Chapter 9

Lyapunov Functions

Introduction

Consider a differential inclusion $x' \in F(x)$, a function $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ and a real-valued function $w(\cdot)$.

The function V is said *to enjoy the Lyapunov property* if and only if for any initial state x_0, there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq 0, \;\; V(x(t)) \;\leq\; w(t)$$

Such inequalities allow us to deduce many properties on the asymptotic behavior of V along the solutions to the differential inclusion (in numerous instances, $w(t)$ goes to 0 when $t \to +\infty$, so that $V(x(t))$ converges also to 0).

Recall that the *epigraph* of V is defined by

$$\mathcal{E}p(V) \;:=\; \{(x, \lambda) \in X \times \mathbf{R} \mid V(x) \leq \lambda\}$$

We see right away that when $w(\cdot)$ is a solution to a differential equation $w' = -\varphi(w)$, we have actually a viability problem in the epigraph of V because the Lyapunov property can be written: For any initial state x_0, there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq 0, \;\; (x(t), w(t)) \;\in\; \mathcal{E}p(V)$$

So that we can apply viability theorems whenever the epigraph of V is closed, i.e., whenever V is lower semicontinuous: V enjoys the

Lyapunov property if and only if its epigraph is a viability domain of the map $(x, w) \rightsquigarrow F(x) \times \{-\varphi(w)\}$.

Therefore, our first task is to study the contingent cone to the epigraph of an extended function V at some point $(x, V(x))$: it is the epigraph of a function denoted $D_\uparrow V(x)$ and called the *contingent epiderivative* of V at x. It is an extension of the concept of directional derivative: If V is Fréchet differentiable at x, then

$$\forall\, u \in X,\ D_\uparrow V(x)(u) \ = \ \langle V'(x), u\rangle$$

It is also an extension of the *lower Dini derivative* when V is locally Lipschitz around x and an extension of the derivative from the right of a convex function. We devote the first section to a minimal presentation of these contingent epiderivatives, which are studied thoroughly in Chapter 6 of SET-VALUED ANALYSIS.

Hence it is no surprise that lower semicontinuous extended functions V which satisfy the Lyapunov property are solutions to the *contingent Hamilton-Jacobi inequality*

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_\uparrow V(x)(v) + \varphi(V(x)) \leq 0$$

We call them *Lyapunov functions (with respect to φ)* because, when V is differentiable and $F \equiv f$ is single-valued, we recognize the classical definition of a Lyapunov function, solution to

$$< V'(x), f(x) > +\varphi(V(x)) \leq 0$$

Therefore, the use of contingent epiderivatives allows lower semicontinuous extended functions to rank among candidates to be solutions to such a contingent Hamilton-Jacobi inequality.

This is of particular importance whenever state constraints are involved, because the restriction of a smooth function to a closed subset is no longer smooth[1].

We prove and exploit these facts in the second section.

The main question we face is *how to construct Lyapunov functions*. Ever since Lyapunov proposed a century ago his second method

[1]By the way, we observe that the indicator function ψ_K of a closed subset K is a Lyapunov function (for $\varphi \equiv 0$) if and only if K is a viability domain.

for studying the behavior of a solution around an equilibrium, finding Lyapunov functions for such and such differential equation (or partial differential equation) has been a source of numerous problems requiring most often many clever tricks. The same difficulty is found here.

However, using the concept of viability kernel, we are able to assert in section 9.3 the existence of a *smallest lower semicontinuous Lyapunov function* $U_\star^\varphi$ *larger than or equal to a given function* U. Hence, starting with any lower semicontinuous function U, we know that there exists a lower semicontinuous Lyapunov function $U_\star^\varphi$ (may be identically equal to $+\infty$) such that

$$\forall\, t \geq 0, \;\; U(x(t)) \leq U_\star^\varphi(x(t)) \leq w(t)$$

whenever the initial state is in the domain of $U_\star^\varphi$.

This may be quite useful when U is the distance function $d_M(\cdot)$ to a subset. For instance, in the case when $\varphi(w) = aw$, the domain of this Lyapunov function $d_{M\star}^a$ provides the set of states (the basin) from which a solution to the differential inclusion converges exponentially to M because

$$\forall\, x_0 \in \mathrm{Dom}(D_{M\star}^a), \;\; d_M(x(t)) \leq d_{M\star}^a(x(t)) \leq d_{M\star}^a(x_0)e^{-at}$$

The results about Lyapunov functions are generalized in the section 9.4 to obtain inequalities of the type

$$\forall\, t \geq s \geq 0, \;\; V(x(t)) - V(x(s)) + \int_s^t W(x(\tau), x'(\tau)d\tau) \leq 0$$

which are very useful for studying the *asymptotic behavior of solutions to differential inclusions and for sufficient conditions for optimality in optimal control.* These important issues are not treated here: we refer to the monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène Frankowska for an exhaustive study of generalized solutions (both contingent and viscosity) to Hamilton-Jacobi equations.

We also show as an example that *gradient inclusions* $x' \in -\partial V(x)$ (where $\partial V(x)$ denotes the generalized gradient) have slow solutions along which V does not increase when V is locally Lipschitz. We refer to Section 3.4 of DIFFERENTIAL INCLUSIONS for the case of lower semicontinuous convex extended functions.

A real-valued function defines the preorder $\succeq$ by

$$x \succeq y \text{ if and only if } V(x) \leq V(y)$$

Since different functions can yield the same preorder, since some (total) preorders cannot be derived from a cost function and since it is needed to consider also any preorder, total or not, in such fields as economics, we address the problem of characterizing preorders satisfying the *Lyapunov property*: for any initial state x_0, there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq s \geq 0, \;\; x(t) \succeq x(s)$$

This problem and the comparison of solutions to two differential inclusions are the topics of section 9.5.

As an application, we touch upon the asymptotic observability problem for differential inclusions in the section 9.6. Here is the problem (for differential equations). We *do not know* the solution $x(\cdot)$ to a differential equation $x' = f(x)$, i.e., its initial value which would allow us to reconstruct it, *but only its observation* $y(t) = h(x(t))$ where $h : X \mapsto Y$ is an observation map.

How can we reconstruct the solution $x(\cdot)$ *knowing only* $y(\cdot)$*?* We investigated this *tracking problem* in Chapter 8.

Here, we address a less demanding problem: we only wish to approximate the solution $x(t)$ for large t's. In other words, we would like to build a differential equation $z'(t) = g(z(t), y(t))$ which yields a solution $z(\cdot)$ such that

$$U(x(t) - z(t)) \leq w(t)$$

where U measures some kind of distance and $w(t)$ goes to 0. This problem is known under the name of *asymptotic observability*.

9.1 Contingent Epiderivatives

9.1.1 Extended Functions and their Epigraphs

A function $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ is called an *extended (real-valued) function*. Its *domain* is the set of points at which V is finite:

$$\mathrm{Dom}(V) \ := \ \{x \in X \ | \ V(x) \neq \pm\infty\}$$

A function is said to be *nontrivial*[2] if its domain is not empty. Any function V defined on a subset $K \subset X$ can be regarded as the extended function V_K equal to V on K and to $+\infty$ outside of K, whose domain is K.

Since the order relation on the real numbers is involved in the definition of the Lyapunov property (as well as in minimization problems), we no longer characterize a real-valued function by its graph, but rather by its *epigraph*

$$\mathcal{E}p(V) \ := \ \{(x, \lambda) \in X \times \mathbf{R} \mid V(x) \leq \lambda\}$$

or by its *hypograph* defined in a symmetric way by

$$\mathcal{H}p(V) \ := \ \{(x, \lambda) \in X \times \mathbf{R} \mid V(x) \geq \lambda\} \ = \ -\mathcal{E}p(-V)$$

The graph of a function is then the intersection of its epigraph and its hypograph.

We also remark that some properties of a function are actually properties of their epigraphs. For instance, *an extended function V is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone).* The epigraph of V is closed if and only if

$$\forall\, x \in X, \ \ V(x) \ = \ \liminf_{y \to x} V(y)$$

For extended functions V which never take the value $-\infty$, this is equivalent to the lower semicontinuity of V. We also observe that any positively homogeneous extended function is non trivial whenever $V(0) \neq -\infty$. In this case, $V(0) = 0$.

[2]Such a function is said to be *proper* in convex and non smooth analysis. We chose this terminology for avoiding confusion with proper maps.

Indicators ψ_K *of subsets* K defined by

$$\psi_K(x) := 0 \text{ if } x \in K \text{ and } +\infty \text{ if not}$$

which characterize subsets (as *characteristic functions* do for other purposes), provide important examples of extended functions.

The indicator ψ_K is lower semicontinuous if and only if K is closed and ψ_K is convex if and only if K is convex. One can regard the sum $V + \psi_K$ as the restriction of V to K.

We recall the convention $\inf(\emptyset) := +\infty$.

9.1.2 Contingent Epiderivatives

Before defining the contingent epiderivatives of a function by taking the contingent cones to its epigraph, we need to prove the following statement:

Proposition 9.1.1 *Let* $V : X \mapsto \mathbf{R}\cup\{\pm\infty\}$ *be a nontrivial extended function and* x *belong to its domain.*

Then the contingent cone to the epigraph of V *at* $(x, V(x))$ *is the epigraph of an extended function denoted* $D_\uparrow V(x)$*:*

$$\mathcal{E}p(D_\uparrow V(x)) = T_{\mathcal{E}p(V)}(x, V(x))$$

equal to[3]*:*

$$\forall\, u \in X, \;\; D_\uparrow V(x)(u) \;=\; \liminf_{h\to 0+, u'\to u} (V(x+hu') - V(x))/h$$

[3]We can reformulate this formula below by saying that *the contingent epiderivative* $D_\uparrow V(x)$ *is the lower epilimit (See Definition 9.2.4) of the differential quotients*

$$u \rightsquigarrow \nabla_h V(x)(u) \;:=\; \frac{V(x+hu) - V(x)}{h}$$

Indeed, we know that the contingent cone

$$T_{\mathcal{E}p(V)}(x, V(x)) \;=\; \mathrm{Limsup}_{h\to 0+} \frac{\mathcal{E}p(V) - (x, V(x))}{h}$$

is the upper limit of the differential quotients $\frac{\mathcal{E}p(V) - (x,V(x))}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathcal{E}p(D_\uparrow V(x)) := T_{\mathcal{E}p(V)}(x, y) \;\;\&\;\; \mathcal{E}p(\nabla_h F(x,y)) = \frac{\mathcal{E}p(V) - (x, V(x))}{h}$$

to conclude.

Proof — Indeed, to say that

$$(u,v) \in T_{\mathrm{Ep}(V)}(x, V(x))$$

amounts to saying that there exist sequences $h_n > 0$ converging to $0+$ and (u_n, v_n) converging to (u,v) satisfying

$$\forall\, n \geq 0, \quad \frac{V(x+h_n u_n) - V(x)}{h_n} \leq v_n$$

This is equivalent to saying that

$$\forall\, u \in X, \quad \liminf_{h \to 0+, u' \to u} (V(x+hu') - V(x))/h \leq v \quad \square$$

Definition 9.1.2 *Let $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be an extended function and $x \in \mathrm{Dom}(V)$. We shall say that the function $D_{\uparrow}V(x)$ is the* contingent epiderivative *of V at x and that the function V is* contingently epidifferentiable *at x if for any $u \in X$, $D_{\uparrow}V(x)(u) > -\infty$ (or, equivalently, if $D_{\uparrow}V(x)(0) = 0$).*

A function V is episleek *(at x) if its epigraph is episleek (at $(x, V(x))$).*

Consequently, *the epigraph of the contingent epiderivative at x is a closed cone. It is then lower semicontinuous and positively homogeneous whenever V is contingently epidifferentiable at x.*

We shall need also the contingent cone to the epigraph of V at points (x,w) where $w > V(x)$:

Proposition 9.1.3 *Let $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and x belong to its domain. For all $w \geq V(x)$,*

$$T_{\mathcal{E}p(V)}(x,w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

and for all $w > V(x)$,

$$\mathrm{Dom}(D_{\uparrow}V(x)) \times \mathbf{R} \subset T_{\mathcal{E}p(V)}(x,w)$$

If the restriction of V to its domain is upper semicontinuous, then, for all $w > V(x)$,

$$T_{\mathcal{E}p(V)}(x,w) = T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

Proof

1. — Fix $w \geq V(x)$. Let us assume that (u, v) belongs to $T_{\mathcal{E}p(V)}(x, w)$. We infer that there exist sequences u_n, v_n and $h_n > 0$ converging to u, v and 0 such that

$$w + h_n v_n \ \geq\ V(x + h_n u_n)$$

We thus deduce that u belongs to the contingent cone to the domain of V at x, and thus, that $T_{\mathcal{E}p(V)}(x, w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$.

2. — If u belongs to the domain of the contingent epiderivative of V at x, if $w > V(x)$ and if v is any real number, we check that (u, v) belongs to $T_{\mathcal{E}p(V)}(x, w)$.

Indeed, there exist sequences of elements $h_n > 0$, u_n and v_n converging to 0, u and $D_{\uparrow}V(x)(u)$ respectively such that

$$(x + h_n u_n, V(x) + h_n v_n) \ \in\ \mathcal{E}p(V)$$

But we can write

$$(x+h_n u_n, w+h_n v) = (x+h_n u_n, V(x)+h_n v_n)+(0, w-V(x)+h_n(v-v_n))$$

Since $w - V(x) + h_n(v - v_n)$ is strictly positive when h_n is small enough, we infer that $(x + h_n u_n, w + h_n v)$ belongs to the epigraph of V, i.e., that (u, v) belongs to the cone $T_{\mathcal{E}p(V)}(x, w)$.

3. — Let w be strictly larger than $V(x)$ and u belong to $T_{\mathrm{Dom}(V)}(x)$. Then there exist sequences u_n and $h_n > 0$ converging to u and 0 such that $V(x + h_n u_n) < +\infty$ for all n.

When V is upper semicontinuous on its domain, for all $\varepsilon \in]0, \frac{w-V(x)}{2}[$, there exists $\eta > 0$ such that, for all $h_n\|u_n\| < \eta$, we obtain

$$V(x + h_n u_n) \ \leq\ V(x) + \varepsilon \ <\ w - \varepsilon$$

Let v be given arbitrarily in $\mathbf{R}$. Then, for any $h_n > 0$ when $v \geq 0$ or for any $h_n \in]0, \frac{\varepsilon}{-v}[$ when $v < 0$, inequality $w - \varepsilon \leq w + h_n v$ implies that $V(x + h_n u_n) \leq w + h_n v$, i.e., that the pair (u, v) belongs to $T_{\mathcal{E}p(V)}(x, w)$. $\square$

We then have to compare contingent derivatives with the contingent epiderivatives and hypoderivatives, defined in a analogous way:

the hypograph of the contingent hypoderivative $D_\downarrow V(x)$ *of* V *at* x *is the contingent cone to the hypograph of* V *at* $(x, V(x))$:

$$\mathcal{E}p(D_\downarrow V(x)) = T_{\mathcal{H}p(V)}(x, V(x))$$

It is equal to

$$\forall\, u \in X, \;\; D_\downarrow V(x)(u) \;=\; \limsup_{h \to 0+, u' \to u} (V(x+hu') - V(x))/h$$

Proposition 9.1.4 *Let* $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be an extended function and* x *belong to its domain. Take any* $u \in \mathrm{Dom}(D_\uparrow V(x)) \cap \mathrm{Dom}(D_\downarrow V(x))$. *Then*

$$\{D_\uparrow V(x)(u), D_\downarrow V(x)(u)\} \;\subset\; D\mathbf{V}(x)(u) \;\subset\; [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$$

Equality

$$D\mathbf{V}(x)(u) \;=\; [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$$

hods true either when V *is continuous on a neighborhood of* x *or when* V *is episleek at* x.

Proof — Since the contingent epiderivative of V at x in the direction u is equal to

$$D_\uparrow V(x)(u) := \liminf_{h \to 0+, u' \to u} \frac{V(x+hu') - V(x)}{h}$$

we see that $D_\uparrow V(x)(u)$ is the limit of a subsequence of $\frac{V(x+hu')-V(x)}{h}$, and thus, that $D_\uparrow V(x)(u) \in D\mathbf{V}(x)(u)$. The same is true with the contingent hypoderivative.

Since $\mathrm{Graph}(V) = \mathcal{E}p(V) \cap \mathcal{H}p(V)$, we deduce that the inclusions

$$T_{\mathrm{Graph}(V)}(x, V(x)) \subset T_{\mathcal{E}p(V)}(x, V(x)) \cap T_{\mathcal{H}p(V)}(x, V(x))$$

can be translated into

$$\mathrm{Graph}(D\mathbf{V}(x)) \subset \mathcal{E}p(D_\uparrow V(x)) \cap \mathcal{H}p(D_\downarrow V(x))$$

from which the inclusion $D\mathbf{V}(x)(u) \subset [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$ follows.

The image $DV(x)(u)$ being convex (and thus, an interval) when V is episleek at x, we infer that $[D_{\uparrow}V(x)(u), D_{\downarrow}V(x)(u)] \subset D\mathbf{V}(x)(u)$.

Assume now that V is continuous on a neighborhood of x. Then, on a neighborhood of $(x, V(x))$, the graph of V is the boundary of both the epigraph and the hypograph of V, so that Theorem 4.3.3 implies that

$$T_{\text{Graph}(V)}(x, V(x)) \subset T_{\mathcal{E}p(V)}(x, V(x)) \cap T_{\mathcal{H}p(V)}(x, V(x))$$

and thus, that $D\mathbf{V}(x)(u) = [D_{\uparrow}V(x)(u), D_{\downarrow}V(x)(u)]$. □

The contingent epiderivative coincides with the directional derivative $< V'(x), u >$ when V is Fréchet differentiable.

If V is Fréchet differentiable around a point $x \in K$, then the *contingent epiderivative of the restriction is the restriction of the derivative to the contingent cone*:

$$D_{\uparrow}(V|_K)(x)(u) := \begin{cases} < V'(x), u > & \text{if } \; u \in T_K(x) \\ +\infty & \text{if } \; \text{not} \end{cases}$$

The formulas become much more simple when V is Lipschitz: the contingent epiderivative coincides with the *lower Dini derivative* :

Proposition 9.1.5 *Let us assume that $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ is Lipschitz around a point x of its domain. Then*

$$D_{\uparrow}V(x)(u) = \liminf_{h \to 0+} (V(x+hu) - V(x))/h \;\; \text{(the lower Dini derivative)}$$

and satisfies for some $l > 0$:

$$\forall\, u \in X, \;\; |D_{\uparrow}V(x)(u)| \;\leq\; l\|u\|$$

Remark — There are other intimate connections between contingent cones and contingent epiderivatives.

Let ψ_K be the *indicator* of a subset K. Then it is easy to check that

$$D_{\uparrow}(\psi_K)(x) = \psi_{T_K(x)}$$

Therefore we can either derive properties of the epiderivatives from properties of the tangent cones through epigraphs or take the opposite approach by using the above formula.

Recall that there is also an obvious link between the contingent cone and the contingent epiderivative of the distance function to K since we can write for every $x \in K$:

$$T_K(x) = \{v \in X \mid D_\uparrow d_K(x)(v) = 0\}$$

and that we used this formula to extend contingent cones to the whole space in Section 5.1. □

9.1.3 Epidifferential Calculus

We begin by computing epiderivatives of the sum and the composition product of functions:

Theorem 9.1.6 *Let us consider two finite dimensional vector-spaces X and Y, a continuous single-valued map $f : X \mapsto Y$, and two extended functions V and W from X and Y to $\mathbf{R} \cup \{+\infty\}$ respectively. Let x_0 belong to the domain of the functions $U := V + W \circ f$. We assume that f is continuously differentiable around x_0, that V and W are contingently epidifferentiable at x_0 and $f(x_0)$ respectively. Inequality*

$$D_\uparrow U(x_0)(u) \geq D_\uparrow V(x_0)(u) + D_\uparrow W(f(x_0))(f'(x_0)u)$$

is always true. If V is episleek at x_0 or W is episleek at $f(x_0)$ and the following transversality condition:

$$\mathrm{Dom}(D_\uparrow W(f(x_0))) - f'(x_0)\mathrm{Dom}(D_\uparrow V(x_0)) = Y$$

holds true, then

$$D_\uparrow U(x_0)(u) = D_\uparrow V(x_0)(u) + D_\uparrow W(f(x_0))(f'(x_0)u)$$

In particular, if K is a closed subset and V is a lower semicontinuous function, if $x_0 \in K \cap \mathrm{Dom}(V)$, if K is sleek at x_0 and V is episleek at x_0 and if the transversality condition

$$\mathrm{Dom}(D_\uparrow V)(x_0) - T_K(x_0) = X$$

holds true, then the contingent epiderivative of the restriction is the restriction of the contingent epiderivative to the contingent cone:

$$\forall\, u \in T_K(x_0),\ D_\uparrow V|_K(x_0)(u) = D_\uparrow V(x_0)(u)$$

Let us consider now a finite family of functions $V_i : X \mapsto \mathbf{R} \cup \{\pm\infty\}$, $(i \in I)$ with which we associate the function U defined by

$$U(x) \ := \ \max_{i \in I} V_i(x)$$

We set $I(x) := \{i \in I \mid V_i(x) = U(x)\}$. The following estimates are always true:

$$\forall\, u \in X, \ \max_{i \in I(x_0)} D_\uparrow V_i(x_0)(u) \ \leq \ D_\uparrow U(x_0)(u)$$

Equality holds true under transversality conditions:

Proposition 9.1.7 *Let us consider n extended lower semicontinuous functions $V_i : X \mapsto \mathbf{R} \cup \{+\infty\}$. If the dimension of X is finite, if the functions U_i are episleek at x_0 and if we posit the transversality assumption at $x_0 \in \mathrm{Dom}(U)$*

$$\forall\, u_i \in X, \ \bigcap_{i=1}^{n} (\mathrm{Dom}(D_\uparrow V_i(x_0)) - u_i) \neq \emptyset$$

then

$$\left\{ \begin{array}{l} \forall\, u \in \bigcap_{i=1}^{n} \mathrm{Dom}(D_\uparrow V_i(x_0)), \\ DU(x_0)(u) \ = \ \max_{i \in I(x_0)} D_\uparrow V_i(x_0)(u) \end{array} \right.$$

Consider finally two normed vector spaces X and Y and an extended function $U : X \times Y \mapsto \mathbf{R} \cup \{\pm\infty\}$, with which we associate the marginal function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ defined by

$$V(x) \ := \ \inf_{y \in Y} U(x, y)$$

Proposition 9.1.8 *Let us consider two normed vector spaces X and Y, an extended function $U : X \times Y \mapsto \mathbf{R} \cup \{\pm\infty\}$, and its marginal function V. Suppose that there exists $y_0 \in Y$ which achieves the minimum of $U(x_0, \cdot)$ on Y:*

$$V(x_0) \ = U(x_0, y_0)$$

Then

$$\forall\, u \in X, \ D_\uparrow V(x_0)(u) = \liminf_{u' \to u} \left(\inf_{v \in Y} D_\uparrow U(x_0, y_0)(u', v) \right)$$

Equality holds true if U is convex.

9.2 Lyapunov Functions

9.2.1 The Characterization Theorem

We consider a differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t)) \tag{9.1}$$

and a time-dependent function $w(\cdot)$ defined as a solution to the differential equation

$$w'(t) = -\varphi(w(t)) \tag{9.2}$$

where $\varphi : \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth. This function φ is used as a parameter in what follows. (The main instance of such a function φ is the affine function $\varphi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a})$.

Our problem is to characterize either functions enjoying the φ -Lyapunov property, i.e., nonnegative extended functions $V : X \to \mathbf{R}_+ \cup \{+\infty\}$ (such that $\mathrm{Dom}(V) \subset \mathrm{Dom}(F)$) satisfying

$$\forall\, t \geq 0, \;\; V(x(t)) \;\leq\; w(t), \quad w(0) = V(x(0)) \tag{9.3}$$

along at least one solution $x(\cdot)$ to differential inclusion (9.1) and a solution $w(\cdot)$ to differential equation (9.2).

Definition 9.2.1 (Lyapunov Functions) *We shall say that a nonnegative contingently epidifferentiable*[4] *extended function V is a* Lyapunov function *of F associated with a function $\varphi(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}$ if and only if V is a solution to the* contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(V), \;\; \inf_{v \in F(x)} D_\uparrow V(x)(v) + \varphi(V(x)) \;\leq\; 0 \tag{9.4}$$

Theorem 9.2.2 *Let V be a nonnegative contingently epidifferentiable lower semicontinuous extended function and $F : X \rightsquigarrow X$ be a Marchaud map. Then V is a Lyapunov function of F associated with $\varphi(\cdot)$ if and only if for any initial state $x_0 \in \mathrm{Dom}(V)$,* there exist solutions *$x(\cdot)$ to (9.1) and $w(\cdot)$ to (9.2) satisfying property (9.3).*

[4]We recall that this means that for all $x \in \mathrm{Dom}(V)$, $\forall\, v \in X$, $D_\uparrow V(x)(v) > -\infty$ and that $D_\uparrow V(x)(v) < \infty$ for at least a $v \in X$.

Proof — We set $G(x,w) := F(x) \times \{-\varphi(w)\}$. Obviously, the system (9.1), (9.2) has a solution satisfying (9.3) if and only if the system of differential inclusions

$$(x'(t), w'(t)) \in G(x(t), w(t)) \tag{9.5}$$

has a solution starting at $(x_0, V(x_0))$ viable in $\mathcal{K} := \mathcal{E}p(V)$. We first observe that $\mathcal{K}$ is a viability domain for G if and only if V is a Lyapunov function for F with respect to φ: If $\mathcal{K}$ is a viability domain of G, by taking $z = (x, V(x))$, we infer that

$$(v, -\varphi(V(x))) \in T_{\mathcal{K}}(x, V(x)) = \mathcal{E}p(D_{\uparrow}V(x))$$

for some $v \in F(x)$, hence (9.4).

Conversely, since $F(x)$ is compact and $v \mapsto D_{\uparrow}V(x)(v)$ is lower semicontinuous, (9.4) implies that there exists $v \in F(x)$ such that the pair $(v, -\varphi(V(x)))$ belongs to $T_{\mathcal{E}p(V)}(x, V(x))$. Hence

$$(x + h_n v_n, V(x) + h_n s_n) \in \mathcal{K}$$

with $h_n \to 0+$, $v_n \to v$ and $s_n \to -\varphi(V(x))$. If $w > V(x)$, this implies that for large n

$$\left\{ \begin{array}{l} (x + h_n v_n, w - h_n\varphi(w)) \;=\; (x + h_n v_n, V(x) + h_n s_n) \\ +(0, w - V(x) - h_n(s_n + \varphi(w))) \in \mathcal{K} + \{0\} \times \mathbf{R}_+ = \mathcal{K} \end{array} \right.$$

so that $(v, -\varphi(w)) \in T_{\mathcal{K}}(x, w)$.

Remark — We can reformulate the viability theorem in the following way:

Corollary 9.2.3 *Let $F : X \rightsquigarrow X$ be a Marchaud map. A closed subset K enjoys the viability property if and only if its indicator ψ_K is a solution to the contingent equation*

$$\inf_{v \in F(x)} D_{\uparrow}\psi_K(x)(v) \;=\; 0$$

Remark — With an extended nonnegative function V, we can associate affine functions $w \to aw - b$ for which V is a solution to the contingent Hamilton-Jacobi inequalities (9.4).

For that purpose, we consider the convex function b defined by

$$b(a) := \sup_{x \in \mathrm{Dom}(F)} \left(\inf_{v \in F(x)} D_{\uparrow}V(x)(v) + aV(x) \right)$$

Then it is clear that V is a solution to the contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(F), \ \inf_{v \in F(x)} D_{\uparrow}V(x)(v) + aV(x) - b(a) \leq 0$$

Therefore, we deduce that there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq 0, \ V(x(t)) \ \leq \ (V(x_0) - \frac{b(a)}{a})e^{-at} + \frac{b(a)}{a}$$

A reasonable choice of a is the largest of the minimizers of $a \in]0, \infty[\to \max(0, b(a)/a)$, for which $V(x(t))$ decreases as fast as possible to the smallest level set $V^{-1}(]0, \frac{b(a)}{a}])$ of V. $\square$

9.2.2 Stability Theorems

We address now a stability question: *Is the limit of a sequence of Lyapunov functions still a Lyapunov function?*

It depends on what we understand as "limit": the appropriate concept is the one of *lower epilimit* defined in the following way:

Definition 9.2.4 *The epigraph of the lower epilimit*

$$\mathrm{lim}^{\sharp}_{\uparrow n \to \infty} V_n$$

of a sequence of extended functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ is the upper limit of the epigraphs:

$$\mathcal{E}p(\mathrm{lim}^{\sharp}_{\uparrow n \to \infty} V_n) \ := \ \mathrm{Limsup}_{n \to \infty} \mathcal{E}p(V_n)$$

One can check that

$$\mathrm{lim}^{\sharp}_{\uparrow n \to \infty} V_n(x_0) \ = \ \liminf_{n \to \infty, x \to x_0} V_n(x)$$

We refer to Chapter 7 of SET-VALUED ANALYSIS for further details on *epigraphical convergence*.

Meanwhile, we deduce from Theorem 3.6.2 that

Theorem 9.2.5 *Let F be a Marchaud map. Then the lower epilimit of a sequence of Lyapunov functions V_n associated with a function φ is still a Lyapunov function of F associated with φ.*

We now consider the case when the functions V_n are Lyapunov functions of maps F_n:

Theorem 9.2.6 (Stability) *Let us consider a sequence of Marchaud maps $F_n : X \times Y \rightsquigarrow X$ with uniform linear growth and their graphical upper limit $F^\sharp$. Then the lower epilimit of a sequence of Lyapunov functions V_n of F_n associated with a function φ is a Lyapunov function of $\overline{\text{co}}F^\sharp$ associated with φ.*

It is an obvious consequence of Theorem 3.6.5.

9.2.3 W-Monotone Set-Valued Maps

Let $W : X \to \mathbf{R}_+ \cup \{+\infty\}$ be a nonnegative extended function. We say that a set-valued map F is W-monotone (with respect to φ) if

$$\forall\, x, y,\ \forall\, u \in F(x), v \in F(y),\ D_\uparrow W(x-y)(v-u) + \varphi(W(x-y)) \le 0 \tag{9.6}$$

We obtain for instance the following consequence:

Corollary 9.2.7 *Let W be a nonnegative contingently epidifferentiable extended lower semicontinuous function and $F : X \rightsquigarrow X$ be a nontrivial Marchaud map such that $-F$ is W-monotone with respect to some φ. Let $\bar{x}$ be an equilibrium of F (i.e., a solution to $0 \in F(\bar{x})$). Then, for any initial state x_0, there exist solutions $x(\cdot)$ and $w(\cdot)$ satisfying*

$$\forall\, t \ge 0,\ \ W(x(t) - \bar{x}) \ \le\ w(t)$$

In particular, for $W(z) := \frac{1}{2}\|z\|^2$, we find the usual concept of monotonicity (with respect to φ):

$$\forall\, x, y,\ \ \forall\, u \in F(x), v \in F(y),\ \ < u-v, x-y > \ \ge\ \varphi\left(\frac{1}{2}\|x-y\|^2\right) \quad \square$$

9.2.4 Attractors

Using distance functions as Lyapunov functions, we can study attractors:

Definition 9.2.8 *We shall say that a closed subset K is an* attractor *of order $\alpha \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(F)$, there exists at least one solution $x(\cdot)$ to differential inclusion (9.1) such that*

$$\forall\, t \geq 0, \;\; d_K(x(t)) \;\leq\; d_K(x_0)e^{-\alpha t}$$

We can recognize attractors by checking whether the distance function to K is a Lyapunov function:

Corollary 9.2.9 *Assume that F is a nontrivial Marchaud map. Then a closed subset $K \subset \mathrm{Dom}(F)$ is an attractor if and only if the function $d_K(\cdot)$ is a solution to the contingent inequalities:*

$$\forall\, x \in \mathrm{Dom}(F), \;\; \inf_{v \in F(x)} D_{\uparrow} d_K(x)(v) + \alpha d_K(x) \;\leq\; 0$$

Example Let us consider a function V defined through a nonnegative function $U : X \times Y \to \mathbf{R}_+ \cup \{+\infty\}$ in the following way:

$$V(x) := \inf_{y \in Y} U(x,y)$$

When we assume that the infimum is achieved at a point y_x, we recall that

$$D_{\uparrow} V(x)(u) \;\leq\; \inf_{v \in Y} D_{\uparrow} U(x, y_x)(u,v)$$

Hence, under the assumptions of Theorem 9.2.2, we deduce that assumption

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{u \in F(x), v \in Y} D_{\uparrow} U(x,y_x)(u,v) + \varphi(U(x,y_x)) \;\leq\; 0$$

implies that there exists a solution $x(\cdot)$ satisfying

$$\forall\, t \geq 0, \;\; \inf_{y \in Y} U(x(t), y) \;\leq\; w(t)$$

We can derive from this inequality and the calculus of contingent epiderivatives many consequences.

9.2.5 Universal Lyapunov Functions

We shall characterize the φ- universal Lyapunov property, for which property (9.3) is satisfied along *all* solutions to (9.1) and all solutions $w(\cdot)$ to (9.2).

We say that V is a *universal Lyapunov function* of F associated with a function φ if and only if V is a solution to the upper contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(V), \quad \sup_{v\in F(x)} D_{\uparrow}V(x)(v) + \varphi(V(x)) \;\leq\; 0 \tag{9.7}$$

In the same way as in Theorem 9.2.2, one can check that the closed subset $\mathcal{E}pV$ is an invariance domain of the set-valued map G if and only if V is a universal Lyapunov function. Then the Invariance Theorem 5.3.4 implies:

Theorem 9.2.10 *Let V be a nonnegative contingently epidifferentiable lower semicontinuous extended function. If F is Lipschitz on the interior of its domain with compact values and*

$$\mathrm{Dom}(V) \subset \mathrm{Int}(\mathrm{Dom}(F))$$

then V is a universal Lyapunov function associated with φ if and only if for any initial state $x_0 \in \mathrm{Dom}(V)$, all solutions *$x(\cdot)$ to (9.1) and $w(\cdot)$ to (9.2) do satisfy this property (9.3).*

If F is Lipschitz on the interior of its domain with compact values and φ is Lipschitz, then a subset $K \subset \mathrm{Dom}(F)$ is invariant under F if and only if its indicator ψ_K is a solution to the contingent equation

$$\sup_{v\in F(x)} D_{\uparrow}\psi_K(x)(v) \;=\; 0$$

We say that a subset $M \subset \mathrm{Dom}(F)$ is a *universal attractor* of order $\alpha \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(F)$, all solutions $x(\cdot)$ to differential inclusion (9.1) satisfy property.

We deduce that if F is Lipschitz with compact images, then K is a universal attractor if and only if

$$\forall\, x \in \mathrm{Dom}(F), \quad \sup_{v\in F(x)} D_{\uparrow}d_K(x)(v) + \alpha d_K(x) \;\leq\; 0$$

9.3 Optimal Lyapunov Functions

9.3.1 Smallest Lyapunov Functions

The functions φ and $U : X \to \mathbf{R}_+ \cup \{+\infty\}$ being given, we shall construct the smallest lower semicontinuous Lyapunov function larger than or equal to U, i.e., the smallest nonnegative lower semicontinuous solution $U_\star^\varphi$ to the contingent Hamilton-Jacobi inequalities (9.4) larger than or equal to U.

Theorem 9.3.1 *Let us consider a Marchaud map $F : X \rightsquigarrow X$, a continuous function $\varphi : \mathbf{R}_+ \to \mathbf{R}$ with linear growth and a proper nonnegative extended function U such that* $\mathrm{Dom}(U) \subset \mathrm{Dom}(F)$.

Then there exists a smallest nonnegative lower semicontinuous solution $U_\star^\varphi : \mathrm{Dom}(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the contingent Hamilton-Jacobi inequalities (9.4) larger than or equal to U (which can be the constant $+\infty$), which thus enjoys the property:

$$\begin{array}{l} \forall\, x_0 \in \mathrm{Dom}(U_\star^\varphi),\ \textit{there exist solutions}\text{ to (9.1) and (9.2)} \\ \quad \text{satisfying } \forall\, t \geq 0,\ \ U(x(t)) \ \leq\ U_\star^\varphi(x(t)) \ \leq\ w(t) \end{array}$$

Consequently, if $U(x_0) < U_\star^\varphi(x_0)$, all solutions $x(\cdot)$ to differential inclusion (9.1) and all solutions $w(\cdot)$ to differential equation (9.2) starting at $(x_0, U(x_0))$ satisfy

$$\left\{ \begin{array}{ll} i) & \forall\, t \geq 0,\ U_\star^\varphi(x(t)) \ >\ w(t) \text{ as long as } U(x(t)) \ \leq\ w(t) \\ \\ ii) & \exists\, T > 0 \ \ \textit{such that}\ \ U(x(T)) \ >\ w(T) \end{array} \right.$$

This happens for any solution $w(\cdot)$ whenever the initial state x_0 does not belong to the domain of $U_\star^\varphi$.

Proof — By Theorem 4.1.2, we know that there exists a largest closed viability domain $\mathcal{K} \subset \mathcal{E}p(U)$ (the viability kernel of the epigraph of U) of the set-valued map $(x,w) \rightsquigarrow G(x,w) := F(x) \times \{-\varphi(w)\}$. If it is empty, it is the epigraph of the constant function equal to $+\infty$.

If not, we have to prove that it is the epigraph of the nonnegative lower semicontinuous function $U_\star^\varphi$ defined by

$$U_\star^\varphi(x) \ := \ \inf_{(x,\lambda)\in\mathcal{K}} \lambda$$

we are looking for. Indeed, the epigraph of any solution $V \geq U$ to the contingent inequalities (9.4) being a closed viability domain of the set-valued map G, is contained in the epigraph of $U_\star^\varphi$, so that $U_\star^\varphi$ is smaller than the lower semicontinuous solutions to (9.4) larger than U. Since

$$\mathcal{E}p(U_\star^\varphi) = \text{Graph}(U_\star^\varphi) + \{0\} \times \mathbf{R}_+ \subset \mathcal{K} + \{0\} \times \mathbf{R}_+$$

it is therefore enough to show that $\mathcal{K} + \{0\} \times \mathbf{R}_+ \subset \mathcal{K}$. In fact, we prove *if* $\mathcal{M} \subset$ *Dom*$(F) \times \mathbf{R}_+$ *is a closed viability domain of* G,, *then so is the subset*

$$\mathcal{M}_0 := \mathcal{M} + \{0\} \times \mathbf{R}_+$$

Obviously, $\mathcal{M}_0$ is closed. To see that $G(x,w) \cap T_{\mathcal{M}_0}(x,w) \neq \emptyset$, let

$$U_{\mathcal{M}}(x) \;:=\; \inf_{(x,\lambda)\in\mathcal{M}} \lambda, \;\; d := -\varphi(U_{\mathcal{M}}(x))$$

By assumption, there exists $v \in F(x)$ such that (v,d) belongs to the contingent cone to $\mathcal{M}$ at the point $(x, U_{\mathcal{M}}(x)) \in \mathcal{M}$. Hence, there exist sequences $h_n > 0$ converging to 0, v_n converging to v and d_n converging to d such that

$$\forall\, n \geq 0, \;\; (x + h_n v_n, U_{\mathcal{M}}(x) + h_n d_n) \;\in\; \mathcal{M}$$

This proves the claim when $w = U_{\mathcal{M}}(x)$ and the case $w > U_{\mathcal{M}}(x)$ follows as in the proof of Theorem 9.2.2. □

Corollary 9.3.2 *We posit the assumptions of Theorem 9.3.1.*

— *The indicator* $\psi_{\text{Viab}(K)}$ *of the viability kernel* Viab(K) *of a closed subset* K *is the smallest nonnegative lower semicontinuous solution to*

$$\forall\, x \in \text{Dom}(V), \;\; \inf_{v\in F(x)} D_\uparrow V(x)(v) \;\leq\; 0 \qquad (9.8)$$

larger than or equal to ψ_K.

— *For all* $a \geq 0$, *there exists a smallest lower semicontinuous function* $d_{M\star}^a : X \to \mathbf{R} \cup \{+\infty\}$ *larger than or equal to* d_M *such that*

$$\forall\, x_0 \in \text{Dom}(d_{M\star}^a), \;\; \text{there exists a solution } x(\cdot) \text{ to (9.1) such that } d_M(x(t)) \leq d_{M\star}^a(x_0)e^{-at}$$

We can regard the subsets $\mathrm{Dom}(d^a_{M\star})$ as the *basins of exponential attraction of* M.

Proof — Let us check that the smallest lower semicontinuous solution U_0 larger than or equal to $U \equiv 0$ is equal to the indicator of $\mathrm{Viab}(K)$. Since it is clear that it is a solution to the above contingent inequalities (9.8), then

$$\forall\, x \in \mathrm{Viab}(K), \quad U_0(x) \;\le \psi_{\mathrm{Viab}(K)}(x)$$

Let x_0 belong to the domain of U_0. Then there exists a solution $x(\cdot)$ to the system of differential inclusions (9.5) starting at $(x_0, U_0(x_0))$ satisfying $U_0(x(t)) \le U_0(x_0)$ since $w(t) \equiv U_0(x_0)$. Therefore x_0 belongs to the largest closed viability domain $\mathrm{Viab}(K)$. Hence $U_0(x_0) \le \psi_{K_0}(x_0) = 0$.

The proof of the second statement is easy. □

Proposition 9.3.3 *We posit the assumptions of Theorem 9.3.1. Assume furthermore that* φ *vanishes at* 0. *Then if* U *vanishes at an equilibrium* $\bar{x}$ *of* F, *so does the function* $U^\varphi_\star$ *associated with* φ.

Let L *be the set-valued map associating to any solution* $x(\cdot)$ *to the differential inclusion (9.1) its limit set and* $\mathcal{S}$ *be the solution map. If* φ *is asymptotically stable, then for any* $x_0 \in \mathrm{Dom}(U^\varphi_\star)$, *there exists a solution* $x(\cdot) \in \mathcal{S}(x_0)$ *such that* $L(x(\cdot)) \subset U^{-1}(0)$.

Proof

— If $\bar{x}$ is an equilibrium of F such that $U(\bar{x}) = 0$, then $(\bar{x}, 0)$ is an equilibrium of G restricted to the epigraph of U (because $\varphi(0) = 0$), so that the singleton $(\bar{x}, 0)$, being a viability domain, is contained in viability kernel of $\mathcal{E}p(U)$, which is the epigraph of $U^\varphi_\star$. Hence $0 \le U(\bar{x}) \le U^\varphi_\star(\bar{x}) \le 0$.

— If φ is asymptotically stable, then the solutions $w(\cdot)$ to the differential equation $w'(t) = -\varphi(w(t))$ do converge to 0 when $t \to +\infty$. Let x_0 belong to the domain of $U^\varphi_\star$ and $x(\cdot)$ be a solution satisfying

$$U(x(t)) \;\le\; U^\varphi_\star(x(t)) \;\le\; w(t)$$

Hence any cluster point ξ of $L(x(\cdot))$, which is the limit of a subsequence $x(t_n)$, belongs to $U^{\varphi^{-1}}_\star(0)$, because the limit $(\xi, 0)$ of the sequence of elements $(x(t_n), w(t_n))$ of the epigraph of $U_\star^\varphi$ belongs to it, for it is closed. Hence $0 \leq U(\xi) \leq U_\star^\varphi(\xi) \leq 0$. □

9.3.2 Smallest Universal Lyapunov Functions

Using the concept of invariance kernels, we can adapt the above results to optimal universal Lyapunov functions:

Theorem 9.3.4 *If F is Lipschitz on the interior of its domain with compact values and φ is Lipschitz, then there exists a smallest nonnegative lower semicontinuous solution $U_\triangleleft^\varphi : \mathrm{Dom}(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the upper contingent Hamilton-Jacobi inequalities (9.7) larger than or equal to U (which can be the constant $+\infty$), which enjoys the property:*

$$\begin{array}{l} \forall\, x_0 \in \mathrm{Dom}(U_\triangleleft^\varphi), \quad \textit{all solutions} \text{ to (9.1) and (9.2) satisfy} \\ \qquad \forall\, t \geq 0, \;\; U(x(t)) \;\leq\; U_\triangleleft^\varphi(x(t)) \;\leq\; w(t) \end{array}$$

Proof — The proof is analogous to the one of Theorem 9.3.1: When F and φ are Lipschitz, Theorem 5.4.2 implies that there exists a largest closed invariance domain $\widetilde{\mathcal{K}}$ contained in the epigraph of U. We prove that it is the epigraph of the smallest lower semicontinuous solution

$$U_\triangleleft^\varphi \;=\; \inf_{(x,\lambda)\in\widetilde{\mathcal{K}}} \lambda$$

to (9.7) we are looking for. This can be checked by showing that *if $\mathcal{M} \subset Dom(F) \times \mathbf{R}_+$ is a closed invariance domain of the set-valued map G, then so is the subset $\mathcal{M} + \{0\} \times \mathbf{R}_+$.* □

We quote the following consequence:

Corollary 9.3.5 *Assume that F is Lipschitz on the interior of its domain with compact values.*

— *The indicator $\psi_{\mathrm{Inv}(K)}$ of the invariant kernel* Inv(K) *of a closed subset K (i.e., the largest closed invariance domain of F contained in K) is the smallest nonnegative lower semicontinuous solution to*

$$\forall\, x \in \mathrm{Dom}(V), \quad \sup_{v\in F(x)} D_\uparrow V(x)(v) \;\leq\; 0 \tag{9.9}$$

larger than or equal to ψ_K.

— *For all $a \geq 0$, there exists a smallest lower semicontinuous function $d_{M\triangleleft}^a : X \to \mathbf{R} \cup \{+\infty\}$ larger than or equal to d_M such that*

$$\begin{array}{c} \forall\, x_0 \in \mathrm{Dom}(d_{M\triangleleft}^a), \;\; \text{any solution } x(\cdot) \text{ to (9.1) satisfies} \\ d_M(x(t)) \leq d_{M\triangleleft}^a(x_0)e^{-at} \end{array}$$

We can regard the subsets $\mathrm{Dom}(d_{M\triangleleft}^a)$ as the basins of universal exponential attraction of M.

9.4 Other Monotonicity Properties

9.4.1 Monotone Solutions

We extend the Lyapunov property to more sophisticated inequalities:

Theorem 9.4.1 *Let $F : X \rightsquigarrow X$ be a Marchaud map,*

$$W : (x, v) \in \mathrm{Graph}(F) \mapsto W(x, v) \in \mathbf{R}$$

a lower semicontinuous function convex with respect to v and $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ a nonnegative extended lower semicontinuous function whose domain is contained in the domain of F.

We assume that there exists a positive constant c such that

$$\begin{cases} \forall\, x \in \mathrm{Dom}(V), \ \ \inf_{v \in F(x)} D_\uparrow V(x)(v) \ \geq \ -c(\|x\| + 1) \\ \\ \forall\, (x, v) \in \mathrm{Graph}(F), \ \ W(x, v) \ \geq \ -c(\|x\| + 1) \end{cases} \tag{9.10}$$

*and that V is a W-*Lyapunov function *in the sense that it is a solution to the contingent Hamilton-Jacobi inequality*

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_\uparrow V(x)(v) + W(x, v) \leq 0 \tag{9.11}$$

Then, for any initial state $x_0 \in \mathrm{Dom}(V)$, there exists a solution to differential inclusion (9.1) satisfying

$$\forall\, t \geq 0, \ \ V(x(t)) - V(x_0) + \int_0^t W(x(\tau), x'(\tau))d\tau \ \leq \ 0 \tag{9.12}$$

Proof — We introduce the set-valued map $G : X \times \mathbf{R} \rightsquigarrow X \times \mathbf{R}$ defined by

$$G(x, w) := \{(v, \lambda) \mid v \in F(x) \ \& \ \lambda \in [-c(\|x\| + 1), -W(x, v)]\}$$

It is clear that the graph of G is closed and its values are convex and nonempty by definition (9.10) of c. Its growth is linear by construction. Furthermore, the epigraph of V is a closed viability domain of G: take $v \in F(x)$ achieving the minimum of the lower semicontinuous function $D_\uparrow V(x)(\cdot) + W(x, \cdot)$ on the compact subset $F(x)$. It satisfies $D_\uparrow V(x)(v) + W(x, v) \leq 0$ by assumption (9.11), so that the pair $(v, -W(x,v))$ belongs to the contingent cone to the epigraph of V at (x,w). This follows from the very definition of the epiderivative when $w := V(x)$ and from Proposition 9.1.3 when $w > V(x)$.

Hence $\mathcal{E}p(V)$ being a closed viability domain of $G(\cdot,\cdot)$, there exists a solution $(x(\cdot), w(\cdot))$ to differential inclusion

$$\text{for almost all } t \geq 0, \quad (x'(t), w'(t)) \in G(x(t), w(t))$$

starting from $(x_0, V(x_0))$ and viable in the epigraph of V. Inequalities

$$w'(\tau) \leq -W(x(\tau), x'(\tau)) \ \& \ V(x(t)) \leq w(t)$$

for almost all $\tau \geq 0$ and all $t \geq 0$ imply by integration from 0 to t inequality (9.12). □

As a consequence, we deduce the following monotonicity theorem:

Theorem 9.4.2 *Let $F : X \rightsquigarrow X$ be a Marchaud map,*

$$W : (x,v) \in \mathrm{Graph}(F) \mapsto W(x,v) \in \mathbf{R}_+$$

a nonnegative continuous function convex with respect to v and $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ a nonnegative extended lower semicontinuous function, continuous on its domain (assumed to be contained in the domain of F). We posit assumptions (9.10) and (9.11).

Then, for any initial sate $x_0 \in \mathrm{Dom}(V)$, there exists a solution to differential inclusion (9.1) satisfying

$$\forall\, t \geq s \geq 0, \quad V(x(t)) - V(x(s)) + \int_s^t W(x(\tau), x'(\tau))d\tau \ \leq \ 0 \quad (9.13)$$

Proof — We associate with $h \to 0+$ the grid jh, $(j = 1, \ldots)$ and we build a solution $x_h(\cdot) \in \mathcal{S}(x_0)$ to differential inclusion (9.4.1) by using Theorem 9.4.1 iteratively: for $j = 0$, we take $x_h(\cdot)$ on the interval $[0,h]$ satisfying (9.12), then we take $x_h(\cdot)$ on $[h, 2h]$ to be a solution starting at $x_h(h)$ and satisfying $V(x_h(t)) - V(x_h(h)) + \int_h^t W(x(\tau), x'(\tau))d\tau \leq 0$, etc.

Since the image $\mathcal{S}(x_0)$ is compact, a subsequence (again denoted) $x_h(\cdot)$ converges to some solution $x(\cdot) \in \mathcal{S}(x_0)$ in the Sobolev space

$W^{1,1}(0,\infty;X;e^{-bt}dt)$. Continuity of W and Proposition 6.3.4 of DIFFERENTIAL INCLUSIONS implies that the functional

$$x(\cdot)\mapsto\int_0^\infty W(x(\tau),x'(\tau))d\tau$$

is lower semicontinuous on $W^{1,1}(0,\infty;X;e^{-bt}dt)$. Hence

$$\int_0^\infty W(x(\tau),x'(\tau))d\tau \;\le\; \liminf_{h\to 0+}\int_0^\infty W(x_h(\tau),x_h'(\tau))d\tau$$

Let $t>s$ be approximated by $j_h h\ge k_h h$ so that

$$V(x_h(j_h h))-V(x_h(k_h h))+\int_{k_h h}^{j_h h} W(x_h(\tau),x_h'(\tau))d\tau \;\le\; 0$$

The function V being continuous on its domain, inequality (9.13) ensues. □

Remark — We refer to Chapter 6 of DIFFERENTIAL INCLUSIONS and above all, to CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène Frankowska for an exposition of the consequences of such an inequality and of generalized solutions (both contingent and viscosity) to Hamilton-Jacobi-Bellman equations.

Let us just mention that F and W being given and satisfying the assumptions of Theorem 9.4.2, the (extended) function V_F defined by

$$V_F(x):=\inf_{x(\cdot)\in\mathcal{S}(x)}\int_0^\infty W(x(\tau),x'(\tau))d\tau$$

is the smallest of the nonnegative solutions to the contingent inequality (9.11). Furthermore, a solution $\widehat{x}(\cdot)\in\mathcal{S}(x_0)$ satisfies inequality (9.13) for V_F if and only if it is a minimal solution to the optimal control problem

$$\int_0^\infty W(\widehat{x}(\tau),\widehat{x}'(\tau))d\tau=\inf_{x(\cdot)\in\mathcal{S}(x)}\int_0^\infty W(x(\tau),x'(\tau))d\tau$$

In this case, it obeys the "optimality principle"

$$\forall\, t\ge 0,\;\; V_F(\widehat{x}(t)) \;=\; \int_t^\infty W(\widehat{x}(\tau),\widehat{x}'(\tau))d\tau \;\;\square$$

For $W\equiv 0$, we obtain the following consequence:

Corollary 9.4.3 *Let $F : X \rightsquigarrow X$ be a Marchaud map and V be a nonnegative lower semicontinuous function satisfying*

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\geq\; -c(\|x\|+1)$$

Then V is a Lyapunov function of F if and only if

$$\inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\leq\; 0$$

Furthermore, if V is continuous on its domain, then, for any initial state $x_0 \in \mathrm{Dom}(V)$, V does not increase along at least one solution $x(\cdot)$ to differential inclusion (9.1).

9.4.2 LaSalle's Theorem

One can find attractors using Lyapunov functions by adapting to the set-valued case a classical result due to Lassale:

Theorem 9.4.4 *Assume that $F : X \rightsquigarrow X$ is a Marchaud map and that V is a nonnegative lower semicontinuous Lyapunov function continuous on its domain and satisfying*

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\geq\; -c(\|x\|+1)$$

We denote by

$$K \;:=\; \left\{ x \in \mathrm{Dom}(F) \mid \sup_{u \in F(x)} D_{\downarrow}V(x)(u) \;\geq\; 0 \right\}$$

If K is closed[5]*, then for any $x_0 \in \mathrm{Dom}(V)$, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ such that its ω-limit set*[6] *is contained in* $\mathrm{Viab}(K)$*:*

$$\omega(x(\cdot)) \;\subset\; \mathrm{Viab}(K)$$

---

[5]This happens whenever F is upper semicontinuous with compact values and $(x,v) \mapsto D_{\uparrow}V(x)(u)$ is upper semicontinuous.

[6]See Definition 3.7.1:

$$\omega(x(\cdot)) \;:=\; \bigcap_{T>0} cl(x([T,\infty[))$$

It is not empty if we assume that V is *inf-compact (or lower semicompact)* (this means that the lower sections $\{x \in X \mid V(x) \leq \lambda\}$ are relatively compact).

Proof — By Corollary 9.4.3, we know that for any $x_0 \in \text{Dom}(V)$, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ such that $t \mapsto V(x(t))$ is nonincreasing and converges to some $a \geq 0$. Let $x_\star \in \omega(x(\cdot))$ be a cluster point of the solution $x(\cdot)$ when $t \to \infty$. There exists a sequence $t_n \to \infty$ such that $x(t_n)$ converges to some $x_\star \in V^{-1}(a)$. The functions $y_n(\cdot)$ defined by $y_n(t) := x(t + t_n)$ belong to $\mathcal{S}(x(t_n))$, so that Theorem 3.5.2 implies that a subsequence (again denoted by) $y_n(\cdot)$ converges to a solution $y(\cdot) \in \mathcal{S}(x_\star)$.

The function V being continuous on its domain, inequalities

$$a \leq V(y_n(t)) = V(x(t+t_n)) \leq V(x(t_n))$$

imply by going to the limit that

$$\forall\, t \geq 0, \;\; V(y(t)) = a \text{ or, equivalently, } (y(t), a) \in \text{Graph}(V)$$

Hence $y(\cdot)$ is viable in $V^{-1}(a)$. The necessary condition of the Viability Theorem implies that

$$\forall\, t \geq 0, \;\; 0 \in DV(y(t))(F(y(t)))$$

By Proposition 9.1.4, we infer that

$$\forall\, t \geq 0, \quad \sup_{u \in F(y(t))} D_\downarrow V(y(t))(u) \geq 0$$

i.e., that $y(\cdot)$ is viable in K. Hence $x_\star = y(0)$ belongs to the viability kernel of K. $\square$

9.4.3 Example: Gradient Inclusions

Consider a locally Lipschitz sleek real-valued function $V : X \mapsto \mathbf{R}$. Since the contingent epiderivative $D_\uparrow V(x)$ is positively homogeneous, convex and lower semicontinuous, it is the support function of the bounded closed convex subset

$$\partial V(x) := \{p \in X^\star \mid \forall\, v \in X, \;\; < p, v > \leq D_\uparrow V(x)(v)\}$$

called the *generalized gradient* $\partial V(x)$. A *gradient inclusion* is the differential inclusion

$$\text{for almost all } t \geq 0, \;\; -x'(t) \in \partial V(x(t))$$

We shall show that a gradient inclusion does have a *slow solution*, i.e., a solution to the differential equation

$$\text{for almost all } t \geq 0, \quad -x'(t) = (\partial V)^{\circ}(x(t)) \subset \partial V(x(t)) \qquad (9.14)$$

(where $\|(\partial V)^{\circ}(x)\| = \min_{v \in \partial V(x)} \|v\|$) along which the function V decreases.

Theorem 9.4.5 *Let us assume that $V : X \mapsto \mathbf{R}$ is a locally Lipschitz episleek real-valued function. Then there exists a slow solution $x(\cdot)$ to the gradient inclusion (9.14) satisfying*

$$\text{for almost all } t \geq 0, \quad D_{\uparrow}V(x(t))(x'(t)) + \|x'(t)\|^2 = 0 \qquad (9.15)$$

Proof — We apply Theorem 9.4.2 above with $F(x) := -\partial V(x)$ and $W(x,v) := \|v\|^2$. Since V is locally Lipschitz, its generalized gradient $\partial V(x)$ is convex and compact. Being episleek, one can prove that the function

$$(x, u) \mapsto D_{\uparrow}V(x)(u) \text{ is upper semicontinuous}$$

Since $D_{\uparrow}V(x)$ is the support function of $\partial V(x)$, we infer that $\partial V(\cdot)$ is upper hemicontinuous. The solution $v \in -\partial V(x)$ to the equation $D_{\uparrow}V(x)(v) + \|v\|^2 \leq 0$ exists and is unique: it is the projection of 0 onto the closed convex $-\partial V(x)$. Therefore, there exists a solution to the gradient inclusion satisfying (9.15), i.e., such that for almost all $t \geq 0$, $-x'(t)$ is the projection of 0 onto $-\partial V(x(t))$. This is a slow solution. We also know that for all $t \geq s \geq 0$,

$$V(t) - V(s) = -\int_s^t \|x'(\tau)\|^2 d\tau$$

and thus, that $V(x(t))$ decreases whenever $x(\cdot)$ is not an equilibrium. □

9.4.4 Feedbacks Regulating Monotone Solutions

The regulation map R_V^W which provides solutions satisfying property (9.13) is defined by

$$R_V^W(x) := \{v \in F(x) \mid D_{\uparrow}V(x)(v) + W(x,v) \leq 0\}$$

Finding closed loop controls, slow solutions, etc., requires that the regulation map is lower semicontinuous with convex values. The following supplies a sufficient condition for this purpose.

Corollary 9.4.6 *We posit the assumptions of Theorem 9.4.2. If F is lower semicontinuous, if $(x,v) \mapsto D_{\uparrow}V(x,v)$ is upper semicontinuous and if*

$$\forall\, x \in \mathrm{Dom}(V),\ \exists\, \bar{v} \in F(x) \mid D_{\uparrow}V(x)(\bar{v}) + W(x,\bar{v}) < 0 \qquad (9.16)$$

then the regulation map is lower semicontinuous and there exists a continuous selection $\tilde{r}$ of R_V^W such that the solutions of differential equation $x'(t) = \tilde{r}(x(t))$ are solutions to differential inclusion (9.1) satisfying property (9.13).

Proof — It is analogous to the proof of Theorem 6.3.2. We first observe that the graph of the set-valued map S defined by $S(x) := \{v \mid D_{\uparrow}V(x)(v)+W(x,v)\}$ is open, then that $x \rightsquigarrow F(x)\cap S(x)$ is lower semicontinuous thanks to the lower semicontinuity of F and thus, that R_V^W is also lower semicontinuous because $R_V^W(x) = \overline{F(x) \cap S(x)}$ and because $F(x) \cap S(x)$ is convex.

Hence the assumptions of Michael's Theorem 6.5.7 are satisfied and there exits a continuous selection of R_V^W. □

Remark — Assumption (9.16) is satisfied for instance when V is both episleek and locally Lipschitz. When it is not satisfied, we can still derive the lower semicontinuity of the regulation map by using Theorem 6.3.1 and the lower semicontinuity of the set-valued map $x \rightsquigarrow T_V^W(x)$ defined by:

$$T_V^W(x) \ := \ \{\, v \in X \mid D_{\uparrow}V(x)(v) + W(x,v) \leq 0 \,\}$$

Proposition 9.4.7 *Let us assume that V is episleek, that the restriction of V to its domain is continuous, that $W(\cdot,\cdot)$ is continuous and convex with respect to the second argument and that F is lower semicontinuous with closed convex values. If for any x, there exists $\bar{v} \in F(x)$ such that*

$$D_{\uparrow}V(x)(\bar{v}) + W(x,\bar{v}) \ < \ 0$$

then $x \rightsquigarrow T_V^W(x)$ is lower semicontinuous at x.

Proof — Let v belong to $T_V^W(x)$ be chosen and a sequence $x_n \in \mathrm{Dom}(D_\uparrow(V))$ converge to x. Since the set-valued map $\mathcal{E}p(D_\uparrow V(\cdot))$ is lower semicontinuous, and since $(v, -W(x,v))$ belongs to $\mathcal{E}p(D_\uparrow V(x))$, there exist a subsequence (again denoted x_n), a sequence v_n converging to v and a sequence $\varepsilon_n \geq 0$ converging to 0 such that

$$(v_n, -W(x_n, v_n) + \varepsilon_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

Let us set $a_0 := -W(x,\bar{v}) - D_\uparrow V(x)(\bar{v}) > 0$. Since by assumption the pair $(\bar{v}, -W(x,\bar{v}) - a_0)$ belongs also to $\mathcal{E}p(D_\uparrow V(x))$, we deduce that there exist sequences $\bar{v}_n$ converging to $\bar{v}$ and $a_n > 0$ converging to a_0 such that

$$(\bar{v}_n, -W(x_n, \bar{v}_n) - a_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

We introduce now $\theta_n := \frac{\varepsilon_n}{2(\varepsilon_n + a_n)} \in [0,1]$ converging to 0, $u_n := (1-\theta_n)v_n + \theta_n \bar{v}_n$ converging to v and $\alpha_n := (1-\theta_n)W_n, v_n) + \theta_n W(x_n, \bar{v}_n) - W(x_n, u_n) \geq 0$ (thanks to the convexity of $W(x_n, \cdot)$). The lower semicontinuity of the contingent cone to the epigraph of V, which is the epigraph of $D_\uparrow V(\cdot)$, implies that these cones are convex. Hence

$$\left\{ \begin{array}{l} (u_n, -W(x_n, u_n) - \varepsilon_n/2 - \alpha_n) \\ = (1-\theta_n)(v_n, -W(x_n, v_n) + \varepsilon_n) + \theta_n(\bar{v}_n, -W(x_n, \bar{v}_n) - a_n) \end{array} \right.$$

belongs to $\mathcal{E}p(D_\uparrow V(x_n))$. This can be written

$$D_\uparrow V(x_n)(u_n) \leq -W(x_n, u_n) - \varepsilon_n - \alpha_n/2 < -W(x_n, u_n)$$

Hence u_n belongs to $T_V^W(x_n)$ and converges to v. $\square$

9.5 Lyapunov Preorders

A given function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ defines the preorder

$$x \succeq y \iff V(x) \leq V(y)$$

i.e., a reflexive ($x \succeq x$ for every x) and transitive ($x \succeq y$ and $y \succeq z$ imply $x \succeq z$) binary relation.

Let us consider more generally a *preorder* $\succeq$ and look for solutions $x(\cdot)$ of differential inclusion (9.1) which do not decrease in the sense that

$$\forall\, t \geq s \geq 0, \;\; x(t) \succeq x(s)$$

For that purpose, it is useful to characterize a preorder by the set-valued map P defined[7] by

$$\forall\, x, \;\; P(x) := \; \{y \mid y \succeq x\}$$

the graph of which is the graph of the preorder.

Conversely, any set-valued map P *reflexive* (in the sense that $x \in P(x)$ for every x) and *transitive* (in the sense that $P(y) \subset P(x)$ for every $y \in P(x)$) defines the preorder $\succeq$ defined by $x \succeq y$ if and only if $x \in P(y)$.

Hence, from now on, we shall represent a preorder by a reflexive and transitive set-valued map.

9.5.1 Monotone solutions with respect to a preorder

Corollary 9.4.3 can be extended to general closed preorders.

Proposition 9.5.1 *Let F be a Marchaud map and P be a preorder with closed graph whose domain is contained in the domain of F.*

The following statements are equivalent:

$$\left\{\begin{array}{ll} i) & \forall\, x \in \mathrm{Dom}(P), \;\; F(x) \cap T_{P(x)}(x) \neq \emptyset \\ \\ ii) & \forall\, (x,y) \in \mathrm{Graph}(P), \;\; F(y) \cap DP(x,y)(0) \neq \emptyset \\ \\ iii) & \forall\, x_0 \in \mathrm{Dom}(P), \; \exists\, x(\cdot) \in \mathcal{S}(x_0) \;\; \textit{such that} \\ & \forall\, t \geq s \geq 0, \;\; x(t) \succeq x(s) \end{array}\right. \tag{9.17}$$

Proof

— Condition (9.17)i) implies (9.17)ii) because, for any $y \in P(x)$, there exists $v \in F(y) \cap T_{P(y)}(y)$, i.e., such that $y + h_n v_n \in P(y) \subset P(x)$ for some sequences $h_n \to 0+$ and $v_n \to v$. Hence,

[7] When the (total) preorder is defined by a function V, the set-valued map P associates with any x the subset $P(x) := \{y \mid V(y) \leq V(x)\}$. Its graph is closed if and only if V is continuous on its domain.

the pair $(x + h_n 0, y + h_n v_n)$ belongs to the graph of P, i.e., $v \in DP(x,y)(0)$.

— Condition (9.17)ii) implies (9.17)iii). First, observing that condition (9.17)ii) means that the graph of P is a closed viability domain of the set-valued map $(x,y) \rightsquigarrow \{0\} \times F(y)$, we infer that for any $(x_0, x_0) \in \text{Graph}(P)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions $x' = 0$ and $y' \in F(y)$ which is viable in Graph(P), i.e., a solution $y(\cdot) \in \mathcal{S}(x_0)$ such that

$$\forall\, t \geq 0,\;\; y(t) \in P(x_0) \tag{9.18}$$

We associate now with $h \to 0+$ the grid jh, $(j = 1, \ldots)$ and we build a solution $x_h(\cdot) \in \mathcal{S}(x_0)$ to differential inclusion (9.1) iteratively: for $j = 0$, we take $x_h(\cdot) = y(\cdot)$ on the interval $[0, h]$ satisfying (9.18), then we take $x_h(\cdot)$ on $[h, 2h]$ to be a solution starting at $x_h(h)$ and satisfying $x_h(t) \in P(x_h(h))$, etc.

Since the image $\mathcal{S}(x_0)$ is compact, a subsequence (again denoted by) x_h converges to some solution $x(\cdot) \in \mathcal{S}(x_0)$ in the Sobolev space $W^{1,1}(0, \infty; X; e^{-bt}dt)$. Let $t > s$ be approximated by $j_h h \geq k_h h$ so that

$$x_h(jh) \;\in\; P(x_h(kh)) \;\text{ or }\; (x_h(kh), x_h(jh) \;\in\; \text{Graph}(P))$$

The graph of P being closed, we infer that $(x(s), x(t)) \in \text{Graph}(P)$, i.e., that $x(t) \in P(x(s))$.

— Condition (9.17)iii) implies (9.17)i) exactly as in the proof of the necessary condition of Haddad's Viability Theorem. □

9.5.2 Comparison of solutions

The same type of proofs yields results dealing with the comparison of solutions to two differential inclusions:

Proposition 9.5.2 *Let $F : X \rightsquigarrow X$ and $G : X \rightsquigarrow X$ be two Marchaud maps and a preorder P with closed graph whose graph is contained in* $\text{Dom}(F) \times \text{Dom}(G)$.

Then the following statements are equivalent:

$$\begin{cases} i) & \forall\,(x,y) \in \mathrm{Graph}(P), \quad G(y) \cap DP(x,y)(F(x)) \neq \emptyset \\ \\ ii) & \forall\, x_0 \in \mathrm{Dom}(P),\ \exists\, x(\cdot) \in \mathcal{S}_F(x_0)\ \ \&\ \ y(\cdot) \in \mathcal{S}_G(x_0) \quad \textit{such that} \\ & \forall\, t \geq 0, \quad y(t) \succeq x(t) \end{cases} \tag{9.19}$$

Proof — Condition (9.19)i) states that the graph of the preorder P is a closed viability domain of the set-valued map

$$(x,y) \in \mathrm{Graph}(P) \rightsquigarrow F(x) \times G(y)$$

and condition (9.19)ii) that it enjoys the viability property. We then apply Viability Theorem 3.3.5. □

Corollary 9.5.3 *Let $F : X \rightsquigarrow X$ and $G : X \rightsquigarrow X$ be two Marchaud maps, $K \subset \mathrm{Dom}(F) \cap \mathrm{Dom}(G)$ be a closed sleek subset and $Q \subset X$ be a closed convex cone*[8] *defining an order relation on X. We assume the transversality condition*

$$\forall\,(x,y) \in K \times K \quad \textit{such that}\ y - x \in Q, \quad T_K(y) - T_K(x) - T_Q(y-x) \ = \ X$$

Then the following statements are equivalent:

$$\begin{cases} i) & \forall\,(x,y) \in K \times K \quad \textit{such that}\ \ y - x \in Q, \\ & 0 \in G(y) - F(x) - T_Q(y-x) \\ \\ ii) & \forall\,(x_0,y_0) \in K \times K \quad \textit{such that}\ \ y_0 - x_0 \in Q, \\ & \exists\, x(\cdot) \in \mathcal{S}_F(x_0)\ \ \&\ \ y(\cdot) \in \mathcal{S}_G(y_0) \quad \textit{such that} \\ & \forall\, t \geq 0,\ y(t) - x(t) \in Q \end{cases}$$

Proof — We define the set-valued map P by

$$\mathrm{Graph}(P) := \{(x,y) \in K \times K \mid y - x \in Q\}$$

Since K is sleek, as well as Q which is convex, we infer from the transversality condition that the contingent derivative of P at (x,y) in the direction u is equal to

$$DP(x,y)(u) := \{v \in T_K(x) \mid v - u \in T_Q(y-x)\} \ \text{ if } \ u \in T_K(x)$$

We then apply Proposition 9.5.2 above. □

[8] We recall that the contingent cone $T_Q(z)$ to Q at z is equal to $\overline{Q + \mathbf{R}z}$.

9.6 Asymptotic Observability of Differential Inclusions

Let us consider a set-valued map F from a finite dimensional vector-space $X := \mathbf{R}^n$ to X and an observation map h from X to another finite dimensional vector-space $Y := \mathbf{R}^p$. We "observe" the evolution

$$\forall\, t \geq 0, \;\; y(t) \;\; := h(x(t))$$

of an unknown solution $x(\cdot)$ to the differential equation (9.1).

The problem is to "simulate asymptotically" at least an unknown state $x(\cdot)$ by a solution $z(\cdot)$ to a control system where the control is the observation of the state

$$z'(t) \;\; = g(z(t), y(t)) \tag{9.20}$$

We shall measure the asymptotic behavior of the error $x(\cdot) - z(\cdot)$ through a nonnegative lower semicontinuous extended function $U : X \mapsto \mathbf{R} \cup \{+\infty\}$ and through a function $w(\cdot)$ from $[0, +\infty]$ to $\mathbf{R}_+$ by inequalities

$$\forall\, t \geq 0, \;\; U(x(t) - z(t)) \;\; \leq \;\; w(t) \tag{9.21}$$

Typically, we would like that $w(t)$ converges to 0 when t goes to $+\infty$ (for instance, $w(t) = ce^{-at}$) and that $U^{-1}(0) = \{0\}$ (for instance, $U(x) := \|x\|^\alpha$) so that we deduce that the error $z(t) - x(t)$ between the observed state $z(t)$ and the unknown state $x(t)$ converges to 0. The bound $w(t)$ which sets an estimate of the measure of the error will be provided by a differential equation

$$w'(t) \;\; = \;\; -\varphi(w(t)), \;\; w(0) \;\; = \;\; U(x(0) - z(0)) \tag{9.22}$$

where $\varphi : [0, +\infty] \;\; \mapsto \;\; \mathbf{R}$ (such as $\varphi(w) = aw$ to obtain exponential decay).

Definition 9.6.1 *Let F, h, φ and U be given. We say that the dynamical system F observed through h is* stabilizable *by g with respect to U and φ if*

$$\forall\, x, z, \quad \inf_{v \in F(x)} D_\uparrow U(x - z)(v - g(z, h(x))) \;\; \leq -\varphi(U(x - z))$$

Proposition 9.6.2 *We assume that F is a Marchaud map, that g, h and φ are continuous with linear growth and that $U : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is contingently epidifferentiable, lower semicontinuous and episleek. If the dynamical system F observed through h is stabilizable by g, then for any initial state x_0 and z_0, there exist solutions $x(\cdot)$ to (9.1), $z(\cdot)$ to (9.20) and $w(\cdot)$ to (9.22) starting at x_0, z_0 and $U(x_0 - z_0)$ respectively and satisfying inequalities (9.21).*

Proof — The conclusion of the theorem amounts to saying that the function $(x,z) \mapsto V(x,z) := U(x-z)$ enjoys the Lyapunov property with respect to φ for the system of differential inclusions

$$\begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & z'(t) = g(z(t), h(x(t))) \end{cases}$$

because, U being episleek, we infer that $D_\uparrow V(x,z)(x',z') = D_\uparrow U(x - z)(x'-z')$. We then apply Theorem 9.2.2. □

We now have to construct stabilizing maps g in various situations.

We begin by providing a first class of examples using (U,φ)-monotone maps. We derive from the definition of U-monotone maps with respect to φ the following obvious statement.

Proposition 9.6.3 *Let us assume that U, φ, f and h being given, we can find a continuous map $c : Y \mapsto X$ such that*

$$\text{the map } x \mapsto c(h(x)) - F(x) \text{ is } (U,\varphi)\text{-monotone}$$

Then for any continuous selection f of F, the single-valued map

$$g(z,y) := f(z) - c(h(z)) + c(y)$$

stabilizes F through h with respect to U and φ.

The problem now is to recognize whether there exist functions U and φ and a map c which make the set-valued map $c \circ h - F$ to be (U,φ)-monotone.

More generally, let us introduce the set-valued map H defined by

$$H(z,x) := \{v \mid \inf_{u \in F(x)} D_\uparrow U(x-z)(u-v) + \varphi(U(x-z)) \leq 0\}$$

The general problem of stabilizing F through h amounts to finding selections g of the set-valued map G defined by

$$\forall\ (z,y), \quad G(z,y) \quad = \quad \bigcap_{h(x)=y} H(z,x)$$

since by construction, such selections are stabilizing f through h. When G is lower semicontinuous with closed convex values, Michael's Theorem guarantees the existence of a continuous selection. Hence, in this case, we can stabilize F, at least in theory, since Michael's Theorem is not constructive.