Outline of the Book

Instead of beginning with viability theorems for differential inclusions, we prefer to sketch in Chapter 1 the role of the concept of viability domain in the much simpler case of differential equations. (The first viability theorem was proved in 1942 by Nagumo.)

For a variety of reasons, an important example of a viability set is the probability simplex. Whenever the state of a system is difficult to model mathematically, one way to overcome this difficulty is to deal with probabilities, frequencies, concentrations, proportions, etc., and the probability simplex then naturally appears. Systems controlled by scalar controls (called flux) of the form

$$x_i'(t) \quad = \quad x_i(t)(f_i(x(t)) - x(t)u), \;\; i = 1, \ldots, n$$

are called *replicator systems.* They are encountered in such diverse fields as biochemistry (Eigen & Schuster's hypercycle), ethology (Maynard-Smith's game for behavioral strategies), population dynamics (Fisher's model of the evolution of genes in a population), ecology, etc. These examples are presented in the first chapter.

We also included in this chapter viability and invariance theorems for *stochastic differential equations*, which provide another way to treat uncertainty.

This chapter can be bypassed by readers mainly interested in differential inclusions and control systems.

Chapter 2 deals with the minimal information about set-valued maps that is needed to prove the viability theorems for differential inclusions. Upper and lower semicontinuous set-valued maps are defined. Then our basic result, *the Convergence Theorem*, is proved.

Since this involves convex-valued maps, some results on support functions of convex subsets are recalled in this chapter. Closed convex processes, which are the set-valued analogues of continuous linear operators, enjoy most of the properties of linear operators, including Banach's closed graph theorem and the uniform boundedness theorem. These results are reviewed, because contingent derivatives of set-valued maps being closed processes, they will be used later.

Chapter 3 is basic: it states and proves the main viability theorems (in locally compact, open and closed viability sets respectively) and shows that the solution map is upper semicontinuous. We also prove a *stability result:* (upper) limits of viability domains are still viability domains and we show that *ω-limit sets* of solutions, limits of solutions when the time goes to infinity (equilibria), trajectories of periodic solutions are examples of closed viability domains.

We adapt Saari's principle on the chaotic behavior of discrete systems to the case of differential inclusions. The viability domain is divided into a number of cells in such a way that each of them can be "visited" in any given way by at least one trajectory of a differential inclusion.

We then proceed in Chapter 4 with further properties of the *viability kernels* of closed subsets: There exists a largest closed viability domain contained in a closed subset, called the viability kernel, which enjoys many properties which are investigated in this chapter. Important concepts of biomathematics such as *permanence* and *fluctuation* can be defined in terms of viability kernels. On the other hand, each closed subset of the viability kernel is contained in a minimal viability domain, called *viability envelope*.

The analysis is refined by introducing *exit time functions* associating with each initial state the first instant at which at least one solution starting from this state leaves the viability set. Viability kernels are the subsets of states with infinite exit time. We then introduce *exit tubes*, which are the subsets of states from which at least one solution satisfies the viability constraints for a prescribed length of time.

We then study the *anatomy of a set* by distinguishing inward and outward areas of the boundary of a set. It is also shown that the boundary of a viability kernel is also a viability domain.

These facts among others are used to study several *viability kernel algorithms*, including the *zero dynamics algorithm*, which converge to viability domains and/or kernels.

We devote the fifth chapter to the study of *invariant subsets*, which are sets K with the property that *all solutions* to a differential inclusion starting from a state in K are viable in K.

We need for that purpose more information on contingent cones, which are involved in a crucial way in the characterization of the viability and invariance properties. For this reason, we review some results about these cones before proceeding any further. We recall some useful formulas of the calculus of contingent cones (proved in the fourth chapter of *Set-Valued Analysis*.)

Several characterizations of invariance are provided, one of which is based on the fundamental Filippov Theorem dealing with differential inclusions with Lipschitz right-hand sides. It implies that the solution map is lower semicontinuous. This latter property is crucial to prove the existence of *invariance kernels*, which are the largest closed invariant domains contained in closed subsets.

It also implies the *semi-permeability* property of the boundary of the viability kernel of a closed subset, which states that no solution can cross the boundary to enter the interior of the viability kernel.

These invariance and viability kernels are needed to define *defeat and victory domains of a target.*

We illustrate these results in the case of *linear differential inclusions*, which are differential inclusions whose right-hand sides are closed convex processes. In this framework, we show that the concepts of invariance and viability domains are dual.

We tackle in Chapter 6 the problem of *regulating control systems by closed loop controls (single-valued feedback controls.)* The problem we have to solve is that of finding selections of the regulation map, possibly continuous. The latter are provided by Michael's Theorem, but in a non constructive way. Hence we have to design *selection procedures* which yield explicit selections, which may not be continuous, but still provide viable solutions when fed back to the differential equation governing the evolution of the control system. These selection procedures provide in particular *slow viable solutions* regulated by controls with minimal norm. For that purpose we need to complete our study of lower semicontinuous maps and provide lower semicontinuity criteria for finite and infinite intersections of lower semicontinuous maps.

Chapter 7 deals with *the inertia principle, heavy viable solutions and "punctuated equilibria", ramp controls, etc.*, which constitute the main motivations of viability theory.

At this point, we need to differentiate the regulation map. Hence this chapter starts with the shortest introduction to derivatives of set-valued maps needed to proceed. It continues with the construction of regulation maps providing viable controls that are almost everywhere differentiable.

Once we know the regulation maps yielding differentiable controls, we can differentiate the regulation law and discover the system of differential inclusions which governs the evolution of both the state and the control of the system. Then, by using the selection procedures introduced in the preceding chapter, we are able to define dynamical closed loops and, among them, the ones which provide heavy viable solutions. Viability problems for second order differential inclusions, which are first order systems in disguise, are also investigated in this chapter.

The *tracking problem*, as well as *observability, decentralization, hierarchical* issues, are studied in Chapter 8 in the framework of viability theory. The common thread of these problems is the connection between two dynamical systems through an *observation map*: Are some or all solutions to these differential inclusions linked by this observation map, in the sense that its graph is a viable or invariant manifold ? The viability theorems applied to the graphs of the observation maps imply that such observation maps are solutions to some *systems of first-order partial differential inclusions*, where the derivatives are taken in the contingent sense.

Derivatives in the sense of distributions do not offer the unique way to describe weak or generalized solutions to partial differential equations and inclusions. Contingent derivatives offer another way to weaken the required properties of a derivative, loosing the linear character of the differential operator, but allowing a pointwise definition. They provide a convenient way to treat hyperbolic problems and also allow us to look for solutions among set-valued maps, since we know how to differentiate them. Set-valued solutions constitute a useful framework to describe shocks for instance.

We study the existence of both single-valued and set-valued solutions to such partial differential inclusions, as well as a variational principle.

Differential inequalities, Lyapunov functions and related matters can also be analyzed in terms of special viability problems where the viability sets are epigraphs of functions or, more generally, graphs of preorders. This allows us to include, among the candidates that enjoy Lyapunov-type inequalities, not only differentiable functions but also lower semicontinuous functions. Thus we derive from viability theorems several generalizations of classical results. Applying to this situation the concept of viability kernel, we infer the existence of the *smallest Lyapunov function larger than a given one.*

Asymptotic stability is treated here in the framework of viability theory. These are explained in Chapter 9.

Chapter 10 gathers miscellaneous issues, such as *variational differential inequalities*. The question is the following: If we take a differential inclusion and a closed subset which is not a viability domain, can we modify the set-valued map F in such a way that K becomes a viability domain for the new map? The method is straightforward: we project the images $F(x)$ onto the contingent cone $T_K(x)$ (and obtain, when K is convex, variational differential inequalities.) By doing so, we lose both the convexity of the images and the upper semicontinuity. However, it is still possible to prove the existence of the projected system and even, under stronger assumptions, the existence of slow solutions.

The second section of chapter 10 deals with *fuzzy differential inclusions*. The right-hand sides of such differential inclusions are *fuzzy* subsets, whose membership functions are cost functions taking their values in $[0, \infty]$ (instead of $[0, 1]$ for membership functions of usual fuzzy sets). The concept of uncertainty involved in differential inclusions becomes more refined, by allowing the velocities not only to depend in a plain multivalued way upon the state of the system, but also in a fuzzy way.

The viability theorems are adapted to fuzzy differential inclusions and to sets of state constraints which are either usual or fuzzy. The existence of a largest closed fuzzy viability domain contained in a given closed fuzzy subset is also provided.

The third section presents a very short introduction to some numerical aspects of differential inclusions. The convergence of solutions to implicit and projected explicit finite-difference schemes to viable solutions of a differential inclusion is proved.

The fourth section deals with the adaptation of continuous Newton's methods for finding an equilibrium of a set-valued map: it happens that this is also a viability problem.

Chapter 11 is devoted to *time-dependent viability sets* $t \rightsquigarrow P(t)$, naturally called *tubes*. Tubes which contain at least one viable solution[8] starting from any initial state $x_0 \in P(t_0)$ at any initial time are viability tubes. These are solutions to a set-valued differential inclusion of the form $F(x) \cap DP(t, x)(1) \neq \emptyset$.

We will study the Cauchy problem, where we look for minimal viability tubes satisfying an initial condition.

One can show that their "limits"[9] when $t \to \infty$ are viability domains, and actually, attractors. If we use such viability tubes to guide a solution towards a target, we see that a necessary condition for a subset to be an asymptotic target is that it is a viability domain.

Of much greater importance for systems arising in biology, economics and cognitive sciences is the case when *both the velocity and viability sets depend upon the history of the evolution of the state.* Delays

$$\forall\, t \geq 0, \;\; x(t) \;\in\; M(x(t-\theta_1), \ldots, x(t-\theta_p))$$

[8]in the sense that $x(t) \in P(t)$ for all t.

[9]in the sense of upper limits. When the tube $P(t) := \{x(t)\}$ is single-valued, this upper limit boils down to the ω-limit set.

accumulated consequences of past evolution

$$\forall\, t \geq 0,\;\; x(t) \;\in\; M\left(\int_{-\infty}^{t} A(t-s)x(s)ds\right)$$

all these features fall under the case called *functional viability*. Here, functional viability sets $\mathcal{K}$ are subsets consisting of time-dependent functions, and viable solutions are the solutions which evolve in such function subsets in the sense that for all $t \geq 0$, $x(t+\cdot) \in \mathcal{K}$. It is the topic of Chapter 12.

Can viability theorems be extended to partial differential equations and inclusions? The answer is positive, at least for elliptic and parabolic type inclusions, as is shown in chapter 13. In this case, viability sets are comprised of spatial functions (functions depending upon the space variable.) The situation becomes more complex, because we have to work with unbounded operators on Hilbert spaces, but still, the statements which are expected to be true can be proved.

Chapter 14 treats differential games, where the controls are regarded as *strategies* used by the players to govern the evolution of the states of the game. Here, intertemporal criteria involved classically in differential games are replaced by viability constraints representing *power relations* among players, describing the constraints imposed by one player on the other. We characterize winability, playability properties adequately defined by *contingent Isaacs' equations*.

We shall prove the existence of continuous single-valued playable feedbacks, as well as more constructive, but discontinuous, playable feedbacks, such as the feedbacks associating in a myopic way optimal strategies in a cooperative framework or minimax strategies in a noncooperative environment.

In other words, *the players can implement playable feedbacks by playing for each state a static game on the strategies.*

We also provide closed loop decision rules, which *operate on the velocities of the strategies*, (regarded as *decisions*).