Chapter 8

Partial Differential Inclusions of Tracking Problems

Introduction

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and the *system of differential inclusions*

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases}$$

We further introduce a set-valued map $H : X \rightsquigarrow Y$, regarded as an *observation map*.

We devote this chapter to several issues related to the following *tracking property*: for every $x_0 \in \mathrm{Dom}(H)$ and every $y_0 \in H(x_0)$, there exist solutions $(x(\cdot), y(\cdot))$ to the system of differential inclusions such that

$$\forall\, t \geq 0, \quad y(t) \in H(x(t))$$

This is a *viability problem*, since we actually look for a solution $(x(\cdot), y(\cdot))$ which remains viable in the graph of the observation map H. So, if the set-valued maps F and G are Marchaud maps and if the graph of H is closed, the Viability Theorem states that the tracking property is equivalent to the fact that the graph of H is a viability domain of $(x,y) \rightsquigarrow F(x,y) \times G(x,y)$.

Recalling that the graph of the contingent derivative $DH(x,y)$ of H at a point (x,y) of its graph is the contingent cone to the graph of H at (x,y), the tracking property is then equivalent to saying that H is a (set-valued contingent) solution to the *system of partial differential inclusions*

$$\forall\ (x,y) \in \mathrm{Graph}(H),\ \ 0\ \in\ DH(x,y)(F(x,y)) - G(x,y)$$

We observe that when F and G are single-valued maps f and g and H is a differentiable single-valued map h, the partial differential inclusion boils down to the more familiar *system of first-order partial differential equations*

$$\forall\, j = 1,\ldots,m,\ \ \sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i(x,h(x)) - g_j(x,h(x))\ =\ 0$$

For special types of systems of differential equations, the graph of such a map h (satisfying additional properties) is called a *center manifold*. Theorems providing the existence of local center manifolds have been widely used for the study of stability near an equilibrium and in control theory.

Since the partial differential inclusion links the three data F, G and H, we can use it in three different ways:

1. — Knowing F and H, find G or selections g of G such that the tracking property holds (observation problem)

2. — Knowing G (regarded as an *exosystem*, following Byrnes & Isidori's terminology) and H, find F or selections f of F such that the tracking property holds (tracking problem)

3. — Knowing F and G, find observation maps H satisfying the tracking property, i.e., solve the above partial differential inclusion.

Furthermore, we can address other specific questions such as:

a) — Find the largest set-valued contingent solution to the partial differential inclusion contained in a given set-valued map (which then, contains all the other ones if any)

b) — Find single-valued contingent solutions h to the partial differential inclusion which then becomes

$$\forall\ x \in K,\ \ 0 \in Dh(x)(F(x,h(x))) - G(x,h(x))$$

In this case, the tracking property states that there exists a solution to the *"reduced" differential inclusion*

$$x'(t) \in F(x(t), h(x(t)))$$

so that $(x(\cdot), y(\cdot) := h(x(\cdot)))$ is a solution to the initial system of differential inclusions starting at $(x_0, h(x_0))$. Knowing h allows us to divide the system by half, so to speak.

It may seem strange to accept set-valued maps as solutions to a system of first-order partial differential inclusions. But this may offer a way to describe shocks by the set-valued character of the solution (which may happen even for maps with smooth graphs, but whose projection leads to set-valued maps.)

Set-valued solutions provide a convenient way to treat hyperbolic problems.

Indeed, looking for "weak" solutions to this partial differential inclusion in Sobolev spaces or other spaces of distributions does not help here since we require *solutions h to be defined through their graph, and thus, solutions which are defined everywhere.*

However, derivatives in the sense of distributions do not offer the unique way to describe weak or generalized solutions.

The contingent derivative $v \mapsto Du(x)(v)$ of single-valued map u at x is obtained by taking *upper graphical limits* when $h \to 0$ of the difference-quotients $v \mapsto \frac{u(x+hv)-u(x)}{h}$ whereas the distributional derivatives are limits of the difference-quotients $x \mapsto \frac{u(x+hv)-u(x)}{h}$ in the space of distributions. In both cases, we use convergences weaker than the pointwise convergence for increasing the possibility for the difference-quotients to converge, and, in doing so, we may lose some properties by passing to these weaker limits. In the first case, the contingent derivative is no longer necessarily a single-valued map, but may be set-valued, whereas in the second case, the derivative may be a distribution. Further, each one of these weaker convergence allows us to differentiate set-valued maps U at (x, y) since one can check that the contingent derivative is the graphical upper limits of the difference quotients $v \rightsquigarrow \frac{U(x+hv)-y}{h}$ and to differentiate a distribution T by taking distributional limits of the difference quotients $\frac{\tau_v^h T - T}{h}$.

We devote the first section to general properties of set-valued contingent solutions to these partial differential inclusions.

We begin by deriving the existence of the largest closed solution contained in a given observation map and by providing a very useful stability theorem stating that graphical upper limits of solutions is still a solution.

The observation and tracking problems are two sides of the same coin because the set-valued map H and its inverse play symmetric roles. This is one of the reasons why we regard a single-valued map as a set-valued map characterized by its graph.

Consider then the observation problem: the idea is to observe solutions of a system $x' \in F(x,y)$ by a system $y' \in G(y)$ where $G : Y \rightsquigarrow Y$ provides simpler dynamics: equilibria, uniform movement, exponential growth, periodic solutions, etc. This allows us to observe complex systems[1] $x' \in F(x)$ in high dimensional spaces X by simpler systems $y' \in G(y)$ — or even better, $y' = g(y)$ — in lower dimensional spaces. We can think of H as an observation map, made of a small number of *sensors* taking into account uncertainty or lack of precision.

For instance, when $G \equiv 0$, we obtain constant observations. Observation maps H such that $F(x) \cap DH(x,y)^{-1}(0) \neq \emptyset$ for all $y \in H(x)$ provide solutions satisfying

$$\forall\, t \geq 0, \;\; x(t) \in H^{-1}(y_0) \;\text{ where }\; y_0 \in H(x_0)$$

In other words, inverse images $H^{-1}(y_0)$ are closed viability domains of F. *Viewed through such an observation map, the system appears to be in equilibrium.*

More generally, if there exists a linear operator $A \in \mathcal{L}(Y,Y)$ such that

$$\forall\, y \in \operatorname{Im}(H),\; \forall\, x \in H^{-1}(y),\;\; F(x) \cap DH(x,y)^{-1}(Ay) \;\neq\; \emptyset$$

[1]We can use this tracking property as a *mathematical metaphor* to model the concept of .... metaphors in epistemology. The simpler system (the model) $y' \in G(y)$ is designed to provide *explanations* of the evolution of the unknown system $x' \in F(x)$ and the tracking property means that the *metaphor* H is valid (*non falsifiable.*) Evolution of knowledge amounts to "increasing" the observation space Y and to *modifying* the system G (replacing the model) and/or the observation map H (obtain more experimental data), checking that the tracking property (the validity or the consistency of the metaphor) is maintained.

then we obtain solutions $x(\cdot)$ satisfying the time-dependent viability condition

$$\forall\, t \geq 0,\ \ x(t) \in H^{-1}(e^{At}y_0) \text{ where } y_0 \in H(x_0)$$

so that we can use the exhaustive knowledge of linear differential equations to derive behavioral properties of the solutions to the original system.

But instead of checking whether such given dynamics G satisfy the tracking property, we can look for systematic ways of finding them. For that purpose, it is natural to appeal to the selection procedures studied in section 4 of Chapter 6.

For instance, the most attractive idea is to choose the minimal selection $(x, y) \mapsto g^\circ(x, y)$ of the set-valued map

$$(x, y) \rightsquigarrow DH(x, y)(F(x, y))$$

which, by construction, satisfies the partial differential inclusion. We shall prove that under adequate assumptions, the system

$$\begin{cases} i) & x'(t) \in F(x(t), y(t)) \\ ii) & y'(t) = g^\circ(x(t), y(t)) \end{cases}$$

has solutions (satisfying automatically the tracking property) even though the minimal selection g° is not necessarily continuous.

The drawback of the minimal selection and the other ones of the same family is that g° depends upon x. We would like to obtain single-valued dynamics g independent of x. They are selections of the set-valued map G_H defined by

$$G_H(y) := \bigcap_{x \in H^{-1}(y)} DH(x, y)(F(x, y))$$

We must appeal to Michael's Continuous Selection Theorem to find continuous selections g of this map, so that the system

$$\begin{cases} i) & x'(t) \in F(x(t), y(t)) \\ ii) & y'(t) = g(y(t)) \end{cases}$$

has solutions satisfying the tracking property.

When $F : X \rightsquigarrow X$ does not depend upon y, the size of the set-valued map G_H *measures in some sense a degree of inadequacy of the observation of the system* $x' \in F(x)$ through H, because the larger the images of G_H, the more dynamics g tracking an evolution of the differential inclusion.

Tracking problems, which are the topic of the second section, are intimately related to the observation problem: Here, the system $y' \in G(y)$, called the *exosystem*, is given. The problem is to *regulate the system* $x'(t) \in F(x(t), y(t))$ for finding solutions $x(\cdot)$ *that match the solutions to the exosystem* $y'(t) \in G(y(t))$ in the sense that $y(t) \in H(x(t))$, or, more to the point, $x(t) \in H^{-1}(y(t))$.

Decentralization of control systems and *decoupling properties* are instances of this problem.

An instance of decentralization can be described as follows: We take $X := Y^n$, $F(x) := \prod_{i=1}^n F_i(x_i)$, and the viability subset is given in the form

$$K := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^n x_i \in M\}$$

so that we observe the individual evolutions $x_i'(t) \in F_i(x_i(t))$ through their sum $y(t) := \sum_{i=1}^n x_i(t)$. Decentralizing the system means solving

— first a differential inclusion $y'(t) \in G(y(t))$ providing a solution $y(\cdot)$ viable in the viability subset $M \subset Y$, and

— second, find solutions to the differential inclusions $x_i'(t) \in F_i(x_i(t))$ satisfying the (time-dependent) viability condition

$$\sum_{i=1}^n x_i(t) = y(t)$$

a condition which does not depend any more on M.

Hierarchical decomposition happens whenever the observation map is a composition product of several maps determining the *successive levels of the hierarchy.* The evolution at each level is linked to the state of the lower level and is regulated by controls depending upon the evolution of state-control of the lower level.

The third section is devoted to existence and comparison theorems of invariant manifolds.

We extend Hadamard's formula of solutions to linear hyperbolic differential equations to the set-valued case. We shall prove the existence of one set-valued contingent solutions $H_\star$ to the *decomposable system*

$$\forall\ (x,y) \in \text{Graph}(H_\star),\ \ Ay \in DH_\star(x,y)(\Phi(x)) - \Psi(x)$$

where $\Phi : X \rightsquigarrow X$ and $\Psi : X \rightsquigarrow Y$ are two Marchaud maps. If we denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x, then the set-valued map $H_\star : X \rightsquigarrow Y$ defined by

$$\forall\ x \in X,\ \ H_\star(x)\ :=\ -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

is the largest contingent solution with linear growth to this partial differential inclusion when $\lambda := \inf_{\|x\|=1}\langle Ax, x\rangle > 0$ is large enough. We also show that it is Lipschitz whenever Φ and Ψ are Lipschitz and compare the solutions associated with maps Φ_i and Ψ_i $(i = 1,2)$.

We then turn our attention to partial differential inclusions of the form

$$\forall\ x \in X,\ \ Ah(x)\ \in\ Dh(x)(f(x,h(x))) - G(x,h(x))$$

when $\lambda > 0$ is large enough, $f : X \times Y \mapsto X$ is Lipschitz, $G : X \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and satisfies the growth condition

$$\forall\ x,y,\ \ \|G(x,y)\| \leq c(1 + \|y\|)$$

When G is single-valued, we obtain a global *Center Manifold Theorem*, stating the existence and uniqueness of an invariant manifold for systems of differential equations with Lipschitz right-hand sides.

We end this section with comparison theorems between single-valued and set-valued solutions to such partial differential inclusions.

We characterize in the fourth section the single-valued contingent solutions to partial differential inclusions as minimizers of a functional. i.e., we provide a *variational principle*.

We apply in the sixth section some of the results obtained so far to the existence of closed-loop controls regulating smooth viable solutions to a control system. In chapter Chapter 7, we saw that closed-loop controls $r : K \mapsto Z$ regulating smooth solutions to a control system (U, f):

$$\begin{cases} i) & \text{for almost all } t, \ x'(t) = f(x(t), u(t)) \\ ii) & \text{where } u(t) \in U(x(t)) \end{cases}$$

in the sense that *for any* $x_0 \in K$, *there exists a solution to the differential equation* $x'(t) = f(x(t), r(x(t)))$ *starting at* x_0 *such that* $u(t) := r(x(t)) \in U(x(t))$ *is absolutely continuous and satisfies the growth condition*

$$\|u'(t)\| \leq \varphi(x(t), u(t))$$

for almost all t.

They are solutions to the following partial differential inclusion

$$\forall\, x \in K, \ \ 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraints

$$\forall\, x \in K, \ \ r(x) \in U(x)$$

This is a tracking problem, where the closed loop control is regarded as an observation map of a system where $F(x, u) := f(x, u)$ and $G(x, u) := \varphi(x, u)B$.

8.1 The Tracking Property

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and a set-valued map $H : X \rightsquigarrow Y$, regarded as (and often called) the *observation map*.

Definition 8.1.1 *We shall say that* F, G *and* H *satisfy the* tracking property *if for any initial state* $(x_0, y_0) \in \text{Graph}(H)$, *there exists at least one solution* $(x(\cdot), y(\cdot))$ *to the* system of differential inclusions

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases} \tag{8.1}$$

starting at (x_0, y_0)*, defined on* $[0, \infty[$ *and satisfying*

$$\forall\, t \geq 0, \;\; y(t) \;\in\; H(x(t))$$

8.1.1 Characterization of the Tracking Property

We now consider the first-order system of the *partial differential inclusion*

$$\forall\, (x,y) \;\in\; \mathrm{Graph}(H), \;\; 0 \in DH(x,y)(F(x,y)) - G(x,y) \qquad (8.2)$$

Definition 8.1.2 *We shall say that a set-valued map* $H : X \rightsquigarrow Y$ *satisfying (8.2) is a solution to the* partial differential inclusion *if its graph is a* closed *subset of* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$*.*

When $H = h : \mathrm{Dom}(h) \mapsto Y$ *is a single-valued map with closed graph contained in* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$*, the partial differential inclusion (8.2) becomes*

$$\forall\, x \in \mathrm{Dom}(h), \;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) \qquad (8.3)$$

We deduce at once from the Viability Theorem 3.3.5 and Theorem 4.1.2 the following:

Theorem 8.1.3 *Let us assume that* $F : X \times Y \rightsquigarrow X$*,* $G : X \times Y \rightsquigarrow Y$ *are Marchaud maps and that the graph of the set-valued map* H *is a closed subset of* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$*.*

1. — The triple (F, G, H) *enjoys the tracking property if and only if* H *is a solution to the* partial differential inclusion *(8.2).*

2. — There exists a largest *solution* $H_\star$ *to the partial differential inclusion (8.2) contained in* H*. It enjoys the following property: whenever an initial state* $y_0 \in H(x_0)$ *does not belong to* $H_\star(x_0)$*, then all solutions* $(x(\cdot), y(\cdot))$ *to the system of differential inclusions (8.1) satisfy*

$$\begin{cases} i) & \forall\, t \geq 0, \quad y(t) \;\notin\; H_\star(x(t)) \text{ as long as } y(t) \;\in\; H(x(t)) \\ ii) & \exists\, T > 0 \quad \textit{such that } y(T) \;\notin\; H(x(T)) \end{cases} \qquad (8.4)$$

3. — Any closed set-valued map $L \subset H_\star$ *is contained in a minimal set-valued map satisfying the tracking property.*

Naturally, the graph of $H_\star$ is the viability kernel of the graph of H.

We now translate in this framework the useful Stability Theorem 3.6.5. We recall that the graph of the *graphical upper limit* $H^\sharp$ of a sequence of set-valued maps $H_n : X \rightsquigarrow Y$ is by definition the graph of the upper limit of the graphs of the maps H_n. (See Chapter 7 of SET-VALUED ANALYSIS.)

Theorem 8.1.4 (Stability) *Let us consider a sequence of Marchaud maps $F_n : X \times Y \rightsquigarrow X$, $G_n : X \times Y \rightsquigarrow Y$ with uniform linear growth*[2] *and their graphical upper limit $F^\sharp$ and $G^\sharp$.*

Consider also a sequence of set-valued maps $H_n : X \rightsquigarrow Y$, solutions to the partial differential inclusions

$$\forall (x,y) \in \text{Graph}(H_n), \quad 0 \in DH_n(x,y)(F_n(x,y)) - G_n(x,y) \quad (8.5)$$

Then the graphical upper limit $H^\sharp$ of the solutions H_n is a solution to

$$\forall (x,y) \in \text{Graph}(H^\sharp), \quad 0 \in DH^\sharp(x,y)(\overline{co}F^\sharp(x,y)) - \overline{co}(G^\sharp(x,y)) \quad (8.6)$$

In particular, if the set-valued maps F_n and G_n converge graphically to maps F and G respectively, then the graphical upper limit $H^\sharp$ of the solutions H_n is a solution of (8.2).

It is an obvious consequence of Theorem 3.6.5.

We recall that graphical convergence of single-valued maps is weaker than pointwise convergence. This is why graphical limits of single-valued maps which are converging pointwise may well be set-valued.

Therefore, for single-valued solutions, the stability property implies the following statement: *Let h_n be single-valued solutions to the*

[2] In the sense that there exists a constant $c > 0$ such that

$$\sup_{n \geq 0} \max\left(\|F_n(x,y)\|, \|G_n(x,y)\|\right) \leq c(\|x\| + \|y\| + 1)$$

contingent partial differential inclusion (8.5). Then their graphical upper limit $h^\sharp$ *is a* (possibly set-valued) solution *to (8.6).*

Although set-valued solutions to first-order systems of partial differential inclusions make sense to describe shock and other phenomena, we may still need sufficient conditions for an upper graphical limit of single-valued maps to remain single-valued. This is the case for instance when *a sequence of continuous solutions* h_n *to the partial differential inclusion (8.5) is equicontinuous and converges pointwise to a function h. Then*[3] *h is a single-valued solution to (8.6).*

Remark — We could also introduce two other kinds of *partial differential inclusions*:

$$\forall\, (x,y) \in \text{Graph}(H),\;\; DH(x,y)(F(x,y)) \;\subset\; G(x,y)$$

and

$$\forall\, (x,y) \in \text{Graph}(H),\;\; G(x,y) \;\subset\; \bigcap_{u \in F(x,y)} DH(x,y)(u)$$

The first inclusion implies obviously that any solution $(x(\cdot), y(\cdot))$ to the viability problem

$$x'(t) \;\in\; F(x(t), y(t)) \;\;\&\;\; x(t) \;\in\; H^{-1}(y(t))$$

parametrized by the absolutely continuous function $y(\cdot)$ is a solution to the differential inclusion

$$y'(t) \;\in\; G(x(t), y(t))$$

The second inclusion states that the graph of H is an invariance domain of the set-valued map $F \times G$. Assume that F and G are Lipschitz with compact values on a neighborhood of the graph of F. By the Invariance Theorem 5.3.4, the second inclusion is equivalent to the following strong tracking property:

For any initial state $(x_0, y_0) \in \text{Graph}(H)$, *every solution* $(x(\cdot), y(\cdot))$ to the system of differential inclusions (8.1) starting at (x_0, y_0) satisfies $y(t) \in H(x(t))$ for all $t \geq 0$. □

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[3]Indeed, a pointwise limit h of single-valued maps h_n is a selection of the graphical upper limit of the h_n. The latter is equal to h when h_n remain in an equicontinuous subset.

8.1.2 Construction of trackers

Any selection of the map Φ defined by

$$\forall\ (x,y) \in \mathrm{Graph}(H), \quad \Phi(x,y) \ := \ DH(x,y)(F(x,y))$$

provides dynamics that satisfy the tracking property, provided that the system has solutions.

Naturally, we can obtain such selections by using selection procedures $G := S_\Phi$ of Φ (see Definition 6.5.2) that have convex values and linear growth, since the solutions to the system

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ \in \ S_\Phi(x(t),y(t)) \end{cases}$$

satisfying the tracking property (which exist by Theorem 8.1.3) are solutions to the system

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ \in \ S(\Phi)(x(t),y(t)) := \Phi(x(t),y(t)) \cap S_\Phi(x(t),y(t)) \end{cases}$$

Let us mention only the case of the minimal selection g° of Φ defined by

$$\begin{cases} i) & g^\circ(x,y) \ \in \ DH(x,y)(F(x,y)) \\ ii) & \|g^\circ(x,y)\| \ = \ \inf_{v\in DH(x,y)(F(x,y))} \|v\| \end{cases}$$

Theorem 8.1.5 *Assume that the Marchaud map F is continuous and that H is a sleek closed set-valued map satisfying, for some constant $c > 0$,*

$$\forall\ (x,y) \in \mathrm{Graph}(H), \quad \|DH(x,y)\| \ \leq c$$

where $\|DH(x,y)\| := \sup_{\|u\|\leq 1} \inf_{v\in DH(x,y)(u)} \|v\|$ denotes the norm of the closed convex process $DH(x,y)$. Then the system observed by the minimal selection g° of $DH(\cdot,\cdot)(F(\cdot,\cdot))$

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ = \ g^\circ(x(t),y(t)) \end{cases}$$

has solutions enjoying the tracking property.

Proof — By Theorem 7.1.3, the set-valued map $(x, y, u) \rightsquigarrow DH(x, y)(u)$ is lower semicontinuous. We deduce then from the lower semicontinuity of F that the set-valued map Φ is also lower semicontinuous. Since $DH(x, y)$ is a convex process, it maps the convex subset $F(x, y)$ to the convex subset $\Phi(x, y)$. Therefore, Φ being lower semicontinuous with closed convex images, its minimal selection S_{Φ}° defined by

$$S_{\Phi}^{\circ}(x, y) := \{v \in Y \mid \|v\| \leq d(0, \Phi(x, y))\}$$

is closed with convex values. Furthermore,

$$\|g^{\circ}(x, y)\| \leq c\|F(x, y)\| \leq c'(\|x\| + \|y\| + 1)$$

since $\|DH(x, y)\| \leq c$ and the growth of F is linear. Then the system

$$\begin{cases} i) & x'(t) \in F(x(t), y(t)) \\ ii) & y'(t) \in S_{\Phi}^{\circ}(x(t), y(t)) \cap c'(\|x(t)\| + \|y(t)\| + 1)B \end{cases}$$

has solutions enjoying the tracking property by Theorem 8.1.3. Therefore for almost all $t \geq 0$,

$$y'(t) \in \Phi(x(t), y(t)) \cap S_{\Phi}^{\circ}(x(t), y(t)) = g^{\circ}(x(t), y(t)) \quad \square$$

8.1.3 The Observation Problem

We consider the important case when $G : Y \rightsquigarrow Y$ does not depend upon x. Hence the partial differential inclusion becomes

$$\forall\, x \in \mathrm{Dom}(H), \forall\; y \in H(x), \quad G(y) \cap DH(x, y)(F(x, y)) \neq \emptyset$$

The behavior of observations of some solutions to the differential inclusion $x' \in F(x, y)$ may be given as the prescribed behavior of solutions to differential equations $y' = g(y)$, where g is a selection of

$$g(y) \in \bigcap_{x \in H^{-1}(y)} DH(x, y)(DF(x, y))$$

In the case when the differential equation $y' = g(y)$ has a unique solution $r(t)y_0$ starting from y_0, the solution $x(\cdot)$ satisfies the condition

$$\forall\, t \geq 0, \;\; x(t) \in H^{-1}(r(t)y(0)), \;\; x(0) \in H^{-1}(y(0))$$

When g is a linear operator $G \in \mathcal{L}(Y,Y)$, it can be written

$$\forall\, t \geq 0,\;\; x(t) \in H^{-1}(e^{Gt}y(0)),\;\; x(0) \in H^{-1}(y(0))$$

When $H \equiv h$ is a single-valued differentiable map, then the map G_H can be written

$$G_H(y) \;:=\; \bigcap_{h(x)=y} h'(x)F(x,y)$$

and a single-valued map g is a selection of G_H if and only if

$$\forall\, x \in \mathrm{Dom}(H),\;\; g(h(x)) \;\in\; h'(x)F(x,h(x))$$

The problem arises as to how to construct such maps g. But before studying it in the next subsection, we consider the particular case when $G \equiv 0$. Therefore, if F is a Marchaud map, H enjoys the tracking property if and only if it is a solution to

$$\forall\, (x,y) \in \mathrm{Graph}(H),\;\; 0 \in DH(x,y)(F(x,y)) \qquad (8.7)$$

Since the tracking property of H amounts to saying that each subset $H^{-1}(y)$ enjoys the viability property for $F(\cdot,y)$, we observe that this condition is also equivalent to condition

$$\forall\, y \in \mathrm{Im}(H),\, \forall\, x \in H^{-1}(y),\;\; F(x,y) \cap T_{H^{-1}(y)}(x) \;\neq\; \emptyset$$

We may say that such a set-valued map H is an *energy map* of F. □

In the general case, the evolution with respect to a parameter y of the viability kernels of the closed subsets $H^{-1}(y)$ under the set-valued map $F(\cdot,y)$ is described in terms of $H_\star$:

Proposition 8.1.6 *Let $F : X \times Y \rightsquigarrow X$ be a Marchaud map and $H : X \rightsquigarrow Y$ be a closed set-valued map. Then there exists a largest solution $H_\star : X \rightsquigarrow Y$ contained in H to (8.7).*

The inverse images $H_\star^{-1}(y)$ are the viability kernels of the subsets $H^{-1}(y)$ under the maps $F(\cdot,y)$:

$$\mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y)) \;=\; H_\star^{-1}(y)$$

The graphical upper limit of energy maps is still an energy map.

Therefore *the graph of the map* $y \rightsquigarrow \text{Viab}_{F(\cdot,y)}(H^{-1}(y))$ *is closed, and thus upper semicontinuous whenever the domain of* H *is bounded.*

When the observation map H is a single-valued map h, the partial differential inclusion becomes[4]:

$$\forall\, x,\ \exists\, u \in F(x, h(x)) \ \text{ such that } \ 0 \ \in \ Dh(x)(u)$$

The largest closed energy map $h_\star$ contained in h is necessarily the restriction of h to a closed subset $K_\star$ of the domain of h. Therefore, for all $y \in \ \text{Im}(h)$, $K_\star \cap h^{-1}(y)$ *is the viability kernel of* $h^{-1}(y)$. The restriction of the differential inclusion $x'(t) \in F(x(t), y)$ to the viability kernel of $h^{-1}(y)$ is what Byrnes and Isidori call *zero dynamics of* F (in the framework of smooth nonlinear control systems.)

Remark — The Equilibrium Map. We associate with each parameter y the set

$$E(y) \ := \ \{x \in H^{-1}(y) \mid 0 \in F(x, y)\}$$

of the equilibria of $F(\cdot, y)$ viable in $H^{-1}(y)$. We say that $E : Y \rightsquigarrow X$ is the *equilibrium map*.

We can derive some information on this equilibrium map from its derivative, which we can compute easily:

Theorem 8.1.7 *Assume that both* $H : X \rightsquigarrow Y$ *and* $F : X \times Y \rightsquigarrow X$ *are closed and sleek and that*

$$\begin{cases} \forall\, (x, y) \in \text{Graph}(H), \ \forall\, (u, v, w) \ \in X \times Y \times X, \\ \exists\, v_1 \in DH(x, y)(u_1) \ \ \textit{such that} \ \ w \in DF(x, y, 0)(u + u_1, v + v_1) \end{cases}$$

---

[4]When $h : X \mapsto \mathbf{R}$ is a continuous real function, we shall see in Chapter 9, Proposition 9.1.4 below, that the values

$$Df(x)(u) = [D_\uparrow f(x)(u), D_\downarrow f(x)(u)]$$

of the contingent derivative are intervals bounded by the epi and hypo contingent derivatives, so that the previous equation becomes a system of two contingent inequalities:

$$\forall\, x, \ \exists\, u \in F(x, h(x)) \ \text{ such that } \ D_\uparrow f(x)(u) \ \leq \ 0 \ \leq \ D_\downarrow f(x)(u)$$

See Hélène Frankowska's CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS for an exhaustive study of contingent inequalities in the framework of Hamilton-Jacobi equations.

Then the contingent derivative of the equilibrium map is the equilibrium map of the contingent derivative of F*:*

$$u \in DE(y,x)(v) \Longleftrightarrow u \in DH(x,y)^{-1}(v) \ \ \& \ \ 0 \in DF(x,y,0)(u,v)$$

Proof — We observe that by setting $\pi(x,y) := (x,y,0)$, the graph of E^{-1} can be written:

$$\text{Graph}(E^{-1}) \ := \ \text{Graph}(H) \cap \pi^{-1}(\text{Graph}(F))$$

and we apply formula (5) of Table 5.2, which states that if the transversality condition: for all $(x,y) \in \text{Graph}(E^{-1})$,

$$\pi\left(T_{\text{Graph}(H)}(x,y)\right) - T_{\text{Graph}(F)}(\pi(x,y)) \ = \ X \times Y \times X$$

holds true, then

$$T_{\text{Graph}(E^{-1})}(x,y) \ := \ T_{\text{Graph}(H)}(x,y) \cap \pi^{-1}\left(T_{\text{Graph}(F)}(\pi(x,y))\right)$$

Recalling that the contingent cone to the graph of a set-valued map is the graph of its contingent derivative, the assumption of our proposition implies the transversality condition. We then observe that the latter equality yields the conclusion of the proposition. □

Using the inverse function and the localization theorems presented in section 5.4 of SET-VALUED ANALYSIS, we can derive the following information. For instance, set

$$Q(y,x) \ := \ \left\{u \in DH(x,y)^{-1}(0) \mid 0 \ \in \ DF(x,y,0)(u,0)\right\}$$

Then, for any equilibrium $x \in E(y)$ and any closed cone P satisfying $P \cap Q(y,x) = \{0\}$, there exists $\varepsilon > 0$ such that

$$E(y) \cap (x + \varepsilon(P \cap B)) \ = \ \{x\}$$

where B denotes the unit ball. In particular, *an equilibrium* $x \in E(y)$ *is locally unique whenever* $0 \in DH(x,y)^{-1}(0)$ *is the unique equilibrium of* $DF(x,y,0)(\cdot,0)$.

Furthermore, if the set $E(y)$ of equilibria is convex, then

$$E(y) \ \subset \ x + Q(y,x) \ \square$$

8.1.4 Construction of Observers

These maps g are selections of the map $G_H : Y \rightsquigarrow Y$ defined by

$$G_H(y) := \bigcap_{x \in H^{-1}(y)} (DH(x,y)(F(x,y)))$$

The set-valued map G_H *measures so to speak a degree of disorder of the system* $x' \in F(x,y)$, because the larger the images of G_H, the more observed dynamics g tracking an evolution of the differential inclusion.

By using Michael's Continuous Selection Theorem, we obtain the following

Theorem 8.1.8 *Assume that the set-valued map F is continuous with convex compact images and linear growth, that H is a sleek closed set-valued map the domain of which is bounded and that there exists a constant $c > 0$ such that*

$$\forall\ (x,y) \in \text{Graph}(H), \quad \|DH(x,y)\| \ \leq c$$

Assume also that there exist constants $\delta > 0$ and $\gamma > 0$ such that, for any map $x \mapsto e(x) \in \gamma B$,

$$\delta B \cap \bigcap_{x \in H^{-1}(y)} (DH(x,y)(F(x,y)) - e(x)) \ \neq \ \emptyset$$

Then there exists a continuous map g such that the solutions of

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ = \ g(y(t)) \end{cases}$$

enjoy the tracking property for any initial state $(x_0, y_0) \in \text{Graph}(H)$.

Proof — The proof of Theorem 8.1.5 showed that the set-valued map Φ is lower semicontinuous with compact convex images. Furthermore, the set-valued map H^{-1} is upper semicontinuous with compact images since we assumed the domain of H bounded. Then the lower semicontinuity criterion Theorem 6.3.3 implies that the set-valued map G_H is also lower semicontinuous with compact convex images. Therefore there exists a continuous selection g of G_H, so that the above system does have solutions viable in the graph of H. □

8.2 The Tracking Problem

8.2.1 Tracking Control Systems

Let $H : X \rightsquigarrow Y$ be an observation map. We consider two control systems

$$\begin{cases} i) & \text{for almost all } t \geq 0, \ x'(t) = f(x(t), u(t)) \\ ii) & \text{where } u(t) \in U(x(t)) \end{cases} \tag{8.8}$$

and

$$\begin{cases} i) & \text{for almost all } t \geq 0, \ y'(t) = g(y(t), v(t)) \\ ii) & \text{where } v(t) \in V(y(t)) \end{cases} \tag{8.9}$$

on the state and observation spaces respectively, where $U : X \rightsquigarrow Z_X$ and $V : Y \rightsquigarrow Z_X$ map X and Y to the control spaces Z_X and Z_Y and where $f : \text{Graph}(U) \mapsto X$ and $g : \text{Graph}(V) \mapsto Y$.

We introduce the set-valued maps $R_H(x, y) : Z_Y \rightsquigarrow Z_X$ defined by

$$R_H(x, y; v) := \{u \in U(x) \mid f(x, u) \in DH(x, y)^{-1}(g(y, v))\}$$

if $v \in V(y)$ and $R_H(x, y; v) := \emptyset$ if $v \notin V(y)$.

Corollary 8.2.1 *Assume that the set-valued maps U and V are Marchaud maps and that the maps f and g are continuous, affine with respect to the controls and with linear growth. The set-valued map H enjoys the tracking property if and only if*

$$\forall \, (x, y) \in \text{Graph}(H), \ \ \text{Graph}(R_H(x, y)) \neq \emptyset$$

Then the system is regulated by the regulation law

$$\text{for almost all } t \geq 0, \ \ u(t) \in R_H(x(t), y(t); v(t))$$

When $H \equiv h$ is single-valued and differentiable and when we set $f(x, u) := c(x) + g(x)u$ and $g(y, v) := d(y) + e(y)v$ where $g(x)\cdot$ and $e(y)\cdot$ are linear operators, we obtain the formula

$$R_h(x; v) := U(x) \cap (h'(x)g(x))^{-1}(d(h(x)) - h'(x)c(x) + e(h(x)v))$$

8.2.2 Decentralization of a control system

We assume that the viability set of the control system (8.8) is defined by constraints of the form $K := L \cap h^{-1}(M)$ where

$$\left\{ \begin{array}{ll} i) & L \subset X \text{ and } M \subset Y \text{ are closed and sleek} \\ ii) & h \text{ is a } \mathcal{C}^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K := L \cap h^{-1}(M), \;\; Y = h'(x)T_L(x) - T_M(h(x)) \end{array} \right. \tag{8.10}$$

We associate with the two systems (8.8), (8.9) the *decoupled viability constraints*

$$\left\{ \begin{array}{lll} i) & \forall t \geq 0, & x(t) \in L \\ ii) & \forall t \geq 0, & h(x(t)) = y(t) \\ iii) & \forall t \geq 0, & y(t) \in M \end{array} \right. \tag{8.11}$$

It is obvious that the *first component* $x(\cdot)$ of any pair of solutions $(x(\cdot), y(\cdot))$ to the system ((8.8),(8.9)) satisfying viability constraints (8.11) is a solution to the initial control system (8.8) viable in the set K defined by (8.10)iii).

On the other hand, solutions to the system (8.8) viable in K can be obtained in two steps:

— First, find a solution $y(\cdot)$ to the control system (8.9) *viable in M*

and then,

— second, find a solution $x(\cdot)$ the control system (8.8) satisfying the viability constraints

$$\left\{ \begin{array}{lll} i) & \forall t \geq 0, & x(t) \in L \\ ii) & \forall t \geq 0, & h(x(t)) = y(t) \end{array} \right. \tag{8.12}$$

which does no longer involve the subset $M \subset Y$ of constraints.

This decentralization problem is a particular case of the observation problem for the set-valued map H defined by

$$H(x) := \left\{ \begin{array}{ll} h(x) & \text{if } x \in L \;\&\; h(x) \in M \\ \emptyset & \text{if not} \end{array} \right.$$

whose contingent derivative is equal under assumptions (8.10) to

$$DH(x)(u) := \begin{cases} h'(x)u & \text{if } u \in T_L(x) \ \& \ h'(x)u \in T_M(h(x)) \\ \emptyset & \text{if not} \end{cases}$$

We know that the regulation map of the initial system (8.8), (8.9) on the subset K defined by (8.10) is equal to

$$R_K(x) = \{u \in U(x) \cap T_L(x) \mid h'(x)f(x,u) \in T_M(h(x))\}$$

The regulation map of the projected control system (8.9) on the subset M is defined by

$$R_M(y) = \{v \in V(y) \mid g(y,v) \in T_M(y)\}$$

We introduce now the set-valued map R_H which is equal to

$$R_H(x,y;v) := \{u \in U(x) \cap T_L(x) \mid h'(x)f(x,u) = g(y,v)\}$$

We observe that

$$\forall x \in K, \ \ R_H(x, h(x); R_M(h(x))) \subset R_K(x)$$

The regulation map regulating solutions to the system ((8.8),(8.9)) satisfying viability conditions (8.11) is equal to

$$x \rightsquigarrow R_H(x, h(x); R_M(h(x)))$$

Therefore, the regulation law feeding back the controls from the solutions are given by: for almost all $t \geq 0$

$$\begin{cases} i) & v(t) \in R_M(y(t)) \\ ii) & u(t) \in R_H(x(t); v(t)) \end{cases}$$

The first law regulates the solutions to the control system (8.9) viable in M and the second regulates the solutions to the control system (8.8) satisfying the viability constraints (8.12).

Remark — The reason why this property is called decentralization lies in the particular case when $X := Y^n$, when $h(x) := \sum_{i=1}^n x_i$ and when the control system (8.8) is

$$\forall\, i = 1, \ldots, n, \ \ x_i'(t) = f_i(x_i(t), u(t)) \ \text{ where } \ u(t) \in U_i(x_i(t))$$

constrained by

$$\forall\, i = 1, \ldots, n, \;\; x_i(t) \in L_i \;\; \& \;\; \sum_{i=1}^{n} x_i(t) \in M$$

We introduce the regulation map R_H defined by

$$\begin{array}{l} R_H(x_1, \ldots, x_n, y; v) \\ := \; \{u \in \bigcap_{i=1}^{n} (U_i(x_i) \cap T_{L_i}(x_i)) \mid \sum_{i=1}^{n} f_i(x_i, u) = g(y, v)\} \end{array}$$

This system can be decentralized first by solving the viability problem for system (8.9) in the viability set M through the regulation law $v(t) \in R_M(y(t))$.

This being done, the state-control $(y(\cdot), v(\cdot))$ being known, it remains in a second step to study the evolution of the n control systems

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \;=\; f_i(x_i, u(t))$$

through the regulation law

$$u(t) \;\in\; R_H(x_1(t), \ldots, x_n(t), \sum_{i=1}^{n} x_i(t); v(t)) \;\; \square$$

Economic Interpretation — We can illustrate this problem with an economic interpretation: the state $x := (x_1, \ldots, x_n)$ describes an allocation of a commodity $y \in M$ among n consumers. The subsets L_i represent the consumptions sets of each consumer and the subset M the set of available commodities. The control u plays the role of the price system of the consumptions goods and v the price of the production goods. Differential equations $x_i' = f_i(x_i, u)$ represent the behavior of each consumer in terms of the consumption price and $y' = g(y, v)$ the evolution of the production process.

The decentralization process allows us to decouple the production problem and the consumption problem. $\square$

8.2.3 Hierarchical Decomposition Property

For simplicity, we restrict ourself here to the case when *the observation map $H \equiv h := h_2 \circ h_1$ is the product of two differentiable single-valued maps $h_1 : X \mapsto Y_1$* and *$h_2 : Y_1 \mapsto Y_2$.*

We address the following issue: Can we observe the evolution of a solution to a control problem (8.8) through $h_2 \circ h_1$ by observing it

— first through h_1 by a control system

$$\begin{cases} i) & \text{for almost all } t \geq 0, \ \ y_1'(t) = g_1(y_1(t), v_1(t)) \\ ii) & \text{where } v_1(t) \in V_1(y_1(t)) \end{cases} \tag{8.13}$$

and then,

— second, observing this system through h_2.

We introduce the maps R_h, R_{h_1} and R_{h_2} defined respectively by

$$\begin{cases} R_h(x;v) & := \{u \in U(x) \mid h'(x)f(x,u) = g(h(x),v) \\ & \quad \text{if } v \in V(h(x))\} \\ R_{h_1}(x;v_1) & = \{u \in U(x) \mid h_1'(x)f(x,u) = g_1(h_1(x),v_1) \\ & \quad \text{if } v_1 \in V(h_1(x))\} \\ R_{h_2}(x_1;v) & = \{v_1 \in V_1(x_1) \mid h_2'(x_1)g_1(x_1,v_1) = g(h_2(x_1),v) \\ & \quad \text{if } v \in V(h_2(x_1))\} \end{cases}$$

and we see at once that

$$R_{h_1}(x; R_{h_2}(h_1(x);v)) \subset R_h(x;v)$$

Therefore, if the graph of $v \rightsquigarrow R_{h_1}(x; R_{h_2}(h_1(x);v))$ is not empty, we can recover from the evolution of a solution $y(\cdot)$ to the control system (8.9) a solution $y_1(\cdot)$ to the control system (8.13) by the tracking law

$$\text{for almost all } t, \ \ v_1(t) \in R_{h_2}(y_1(t), v(t))$$

and then, a solution $x(\cdot)$ to the control system (8.8) by the tracking law

$$\text{for almost all } t, \ \ u(t) \in R_{h_1}(x(t), v_1(t))$$

This can illustrate hierarchical organization *which is found in the evolution of so many macrosystems. The decomposition of the observation map as a product of several maps determines the successive levels of the hierarchy. The evolution at each level obeys the constraint binding its state to the state of the lower level. It is regulated by controls determined (in a set-valued way) by the evolution of the state-control of the lower level.*

8.3 Partial Differential Inclusions

We shall begin by the decomposable case (or the set-valued linear systems) for which we have explicit formulas, that we next use to solve the general problem of finding a contingent solution to the system of partial differential inclusions

$$\forall\, x \in X, \quad Ah(x) \;\in\; Dh(x)(f(x,h(x))) - G(x,h(x))$$

(where $A \in \mathcal{L}(Y,Y)$) whose graph is a viable manifold.

If $h : X \mapsto Y$, we set

$$\|h\|_\infty \;:=\; \sup_{x\in X} \|h(x)\| \;\;\&\;\; \|h\|_\Lambda \;:=\; \sup_{x\neq y} \frac{\|h(x)-h(y)\|}{\|x-y\|}$$

When G is Lipschitz with nonempty closed images, we denote by $\|G\|_\Lambda$ its Lipschitz constant, the smallest of the constants l satisfying

$$\forall\, z_1, z_2, \quad G(z_1) \;\subset\; G(z_2) + l\,\|z_1 - z_2\|\,B$$

where B is the unit ball.

8.3.1 Decomposable Case

Let $K \subset X$, $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ be set-valued maps and $A \in \mathcal{L}(Y,Y)$. We set

$$\lambda \;:=\; \inf_{\|x\|=1} \langle Ax, x\rangle$$

and we recall that[5]

$$\forall\, y \in Y, \quad \left\|e^{-At}y\right\| \;\leq\; e^{-\lambda t}\|y\|$$

[5]Indeed, $y(t) := e^{-At}y$ being a solution to the differential equation $y'(t) = -Ay(t)$ starting at y, we infer that

$$\frac{d}{dt}\|y(t)\|^2 \;=\; 2\langle y(t), -Ay(t)\rangle \;\leq\; -2\lambda\|y(t)\|^2$$

so that $\|y(t)\| \leq e^{-\lambda t}\|y\|$.

Consider the *decomposable* system of differential inclusions

$$\begin{cases} x'(t) \in \Phi(x(t)) \\ y'(t) \in Ay(t) + \Psi(x(t)) \end{cases} \tag{8.14}$$

which extends to the set-valued case the characteristic system of linear hyperbolic systems

$$\forall\,(x,y) \in \mathrm{Graph}(H_\star),\quad Ay \in DH_\star(x,y)(\Phi(x)) - \Psi(x) \tag{8.15}$$

the solutions of which are the maps satisfying the tracking property.

We denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and viable in K.

We define the set-valued map $H_\star : K \rightsquigarrow Y$ by[6]

$$\forall\, x \in K,\quad H_\star(x) \;:=\; -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt \tag{8.16}$$

Theorem 8.3.1 *Assume that* $\Phi : K \rightsquigarrow X$ *and* $\Psi : K \rightsquigarrow Y$ *are Marchaud maps and that* K *is a closed viability domain*[7] *of* Φ*. If* λ *is large enough, then* $H_\star : K \rightsquigarrow Y$ *defined by (8.16) is the* largest solution with linear growth *to inclusion (8.15) and is bounded whenever* Ψ *is bounded.*

More precisely, if there exist positive constants α, β *and* γ *such that*

$$\forall\, x \in K,\quad \|\Phi(x)\| \;\le\; \alpha(\|x\|+1) \;\;\&\;\; \|\Psi(x)\| \;\le\; \beta + \gamma\|x\|$$

and if $\lambda > \alpha$*, then*

$$\forall\, x \in K,\quad \|H_\star(x)\| \;\le\; \frac{\beta}{\lambda} + \frac{\gamma}{\lambda-\alpha}(\|x\|+1) \tag{8.17}$$

---

[6]By definition of the integral of a set-valued map (see Chapter 8 of SET-VALUED ANALYSIS for instance), this means that for every $y \in H_\star(x)$, there exists a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and $z(t) \in \Psi(x(t))$ such that

$$y \;:=\; -\int_0^\infty e^{-At}z(t)dt \;\in\; H_\star(x)$$

[7]If K is closed, then $H_\star$ is defined on the viability kernel $\mathrm{Viab}_\Phi(K)$.

Moreover, if $K := X$ and Φ, Ψ are Lipschitz, then $H_\star : X \rightsquigarrow Y$ is also Lipschitz (with nonempty values) whenever λ is large enough:

$$\text{If } \lambda > \|\Phi\|_\Lambda, \; H_\star(x_1) \subset H_\star(x_2) + \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda} \|x_1 - x_2\| B$$

Formula (8.16) shows also that the graph of $H_\star$ is convex (respectively $H_\star$ is a closed convex process) whenever the graphs of the set-valued maps Φ and Ψ are convex (respectively Φ and Ψ are closed convex processes).

Proof

1. — We prove first that the graph of $H_\star$ satisfies contingent inclusion (8.15).

Indeed, choose an element y in $H_\star(x)$. By definition of the integral of a set-valued map, this means that there exist a solution $x(\cdot) \in \mathcal{S}_\Phi(x, \cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and viable in K and $z(t) \in \Psi(x(t))$ such that

$$y := -\int_0^\infty e^{-At} z(t) dt \in H_\star(x)$$

We check that for every $\tau > 0$

$$-\int_0^\infty e^{-At} z(t+\tau) dt \in H_\star(x(\tau)) = H_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right)$$

By observing that

$$\begin{cases} \frac{1}{\tau}\int_0^\infty e^{-At}\left(z(t) - z(t+\tau)\right) dt \\ = -\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At} z(t) dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At} z(t) dt \end{cases}$$

we deduce that

$$\begin{cases} y + \tau\left(-\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At} z(t)dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At} z(t)dt\right) \\ \in H_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right) \end{cases}$$

Since Φ is upper semicontinuous, we know that for any $\varepsilon > 0$ and t small enough, $\Phi(x(t)) \subset \Phi(x) + \varepsilon B$, so that $x'(t) \in \Phi(x) + \varepsilon B$ for almost all small t. Therefore, $\Phi(x)$ being closed and convex, we infer that for $\tau > 0$ small enough, $\frac{1}{\tau}\int_0^\tau x'(t)dt \in \Phi(x) + \varepsilon B$ thanks to the Mean-Value Theorem. This latter set being compact, there exists a sequence of $\tau_n > 0$ converging to 0 such that $\frac{1}{\tau_n}\int_0^{\tau_n} x'(t)dt$ converges to some $u \in \Phi(x)$.

In the same way, Ψ being upper semicontinuous, $\Psi(x(t)) \subset \Psi(x) + \varepsilon B$ for any $\varepsilon > 0$ and t small enough, so that $z(t) \in \Psi(x) + \varepsilon B$ for almost all small t. The Mean-Value Theorem implies that

$$\forall\, n > 0, \quad z_n \;:=\; \frac{1}{\tau_n}\int_0^{\tau_n} z(t)dt \;\in\; \Psi(x) + \varepsilon B$$

since this set is compact and convex. Furthermore, there exists a subsequence of z_n converging to some $z_0 \in \Psi(x)$. Hence, since

$$\frac{1}{\tau_n}\int_0^{\tau_n} \left(e^{-At} - 1\right) z(t)dt \;\to\; 0$$

we infer that

$$Ay + z_0 \;\in\; DH_\star(x,y)(u)$$

so that $Ay \;\in\; DH_\star(x,y)(\Phi(x)) - \Psi(x)$.

2. — Let us prove now that the graph of $H_\star$ is closed when λ is large enough. Consider for that purpose a sequence of elements (x_n, y_n) of the graph of $H_\star$ converging to (x, y). There exist solutions $x_n(\cdot) \in \mathcal{S}_\Phi(x_n, \cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at x_n and viable in K and measurable selections $z_n(t) \in \Psi(x_n(t))$ such that

$$y_n \;:=\; -\int_0^\infty e^{-At} z_n(t)dt \in H_\star(x_n)$$

The growth of Φ being linear, there exists $\alpha > 0$ such that the solutions $x_n(\cdot)$ obey the estimate

$$\|x_n(t)\| \le (\|x_n\| + 1)e^{\alpha t} \;\;\&\;\; \|x_n'(t)\| \le \alpha(\|x_n\| + 1)e^{\alpha t}$$

By Theorem 3.5.1, we know that there exists a subsequence (again denoted by) $x_n(\cdot)$ converging uniformly on compact intervals to a solution $x(\cdot) \in \mathcal{S}_\Phi(x, \cdot)$.

The growth of Ψ also being linear, we deduce that, setting $u_n(t) := e^{-At}z_n(t)$,

$$\|z_n(t)\| \leq \beta + \gamma(\|x_n\|+1)e^{\alpha t} \;\&\; \|u_n(t)\| \leq \beta e^{-\lambda t} + \gamma(\|x_n\|+1)e^{-(\lambda-\alpha)t}$$

When $\lambda > \alpha$, Dunford-Pettis' Theorem implies that a subsequence (again denoted by) $u_n(\cdot)$ converges weakly to some function $u(\cdot)$ in $L^1(0,\infty;Y)$. This implies that $z_n(\cdot)$ converges weakly to some function $z(\cdot)$ in $L^1(0,\infty;Y;e^{-\lambda t}dt)$. The Convergence Theorem 2.4.4 states that $z(t) \in \Psi(x(t))$ for almost every t. Since the integrals y_n converge to $-\int_0^\infty e^{-At}z(t)dt$, we have proved that

$$y = -\int_0^\infty e^{-At}z(t)dt \in H_\star(x)$$

3. — Estimate (8.17) is obvious since any solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ satisfies

$$\forall\, t \geq 0, \quad \|x(t)\| \leq (\|x\|+1)e^{\alpha t}$$

so that, if $\lambda > \alpha$,

$$\|H_\star(x)\| \leq \int_0^\infty e^{-\lambda t}\left(\beta + \gamma(\|x\|+1)e^{\alpha t}\right)dt = \frac{\beta}{\lambda} + \frac{\gamma}{\lambda-\alpha}(\|x\|+1)$$

Assume now that $M : K \rightsquigarrow Y$ is any set-valued contingent solution to inclusion (8.15) with linear growth: there exists $\delta > 0$ such that for all $x \in X$, $\|M(x)\| \leq \delta(\|x\|+1)$. Since M enjoys the tracking property, we know that for any $(x,y) \in \text{Graph}(M)$, there exists a solution $(x(\cdot),y(\cdot))$ to the system of differential inclusions

$$\begin{cases} i) \quad x'(t) \in \Phi(x(t)) \\ ii) \quad y'(t) - Ay(t) \in \Psi(x(t)) \end{cases} \tag{8.18}$$

starting at (x,y) such that $y(t) \in M(x(t))$ for all $t \geq 0$. We also know that $\|x(t)\| \leq (\|x\|+1)e^{\alpha t}$ so that $\|y(t)\| \leq \delta(1+(\|x\|+1)e^{\alpha t})$. The second differential inclusion of the above system implies that $z(t) := y'(t) - Ay(t)$ is a measurable selection of $\Psi(x(t))$ satisfying the growth condition

$$\|z(t)\| \leq \beta + \gamma(\|x\|+1)e^{\alpha t}$$

Therefore, if $\lambda > \alpha$, the function $e^{-At}z(t)$ is integrable. On the other hand, integrating by parts $e^{-At}z(t) := e^{-At}y'(t) - e^{-At}Ay(t)$, we obtain

$$e^{-AT}y(T) - y = \int_0^T e^{-At}z(t)dt$$

which implies that

$$y = -\int_0^\infty e^{-At}z(t)dt \in H_\star(x)$$

by letting $T \mapsto \infty$. Hence we have proved that[8] $M(x) \subset H_\star(x)$.

4. — Assume now that Φ and Ψ are Lipschitz, take any pair of elements x_1 and x_2 and choose $y_1 = -\int_0^\infty e^{-At}z_1(t)dt \in H_\star(x_1)$, where

$$x_1(\cdot) \in \mathcal{S}_\Phi(x_1, \cdot) \;\&\; z_1(t) \in \Psi(x_1(t))$$

By the Filippov Theorem 5.3.1, there exists a solution $x_2(\cdot) \in \mathcal{S}_\Phi(x_2, \cdot)$ such that

$$\forall\, t \geq 0, \;\; \|x_1(t) - x_2(t)\| \leq e^{\|\Phi\|_\Lambda t}\|x_1 - x_2\|$$

We denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi(x_2(t))$, which is measurable thanks to Corollary 8.2.13 of SET-VALUED ANALYSIS and which satisfies

$$\forall\, t \geq 0, \; \|z_1(t)-z_2(t)\| \leq \|\Psi\|_\Lambda\|x_1(t)-x_2(t)\| \leq \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda t}\|x_1-x_2\|$$

Therefore, if $\lambda > \|\Phi\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At}z_2(t)dt$ belongs to $H_\star(x_2)$ and satisfies

$$\|y_1-y_2\| \leq \int_0^\infty \|\Psi\|_\Lambda e^{-t(\lambda-\|\Phi\|_\Lambda)}\|x_1-x_2\|dt \leq \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda}\|x_1-x_2\| \;\square$$

[8]This proof actually implies that any set-valued contingent solution M with polynomial growth in the sense that for some $\rho \geq 0$,

$$\forall\, x \in X, \;\; \|M(x)\| \leq \delta(\|x\|^\rho + 1)$$

is contained in $H_\star$ if $\lambda > \alpha\rho$, i.e., that there is no contingent solution with polynomial growth other than with linear growth (and bounded when $\gamma = 0$).

We prove now a comparison result between solutions to two decomposable partial differential inclusions.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L,M) := \sup_{y \in L} \inf_{z \in M} d(y,z) = \sup_{y \in L} d(y,M)$$

their *semi-Hausdorff distance*[9], and recall that $\Delta(L,M) = 0$ if and only if $L \subset M$. If Φ and Ψ are two set-valued maps, we set

$$\Delta(\Phi,\Psi)_\infty = \sup_{x \in X} \Delta(\Phi(x),\Psi(x)) := \sup_{x \in X} \sup_{y \in \Phi(x)} d(y,\Psi(x))$$

Theorem 8.3.2 *Consider now two pairs (Φ_1,Ψ_1) and (Φ_2,Ψ_2) of Marchaud maps defined on X and their associated solutions*

$$\forall\, x \in X, \quad H_{\star i}(x) := -\int_0^\infty e^{-At}\Psi_i(\mathcal{S}_{\Phi_i}(x,t))dt \quad (i = 1,2)$$

If the set-valued maps Φ_2 and Ψ_2 are Lipschitz, and if $\lambda > \|\Phi_2\|_\Lambda$, then

$$\Delta(H_{\star_1},H_{\star_2})_\infty \le \frac{1}{\lambda}\Delta(\Psi_1,\Psi_2)_\infty + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)}\Delta(\Phi_1,\Phi_2)_\infty$$

Proof — Consider the two pairs (Φ_1,Ψ_1) and (Φ_2,Ψ_2) of set-valued maps and choose $y_1 = -\int_0^\infty e^{-At}z_1(t)dt \in H_{\star 1}(x)$ where

$$x_1(\cdot) \in \mathcal{S}_{\Phi_1}(x,\cdot) \ \& \ z_1(t) \in \Psi_1(x_1(t))$$

In order to compare $x_1(\cdot)$ with the solution-set $\mathcal{S}_{\Phi_2}(x,\cdot)$ via the Filippov Theorem, we use the estimate

$$d(x_1'(t),\Phi_2(x_1(t))) \le \sup_{z \in \Phi_1(x_1(t))} d(z,\Phi_2(x_1(t)))) \le \Delta(\Phi_1,\Phi_2)_\infty$$

Therefore, there exists a solution $x_2(\cdot) \in \mathcal{S}_{\Phi_2}(x,\cdot)$ such that

$$\forall\, t \ge 0, \quad \|x_1(t) - x_2(t)\| \le \Delta(\Phi_1,\Phi_2)_\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda}$$

[9]The *Hausdorff distance* between L and M is $\max(\Delta(L,M),\Delta(M,L))$, which may be equal to ∞.

by Filippov's Theorem. As before, we denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi_2(x_2(t))$, which is measurable and satisfies

$$\begin{cases} \forall\, t \geq 0, \ \ \|z_1(t) - z_2(t)\| \ \leq \ \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \|x_1(t) - x_2(t)\| \\ \leq \ \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda} \end{cases}$$

Therefore $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $H_{\star 2}(x)$ and satisfies

$$\begin{cases} \|y_1 - y_2\| \\ \leq \ \int_0^\infty e^{-\lambda t}\Delta(\Psi_1, \Psi_2)_\infty dt + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \int_0^\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda} e^{-\lambda t} dt \\ \leq \ \frac{\Delta(\Psi_1, \Psi_2)_\infty}{\lambda} + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)} \Delta(\Phi_1, \Phi_2)_\infty \ \ \square \end{cases}$$

When Φ, Ψ are single-valued, we obtain:

Proposition 8.3.3 *Assume that φ and ψ are Lipschitz and that ψ is bounded. Then if $\lambda > 0$, the map $h := \Gamma(\varphi, \psi)$ defined by*

$$h(x) \ = - \int_0^\infty e^{-At} \psi(S_\varphi(x,t))dt$$

is the unique bounded single-valued solution to the contingent inclusion

$$Ah(x) \ \in \ Dh(x)(\varphi(x)) - \psi(x) \tag{8.19}$$

and satisfies

$$\|h\|_\infty \ \leq \ \frac{\|\psi\|_\infty}{\lambda} \ \ \& \ \forall\, \lambda > \|\varphi\|_\Lambda, \ \ \|h\|_\Lambda \ \leq \ \frac{\|\psi\|_\Lambda}{\lambda - \|\varphi\|_\Lambda} \tag{8.20}$$

The map $(\varphi, \psi) \mapsto \Gamma(\varphi, \psi)$ is continuous from $\mathcal{C}(X, X) \times \mathcal{C}(X, Y)$ to $\mathcal{C}(X, Y)$:

$$\|\Gamma(\varphi_1, \psi_1) - \Gamma(\varphi_2, \psi_2)\|_\infty \ \leq \ \frac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \frac{\|\psi_2\|_\Lambda}{\lambda(\lambda - \|\varphi_2\|_\Lambda)} \|\varphi_1 - \varphi_2\|_\infty$$

The proof follows Theorems 8.3.1 and 8.3.2.

8.3.2 Existence of a Lipschitz Contingent Solution

We shall now prove the existence of a contingent single-valued solution to inclusion

$$\forall\, x \in X,\;\; Ah(x) \;\in\; Dh(x)(f(x,h(x))) - G(x,h(x)) \qquad (8.21)$$

Theorem 8.3.4 *Assume that the map $f : X \times Y \mapsto X$ is Lipschitz, that $G : X \times Y \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and that*

$$\forall\, x,\; y,\;\; \|G(x,y)\| \le c(1 + \|y\|)$$

Let $A \in \mathcal{L}(Y,Y)$ such that $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ (where ν is the dimension of X). Then there exists a bounded Lipschitz contingent solution to the partial differential inclusion (8.21).

Proof — Since for every Lipschitz single-valued map $s(\cdot)$, $x \rightsquigarrow G(x,s(x))$ is Lipschitz (with constant $\|G\|_\Lambda\,(1 + \|s\|)_\Lambda)$ and has convex compact values, Theorem 9.4.3 of SET-VALUED ANALYSIS implies that the subset G_s of Lipschitz selections ψ of the set-valued map $x \rightsquigarrow G(x,s(x))$ with Lipschitz constant less than $\nu\,\|G\|_\Lambda\,(1+\|s\|_\Lambda)$ is not empty (where ν denotes the dimension of X.) We denote by φ_s the Lipschitz map defined by $\varphi_s(x) := f(x,s(x))$, with Lipschitz constant equal to $\|f\|_\Lambda(1 + \|s\|_\Lambda)$.

The solutions h to inclusion (8.21) are the fixed points to the set-valued map $\mathcal{R} : \mathcal{C}(X,Y) \rightsquigarrow \mathcal{C}(X,Y)$ defined by

$$\mathcal{R}(s) \;:=\; \{\Gamma(\varphi_s,\psi)\}_{\psi\in G_s} \qquad (8.22)$$

Indeed, if $h \in \mathcal{R}(h)$, there exists a selection $\psi \in G_h$ such that

$$Ah(x) \;\in\; Dh(x)(f(x,h(x)))-\psi(x) \;\subset\; Dh(x)(f(x,h(x)))-G(x,h(x))$$

Since $\|G(x,y)\| \le c(1 + \|y\|)$, we deduce that any selection $\psi \in G_s$ satisfies

$$\|\psi\|_\infty \;\le\; c(1 + \|s\|_\infty)$$

Therefore, Proposition 8.3.3 implies that

$$\forall\, h \in \mathcal{R}(s),\;\; \|h\|_\infty \;\le\; \frac{c}{\lambda}(1+\|s\|_\infty) \;\;\&\;\; \|h\|_\Lambda \;\le\; \frac{\nu\|G\|_\Lambda(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)}$$

We first observe that when $\lambda > c$,

$$\forall\, s \in \mathcal{C}(X,Y) \;\text{ such that }\; \|s\|_\infty \le \frac{c}{\lambda - c},\; \forall\, h \in \mathcal{R}(s),\; \|h\|_\infty \le \frac{c}{\lambda - c}$$

When $\lambda > 4\nu\|f\|_\Lambda \,\|G\|_\Lambda$, we denote by

$$\rho_\lambda := \frac{\lambda - \|f\|_\Lambda - \nu\|G\|_\Lambda - \sqrt{\lambda^2 - 2\lambda(\|f\|_\Lambda + \nu\|G\|_\Lambda) + (\|f\|_\Lambda - \nu\|G\|_\Lambda)^2}}{2\|f\|_\Lambda}$$

the smallest root of the equation

$$\lambda\rho \;=\; \|f\|_\Lambda \rho^2 + (\|f\|_\Lambda + \nu\|G\|_\Lambda)\rho + \nu\|G\|_\Lambda$$

which is positive. We observe that

$$\lim_{\lambda \to +\infty} \lambda\rho_\lambda \;=\; \nu\|G\|_\Lambda$$

and infer that

$$\forall\, s \in \mathcal{C}(X,Y) \;\text{ such that }\; \|s\|_\Lambda \le \rho_\lambda,\; \forall\, h \in \mathcal{R}(s),\; \|h\|_\Lambda \le \rho_\lambda$$

because h being of the form $\Gamma(\varphi_s, \psi_s)$, satisfies by Proposition 8.3.3:

$$\|h\|_\Lambda \le \frac{\|\psi_s\|_\Lambda}{\lambda - \|\varphi_s\|_\Lambda} \le \frac{\nu\,\|G\|_\Lambda\,(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)} \le \frac{\nu\,\|G\|_\Lambda\,(1 + \rho_\lambda)}{\lambda - \|f\|_\Lambda(1 + \rho_\lambda)} = \rho_\lambda$$

Let us denote by $B^1_\infty(\lambda)$ the subset defined by

$$B^1_\infty(\lambda) \;:=\; \left\{ h \in \mathcal{C}(X,Y) \;\middle|\; \|h\|_\infty \le \frac{c}{\lambda - c} \;\;\&\;\; \|h\|_\Lambda \le \rho_\lambda \right\}$$

which is compact (for the compact convergence topology) thanks to Ascoli's Theorem.

We have therefore proved that when $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$, the set-valued map $\mathcal{R}$ sends the compact subset $B^1_\infty(\lambda)$ to itself.

It is obvious that the values of $\mathcal{R}$ are convex. Kakutani's Fixed-Point Theorem implies the existence of a fixed point $h \in \mathcal{R}(h)$ if we prove that the graph of $\mathcal{R}$ is closed.

Actually, the graph of $\mathcal{R}$ is compact. Indeed, let us consider any sequence $(s_n, h_n) \in \text{Graph}(\mathcal{R})$. Since $B^1_\infty(\lambda)$ is compact, a subsequence (again denoted by) (s_n, h_n) converges to some function

$$(s,h) \;\in\; B^1_\infty(\lambda) \times B^1_\infty(\lambda)$$

But there exist bounded Lipschitz selections $\psi_n \in G_{s_n}$ with Lipschitz constant $\nu\|G\|_\Lambda(1+\rho_\lambda)$ such that

$$\forall\, n \geq 0,\;\; h_n \;=\; \Gamma(\varphi_{s_n}, \psi_n)$$

Therefore a subsequence (again denoted by) ψ_n converges to some function $\psi \in G_s$. Since φ_{s_n} converges obviously to φ_s, we infer that h_n converges to $\Gamma(\varphi_s, \psi)$ where $\psi \in G_s$, i.e., that $h \in \mathcal{R}(s)$, since Γ is continuous by Proposition 8.3.3. □

8.3.3 Comparison Results

The point of this section is to compare two solutions to inclusion (8.21), or even, a single-valued solution and a contingent set-valued solution $M : X \rightsquigarrow Y$.

We first deduce from Theorem 8.3.2 the following "localization property":

Theorem 8.3.5 *We posit the assumptions of Theorem 8.3.4, with* $A \in \mathcal{L}(Y,Y)$ *such that* $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ *(where* ν *is the dimension of* X*). Let* $\Phi : X \rightsquigarrow X$ *and* $\Psi : X \rightsquigarrow Y$ *be two Lipschitz and Marchaud maps with which we associate the set-valued map* $H_\star$ *defined by*

$$\forall\, x \in X,\;\; H_\star(x) \;:=\; -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

Then any bounded single-valued contingent solution $h(\cdot)$ *to inclusion (8.21) satisfies the following estimate*

$$\begin{cases} \forall\, x \in X,\;\; d(h(x), H_\star(x)) \;\leq\; \frac{1}{\lambda}\sup_{x\in X}\Delta(G(x,h(x)),\Psi(x)) \\ +\frac{\|\Psi\|_\Lambda}{\lambda(\lambda-\|\Phi\|_\Lambda)}\sup_{x\in X} d(f(x,h(x)),\Phi(x)) \end{cases}$$

In particular, if we assume that

$$\forall\, y \in Y,\;\; f(x,y) \;\in\; \Phi(x) \;\;\&\;\; G(x,y) \;\subset\; \Psi(x)$$

then all bounded single-valued contingent solutions $h(\cdot)$ *to inclusion (8.21) are selections of* $H_\star$*.*

Proof — Let h be any bounded single-valued contingent solution to inclusion (8.21). One can show that h can be written in the form

$$h(x) = -\int_0^\infty e^{-At} z(t)dt \text{ where } z(t) \in G(x(t), h(x(t)))$$

by using the same arguments as in the first part of the proof of Theorem 8.3.1.

We also adapt the proof of Theorem 8.3.2 with $\Phi_1 := f(x, h(x))$, $z_1(t) := z(t)$, $\Phi_2 := \Phi$ and $\Psi_2 := \Psi$, to show that the estimates stated in the theorem hold true. □

8.4 The Variational Principle

We characterize in this section solutions to the partial differential inclusion (8.3) through a *variational principle*. For that purpose, we recall that

$$\sigma(M,p) := \sup_{z \in M} < p, z > \quad \& \quad \sigma^\flat(M,p) := \inf_{z \in M} < p, z >$$

denote the support functions of $M \subset X$ and $B_\star$ the unit ball of $Y^\star$. We also need the following

Definition 8.4.1 *Let $H : X \rightsquigarrow Y$ be a set-valued map and (x,y) belong to its graph. We shall say that the transpose $DH(x,y)^\star$: $Y^\star \rightsquigarrow X^\star$ of the contingent derivative $DH(x,y)$ is the* codifferential of H at (x,y). *When $H := h$ is single-valued, we set $Dh(x)^\star := Dh(x, h(x))^\star$.*

8.4.1 Definition of the Functional

Consider a closed subset $K \subset X$. We introduce the nonnegative functional Φ defined on the space $\mathcal{C}(K,Y)$ of continuous maps by

$$\Phi(h) := \sup_{q \in B_\star} \sup_{x \in K} \sup_{p \in Dh(x)^\star(q)} \left(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q)\right)$$

Theorem 8.4.2 (Variational Principle) *Let the set-valued maps F and G be upper semicontinuous with convex and compact values.*

Let $c > 0$. Then a single-valued map $h : K \mapsto Y$ is a solution to the partial differential inclusion

$$\forall\, x \in K,\;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

if and only if $\Phi(h) \leq c$.

Consequently, h is a solution to the partial differential inclusion (8.3) if and only if $\Phi(h) = 0$.

Proof — The first inclusion is easy: let $u \in F(x,h(x))$, $v \in G(x,h(x))$ and $e \in cB$ be such that $v - e \in Dh(x)(u)$. Then, for any $q \in B_\star$ and $p \in Dh(x)^\star(q)$, we know that

$$< p, u > - < q, v - e > \leq\; 0$$

so that

$$\left\{ \begin{array}{l} \sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \\ \\ \leq < p,u > - < q,v > \leq < q,e > \leq\; c \end{array} \right.$$

By taking the supremum with respect to $x \in K$, $q \in B_\star$ and $p \in Dh(x)^\star(q)$, we infer that $\Phi(h) \leq c$.

Conversely, we can write inequality $\Phi(h) \leq c$ in the form of the minimax inequality: for any $x \in K$, $q \in Y^\star$,

$$\sup_{p \in Dh(x)^\star(q)} \inf_{u \in F(x,h(x))} \inf_{v \in G(x,h(x))} (< p,u > - < q,v >) \leq c\|q\|$$

Noticing that $c\|q\| = \sigma(cB,q)$ and setting

$$\beta(p,q;u,v,e) \;:= < p,u > - < q, v - e >$$

this inequality can be written in the form: for every $x \in K$,

$$\sup_{(p,-q) \in \mathrm{Graph}(Dh(x))^-} \inf_{(u,v,e) \in F(x,h(x)) \times G(x,h(x)) \times cB} \beta(p,q;u,v,e) \leq 0$$

Since the set $F(x,h(x)) \times G(x,h(x)) \times cB$ is convex compact and since the negative polar cone to the graph of $Dh(x)$ is convex, the

Lop-Sided Minimax Theorem 3.7.10 implies the existence of $u_0 \in F(x, h(x))$, $v_0 \in G(x, h(x))$ and $e_0 \in cB$ such that

$$\sup_{(p,-q)\in \text{Graph}(Dh(x))^-} (< p, u_0 > - < q, v_0 - e_0 >) =$$

$$\sup_{(p,-q)\in \text{Graph}(Dh(x))^-} \inf_{(u,v,e)\in F(x,h(x))\times G(x,h(x))\times cB} \beta(p, q; u, v, e)$$

$$\leq 0$$

This means that $(u_0, v_0 - e_0)$ belongs to the bipolar of the graph of $Dh(x)$, i.e., its closed convex hull $\overline{co}(\text{Graph}(Dh(x)))$. In other words, we have proved that

$$(F(x, h(x)) \times (G(x, h(x)) + cB)) \cap \overline{co}\left(T_{\text{Graph}(h)}(x, h(x))\right) \neq \emptyset$$

But by Theorem 3.2.4, this is equivalent to the condition

$$(F(x, h(x)) \times (G(x, h(x)) + cB)) \cap T_{\text{Graph}(h)}(x, h(x)) \neq \emptyset$$

i.e., h is a solution to the partial differential inclusion. □

Theorem 8.4.3 *Assume that the set-valued maps F and G are upper semicontinuous with nonempty convex compact images. Let $\mathcal{H} \subset \mathcal{C}(K, Y)$ be a compact subset for the compact convergence topology.*

Assume that $c := \inf_{h\in\mathcal{H}} \Phi(h) < +\infty$. Then there exists a solution $h \in \mathcal{H}$ to the partial differential inclusion

$$0 \in Dh(x)(F(x, h(x))) - G(x, h(x)) + cB$$

Since $\mathcal{H}$ is a compact subset for the compact convergence topology, it is sufficient to prove that the functional Φ is lower semicontinuous on the space $\mathcal{C}(K, Y)$ for this topology: If it is proper (i.e., different from the constant $+\infty$), it achieves its minimum at some $h \in \mathcal{H}$, which is a solution to the above partial differential inclusion thanks to Theorem 8.4.2. So Theorem 8.4.3 follows from Proposition 8.4.4 below:

Proposition 8.4.4 *Assume that the set-valued maps F and G are upper semicontinuous with nonempty convex compact images. Then the functional Φ is lower semicontinuous on equicontinuous subsets of the space $\mathcal{C}(K, Y)$ for the compact convergence topology.*

To prove this result, we need more information about the convergence properties of the codifferentials.

8.4.2 Convergence Properties of the Codifferentials

Proposition 8.4.5 *Let X, Y be finite dimensional vector-spaces and $K \subset X$ be a closed subset. Assume that h is the pointwise limit of an equicontinuous family of maps $h_n : K \mapsto Y$. Let $x \in K$ and $p \in Dh(x)^\star(q)$ be fixed. Then there exist subsequences of elements $x_{n_k} \in K$ converging to x, q_{n_k} converging to q and $p_{n_k} \in Dh_{n_k}(x_{n_k})^\star(q_{n_k})$ converging to p.*

If the functions h_n are differentiable, we deduce that there exist subsequences of elements $x_{n_k} \in K$ converging to x and q_{n_k} converging to q such that $h'_{n_k}(x_{n_k})^\star(q_{n_k})$ converges to p.

Proof — We can reformulate the statement in the following way: we observe that $p \in Dh(x)^\star(q)$ if and only if

$$(p, -q) \in \left(T_{\mathrm{Graph}(h)}(x, h(x))\right)^-$$

so that we have to prove that there exist subsequences $x_{n_k} \in K$ and

$$(p_{n_k}, -q_{n_k}) \in \left(T_{\mathrm{Graph}(h_{n_k})}(x_{n_k}, h_{n_k}(x_{n_k}))\right)^-$$

converging to x and $(p, -q)$ respectively. Therefore the proposition follows from

Theorem 8.4.6 (Frankowska) *Let us consider a sequence of closed subsets K_n and an element $x \in \mathrm{Liminf}_{n\to\infty} K_n$ (assumed to be nonempty.) Set $K^\sharp := \mathrm{Limsup}_{n\to\infty} K_n$.*

Then, for any $p \in (T_{K^\sharp}(x))^-$, there exist subsequences of elements $x_{n_k} \in K_{n_k}$ and $p_{n_k} \in \left(T_{K_{n_k}}(x_{n_k})\right)^-$ converging to p and x respectively:

$$(T_{K^\sharp}(x))^- \subset \mathrm{Limsup}_{n\to\infty,\, x_n \to_{K_n} x} (T_{K_n}(x_n))^-$$

Proof — First, it is sufficient to consider the case when x belongs to the intersection $\bigcap_{n=1}^\infty K_n$ of the subsets K_n. If not, we set $\widehat{K_n} := K_n + x - u_n$ where $u_n \in K_n$ converges to x. We observe that $x \in \bigcap_{n=1}^\infty \widehat{K_n}$ and that $T_{\widehat{K_n}}(x_n) = T_{K_n}(x_n - x + u_n)$.

Let $p \in (T_{K^\sharp}(x))^-$ be given with norm 1. We associate with any positive λ the projection x_n^λ of $x + \lambda p$ onto K_n:

$$\left\|x + \lambda p - x_n^\lambda\right\| = \min_{x_n \in K_n} \|x + \lambda p - x_n\| \tag{8.23}$$

and set

$$v_n^\lambda := \frac{x_n^\lambda - x}{\lambda} \;\; \& \;\; p_n^\lambda := p - v_n^\lambda \in \left(T_{K_n}(x_n^\lambda)\right)^-$$

because $x+\lambda p - x_n^\lambda = \lambda(p - v_n^\lambda)$ belongs to the polar cone $\left(T_{K_n}(x_n^\lambda)\right)^-$ to the contingent cone $T_{K_n}(x_n^\lambda)$ by Proposition 3.2.3.

Let us fix for the time $\lambda > 0$. By taking $x_n = x \in K_n$ in (8.23), we infer that $\|v_n^\lambda\| \leq 2$. Therefore, the sequences x_n^λ and v_n^λ being bounded, some subsequences $x_{n'}^\lambda$ and $v_{n'}^\lambda$ converge to elements $x^\lambda \in K^\sharp$ and $v^\lambda = \frac{x^\lambda - x}{\lambda}$ respectively.

Furthermore, there exists a sequence $\lambda_k \to 0+$ such that v^{λ_k} converge to some $v \in T_{K^\sharp}(x)$ because $\|v^\lambda\| \leq 2$ and because for every λ,

$$x^\lambda = x + \lambda v^\lambda \in K^\sharp$$

Therefore $\langle p, v\rangle \leq 0$ since $p \in (T_{K^\sharp}(x))^-$.

On the other hand, we deduce from (8.23) the inequalities

$$\left\|p - v_n^\lambda\right\|^2 = \|p\|^2 + \|v_n^\lambda\|^2 - 2\langle p, v_n^\lambda\rangle \leq \|p\|^2$$

which imply, by passing to the limit, that $\|v\|^2 \leq 2\langle p, v\rangle \leq 0$.

We have proved that a subsequence v^{λ_k} converges to 0, and thus, that a subsequence $v_{n_k}^{\lambda_k} = p - p_{n_k}^{\lambda_k}$ converges also to 0. The lemma ensues. □

Proof of Proposition 8.4.4 — Assume that Φ is proper. Let h_n be a sequence of Φ satisfying for any n, $\Phi(h_n) \leq c$ and converging to some map h. We have to check that $\Phi(h) \leq c$. Indeed, fix $x \in K$, $q \in B_\star$ and $p \in Dh(x)^\star(q)$. By Proposition 8.4.5, there exist subsequences (again denoted by) $x_n \in K$ converging to x, q_n converging to q and $p_n \in Dh_n(x_n)^\star(q_n)$ converging to p such that $h_n(x_n)$ converges to $h(x)$.

We can always assume that $\|q_n\| \leq 1$. If not, we replace q_n by $\widehat{q}_n := \frac{\|q\|}{\|q_n\|} q_n$ and p_n by

$$\widehat{p}_n := \frac{\|q\|}{\|q_n\|} p_n \in Dh_n(x_n)^\star(\widehat{q}_n)$$

Since F and G are upper semicontinuous with compact values, we know that for any (p, q) and $\varepsilon > 0$, we have

$$\begin{cases} \sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q) \\ \leq \sigma^\flat(F(x_n, h_n(x_n)), p_n) - \sigma(G(x_n, h_n(x_n)), q) + \varepsilon \leq \Phi(h_n) + \varepsilon \end{cases}$$

for n large enough. Hence, by letting n go to ∞, we infer that for any $\varepsilon > 0$,

$$\sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q) \leq c + \varepsilon$$

Letting ε converge to 0 and taking the supremum on $q \in B_\star$, $x \in K$ and $p \in Dh(x)^\star(q)$, we infer that $\Phi(h) \leq c$. □

8.5 Feedback Controls Regulating Smooth Evolutions

Consider a control system (U, f):

$$\begin{cases} i) & \text{for almost all } t, \ x'(t) = f(x(t), u(t)) \\ ii) & \text{where } u(t) \in U(x(t)) \end{cases} \tag{8.24}$$

Let $(x, u) \to \varphi(x, u)$ be a nonnegative continuous function with linear growth.

We have proved in Chapter 7 that there exists a closed regulation map $R^\varphi \subset U$ larger than any closed regulation map $R : K \rightsquigarrow Z$ contained in U and enjoying the following viability property: *For any initial state $x_0 \in \mathrm{Dom}(R)$ and any initial control $u_0 \in R(x_0)$, there exists a solution $(x(\cdot), u(\cdot))$ to the control system (8.24) starting at (x_0, u_0) such that*

$$\forall\, t \geq 0, \ u(t) \in R(x(t))$$

and

$$\text{for almost all } t \geq 0, \ \|u'(t)\| \ \leq \ \varphi(x(t), u(t))$$

Let $K \subset \mathrm{Dom}(U)$ be a closed subset. We also recall that *a closed set-valued map $R : K \rightsquigarrow Z$ is a feedback control regulating viable solutions to the control problem satisfying the above growth condition if and only if R is a solution to the* partial differential inclusion

$$\forall\, x \in K, \ \ 0 \in DR(x,u)(f(x,u)) - \varphi(x,u)B$$

satisfying the constraint

$$\forall\, x \in K, \ \ R(x) \ \subset \ U(x)$$

In particular, a *closed graph single-valued regulation map* $r : K \mapsto Z$ is a solution to the *partial differential inclusion*

$$\forall\, x \in K, \ \ 0 \in Dr(x)(f(x,r(x))) - \varphi(x,r(x))B \tag{8.25}$$

satisfying the constraint

$$\forall\, x \in K, \ \ r(x) \ \in \ U(x)$$

Such a solution can be obtained by a variational principle:
We introduce the functional Φ defined by

$$\Phi(r) \ := \ \sup_{q \in B_\star} \sup_{x \in K} \sup_{p \in Dr(x)^\star(q)} (< p, f(x,r(x)) > -\varphi(x,r(x))\|q\|)$$

Theorem 8.5.1 *Let $\mathcal{R} \subset \mathcal{C}(K,Y)$ be a nonempty compact subset of selections of the set-valued map U (for the compact convergence topology.)*

Suppose that the functions f and φ are continuous and that

$$c \ := \ \inf_{r \in \mathcal{R}} \Phi(r) \ < \ +\infty$$

Then there exists a solution $r(\cdot)$ to the partial differential inclusion

$$\forall\, x \in K, \ \ 0 \in Dr(x)(f(x,r(x))) - (\varphi(x,r(x)) + c)B$$