Chapter 14

Differential Games

Introduction

We consider two players, Xavier and Yvette, and a differential game whose dynamics are described by

$$\begin{cases} a) & \begin{cases} i) & x'(t) & = & f(x(t), y(t), u(t)) \\ ii) & u(t) & \in & U(x(t), y(t)) \end{cases} \\ \\ b) & \begin{cases} i) & y'(t) & = & g(x(t), y(t), v(t)) \\ ii) & v(t) & \in & V(x(t), y(t)) \end{cases} \end{cases}$$

where u, v, the controls, are regarded as *strategies* used by the players to govern the evolution of the states x, y of the game.

The rules of the game are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$, describing the constraints imposed by one player on the other. They replace the traditional intertemporal optimality and/or end-point criteria used in differential games.

The *playability domain* of the game $K \subset X \times Y$ is defined by:

$$K := \{(x, y) \in X \times Y \mid x \in P(y) \text{ and } y \in Q(x)\}$$

(We consider only the time-independent case for the sake of simplicity). We single out the following properties:

— The **playability property:** it states that for any initial state $(x_0, y_0) \in K$, there exists a solution to the differential game which is playable in the sense that

$$\forall\, t \geq 0, \quad x(t) \ \in \ P(y(t)) \ \ \& \ \ y(t) \ \in \ Q(x(t))$$

— **Xavier's discriminating property:** It states that for any initial state $(x_0, y_0) \in K$ and for any continuous closed loop strategy $\tilde{v}(\cdot,\cdot)$ played by Yvette, there exists a playable solution to the differential game.

— **Xavier's leading property:** It states that there exists a continuous closed loop strategy $\tilde{u}(\cdot,\cdot)$ played by Xavier such that for any initial state $(x_0, y_0) \in K$, there exists a playable solution to the differential game.

Our first task is to characterize the rules satisfying such properties as somewhat generalized solutions to Isaacs equations. Since the rules are set-valued maps and not functions, we may characterize them by the indicators Ψ_P and Ψ_Q of their graphs, defined by $\Psi_P(x,y) := 0$ when $x \in P(y)$ and $\Psi_P(x,y) := +\infty$ when $x \notin P(y)$. But these functions, which are only lower semicontinuous (when the graphs are closed) are not differentiable in the usual sense. Hence we must replace the concept of derivative by the one of contingent epiderivative in the Isaacs equations.

This being done, we shall interpret the solutions to contingent Isaacs equations in game theoretical terms and characterize the above properties of the rules P and Q by checking whether the function $\max(\Psi_P, \Psi_Q)$ is a solution to the corresponding contingent Isaacs equation.

We focus our attention in the second section to the playability property.

We shall characterize it by constructing *retroaction rules*

$$(x,y,v) \rightsquigarrow C(x,y;v) \ \ \& \ \ (x,y,u) \rightsquigarrow D(x,y;u)$$

which involve the contingent derivatives of the set-valued maps P and Q, with which we build the *regulation map* R mapping each $(x,y) \in K$ to the regulation set

$$R(x,y) \ \ = \ \ \{ \ (u,v) \mid u \in C(x,y;v) \ \text{ and } \ v \in D(x,y;u) \ \}$$

The strategies belonging to $R(x,y)$ are called *playable.*

The Playability Theorem states that under technical assumptions, the playability property holds true if and only if

$$\forall\,(x,y)\in K,\ \ R(x,y)\ \neq\ \emptyset$$

and that playable solutions to the game are regulated by the *regulation law*:

$$\forall\, t\geq 0,\quad u(t)\ \in\ C(x(t),y(t);v(t))\ \ \&\ \ v(t)\ \in\ D(x(t),y(t);u(t))$$

We then deal in the third section with the construction of single-valued *playable feedbacks* $(\tilde{u},\tilde{v})$, such that the differential system

$$\begin{cases} x'(t) & = & f(x(t),y(t),\tilde{u}(x(t),y(t))) \\ y'(t) & = & g(x(t),y(t),\tilde{v}(x(t),y(t))) \end{cases}$$

has playable solutions for each initial state. By the Playability Theorem, they must be selections of the regulation map R in the sense that

$$\forall\,(x,y)\ \in\ K,\ \ (x,y)\mapsto(\tilde{u}(x,y),\tilde{v}(x,y))\in R(x,y)$$

We shall prove the existence of such continuous single-valued playable feedbacks, as well as more constructive, but discontinuous, playable feedbacks, such as the feedbacks associating the strategies of $R(x,y)$ with minimal norm (the playable slow feedbacks, as in Chapter 6). More generally, we shall show the existence of feedbacks (possibly set-valued) associating with any $(x,y)\in K$ the set of strategies $(u,v)\in R(x,y)$ which are solutions to a (static) optimization problem of the form:

$$(u,v)\in R(x,y)\ \ |\ \ \sigma(x,y;u,v)\leq\inf_{u',v'\in R(x,y)}\sigma(x,y;u',v')$$

or solutions to a noncooperative game of the form:

$$\forall(u',v')\in R(x,y),\ \ a(x,y;u,v')\leq a(x,y;u,v)\leq a(x,y;u',v)$$

In other words,

the players can implement playable solutions to the differential game by playing for each state $(x, y) \in K$ a static game on the strategies of the regulation subset $R(x, y)$.

We also consider in the fourth section the issue of finding *discriminating feedbacks* by providing for instance sufficient conditions implying that for all continuous feedback $\tilde{v}(x, y) \in V(x, y)$ played by Yvette, Xavier can find a feedback (continuous or of minimal norm) $\tilde{u}(x, y)$ such that the differential equation above has playable solutions for each initial state.

We address the question of whether Xavier has a leading role, i.e., the problem of constructing continuous *pure feedbacks* $\tilde{u}(x, y)$ which have the property of yielding playable solutions to the above differential game whatever the strategy played by Yvette.

The last section is devoted to closed loop decision rules, which *operate on the velocities of the strategies* (regarded as *decisions*) rather than on the controls. We need to provide first regulation maps which yield absolutely continuous strategies which are then almost everywhere differentiable. We distinguish among them the ones which guarantee or which allow victory or defeat of players adequately defined. The indicator functions of their graphs are characterized as solutions of contingent partial differential inequalities. We apply analogous selection procedures which yield closed loop decision rules allowing, say, a game to remain stable.

14.1 Contingent Isaacs Equations

Let us consider two players, Xavier and Yvette. Xavier acts on a state space X and Yvette on a state space Y. For doing so, they have access to some knowledge about the global state (x, y) of the system and are allowed to choose strategies u in a global state-dependent set $U(x, y)$ and v in a global state-dependent set $V(x, y)$ respectively.

But Xavier does not know Yvette's choice of controls v nor is Yvette assumed to know Xavier's controls.

Their actions on the state of the system are governed by the system of differential inclusions:

$$\left\{\begin{array}{ll} a) & \left\{\begin{array}{llcl} i) & x'(t) & = & f(x(t),y(t),u(t)) \\ ii) & u(t) & \in & U(x(t),y(t)) \end{array}\right. \\ & \\ b) & \left\{\begin{array}{llcl} i) & y'(t) & = & g(x(t),y(t),v(t)) \\ ii) & v(t) & \in & V(x(t),y(t)) \end{array}\right. \end{array}\right. \tag{14.1}$$

We now describe the influences (power relations) that Xavier exerts on Yvette and vice versa through *rules of the game.* They are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$ which are interpreted in the following way. When the state of Yvette is y, Xavier's choice is constrained to belong to $P(y)$. In a symmetric way, the set-valued map Q assigns to each state x the set $Q(x)$ of states y that Yvette can implement[1].

Hence, the *playability subset* of the game is the subset $K \subset X \times Y$ defined by:

$$K \ := \ \{ (x,y) \in X \times Y \ | \ x \in P(y) \ \text{and} \ y \in Q(x) \} \tag{14.2}$$

Naturally, we must begin by providing sufficient conditions implying that the playability subset is nonempty. Since the playability subset is the subset of fixed-points (x,y) of the set-valued map $(x,y) \rightsquigarrow P(y) \times Q(x)$, we can use one of the many fixed point theorems to answer these types of questions[2].

From now on, *we shall assume that the playability subset associated with the rules P and Q is not empty.*

We can reformulate this differential game in a more compact form, by denoting

- by $z := (x,y) \in Z := X \times Y$ the global state,
- by $h(z,u,v) := (f(x,y,u,v), g(x,y,u,v))$ the values of the map $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ describing the dynamics of the game,

[1] We can extend the results below to the time-dependent case using the methods of Chapter 11.

[2] For instance, Kakutani's Fixed Point Theorem 3.7.7 furnishes such conditions: *Let $L \subset X$ and $M \subset Y$ be compact convex subsets and $P : M \rightsquigarrow L$ and $Q : L \rightsquigarrow M$ be closed maps with nonempty convex images. Then the playability subset is not empty.*

- by $L := \text{Graph}(P)$ Xavier's closed domain of definition,
- by $M := \text{Graph}(Q^{-1})$ Yvette's one and by $K := L \cap M$ the playability subset.

We shall also identify the set-valued maps U and V with their restrictions to L and M respectively by setting $U(z) := \emptyset$ whenever $z \notin L$ and $V(z) := \emptyset$ when $z \notin M$.

Hence the differential game can be written in the compact form

$$\left\{ \begin{array}{lrcl} i) & z'(t) & = & h(z(t), u(t), v(t)) \\ ii) & u(t) & \in & U(z(t)) \\ iii) & v(t) & \in & V(z(t)) \end{array} \right. \tag{14.3}$$

We denote by $\mathcal{S}(z_0)$ the subset of solutions $z(\cdot)$ to (14.3) starting at z_0.

Let us associate with this differential game the following four Hamilton-Jacobi-Isaacs partial differential equations:

$$\left\{ \begin{array}{llcl} i) & \inf_{u \in U(z)} \inf_{v \in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \\ \\ ii) & \sup_{u \in U(z)} \sup_{v \in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \\ \\ iii) & \sup_{v \in V(z)} \inf_{u \in U(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \\ \\ iv) & \inf_{u \in U(z)} \sup_{v \in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \end{array} \right.$$

We would like to study the properties of the solutions to these partial differential equations, and in particular, characterize the solutions which are indicators of closed subsets L. Hence we are led to weaken the concept of usual derivatives involved in these partial differential equations by replacing them by contingent epiderivatives[3].

[3]since any extended function $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ has contingent epiderivative, and in particular, indicators, for which we have the relation

$$D_{\uparrow} \Psi_L(z)(v) \; = \; \Psi_{T_L(z)}(v) \; := \left\{ \begin{array}{ll} 0 & \text{if } v \in T_L(z) \\ +\infty & \text{if } v \notin T_L(z) \end{array} \right.$$

Theorem 14.1.1 *Let us assume at least that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps* U *and* V *are closed with linear growth.*

We assume that all extended functions Φ *are nonnegative and contingently epidifferentiable*[4] *and that their domains are contained in the intersection* K *of the domains of* U *and* V.

1 — If the values of the set-valued maps U *and* V *are convex and if* h *is affine with respect to the controls,* Φ *is a solution to the contingent inequality*

$$\inf_{u\in U(z)} \inf_{v\in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \leq 0 \tag{14.4}$$

if and only if

$$\forall\, z \in \mathrm{Dom}(\Phi),\ \exists\, z(\cdot) \in \mathcal{S}(z) \mid \forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$$

2 — Assume that h *is uniformly Lipschitz with respect to* x. *Then* Φ *is a solution to the contingent inequality*

$$\sup_{u\in U(z)} \sup_{v\in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \leq 0 \tag{14.5}$$

if and only if

$$\forall\, z \in \mathrm{Dom}(\Phi),\ \forall\ z(\cdot) \in \mathcal{S}(z), \forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$$

3 — Assume that V *is lower semicontinuous, that the values of* U *and* V *are convex and that* h *is affine with respect to* u. *Then* Φ *is a solution to the contingent inequality*

$$\sup_{v\in V(z)} \inf_{u\in U(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \leq 0 \tag{14.6}$$

if and only if for any continuous closed-loop strategy $\tilde{v}(z) \in V(z)$ *played by Yvette and any initial state* $z \in \mathrm{Dom}(\Phi)$, *there exists a solution* $z(\cdot)$ *to Xavier's control problem*

$$\begin{cases} i) & z'(t) = h(z(t), u(t), \tilde{v}(z(t))) \\ ii) & u(t) \in U(z(t)) \end{cases}$$

[4]This means that for all $z \in \mathrm{Dom}(\Phi)$, $\forall\, v \in X$, $D_{\uparrow}\Phi(z)(v) > -\infty$ and that $D_{\uparrow}\Phi(z)(v) < \infty$ for at least one $v \in X$.

starting at z and satisfying $\forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$.

4 — *Assume that V is lower semicontinuous with convex values. Then* Φ *is a solution to the contingent inequality*

$$\inf_{u\in U(z)}\sup_{v\in V(z)} D_\uparrow\Phi(z)(h(z,u,v)) \leq 0 \tag{14.7}$$

if and only if Xavier can play a closed-loop strategy $\tilde{u}(z) \in U(z)$ *such that, for any continuous closed-loop strategy* $\tilde{v}(z) \in V(z)$ *played by Yvette and for any initial state* $z \in \mathrm{Dom}(\Phi)$, *there exists a solution* $z(\cdot)$ *to*

$$z'(t) = h(z(t), \tilde{u}(z(t)), \tilde{v}(z(t))) \tag{14.8}$$

starting at z and satisfying for all $t \geq 0$, $\Phi(z(t)) \leq \Phi(z)$. *The converse is true if*

$$\begin{cases} B_\Phi(z) := \{\bar{u} \in U(z) \quad \textit{such that} \\ \sup_{v\in V(z)} D_\uparrow\Phi(z)(h(z,\bar{u},v)) \\ = \inf_{u\in U(z)}\sup_{v\in V(z)} D_\uparrow\Phi(z)(h(z,u,v))\} \end{cases}$$

is lower semicontinuous with closed convex values.

Proof

— The two first statements are translations of the theorems characterizing Lyapunov and global Lyapunov functions (see Chapter 9) applied to the differential inclusion $z'(t) \in H(z(t))$ where $H(z) := f(z, U(z), V(z))$.

— Let us prove the third one. Assume that Φ satisfies the stated property. Since V is lower semicontinuous with convex values, Michael's Theorem 6.5.7 implies that for all $z_0 \in \mathrm{Dom}(V)$ and $v_0 \in V(z_0)$, there exists a continuous selection $\tilde{v}(\cdot)$ of V such that $v(z_0) = v_0$. Then Φ enjoys the Lyapunov property for the set-valued map $H_{\tilde{v}}(z) := h(z, U(z), \tilde{v}(z))$, and thus, there exists $u_0 \in U(z_0)$ such that

$$D_\uparrow\Phi(z_0)(h(z_0, u_0, \tilde{v}(z_0))) \leq 0$$

Hence Φ is a solution to (14.6).

Conversely, assume that Φ is a solution to (14.6). Then for any closed-loop strategy $\tilde{v}$, the set-valued map $H_{\tilde{v}}$ satisfies the assumptions of the theorem characterizing Lyapunov functions, so that there exists a solution to the inclusion $z' \in H_{\tilde{v}}(z)$ for any initial state $z \in \mathrm{Dom}(\Phi)$ satisfying for all $t \geq 0$, $\Phi(z(t)) \leq \Phi(z)$.

— Consider finally the fourth statement. Assume that Xavier can find a continuous closed-loop strategy $\tilde{u}$ such that for any closed-loop strategy $\tilde{v}$, Φ enjoys the stated property. Since V is lower semicontinuous with convex values, Michael's Theorem implies that for all $z_0 \in \mathrm{Dom}(V)$ and $v_0 \in V(z_0)$, there exists a continuous selection $\tilde{v}(\cdot)$ of V such that $v(z_0) = v_0$. Since for any continuous closed-loop strategy $\tilde{v}(\cdot)$, Φ enjoys the Lyapunov property for the single-valued map $z \to h(z, \tilde{u}(z), \tilde{v}(z))$, we deduce that for all $z_0 \in \mathrm{Dom}(\Phi)$, there exists $u := \tilde{u}(z)$ such that for all $v \in V(z)$, $D_{\uparrow}\Phi(z)(h(x,u,v)) \leq 0$, so that Φ is a solution to (14.6).

Conversely, assume that the set-valued map B_{Φ} is lower semicontinuous with closed convex values. Hence Michael's Theorem implies that there exists a continuous selection $\tilde{u}$ of B_{Φ}. Then for any continuous closed-loop strategy $\tilde{v}(\cdot) \in V(\cdot)$, we deduce from (14.7) that Φ is a Lyapunov function for the single-valued map $z \to h(z, \tilde{u}(z), \tilde{v}(z))$, so that, for all $z \in \mathrm{Dom}(\Phi)$, there exists a solution $z(\cdot)$ to the system (14.8) satisfying for all $t \geq 0$, $\Phi(z(t)) \leq \Phi(z)$. □

Let L be a closed subset of the intersection K of the domains of U and V. The problem we investigate is of finding that one (or all) solution(s) $z(\cdot)$ of the game is (are) viable in L. There are several ways to achieve that purpose, according to the cooperative or noncooperative behavior of the players. Here, we shall investigate several of them.

Definition 14.1.2 *We shall say the a subset L enjoys:*

1 — *the "playability property" if and only if*

$$\forall\, z \in L,\ \exists\, z(\cdot) \in \mathcal{S}(z) \mid \forall\, t \geq 0,\ \ z(t) \in L$$

2 — *the "winability property" if and only if*

$$\forall\, z \in L,\ \forall\ \ z(\cdot) \in \mathcal{S}(z), \forall\, t \geq 0,\ \ z(t) \in L$$

3 — *"Xavier's discriminating property" if and only if for any continuous closed-loop strategy $\tilde{v}(z) \in V(z)$ played by Yvette*

and any initial state $z \in L$, there exists a solution $z(\cdot)$ to Xavier's control problem

$$\begin{cases} i) & z'(t) = h(z(t), u(t), \tilde{v}(z(t))) \\ ii) & u(t) \in U(z(t)) \end{cases}$$

starting at z and viable in L.

4 — "Xavier's leading property" if and only if Xavier can play a closed-loop strategy $\tilde{u}(z) \in U(z)$ such that, for any continuous closed-loop strategy $\tilde{v}(z) \in V(z)$ played by Yvette and for any initial state $z \in L$, there exists a solution $z(\cdot)$ to (14.8) starting at z and viable in L.

We shall characterize these properties: for that purpose we associate with L the following set-valued maps:

— The regulation map R_L defined by

$$\forall\, z \in L,\ R_L(z) := \{\ (u,v) \in U(z) \times V(z) \mid h(z,u,v) \in T_L(z)\ \}$$

— Xavier's discriminating map A_L defined by

$$\forall\, z \in L,\ A_L(z,v) := \{\ u \in U(z) \mid (u,v) \in R_L(z)\ \}$$

— Xavier's leading map B_L defined by

$$\forall\, z \in L,\ \ B_L(z) \ :=\ \bigcap_{v \in V(z)} A_L(z,v)$$

Definition 14.1.3 *We shall say that*

— *L is a playability domain if $\forall\, z \in L,\ R_L(z) \neq \emptyset$*
— *L is a winability domain if $\forall\, z \in L,\ R_L(z) := U(z) \times V(z)$*
— *L is a Xavier's discriminating domain if*

$$\forall\, z \in L, \forall\, v \in V(z),\ A_L(z,v) \neq \emptyset \tag{14.9}$$

— *L is a Xavier's leading domain if $\forall\, z \in L,\ B_L(z) \neq \emptyset$*

We begin by translating these properties in terms of contingent Isaacs' equations:

Proposition 14.1.4 *Let us assume that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps* U, V *are closed with linear growth.*

— L *is a playability domain if and only if* Ψ_L *is a solution to (14.4)*

— L *is a winability domain if and only if* Ψ_L *is a solution to (14.5)*

— L *is a discriminating domain for Xavier if and only if* Ψ_L *is a solution to (14.6)*

— L *is a leading domain for Xavier if and only if* Ψ_L *is a solution to (14.7)*

Therefore, Theorem 14.1.1 implies the following characterization of these domains:

Corollary 14.1.5 *Let us assume at least that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps are closed with linear growth.*

1 — If the values of the set-valued maps U *and* V *are convex and if* h *is affine with respect to the controls, then* L *enjoys the playability property if and only if it is a playability domain.*

2 — Assume that h *is uniformly Lipschitz with respect to* x*. Then* L *enjoys the winability property if and only if it is a winability domain.*

3 — Assume that V *is lower semicontinuous, that the values of* U *and* V *are convex and that* h *is affine with respect to* u*. Then* L *enjoys Xavier's discriminating property if and only if it is a discriminating domain for Xavier.*

4 — Assume that V *is lower semicontinuous with convex values. If* L *enjoys Xavier's leading property, then it is a leading domain for him. The converse is true if* B_L *is lower semicontinuous with closed convex values.*

The existence theorems of the viability and invariance kernels imply the following consequence:

Proposition 14.1.6 *Let us assume that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps are closed with linear growth.*

1 — If the values of the set-valued maps U and V are convex and if h is affine with respect to the controls, then there exists a largest closed playability domain contained in L, whose indicator is the smallest lower semicontinuous solution to (14.4) larger than or equal to the indicator Ψ_L of L.

2 — Assume that h is uniformly Lipschitz with respect to x. Then there exists a largest closed winability domain contained in L, whose indicator is the smallest lower semicontinuous solution to (14.5) larger than or equal to the indicator Ψ_L of L.

14.2 Playable Differential Games

We now proceed with the case of the game described by (14.1), where the playability domain is defined from rules P and Q by

$$K := \{ (x,y) \in X \times Y \mid x \in P(y) \text{ and } y \in Q(x) \}$$

enjoys the *playability property*, which becomes in this case: for any initial state $(x_0, y_0) \in K$, there exists a solution to the differential game (14.1) which is *playable* in the sense that

$$\forall\, t \geq 0, \quad x(t) \in P(y(t)) \;\&\; y(t) \in Q(x(t))$$

We now need to define *playable rules.* For that purpose, we associate with the rules P and Q acting on the states *retroaction rules* C and D acting on the strategies defined in the following way:

Definition 14.2.1 *Xavier's* retroaction rule *is the set-valued map C defined by*

$$\begin{cases} C(x,y;v) \\ = \{ u \in U(x,y) \mid f(x,y,u) \in DP(y,x)(g(x,y,v)) \} \end{cases}$$

and Yvette's retroaction rule *is the set-valued map D defined by*

$$\begin{cases} D(x,y;u) \\ = \{ v \in V(x,y) \mid g(x,y,v) \in DQ(x,y)(f(x,y,u)) \} \end{cases}$$

We associate with them the regulation map *R defined by*

$$R(x,y) = \{ (u,v) \mid u \in C(x,y;v) \;\&\; v \in D(x,y;u) \} \quad (14.10)$$

The subset $R(x,y)$ *is called the* regulation set *and its elements are called* playable controls.

In other words, we have associated with each state (x,y) of the playability domain a static game on the strategies defined by the retroaction rules. This new game on strategies is playable if the subset $R(x,y)$ is nonempty. This property deserves a definition.

Definition 14.2.2 *We shall say that* P *and* Q *are* playable rules *if their graphs are closed, the playability domain* K *defined by (14.2) is nonempty and if for all pairs* $(x,y) \in K$*, the values* $R(x,y)$ *of the regulation map are nonempty.*

We still need a definition of transversality of the rules before stating an adequate characterization of playability.

Definition 14.2.3 *We shall say that the rules* P *and* Q *are* transversal *if for all* $(x,y) \in K$ *and for all perturbations* $(e,f) \in X \times Y$*, there exists* (u,v) *satisfying*

$$\begin{cases} i) & u \in DP(y,x)(v) + e \\ \\ ii) & v \in DQ(x,y)(u) + f \end{cases}$$

We shall say that they are strongly transversal *if*

$$\begin{cases} \forall (x,y) \in K,\ \exists c > 0, \delta > 0 \ \textit{such that} \ \forall (x',y') \in B_K((x,y),\delta), \\ \forall\ (e,f) \in X \times Y,\ \ \textit{there exist solutions } (u,v) \textit{ to the system} \\ \begin{cases} i) & u \in DP(y',x')(v) + e \\ ii) & v \in DQ(x',y')(u) + f \end{cases} \\ \textit{satisfying} \\ \max(\|u\|, \|v\|) \ \le \ \max(\|e\|, \|f\|) \end{cases}$$

We also assume that the rules are *sleek* (See Definition 5.1.4).

We shall now derive from Corollary 14.1.5 a characterization of the playability property.

Theorem 14.2.4 (Playability Theorem) *Let us assume that the functions* f *and* g *are continuous, affine with respect to the strategies and have a linear growth, that the feedback maps* U *and* V *are upper* *semicontinuous with compact convex images and have a linear growth and that the rules P and Q are sleek and transversal.*

Then the rules P and Q enjoy the playability property if and only if they are playable. *Furthermore, the strategies* $u(\cdot)$ *and* $v(\cdot)$ *which provide playable solutions obey the following* regulation law*: for every* $t \geq 0$,

$$u(t) \in C(x(t), y(t); v(t)) \ \& \ v(t) \in D(x(t), y(t); u(t)) \quad (14.11)$$

Proof — We apply Corollary 14.1.5 and prove that the playability subset of the differential game is a playability domain, i.e., that for any global state $(x, y) \in K$ of the system, there exist strategies u and v such that the pair $(f(x, y, u), g(x, y, v))$ belongs to the contingent cone $T_K(x, y)$.

Since K is the intersection of the graphs of Q and P^{-1}, we need to use a sufficient condition for the contingent cone to an intersection to be equal to the intersection of the contingent cones.

The graphs of Q and P^{-1} are sleek because the rules of the game are supposed to be so. Furthermore,

$$T_{\text{Graph}(P^{-1})}(x, y) - T_{\text{Graph}(Q)}(x, y) = X \times Y$$

because the maps P and Q are transversal: For any $(e, f) \in X \times Y$, there exists (u, v) such that $(u, v - f)$ belongs to the graph of Q and $(u + e, v)$ to the graph of P^{-1}, i.e., that $(e, f) = (u + e, v) - (u, v - f)$. We deduce that

$$\begin{cases} T_K(x, y) & = T_{\text{Graph}(P^{-1})}(x, y) \cap T_{\text{Graph}(Q)}(x, y) \\ & = \text{Graph}(DP(y, x))^{-1} \cap \text{Graph}(DQ(x, y)) \end{cases}$$

Therefore, K is a viability domain if and only if the regulation map R has nonempty values, i.e., if and only if the rules of the game are playable. □

The regulation law (14.11) describes how the players must behave to keep the state of the system playable. A first question arises: Do the domains of the set-valued maps

$$\begin{cases} i) & C(x, y) : v \rightsquigarrow C(x, y; v) \\ ii) & D(x, y) : u \rightsquigarrow D(x, y; u) \end{cases}$$

coincide with $U(x,y)$ and $V(x,y)$ respectively?

Proposition 14.2.5 *We posit the assumptions of Theorem 14.2.4. Let us assume that for all* $(x,y) \in K$,

$$\begin{cases} i) & \mathrm{Dom}(C(x,y)) = V(x,y) \\ ii) & \mathrm{Dom}(D(x,y)) = U(x,y) \end{cases} \tag{14.12}$$

Then the rules are playable.

Proof — We deduce it from Kakutani's Fixed Point Theorem, since the set $R(x,y)$ is the set of fixed points of the set-valued map

$$(u,v) \rightsquigarrow C(x,y;v) \times D(x,y;u)$$

defined on the convex compact subset $U(x,y) \times V(x,y)$ to itself. This set-valued map has non empty values by assumption, which are moreover convex since the rules P and Q being sleek, the graphs of the contingent derivatives $DP(x,y)$ and $DQ(x,y)$ are convex. They are also closed. This implies that the graph of $(u,v) \rightsquigarrow C(x,y;v) \times D(x,y;u)$ is closed. Hence we can apply Kakutani's Fixed Point Theorem[5]. □

14.3 Feedback Solutions

When we know the regulation law (14.11), *playing the game* amounts to choosing for each pair $(x,y) \in K$ playable strategies (u,v) in the regulation set $R(x,y)$ through *playable feedbacks*.

We begin by looking for single-valued playable feedbacks $(\tilde{u},\tilde{v})$, which are selections of the regulation map R in the sense that

$$\forall (x,y) \in K, \quad (x,y) \mapsto (\tilde{u}(x,y), \tilde{v}(x,y)) \in R(x,y)$$

[5]We can also use Theorem 3.7.11 and replace condition (14.12) by a sufficient condition of the form:

$$\begin{cases} \forall (u,v) \in U(x,y) \times V(x,y), \\ 0 \in (f(x,y;u), g(x,y;v)) - T_K(x,y) - A(T_{U(x,y)}(u) \times T_{V(x,y)}(v)) \end{cases}$$

where A is a linear operator from $Z_X \times Z_Y$ to $X \times Y$. This provides many sufficient conditions for playability.

or, equivalently, solutions to the system

$$\forall\ (x,y) \in K, \quad \begin{cases} \tilde{u}(x,y) & \in & C(x,y;\tilde{v}(x,y)) \\ \text{and} & & \\ \tilde{v}(x,y) & \in & D(x,y;\tilde{u}(x,y)) \end{cases}$$

For instance, continuous selections of the set-valued map R provide continuous playable feedbacks $(\tilde{u},\tilde{v})$ such that the system of differential equations

$$\begin{cases} x'(t) & = & f(x(t),y(t),\tilde{u}(x(t),y(t))) \\ y'(t) & = & g(x(t),y(t),\tilde{v}(x(t),y(t))) \end{cases} \tag{14.13}$$

does have solutions which are playable.

Michael's Continuous Selection Theorem, as well as other selection procedures we shall use, require the lower semicontinuity of the regulation map R.

Our next objective is then to provide criteria under which the regulation map is lower semicontinuous. For that purpose, we need to strengthen the concept of playable rules.

Definition 14.3.1 *We associate with any perturbation* (e,f) *the* retroaction rules $C_{(e,f)}$ *and* $D_{(e,f)}$ *defined by:*

$$\begin{cases} C_{(e,f)}(x,y;v) \\ = \{\, u \in U(x,y) \mid f(x,y;u) \in DP(y,x)(g(x,y,v)-f)+e \,\} \end{cases}$$

and

$$\begin{cases} D_{(e,f)}(x,y;u) \\ = \{\, v \in V(x,y) \mid g(x,y,v) \in DQ(x,y)(f(x,y;u)-e)+f \,\} \end{cases}$$

and regulation map $R_{(e,f)}$ *defined by*

$$R_{(e,f)}(x,y) \quad = \quad \{\, (u,v) \mid u \in C_{(e,f)}(x,y;v) \ \ \& \ \ v \in D_{(e,f)}(x,y;u) \,\}$$

We shall say that the rules P *and* Q *are* strongly playable *if*

$$\begin{cases} \forall (x,y) \in K,\ \exists \gamma > 0, \delta > 0 \ \ \textit{such that}\ \forall (x',y') \in B_K((x,y),\delta), \\ \forall\ (e,f) \in \gamma B,\ \ R_{(e,f)}(x',y') \ \neq \ \emptyset \end{cases}$$

Theorem 14.3.2 *Let us assume that the functions f and g are continuous, affine with respect to the strategies and have a linear growth, that the feedback maps U and V are upper semicontinuous with compact convex images and have a linear growth and that the rules P and Q are sleek, strongly transversal and strongly playable.*

Then the regulation map R is lower semicontinuous *with closed convex images.*

Consequently, there exist continuous playable feedbacks $(\tilde{u}, \tilde{v})$.

Proof — We use the Lower Semicontinuity Criterion of the intersection and the inverse image of lower semicontinuous set-valued maps (see Theorem 6.3.1).

First, we need to prove that the set-valued map

$$(x,y) \rightsquigarrow T_K(x,y) \;:=\; \mathrm{Graph}(DP(y,x)^{-1}) \cap \mathrm{Graph}(DQ(x,y))$$

is lower semicontinuous. But this follows from the strong transversality of the rules P and Q and the Lower Semicontinuity Criterion.

We observe that $U \times V$ being upper semicontinuous with compact values, it maps a neighborhood of each point to a compact set. Since we can write

$$R(x,y) = \{(u,v) \in (U \times V)(x,y) \mid (f(x,y;u), g(x,y;v)) \in T_K(x,y)\}$$

and since both $U \times V$ and T are lower semicontinuous with convex images, strong playability of the retroaction rules implies that the regulation map R is lower semicontinuous. □

Unfortunately, the proof of Michaels's Continuous Selection Theorem is not constructive. We would rather trade the continuity of the playable control with some explicit and computable property, such as $u^\circ(x,y)$ being the element of minimal norm in $R(x,y)$, or other properties. Hence we need to prove the existence of a solution to the differential equation (14.13) for such discontinuous feedbacks.

Theorem 6.6.6 on the regulation of control systems becomes

Theorem 14.3.3 *We posit the assumptions of Theorem 14.2.4 and we suppose that K is a playability domain.*

Let S_R be a selection procedure with convex images of the regulation map R. Then, for any initial state $(x_0, y_0) \in K$, there exists a playable solution starting at (x_0, y_0) to the differential inclusion

$$\begin{cases} i) & x'(t) = f(x(t), y(t); u(t)) \\ ii) & y'(t) = g(x(t), y(t); v(t)) \\ iii) & \text{for almost all } t, \ (u(t), v(t)) \in S(R(x(t), y(t))) \end{cases}$$

In particular, if for every (x, y) the intersection

$$S_R(x, y) \cap R(x, y) := (\tilde{u}(x, y)), \tilde{v}(x, y))$$

is single-valued, then the strategies $(x, y) \mapsto (\tilde{u}(x, y)), \tilde{v}(x, y))$ are single-valued playable feedback controls.

We can now multiply the possible corollaries, by supplying several instances of selection procedures of set-valued maps.

We begin by cooperative procedures, where the players agree on criteria $\sigma(x, y; \cdot, \cdot)$ for selecting strategies in the regulation sets $R(x, y)$.

Example— COOPERATIVE BEHAVIOR

Proposition 14.3.4 *We posit the assumptions of Theorem 14.3.2. Let σ be continuous on* Graph(R) *and convex with respect to the pair (u, v). Then, for any initial state $(x_0, y_0) \in K$, there exist a playable solution starting at (x_0, y_0) and playable strategies to the differential game (14.1) which are regulated by:*

$$\begin{cases} \text{for almost all } t \geq 0, \ \ (u(t), v(t)) \in R(x(t), y(t)) \ \text{ and} \\ \\ \sigma(x(t), y(t); u(t), v(t)) = \inf_{u', v' \in R(x(t), y(t))} \sigma(x(t), y(t); u', v') \end{cases}$$

In particular, the game can be played by the slow feedbacks *of minimal norm:*

$$\begin{cases} (u^\circ(x, y), v^\circ(x, y)) \in R(x, y) \\ \|u^\circ(x, y)\|^2 + \|v^\circ(x, y)\|^2) = \min_{(u,v) \in R(x,y)} (\|u\|^2 + \|v\|^2) \end{cases}$$

Proof — We introduce the set-valued map S_R defined by:

$$S_R(x,y) := \{(u,v) \mid \sigma(x,y;u,v) \leq \inf_{(u',v')\in R(x,y)} \sigma(x,y;u',v')\}$$

which is a convex-valued *selection procedure* of R since R is lower semicontinuous (see Theorem 6.5.3). We then apply Theorem 14.3.3. We observe that when we take

$$\sigma(x,y;u,v) \ := \ \|u\|^2 + \|v\|^2$$

the selection procedure yields the elements of minimal norm. □

Example— NONCOOPERATIVE BEHAVIOR We can also choose strategies in the regulation sets $R(x,y)$ in a non cooperative way, as saddle points of a function $a(x,y;\cdot,\cdot)$.

Proposition 14.3.5 *We posit the assumptions of Theorem 14.3.2 and we suppose that K is a playability domain. Let us assume that $a : X \times Y \times U \times V \to \mathbf{R}$ satisfies*

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall (x,y,v) \in X \times V, \ \ u \mapsto a(x,y;u,v) \ \text{ is convex} \\ iii) & \forall (x,y;u) \in X \times U, \ \ v \mapsto a(x,y;u,v) \ \text{ is concave} \end{cases}$$

Then, for any initial state $(x_0,y_0) \in K$, there exist a playable solution starting at (x_0,y_0) and playable strategies to the differential game (14.1) which are regulated by: for almost all $t \geq 0$,

$$\begin{cases} i) & (u(t),v(t)) \in R(x(t),y(t)) \\ ii) & \forall\, (u',v') \in R(x(t),y(t)), \\ & a(x(t),y(t);u(t),v') \leq a(x(t),y(t);u(t),v(t)) \\ & \leq a(x(t),y(t);u',v(t)) \end{cases}$$

Proof — The set-valued map S_R associating with any $(x,y) \in K$ the subset

$$\begin{array}{l} S_R(x,y) \ := \{(u,v) \ \text{such that} \\ \forall (u',v') \in R(x,y), \ \ a(x,u,v') \leq a(x,u',v)\} \end{array}$$

is a convex-valued selection procedure of R. The associated selection map $S(R(\cdot))$ associates with any $(x, y) \in X \times Y$ the subset

$$S(R(x,y)) \ := \ \{ (u,v) \in R(x,y) \ \text{such that} \ \forall (u',v') \in R(x,y), \ a(x,y;u,v') \le a(x,y;u',v) \}$$

of saddle-points of $a(x, y; \cdot, \cdot)$ in $R(x, y)$. We then apply Theorem 14.3.3. □

14.4 Discriminating and Leading Feedbacks

We now address the question of finding criteria for the playability domain K to be Xavier's discriminating domain, and for finding Xavier's feedback strategies which are selections of the set-valued map $(x, y, v) \rightsquigarrow A(x, y, v) \subset U(x, y)$ defined by

$$A(x,y;v) \ := \ \{ u \in U(x,y) \mid (u,v) \in R(x,y) \}$$

Such feedbacks are called *discriminating feedbacks*. If we assume that Xavier has access to the strategies chosen by Yvette, he can keep the states of the system playable by "playing" a discriminating control whatever the choice of Yvette through a discriminating feedback. Then, we shall investigate whether we can find (possibly, single-valued) selections of such a set-valued map A, and for that, provide sufficient conditions for A to be lower semicontinuous.

We first observe that A can be written in the form

$$A(x,y;v) \ := \ C(x,y;v) \cap (D(x,y))^{-1}(v)$$

The first assumption we must make for obtaining discriminating feedbacks for Xavier is that the domain of the set-valued maps $A(x, y; \cdot))$ are not empty. i.e., that

$$\begin{cases} \forall\, v \in V(x,y), \ \exists\, u \in U(x,y) \ \text{ such that} \\ f(x,y;u) \in DP(y,x)(g(x,y;v)) \cap DQ(x,y)^{-1}(g(x,y;v)) \end{cases}$$

We shall actually strengthen it a bit to get the lower semicontinuity of A, by assuming that

$$\begin{cases} \forall\,(x,y)\in K,\ \forall\,v\in V(x,y),\ \exists\,\delta>0,\ \exists\,\gamma>0\ \text{ such that} \\ \forall\,(x',y')\in B_K(x,y,\delta),\ \forall\,v'\in B(v,\delta)\cap V(x',y'), \forall\,\|e_i\|\leq\gamma \\ (i=1,2),\ \exists\,u\in U(x',y')\ \text{ such that }\ f(x',y';u)\ \text{belongs to} \\ (DP(y',x')(g(x',y';v'))-e_1)\cap(DQ(x',y')^{-1}(g(x',y';v'))-e_2) \end{cases} \tag{14.14}$$

Proposition 14.4.1 *We posit the assumptions of Theorem 14.3.2, where we replace strong playability by assumption (14.14), and we assume further that the norms of the closed convex processes $DP(y,x)$ and $DQ(x,y)^{-1}$ are bounded. Then the set-valued map A is lower semicontinuous.*

Proof — First, we have to prove that C is lower semicontinuous, and, for that purpose, that $(x,y,w)\rightsquigarrow DP(y,x)(w)$ is lower semicontinuous.

By Theorem 2.5.7, we know that it is sufficient to prove that

$$(x,y)\rightsquigarrow \text{Graph}(DP(y,x))\ \text{ is lower semicontinuous}$$

and that

$$\|DP(y,x)\| \;:=\; \sup_{\|w\|\leq 1}\ \inf_{u\in DP(y,x)(w)} \|u\| \;<\; +\infty$$

This is the case because P is assumed to be sleek and because we have assumed that the norms of the derivatives are bounded.

Therefore, the set-valued map

$$(x,y,v)\rightsquigarrow DP(y,x)(g(x,y;v))$$

is also lower semicontinuous.

The Lower Semicontinuity Criterion and assumption (14.14) imply that $(x,y,v)\rightsquigarrow C(x,y;v)$ is lower semicontinuous.

The same proof shows that the map $(x,y,v)\rightsquigarrow DQ(x,y)^{-1}(v)$ is also lower semicontinuous. Since A is the intersection of these two set-valued maps, we apply again the Lower Semicontinuity Criterion to deduce that A is lower semicontinuous, which is possible thanks to assumption (14.14). □

Theorem 14.4.2 *We posit the assumptions of Theorem 14.2.4. For any continuous feedback control* $(x, y) \mapsto \tilde{v}(x, y)$ *played by Yvette, there exits a continuous single-valued feedback* $\tilde{u}(x, y)$ *played by Xavier such that the differential equation (14.13) has playable solutions for any initial state* $(x_0, y_0) \in K$.

More generally, let S_A *be a convex-valued selection procedure of the set-valued map* A. *Then, for any continuous feedback control* $(x, y) \mapsto \tilde{v}(x, y)$ *played by Yvette, for any initial state* $(x_0, y_0) \in K$, *there exists a playable solution starting at* (x_0, y_0) *to the differential game*

$$\begin{cases} i) & x'(t) = f(x(t), y(t); u(t)) \\ ii) & y'(t) = g(x(t), y(t); \tilde{v}(x(t), y(t))) \\ iii) & u(t) \in S(A(x(t), y(t); \tilde{v}(x(t), y(t)))) \end{cases}$$

where

$$S(A(x, y; \tilde{v}(x, y))) := S_A(x, y; \tilde{v}(x, y)) \cap A(x, y; \tilde{v}(x, y))$$

In particular, if the selection procedure yields single-valued selections, then the control $\tilde{u}(x, y)$ *defined by*

$$\tilde{u}_{\tilde{v}}(x, y) := S(A(x, y; \tilde{v}(x, y)))$$

is a single-valued feedback control.

This is the case, for instance, when we posit the assumptions of Proposition 14.4.1 and when Xavier plays the feedback control $u^\circ_{\tilde{v}}(x, y)$ *of minimal norm in the set* $A(x, y; \tilde{v}(x, y))$. *In this case, there exists also a continuous control* $\tilde{u}(x, y) \in A(x, y; \tilde{v}(x, y))$

Proof — Whenever Yvette plays a continuous feedback $\tilde{v}(x, y)$, K remains a playability domain for the system

$$\begin{cases} i) & x'(t) = f(x(t), y(t); u(t)) \\ ii) & y'(t) = g(x(t), y(t); \tilde{v}(x(t), y(t))) \\ iii) & u(t) \in S_A(x(t), y(t); \tilde{v}(x(t), y(t))) \end{cases}$$

So playable solutions to this system satisfy also the condition

$$u(t) \in A(x(t), y(t); \tilde{v}(x(t), y(t)))$$

so that actually,

$$u(t) \in S(A(x(t), y(t); \tilde{v}(x(t), y(t))))$$

When the set-valued map $(x, y) \rightsquigarrow A(x, y; \tilde{v}(x, y))$ is lower semicontinuous, it contains continuous selections $\tilde{u}(x, y)$ which therefore yield playable selections.

We can also use more constructive selection procedures of the set-valued map $(x, y) \rightsquigarrow A(x, y; \tilde{v}(x, y))$ with convex values and deduce that Xavier can implement playable solutions by playing strategies $u(t)$ in the selection $S(A(x(t), y(t); \tilde{v}(x(t), y(t))))$. □

A much better situation for Xavier occurs when he can find feedback strategies $\tilde{u}$ which are selections of the set-valued map B defined by

$$B(x, y) := \bigcap_{v \in V(x,y)} A(x, y; v)$$

In other words, such a feedback allows him to implement playable solutions whatever the control $v \in V(x, y)$ chosen by Yvette, since in this case the pair (u, v) belongs to the regulation set $R(x, y)$ for any v. Such feedbacks are called *pure feedbacks*.

In order to obtain continuous single-valued feedbacks, we need to prove the lower semicontinuity of the set-valued map B, which is an infinite intersection of lower semicontinuous set-valued maps.

Theorem 14.4.3 *We posit the assumptions of Proposition 14.4.1. We assume further that there exist positive constants δ and γ such that for all $(x', y') \in B_K((x, y), \delta)$, we have*

$$\begin{cases} \forall\, v \in V(x', y'),\ \forall\, e_v^i \in \gamma B,\ (i = 1, 2),\ \exists\, u \in U(x', y') \quad \textit{such that} \\ \\ f(x', y'; u) \in DP(y', x'; v) + e_v^1 \\ \textit{and} \\ g(x', y'; v) \in DQ(x', y'; u) + e_v^2 \end{cases} \tag{14.15}$$

Then the set-valued map B is lower semicontinuous and there exist continuous single-valued pure feedback strategies for Xavier.

Proof — We observe that V is upper semicontinuous with compact values, that A is lower semicontinuous and has its images in a fixed compact set, and that assumption (14.15) implies obviously that there exist positive constants δ and γ such that for all $(x', y') \in B_K((x,y),\delta)$, we have

$$cB \cap \bigcap_{y \in H(x'), z \in \gamma B} (F(x', y) - z) \neq \emptyset$$

This theorem follows then from Theorem 6.3.3 on the lower semicontinuity of an infinite intersection of lower semicontinuous set-valued maps. □

14.5 Closed Loop Decision Rules

Actually, although differential games can be played through retroaction rules, there are many games where players *act on the velocities of the strategies* regarded as *decisions* of players.

This leads us to introduce the following definition: We shall call *decisions* the derivatives of the strategies.

Then, in order to deal with decisions defined in such a sense, we must now assume that players use open-loop strategies $u(\cdot)$ and $v(\cdot)$ which are *absolutely continuous* and obey a growth condition of the type[6]

$$\begin{cases} i) & \|u'(t)\| \leq \rho(\|u(t)\| + 1) \\ ii) & \|v'(t)\| \leq \sigma(\|v(t)\| + 1) \end{cases} \qquad (14.16)$$

We shall refer to them as "smooth open-loop controls", the non-negative parameters[7] ρ and σ being fixed once and for all. We denote by $\mathcal{K}$ the subset

$$\begin{cases} (z, u, v) \in \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \text{ such that} \\ u \in U(z) \ \& \ v \in V(z) \end{cases}$$

Instead of finding largest playability or winability domains in the state space, we shall look for analogous concepts in the state-strategy

[6] one can replace $\rho(\|u\|+1)$ by any continuous function $\varphi(u)$ with linear growth.
[7] or any other linear growth condition $\varphi(\cdot)$ or $\psi(\cdot)$.

space. We shall determine set-valued maps which allow players to win in the sense that either property

$$\forall\, t \geq 0, \;\; u(t) \;\in\; U(z(t)) \tag{14.17}$$

or property

$$\forall\, t \geq 0, \;\; v(t) \;\in\; V(z(t)) \tag{14.18}$$

or both hold. Roughly speaking, Xavier may win as long as his opponent allows him to choose at each instant $t \geq 0$ strategies $u(t)$ in the subset $U(z(t))$, and must lose if for any choice of open-loop controls, there exists a time $T > 0$ such that $u(T) \notin U(z(T))$.

Definition 14.5.1 *Let (u_0, v_0, z_0) be an initial situation such that initial strategies $u_0 \in U(z_0)$ and $v_0 \in V(z_0)$ of the two players are consistent with the initial state z_0.*

We shall say that

— *Xavier* must win *if and only if for all smooth open-loop strategies $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0, there exists a solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0 such that (14.17) is satisfied.*

— *Xavier* may win *if and only if there exist smooth open-loop strategies $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and a solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0 such that (14.17) is satisfied.*

— *Xavier* must lose *if and only if for any smooth open-loop strategy $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0, there exists a time $T > 0$ such that*

$$u(T) \;\notin\; U(z(T))$$

— *The initial situation is* stable *if and only if there exist open-loop strategies $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and a solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0 satisfying* both *relations (14.17) and (14.18).*

Naturally, if both Xavier and Yvette must win, then both relations (14.17) and (14.18) are satisfied. This is not necessarily the case when both Xavier and Yvette may win, and this is the reason why we need to introduce the concept of stability.

Table 14.1: The 10 areas of the domain of the differential game

$(z_0, u_0, v_0) \in$	Graph(S_U)	Graph(R_U)	$\mathcal{K}\backslash$Graph(R_U)
Graph(S_V)	Xavier must win Yvette must win	Xavier may win Yvette must win	Xavier must lose Yvette must win
Graph(R_V)	Xavier must win Yvette may win	? ? ? ? STABILITY ? ? ? ?	Xavier must lose Yvette may win
$\mathcal{K}\backslash$Graph(R_V)	Xavier must win Yvette must lose	Xavier may win Yvette must lose	Xavier must lose Yvette must lose

Theorem 14.5.2 *Let us assume that h is continuous with linear growth and that the graphs of U and V are closed. Let the growth rates ρ and σ be fixed.*

There exist five (possibly empty) closed set-valued feedback maps from $\mathbf{R}^n$ to $\mathbf{R}^p \times \mathbf{R}^q$ having the following properties:

— *$R_U \subset U$ is such that whenever $(u_0, v_0) \in R_U(z_0)$, Xavier* may win *and that whenever $(u_0, v_0) \notin R_U(z_0)$, Xavier* must lose

— *If h is Lipschitz, $S_U \subset R_U$ is the largest closed set-valued map such that whenever $(u_0, v_0) \in S_U(z_0)$, Xavier must win.*

— *$S_V \subset R_V \subset V$, which have analogous properties.*

— *$R_{UV} \subset R_U \cap R_V$ is the largest closed set-valued map such that any initial situation satisfying $(u_0, v_0) \in R_{UV}(z_0)$ is stable.*

Knowing these five set-valued feedback maps, we can split the domain $\mathcal{K}$ of initial situations in ten areas which describe the behavior of the differential game according to the position of the initial situation.

In particular, the complement of the graph of R_{UV} in the intersection of the graphs of R_U and R_V is the instability region, where either Xavier or Yvette may win, but not both together.

The problem is to characterize these five set-valued maps, the existence of which is now guaranteed, by solving the "contingent extension" of the partial differential equation[8]

[8]If Φ is a solution to this partial differential equation, one can check that

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)-\rho(\|u\|+1)\left\|\frac{\partial\Phi}{\partial u}\right\|-\sigma(\|v\|+1)\left\|\frac{\partial\Phi}{\partial v}\right\|\leq 0 \quad (14.19)$$

which can be written in the following way:

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)+\inf_{\|u'\|\leq\rho(\|u\|+1)}\frac{\partial\Phi}{\partial u}\cdot u'+\inf_{\|v'\|\leq\sigma(\|v\|+1)}\frac{\partial\Phi}{\partial v}\cdot v'\leq 0$$

We shall also introduce the partial differential equation[9]

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)+\rho(\|u\|+1)\left\|\frac{\partial\Phi}{\partial u}\right\|+\sigma(\|v\|+1)\left\|\frac{\partial\Phi}{\partial v}\right\|\leq 0 \quad (14.20)$$

which can be written in the following way:

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)+\sup_{\|u'\|\leq\rho(\|u\|+1)}\frac{\partial\Phi}{\partial u}\cdot u'+\sup_{\|v'\|\leq\sigma(\|v\|+1)}\frac{\partial\Phi}{\partial v}\cdot v'\leq 0$$

The link between the feedback maps and the solutions to the solutions to these partial differential equations is provided by the indicators of the graphs: we associate with the set-valued maps S_U, R_U and R_{UV} the functions Φ_U, Ψ_U and Ψ_{UV} from $\mathbf{R}^n\times\mathbf{R}^p\times\mathbf{R}^q$ to $\mathbf{R}_+\cup\{+\infty\}$ defined by

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for any initial situation $(z_0,u_0,v_0)\in\text{Dom}(\Phi)$, there exists a smooth solution $(z(\cdot),u(\cdot),v(\cdot))$ such that

$$t\to\Phi(z(t),u(t),v(t)) \text{ is nonincreasing}$$

This property remains true for the solutions to the contingent partial differential equation (14.22).

[9] We can check that if h is Lipschitz and Φ is a solution to this partial differential equation, for any initial situation $(z_0,u_0,v_0)\in\text{Dom}(\Phi)$, any smooth solution $(z(\cdot),u(\cdot),v(\cdot))$ satisfies

$$t\to\Phi(z(t),u(t),v(t)) \text{ is non increasing}$$

This property remains true for the solutions to the contingent partial differential equation (14.23).

$$\left\{\begin{array}{llll} i) & \Phi_U(z,u,v) & := & \left\{\begin{array}{lll} 0 & \text{if} & (u,v)\in S_U(z) \\ +\infty & \text{if} & (u,v)\notin S_U(z) \end{array}\right. \\ ii) & \Psi_U(z,u,v) & := & \left\{\begin{array}{lll} 0 & \text{if} & (u,v)\in R_U(z) \\ +\infty & \text{if} & (u,v)\notin R_U(z) \end{array}\right. \\ iii) & \Psi_{UV}(z,u,v) & := & \left\{\begin{array}{lll} 0 & \text{if} & (u,v)\in R_{UV}(z) \\ +\infty & \text{if} & (u,v)\notin R_{UV}(z) \end{array}\right. \end{array}\right. \tag{14.21}$$

and the functions Ψ_V and Φ_V associated to the set-valued map R_V and S_V in an analogous way.

These functions being only lower semicontinuous, but not differentiable, cannot be solutions to either partial differential equations (14.19) and (14.20). But we can use the *contingent epiderivatives* of any function $\Phi : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R} \cup \{+\infty\}$ and replace the partial differential equations (14.19) and (14.20) by the contingent partial differential equations

$$\inf_{\substack{\|u'\|\leq\rho(\|u\|+1) \\ \|v'\|\leq\sigma(\|v\|+1)}} D_\uparrow\Phi(z,u,v)(h(z,u,v),u',v') \leq 0 \tag{14.22}$$

and

$$\sup_{\substack{\|u'\|\leq\rho(\|u\|+1) \\ \|v'\|\leq\sigma(\|v\|+1)}} D_\uparrow\Phi(z,u,v)(h(z,u,v),u',v') \leq 0 \tag{14.23}$$

respectively.

Let Ω_U and Ω_V be the indicators of the graphs of the set-valued maps U and V defined by

$$\left\{\begin{array}{ll} i) & \Omega_U(z,u,v) := \left\{\begin{array}{lll} 0 & \text{if} & u\in U(z) \\ +\infty & \text{if} & u\notin U(z) \end{array}\right. \\ ii) & \Omega_V(z,u,v) := \left\{\begin{array}{lll} 0 & \text{if} & v\in V(z) \\ +\infty & \text{if} & v\notin V(z) \end{array}\right. \end{array}\right.$$

Theorem 14.5.3 *We posit the assumptions of Theorem 14.5.2. Then*

— Ψ_U *is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to* Ω_U

— Ψ_V *is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to* Ω_V

— Ψ_{UV} *is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to* $\max(\Omega_U, \Omega_V)$

— *If h is Lipschitz,* Φ_U *is the smallest lower semicontinuous solution to the contingent partial differential equation (14.23) larger than or equal to* Ω_U

— *If h is Lipschitz,* Φ_V *is the smallest lower semicontinuous solution to the contingent partial differential equation (14.23) larger than or equal to* Ω_V

If any of the above solutions is the constant $+\infty$, the corresponding feedback map is empty.

Proof of Theorem 14.5.2 — Let us denote by B the unit ball and introduce the set-valued map F defined by

$$H(z,u,v) \ := \ \{h(z,u,v)\} \times \rho(\|u\|+1)B \times \sigma(\|v\|+1)B$$

The evolution of the differential game described by equations (14.3) and (14.16) is governed by the differential inclusion

$$(z'(t), u'(t), v'(t)) \ \in \ H(z(t), u(t), v(t))$$

— Since the graph of U is closed, we take the graph of R_U to be the viability kernel of $\mathrm{Graph}(U) \times \mathbf{R}^q$. Indeed, if $(u_0, v_0) \in R_U(z_0)$, there exists a solution to the differential inclusion remaining in the graph of U, i.e., Xavier may win. If not, all solutions starting at (z_0, u_0, v_0) must leave this domain in finite time.

The set-valued feedback map R_V is defined in an analogous way.

— For the same reasons, the graph of the set-valued feedback map R_{UV} is the viability kernel of the set $\mathcal{K}$ of initial situations.

— When h is Lipschitz, so is H. We define the graph of S_U as the invariance kernel of $\mathrm{Graph}(U) \times \mathbf{R}^q$. □

Proof of Theorem 14.5.3 — We recall that thanks to the viability Theorem, a subset $L \subset \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ is a viability domain of F if and only if

$$\forall\ (z,u,v) \in L,\ T_L(z,u,v) \cap H(z,u,v) \ \neq \ \emptyset$$

Let Ψ_L denote the indicator of L. We know that the Viability Theorem can be reformulated in the following way:

The set L is a closed viability domain if and only if its indicator function Ψ_L is a solution to the contingent partial differential equation (14.22).

— Hence to say that the graph of R_U is the largest closed viability domain contained in the graph of U amounts to saying that its indicator Ψ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to the indicator Ω_U of Graph$(U) \times \mathbf{R}^q$. The same reasoning shows that indicator Ψ_V of R_V is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to Ω_V and that the indicator Ψ_{UV} of the graph of R_{UV} is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to the indicator of $\mathcal{K}$, which is equal to $\max(\Omega_U, \Omega_V)$.

— We know that a closed subset $L \subset \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ is "invariant" by a Lipschitz set-valued map F if and only if

$$\forall\, (z,u,v) \in L,\ T_L(z,u,v) \ \subset\ H(z,u,v)$$

This condition can be reformulated in terms of contingent epiderivative of the indicator function Ψ_L of L by saying that

$$\forall\, (z,u,v) \in L, \quad \sup_{w \in H(z,u,v)} \ D_\uparrow \Psi_L(z,u,v)(w) = 0$$

Hence to say that the graph of S_U is the largest closed invariance domain contained in the graph of U amounts to saying that its indicator Φ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (14.23) larger than or equal to the indicator Ω_U of Graph$(U) \times \mathbf{R}^q$. □

Let us denote by R one of the feedback maps R_U, R_V, R_{UV} and assume that the initial situation belongs to the graph of the set-valued feedback map R (when it is not empty). The theorem states only that there exists at least a solution $(z(\cdot), u(\cdot), v(\cdot))$ to the differential game such that

$$\forall\, t \geq 0,\ (u(t), v(t)) \ \in\ R(z(t))$$

To implement these strategies, players must *make decisions, i.e., to choose velocities of controls* in an adequate way:

We observe that stable solutions

Proposition 14.5.4 *The solutions to the game satisfying*

$$\forall\, t \geq 0,\ (u(t), v(t)) \ \in \ R(z(t))$$

are the solutions to the system of differential inclusions

$$\begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & (u'(t), v'(t)) \in G_R(z(t), u(t), v(t)) \end{cases} \tag{14.24}$$

where we have denoted by G_R *the* R-decision map *defined by*

$$G_R(z, u, v) \ := \ DR(z, u, v)(h(z, u, v)) \cap (\rho(\|u\| + 1)B \times \sigma(\|v\| + 1)B)$$

For simplicity, we shall set $G := G_R$ whenever there is no ambiguity.

Proof — Indeed, since the function $(z(\cdot), u(\cdot), v(\cdot))$ takes its values into Graph(R) and is absolutely continuous, then its derivative $(z'(\cdot), u'(\cdot), v'(\cdot))$ belongs almost everywhere to the contingent cone

$$T_{\text{Graph}(R)}(z(t), u(t), v(t)) \ := \ \text{Graph}(DR(z(t), u(t), v(t)))$$

We then replace $z'(t)$ by $h(z(t), u(t), v(t))$.

The converse holds true because equation (14.24) makes sense only if $(z(t), u(t), v(t))$ belongs to the graph of R. □

The question arises whether we can construct selection procedures of the decision components of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 14.5.5 (Closed Loop Decision Rules) *We say that a selection* $(\tilde{c}, \tilde{d})$ *of the contingent derivative of the smooth regulation map* R *in the direction* h *defined by: for all* $(z, u, v) \in$ Graph(R).

$$(\tilde{c}(z, u, v), \tilde{d}(z, u, v)) \in DR(z, u, v)(h(z, u, v) \tag{14.25}$$

is a closed loop decision rule.

The system of differential equations

$$\begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & u'(t) = c(z(t), u(t), v(t)) \\ iii) & v'(t) = d(z(t), u(t), v(t)) \end{cases} \tag{14.26}$$

is called the associated closed loop decision game.

Therefore, closed loop decision rules being given for each player, the closed loop decision system is just a system of ordinary differential equations.

It has solutions whenever the maps c and d are continuous (and if such is the case, they will be continuously differentiable).

But they also may exist when c or d or both are no longer continuous. This is the case when the decision map is lower semicontinuous thanks to Michael's Theorem:

Theorem 14.5.6 *Let us assume that the decision map $G := G_R$ is lower semicontinuous with non empty closed convex values on the graph of R. Then there exist continuous decision rules c and d, so that the decision system (14.26) has a solution whenever the initial situation $(u_0, v_0) \in R(z_0)$*

By using selection procedures (see Definition 6.5.2), we can obtain explicit decision rules which are not necessarily continuous, but for which the decision system (14.26) still has a solution.

Hence, we also obtain the following existence theorem for closed loop decision rules obtained through convex-valued selection procedures, which is analogous to Theorem 7.6.4.

Theorem 14.5.7 *Let S_G be a selection of the set-valued map G with convex values. Then, for any initial state $(z_0, u_0, v_0) \in$ Graph(R), there exists a solution starting at (z_0, u_0, v_0) to the associated system of differential inclusions*

$$\begin{cases} z'(t) = h(z(t), u(t), v(t)) \\ (u'(t), v'(t)) \in G(z(t), u(t), v(t)) \cap S_G(z(t), u(t), v(t)) \end{cases} \tag{14.27}$$

In particular, if for every (z, u, v) the intersection

$$S(G(z, u, v)) := (\tilde{c}(z, u, v), \tilde{d}(z, u, v))$$

is single-valued, then the strategies $(x, y) \mapsto (\tilde{c}(z, u, v), \tilde{d}(z, u, v))$ are single-valued closed-loop decision rules, for which decision system 14.26 has a solution for any initial state $(z_0, u_0, v_0) \in \text{Graph}(R)$.

We can now multiply the possible corollaries, since we have given several instances of selection procedures of set-valued maps.

Example— COOPERATIVE BEHAVIOR
Let $\sigma : \text{Graph}(G) \mapsto \mathbf{G}$ be continuous.

Corollary 14.5.8 *Let us assume that the set-valued map G is lower semicontinuous with nonempty closed convex images on* Graph(R). *Let σ be continuous on* Graph(G) *and convex with respect to the pair (u, v). Then, for all initial situations $(u_0, v_0) \in R(z_0)$, there exists a solution starting at (z_0, u_0, v_0) to the differential game (14.3)-(14.16) which are regulated by:*

$$\begin{cases} \text{for almost all } \geq 0, \quad (u'(t), v'(t)) \in G(z(t), u(t), v(t)) \text{ and} \\ \sigma(z(t), u(t), v(t), u'(t), v'(t)) \\ = \inf_{u', v' \in G(z(t), u(t), v(t))} \sigma(z(t), u(t), v(t), u', v') \end{cases}$$

In particular, the game can be played by the heavy decision *of minimal norm:*

$$\begin{cases} (c^\circ(z, u, v), d^\circ(z, u, v)) \in G(z, u, v) \\ \|c^\circ(z, u, v)\|^2 + \|d^\circ(z, u, v)\|^2) = \min_{(u', v') \in G(z, u, v)} (\|u'\|^2 + \|v'\|^2) \end{cases}$$

Example— NONCOOPERATIVE BEHAVIOR
We can also choose strategies in the regulation sets $G(z, u, v)$ in a noncooperative way, as saddle points of a function $a(z, u, v, \cdot, \cdot)$.

Corollary 14.5.9 *Let us assume that the set-valued map G is lower semicontinuous with nonempty closed convex images on* Graph(R)

and that $a : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}$ *satisfies*

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall (z,u,v,d),\ \ c \mapsto a(z,u,v,c,d) \text{ is convex} \\ iii) & \forall (z,u,v,c),\ \ d \mapsto a(z,u,v,c,d) \text{ is concave} \end{cases}$$

Then, for all initial situation $(u_0, v_0) \in R(z_0)$*, there exist solutions to the differential game (14.3)-(14.16) starting at* (z_0, u_0, v_0) *which are regulated by: for almost all* $t \geq 0$,

$$\begin{cases} i) & (u'(t), v'(t)) \in G(z(t), u(t), v(t)) \\ ii) & \forall\ (u', v') \in G(z(t), u(t), v(t)),\ \ a(z(t), u(t), v(t), u'(t), v') \\ & \leq a(z(t), u(t), v(t), u'(t), v'(t)) \leq a(z(t), u(t), v(t), u', v'(t)) \end{cases}$$