

Therefore, v_n lies in a bounded subset of a finite dimensional vector space, so that a subsequence (again denoted) v_n converges to some $v \in X$ satisfying

$$\forall\, p \in X^\star,\ \ < p, v > \leq\ \sigma(F(x_0), p) + \varepsilon \|p\|_\star$$

By letting ε converge to 0, we deduce that v belongs to the closed convex hull of $F(x_0)$.

On the other hand, since for any n, $x(t_n) = x_0 + t_n v_n$ belongs to K, we infer that v belongs to the contingent cone $T_K(x_0)$ since

$$\begin{cases} \liminf_{n\to\infty} d_K(x_0 + hv)/h \\ \leq \lim_{n\to\infty} \|x_0 + t_n v - x(t_n)\|/t_n \ = \ \lim_{n\to\infty} \|v_n - v\| \ = \ 0 \end{cases}$$

The intersection $F(x_0) \cap T_K(x_0)$ is then nonempty, so that the necessary condition ensues. □

Proposition 3.4.2 (Sufficient Condition) *Let us assume that*

$$\begin{cases} i) & F : X \rightsquigarrow X \text{ is upper semicontinuous} \\ ii) & \text{the images of } F \text{ are convex and compact} \end{cases}$$

Let K be a locally compact subset of the domain of F and $K_0 \subset K$ be a compact neighborhood of x_0 such that

$$\forall\, x \in K_0,\ \ F(x) \cap T_K(x) \ \neq\ \emptyset$$

Then there exist $T > 0$ and a solution to differential inclusion (3.7) starting at x_0 and viable in K on $[0, T]$.

Proof — We adapt the proof of Nagumo's Theorem 1.2.1 to the case of differential inclusion by following the same strategy: construct approximate solutions by modifying Euler's method to take into account the viability constraints, then deduce from available estimates that a subsequence of these solutions converges in some sense to a limit, and finally, check that this limit is a viable solution to differential inclusion (3.7). The two first steps are slight variations of the analogous steps of the proof of Nagumo's Theorem. The third step, which is specific to the set-valued case, uses the Convergence Theorem 2.4.4.

1. — Construction of Approximate Solutions

By assumption, there exists $r > 0$ such that the compact neighborhood $K_0 \subset K$ contains the ball $B_K(x_0, r) := K \cap (x_0 + rB)$. We set

$$C := F(K_0) + B, \quad T := r/\|C\|$$

We observe that C *is bounded.* We begin by proving

Lemma 3.4.3 *We posit the assumptions of Proposition 3.4.2. For any integer m, there exists $\theta_m \in]0, 1/m[$ such that for any $x \in K_0$, there exist $h \in [\theta_m, 1/m]$ and $u \in X$ satisfying*

$$\begin{cases} i) & u \in C \\ ii) & x + hu \in K \\ iii) & (x, u) \in B(\text{Graph}(F), 1/m) \end{cases}$$

Proof of Lemma 3.4.3 — By assumption, we know that for all $y \in K_0$, there exists an element $f(y) \in F(y) \cap T_K(y)$. By definition of the contingent cone, there exists $h_y \in]0, 1/m[$ such that

$$d_K(y + h_y f(y)) < h_y/2m$$

We introduce the subsets

$$N(y) := \{x \in K_0 \mid d_K(x + h_y f(y)) < h_y/2m\}$$

These subsets are obviously *open*. Since y belongs to $N(y)$, there exists $\eta_y \in]0, 1/m[$ such that $B(y, \eta_y) \subset N(y)$. The compactness of K_0 implies that it can be covered by q such balls $B(y_j, \eta_j)$, $j = 1, \ldots, q$. We set

$$\theta_m := \min_{j=1,\ldots,q} h_{y_j}$$

Let us choose any $x \in K_0$. Since it belongs to one of the balls $B(y_j, \eta_j) \subset N(y_j)$, there exists $z_j \in K$ such that

$$\begin{cases} \|x + h_{y_j} f(y_j) - z_j\| / h_{y_j} \\ \leq d_K(x + h_{y_j} f(y_j))/h_{y_j} + 1/2m \leq 1/m. \end{cases}$$

Let us set

$$u_j := \frac{z_j - x}{h_{y_j}}$$

We see that $\|x - y_j\| \leq \eta_j \leq 1/m$, that $x + h_{y_j} u_j = z_j \in K$ and that $\|u_j - f(y_j)\| \leq 1/m$. Hence,

$$(x, u_j) \in B((y_j, f(y_j)), 1/m) \subset B(\text{Graph}(F), 1/m)$$

and $u_j \in B(F(K_0), 1/m) \subset C$. Hence the Lemma ensues. □

We can now construct by induction a sequence of positive numbers $h_j \in]\theta_m, 1/m[$ and a sequence of elements $x_j \in K_0$ and $u_j \in C$ such that

$$\begin{cases} i) \quad x_{j+1} := x_j + h_j u_j \in K_0, \quad u_j \in C \\ ii) \quad (x_j, u_j) \in B(\text{Graph}(F), 1/m) \end{cases}$$

as long as $\sum_{i=0}^{j-1} h_i \leq T$.

Indeed, the elements x_j belong to K_0, since

$$\|x_j - x\| \leq \sum_{i=0}^{i=j-1} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{i=j-1} h_i \|C\| \leq T\|C\| = r$$

Since the h_j's are larger than or equal to $\theta_m > 0$, there exists J such that

$$h_1 + \cdots + h_{J-1} \leq T < h_1 + \cdots + h_J$$

We introduce the nodes $\tau_m^j := h_0 + \cdots + h_{j-1}$, $j = 1, \ldots, J+1$ and we interpolate the sequence of elements x_j at the nodes τ_m^j by the piecewise linear functions $x_m(t)$ defined on each interval $[\tau_m^j, \tau_m^{j+1}[$ by

$$\forall \, t \in [\tau_m^j, \tau_m^{j+1}[, \ x_m(t) := x_j + (t - \tau_m^j) u_j$$

We observe that this sequence satisfies the following estimates

$$\begin{cases} i) & \forall\, t \in [0,T], \;\; x_m(t) \in \text{co}(K_0) \\ ii) & \forall\, t \in [0,T], \;\; \|x'_m(t)\| \le \|C\| \end{cases} \tag{3.10}$$

Let us fix $t \in [\tau_m^j, \tau_m^{j+1}[$. Since $\|x_m(t) - x_m(\tau_m^j)\| \le h_j \|u_j\| \le \|C\|/m$, and since $(x_j, u_j) \in B(\text{Graph}(F), 1/m)$ by Lemma 3.4.3, we deduce that these functions are approximate solutions in the sense that

$$\begin{cases} i) & \forall\, t \in [0,T], \;\; x_m(t) \in B(K_0, \varepsilon_m) \\ ii) & \forall\, t \in [0,T], \;\; (x_m(t), x'_m(t)) \in B(\text{Graph}(F), \varepsilon_m) \end{cases} \tag{3.11}$$

where $\varepsilon_m := (\|C\| + 1)/m$ converges to 0.

2. — Convergence of the Approximate Solutions

Estimates (3.10) imply that for all $t \in [0,T]$, the sequence $x_m(t)$ remains in the compact subset $\text{co}(K_0)$ and that the sequence $x_m(\cdot)$ is *equicontinuous*, because the derivatives $x'_m(\cdot)$ are bounded. We then deduce from Ascoli's Theorem that it remains in a compact subset of the Banach space $\mathcal{C}(0,T;X)$, and thus, that a subsequence (again denoted) $x_m(\cdot)$ converges uniformly to some function $x(\cdot)$.

Furthermore, the sequence $x'_m(\cdot)$ being bounded in the dual of the Banach space $L^1(0,T;X)$ which is equal to $L^\infty(0,T;X)$, it is weakly relatively compact thanks to Alaoglu's Theorem[7]. But since T is finite, the Banach space $L^\infty(0,T;X)$ is contained in $L^1(0,T;X)$ with a stronger topology[8]. The identity map being continuous for the

[7] Alaoglu's Theorem states that any *bounded subset of the dual of a Banach space is weakly compact.*

[8] Since the Lebesgue measure on $[0,T]$ is finite, we know that

$$L^\infty(0,T;X) \subset L^1(0,T;X)$$

with a stronger topology. The weak topology $\sigma(L^\infty(0,T;X),\ L^1(0,T;X))$ (weak-star topology) is stronger than the weakened topology $\sigma(L^1(0,T;X),\ L^\infty(0,T;X))$ since the canonical injection is continuous. Indeed, we observe that the seminorms of the weakened topology on $L^1(0,T;X)$, defined by finite sets of functions of $L^\infty(0,T;X)$, are seminorms for the weak-star topology on $L^\infty(0,T;X)$), since they are defined by finite sets of functions of $L^1(0,T;X)$.

norm topologies, is still continuous for the weak topologies. Hence the sequence $x'_m(\cdot)$ is weakly relatively compact in $L^1(0,T;X)$ and a subsequence (again denoted) $x'_m(\cdot)$ converges weakly to some function $v(\cdot)$ belonging to $L^1(0,T;X)$. Equations

$$x_m(t) - x_m(s) \quad = \quad \int_s^t x'_m(\tau)d\tau$$

imply that this limit $v(\cdot)$ is actually the weak derivative $x'(\cdot)$ of the limit $x(\cdot)$.

In summary, we have proved that

$$\begin{cases} i) & x_m(\cdot) \text{ converges uniformly to } x(\cdot) \\ ii) & x'_m(t) \text{ converges weakly to } \; x'(\cdot) \text{ in } L^1(0,T;X) \end{cases}$$

3. — The Limit is a Solution

Condition (3.11)i) implies that

$$\forall\, t \in [0,T], \;\; x(t) \;\in\; K_0$$

i.e., that $x(\cdot)$ is viable. The Convergence Theorem 2.4.4 and properties (3.11)ii) imply that

$$\text{for almost all } \; t \in [0,T], \;\; x'(t) \;\in\; F(x(t))$$

i.e., that $x(\cdot)$ is a solution to differential inclusion (3.7). □

Proof of Theorem 3.3.4 — First, K is locally compact since it is closed and the dimension of X is finite.

Second, we claim that starting from any x_0, there exists a maximal solution. Indeed, denote by $\mathcal{S}_{[0,T[}(x_0)$ the set of solutions to the differential inclusion defined on $[0,T[$.

We introduce the set of pairs $\{(T,x(\cdot))\}_{T>0,\; x(\cdot)\in\mathcal{S}_{[0,T[}(x_0)}$ on which we consider the order relation $\prec$ defined by

$$(T,x(\cdot)) \prec (S,y(\cdot)) \text{ if and only if } T \;\leq\; S \;\;\&\;\; \forall\, t \in [0,T[, \;\; x(t) = y(t)$$

Since every totally ordered subset has obviously a majorant, Zorn's Lemma implies that any solution $y(\cdot) \in \mathcal{S}_{[0,S[}(x_0)$ defined on some interval $[0, S[$ can be extended to a solution $x(\cdot) \in \mathcal{S}_{[0,T[}(x_0)$ defined on a maximal interval $[0, T[$.

Third, we have to prove that if T is finite, we cannot have

$$c \; := \; \limsup_{t \to T-} \|x(t)\| \; < \; +\infty$$

Indeed, if $c < +\infty$, there would exist a constant $\eta \in]0, T[$ such that

$$\forall \, t \in [T - \eta, T[, \quad \|x(t)\| \; \leq \; c + 1$$

Since F is upper semicontinuous with compact images on the compact subset $K \cap (c+1)B$, we infer that

$$\forall \, t \in [T - \eta, T[, \quad x'(t) \in F(K \cap (c+1)B), \quad \text{which is compact}$$

and thus bounded by a constant ρ. Therefore, for all $\tau, \sigma \in [T - \eta, T[$, we obtain:

$$\|x(\tau) - x(\sigma)\| \; \leq \; \int_\sigma^\tau \|x'(s)ds\| \; \leq \; \rho|\tau - \sigma|$$

Hence the Cauchy criterion implies that $x(t)$ has a limit when $t \to T-$. We denote by $x(T)$ this limit, which belongs to K because it is closed. Equalities

$$x(T_k) \; = \; x_0 + \int_0^{T_k} x'(\tau)d\tau$$

and Lebesgue's Theorem imply that by letting $k \to \infty$, we obtain:

$$x(T) \; = \; x_0 + \int_0^{T} x'(\tau)d\tau$$

This means that we can extend the solution up to T and even beyond, since Theorem 3.3.2 allows us to find a viable solution starting at $x(T)$ on some interval $[T, S]$ where $S > T$. Hence c cannot be finite. □

Proof of Theorem 3.3.5 — Since the growth of F is linear,

$$\exists \, c \geq 0, \quad \text{such that } \forall \, x \in \mathrm{Dom}(F), \quad \|F(x)\| \leq c(\|x\| + 1)$$

Therefore, any solution to differential inclusion (3.7) satisfies the estimate:

$$\|x'(t)\| \leq c(\|x(t)\| + 1)$$

The function $t \to \|x(t)\|$ being locally Lipschitz, it is almost everywhere differentiable. Therefore, for any t where $x(t)$ is different from 0 and differentiable, we have

$$\frac{d}{dt}\|x(t)\| = \left\langle \frac{x(t)}{\|x(t)\|}, x'(t) \right\rangle \leq \|x'(t)\|$$

These two inequalities imply the estimates:

$$\|x(t)\| \leq (\|x_0\| + 1)e^{ct} \;\; \& \;\; \|x'(t)\| \leq c(\|x_0\| + 1)e^{ct} \tag{3.12}$$

Hence, for any $T > 0$, we infer that

$$\limsup_{t \to T-} \|x(t)\| < +\infty$$

Theorem 3.3.4 implies that we can extend the solution on the interval $[0, \infty[$.

Furthermore, estimates (3.12) imply that for $b > c$, the solution $x(\cdot)$ belongs to the weighted Sobolev space $W^{1,1}(0, \infty; X; e^{-bt}dt)$ since the multiplication by $e^{-(b-c)t}$ is continuous from $L^\infty(0, \infty; X)$ to $L^1(0, \infty; X)$. □

Proof of Theorem 3.3.6 — We introduce the set-valued map G defined on K by

$$G(x) := F(x) \cap c(\|x\| + 1)B$$

Corollary 2.2.3 implies that G is a Marchaud map. Assumption (3.8) implies that K is a viability domain of G. Therefore by Theorem 3.3.5, we know that for any $x_0 \in K$, there exists a viable solution to differential inclusion

$$x'(t) \in G(x(t))$$

on $[0, \infty[$, which is also a solution to differential inclusion (3.7) viable in K. □

3.5 Solution Map

We denote by $\mathcal{S}(x_0)$ or by $\mathcal{S}_F(x_0)$ the (possibly empty) set of solutions to differential inclusion (3.7.)

Definition 3.5.1 (Solution Map) *We shall say that the set-valued map $\mathcal{S}$ defined by* $\mathrm{Dom}(F) \ni x \mapsto \mathcal{S}(x)$ *is the* solution map *of F (or of differential inclusion (3.7).)*

When a closed subset K is viable under F, we denote

$$\mathcal{S}_F^K(x) \ := \ \mathcal{S}_F(x)(K) \cap \mathcal{K}$$

the set of solutions starting from $x \in K$ and viable in K.

We shall devote this section to the study of the solution map.

3.5.1 Upper Semicontinuity of Solution Maps

We recall that the space $W^{1,1}(0,\infty;X;e^{-bt}dt)$ is supplied with the topology for which a sequence $x_n(\cdot)$ converges to $x(\cdot)$ if and only if

$$\begin{cases} i) & x_n(\cdot) \text{ converges } \textit{uniformly} \text{ to } x(\cdot) \text{ on compact sets} \\ ii) & x_n'(\cdot) \text{ converges } \textit{weakly} \text{ to } x'(\cdot) \text{ in } L^1(0,T;X,e^{-bt}dt) \end{cases}$$

Theorem 3.5.2 (Continuity of the Solution Map) *Let us consider a finite dimensional vector space X and a* Marchaud map $F : X \rightsquigarrow X$. *We set*

$$c := \sup_{x \in \mathrm{Dom}(F)} \frac{\|F(x)\|}{\|x\|+1}$$

Then the solution map $\mathcal{S}$ is upper semicontinuous with compact images from its domain to the space $\mathcal{C}(0,\infty;X)$ supplied with the compact convergence topology.

Actually, for $b > c$, the solution map $\mathcal{S}$ is upper semicontinuous with compact images from its domain to the space $W^{1,1}(0,\infty;X;e^{-bt}dt)$.

Furthermore, the graph of the restriction of $\mathcal{S}|_L$ to any compact subset L of $\mathrm{Dom}(F)$ *is compact in $X \times W^{1,1}(0,\infty;X;e^{-bt}dt)$.*

Proof — We shall show that the graph of the restriction $\mathcal{S}|_L$ of the solution map $\mathcal{S}$ to a compact subset $L \subset \mathrm{Dom}(F)$ (assumed to be nontrivial) is compact.

Let us choose a sequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of the solution map $\mathcal{S}$. They satisfy:

$$x_n'(t) \in F(x_n(t)) \;\; \& \;\; x_n(0) = x_{0_n} \in L$$

A subsequence (again denoted) x_{0_n} converges to some $x_0 \in L$ because L is compact.

Then inequalities

$$\text{for almost all } t \geq 0, \quad \frac{d}{dt}\|x_n(t)\| \leq \|x_n'(t)\| \leq c(\|x_n(t)\| + 1)$$

imply that

$$\forall\, n \geq 0, \;\; \|x_n(t)\| \;\leq\; (\|x_{0_n}\| + 1)e^{ct} \;\; \& \;\; \|x_n'(t)\| \;\leq\; c(\|x_{0_n}\| + 1)e^{ct}$$

Therefore, by Ascoli's Theorem, the sequence $x_n(\cdot)$ is relatively compact in the Fréchet space $\mathcal{C}(0,\infty;X)$ and by Alaoglu's Theorem, the sequence $x_n'(\cdot)e^{-ct}$ is weakly relatively compact in $L^\infty(0,\infty;X)$.

Let us take $b > c$. Since the multiplication by $e^{-(b-c)t}$ is continuous from $L^\infty(0,\infty;X)$ to $L^1(0,\infty;X)$, it remains continuous when these spaces are supplied with weak topologies[9].

We have proved that the sequence $x_n'(\cdot)$ is *weakly relatively compact* in the weighted space $L^1(0,\infty;X;e^{-bt}dt)$.

We thus deduce that a subsequence (again denoted) x_n converges to x in the sense that:

$$\begin{cases} i) & x_n(\cdot) \text{ converges uniformly to } x(\cdot) \text{ on compact sets} \\ ii) & x_n'(\cdot) \text{ converges weakly to } x'(\cdot) \text{ in } L^1(0,\infty;X;e^{-bt}dt) \end{cases}$$

[9]If u_n converges weakly to u in $L^\infty(0,\infty;X)$, then $e^{-(b-c)t}u_n$ converges weakly to $e^{-(b-c)t}u$ in $L^1(0,\infty;X)$, because, for every $\varphi \in L^\infty(0,\infty;X) = L^1(0,\infty;X)^\star$, the values

$$< u_n, e^{-(b-c)t}\varphi >:= \int_0^\infty e^{-(b-c)t}u_n(t)\varphi(t)dt$$

converge to

$$< u, e^{-(b-c)t}\varphi >:= \int_0^\infty e^{-(b-c)t}u(t)\varphi(t)dt$$

since $e^{-(b-c)t}\varphi(\cdot)$ belongs to $L^1(0,\infty;X)$.

Inclusions

$$\forall\, n > 0, \quad (x_n(t), x_n'(t)) \in \text{Graph}(F)$$

imply that

$$\text{for almost all } t > 0, \quad x'(t) \in F(x(t))$$

thanks to the Convergence Theorem 2.4.4.

We thus have proved that a subsequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of $\mathcal{S}|_L$ converges to an element $(x_0, x(\cdot))$ of this graph. This shows that it is compact, and thus, that the solution map $\mathcal{S}$ is upper semicontinuous with compact images. □

Remark — We shall prove in Chapter 4 that the domain of the solution map $\mathcal{S}_F$ associated with a Marchaud map is a closed subset, called the *viability kernel* of Dom(F). Chapter 4 is devoted to the study of viability kernels. □

Remark — The "contingent derivative" of the solution map is contained in the solution map of the "variational inclusion", which is a "set-valued linearization" of the differential inclusion. (See Section 5, Chapter 10 of SET-VALUED ANALYSIS.) □

3.5.2 Closure of a Viability Domain

The first application of the upper semicontinuity of the solution map is that the closure of any subset viable under F is a viability domain:

Proposition 3.5.3 *Let us consider a* Marchaud map $F : X \rightsquigarrow X$ *and a subset* $\Omega \subset$ Dom(F) *viable under* F. *Then its closure* $\overline{\Omega}$ *is still viable under* F.

Proof — Indeed, let a sequence $x_n \in \Omega$ converge to x given in $\overline{\Omega}$. It remains in a compact subset L of the finite dimensional vector space X. Let us choose a sequence of solutions $x_n(\cdot) \in \mathcal{S}_F(x_n)$ viable in Ω, which exist by assumption.

Since the graph of the restriction $\mathcal{S}_F|_L$ of $\mathcal{S}_F$ to the compact subset L is compact, Theorem 3.5.2 implies that $(x_n, x_n(\cdot))$ belongs to the compact subset $\mathrm{Graph}(\mathcal{S}_F|_L)$. Therefore a subsequence converges to some $(x, x(\cdot))$ of the graph of $\mathcal{S}_F|_L$, so that $x(\cdot)$ belongs to $\mathcal{S}_F(x)$. Since $x_n(t) \in \Omega$ for all $t \geq 0$, we infer the limit $x(\cdot)$ is viable in $\overline{\Omega}$. □

3.5.3 Reachable Map

We associate with the solution map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ of the differential inclusion (3.7) the *reachable map*, (or flow, or set-valued semi-group) defined in the following way:

Definition 3.5.4 *For any $t \geq 0$, we denote by $R_F(t)(x) := (\mathcal{S}_F(x))(t)$ the set of states $x(t)$ reached from x through differential inclusion (3.7), by*

$$\begin{cases} R_F^K(t) := (\mathcal{S}_F(K))(t) \\ \\ Q_F^K(t) := (\mathcal{S}_F^K(K))(t) \end{cases}$$

the set of states $x(t)$ reached from K by solutions $x(\cdot) \in \mathcal{S}_F(x)$ and by solutions $x(\cdot) \in \mathcal{S}_F^K(x)$ viable in K respectively. They are called the reachable map *and* viable reachable map *respectively.*

The reachable map $R_F(t)(x)$ enjoys the *semigroup property*:

$$\forall\, t, s \,\geq\, 0, \;\; R_F(t+s)(x) \;=\; R_F(t)\left(R_F(s)(x)\right)$$

The maps $t \rightsquigarrow R_F^K(t)$ and $t \rightsquigarrow Q_F^K(t)$ are examples of *viability tubes* which shall be studied in Chapter 11. For the time, let us prove that these maps are closed:

Proposition 3.5.5 *Assume that $F : X \rightsquigarrow X$ is a Marchaud map and that a closed subset K is contained in the domain of $\mathcal{S}_F$. Then the graphs of the maps $t \rightsquigarrow R_F^K(t)$ and $t \rightsquigarrow Q_F^K(t)$ are closed.*

Proof — Let us consider a sequence (t_n, x_n) of the graph of $R_F^K(\cdot)$ converging to (t, x). By definition, there exist solutions $x_n(\cdot) \in \mathcal{S}_F(x_{0n})$ such that $x_{0n} \in K$ and $x_n(t_n) = x_n$. Since the sequence x_n is bounded, so that a subsequence converges to some $x \in K$, a slight modification of the proof of Theorem 3.5.2 obtained by writing that

$$x_n(t) \;=\; x_n + \int_{t_n}^{t} x_n'(\tau) d\tau$$

implies that a subsequence converges to some solution $x_\star(\cdot)$ such that $x_{n'} = x_{n'}(t_{n'})$ converges to $x = x_\star(t)$. Since a subsequence of x_{0n} converges to $x_\star(0)$, hence $x_\star(0)$ belongs to K and we deduce that $x \in R_F^K(t)$. □

The reachable maps play an important role in control theory. One can state that under adequate assumptions, $co(F)$ is its *infinitesimal generator*:

Theorem 3.5.6 (Frankowska) *If F is continuous with compact values, then*

$$\lim_{h \to 0+} \frac{R_F(h)(x) - x}{h} = co(F(x))$$

We shall see in Chapter 7 that the left-hand side of this formula is the *derivative* of the reachable map $R_F(\cdot)(x)$ at $(0,x)$, so that this theorem states that the when F is continuous, $co(F(x))$ is the derivative of the reachable map at $(0,x)$.

We refer to Hélène Frankowska's monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS for a proof of this basic theorem which plays a very important role for studying local controllability and value functions in optimal control.

3.5.4 Representation Property

When the viability subset is convex, we can represent the set of viable solutions in K as a countable intersection of solution sets to *unconstrained differential inclusions*, a property which is analogous to the duality property in convex minimization.

Theorem 3.5.7 (Kurzhanski) *Consider a set-valued map $F : X \rightsquigarrow X$ with nonempty compact values. Assume that $K := A^{-1}(M)$ is the inverse image of a closed convex subset $M \subset Y$ by a surjective linear operator $A \in \mathcal{L}(X,Y)$. Denote by $F|_K$ the restriction of F to K. Then, for any right-inverse $B \in \mathcal{L}(Y,X)$ of A,*

$$\forall\, x \in X, \quad F|_K(x) = \bigcap_{n \in \mathbf{Z}} (F(x) + nBAx - nB(M))$$

Consequently, for any $x \in K$, the set $\mathcal{S}|_K(x)$ of solutions to the differential inclusion $x'(t) \in F(x(t))$ viable in K is the intersection of *the sets of solutions $\mathcal{S}_n(x)$ to the unconstrained differential inclusions $x'(t) \in F(x(t)) + nBAx(t) - nB(M)$ when n ranges over* $\mathbf{Z}$.

Proof — Consider first the case when $x \in K$. Since $F(x) = F(x) + 0B(Ax - M)$, the intersection of the subsets $F(x) + nBAx - nB(M)$ is contained in $F(x)$. On the other hand, 0 belonging to $Ax - M$, we infer that $F(x) \subset F(x) + nB(Ax - M)$ for any $n \in \mathbf{Z}$, so that

$$F(x) \subset \bigcap_{n\in\mathbf{Z}} (F(x) + nBAx - nB(M))$$

Consider now the case when $x \notin K$ and let us show that

$$\bigcap_{n\in\mathbf{Z}} (F(x) + nBAx - nB(M)) = \emptyset$$

Since any right inverse B of A is injective, 0 does not belong to the closed convex subset $B(Ax - M)$, and thus can be separated from 0: There exist $p \in X^\star$ and $\varepsilon > 0$ such that

$$\sigma(B(Ax - M), p) = -\varepsilon < 0$$

Now, we observe that $F(x)$ being bounded, the support function $\sigma(F(x) - F(x)), p)$ is nonnegative and bounded. We claim that for any $n > (\sigma(F(x) - F(x)), p)/2\varepsilon$,

$$(F(x) - nB(Ax - M)) \cap (F(x) + nB(Ax - M)) = \emptyset$$

Otherwise, there would exist u_1 and u_2 in $F(x)$ such that $u_1 - u_2$ would belong both to $F(x) - F(x)$ and to $2nB(Ax - M)$, so that we would obtain the contradiction

$$-\sigma(F(x) - F(x), p) \leq < p, u_1 - u_2 > \leq -2n\varepsilon \quad \square$$

3.6 Stability of Viability Domains

Let us recall the definition of Painlevé-Kuratowski upper limit[10] of sets:

[10]The concepts of upper and lower limits of sets were introduced by Painlevé in 1902 and popularized by Kuratowski in his famous book TOPOLOGIE, to the point that they are often Christened *Kuratowski upper limits*. See the first chapter of

Definition 3.6.1 *Let K_n be a sequence of subsets of a metric space X. we say that*

$$K^\sharp := \text{Limsup}_{n\to\infty} K_n := \{y \in Y \mid \liminf_{n\to\infty} d(y, K_n) = 0\}$$

is its upper limit.

In other words, *it is the closed subset of cluster points of sequences of elements $x_n \in K_n$.*

We observe that the *contingent cone*

$$T_K(x) \;=\; \text{Limsup}_{h\to 0+} \frac{K - x}{h}$$

is the upper limit of the differential quotients $\frac{K-x}{h}$ when $h \to 0+$.

Let us consider now a sequence of closed subsets K_n viable under a set-valued map F. *Is the upper limit of these closed subsets still viable under F?* The answer is positive.

Theorem 3.6.2 *Let us consider a* Marchaud map $F : X \rightsquigarrow X$. *Then the upper limit of a sequence of closed subsets viable under F is still viable under F.*

In particular, the intersection of a decreasing family of closed viability domains is a closed viability domain.

Proof — We shall prove that the upper limit $K^\sharp$ of a sequence of subsets K_n viable under F is still viable under F.

Let x belong to $K^\sharp$. It is the limit of a subsequence $x_{n'} \in K_{n'}$. Since the subsets K_n are viable under F, there exist solutions $y_{n'}(\cdot)$ to differential inclusion $x' \in F(x)$ starting at $x_{n'}$ and viable in $K_{n'}$. The upper semicontinuity of the solution map implies that a subsequence (again denoted) $y_{n'}(\cdot)$ converges uniformly on compact intervals to a

SET-VALUED ANALYSIS for an exhaustive study of these upper and lower limits of sequences of sets. Recall only that if the space X is *compact* , then the upper limit $K^\sharp$ enjoys

$$\text{for all neighborhood } \mathcal{U} \text{ of } K^\sharp, \;\; \exists N \;\; \text{such that } \forall n > N, \;\; K_n \subset \mathcal{U}$$

solution $y(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x. Since $y_{n'}(t)$ belongs to $K_{n'}$ for all n', we deduce that $y(t)$ does belong to $K^\sharp$ for all $t > 0$.

When the sequence K_n is decreasing, we know that its upper limit is equal to the intersection of the K_n. □

What happens if we deal with the upper limit $K^\sharp$ of a sequence of closed viability domains K_n of set-valued maps F_n?

For that purpose, we introduce the concept of *graphical upper limit* of a sequence of set-valued maps F_n.

Definition 3.6.3 *We shall say that the set-valued maps* $\mathrm{Lim}^\sharp{}_{n\to\infty}F_n$ *from X to X defined by*

$$\mathrm{Graph}(\mathrm{Lim}^\sharp{}_{n\to\infty}F_n) \;:=\; \mathrm{Limsup}_{n\to\infty}\mathrm{Graph}(F_n)$$

is the graphical upper limit of the set-valued maps F_n.

For simplicity, we set $F^\sharp := \mathrm{Lim}^\sharp{}_{n\to\infty}F_n$. One can find more details on graphical limits in Chapter 7 of SET-VALUED ANALYSIS.

The question then arises whether the upper limit $K^\sharp$ of a sequence of closed subsets K_n viable under set-valued maps F_n is viable under the closed convex hull of the upper graphical limit $\overline{\mathrm{co}}F^\sharp$ of the set-valued maps F_n?

Theorem 3.6.4 (Stability of Solution Maps) *Let us consider a sequence of nontrivial set-valued maps $F_n : X \rightsquigarrow X$ satisfying a uniform linear growth: there exists $c > 0$ such that*

$$\forall\, x \in X,\;\; \|F_n(x)\| \;\leq\; c(\|x\| + 1)$$

Then

1. — The upper limit of the solution maps $\mathcal{S}_{F_n}$ is contained in the solution map $\mathcal{S}_{\overline{\mathrm{co}}(F^\sharp})$ of the convex hull of the graphical upper limit of the set-valued maps F_n

2. — If the subsets $K_n \subset \mathrm{Dom}(F_n)$ are viable under the set-valued maps F_n, then the upper limit $K^\sharp$ is viable under $\overline{\mathrm{co}}(F^\sharp)$.

It follows from the adaptation of the Convergence Theorem to limits of set-valued maps:

Theorem 3.6.5 *Let X be a topological vector space, Y be a finite dimensional vector space and F_n be a sequence of nontrivial set-valued maps from X to Y satisfying a uniform linear growth.*

Let us consider measurable *functions x_m and y_m from $[0,\infty[$ to X and Y respectively, satisfying:*

for almost all $t \in [0,\infty[$ and for all neighborhood $\mathcal{U}$ of 0 in the product space $X \times Y$, there exists $M := M(t,\mathcal{U})$ such that

$$\forall\, m > M,\ (x_m(t), y_m(t)) \in \text{Graph}(F_m) + \mathcal{U} \tag{3.13}$$

If we assume that

$$\begin{cases} i) & x_m(\cdot) \text{ converges almost everywhere to a function } x(\cdot) \\ ii) & y_m(\cdot) \in L^1(0,\infty,Y;a) \text{ and converges } \textit{weakly} \text{ in } L^1(0,\infty,Y;a) \\ & \text{to a function } y(\cdot) \in L^1(0,\infty,Y;a) \end{cases}$$

then,

$$\text{for almost all } t \in [0,\infty[,\ \ y(t) \in \overline{co}(F^\sharp(x(t)))$$

We refer to Theorem 7.2.1 of SET-VALUED ANALYSIS for a proof.

3.7 ω-Limit Sets and Equilibria

3.7.1 ω-Limit Sets

The ω-limit sets of the solutions to differential inclusion

$$\text{for almost all } t \geq 0,\ \ x'(t) \in F(x(t)) \tag{3.14}$$

provide examples of closed viability domains:

Definition 3.7.1 (ω-Limit set) *Let $x(\cdot)$ be a function from $[0,\infty[$ to X. We say that the subset*

$$\omega(x(\cdot)) := \bigcap_{T>0} cl(x([T,\infty[)) = \text{Limsup}_{t\to+\infty}\{x(t)\}$$

of its cluster points when $t \to \infty$ is the ω-limit set *of $x(\cdot)$.*

If F is a set-valued map, K a subset of $\text{Dom}(\mathcal{S}_F)$ *and $R_F^K(\cdot)$ the reachable map, we denote by*

$$\omega_F(K) := \text{Limsup}_{t\to+\infty} R_F^K(t)$$

the ω-limit set of the subset K. If K is a closed subset viable under F, the viable ω-limit set of K *is defined by*

$$\omega_F^K(K) := \text{Limsup}_{t\to+\infty} Q_F^K(t)$$

Being upper limits, *the ω-limit sets of solutions and sets are closed subsets.* They also are viable under F. We begin with the case of ω-limit sets of solutions:

Theorem 3.7.2 (ω-Limit sets are viability domains) *Let us consider a* Marchaud map $F : X \rightsquigarrow X$. *Then the ω-limit set of a solution to the differential inclusion (3.14) is a* closed viability domain[11].

In particular, the limits of solutions to the differential inclusion (3.14), when they exist, are equilibria *of F and the trajectories of periodic solutions to the differential inclusion (3.14) are also closed viability domains.*

If K is a viability domain of F, then the ω-limit sets of viable solutions are contained in K.

Proof — Let $\bar{x}$ belong to the ω-limit set of a solution $x(\cdot)$. It is the limit of a sequence of elements $x(t_n)$ when $t_n \to \infty$. We then introduce the functions $y_n(\cdot)$ defined by $y_n(t) := x(t + t_n)$. They are solutions to the differential inclusion (3.14) starting at $x(t_n)$. By Theorem 3.5.2 on the upper semicontinuity of the solution map, a subsequence (again denoted) $y_n(\cdot)$ converges uniformly on compact intervals to a solution $y(\cdot)$ to the differential inclusion (3.14) starting at $\bar{x}$. On the other hand, for all $t > 0$,

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} x(t + t_n) \in \omega(x(\cdot))$$

i.e., $y(\cdot)$ is *viable* in the ω-limit set $\omega(x(\cdot))$. Hence the ω-limit set is viable under F. The necessary condition of the Viability Theorem 3.3.2 implies that this ω-limit set is a viability domain.

[11] which is *connected when $\omega(x(\cdot))$ is compact.* If not, $\omega(x(\cdot))$ would be covered by two nonempty disjoint closed subsets K_1 and K_2. So, they can be separated by two disjoint open neighborhoods $U_1 \supset K_1$ and $U_2 \supset K_2$.

Since $U_1 \cup U_2$ is a neighborhood of the compact subset $\omega(x(\cdot))$, there exists T such that the subset $\Gamma := \{x(t)\}_{t\geq T}$ is contained in $U_1 \cup U_2$. This set is connected as the continuous image of $[T, \infty[$. We observe that the subsets $\Gamma_i := \Gamma \cap U_i$ are not empty, open, disjoint and cover Γ: this is a contradiction of the connectedness of Γ.

When a solution has a limit $\overline{x}$ when $t \to \infty$, the subset $\{\overline{x}\}$ is a viability domain, and thus, $\overline{x}$ is an equilibrium. □

We consider now the case of ω-limit sets of closed subsets:

Proposition 3.7.3 *Let us consider a Marchaud map $F : X \leadsto X$ and a closed subset K of the domain of $\mathcal{S}_F$. Then the ω-limit set $\omega_F(K)$ is viable under F.*

If there exists $T \geq 0$ such that $\bigcup_{t \geq T} R_F^K(t)$ is bounded, then $\omega_F(K)$ is an universal attractor *in the sense that*

$$\forall\, x \in K,\ \forall\, x(\cdot) \in \mathcal{S}_F(x),\ \lim_{t \to \infty} d(x(t), \omega_F(K)) \;=\; 0$$

If K is viable under F, then the viable ω-limit set $\omega_F^K(K)$ is a closed viability domain contained in K.

If K is compact, it is an attractor *in the sense that*

$$\forall\, x \in K,\ \exists\ x(\cdot) \in \mathcal{S}_F^K(x) \quad \textit{such that} \ \lim_{t \to \infty} d(x(t), \omega_F^K(K)) \;=\; 0$$

Proof — The closed subset $\omega_F(K)$ is viable under F. Indeed, let ξ belong to $\omega_F(K)$. Then $\xi = \lim \xi_n$ where $\xi_n \in R_F^K(t_n)$. We associate with the solutions $x_n(\cdot)$ to the differential inclusion

$$x_n'(t) \;\in F(x_n(t)),\ \ x_n(t_n) \;=\; \xi_n$$

the functions $y_n(\cdot)$ defined by $y_n(t) := x_n(t + t_n)$ which are solutions to

$$y_n'(t) \;\in\; F(y_n(t)),\ \ y_n(0) \;=\; \xi_n$$

Theorem 3.5.2 implies that these solutions remain in a compact subset of $\mathcal{C}(0, \infty; X)$. Therefore, a subsequence (again denoted by) $y_n(\cdot)$ converges to $y(\cdot)$, which is a solution to

$$y'(t) \;\in\; F(y(t)),\ \ y(0) \;=\; \xi$$

Furthermore, this solution is viable in $\omega_F(K)$ since for all $t \geq 0$, $y(t)$ is the limit of a subsequence of $y_n(t) = x_n(t + t_n) \in R_F^K(t + t_n)$, and thus belongs to $\omega_F(K)$.

Let us prove now that $\omega_F(K)$ is an universal attractor. If not, there would exist $x_0 \in K$, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$, $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$\forall\, n \geq 0,\;\; d\,(x(t_n), \omega_F(K)) \;\geq\; \delta \;>\; 0$$

Since the closure of $\bigcup_{t \geq T} R_F^K(t)$ is compact by assumption, a subsequence (again denoted by) $x(t_n)$ converges to some $x_\star$ which belongs to the ω-limit set $\omega_F(K)$. We thus obtain a contradiction.

The proofs of the statements about $\omega_F^K(K)$ are analogous. □

We shall see in Chapter 11 that *upper limits of viability tubes* $t \mapsto P(t)$ *when* $t \to \infty$ *are closed subsets viable under* F *which are attractors* when $\bigcup_{t \geq T} P(t)$ is relatively compact. If we regard such ω-limit sets as *"asymptotic targets"* (because they are made of cluster points of solutions viable in such tubes), we must look for asymptotic targets among the closed subsets viable under F. □

3.7.2 Cesaro means of the velocities

The property of the Cesaro means described in the assumptions of the next theorem implies the existence of an equilibrium:

Theorem 3.7.4 *Let us assume that* F *is upper hemicontinuous with closed convex images and that* $K \subset \mathrm{Dom}(F)$ *is compact. If there exists a solution* $x(\cdot)$ *viable in* K *such that*

$$\inf_{t>0} \frac{1}{t} \int_0^t \|x'(\tau)\| d\tau \;=\; 0$$

then there exists a viable equilibrium $\bar{x}$*, i.e., a state* $\bar{x} \in K$ *solution to the inclusion* $0 \in F(\bar{x})$.

Proof — Let us assume that there is no viable equilibrium, i.e., that for any $x \in K$, 0 does not belong to $F(x)$. Since the images of F are closed and convex, the Separation Theorem implies that there exists $p \in \Sigma$, the unit sphere, and $\varepsilon_p > 0$ such that

$\sigma(F(x), -p) < -\varepsilon_p$. In other words, we can cover the compact subset K by the subsets

$$\mathcal{V}_p := \{ x \in K \mid \sigma(F(x), -p) < -\varepsilon_p \}$$

when p ranges over Σ. They are open thanks to the upper hemicontinuity of F, so that the compact subset K can be covered by q open subsets $\mathcal{V}_{p_j}$. Set $\varepsilon := \min_{i=1,\ldots,q} \varepsilon_{p_i} > 0$.

Consider now any viable solution to differential inclusion (3.14). Hence, for any $t \geq 0$, $x(t)$ belongs to some $\mathcal{V}_{p_j}$, so that

$$-\|x'(t)\| \leq \langle -p_j, x'(t) \rangle \leq \sigma(F(x(t)), -p_j) < -\varepsilon$$

and thus, by integrating from 0 to t, we have proved that there exists $\varepsilon > 0$ such that, for all $t > 0$,

$$\varepsilon < \frac{1}{t} \int_0^t \|x'(\tau)\| d\tau$$

a contradiction of the assumption of the theorem. □

3.7.3 Viability implies Stationarity

When K is a compact viability domain, then the convexity of either $F(K)$ or of K implies the existence of a viable equilibrium.

Theorem 3.7.5 *Let F be a Marchaud map. If $K \subset \mathrm{Dom}(F)$ is a compact viability domain and if $F(K)$ is convex, then there exists an equilibrium.*

Proof — Assume that there is no equilibrium. Hence, this means that 0 does not belong to the closed convex subset $F(K)$, so that the Separation Theorem implies the existence of some $p \in X^\star$ and $\varepsilon > 0$ such that

$$\sup_{x \in K, v \in F(x)} < v, -p > = \sigma(F(K), -p) < -\varepsilon$$

Hence, let us take any viable solution $x(\cdot)$ to differential inclusion (3.14), which exists by the Viability Theorem. We deduce that

$$\forall\, t \geq 0, \ < -p, x'(t) > \leq -\varepsilon$$

so that, integrating from 0 to t, we infer that

$$\varepsilon\, t \;\leq\; < p, x(t) - x(0) >$$

But K being bounded, we thus derive a contradiction. □

We shall state now that any convex compact viability domain contains an equilibrium.

Theorem 3.7.6 (Equilibrium Theorem) *Let X be a Banach space*[12] *and $F : X \rightsquigarrow X$ be an upper hemicontinuous set-valued map with closed convex images.*

If $K \subset X$ is a convex compact viability domain of F, then it contains an equilibrium of F.

This theorem is equivalent to the Kakutani and Brouwer Fixed Point Theorems; we shall not prove this equivalence here[13].

We show only that the Equilibrium Theorem 3.7.6 implies the Kakutani Fixed Point Theorem[14], which is the set-valued version of the Brouwer fixed Point Theorem.

Theorem 3.7.7 (Kakutani Fixed Point Theorem) *Let K be a convex compact subset of a Banach space X and $G : K \rightsquigarrow K$ be a strict upper hemicontinuous set-valued map with closed convex values. Then G has a fixed point*[15] *$\bar{x} \in K \cap G(\bar{x})$.*

Proof — We set $F(x) := G(x) - x$, which is also upper hemicontinuous with convex values. Since K is convex, then $K - x \subset T_K(x)$, and since $G(K) \subset K$, we deduce that K is a viability domain of F because $F(x) \subset T_K(x)$. Hence there exists a viable equilibrium $\bar{x} \in K$ of F, which is a fixed point of G. □

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[12] Actually, this theorem remains true for any Hausdorff locally convex topological vector space and in particular, for spaces endowed with weak topologies.

[13] See Appendix C of MATHEMATICAL METHODS OF GAME AND ECONOMIC THEORY for a proof of the Brouwer Fixed Point Theorem based on Sperner's Lemma and the second chapter of APPLIED NONLINEAR ANALYSIS for a proof based on differential geometry. We refer to these books or SET-VALUED ANALYSIS for a proof of the equivalence between these statements and the Ky Fan Inequality.

[14] called Ky Fan's Fixed Point Theorem in infinite dimensional spaces.

[15] which can be regarded as an equilibrium for the discrete set-valued dynamical system $x_{n+1} \in G(x_n)$.

Actually, Equilibrium Theorem 3.7.6 can be derived from the Brouwer Fixed-Point Theorem via the *Ky Fan Inequality.* We recall it below not only because we shall use it later, but because of its efficiency for proving many results of nonlinear analysis.

Theorem 3.7.8 (Ky Fan Inequality) *Let K be a compact convex subset of a Banach space and $\varphi : K \times K \mapsto \mathbf{R}$ be a function satisfying*

$$\left\{ \begin{array}{lll} i) & \forall\, y \in K, & x \mapsto \varphi(x,y) \text{ is lower semicontinuous} \\ ii) & \forall\, x \in K, & y \mapsto \varphi(x,y) \text{ is concave} \\ iii) & \forall\, y \in K, & \varphi(y,y) \;\leq\; 0 \end{array} \right. \tag{3.15}$$

Then, there exists $\bar{x} \in K$, a solution to

$$\forall\, y \in K, \;\; \varphi(\bar{x}, y) \;\leq\; 0 \tag{3.16}$$

The Ky Fan inequality implies readily the von Neumann Minimax Theorem:

Theorem 3.7.9 (Minimax) *Let X and Y be Banach spaces*[16]*, $L \subset X$ and $M \subset Y$ be compact convex subsets and $f : L \times M \mapsto \mathbf{R}$ be a real valued function satisfying*

$$\left\{ \begin{array}{ll} i) & \forall\, y \in M, \;\; x \;\mapsto\; f(x,y) \text{ is lower semicontinuous and convex} \\ & \\ ii) & \forall\, x \in L, \;\; y \;\mapsto\; f(x,y) \text{ is upper semicontinuous and concave} \end{array} \right.$$

Then there exists a saddle point $(\bar{x}, \bar{y}) \in L \times M$ *of f:*

$$\forall\, (x,y) \in L \times M, \;\; f(\bar{x}, y) \;\leq\; f(\bar{x}, \bar{y}) \;\leq\; f(x, \bar{y})$$

Proof — We apply the Ky Fan Inequality with $K := L \times M$ and φ defined by

$$\varphi((\bar{x}, \bar{y}), (x,y)) \;:=\; f(\bar{x}, y) - f(x, \bar{y}) \;\; \square$$

Actually, we often need a weaker version of the Minimax Theorem, called the Lop-Sided Minimax Theorem. We recall its statement:

[16] actually, Hausdorff locally convex topological vector spaces.

Theorem 3.7.10 (Lop-Sided Minimax Theorem) *Let X and Y be Banach spaces*[17]*, $L \subset X$ be a compact convex subset, $M \subset Y$ be a convex subset and $f : L \times M \mapsto \mathbf{R}$ be a real valued function satisfying*

$$\begin{cases} i) & \forall\, y \in M, \quad x \mapsto f(x,y) \text{ is lower semicontinuous and convex} \\ ii) & \forall\, x \in L, \quad y \mapsto f(x,y) \text{ is and concave} \end{cases}$$

Then there exists $\bar{x} \in L$ satisfying

$$\forall\, y \in M, \quad f(\bar{x},y) \;\leq\; \inf_{x \in L} \sup_{y \in M} f(x,y) \;=\; \sup_{y \in M} \inf_{x \in L} f(x,y)$$

Proof — We refer to Theorem 6.2.7. of APPLIED NONLINEAR ANALYSIS for an instance of proof using only the Separation Theorem. □

Remark — A slight modification of the proof of the Equilibrium Theorem yields a whole family of sufficient conditions for the existence of zeros of a set-valued map from $K \subset X$ to another space Y. □

Theorem 3.7.11 *Let K be a convex compact subset of a Banach space X and F be a nontrivial upper hemicontinuous set-valued map with closed convex values from X to another Banach space Y.*

Let us consider also a continuous map $B : K \to \mathcal{L}(X,Y)$. If K, F and B are related by the condition

$$\forall\, x \in K, \quad F(x) \cap \overline{B(x)T_K(x)} \;\neq\; \emptyset$$

then

$$\begin{cases} i) & \exists\, \bar{x} \in K \text{ such that } 0 \;\in\; F(\bar{x}) \\ ii) & \forall\, y \in K, \; \exists\, \hat{x} \in K \text{ such that } B(\hat{x})y \;\in\; B(\hat{x})\hat{x} - F(\hat{x}) \end{cases}$$

As an example, we derive the existence of a solution to the equation $f(\bar{x}) = 0$ where the solution $\bar{x}$ must belong to a compact convex subset K:

[17] or, more generally, an Hausdorff locally convex topological vector spaces.

Theorem 3.7.12 *Let X and Y be Banach spaces, $K \subset X$ be a compact convex subset, $\Omega \supset K$ be an open neighborhood of K and $f : \Omega \mapsto Y$ be a continuously differentiable single-valued map. Assume that*

$$\forall\, x \in K, \;\; -f(x) \;\in\; f'(x)T_K(x)$$

Then there exists a solution $\bar{x} \in K$ to the equation $f(\bar{x}) = 0$. In particular, when $x_0 \in K$ is given, there exists a sequence of elements $x_n \in K$ satisfying

$$\forall\, n \geq 0, \;\; f'(x_n)(x_n - x_{n-1}) \;=\; -f(x_n)$$

i.e., the implicit version of the Newton algorithm, studied in more details in Chapter 10.

Proof — We take $F(x) := \{f(x)\}$ and $B(x) = -f'(x)$ in Theorem 3.7.11. □

3.8 Chaotic Solutions to Differential Inclusions

Let $F : X \rightsquigarrow X$ be a Marchaud map, describing the dynamics of the differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t)) \tag{3.17}$$

Theorem 3.8.1 (Chaotic Behavior) *Let us assume that a compact viability domain K of the Marchaud map F is covered by a family of closed subsets K_a ($a \in \mathcal{A}$) such that the following* controllability assumption *holds true: There exists $T < \infty$ such that*

$$\forall\, a \in \mathcal{A}, \forall\, y \in K, \;\; \exists\, x \in K_a, \; x(\cdot) \in \mathcal{S}(x) \;\&\; t \in [0,T[\text{ with } x(t) = y$$

Then, for any sequence $a_0, a_1, \ldots, a_n, \ldots$, there exists at least one solution $x(\cdot) \in \mathcal{S}(x)$ to differential inclusion (3.17) and a sequence of elements $t^j \geq 0$ such that $x(t^j) \;\in\; K_{a_j}$ for all $j \geq 0$.

Proof — Let $M \subset K$ be any closed subset. We associate with any solution $x(\cdot)$ starting at $x \in K$ and intersecting M at some time $t \in [0,T]$ the number $\tau_M := \inf(t \in [0,T] \mid x(t) \in M)$.

We associate with the sequence $a_0, a_1, \ldots$ the subsets $M_{a_0a_1\cdots a_n}$ defined by induction by $M_{a_n} := K_{a_n}$,

$$M_{a_{n-1}a_n} := \{x \in K_{a_{n-1}} \mid \exists\, x(\cdot) \in \mathcal{S}(x) \text{ such that } x(\tau_{M_{a_n}}) \in K_{a_n}\}$$

and, for $j = n-2, \ldots, 0$, by:

$$\begin{cases} M_{a_ja_{j+1}\cdots a_n} := \{x \in K_{a_j} \mid \exists\, x(\cdot) \in \mathcal{S}(x) \\ \text{such that } x(\tau_{M_{a_{j+1}\cdots a_n}}) \in M_{a_{j+1}\cdots a_n}\} \end{cases}$$

The controllability assumption implies that they are nonempty. They are closed thanks to Theorem 3.5.2. Since the family of subsets $M_{a_0a_1\cdots a_n}$ form a nonincreasing family and since K is compact, the intersection $K_\infty := \bigcap_{n=0}^{\infty} M_{a_0a_1\cdots a_n}$ is nonempty.

Let us take an initial state x in K_∞ and fix n. Hence there exists $x_n(\cdot) \in \mathcal{S}(x)$ and a sequence of $t_n^j \in [0, jT]$ such that

$$\forall\, j = 1, \ldots, n, \;\; x_n(t_n^j) \in M_{a_j\cdots a_n} \subset K_{a_j}$$

Indeed, there exist $y_1 \in \mathcal{S}(x)$ and $\tau_{M_{a_1\cdots a_n}} \in [0, T]$ such that $y_1(\tau_{M_{a_1\cdots a_n}})$ belongs to $M_{a_1\cdots a_n}$. We set $t_n^1 := \tau_{M_{a_1\cdots a_n}}$ and $x_n(t) := y_1(t)$ on $[0, t_n^1]$.

Assume that we have built $x_n(\cdot)$ on the interval $[0, t_n^k]$ such that $x_n(t_n^j) \in M_{a_j\cdots a_n} \subset K_{a_j}$ for $j = 1, \ldots, k$. Since $x_n(t_n^k)$ belongs to $M_{a_k\cdots a_n}$, there exist $y_{k+1} \in \mathcal{S}(x_n(t_n^k))$ and $\tau_{M_{a_{k+1}\cdots a_n}} \in [0, T]$ such that $y_{k+1}(\tau_{M_{a_{k+1}\cdots a_n}})$ belongs to $M_{a_{k+1}\cdots a_n}$. We set

$$t_n^{k+1} := t_n^k + \tau_{M_{a_{k+1}\cdots a_n}} \;\;\&\;\; x_n(t) := y_{k+1}(t + \tau_{M_{a_{k+1}\cdots a_n}})$$

on $[t_n^k, t_n^{k+1}]$.

Since for some $b > 0$, the sequence $x_n(\cdot) \in \mathcal{S}(x)$ is compact in the space $W^{1,1}(0, \infty; X; e^{-bt}dt)$, a subsequence (again denoted $x_n(\cdot)$) converges to some solution $x(\cdot) \in \mathcal{S}(x)$ to the differential inclusion. By extracting successive converging subsequences of $t_{n_1}^1, \ldots, t_{n_j}^j, \ldots$, we infer the existence of t_j's in $[0, jT]$ such that $x_{n_j}(t_{n_j}^j)$ converges to $x(t_j) \in K_{a_j}$, because the functions $x_n(\cdot)$ remain in an equicontinuous subset. □