Chapter 12

Functional Viability

Introduction

Differential equations and inclusions describe the evolution of systems where, at each instant, the velocity of the state depends upon the value of the state at this very instant (in a single or multivalued way).

Differential inclusions with memory, also called *functional differential inclusions*, express that at each instant, the velocity of the state depends upon the history of its evolution up to this instant.

By *functional viability*, we mean viability constraints which also depend upon the history of the evolution of the state of the system, or even, when the constraints act not only on the state of the system, but on its past evolution.

This allows us to take into account delays, anticipations, cumulated consequences of the past, etc., in both the dynamics of the system and the viability constraints.

We shall adapt the techniques devised for the usual viability problems for differential inclusions to functional viability problems for differential inclusions with memory.

This will leas to a characterization of the functional viability property by a "functional tangential condition" stating that for any past evolution, there exists at least a velocity "tangent" to the set of past evolutions satisfying the functional viability constraints.

This characterization does not solve completely the problem, since, for concrete examples, we have to prove that it is satisfied. It is well

J.-P. Aubin, *Viability Theory*

known that invariance problems for differential equations with delays are difficult to solve.

But as in the case of differential equations and inclusions, the characterization of functional viability by functional tangential conditions offers easier routes to solve the problem since these conditions do not require the resolution of the functional differential inclusion.

The first section is devoted to the definitions and the presentation of the main classes of examples (differential inclusion with delays, Volterra type differential inclusions, etc.) and Haddad's functional viability theorem is proved in the third section.

We treat in the third section the particular cases of functional viability constraints of the form

$$\forall\, t \geq 0, \;\; x(t) \;\in\; M\left(\int_{-\infty}^{t} A(t-s)x(s)d\mu(s)\right)$$

and sufficient conditions involving the derivative of the set-valued map M are presented.

We end this chapter by adapting to the functional viability case the concepts of viability kernels and viability tubes.

12.1 Definitions and Examples

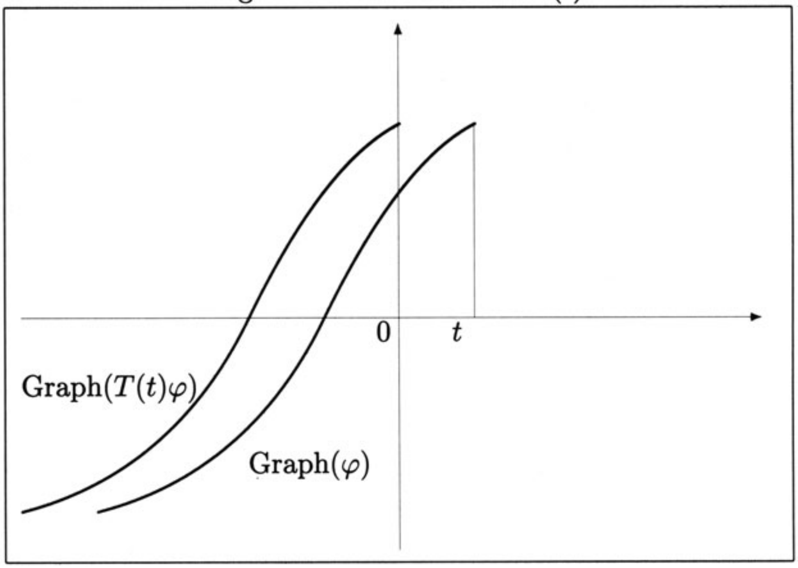
Our first task is to translate the concept of history of the evolution of the state up to the instant $t > 0$. We achieve this purpose by using the operator $T(t)$ from the Fréchet space $\mathcal{C}(-\infty,+\infty;X)$ to $\mathcal{C} := \mathcal{C}(-\infty,0;X)$ which associates with any continuous function $x(\cdot)$ its *history* $T(t)x$ *up to time* t defined by:

$$\forall\, \tau \in]-\infty,0], \;\; T(t)x(\tau) \;:=\; x(t+\tau)$$

A differential inclusion with memory describes in the following way the link between the velocity $x'(t)$ and the history $T(t)x$ up to time t through a set-valued map $\mathcal{F}$ from $\mathcal{C}$ to X in the following manner:

$$\text{for almost all } t \in [0,\infty[, \;\; x'(t) \in \;\mathcal{F}(T(t)x) \qquad (12.1)$$

Figure 12.1: Translation $T(t)$



Initial conditions express that the history of the evolution of the state up to the initial state 0 is known: it is a function $\varphi \in \mathcal{C}$. Hence the initial condition is written in the form:

$$T(0)x \quad = \quad \varphi \tag{12.2}$$

The viability constraints bear not only on the state of the system, but on its evolution, by requiring that at each instant,

$$\forall\, t \in [0, \infty[, \;\; T(t)x \;\in\; \mathcal{K} \tag{12.3}$$

where $\mathcal{K} \subset \mathcal{C}$ is a given closed subset of state evolutions.

Definition 12.1.1 *We shall say that a subset $\mathcal{K} \subset \mathcal{C}$ is* viable under *$\mathcal{F}$ (or enjoys the* viability property *for $\mathcal{F} : \mathcal{C} \rightsquigarrow X$) if and only if for any initial evolution $\varphi \in \mathcal{K}$, there exists a solution $x(\cdot)$ to (12.1) starting at φ (in the sense of (12.2)) and viable in $\mathcal{K}$ (in the sense of (12.3))*

We first observe that by taking

$$\begin{cases} i) & \mathcal{F}(\varphi) := F(\varphi(0)) \\ \\ ii) & \mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in K \} \end{cases}$$

where $K \subset X$ and $F : X \rightsquigarrow X$, usual viability problems are particular cases of functional viability problems because

$$\begin{cases} i) & x'(t) \in \mathcal{F}(T(t)x) = F((T(t)x)(0)) = F(x(t)) \\ \\ ii) & x(t) = (T(t)x)(0) \in K \iff T(t)x \in \mathcal{K} \end{cases}$$

We can also extend this time-independent functional viability problem to the time-dependent case. We introduce for that purpose

$$\begin{cases} i) & \text{a set-valued map } \mathcal{P} : \mathbf{R} \rightsquigarrow \mathcal{C} \\ ii) & \text{a set-valued map } \mathcal{F} : \mathrm{Graph}(\mathcal{P}) \rightsquigarrow X \end{cases}$$

We thus say that $\mathcal{P}$ enjoys the functional viability property if and only if for any t_0 and $\varphi \in \mathcal{P}_{t_0}$, there exists a solution $x(\cdot)$ to

$$\text{for almost all } t \geq t_0, \ \ x'(t) \in \mathcal{F}(t, T(t)x) \tag{12.4}$$

satisfying the initial condition $T(t_0)x = \varphi$, and which is viable in the sense that:

$$\forall\, t \geq t_0, \ \ T(t)x \in \mathcal{P}(t)$$

Before characterizing closed subsets $\mathcal{K}$ enjoying the viability property, we show that this class of viability problems covers many examples.

Example 1. Viability problems with delays. We consider p delay functions r_i from $[0,\infty[$ to $[0,\infty[$. A *differential inclusion with delays* is described by a set-valued map $F : X^p \rightsquigarrow X$ in the following way:

$$x'(t) \in F(x(t - r_1(t)), \ldots, x(t - r_p(t)))$$

In the same way, viability constraints with delays are described by q delay functions s_i from $[0,\infty[$ to $[0,\infty[$ and a set-valued map $D : X^q \rightsquigarrow X$:

$$\forall\, t \in [0,\infty[, \ \ x(t) \in D(x(t - s_1(t)), \ldots, x(t - s_q(t)))$$

This viability problem fits the general framework of functional viability by taking

$$\begin{cases} i) & \mathcal{F}(\varphi) := F(\varphi(-r_1(t)), \ldots, \varphi(-r_p(t))) \\ ii) & \mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in D(\varphi(-s_1(t)), \ldots, \varphi(-s_q(t))) \} \end{cases} \square$$

Example 2. **Volterra viability problems.** We consider a function $k : \mathbf{R} \times \mathbf{R} \times X \to Y$ (called a *kernel*) which expresses the cumulated consequences $\int_{-\infty}^{t} k(t, s, x(s))ds$ in Y of the evolution of the solution up to t.

A *Volterra differential inclusion* is described by a set-valued map $F : Y \rightsquigarrow X$ in the following way:

$$\text{for almost all } t \in [0, \infty[, \quad x'(t) \in F\left(\int_{-\infty}^{t} k(t, s, x(s))ds\right)$$

In the same way, Volterra viability constraints are described by a kernel $l : \mathbf{R} \times \mathbf{R} \times X \to Z$ and a set-valued map $D : Z \rightsquigarrow X$ through

$$\forall\, t \in [0, \infty[, \quad x(t) \in D\left(\int_{-\infty}^{t} l(t, s, x(s))ds\right)$$

Volterra viability problems are particular cases of functional viability problems when we take

$$\begin{cases} i) & \mathcal{F}(\varphi) := F(\int_{-\infty}^{0} k(t, t+s, \varphi(s))ds) \\ ii) & \mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in \int_{-\infty}^{0} l(t, t+s, \varphi(s))ds \} \end{cases} \square$$

12.2 Functional Viability Theorem

To proceed, we have to adapt to functional viability problems the concept of viability domains;

Definition 12.2.1 (Functional Viability Domains) *Let φ be given in a subset $\mathcal{K} \subset \mathcal{C}$. We denote by $\mathcal{D}_{\mathcal{K}}(\varphi) \subset X$ the subset of* *elements $v \in X$ such that, for any $\varepsilon > 0$, there exist $h \in]0, \varepsilon]$ and $\varphi_h \in \mathcal{C}(-\infty, h; X)$ satisfying*

$$\begin{cases} i) & T(0)\varphi_h = \varphi, \quad T(h)\varphi_h \in \mathcal{K} \\ ii) & (\varphi_h(h) - \varphi_h(0))/h \in v + \varepsilon B \end{cases} \tag{12.5}$$

Let $\mathcal{F} : \mathcal{C} \rightsquigarrow X$ be a set-valued map. We shall say that $\mathcal{K} \subset$ Dom$(\mathcal{F})$ *is a* functional viability domain *of $\mathcal{F}$ if and only if*

$$\forall \varphi \in \mathcal{K}, \quad \mathcal{F}(\varphi) \cap \mathcal{D}_{\mathcal{K}}(\varphi) \neq \emptyset \tag{12.6}$$

We denote by $\mathcal{C}_\lambda$ the closed convex subset of λ-Lipschitz functions from $]-\infty, 0]$ to X. Ascoli's Theorem states that a *closed subset $\mathcal{K} \subset \mathcal{C}_\lambda$ is compact if and only if $\mathcal{K}(0) := \{\varphi(0)\}_{\varphi \in \mathcal{K}}$ is bounded*, because it is closed and equicontinuous (by assumption) and pointwise bounded because, for all $\psi \in \mathcal{K}$ and $\tau \leq 0$,

$$\|\psi(\tau)\| \leq \|\psi(\tau) - \psi(0)\| + \|\psi(0)\| \leq \lambda|\tau| + \|\mathcal{K}(0)\|$$

Theorem 12.2.2 *[Haddad] Let $\mathcal{F} : \mathcal{C}_\lambda \rightsquigarrow X$ be a Marchaud map and $\mathcal{K} \subset$ Dom$(\mathcal{F})$ a closed subset of $\mathcal{C}_\lambda$.*

Then $\mathcal{K}$ enjoys the functional viability property if and only if it is a functional viability domain.

Remark — We observe that when $\mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in K \}$, then

$$\mathcal{D}_{\mathcal{K}}(\varphi) = T_K(\varphi(0))$$

and that when $\mathcal{F}(\varphi) := F(\varphi(0))$, $\mathcal{K}$ is a functional viability domain of $\mathcal{F}$ if and only if K is a viability domain of F. Hence the Viability Theorem for differential inclusions is a consequence of Theorem 12.2.2. □

Proof of the necessary condition — Assume that a solution $x(\cdot)$ to the functional differential inclusion (12.1) satisfies: there exists a sequence t_n converging to 0 such that $T(t_n)x \in \mathcal{K}$.

Since $\mathcal{F}$ is upper hemicontinuous at φ, we can associate with any $p \in X^\star$ and $\varepsilon > 0$ a neighborhood $\mathcal{V}$ of 0 in $\mathcal{C}$ such that

$$\forall \psi \in \varphi + \mathcal{V}, \quad \sigma(\mathcal{F}(\psi), p) \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

Since $T(0)x = \varphi$, there exists $\eta > 0$ such that $T(\tau)x - \varphi \in \mathcal{V}$ for $|\tau| \leq \eta$. Hence, integrating inequalities

$$< x'(\tau), p > \leq \sigma(\mathcal{F}(T(\tau)x), p) \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

from 0 to t_n, we obtain

$$\forall\, p \in X^\star, \;\; < p, \frac{x(t_n) - x_0}{t_n} > \leq \; \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

This implies that the sequence $v_n := \frac{x(t_n)-x_0}{t_n}$ is relatively compact in X. Therefore, a subsequence (again denoted by) v_n converges to some $v \in X$. Since for all $p \in X^\star$, for n large enough,

$$< p, v_n > \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

we deduce that the limit v satisfies

$$\forall\, p \in X^\star, \; < p, v > \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

Letting ε converge to 0, we obtain

$$\forall\, p \in X^\star, \; < p, v > \leq \sigma(\mathcal{F}(\varphi), p)$$

so that v belongs to the closed convex hull of $\mathcal{F}(\varphi)$, which is equal to $\mathcal{F}(\varphi)$ because it is closed and convex.

It remains to show that v belongs to $\mathcal{D}_{\mathcal{K}}(\varphi)$. Indeed, $T(t_n)x \in \mathcal{K}$ by assumption, $T(0)x = \varphi$, so that condition (12.5) is satisfied with $\varphi_h := x(\cdot)$.

Proof of the sufficient condition — Let us consider an initial evolution φ and choose $T := 1$. We shall construct a viable solution to (12.1) on $[0, 1]$, so that it will be possible to extend it on $[0, \infty[$. Let us set

$$\mathcal{K}_0 \;:=\; \{\, \psi \in \mathcal{K} \mid \|\psi(0) - \varphi(0)\| \leq 2\lambda \,\}$$

Since $\mathcal{K} \subset \mathcal{C}_\lambda$ and $\mathcal{K}_0(0)$ is bounded, we deduce that this subset $\mathcal{K}_0$ is compact thanks to Ascoli's Theorem. Since $\mathcal{F}$ is upper semicontinuous with compact images, we know that $\mathcal{F}(\mathcal{K}_0)$ is bounded. We set $C \;:=\; \mathcal{F}(\mathcal{K}_0) + B$ which is bounded.

For any integer m, we denote by $\mathcal{V}_m^a$ the neighborhood of $\mathcal{C}$ defined by

$$\mathcal{V}_m^a := \{ \psi \in \mathcal{C} \mid \sup_{\tau \in [-m,0]} \|\psi(\tau)\| \leq a \}$$

We shall construct a sequence of approximate solutions in a first step, show that this sequence converges to some limit in a second step and prove that this limit is a viable solution in the third one.

— **Construction of approximate solutions.**

We begin by proving

Lemma 12.2.3 *There exists $\theta_m \in]0, \frac{1}{m}[$ such that, for any $\chi \in \mathcal{K}_0$, we can find $h \in [\theta_m, \frac{1}{m}]$, $\psi \in \mathcal{C}(-\infty, +\infty; X)$ and $v \in \mathcal{F}(\mathcal{K}_0)$ satisfying*

$$\begin{cases} i) & T(0)\psi \in \mathcal{K}, \;\; T(h)\psi \in \mathcal{K}, \;\; (\psi(h) - \psi(0))/h \in v + \frac{1}{m}B \\ ii) & T(0)\psi \in \chi + \mathcal{V}_m^{h/m} \\ iii) & (T(0)\psi, v) \; \in \; \mathrm{Graph}(\mathcal{F}) \end{cases} \tag{12.7}$$

Proof — Condition (12.6) allows us to associate with any $\psi \in \mathcal{K}_0$ elements $v \in \mathcal{F}(\psi)$, $h_\psi \in]0, \frac{1}{m}[$ and $\psi_h \in \mathcal{C}$ such that

$$T(0)\psi_h = \psi, \;\; T(h_\psi)\psi_h \in \mathcal{K} \;\; \& \;\; \frac{\psi_h(h_\psi) - \psi(0)}{h_\psi} \in v + \frac{1}{m}B$$

We point out that the Lipschitz constant of ψ_h on the interval $]-\infty, h_\psi]$ is less than or equal to λ because $T(h_\psi)\psi_h$ belongs to $\mathcal{K} \subset \mathcal{C}_\lambda$.

Since $\mathcal{K}_0$ is compact, it can be covered by q neighborhoods $\psi_j + \mathcal{V}_m^{h_{\psi_j}/m}$. We set $\theta_m := \min_{1 \leq i \leq q} h_{\psi_i} \in]0, \frac{1}{m}]$.

Let us take any $\chi \in \mathcal{K}_0$. It belongs to one of these neighborhoods: then there exist elements ψ_i, $h_i := h_{\psi_i} > 0$, ψ_{h_i} and $v_i \in F(\psi_{h_i})$ satisfying properties (12.7). Hence the lemma ensues with $h := h_i$, $\psi := \psi_{h_i}$ and $v := v_i$. $\square$

We take $m \geq 1/\lambda$. We thus deduce

Lemma 12.2.4 *There exist a finite sequence of $h_j \in [\theta_m, \frac{1}{m}]$, of functions $\psi_j \in \mathcal{C}(-\infty, +\infty; X)$ and elements $v_j \in \mathcal{F}(\mathcal{K}_0)$ such that $\psi_0 = \varphi, h_0 = 0$ and*

$$\left\{\begin{array}{ll} i) & T(0)\psi_j \in \mathcal{K}_0, \;\; T(h_j)\psi_j \in \mathcal{K}_0, \\ & (\psi(h_j) - \psi_j(0))/h_j \in v_j + \frac{1}{m}B \\ & \\ ii) & T(0)\psi_j \in T(h_{j-1})\psi_{j-1} + \mathcal{V}_m^{h_j/m} \\ & \\ & (\text{i.e.}, \sup_{-m\leq\tau\leq 0} \|\psi_j(\tau) - \psi_{j-1}(\tau + h_{j-1})\| \leq h_j/m) \\ & \\ iii) & (T(0)\psi_j, v_j) \;\in\; \text{Graph}(\mathcal{F}) \end{array}\right. \tag{12.8}$$

Proof — We proceed by induction. By Lemma 12.2.3, starting with $\psi_0 := \varphi$, there exist $h_1 \in [\theta_m, \frac{1}{m}]$, ψ_1 and $v_1 \in C$ such that the above properties (12.7) hold true. It remains to check that $T(h_1)\psi_1$ belongs to $\mathcal{K}_0$ to deduce that properties (12.8) are also satisfied, i.e., that $\|T(h_1)\psi_1(0) - \varphi(0)\| \leq 2\lambda$. This follows from the fact that

$$\left\{\begin{array}{l} \|T(h_1)\psi_1(0) - \varphi(0)\| \;\leq\; \|\psi_1(h_1) - \psi_1(0)\| + \|\psi_1(0) - \varphi(0)\| \\ \\ \leq\; \lambda h_1 + \frac{1}{m}h_1 \;\leq\; 2\lambda h_1 \end{array}\right.$$

We apply Lemma 12.2.3 to the function $\chi := T(h_1)\psi_1$ and infer the existence of $h_2 \in [\theta_m, \frac{1}{m}]$, ψ_2 and v_2 satisfying properties (12.7) and we verify that

$$\|T(h_2)\psi_2(0) - \varphi(0)\| \leq \lambda h_2 + \frac{1}{m}h_2 + (\lambda + \frac{1}{m})h_1 \leq (\lambda + \frac{1}{m})(h_1 + h_2)$$

We proceed until the index J such that

$$(\lambda + \frac{1}{m})(h_1 + h_2 + \cdots + h_{J-1}) \;\leq 2\lambda \;<\; (\lambda + \frac{1}{m})(h_1 + h_2 + \cdots + h_J) \;\; \square$$

We set $\tau_m^0 := 0, \tau_m^1 = h_1, \ldots, \tau_m^J := \sum_{j=1}^J h_j$ so that $\tau_m^J \geq 1$. We define the functions $y_m(\cdot)$ on $]-\infty, \tau_m^p[$ by

$$\left\{\begin{array}{ll} i) & y_m(t) := \varphi(t) \;\text{ if }\; t \leq 0 \\ & \\ ii) & y_m(t) := \psi_{j+1}(t - \tau_m^j) + \sum_{i=0}^j (\psi_i(h_i) - \psi_{i+1}(0)) \\ & \text{if } \; t \in [\tau_m^j, \tau_m^{j+1}[\end{array}\right.$$

and their values

$$x_j := y_m(\tau_m^j) = \psi_{j+1}(0) + \sum_{i=0}^{j}(\psi_i(h_i) - \psi_{i+1}(0))$$

We interpolate this sequence by piecewise linear functions defined on each interval $[\tau_m^j, \tau_m^{j+1}[$ by

$$x_m(t) := x_j + (t - \tau_m^j)(x_{j+1} - x_j)/h_{j+1}$$

and we set $x_m(\tau) := y_m(\tau) := \varphi(\tau)$ when $\tau \leq 0$.

Properties of the functions $x_m(\cdot)$ are summarized in the following

Lemma 12.2.5 *The functions $x_m(\cdot) :]-\infty, 1]$ are λ-Lipschitz, satisfy*

$$\begin{cases} i) & \forall\, t \in [0,1], \;\; x'_m(t) \in C \\ ii) & \forall\, t \in [0,1], \;\; \|x_m(t) - \varphi(0)\| \leq \lambda t \end{cases} \tag{12.9}$$

and

$$\begin{cases} i) & T(0)x_m \;=\; \varphi \\ ii) & \forall\, t \in]-\infty, 1], \;\; (T(t)x_m, x'_m(t)) \in \mathrm{Graph}(\mathcal{F}) + \left(\mathcal{V}_m^{\varepsilon_m} \times \frac{1}{m}B\right) \\ iii) & \forall\, t \in]-\infty, 1], \;\; T(t)x_m \in \mathcal{K} + \mathcal{V}_m^{\varepsilon_m} \end{cases} \tag{12.10}$$

where ε_m converges to 0.

Proof — The functions ψ_j being λ-Lipschitz, as translations of functions of $\mathcal{K}$, so are the functions $y_m(\cdot)$ and $x_m(\cdot)$.

The velocities of the approximate solutions belong to C because

$$\begin{cases} x'_m(t) = \frac{x_{j+1}-x_j}{h_{j+1}} = \frac{y_m(\tau_m^{j+1})-y_m(\tau_m^j)}{h_{j+1}} \\ = \frac{\psi_{j+1}(h_j)-\psi_{j+1}(0)}{h_{j+1}} \in v_{j+1} + \frac{1}{m}B \end{cases}$$

On the other hand, since $T(0)x_m = \varphi$, we deduce that

$$\|x_m(t) - \varphi(0)\| = \|x_m(t) - x_m(0)\| \leq \lambda t$$

It remains to prove properties (12.10).

For that purpose, we shall prove by induction that for any $j = 0, \ldots, J-1$, we have

$$\sup_{-m \leq \tau \leq 0} \|T(\tau_m^{j+1}) y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \leq \frac{\tau_m^{j+1}}{m} \tag{12.11}$$

For $j = 0$ and $\tau \in [-h_1, 0]$, we obtain

$$\begin{cases} \|T(h_1)y_m(\tau) - T(h_1)\psi_1(\tau)\| = \|y_m(\tau + h_1) - \psi_1(\tau + h_1)\| \\ = \|\psi_1(\tau + h_1 + 0) + \psi_0(0) - \psi_1(0) - \psi_1(\tau + h_1)\| = \|\psi_1(0) - \psi_0(0)\| \\ = \|\psi_1(0) - \varphi(0)\| \leq \frac{h_1}{m} \end{cases}$$

When $\tau \in [-m, -h_1]$, then

$$y_m(\tau + h_1) - \psi_1(\tau + h_1) = \varphi(\tau + h_1) - \psi_1(\tau + h_1)$$

By (12.8)ii) with $j = 1$, we know that $T(h_1)\psi_1 \in \varphi + \mathcal{V}_m^{h_1/m}$. Then property (12.11) is satisfied for $j = 0$. Assume that it is satisfied for $j-1$ and prove that it holds true for j.

First, when $\tau \in [-h_{j+1}, 0]$, we get

$$\begin{cases} \|T(\tau_m^{j+1}) y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \\ = \|\psi_{j+1}(\tau + \tau_m^{j+1} - \tau_m^j) - \psi_{j+1}(\tau + h_{j+1}) + \sum_{i=0}^{j}(\psi_i(h_i) - \psi_{i+1}(0))\| \\ \leq \sum_{i=0}^{j} \|\psi_i(h_i) - \psi_{i+1}(0)\| \leq \sum_{i=0}^{j} \frac{h_i}{m} = \frac{\tau_m^j}{m} \leq \frac{\tau_m^{j+1}}{m} \end{cases}$$

When $\tau \in [-m, -h_{j+1}]$, we obtain

$$\begin{cases} \|T(\tau_m^{j+1}) y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \\ \leq \|T(\tau_m^j) y_m(\tau + h_{j+1}) - T(h_j)\psi_j(\tau + h_{j+1})\| \\ + \|T(h_j)\psi_j(\tau + h_{j+1}) - \psi_{j+1}(\tau + h_{j+1})\| \quad \square \end{cases}$$

Induction hypothesis (12.11) and condition (12.8)iii) imply that for all $\tau \in [-m, -h_{j+1}]$,

$$\|T(\tau_m^{j+1}) y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \leq \frac{\tau_m^j}{m} + \frac{h_{j+1}}{m} = \frac{\tau_m^{j+1}}{m}$$

Hence property (12.11) is established, from which we have to deduce properties (12.10).

We observe that

$$\sup_{t \le \tau_m^J} \|x_m(t) - y_m(t)\| \le \frac{2\lambda}{m} \tag{12.12}$$

This is obvious when $t \le 0$ because these functions are equal in this case. Otherwise, when $t \in [\tau_m^j, \tau_m^{j+1}[$, we obtain

$$\|x_m(t) - y_m(t)\| \le \|x_m(t) - x_j\| + \|x_j - y_m(t)\| \le \frac{2\lambda}{m}$$

Therefore, inequalities (12.12) and (12.11) imply that

$$\|T(\tau_m^{j+1})x_m - T(h_{j+1})\psi_{j+1}\| \le (2\lambda + \tau_m^{j+1})/m$$

and thus, that for all $t \in [\tau_m^j, \tau_m^{j+1}]$,

$$\begin{cases} \|T(t)x_m - T(h_{j+1})\psi_{j+1}\| \\ \le \|T(t)x_m - T(\tau_m^{j+1})x_m\| + \|T(\tau_m^{j+1})x_m - T(h_{j+1})\psi_{j+1}\| \\ \le \lambda h_{j+1} + (2\lambda + \tau_m^{j+1})/m \le 3(\lambda+1)/m =: \varepsilon_m \end{cases}$$

Consequently, when $t \in [\tau_m^j, \tau_m^{j+1}]$,

$$T(t)x_m \ \in T(h_{j+1})\psi_{j+1} + \mathcal{V}_m^{\varepsilon_m} \subset \mathcal{K} + \mathcal{V}_m^{\varepsilon_m}$$

and

$$\begin{cases} (T(t)x_m, x_m'(t)) \ \in \ (T(h_{j+1})\psi_{j+1}, v_{j+1}) + \mathcal{V}_m^{\varepsilon_m} \times \frac{1}{m}B \\ \subset \mathrm{Graph}(\mathcal{F}) + \mathcal{V}_m^{\varepsilon_m} \times \frac{1}{m}B \ \square \end{cases}$$

— **Convergence of approximate solutions**

Conditions (12.9) of Lemma 12.2.5 allow us to apply Ascoli's Theorem. Hence a subsequence (again denoted by) x_m converges uniformly on every compact interval to a continuous function $x(\cdot)$: $]-\infty, 1] \to X$, so that for all $t \ge 0$, $T(t)x_m$ converges to $T(t)x$

in $\mathcal{C}$. Condition (12.9)i) and Alaoglu's Theorem imply also that a subsequence (again denoted by) $x'_m(\cdot)$ converges weakly to $x'(\cdot)$ in $L^1(0,1;X)$ for some positive constant b.

— **The limit is a solution**

Conditions (12.10) of Lemma 12.2.5 allow us to apply the Convergence Theorem 2.4.4, where $\mathcal{C}$ plays the role of X, X the role of Y, $T(t)x_m$ the role of $x_m(t)$ and $x'_m(\cdot)$ the role of $y_m(\cdot)$. Hence the limit $x(\cdot)$ is a solution to the functional differential inclusion (12.1), which is viable since $\mathcal{K}$ is closed. The proof of the Functional Viability Theorem is completed. □

12.3 History-dependent Viability Constraints

We consider the case when

$$\mathcal{K} := \{\varphi \in \mathcal{C} \text{ such that } \varphi(0) \in M(U(\varphi))\} \tag{12.13}$$

where $U \in \mathcal{L}(\mathcal{C}, Y)$ is a continuous linear operator and where $M : Y \rightsquigarrow X$ is a closed set-valued map.

We introduce the affine subspace $\Gamma(x) \subset \mathcal{C}(0,1;X)$ of functions $\psi \in \mathcal{C}_\lambda(0,1;X)$ satisfying $\psi(0) = x$. With any $\varphi \in \mathcal{C}_\lambda$ and $\psi \in \Gamma(\varphi(0))$ we associate the concatenated function $\varphi \vee \psi \in \mathcal{C}(-\infty,1;X)$ equal to φ on $]-\infty,0]$ and to ψ on $[0,1]$.

We denote by $\Lambda : \mathcal{C} \rightsquigarrow \mathcal{C}$ the set-valued map[1] associating with any $\varphi \in \mathcal{C}$ the subset $\Lambda\varphi$ of functions $\nu \in \mathcal{C}$ such that there exist sequences of $h_n > 0$ converging to $0+$, of ν_n converging to ν in $\mathcal{C}$ and of functions $\varphi_n \in \Gamma(\varphi(0))$ such that

$$\forall\, n \geq 0,\ \ T(h_n)(\varphi \vee \varphi_n) \ = \ \varphi + h_n \nu_n$$

Observe that if φ is the restriction to $]-\infty,0]$ of a differentiable function $\widetilde{\varphi}$ defined on $]-\infty,1]$, then $\Lambda\varphi := \widetilde{\varphi}'$ and that $\Lambda\varphi \neq \emptyset$ whenever φ is the restriction to $]-\infty,0]$ of a Lipschitz function $\widetilde{\varphi}$ defined on $]-\infty,1]$. In this case, every selection $\nu \in \Lambda\varphi$ is almost everywhere equal to φ':

$$\text{for almost all } t \geq 0,\ \ \nu(t) \ = \ \varphi'(t)$$

[1] We can regard Λ as a *contingent infinitesimal generator of the semi-group* $T(t)$.

We introduce now the *adjacent derivative* $D^\flat M(y,x)$ of M at (y,x) defined[2] in the following way: $u \in D^\flat M(y,x)(v)$ if and only if for all sequences $h_n > 0$ converging to 0, there exist sequences u_n converging to u and v_n converging to v such that

$$\forall\, n \geq 0, \;\; x + h_n u_n \;\in\; M(y + h_n v_n)$$

If M is sleek at (y,x), then both contingent and adjacent derivatives at (y,x) coincide. This is then the case when the graph of M is either convex or a smooth manifold. See Chapter 5 of SET-VALUED ANALYSIS for further details on adjacent derivatives of set-valued maps.

We provide more and more general sufficient conditions for subsets $\mathcal{K}$ defined by (12.13) to be viability domains.

Theorem 12.3.1 *We posit the following "surjectivity condition" on U: there exists a constant $c > 0$ such that for all $h > 0$,*

$$\begin{cases} \forall\, (v,u) \in Y \times X, \;\; \exists\, \psi_h \in \mathcal{C}(0,1;X) \quad \textit{such that} \\[2ex] \psi_h(0) = 0, \;\; \psi_h(h) = u, \;\; UT(h)(\varphi \vee \psi_h) = v \\[2ex] \text{and satisfying} \\[2ex] \|\psi_h\|_{\mathcal{C}(0,1;X)} \;\leq\; c(\|u\| + \|v\|) \end{cases}$$

Therefore

$$D^\flat M(U\varphi, \varphi(0))(U(\Lambda\varphi)) \;\subset\; \mathcal{D}_{\mathcal{K}}(\varphi)$$

The next statement trades surjectivity condition on U with restrictions on the size of the norm of U and the norm $\|D^\flat M(y,x)\|$

---

[2]Recall that the *adjacent tangent cone* $T_K^\flat(z)$ to a subset K at $z \in K$ is defined by

$$T_K^\flat(z) \;:=\; \text{Liminf}_{h \to 0+} \left(\frac{K-z}{h}\right) \;=\; \left\{ v \mid \lim \frac{d_K(z+hv)}{h} \;=\; 0 \right\}$$

Then

$$\text{Graph}(D^\flat M(y,x)) \;:=\; T^\flat_{\text{Graph}(M)}(y,x)$$

defined by

$$\left\|D^\flat M(y,x)\right\| := \inf_{u \in D^\flat M(y,x)(v)} \frac{\|u\|}{\|v\|}$$

Theorem 12.3.2 *Assume that for any* $(y,x) \in \text{Graph}(M)$*, the domain* $\text{Dom}(D^\flat M(y,x)) = Y$ *and that*

$$\forall\, (y,x) \in \text{Graph}(M), \quad \left\|D^\flat M(y,x)\right\| \le \beta < +\infty$$

and that there exists $\gamma > 0$ *such that*

$$\|UT(h)(0 \vee \psi)\| \le \gamma h \|\psi\|$$

Therefore

$$D^\flat M(U\varphi, \varphi(0))(U(\Lambda\varphi)) \subset \mathcal{D}_K(\varphi)$$

These results follow from the more general sufficient condition, which looks quite involved, but which is flexible enough to cover a wide variety of examples.

Theorem 12.3.3 *We posit that the following "stability condition" linking* U *and* M*: there exist constants* $c, l > 0$ *and* $\alpha \in]0,1[$ *such that for all* $h > 0$,

$$\begin{cases} \forall\, (y,x) \in \text{Graph}(M),\ \forall\, (v,u) \in Y \times X, \\ \exists\, \psi_h \in \mathcal{C}(0,1;X),\ u_\alpha \in X,\ v_\alpha \in Y \quad \textit{such that } \psi_h(0) = 0\ \ \& \\ \\ \psi_h(h) \in D^\flat M(U\varphi, \varphi(0))(UT(h)(0 \vee \psi_h) - v - v_\alpha) + u + u_\alpha \\ \\ \text{and satisfying} \\ \\ \|\psi_h\|_{\mathcal{C}(0,1;X)} \le c(\|u\| + \|v\|), \quad \|u_\alpha\| + \|v_\alpha\| \le \alpha(\|u\| + \|v\|) \end{cases}$$

Therefore

$$D^\flat M(U\varphi, \varphi(0))(U(\Lambda\varphi)) \subset \mathcal{D}_K(\varphi)$$

Proof — Let us pick $\nu \in \Lambda\varphi$ and $u \in D^\flat M(U\varphi, \varphi(0))(U\nu)$ and check that u belongs to $\mathcal{D}_K(\varphi)$.

We know that there exist sequences $h_n > 0$ converging to $0+$, u_n converging to v and v_n converging to $U\nu$ such that

$$\forall\, n \ge 0, \quad \varphi(0) + h_n u_n \in M(U\varphi + h_n v_n)$$

But we also know by definition of Λ that there exist sequences ν_n converging to ν and $\varphi_n \in \Gamma(\varphi(0))$ such that

$$\forall\, n \geq 0,\ \ T(h_n)(\varphi \vee \varphi_n) \ = \ \varphi + h_n \nu_n$$

Denote by A_n the Fréchet differentiable operator from $\mathcal{C}(0,1;X) \times Y \times X$ to $Y \times X$ defined by

$$A_n(\psi, y, x) \ := \ (UT(h_n)(\varphi \vee \psi) - y, \psi(h_n) - x)$$

We observe that

$$A_n(\varphi_n, U\varphi + h_n v_n, \varphi(0) + h_n u_n) \ = \ h_n\,(U\nu_n - v_n, \nu_n(0) - u_n)$$

and that

$$A_n'(\psi, y, x)(\xi, v, u) \ = \ (UT(h_n)(0 \vee \xi) - y, \xi(h_n) - x)$$

We the apply Theorem 3.4.5 of SET-VALUED ANALYSIS which we now recall:

Theorem 12.3.4 *Let X be a Banach space, $L \subset X$ be a closed subset and Y a normed space. Consider a sequence of Fréchet differentiable operators A_n from X to Y and elements $x_{0n} \in L$ such that x_{0n} converges to $x_0 \in L$ and $A_n(x_{0n})$ to y_0.*

We assume that A_n verify the following stability assumption*: there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$\forall\, x \ \in \ L \cap B(x_0, \eta),\ \ B_Y \ \subset \ A_n'(x)(T_L(x) \cap cB_X) + \alpha B_Y \quad (12.14)$$

Then there exist $l > 0$ and $\gamma > 0$ such that

$$\forall\, y_n \ \in \ B(y_0, \gamma),\ \ d\left(x_{0n}, A_n^{-1}(y_n) \cap L\right) \ \leq \ l\|y_n - A_n(x_{0n})\|$$

We apply this theorem with

$$\begin{cases} X := \mathcal{C}(0,1;X) \times Y \times X, \ \ Y := Y \times X \\ L := \Gamma(\varphi(0)) \times \mathrm{Graph}(M) \\ y_n := 0 \ \ \& \ \ x_{0n} := (\varphi_n, U\varphi + h_n v_n, \varphi(0) + h_n u_n) \end{cases}$$

We have seen that $A_n(x_{0n})$ converges to $y_0 := 0$.

By the stability assumption, there exist $c > 0$ and $\alpha \in]0,1[$ such that for any (ψ, y, x) and any (v, u), there exists a solution

$$\begin{cases} (\psi_h, y_h, x_h) \in \\ \Gamma(0) \times T_{\mathrm{Graph}(M)}(U\varphi, \varphi(0)) \; = \; T_{\Gamma(\varphi(0))\times \mathrm{Graph}(M)}(\varphi, U\varphi, \varphi(0)) \end{cases}$$

to the equation $A_h'(\psi, y, x)((\psi_h, y_h, x_h)) = (v, u)$ satisfying the above estimates.

Then we can apply Theorem 12.3.4: for each n, there exists a solution (ψ_n, y_n, x_n) to the equation $A_n(\mu_n, y_n, x_n) = 0$ belonging to

$$(\mu_n, y_n, x_n) \; \in \; \Gamma(\varphi(0)) \times \mathrm{Graph}(M)$$

and satisfying the inequalities

$$\begin{cases} \|\mu_n - \varphi_n\|_{\mathcal{C}(0,1;X)} + \|y_n - U\varphi - h_n v_n\| + \|x_n - \varphi(0) - h_n u_n\| \\ \leq \; h_n(\|U\nu_n - v_n\| + \|\nu_n(0) - u_n\|) \end{cases}$$

This implies in particular that there exists a sequence e_n converging to 0 such that

$$\mu_n(h_n) = x_n = \varphi(0) + h_n u_n + h_n e_n \; \in \; M(UT(h_n)(\varphi \vee \mu_n))$$

and such that

$$\frac{\mu_n(h_n) - \varphi(0)}{h_n} \; = \; u_n + e_n \; \text{ converges to } \; u$$

Since μ_n belongs to $\Gamma(\varphi(0))$, we infer that the function $\varphi_n := \varphi \vee \mu_n$ satisfies the properties

$$T(h_n)\varphi_n \; \in \; \mathcal{K} \;\; \& \;\; \varphi_n(h_n) \; = \; \varphi(0) + h_n(u_n + e_n)$$

where $u_n + e_n$ converges to 0. We thus conclude that u belongs to $\mathcal{D}_{\mathcal{K}}(\varphi)$. □

Proof of Theorem 12.3.2 — We have to prove that the stability condition of Theorem 12.3.3 holds true. We take $\psi_h \in \mathcal{C}(0,1;X)$ defined by $\psi_h(t) := tu/h$ if $t \in [0,h]$ and $\psi_h(t) := (1 -$

$t)u/((1-h))$ if $t \in [h,1]$, so that $\|\psi_h\| \leq \|u\|$ satisfies $\psi_h(0) = 0$ and $\psi_h(h) = u$. Let us set $\rho := \frac{\beta}{1+\beta}$. We then take

$$v_\alpha \;:=\; UT(h)(0 \vee \psi_h) - \rho v$$

and

$$u_\alpha \in (1-\rho)D^\flat M(y,x)(-v)$$

with minimal norm. We thus see that

$$\psi_h(h) \;\in\; D^\flat M(y,x)(UT(h)(0 \vee \psi_h) - v - v_\alpha) + u - u_\alpha$$

and that

$$\|v_\alpha\| \;\leq\; \gamma h\|\psi_h\| + \rho\|v\| \;\;\&\;\; \|u_\alpha\| \;\leq\; (1-\rho)\beta\|v\| = \rho\|v\| \quad (12.15)$$

Therefore the stability assumption is satisfied with $\alpha \in]\rho, 1[$ for h small enough.

Proof of Theorem 12.3.1 — The stability assumption is obviously satisfied with $u_\alpha = 0$ and $v_\alpha = 0$. □

12.3.1 Viability constraints with delays

Here, we take $Y := X^p$ and $U(\varphi) := (\varphi(-\theta_1), \ldots, \varphi(-\theta_p))$. Then the surjectivity assumption of Theorem 12.3.1 is satisfied, so that we obtain the following consequence:

Corollary 12.3.5 *Let us consider p positive delays $\theta_1, \ldots, \theta_p$. Assume that*

$$\mathcal{K} \;:=\; \{\varphi \in \mathcal{C} \quad \textit{such that } \varphi(0) \;\in\; M(\varphi(-\theta_1), \ldots, \varphi(-\theta_p))\}$$

Then

$$D^\flat M(\varphi(-\theta_1), \ldots, \varphi(-\theta_p), \varphi(0))((\Lambda\varphi)(-\theta_1), \ldots, (\Lambda\varphi)(-\theta_p)) \;\subset\; \mathcal{D}_{\mathcal{K}}(\varphi)$$

12.3.2 Volterra Viability constraints

Let us consider a finite dimensional space Y, a set-valued map $M : Y \rightsquigarrow X$ and $A \in L^1(0,\infty;\mathcal{L}(X,Y))$. We consider the case when U is defined by

$$U\varphi := \int_{-\infty}^{0} A(-s)\varphi(s)ds$$

Corollary 12.3.6 *Let us consider $A \in L^1(0,\infty;\mathcal{L}(X,Y))$ satisfying*

$$\sup_{t\in[0,1]} \|A(t)\|_{\mathcal{L}(Y,X)} \leq \gamma < +\infty$$

Assume that for all $(y,x) \in \mathrm{Graph}(M)$, $\mathrm{Dom}(D^\flat M(y,x)) = Y$ and that

$$\sup_{(y,x)\in\mathrm{Graph}(M)} \|D^\flat M(y,x)\| \leq \beta < +\infty$$

Let $\mathcal{K}$ be the subset defined by

$$\mathcal{K} := \left\{\varphi \in \mathcal{C} \quad \textit{such that } \varphi(0) \in M(\int_{-\infty}^{0} A(-s)\varphi(s)ds)\right\}$$

Then

$$D^\flat M\left(\int_{-\infty}^{0} A(-s)\varphi(s)ds, \varphi(0)\right)\left(\int_{-\infty}^{0} A(-s)\varphi'(s)ds\right) \subset \mathcal{D}_{\mathcal{K}}(\varphi)$$

Proof — It follows from Theorem 12.3.2. □

12.4 Functional Viability Kernel

The proof of Theorem 12.2.1 shows also that the solution map is upper semicontinuous and that there exist functional viability kernels of closed subsets $\mathcal{K} \subset \mathcal{C}_\lambda$.

We denote by $\mathcal{S}(\varphi)$ or by $\mathcal{S}_{\mathcal{F}}(\varphi)$ the (possibly empty) set of solutions to differential inclusion (12.1) starting from the initial evolution φ. We shall say that the set-valued map $\mathcal{S}$ defined by

$$\mathrm{Dom}(\mathcal{F}) \ni \varphi \longmapsto \mathcal{S}(\varphi)$$

is the *solution map* of $\mathcal{F}$ (or of functional differential inclusion (12.1).)

We shall say that $\mathcal{F}$ is a *Marchaud map* if it is a nontrivial upper hemicontinuous map with nonempty compact convex images and with linear growth in the sense that there exists $c > 0$ such that

$$\forall \varphi \in \mathcal{C}, \ \ \|\mathcal{F}(\varphi)\| \ \leq \ c(\|\varphi(0)\| + 1)$$

Theorem 12.4.1 (Continuity of the Solution Map) *Let us consider a Marchaud map $\mathcal{F} : \mathcal{C}_\lambda \rightsquigarrow X$.*

The solution map $\mathcal{S}$ is upper semicontinuous with compact images from its domain to the space $\mathcal{C}(-\infty, +\infty; X)$.

Actually, the graph of the restriction of $\mathcal{S}$ to any compact subset $\mathcal{K}$ of $\mathcal{C}_\lambda$ is compact.

Proof — We shall show that for all $\varphi \in \text{Dom}(\mathcal{F})$ and for all $\eta > 0$, the restriction to a compact subset $\mathcal{K} \subset \text{Dom}(\mathcal{F})$ of the set-valued map $\mathcal{S}$ is compact.

Let us choose a sequence of elements $(\varphi_n, x_n(\cdot))$ of the graph of the solution map $\mathcal{S}$. They satisfy:

$$\begin{cases} i) & x_n'(t) \ \in \ \mathcal{F}(T(t)x_n) \\ ii) & T(0)x_n \ = \ \varphi_n \end{cases}$$

The linear growth of $\mathcal{F}$ implies that

$$\|x_n'(t)\| \ \leq \ c(\|x_n(t)\| + 1)$$

and thus, that

$$\forall\, n \geq 0, \ \ \|x_n(t)\| \ \leq \ (\|\varphi_n(0)\| + 1)e^{ct} \ \ \& \ \ \|x_n'(t)\| \ \leq \ c(\|\varphi_n(0)\| + 1)e^{ct}$$

Therefore, since the sequence of $\varphi_n(0)$ is bounded, the sequence $x_n(\cdot)$ is relatively compact in the Fréchet space $\mathcal{C}(0, \infty; X)$ by Ascoli's Theorem, and the sequence $x_n'(\cdot)e^{-ct}$ is weakly relatively compact in $L^\infty(0, \infty; X)$ by Alaoglu's Theorem. Let us take $b > c$.

Hence a subsequence (again denoted by) x_n converges to x in the sense that:

$$\begin{cases} i) & x_n(\cdot) \text{ converges to } x(\cdot) \text{ uniformly on compact intervals} \\ \\ ii) & x_n'(\cdot) \text{ converges to } x'(\cdot) \text{ weakly in } L^1(0, \infty; X; e^{-b\cdot}) \end{cases}$$

Inclusions

$$\forall n > 0, \quad (T(t)x_n, x_n'(t)) \in \mathrm{Graph}(\mathcal{F})$$

imply that

$$\text{for almost all } t > 0, \ \ x'(t) \ \in \ \mathcal{F}(T(t)x)$$

thanks to the Convergence Theorem 2.4.4.

We thus have proved that a subsequence of the elements $(\varphi_n, x_n(\cdot))$ of the graph of $\mathcal{S}$ restricted to $\mathcal{K}$ converges to an element $(\varphi, x(\cdot))$ of this graph. This shows that it is compact, and thus, that the solution map $\mathcal{S}$ is upper semicontinuous with compact images. □

Definition 12.4.2 (Functional Viability Kernels) *Let $\mathcal{K} \subset \mathcal{C}_\lambda$ be a subset of the domain of a set-valued map $\mathcal{F} : \mathcal{C} \rightsquigarrow X$. We shall say that the largest closed functional viability domain contained in $\mathcal{K}$ (which may be empty) is the* viability kernel *of $\mathcal{K}$ and denote it by* $\mathrm{Viab}_{\mathcal{F}}(\mathcal{K})$ *or, simply,* $\mathrm{Viab}(\mathcal{K})$.

We can adapt to the functional case the existence theorem of a viability kernel.

Theorem 12.4.3 *Let us consider a Marchaud map $\mathcal{F} : \mathcal{C} \rightsquigarrow X$ with compact convex images. Then the viability kernel of $\mathcal{K}$ does exist and is the subset of initial evolutions $\varphi \in \mathcal{K}$ such that at least one solution starting from φ is viable in $\mathcal{K}$.*

12.5 Functional Viability Tubes

We can now extend this time-independent functional viability theorem to the time-dependent case. We consider

$$\begin{cases} i) & \text{a set-valued map } \mathcal{P} : \mathbf{R} \rightsquigarrow \mathcal{C}_\lambda \\ ii) & \text{a set-valued map } \mathcal{F} : \mathrm{Graph}(\mathcal{P}) \rightsquigarrow X \end{cases}$$

Definition 12.5.1 *For any $\varphi \in \mathcal{P}(t)$, we denote by $D\mathcal{P}(t, \varphi)(1) \subset X$ the subset of elements $v \in X$ such that, for any $\varepsilon > 0$, there exist*

$h \in]0, \varepsilon]$ *and* $\varphi_h \in \mathcal{C}(-\infty, t+h)$ *satisfying*

$$\begin{cases} i) & T(t)\varphi_h \;=\; \varphi \\ ii) & T(t+h)\varphi_h \;\in\; \mathcal{P}(t+h) \\ iii) & (\varphi_h(t+h) - \varphi_h(t))/h \;\in\; v + \varepsilon B \end{cases} \tag{12.16}$$

We shall say that the set-valued map $\mathcal{P}$ *is a* functional viability tube *if and only if*

$$\forall\, t,\ \varphi \in \mathcal{P}(t),\ \ F(t,\varphi) \cap \mathcal{DP}(t,\varphi)(1) \;\neq\; \emptyset$$

Theorem 12.5.2 *Assume that the set-valued map* $\mathcal{P} : \mathbf{R} \leadsto \mathcal{C}_\lambda$ *takes its values into* λ*-Lipschitz functions and that its graph is closed.*

Assume also that $\mathcal{F}$ *is a Marchaud map. Then* $\mathcal{P}$ *enjoys the functional viability property: for any* t_0 *and* $\varphi \in \mathcal{P}_{t_0}$*, there exists a solution* $x(\cdot)$ *to*

$$\text{for almost all } t \geq t_0,\ \ x'(t) \;\in\; \mathcal{F}(t, T(t)x) \tag{12.17}$$

satisfying the initial condition $T(t_0)x = \varphi$ *which is viable in the sense that:*

$$\forall\, t \geq t_0,\ \ T(t)x \;\in\; \mathcal{P}(t)$$

if and only if is a functional viability tube.

Proof — The proof of the necessary condition is fully analogous to the time-independent case. We deduce the sufficient condition from the time-independent case by observing that the functional viability property for the new system

$$\begin{cases} i) & (s'(t), x'(t)) \;\in\; \{1\} \times \mathcal{F}((T(t)s)(0), T(t)x) \\ ii) & T(t_0)(s,x) \;=\; (t_0, \varphi) \end{cases}$$

and the closed subset $\mathcal{L}$ defined by

$$\mathcal{L} \;=\; \{\, \mathcal{C}(-\infty, 0; \mathbf{R} \times X)_{1,\lambda} \mid \varphi \in \mathcal{P}(s(0))\, \}$$

is equivalent to the functional viability property of the time-dependent system (12.4).

The assumptions of the Functional Viability Theorem 12.2.2 are satisfied since the set-valued map $\mathcal{G}$ defined by $\mathcal{G}(s,\varphi) := \mathcal{F}(s(0),\varphi)$ is upper semicontinuous with compact convex images, taking its values in the subset of $\max(1,\lambda)$-Lipschitz functions.

It remains to check that $\mathcal{L}$ is a functional viability domain of $\mathcal{G}$ if and only if $\mathcal{P}$ is a functional viability tube of $\mathcal{F}$.

Indeed, take $\varepsilon > 0$ and $v \in \mathcal{F}(t,\varphi) \cap \mathcal{DP}(t,\varphi)(1)$ and prove that $1 \times v$ belongs to the intersection of $1 \times \mathcal{G}(s,\varphi)$ and $\mathcal{D}_{\mathcal{L}}(s,\varphi)$ for any function $s(\cdot)$ such that $s(0) = t$. Then $(s,\varphi) \in \mathcal{L}$ since $\varphi \in \mathcal{P}(s(0)) = \mathcal{P}(t)$.

We know that there exist $h \in]0,\varepsilon]$ and $\varphi_h \in \mathcal{C}(-\infty, t+h)$ such that properties (12.16) are satisfied. Let us define the functions s_h and ψ_h on $]-\infty, h]$ by

$$\begin{cases} i) \quad s_h(\tau) = s(\tau) \qquad \text{if } \tau \leq 0 \quad \text{and} \quad s_h(\tau) = t+\tau \quad \text{if } \tau \in [0,h] \\ ii) \quad \psi_h(\tau) = \varphi_h(\tau + t) \end{cases}$$

Then, properties

$$T(0)(s_h,\psi_h) \;=\; (s,\varphi) \;\&\; T(h)(s_h,\psi_h) \;\in\; \mathcal{L}$$

(because $T(h)\psi_h = T(t+h)\varphi_h \in \mathcal{P}(t+h) = \mathcal{P}((T(h)s_h)(0)))$ and

$$\frac{s_h(h) - s_h(0)}{h} = 1 \;\&\; \frac{\psi_h(h) - \psi_h(0)}{h} = \frac{\varphi_h(t+h) - \varphi_h(t)}{h} \in v + \varepsilon B$$

imply that $1 \times v$ belongs to $\mathcal{D}_{\mathcal{L}}(s,\varphi)$ □