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Viability Theory

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This book is dedicated to
Hélène Frankowska

with love

Epigraph

Viability theory is a mathematical theory that offers *mathematical metaphors*[1] *of evolution of macrosystems* arising in biology, economics, cognitive sciences, games, and similar areas, as well as in nonlinear systems of control theory.

We shall specifically be concerned with three main common features:

- A nondeterministic (or contingent) engine of evolution, providing several (and even many) opportunities to explore the environment,

- Viability constraints that the state of the system must obey at each instant under "death penalty",

- An inertia principle stating that the "controls" of the system are changed only when viability is at stake.

The first two features are best summarized by the deeply intuitive statement attributed to Democritus by Jacques Monod: *"Everything that exists in the Universe is due to Chance and Necessity"*. The inertia principle is a mathematical formulation of the concept of *punctuated equilibrium* introduced recently in paleontology by Elredge and Gould. It runs against the teleological trend assigning aims to

[1]Like other means of communications (languages, painting, music, etc.), mathematics provides *metaphors* that can be used to explain a given phenomenon by associating it with some other phenomenon that is more familiar, or at least is felt to be more familiar. This feeling of familiarity, individual or collective, inborn or acquired, is responsible for the inner conviction that this phenomenon is understood.

be achieved (in even an optimal way) by the state of the system and the belief that actors control the system for such purposes.

— **Nondeterminism:** We shall mean by this term that *les jeux ne sont jamais faits*, in the sense that at each instant, there are several available, or feasible, evolutions which depend upon the state, or even the history of the evolution of the state of the system up to this time. Therefore, the concept of evolution borrowed from Newtonian mechanics is no longer adequate for such systems. It has led to the misleading identification of mathematics with a *deterministic* paradigm, which implies that *the evolution of macrosystems can be predicted*. Even if we were to accept the existence of deterministic mechanisms[2] underlying the evolution of biological, economic and social macrosystems, we know that such systems often can be inherently unstable - and this places the actual computation of their solutions beyond the capabilities of even the most sophisticated of present-day computers! To "run" models which have some inbuilt structural instability can serve no useful purpose.

Thus, we suppose here that the dynamics responsible for the evolution are not deterministic. This lack of determinism has many different features: it may be due to nonstochastic "uncertainty"[3], to "disturbances"and "perturbations" of various kinds, or to errors in modeling due to the impossibility of a comprehensive description of the dynamics of the system.

In several instances, the dynamics of the system are related to certain "controls", which, in turn, are restricted by state-dependent constraints (closed systems.) Such controls, which we do not dare to call *regulees* instead of controls, are typically

1. *prices or other fiduciary goods* in economics (when the evolution of commodities and services is regulated by Adam Smith's invisible hand or the market, the planning bureau, ...),

[2] And now we discover that some of our "perfectly deterministic" models can exhibit all sorts of different trajectories. These are *chaotic* systems, making prediction virtually impossible.

[3] No a priori knowledge of an underlying probability law on the state of events is made. *Fuzzy viability* provides models where the available velocities can be ranked through a membership cost function to take into account that some velocities are more likely to be chosen than others.

2. *genotypes* or *fitness matrices* in genetics and population genetics (when the evolution of *phenotypes* of a population is regulated by sexual reproduction and mutations),

3. *conceptual controls* or *synaptic matrices* in pattern recognition mechanisms and neural networks (when the sensory-motor state is regulated by learning processes),

4. *affinity matrices* in immunological systems,

5. *strategies* in differential games (when the state of the system is regulated by the decision rules for the players),

6. *coalitions* in cooperative games,

7. *cultural codes* in sociology (when the evolution of societies is regulated by every individual believing and obeying such codes), etc..

— **Viability:** For a variety of reasons, not all evolutions are possible. This amounts to saying that the state of the system must obey constraints, called *viability constraints*. These constraints include homeostatic constraints in biological regulation, scarcity constraints in economics, state constraints in control, power constraints in game theory, ecological constraints in genetics, sociability constraints in sociology, etc. Therefore, the goal is to select solutions which are *viable in the sense that they satisfy, at each instant, these constraints.*

Viability theorems thus yield selection procedures of viable evolutions, i.e., characterize the connections between the dynamics and the constraints for guaranteeing the existence of at least one viable solution starting from any initial state. These theorems also provide the *regulation processes (feedbacks*[4]*)* that maintain viability, or, even as time goes by, *improve* the state according to some *preference relation.*

Contrary to *optimal control theory*, viability theory does not require any single decision-maker (or actor, or player) to "guide" the

[4]thus providing the central concept of cybernetics as a *solution* to the regulation problem.

system by optimizing an *intertemporal* optimality criterion[5].

Furthermore, the choice (even conditional) of the controls is not made *once and for all* at some initial time, but *they can be changed at each instant so as to take into account possible modifications of the environment of the system*, allowing therefore for *adaptation* to viability constraints.

Finally, by not appealing to intertemporal criteria, *viability theory does not require any knowledge of the future*[6] (even of a stochastic nature.) This is of particular importance when experimentation[7] is not possible or when the phenomenon under study is not periodic. For example, in biological evolution as well as in economics and in the other systems we shall investigate, *the dynamics of the system disappear and cannot be recreated.*

Hence, *forecasting or prediction of the future are not the issues which we shall address in this book.*

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation of some behaviors which do not fit any reasonable optimality criterion.

Therefore, instead of using intertemporal optimization[8] that involves the future, viability theory provides selection procedures of *viable evolutions* obeying, at each instant, state constraints which depend upon the *present or the past.* (This does not exclude *anticipations*, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

[5]the choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

[6]Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

[7]Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to "know" the future evolution of the system.

[8]which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

Nonetheless, selection through viability constraints may not be discriminating enough. Starting from any state at any instant, several viable solutions may be implemented by the system, including equilibria, which are stationary evolutions[9].

Thus further selection mechanisms need to be devised or discovered. We advocate here a third feature to which a selection procedure must comply, the *Inertia Principle.*

— **Inertia Principle:** which states that *"the controls are kept constant as long as viability of the system is not at stake".*

Indeed, as long as the state of the system lies in the interior of the viability set (the set of states satisfying viability constraints), any regularity control will work. Therefore, the system can maintain the control inherited from the past. This happens if the system obeys the inertia principle. Since the state of the system may evolve while the control remains constant, it may reach the viability boundary with an "outward" velocity. This event corresponds to a period of *crisis*: To survive, the system must find another regulatory control such that the new associated velocity forces the solution back inside the viability set. (See Figure 1.) Alternatively, if the viability constraints can evolve, another way to resolve the crisis is to relax the constraints so that the state of the system lies in the interior of the new viability set. When this is not possible, *strategies for structural change fail*: by design, this means the solution leaves the viability set and "dies".

Naturally, there are several procedures for selecting a viable control when viability is at stake. For instance, the selection at each instant of the controls providing viable evolutions with *minimal velocity* is an example that obeys this inertia principle. They are called *"heavy" viable evolutions*[10] in the sense of heavy trends in economics.

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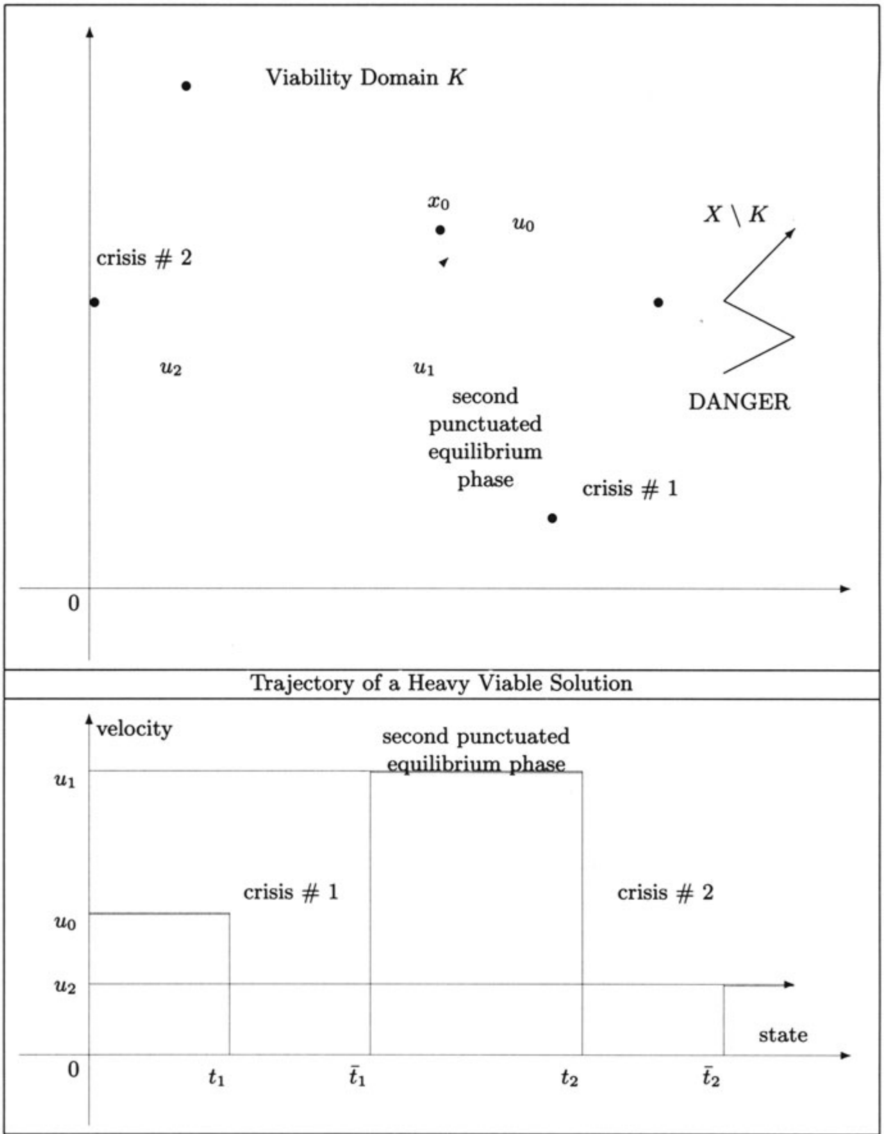
[9]This touches on another aspect of viability theory - that concerned with complexity and robustness: It may be observed that the state of the system becomes increasingly robust the further it is from the boundary of the viability set. Therefore, after some time has elapsed, only the parts of the trajectories furthest away from the viability boundary will remain. This fact may explain the apparent discontinuities ("missing links") and hierarchical organization arising from evolution in certain systems.

[10]When the controls are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.

Figure 0.1: Heavy Viable Solutions



Punctuated Evolution of Velocities. Starting from x_0 with the constant velocity u_0, the solution evolves in K until time t_1, (first *punctuated equilibrium phase*) when the velocity. Then a *crisis* happens during which velocities also evolve (as slowly as possible) to maintain viability, until time $\bar{t}_1$ where the velocities are contingent to K during the *second punctuated equilibrium phase.*

Heavy viable evolutions can be viewed as providing mathematical metaphors for the concept of *punctuated equilibrium*[1] introduced recently in paleontology by Elredge and Gould.

In a nutshell, *the main purpose of viability theory is to explain the evolution of a system, determined by given nondeterministic dynamics and viability constraints, to reveal the concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementing them.*

It assumes implicitly an "opportunistic" and "conservative" behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

On the mathematical side, viability theory contributed to vigorous renewed interest in the field of "differential inclusions", as well as an engine for the development of a differential calculus of set-valued maps[2]. Indeed, as it often occurs in mathematics, these techniques have already found applications to other domains, for instance, to nonlinear systems theory (tracking, zero dynamics, local controllability and observability[3], control under state constraints, etc.) and

[1] Excavations at Kenya's Lake Turkana have provided clear evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolution journey that seems to have proceeded in fits and starts. Examination of more than 3,000 fossils by P. Williamson showed how 13 species evolved. The record indicated that the animals stayed much the same for immensely long stretches of time. But twice, about two million years ago and then, 700,000 years ago, the pool of life seemed to explode — set off, apparently, by a drop in the lake's water level. Intermediate forms appeared very quickly, new species evolving in 5,000 to 50,000 years, after millions of years of constancy, leading paleontologists to challenge the accepted idea of continuous evolution.

[2] One can say that by now the main results of functional analysis have their counterpart in what can be called *Set-Valued Analysis*. Only the results needed in this book will be presented. An exposition of Set-Valued Analysis can be found in the companion monograph SET-VALUED ANALYSIS by Hélène Frankowska and the author.

[3] These topics will be not developed here. The forthcoming monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène

Artificial Intelligence (qualitative physics, learning processes, etc.) These techniques can be efficiently used as mathematical tools and have been related to other questions (such as Lyapunov's second method, variational differential equations, etc..)

This is a book of *motivated mathematics*[4], which searches for new sources of mathematical metaphors.

Unfortunately, the length of the theoretical part of viability theory did not allow us to include in this volume the discussion of the motivating problems. Some problems arising in Artificial Intelligence, economics, game theory, biology, cognitive sciences, etc., which have spawned many of the mathematical questions treated below, will be investigated in forthcoming additional volumes.

By looking at common features of otherwise very different systems and looking at shared consequences, it was necessary to set our mathematical metaphors at a fairly high level of abstraction, yielding an amount of information inversely proportional to the height of this level so to speak.

For the time being at least, this theory is still far from providing an ideal description of the evolution of macrosystems. Some potential users (economists, biologists, ...) should not be disappointed or discouraged by the results obtained so far — for it is too early for such a theory to be "applied" in the engineering sense.

However, the available results may explain a portion of "reality"

Frankowska provides an exhaustive treatment of Control Theory using set-valued analysis and differential inclusions.

[4]We have already mentioned a mathematical metaphor as a means of associating a particular mathematical theory with a certain observed phenomenon. This association can arise in two different ways. The first possibility is to look for an existing mathematical theory which seems to provide a good explanation of the phenomenon under consideration. This is usually regarded as the domain of applied mathematics. However, it is also possible to approach the problem from the opposite direction. Other fields provide mathematicians with metaphors, and this is the domain of what can be called "motivated mathematics".

The ancients divided *analysis* into two forms: *zetetic*, which corresponds to what we mean by motivated mathematics or modeling, and *poristic*, which corresponds to applied mathematics, a procedure by which the validity of the model is confirmed. It is much later, in 1591, that F. Viète added a third form, *rhetic* or *exegetic*, which would correspond to our pure mathematics.

in the extent where *the degree of reality for a social group at a given time is understood in terms of the consensus*[5] *interpretations of the group member's perceptions of their physical, biological, social and cultural environments.*

I hope that this book may help readers from different scientific areas to find a common ground for comparing the behaviors of the systems they study and for asking new questions. Anyhow, whatever the ultimate outcome, the motivation provided by the viability problems has already benefited mathematics by suggesting new concepts and lines of argument, by giving some inkling of possible solutions, or by developing new modes of intuition, leading many mathematicians to revive and enrich the theory of dynamical systems and set-valued analysis. The history of mathematics is full of instances in which mathematical techniques motivated by problems encountered in one scientific field have found applications in many others. *It is this "universality" which renders mathematics so fascinating.*

Jean-Pierre Aubin
Paris, May 12, 1990

[5]Since our brains are built according to the same biological blueprint, and since the general acceptance of local cultural codes seems to be an innate and universal phenomenon, it is highly probable that the individuals comprising a social group arrive at a consensus wide enough for a reasonably believable concept of reality to emerge. However, the prophets and scholars of each group continually question the validity of the metaphors on which this consensus is based, while the high priests and other guardians of ideological purity ultimately try to transform it into dogma and impose it on the other members of the group. (It often happens that the prophets and scholars themselves eventually become high priests ; movement in the reverse direction is much less common.) It is through this permanent struggle that knowledge evolves. But there is an important difference between the metaphors of science and those of, say, religion or ideology : a metaphor that claims scientific validity must be limited, even narrow, in scope. The more "applied" a scientific study, the narrower it must necessarily be. Scientific theories — scientific metaphors — must be capable of logical refutation (as in mathematics) or of experimental falsification (which of course requires that theories be falsifiable.) Ideologies escape these requirements : the "broader" they are, the more seductive they appear, the more dangerous they can be.

Acknowledgments

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This book is dedicated to Hélène Frankowska, for many reasons. She actually should have been the coauthor of this book as well as of Set-Valued Analysis, thanks to her contributions (many of the results are due to her), daily remarks and continuous ideas, corrections, suggestions and challenges to improve the redaction of this monograph. Without her, this book would never have reached this stage. With her, it would have been much better.

[1] among whom my old friends A. Bensoussan, I. Ekeland, the members of the group Viabilité et Contrôle who contributed so much to this book, without forgetting more transient colleagues, P. Bernhard, A. Cellina, F. Clarke to name a few.

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Introduction

Consider the evolution of a control system with (multivalued) feedbacks:

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in U(x(t)) \end{cases}$$

where the state $x(\cdot)$ ranges over a finite dimensional vector-space X and the control $u(\cdot)$ ranges over another finite dimensional vector-space Z. Here, the first equation describes how the control — regarded as an *input* to the system — yields the state of the system[1] — regarded as an *output* — whereas the second inclusion shows how the state-output "feeds back" to the control-input. The set-valued map $U : X \rightsquigarrow Z$ may be called an "a priori feedback". It describes the *state-dependent constraints on the controls*. A solution to this system is a function $t \to x(t)$ satisfying this system for some control $t \to u(t)$.

Viability constraints are described by a closed subset[2] K of the state space: These are intended to describe the "viability" of the system because outside of K, the state of the system is no longer viable.

A subset K *is viable* under the control system described by f and U if for every initial state $x_0 \in K$, there exists at least one solution to the system starting at x_0 which is *viable* in the sense that

$$\forall\, t \geq 0, \quad x(t) \in K$$

[1] once the initial state is fixed.

[2] We shall naturally investigate in the book the cases when K depends upon the time, the state, the history of the evolution of the space. We shall also cover the case of solutions *which improve a reference preorder when time evolves*.

The first task is to characterize the subsets having this property. To be of value, this task must be done without solving the system and then checking the existence of viable solutions for each initial state.

An immediate intuitive idea jumps to the mind: at each point on the boundary of the viability set, where the viability of the system is at stake, there should exist a velocity which is in some sense *tangent* to the viability domain and serves to allow the solution to bounce back and remain inside it. This is, in essence, what the Viability Theorem states. But, first, the mathematical implementation of the concept of tangency must be made.

We cannot be content with viability sets that are smooth manifolds, because inequality constraints would thereby be ruled out. So, we need to "implement" the concept of a direction v tangent to K at $x \in K$, which should mean that starting from x in the direction v, we do not go too far from K.

To convert this intuition into mathematics, we shall choose from among the many ways there are to translate what it means to be "not too far" the one suggested by Bouligand fifty years ago: a direction v is *contingent to K at $x \in K$* if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such directions, which are in some sense "inward", constitutes a closed cone $T_K(x)$, called the *contingent cone*[3] to K at x. Naturally, except if K is a smooth manifold, we lose the fact that the set of contingent vectors is a vector-space.

We then associate with the dynamical system (described by f and U) and with the viability constraints (described by K) the *(set-valued) regulation map R_K*. It maps any state x to the subset $R_K(x)$ consisting of controls $u \in U(x)$ which are *viable* in the sense that

$$f(x,u) \quad \text{is contingent to } K \text{ at } x$$

If, for every $x \in K$, there exists at least one viable control $u \in R_K(x)$, we then say that K is a *viability domain* of the control system with dynamics described by both f and U.

---

[3] replacing the linear structure underlying the use of tangent spaces by the contingent cone is at the root of *Set-Valued Analysis*.

The Viability Theorem we mentioned earlier holds true for a rather large class of systems, called *Marchaud systems*: Beyond imposing some weak technical conditions, the only severe restriction is that, for each state x, the set of velocities $f(x, u)$ when u ranges over $U(x)$ is *convex*[4]. From now on, we assume that the systems under investigation are Marchaud systems.

The basic viability theorem states that for such systems,

a closed subset K is viable under a Marchaud system if and only if K is a viability domain of this system.

Many of the traditional interesting subsets such as *equilibrium points, trajectories of periodic solutions, the ω-limit sets of solutions, are examples of closed viability domains.* Actually, equilibrium points $\overline{x}$, which are solutions to

$$f(\overline{x}, \overline{u}) = 0 \text{ for some } \overline{u} \in U(\overline{x})$$

are the smallest viability domains, the ones reduced to a single point. This is because being *stationary states*, the velocities $f(\overline{x}, \overline{u})$ are equal to zero. Furthermore, there exists a basic and curious link between viability theory and general equilibrium theory:

every compact convex viability domain contains an equilibrium point.

This statement is an equivalent version of the 1910 *Brouwer Fixed Point Theorem*, the cornerstone of nonlinear analysis, which finds here a particularly relevant formulation (viability implies stationarity.)

What happens if a closed subset K is not a viability domain?

First, we characterize the points of the boundary from which some, or all solutions enter or leave the subset (anatomy of a set).

---

[4]This happens for the class of control systems of the form

$$x'(t) = f(x(t)) + G(x(t))u(t)$$

where $G(x)$ are linear operators from the control space to the state space and when the control set U (or the images $U(x)$) are convex.

Second, we also look for closed subsets of K which are viability domains. We shall prove that

there exists a largest closed viability domain contained in K.

This domain will be denoted $\text{Viab}(K)$ and called the *viability kernel*[5] of K. It may be empty (in this case, the subset K is some kind of "repeller".) Furthermore, every closed subset of the viability kernel is contained in a minimal viability domain, called *viability envelope.*

Third, one can also keep the set of constraints and change the dynamics, as it is done in mechanics of unilateral constraints (variational differential equations).

The Viability Theorem also provides a *regulation law* for regulating the system in order to maintain the viability of a solution: The viable solutions $x(t)$ are regulated by viable "open loop controls" $u(t)$ through the regulation law:

$$\text{for almost all } t, \quad u(t) \in R_K(x(t))$$

The multivaluedness of the regulation map (this means that several controls $u(t)$ may exist in $R_K(x(t))$) is an indicator of the "robustness" of the system: *The larger the set $R_K(x(t))$, the larger the set of disturbances which do not destroy the viability of the system !*

Observe that solutions to a control system are solutions to the differential inclusion $x'(t) \in F(x(t))$ where, for each state x, $F(x) := f(x, U(x))$ is the subset of feasible velocities, Conversely, a differential inclusion is an example of a control system in which the controls are the velocities ($f(x,u) = u$ & $U(x) = F(x)$.)

As far as servomechanisms are concerned, the question arises of how to build mechanisms for selecting a *unique* control $\widehat{u}(x)$ in $R_K(x)$ for each state x. Such a map $\widehat{u}(\cdot)$, associating with every x a single control $\widehat{u}(x)$ is called a *closed loop* control (or single-valued feedback.) This is because it allows the system to *automatically associate with*

[5]This concept of viability kernel happens to be a quite efficient mathematical tool that we shall use often.

It is also closely related to the concept of *zero dynamics* introduced recently by Byrnes and Isidori in control theory.

any state $x(t)$ *the control* $\widehat{u}(x(t))$ which produces a viable solution through the differential equation

$$x'(t) = f(x(t), \widehat{u}(x(t)))$$

An interesting example of closed loop control is provided by *slow solutions*. These are the solutions regulated by the controls $u^0(x) \in R_K(x)$ with minimal norm. Despite the fact that $u^0(\cdot)$ is not necessarily continuous, we shall prove that the above differential equation still has solutions. For instance, when the controls are the velocities of the system, viable solutions with *velocities of minimal norm* are implemented by such a selection procedure. This is why they are called *slow solutions*.

Such selection procedures by closed loop controls answer many engineering control problems. But they may not be adequate for the type of systems arising in economic, social, biological and cognitive sciences, as well as in some areas of engineering where the controls must evolve continuously. Here, we are looking for selection procedures which obey the *inertia principle*: keep the control constant as long as the viability of the system is not at stake.

We can reformulate the inertia principle by saying that if the derivative of a viable open loop control $u(\cdot)$ is equal to 0, then this control is the one which is chosen and implemented.

This raises several questions.

— The first one concerns controls which are smooth (at least, differentiable almost everywhere.) (This issue may be relevant for engineering problems, where the lack of continuity of controls $u(t) := \widehat{u}(x(t))$ can be damaging.)

— The second one deals with the problem of differentiating the regulation law.

— The third is to find selections (called *dynamical closed loops*) of the derivative of the regulation map, with which we obtain a system of differential equations which govern the *smooth* viable evolution of both the state and the control.

— The fourth is to find some feedback controls as solutions to systems of first-order partial differential inclusions.

We see at once that this programme requires a concept of derivative of a set-valued map and a chain rule formula in order to differentiate the regulation law.

The idea behind the construction of a differential calculus of set-valued maps is simple and goes back to the very origins of differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of a tangent to the graph of a function:

the tangent space to the graph of a function f at a point (x, y) of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \mapsto f'(x)u$

Consider now a set-valued map $F : X \rightsquigarrow Y$, which is characterized by its graph (the subset of pairs (x, y) such that y belongs to $F(x)$.)

The contingent cone to the graph of F at the point (x, y) of its graph is the graph of the contingent derivative of the set-valued map F at a point (x, y)

The contingent derivative at (x, y) is a set-valued map from X to Y denoted by $DF(x, y)$.

Contingent derivatives keep enough properties of the derivatives of smooth functions to be quite efficient. They enjoy a rich calculus, and they enable such basic theorems of analysis as the inverse function theorem to be extended to the set-valued case.

The chain rule is an example of a property which is still true in this framework: Assume that we start from a "smooth state", producing a viable solution $x(t)$ and a viable control $u(t)$ which are both differentiable (almost everywhere.) Then we can "differentiate" the regulation law to obtain a "first order regulation law":

$$\text{for almost all } t, \quad u'(t) \in DR_K(x(t), u(t))(x'(t))$$

Heavy viable solutions are the ones regulated by the controls whose velocities have minimal norm in the set

$$DR_K(x(t), u(t))(f(x(t), u(t)))$$

For instance, when the controls are the velocities of the system, we choose viable solutions with *acceleration of minimal norm*, i.e., accelerations with maximum inertia. This is why these solutions are called *heavy solutions.* This point of view leads to the introduction of *viability niches* $N(u)$ associated with controls u. These are (possibly empty) subsets consisting of states x such that the zero velocity belongs to $DR_K(x,u)(f(x,u))$. In such a *viability niche* $N(u)$, *the state can evolve while being regulated by the stationary control* u.

Finally, using the concept of contingent derivative, we can obtain feedbacks as solutions of partial differential inclusions.

Let us conclude this introduction with some motivational comments.

In economics, the viability constraints are the scarcity constraints. We can replace the fundamental Walrasian model[6] of resource allocations by a decentralized dynamical model in which the role of the controls is played by the prices[7] (as well as coalitions of consumers, interest rates, and so forth). The regulation law can be interpreted as the behavior of Adam Smith's invisible hand choosing the prices as a function of the allocations. It is possible that among these viable prices, the market (or even a planning bureau) would have a tendency to choose heavy solutions.

In cooperative games, coalitions of player may play the role of controls: each coalition acts on the environment by changing it through a dynamical system. Here, a coalition is described by the players's rate of participation, positive or negative, according to their cooperative or anti-cooperative behavior. The regulation law provides, in this case, an explanation of the evolution of coalitions and alliances.

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[6]Most static models of mathematical economics are based in the last analysis on *general equilibrium theory*. They can be reformulated in a dynamical framework, by slightly changing the underlying dynamical system. (Walrasian tâtonnement, which does not produce viable solutions, except when they reach an equilibrium.)

[7]and other fiduciary goods for which the scarcity constraint can be transgressed. Unlike physical goods, they are limited only by measures dictated by the trust (or, rather, the tolerance) of the agents. Any disequilibrium that cannot exist in physical goods can then be transferred to the fiduciary goods.

In noncooperative games, viability constraints describe power relations among players. Each players associates with each state a subset in which the other players are confined to choosing their own states. Strategies take the role of controls, through which the players act on the state according to some differential equations. We often observe that the inertia principle is operative. The choice of viable strategies (or of their velocities) can be made, at each instant and in a myopic way, by standard game theoretical mechanisms, in such a way as to comply with the inertia principle.

In genetics and population genetics, the viability constraints are the ecological constraints, the state describes the phenotype and the controls are genotypes or fitness matrices. The regulation law may explain the evolution of genotypes or fitness matrices derived from the dynamics and the ecological constraints.

In sociology, a society can be interpreted as a set of individuals subject to viability constraints. They correspond to what is necessary to the survival of the social organization. Laws and other cultural codes are then devised to provide each individual with psychological and economical means of survival as well as guidelines for avoiding conflicts. These cultural codes play the role of controls. The regulation law may represent the evolution of cultural codes for maintaining society's viability, the evolution of which obeys the inertia principle. Such a metaphor may account for the small number of them and the robustness of religions, ideologies and scientific paradigms. It may also explain the phenomena of massive conversions to new cultural codes.

In cognitive sciences, the state describes the sensory-motor couple of the cognitive system, while the control translates into what could be called a conceptual control (which is the synaptic matrix in neural networks.) The state and control are related by a pattern recognition mechanism, which recognizes the (variations of) the perception of the action of the automaton on the environment. The regulation law provides a learning process, that goes beyond simple stimulus-response processes: it associates with each sensory-motor state a subset of (learned) conceptual controls. It seems that in this case, again, the inertia principle applies.

Outline of the Book

Instead of beginning with viability theorems for differential inclusions, we prefer to sketch in Chapter 1 the role of the concept of viability domain in the much simpler case of differential equations. (The first viability theorem was proved in 1942 by Nagumo.)

For a variety of reasons, an important example of a viability set is the probability simplex. Whenever the state of a system is difficult to model mathematically, one way to overcome this difficulty is to deal with probabilities, frequencies, concentrations, proportions, etc., and the probability simplex then naturally appears. Systems controlled by scalar controls (called flux) of the form

$$x_i'(t) \quad = \quad x_i(t)(f_i(x(t)) - x(t)u), \;\; i = 1, \ldots, n$$

are called *replicator systems.* They are encountered in such diverse fields as biochemistry (Eigen & Schuster's hypercycle), ethology (Maynard-Smith's game for behavioral strategies), population dynamics (Fisher's model of the evolution of genes in a population), ecology, etc. These examples are presented in the first chapter.

We also included in this chapter viability and invariance theorems for *stochastic differential equations*, which provide another way to treat uncertainty.

This chapter can be bypassed by readers mainly interested in differential inclusions and control systems.

Chapter 2 deals with the minimal information about set-valued maps that is needed to prove the viability theorems for differential inclusions. Upper and lower semicontinuous set-valued maps are defined. Then our basic result, *the Convergence Theorem*, is proved.

Since this involves convex-valued maps, some results on support functions of convex subsets are recalled in this chapter. Closed convex processes, which are the set-valued analogues of continuous linear operators, enjoy most of the properties of linear operators, including Banach's closed graph theorem and the uniform boundedness theorem. These results are reviewed, because contingent derivatives of set-valued maps being closed processes, they will be used later.

Chapter 3 is basic: it states and proves the main viability theorems (in locally compact, open and closed viability sets respectively) and shows that the solution map is upper semicontinuous. We also prove a *stability result:* (upper) limits of viability domains are still viability domains and we show that *ω-limit sets* of solutions, limits of solutions when the time goes to infinity (equilibria), trajectories of periodic solutions are examples of closed viability domains.

We adapt Saari's principle on the chaotic behavior of discrete systems to the case of differential inclusions. The viability domain is divided into a number of cells in such a way that each of them can be "visited" in any given way by at least one trajectory of a differential inclusion.

We then proceed in Chapter 4 with further properties of the *viability kernels* of closed subsets: There exists a largest closed viability domain contained in a closed subset, called the viability kernel, which enjoys many properties which are investigated in this chapter. Important concepts of biomathematics such as *permanence* and *fluctuation* can be defined in terms of viability kernels. On the other hand, each closed subset of the viability kernel is contained in a minimal viability domain, called *viability envelope*.

The analysis is refined by introducing *exit time functions* associating with each initial state the first instant at which at least one solution starting from this state leaves the viability set. Viability kernels are the subsets of states with infinite exit time. We then introduce *exit tubes*, which are the subsets of states from which at least one solution satisfies the viability constraints for a prescribed length of time.

We then study the *anatomy of a set* by distinguishing inward and outward areas of the boundary of a set. It is also shown that the boundary of a viability kernel is also a viability domain.

These facts among others are used to study several *viability kernel algorithms*, including the *zero dynamics algorithm*, which converge to viability domains and/or kernels.

We devote the fifth chapter to the study of *invariant subsets*, which are sets K with the property that *all solutions* to a differential inclusion starting from a state in K are viable in K.

We need for that purpose more information on contingent cones, which are involved in a crucial way in the characterization of the viability and invariance properties. For this reason, we review some results about these cones before proceeding any further. We recall some useful formulas of the calculus of contingent cones (proved in the fourth chapter of *Set-Valued Analysis*.)

Several characterizations of invariance are provided, one of which is based on the fundamental Filippov Theorem dealing with differential inclusions with Lipschitz right-hand sides. It implies that the solution map is lower semicontinuous. This latter property is crucial to prove the existence of *invariance kernels*, which are the largest closed invariant domains contained in closed subsets.

It also implies the *semi-permeability* property of the boundary of the viability kernel of a closed subset, which states that no solution can cross the boundary to enter the interior of the viability kernel.

These invariance and viability kernels are needed to define *defeat and victory domains of a target.*

We illustrate these results in the case of *linear differential inclusions*, which are differential inclusions whose right-hand sides are closed convex processes. In this framework, we show that the concepts of invariance and viability domains are dual.

We tackle in Chapter 6 the problem of *regulating control systems by closed loop controls (single-valued feedback controls.)* The problem we have to solve is that of finding selections of the regulation map, possibly continuous. The latter are provided by Michael's Theorem, but in a non constructive way. Hence we have to design *selection procedures* which yield explicit selections, which may not be continuous, but still provide viable solutions when fed back to the differential equation governing the evolution of the control system. These selection procedures provide in particular *slow viable solutions* regulated by controls with minimal norm. For that purpose we need to complete our study of lower semicontinuous maps and provide lower semicontinuity criteria for finite and infinite intersections of lower semicontinuous maps.

Chapter 7 deals with *the inertia principle, heavy viable solutions and "punctuated equilibria", ramp controls, etc.*, which constitute the main motivations of viability theory.

At this point, we need to differentiate the regulation map. Hence this chapter starts with the shortest introduction to derivatives of set-valued maps needed to proceed. It continues with the construction of regulation maps providing viable controls that are almost everywhere differentiable.

Once we know the regulation maps yielding differentiable controls, we can differentiate the regulation law and discover the system of differential inclusions which governs the evolution of both the state and the control of the system. Then, by using the selection procedures introduced in the preceding chapter, we are able to define dynamical closed loops and, among them, the ones which provide heavy viable solutions. Viability problems for second order differential inclusions, which are first order systems in disguise, are also investigated in this chapter.

The *tracking problem*, as well as *observability, decentralization, hierarchical* issues, are studied in Chapter 8 in the framework of viability theory. The common thread of these problems is the connection between two dynamical systems through an *observation map*: Are some or all solutions to these differential inclusions linked by this observation map, in the sense that its graph is a viable or invariant manifold ? The viability theorems applied to the graphs of the observation maps imply that such observation maps are solutions to some *systems of first-order partial differential inclusions*, where the derivatives are taken in the contingent sense.

Derivatives in the sense of distributions do not offer the unique way to describe weak or generalized solutions to partial differential equations and inclusions. Contingent derivatives offer another way to weaken the required properties of a derivative, loosing the linear character of the differential operator, but allowing a pointwise definition. They provide a convenient way to treat hyperbolic problems and also allow us to look for solutions among set-valued maps, since we know how to differentiate them. Set-valued solutions constitute a useful framework to describe shocks for instance.

We study the existence of both single-valued and set-valued solutions to such partial differential inclusions, as well as a variational principle.

Differential inequalities, Lyapunov functions and related matters can also be analyzed in terms of special viability problems where the viability sets are epigraphs of functions or, more generally, graphs of preorders. This allows us to include, among the candidates that enjoy Lyapunov-type inequalities, not only differentiable functions but also lower semicontinuous functions. Thus we derive from viability theorems several generalizations of classical results. Applying to this situation the concept of viability kernel, we infer the existence of the *smallest Lyapunov function larger than a given one.*

Asymptotic stability is treated here in the framework of viability theory. These are explained in Chapter 9.

Chapter 10 gathers miscellaneous issues, such as *variational differential inequalities*. The question is the following: If we take a differential inclusion and a closed subset which is not a viability domain, can we modify the set-valued map F in such a way that K becomes a viability domain for the new map? The method is straightforward: we project the images $F(x)$ onto the contingent cone $T_K(x)$ (and obtain, when K is convex, variational differential inequalities.) By doing so, we lose both the convexity of the images and the upper semicontinuity. However, it is still possible to prove the existence of the projected system and even, under stronger assumptions, the existence of slow solutions.

The second section of chapter 10 deals with *fuzzy differential inclusions*. The right-hand sides of such differential inclusions are *fuzzy* subsets, whose membership functions are cost functions taking their values in $[0, \infty]$ (instead of $[0, 1]$ for membership functions of usual fuzzy sets). The concept of uncertainty involved in differential inclusions becomes more refined, by allowing the velocities not only to depend in a plain multivalued way upon the state of the system, but also in a fuzzy way.

The viability theorems are adapted to fuzzy differential inclusions and to sets of state constraints which are either usual or fuzzy. The existence of a largest closed fuzzy viability domain contained in a given closed fuzzy subset is also provided.

The third section presents a very short introduction to some numerical aspects of differential inclusions. The convergence of solutions to implicit and projected explicit finite-difference schemes to viable solutions of a differential inclusion is proved.

The fourth section deals with the adaptation of continuous Newton's methods for finding an equilibrium of a set-valued map: it happens that this is also a viability problem.

Chapter 11 is devoted to *time-dependent viability sets* $t \rightsquigarrow P(t)$, naturally called *tubes*. Tubes which contain at least one viable solution[8] starting from any initial state $x_0 \in P(t_0)$ at any initial time are viability tubes. These are solutions to a set-valued differential inclusion of the form $F(x) \cap DP(t, x)(1) \neq \emptyset$.

We will study the Cauchy problem, where we look for minimal viability tubes satisfying an initial condition.

One can show that their "limits"[9] when $t \to \infty$ are viability domains, and actually, attractors. If we use such viability tubes to guide a solution towards a target, we see that a necessary condition for a subset to be an asymptotic target is that it is a viability domain.

Of much greater importance for systems arising in biology, economics and cognitive sciences is the case when *both the velocity and viability sets depend upon the history of the evolution of the state.* Delays

$$\forall\, t \geq 0,\ \ x(t) \ \in\ M(x(t - \theta_1), \ldots, x(t - \theta_p))$$

[8]in the sense that $x(t) \in P(t)$ for all t.

[9]in the sense of upper limits. When the tube $P(t) := \{x(t)\}$ is single-valued, this upper limit boils down to the ω-limit set.

accumulated consequences of past evolution

$$\forall\, t \geq 0, \;\; x(t) \;\in\; M\left(\int_{-\infty}^{t} A(t-s)x(s)ds\right)$$

all these features fall under the case called *functional viability*. Here, functional viability sets $\mathcal{K}$ are subsets consisting of time-dependent functions, and viable solutions are the solutions which evolve in such function subsets in the sense that for all $t \geq 0$, $x(t+\cdot) \in \mathcal{K}$. It is the topic of Chapter 12.

Can viability theorems be extended to partial differential equations and inclusions? The answer is positive, at least for elliptic and parabolic type inclusions, as is shown in chapter 13. In this case, viability sets are comprised of spatial functions (functions depending upon the space variable.) The situation becomes more complex, because we have to work with unbounded operators on Hilbert spaces, but still, the statements which are expected to be true can be proved.

Chapter 14 treats differential games, where the controls are regarded as *strategies* used by the players to govern the evolution of the states of the game. Here, intertemporal criteria involved classically in differential games are replaced by viability constraints representing *power relations* among players, describing the constraints imposed by one player on the other. We characterize winability, playability properties adequately defined by *contingent Isaacs' equations*.

We shall prove the existence of continuous single-valued playable feedbacks, as well as more constructive, but discontinuous, playable feedbacks, such as the feedbacks associating in a myopic way optimal strategies in a cooperative framework or minimax strategies in a noncooperative environment.

In other words, *the players can implement playable feedbacks by playing for each state a static game on the strategies.*

We also provide closed loop decision rules, which *operate on the velocities of the strategies*, (regarded as *decisions*).

Chapter 1

Viability Theorems for Ordinary and Stochastic Differential Equations

Introduction

This chapter is meant to be an *independent* introduction to the basic theorems of viability theory in the simple framework of ordinary differential equations $x' = f(x)$ and stochastic differential equations

$$d\xi \;=\; f(\xi(t))dt + g(\xi(t))dW(t)$$

It can be omitted by readers who are only interested in the theory for differential inclusions.

So, we begin by tackling the viability issue by isolating it in the framework of ordinary differential equations in the first section. A function $[0, T] \ni t \to x(t)$ is said to be *viable* in a given subset K on $[0, T]$ if, for any $t \in [0, T]$, the state $x(t)$ remains in K.

Actually, even in the simpler situation of differential equations, we have to be careful and make a distinction between two neighboring concepts: *viability property and invariance property*. The first one requires that, starting from any initial point of K, *at least one* *solution* to the differential equation is viable in K whereas the second one demands that *all solutions* are viable in K.

We shall characterize the first one by saying that K is a *viability domain*, i.e., that for any state x in the boundary of K, the velocity is tangent in some sense to K at x.

We may require for that purpose that K is a smooth manifold and therefore, that f is a vector field, the velocity $f(x)$ lying in the tangent space.

But first, we do not need the fact that the space of "tangent" directions (adequately defined) is a vector space. The added luxury of linearity does not compensate for its fragility, in the sense that, for instance, the intersection of two smooth manifolds is no longer smooth. Since we shall regard in most of our applications the subset K as a subset defined by constraints (and above all, inequality constraints), then it is very exceptional that such a subset is smooth. As in optimization, we are quickly led to assume that K is convex, since convex subsets are defined by linear inequality constraints. But, here again, it would be nice to dispense with this assumption if this is possible (with no added mathematical cost), for it allows us to consider also union of convex subsets, for instance.

As we know since 1942[1], one can characterize such a viability property for any closed subset K, with an adequate mathematical implementation of the concept of tangency. The one chosen is actually equivalent to the concept of "contingency" introduced ten years earlier by Bouligand. We can then define the contingent cone to K at $x \in K$ for any subset K, the price to pay being that the set of tangent directions (the contingent cone) is a closed cone instead of a vector space.

We shall only give in this introductory chapter the definition of the contingent cone and provide further properties in Chapter 5 after the presentation of the viability theorems for both ordinary differential equations, stochastic differential equations and differential inclusions.

Nagumo's Theorem states that when f is continuous, a *closed*

[1] In a seminal paper written in German by the Japanese mathematician M. Nagumo. As it could be expected, this theorem was forgotten and rediscovered (at least) fourteen times up to 1968, in different contexts, with various concepts of tangency.

subset K enjoys the viability property if and only if it is a viability domain. We prove only this theorem, and shall derive the other properties as corollaries of statements we shall prove later in the case of differential inclusions.

Many proofs of viability theorems are now available: we chose the most elementary (which is not the shortest) for several reasons: it is the prototype of the extensions of the viability theorems (to functional differential inclusions, partial differential inclusions, ...) we shall present later in this book. It is just a modification of the Euler method of approximating a solution by piecewise linear functions (polygonal lines) in order to force the solution to remain in K. Despite its "constructionist" look, this method is not a finite difference scheme (explicit or implicit.) We shall present a rudimentary numerical introduction in the third section, but are forced to postpone the proofs to chapter 10, because they use more properties of the contingent cones which are presented later in Chapter 5.

The fourth section is dedicated to the "replicator systems". This is because the most popular viability domain is the *probability simplex.* Indeed, it is often too difficult to provide a mathematical description of the state space of problems arising in biology, economics, etc. So, this difficulty is bypassed by studying instead of the evolution of the state itself, the evolution of frequencies, concentrations, probabilities, ..., of the states (without forgetting *mixed strategies* in game theory), which all range over the probability simplex $S^n \subset R^n$.

Replicator systems

$$x_i'(t) \;=\; x_i(t)(g_i(x(t)) - \tilde{u}(x(t)))$$

are the differential equations derived from evolutions

$$x_i'(t) \;=\; x_i(t)g_i(x(t)), \;\; (i = 1, \ldots, n)$$

governed by specific growth rates $g_i(\cdot)$, corrected by subtracting the closed-loop control

$$\tilde{u}(x) \;:=\; \sum_{j=1}^{n} x_j g_j(x)$$

for obeying the viability constraints. The celebrated logistic equation belongs to this class (for constant growth rates.) Dynamical models arising in population genetics, prebiotic evolution, sociobiology and population ecology devised independently are replicator systems for specific linear growth rates[2].

Finally, we conclude this introductory chapter with a brief presentation of viability and/or invariance properties of closed subsets for stochastic differential equations.

Let us consider Lipschitz maps f and g and the stochastic differential equation

$$d\xi \;=\; f(\xi(t))dt + g(\xi(t))dW(t)$$

the solution of which is given by the formula

$$\xi(t) \;=\; \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s)$$

We want to characterize the *(stochastic) viability property* of a closed convex subset K of X with respect to the pair (f,g): for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at x which is *viable in* K, in the sense that

$$\forall\, t \in [0,T], \;\; \text{for almost all } \omega \in \Omega, \;\; \xi_\omega(t) \;\in\; K$$

For that purpose, we adapt to the stochastic case the concept of contingent cone to a subset K at a random variable $x \in K$ as the set $\mathcal{T}_K(t,x)$ of pairs (γ, v) of random variables satisfying the following property: There exist sequences of $h_n > 0$ converging to 0 and of measurable random variables a^n and b^n satisfying for almost all $\omega \in \Omega$,

$$\forall\, n \geq 0, \;\; x_\omega + v_\omega(W_\omega(t+h_n) - W(t)) + h_n\gamma_\omega + h_n a_\omega^n + \sqrt{h_n}b_\omega^n \in K$$

and converging to 0 in some sense.

Then we shall prove in essence that the following conditions are equivalent:

1. — The subset K enjoys the viability property with respect to the pair (f,g)

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[2]Replicator systems are the central theme of the monograph THE THEORY OF EVOLUTION AND DYNAMICAL SYSTEMS by J. Hofbauer and K. Sigmund.

2. — for every $\mathcal{F}_t$-random variable x viable in K,

$$(f(x), g(x)) \in \mathcal{T}_K(t, x)$$

For instance, this condition means that for every $\mathcal{F}_t$-random variable x viable in K

$$f(x) \in K \ \& \ g(x) \in K$$

when K is a vector subspace, that

$$\langle x, g(x) \rangle = 0 \ \& \ \langle x, f(x) \rangle + \frac{1}{2} \|g(x)\|^2 = 0$$

when K is the unit sphere and that

$$\langle x, g(x) \rangle = 0 \ \& \ \langle x, f(x) \rangle + \frac{1}{2} \|g(x)\|^2 \leq 0$$

when K is the unit ball.

1.1 Viability & Invariance Properties

Definition 1.1.1 (Viable functions) *Let K be a subset of a finite dimensional vector-space*[3] *X. We shall say that a function $x(\cdot)$ from $[0, T]$ to X is* viable in K on $[0, T]$ *if* $\forall\, t \in [0, T], \ x(t) \in K$.

Let us describe the (deterministic) dynamics of the system by a (single-valued) map f from some open subset Ω of X to X. We consider the initial value problem (or Cauchy problem) associated with the differential equation

$$\forall\, t \in [0, T], \ x'(t) = f(x(t)) \tag{1.1}$$

satisfying the initial condition $x(0) = x_0$.

Definition 1.1.2 (Viability & Invariance Properties) *Let K be a subset of Ω. We shall say that K* is locally viable under *f (or enjoys the local* viability property *for the map f) if for any initial state x_0 of K,* there exist *$T > 0$ and a* viable *solution on $[0, T]$ to differential*

[3]or even, a normed space.

equation (1.1) starting at x_0. It is said to be (globally) viable under f (or to enjoy the global *viability property or, simply, the viability property) if we can always take $T = \infty$.*

The subset K is said to be invariant under f *(or enjoy the* invariance property*) if for any initial state x_0 of K,* all *solutions to differential equation (1.1) (a priori defined on Ω) are* viable in K.

Remark — We should emphasize that the concept of invariance *depends upon the behavior of f on the domain Ω outside K.* But we observe that viability property depends only on the behavior of f on K. □

So, the viability property requires only the existence of at least one viable solution whereas the invariance property demands that all solutions are viable.

We shall begin by characterizing the subsets K which are viable under f. The idea is simple, intuitive and makes good sense: *A subset K is viable under f if at each state x of K, the velocity $f(x)$ is "tangent" to K at x, so to speak, for bringing back a solution to the differential equation inside K.*

But we do not want to restrict ourselves to the case of *smooth* domains (i.e., differential manifolds) only for the pleasure of obtaining a vector space of tangent directions or to conform to tradition. There are many reasons for this, the first one being that simple operations on subsets — such as the intersection of manifolds — destroy their smoothness. Since we shall perform operations on viability subsets, we have to dispense with this requirement and look for other ways of implementing the idea of tangency to any subset. In economics and ecology, for instance, viability subsets are defined by a family of equality or inequality constraints. They are not differential manifolds. The best we can hope for is that they are convex, which happens, for instance, when the constraints are linear.

Naturally, by trying to define adequate concepts of tangency to nonsmooth subsets, we expect to lose some nice properties of the tangent space, and, among them, the fact that tangent spaces are vector spaces. The price to pay is then to deal with *closed cones* instead. Actually, under some regularity conditions, we shall do even better, and obtain, closed *convex*[4] cones.

We shall postpone the study of tangent cones to Chapter 5, when we will need them, after having provided a strong justification of their usefulness. An exhaustive presentation can be found in Chapter 4 of SET-VALUED ANALYSIS.

Meanwhile, we shall just provide the definition of the *contingent cone*, introduced by Bouligand in the early thirties, with which we shall characterize the viability property by following our intuitive idea.

Definition 1.1.3 *Let X be a normed space, K be a nonempty subset of X and x belong to K. The* contingent cone to K at x *is the set*

$$T_K(x) = \left\{ v \in X \mid \liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0 \right\}$$

where $d_K(y)$ denotes the distance of y to K, defined by

$$d_K(y) \ := \ \inf_{z \in K} \|y - z\|$$

In other words, v *belongs to* $T_K(x)$ *if and only if there exist a sequence of* $h_n > 0$ *converging to* $0+$ *and a sequence of* $v_n \in X$ *converging to* v *such that*

$$\forall\, n \geq 0, \ \ x + h_n v_n \ \in \ K$$

We see easily that

$$\forall\, x \in \mathrm{Int}(K), \ \ T_K(x) = X \tag{1.2}$$

Therefore, *when K is open, the contingent cone to K at any point $x \in K$ is always equal to the whole space.* The converse is not true.

We also observe that when K is a differential manifold, the contingent cone $T_K(x)$ coincides with the tangent space to K at x, and we shall check later that when K is convex, it coincides with the tangent cone of convex analysis. The lemma below shows right away why these cones will play a crucial role: they appear naturally whenever we wish to differentiate viable functions.

[4]In this case, we will be able to use duality, by associating biunivocally *polar* cones to closed convex cones, and use the *bipolar Theorem* (Theorem 2.3.3.)

Lemma 1.1.4 *Let $x(\cdot)$ be a differentiable* viable *function from $[0,T]$ to K. Then*

$$\forall\, t \in [0,T[, \quad x'(t) \in T_K(x(t))$$

Definition 1.1.5 (Viability Domain) *Let K be a subset of Ω. We shall say that K is a* viability domain *of the map $f : \Omega \mapsto X$ if*

$$\forall\, x \in K, \;\; f(x) \in T_K(x) \tag{1.3}$$

Example — We first give the simple example of finite dimensional vector-spaces which are viability domains of linear operators.

Definition 1.1.6 *Let A be a linear operator from a finite dimensional vector-space X to itself. We shall say that a finite dimensional vector-subspace K is* invariant *under A if*

$$A(K) \subset K$$

The following statement is naturally obvious.

Proposition 1.1.7 *Let us consider a linear operator A from a finite dimensional vector-space X to itself, elements $b, c \in X$ and a subspace K of X.*
The affine space $K + c$ is a viability domain of the affine operator $x \to Ax + b$ if and only if

$$\begin{cases} i) & K \text{ is invariant under } A \\ ii) & Ac + b \in K \quad \square \end{cases}$$

1.2 Nagumo Theorem

Nagumo was the first one to prove the viability theorem for ordinary differential equations in 1942. This theorem was apparently forgotten, for it was rediscovered many times during the next twenty years[5].

[5] This does not prove that the statement is true...

Theorem 1.2.1 (Nagumo) *Let us assume that*

$$\begin{cases} i) & K \text{ is locally compact} \\ ii) & f \text{ is continuous from } K \text{ to } X \end{cases} \quad (1.4)$$

Then K is locally viable under f if and only if K is a viability domain of f.

Since the contingent cone to an open subset is equal to the whole space (see (1.2)), an open subset is a viability domain of any map. So, it enjoys the viability property because any open subset of a finite dimensional vector-space is locally compact. The Peano existence theorem is then a consequence of Theorem 1.2.1.

Theorem 1.2.2 (Peano) *Let Ω be an open subset of a finite dimensional vector-space X and $f : \Omega \mapsto X$ be a continuous map.*

Then, for every $x_0 \in \Omega$, there exists $T > 0$ such that differential equation (1.1) has a solution on the interval $[0, T]$ starting at x_0.

The interesting case from the viability point of view is the one when the viability subset is *closed.* In this case, we derive from Theorem 1.2.1 a more precise statement.

Theorem 1.2.3 (Viability) *Let us consider a* closed *subset K of a finite dimensional vector-space X and a* continuous *map f from K to X.*

If K is a viability domain, *then for every initial state $x_0 \in K$, there exist a positive T and a* viable *solution on $[0, T]$ to differential equation (1.1) starting at x_0 such that*

$$\begin{cases} \textit{either} & T = \infty \\ \quad \textit{or} & T < \infty \quad \textit{and} \quad \limsup_{t \to T-} \|x(t)\| = \infty \end{cases} \quad (1.5)$$

Further adequate information — a priori estimates on the growth of f — allows us to exclude the case when $\limsup_{t \to T-} \|x(t)\| = \infty$.

This is the case for instance when f is bounded on K, and, in particular, when K is bounded.

More generally, we can take $T = \infty$ when f enjoys linear growth:

Theorem 1.2.4 *Let us consider a subset K of a finite dimensional vector-space X and a map f from K to X. We assume that the map f is* continuous *from K to X, that*

$$\exists\, c > 0 \quad \textit{such that } \forall\, x \in K, \ \ \|f(x)\| \ \le \ c(\|x\| + 1)$$

and that

$$K \text{ is a closed viability domain of } f$$

Then K is viable under f: for every initial state $x_0 \in K$, there exists a viable solution on $[0, \infty]$ *to differential equation (1.1) starting at x_0.*

We shall prove only Theorem 1.2.1. The proofs of the other theorems are classical and are the same as the ones for analogous statements for differential inclusions (see Chapter 3).

Proof of Theorem 1.2.1

a) — **Necessary Condition**

Let us consider a viable solution $x(\cdot)$ to differential equation (1.1.) It is easy to check that $f(x_0) = x'(0)$ belongs to the contingent cone $T_K(x_0)$ because $x(h)$ belongs to K and consequently, the inequality

$$d_K(x_0 + hf(x_0))/h \le \|x(0) + hx'(0) - x(h)\|/h$$

implies that

$$\lim_{h \to 0+} d_K(x_0 + hf(x_0))/h \ = \ 0$$

Hence K is a viability domain.

b) — **Sufficient Condition**

As quite often happens in analysis, the existence proof can be split into three steps. We begin by constructing approximate solutions by modifying Euler's method to take into account the viability constraints, we then deduce from available estimates that a subsequence of these solutions converges uniformly to a limit, and finally check that this limit is a viable solution to differential equation (1.1.)

1. — **Construction of Approximate Solutions**

Since K is locally compact, there exists $r > 0$ such that the ball $B_K(x_0, r) := K \cap (x_0 + rB)$ is *compact*. When C is a subset, we set

$$\|C\| := \sup_{v \in C} \|v\|$$

and

$$K_0 := B_K(x_0, r), \; C := B(f(K_0), 1), \; T := r/\|C\|$$

We observe that *C is bounded* since *K_0 is compact.* We begin by proving

Lemma 1.2.5 *For any integer m, there exists $\theta_m \in]0, 1/m[$ such that for all $x \in K_0$, there exist $h \in [\theta_m, 1/m]$ and $u \in X$ satisfying*

$$\begin{cases} i) & u \in C \\ ii) & x + hu \in K \\ iii) & (x, u) \in B(\mathrm{Graph}(f), 1/m) \end{cases}$$

Proof of Lemma 1.2.5 — Since K is a *viability domain* of f, we know that for all $y \in K$, $f(y)$ belongs to $T_K(y)$. By definition of the contingent cone, there exists $h_y \in]0, 1/m[$ such that

$$d_K(y + h_y f(y)) < h_y/2m$$

We introduce the subsets

$$N(y) := \{x \in K \mid d_K(x + h_y f(y)) < h_y/2m\}$$

These subsets are obviously *open.* Since y belongs to $N(y)$, there exists $\eta_y \in]0, 1/m[$ such that $B(y, \eta_y) \subset N(y)$. The compactness of K_0 implies that it can be covered by q such balls $B(y_j, \eta_j)$, $j = 1, \ldots, q$. We set

$$\theta_m := \min_{j=1,\ldots,q} h_{y_j} > 0$$

Let us choose any $x \in K_0$. Since it belongs to one of the balls $B(y_j, \eta_j) \subset N(y_j)$, there exists $z_j \in K$ such that

$$\begin{aligned} & \|x + h_{y_j} f(y_j) - z_j\|/h_{y_j} \\ & \leq d_K(x + h_{y_j} f(y_j))/h_{y_j} + 1/2m \leq 1/m. \end{aligned}$$

Let us set $u_j := \frac{z_j - x}{h_{y_j}}$. We see that $\|x - y_j\| \leq \eta_j \leq 1/m$, that $x + h_{y_j} u_j = z_j \in K$ and that $\|u_j - f(y_j)\| \leq 1/m$. Therefore,

$$(x, u_j) \in B((y_j, f(y_j)), 1/m) \subset B(\text{Graph}(f), 1/m)$$

and $u_j \in B(f(K_0), 1/m) \subset C$. The Lemma ensues. □

We can now construct by induction a sequence of positive numbers $h_j \in]\theta_m, 1/m[$ and a sequence of elements $x_j \in K_0$ and $u_j \in C$ such that

$$\begin{cases} i) \quad x_{j+1} := x_j + h_j u_j \in K_0, \; u_j \in C \\ ii) \quad (x_j, u_j) \in B(\text{Graph}(f), 1/m) \end{cases}$$

so long as $\sum_{i=0}^{j-1} h_i \leq T$.

Indeed, the elements x_j belong to K_0, since

$$\begin{array}{l} \|x_j - x\| \leq \sum_{i=0}^{i=j-1} \|x_{i+1} - x_i\| \\ \leq \sum_{i=0}^{i=j-1} h_i \|C\| \leq T\|C\| = r \end{array}$$

Since the h_j's are larger than or equal to $\theta_m > 0$, there exists J such that

$$h_1 + \cdots + h_{J-1} \leq T \leq h_1 + \cdots + h_J$$

We introduce the nodes $\tau_m^j := h_0 + \cdots + h_{j-1}, j = 1, \ldots, J+1$ and we interpolate the sequence of elements x_j at the nodes τ_m^j by the piecewise linear functions $x_m(t)$ defined on each interval $[\tau_m^j, \tau_m^{j+1}[$ by

$$\forall\, t \in [\tau_m^j, \tau_m^{j+1}[, \; x_m(t) := x_j + (t - \tau_m^j) u_j$$

We observe that this sequence satisfies the following estimates

$$\begin{cases} i) \quad \forall\, t \in [0, T], \; x_m(t) \in \text{co}(K_0) \\ ii) \quad \forall\, t \in [0, T], \; \|x'_m(t)\| \leq \|C\| \end{cases} \tag{1.6}$$

Let us fix $t \in [\tau_m^j, \tau_m^{j+1}[$. Since $\|x_m(t) - x_m(\tau_m^j)\| = h_j \|u_j\| \leq \|C\|/m$, and since (x_j, u_j) belongs to $B(\text{Graph}(f), 1/m)$ by Lemma 1.2.5, we deduce that these functions are approximate solutions in the sense that

$$\begin{cases} i) & \forall\, t \in [0,T],\ \ x_m(t) \in B(K_0, \varepsilon_m) \\ ii) & \forall\, t \in [0,T],\ (x_m(t), x'_m(t)) \in B(\mathrm{Graph}(f), \varepsilon_m) \end{cases} \tag{1.7}$$

where $\varepsilon_m := (\|C\| + 1)/m$ converges to 0.

2. — **Convergence of the Approximate Solutions**

Estimates (1.6) imply that for all $t \in [0,T]$, the sequence $x_m(t)$ remains in the compact subset $\mathrm{co}(K_0)$[6] and that the sequence $x_m(\cdot)$ is *equicontinuous*, because the derivatives $x'_m(\cdot)$ are bounded. We then deduce from Ascoli's Theorem[7] that it remains in a compact subset of the Banach space $\mathcal{C}(0,T;X)$, and thus, that a subsequence (again denoted) $x_m(\cdot)$ converges uniformly to some function $x(\cdot)$. Furthermore, the sequence $x'_m(\cdot)$ also converges to $x'(\cdot)$ because $x'_m(t) = f(x_m(t))$ and f is uniformly continuous on the compact $\mathrm{co}(K_0)$.

3. — **The Limit is a Solution**

Condition (1.7)i) implies that

$$\forall\, t \in [0,T],\ \ x(t) \in K_0$$

[6]The *(closed) convex hull* of a subset is the intersection of the (closed) convex subsets which contain it. The convex hull of a compact subset is also compact.

[7]Let us recall that a subset $\mathcal{H}$ of continuous functions of $\mathcal{C}(0,T;X)$ is *equicontinuous* if and only if

$$\forall\, t \in [0,T],\ \forall\, \varepsilon > 0,\ \exists\, \eta\ :=\ \eta(\mathcal{H}, t, \eta)\ |\ \forall\, s \in [t-\eta, t+\eta],\ \sup_{x(\cdot)\in\mathcal{H}} \|x(t) - x(s)\| \le \varepsilon$$

Locally Lipschitz functions with the same Lipschitz constant form an equicontinuous set of functions. In particular, a subset of differentiable functions satisfying

$$\sup_{t\in[0,T]} \|x'(t)\| \le c < +\infty$$

is equicontinuous.

Ascoli's Theorem states that a subset $\mathcal{H}$ of functions is *relatively compact* in $\mathcal{C}(0,T;X)$ if and only if it is equicontinuous and satisfies

$$\forall\, t \in [0,T],\ \mathcal{H}(t)\ :=\ \{x(t)\}_{x(\cdot)\in\mathcal{H}} \text{ is compact.}$$

i.e., that $x(\cdot)$ is viable.

Since f is uniformly continuous on K_0, then for all $\varepsilon > 0$, there exists $\eta \in]0, \varepsilon[$ such that

$$\|f(x) - f(y)\| \leq \varepsilon \text{ whenever } \|x - y\| \leq \eta$$

Since the sequence $x_m(\cdot)$ converges uniformly to $x(\cdot)$ and since property (1.7)ii) holds true, we deduce that for large m and for all $t \in [0, T]$, there exists $u_m^t \in X$ such that

$$\begin{cases} \|x'_m(t) - f(x(t))\| \\ \leq \|x'_m(t) - f(u_m^t)\| + \|f(u_m^t) - f(x_m(t))\| + \|f(x_m(t)) - f(x(t))\| \\ \leq 3\varepsilon \end{cases}$$

so that

$$\left\| x_m(t) - x_0 - \int_0^t f(x(s))ds \right\| \leq \int_0^t \|x'_m(s) - f(x(s))\| ds \leq 3\varepsilon t$$

By letting m go to ∞, these inequalities imply that

$$\forall\, t \in [0, T], \;\; x(t) = x_0 + \int_0^t f(x(s))ds$$

Hence the limit $x(\cdot)$ is a solution to differential equation (1.1), and thus, K enjoys the viability property. □

1.3 Numerical Schemes

A natural approximation scheme for approximating viable solutions to differential equations is the projected explicit difference scheme

$$x_{j+1} = \pi_K(x_j + hf(x_j))$$

where $h > 0$ is fixed and where π_K denotes a selection of the projector of best approximation Π_K defined by

$$y \in \Pi_K(x) \iff y \in K \;\; \& \;\; \|y - x\| = d_K(x)$$

Let us observe that π_K satisfies the property

$$\forall\, z \in K, \; \forall\, x \in X, \;\; \|\pi_K(x) - z\| \leq 2\|x - z\|$$

because, whenever $y \in \Pi_K(x)$,

$$\|y - z\| \leq \|y - x\| + \|x - z\| = d_K(x) + \|x - z\| \leq 2\|x - z\|$$

When K is convex, Π_K is a Lipschitz single-valued map (with Lipschitz constant equal to 1.)

Projectors of best approximation are instances of *quasi-projectors*:

Definition 1.3.1 *We shall say that a map Γ_K from X onto K satisfying*

$$\begin{cases} i) & \forall\, z \in K, \;\; \Gamma_K(z) = z \\ \\ ii) & \exists\, \lambda > 0 \;\; \textit{such that} \\ & \forall\, x \in X, \forall\, z \in K, \;\; \|\Gamma_K(x) - z\| \;\leq\; \lambda\|x - z\| \end{cases}$$

is a quasi-projector onto K.

There are many other examples of quasi-projectors. They enjoy the following property:

Lemma 1.3.2 *Let Γ_K be a quasi-projector from X onto K. Then $\|\Gamma_K(x + hv) - x - hv\| \leq (\lambda + 1)d_K(x + hv)$ so that, for instance,*

$$\forall\, v \in T_K(x), \;\; \liminf_{h \to 0+} \frac{\|\Gamma_K(x + hv) - x - hv\|}{h} = 0$$

We can associate with any quasi-projector a projected explicit difference scheme providing a sequence x_j starting from x_0 and defined by

$$x_{j+1} \;:=\; \Gamma_K(x_j + hf(x_j)) \tag{1.8}$$

and an approximate viable solution $x_h(\cdot)$ which is the piecewise linear function interpolating this sequence on the nodes $\tau_h^j := jh$ defined by $x_h(t) := x_j + (t - jh)(x_{j+1} - x_j)/h$ on the intervals $[jh, (j + 1)h[$.

Theorem 1.3.3 *Let us consider a continuous map f from a compact subset $K \subset X$ to X such that, for every $x \in K, f(x) \in T_K(x)$, and a quasi-projector Γ_K. Then, starting from $x_0 \in K$, the solutions to the projected explicit difference scheme (1.8) converge to a viable solution to differential equation $x' = f(x)$ when $h \to 0+$, in the sense that a subsequence of the piecewise linear functions x_h which interpolates the x_j's on the nodes jh converges uniformly to a viable solution $x(\cdot)$.*

This is a corollary of the set-valued version Theorem 10.3.2 of our statement.

Remark — Actually, when f is not continuous, the proof shows that the solutions to the projected explicit scheme converge to viable solutions so long as property

$$\forall\, x \in K, \quad \lim_{h \to 0+, K \ni y \to x} \frac{d_K(y + hf(y))}{h} = 0 \quad \square \tag{1.9}$$

(which is a consequence of the continuity of f) holds true.

Remark — When the viability domain K of f is convex and compact, we can derive from the Equilibrium Theorem 3.7.6 below that there exists a viable solution to the implicit finite difference scheme

$$x_{j+1} \;=\; x_j + hf(x_{j+1}) \;\;\&\;\; x_{j+1} \in K$$

starting from x_0. $\square$

1.4 Replicator Systems

We begin by studying the viability property of the probability simplex

$$S^n := \left\{ x \in \mathbf{R}^n_+ \mid \sum_{i=1}^n x_i = 1 \right\}$$

This is the most important instance of a viability set, because, in many problems, it is too difficult to describe the state of the system mathematically. We shall provide examples later in this section.

But for recognizing whether the simplex is the viability domain of some differential equation, we need to compute its contingent cones.

Lemma 1.4.1 *The contingent cone $T_{S^n}(x)$ to S^n at $x \in S^n$ is the cone of elements $v \in \mathbf{R}^n$ satisfying*

$$\sum_{i=1}^n v_i = 0 \;\;\&\;\; v_i \geq 0 \;\text{ whenever }\; x_i = 0 \tag{1.10}$$

We provide a direct proof of this lemma, which is a consequence of the calculus of contingent cones.

Proof — Let us take $v \in T_{S^n}(x)$. There exist sequences $h_p > 0$ converging to 0 and v_p converging to v such that $y_p := x + h_p v_p$ belongs to S^n for any $p \geq 0$. Then

$$\sum_{i=1}^{n} v_{p_i} = \frac{1}{h_p}\left(\sum_{i=1}^{n} y_{p_i} - \sum_{i=1}^{n} x_{p_i}\right) = 0$$

so that $\sum_{i=1}^{n} v_i = 0$. On the other hand, if $x_i = 0$, then $v_{p_i} = y_{p_i}/h_p \geq 0$, so that $v_i \geq 0$.

Conversely, let us take v satisfying (1.10) and deduce that $y := x + hv$ belongs to the simplex for h small enough. First, the sum of the y_i is obviously equal to 1. Second, $y_i \geq 0$, either when $x_i = 0$ because in this case v_i is nonnegative, or when $x_i > 0$, because it is sufficient to take $h < x_i/|v_i|$ for having $y_i \geq 0$. Hence y does belong to the simplex. □

We shall investigate now how to make viable the evolution of a system for which we know the growth rates $g_i(\cdot)$ of the evolution without constraints (also called "specific growth rates"):

$$\forall\ i = 1, \ldots, n, \quad x_i'(t) = x_i(t) g_i(x(t))$$

There are no reasons[8] for the solutions to this system of differential equations to be viable in the probability simplex.

But we can correct it by subtracting to each initial growth rate the common "feedback control $\tilde{u}(\cdot)$" (also called "global flux" in many applications) defined as the weighted mean of the specific growth rates

$$\forall\ x \in S^n, \quad \tilde{u}(x) := \sum_{j=1}^{n} x_j g_j(x)$$

Indeed, the probability simplex S^n is obviously a viability domain of the new dynamical system, called *replicator system* (or system *under*

[8]By Nagumo's Theorem and Lemma 1.4.1, the functions g_i should be continuous and satisfy:

$$\forall\, x \in S^n, \quad \sum_{i=1}^{n} x_i g_i(x) = 0$$

constant organization):

$$\begin{cases} \forall\ i = 1,\dots,n,\ x_i'(t) \ = \ x_i(t)(g_i(x(t)) - \tilde{u}(x(t))) \\ \\ = \ x_i(t)(g_i(x(t)) - \sum_{j=1}^n x_j(t)g_j(x(t))) \end{cases} \tag{1.11}$$

As we shall see at the end of the section, these equations come up in many biological models related to the concept of "replicator" in the sense of Dawkins, who coined the term. They lead to many mathematical problems.

Remark — There are other methods for correcting a dynamical system to make a given closed subset a viability domain. A general method consists in projecting the dynamics onto the contingent cone (see chapter 10.) Here, we have taken advantage of the particular nature of the simplex. □

An equilibrium α of the replicator system (1.11) is a solution to the system

$$\forall\, i = 1,\dots,n,\ \ \alpha_i(g_i(\alpha) - \tilde{u}(\alpha)) = 0$$

(Such an equilibrium does exist, thanks to Equilibrium Theorem 3.7.6 below.) These equations imply that either $\alpha_i = 0$ or $g_i(\alpha) = \tilde{u}(\alpha)$ or both, and that $g_{i_0}(\alpha) = \tilde{u}(\alpha)$ holds true for at least one i_0. We shall say that an *equilibrium α is nondegenerate* if

$$\forall\, i = 1,\dots,n, \quad g_i(\alpha) \ = \ \tilde{u}(\alpha) \tag{1.12}$$

Equilibria α which are strongly positive (this means that $\alpha_i > 0$ for all $i = 1,\dots,n$) are naturally non degenerate.

We associate with any $\alpha \in S^n$ the function V_α defined[9] on the simplex S^n by

$$V_\alpha(x) \ := \ \prod_{i=1}^n x_i^{\alpha_i} \ := \ \prod_{i\in I_\alpha} x_i^{\alpha_i}$$

[9] The reason why we introduce this function is that α is the unique maximizer of V_α on the simplex S^n. This follows from the concavity of the function $\varphi :=$ log: Setting $0\log 0 = 0\log\infty = 0$, we get

$$\sum_{i=1}^n \alpha_i \log\frac{x_i}{\alpha_i} \ = \ \sum_{\alpha_i>0} \alpha_i \log\frac{x_i}{\alpha_i} \ \le \ \log\left(\sum_{\alpha_i>0} x_i\right) \ \le \ \log 1 \ = \ 0$$

where we set $0^0 := 1$ and $I_\alpha := \{i = 1, \ldots, n \mid \alpha_i > 0\}$.

Let us denote by S^I the subsimplex of elements $x \in S^n$ such that $x_i > 0$ if and only if $i \in I$.

Theorem 1.4.2 *Let us consider n continuous growth rates g_i. For every initial state $x_0 \in S^n$, there exists a solution to replicator system (1.11) starting from x_0 and which is viable in the subsimplex $S^{I_{x_0}}$.*

The viable solutions satisfy

$$\forall\, t \geq 0, \quad \sum_{i=1}^{n} g_i(x(t))x_i'(t) \;\geq\; 0 \tag{1.13}$$

and, whenever $\alpha \in S^n$ is a nondegenerate equilibrium,

$$\frac{d}{dt}V_\alpha(x(t)) \;=\; -V_\alpha(x(t))\sum_{i=1}^{n}(x_i(t) - \alpha_i)(g_i(x(t)) - g_i(\alpha)) \tag{1.14}$$

Proof — We first observe that

$$\forall\, x \in S^{I_{x_0}}, \quad \sum_{i \in I_{x_0}} x_i(g_i(x) - \tilde{u}(x)) \;=\; 0$$

because, $x_i = 0$ whenever $i \notin I_{x_0}$, i.e., whenever $x_{0_i} = 0$. Therefore, the subsimplex $S^{I_{x_0}}$ is a viability domain of the replicator system (1.11.)

Inequality (1.13) follows from the Cauchy-Schwarz inequality because

$$\left(\sum_{i=1}^{n} x_i g_i(x)\right)^2 \;\leq\; \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_i g_i(x)^2\right) \;=\; \sum_{i=1}^{n} x_i g_i(x)^2$$

We deduce formula (1.14) from

$$\begin{cases} \frac{d}{dt}V_\alpha(x(t)) \;=\; \sum_{i \in I_\alpha} \frac{\partial}{\partial x_i} V_\alpha(x(t))x_i'(t) \\ \;=\; V_\alpha(x(t)) \sum_{i \in I_\alpha} \alpha_i \frac{x_i'(t)}{x_i(t)} \end{cases}$$

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so that

$$\sum_{i=1}^{n} \alpha_i \log x_i \;\leq\; \sum_{i=1}^{n} \alpha_i \log \alpha_i$$

and thus, $V_\alpha(x) \leq V_\alpha(\alpha)$ with equality if and only if $x = \alpha$.

and from

$$\sum_{i=1}^{n} \alpha_i \frac{x_i'(t)}{x_i(t)} = \sum_{i=1}^{n} (\alpha_i - x_i(t)) g_i(x(t))$$

Then we take into account that α being a non degenerate equilibrium, equation (1.12) implies that

$$\sum_{i=1}^{n} (\alpha_i - x_i(t)) g_i(\alpha) = 0 \quad \square$$

Remark — When the specific growth rates are derived from a differentiable potential function U by

$$\forall\, i = 1, \ldots, n, \quad g_i(x) := \frac{\partial U}{\partial x_i}(x)$$

condition (1.13) implies that

$$\forall\, t \geq 0, \quad \frac{dU}{dt}(x(t)) \geq 0$$

because

$$\frac{dU}{dt}(x(t)) = \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}(x(t)) x_i'(t) = \sum_{i=1}^{n} g_i(x(t)) x_i'(t) \geq 0$$

Therefore *the potential function U does not decrease along the viable solutions to the replicator system* (1.11.)

Furthermore, when this potential function U is homogeneous with degree p, Euler's formula implies that

$$\tilde{u}(x) = pU(x)$$

(because $\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} U(x) = pU(x)$) so that in this case, *the global flux $\tilde{u}(x(t))$ also does not decrease along the viable solutions to the replicator system* (1.11.)

On the other hand, if we assume that the growth rates $-g_i$ are "monotone " in the sense that

$$\forall\, x, y \in S^n, \quad \sum_{i=1}^{n} (x_i - y_i)(g_i(x) - g_i(y)) \leq 0$$

then inequality (1.14) implies that for any non degenerate equilibrium $\alpha \in S^n$,

$$\forall\, t \geq 0, \quad \frac{dV_\alpha}{dt}(x(t)) \;\geq\; 0$$

When $g(x) := U'(x)$ is derived from a concave differentiable potential U, it is decreasing so that, *for a concave potential, both* $U(x(\cdot))$ *and* $V_\alpha(x(\cdot))$ *are not decreasing.* □

Example: Replicator systems for constant growth rates. The simplest example is the one where the specific growth rates $g_i(\cdot) \equiv a_i$ are constant. Hence we correct constant growth systems $x_i' = a_i x_i$ whose solutions are exponential $x_{0_i} e^{a_i t}$, by the 0-order replicator system

$$\forall\, i = 1, \ldots, n, \;\; x_i'(t) \;=\; x_i(t)(a_i - \sum_{j=1}^{n} a_j x_j(t))$$

whose solutions are given explicitly by:

$$x_i(t) \;=\; \frac{x_{0_i} e^{a_i t}}{\sum_{j=1}^n x_{0_j} e^{a_j t}} \;\; \text{whenever } x_{0_i} > 0$$

(and $x_i(t) \equiv 0$ whenever $x_{0_i} = 0$.)

Furthermore, *the functions* $\sum_{i=1}^n a_i x_i(\cdot)$ *are increasing* and converge to α defined by

$$\alpha_i \;=\; \begin{cases} 0 & \text{if } i \notin J^a \\ x_{0_i} / \sum_{j \in J^a} x_{0_j} & \text{if } i \in J^a \end{cases}$$

where $J^a := \{i = 1, \ldots, n, \mid a_i := \max_{j=1,\ldots,n} a_j\}$. Indeed, set $a_0 := \max_{j=1,\ldots,n} a_j$; the above claim follows obviously from formula

$$x_i(t) \;=\; \frac{x_{0_i} e^{-(a_0 - a_i)t}}{\sum_{j=1}^n x_{0_j}} e^{-(a_0 - a_j)t}$$

Observe that the limit points of the viable solutions achieve the maximum of the function $x \to \sum_{i=1}^n a_i x_i$ on S^n, since any $\alpha \in S^{J^a}$

achieves the maximum of this linear functional[10]. Observe also that the elements $\alpha \in S^{J^a}$ are equilibria of the 0-order replicator system. Actually, the equilibria of the 0-order replicator system are the elements of the each subsimplex S^{L_k} where $L_k := \{j \mid a_j = a_k\}$[11].

When $n = 2$, after setting $x(\cdot) := x_1(\cdot)$ and $r := a_1 - a_2$, we obtain the celebrated Verhust-Pearl's *logistic equation*

$$\forall\, t \geq 0, \;\; x'(t) \;=\; rx(t)(1 - x(t))$$

the solutions of which are the logistic curves (the S-curves)

$$x(t) \;:=\; \frac{1}{1 + ce^{-bt}}$$

The logistic equation played an important role in *population dynamics*. In the simplest case, assume that the growth rate of an organism is constant whenever there are no constraints on the resources needed for growth. This is no longer valid when the resources are limited for whatever reason. This fact is translated by saying that the growth rate becomes negative for large populations: the larger the population, the more severe the inhibition on further growth. The simplest growth rate fitting these requirements is the function $r(1 - x)$, so that the evolution of the population obeys the logistic equation. □

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[10] Since $a_0 := \max_{j=1,\ldots,n} a_j$, we deduce that

$$\begin{cases} \sum_{i=1}^n a_i(\alpha_i - x_i) \;=\; \sum_{i \in J^a} a_i(\alpha_i - x_i) - \sum_{i \notin J^a} a_i(\alpha_i - x_i) \\ \;=\; a_0(1 - \sum_{i \in J^a} x_i) - \sum_{i \notin J^a} a_i x_i = \sum_{i \in J^a}(a_0 - a_i)x_i \geq 0 \end{cases}$$

[11] Indeed, if $\alpha \in S^{L_k}$, then $\sum_{i=1}^n a_j\alpha_j = \sum_{j \in L_k} a_j\alpha_j = a_k$ and thus,

$$\alpha_i(a_i - \sum_{j=1}^n a_j\alpha_j) \;=\; 0$$

for any $i = 1, \ldots, n$.

Example: Replicator systems for linear growth rates. The next class of examples is provided by linear growth rates

$$\forall\, i = 1,\ldots,n,\;\; g_i(x) \;:=\; \sum_{j=1}^{n} a_{ij}x_j$$

Let A denote the matrix the entries of which are the above a_{ij}'s. Hence the global flux can be written

$$\forall\, x \in S^n,\;\; \tilde{u}(x) \;=\; \sum_{k,l=1}^{n} a_{kl}x_kx_l \;=< Ax, x >$$

Therefore, first order replicator systems can be written[12].

$$\forall\, i = 1,\ldots,n,\;\; x_i'(t) \;=\; x_i(t)(\sum_{j=1}^{n} a_{ij}x_j(t) - \sum_{k,l=1}^{n} a_{kl}x_k(t)x_l(t))$$

Such systems have been investigated independently in

— *population genetics* (allele frequencies in a gene pool)

— *theory of prebiotic evolution* of self replicating polymers (concentrations of polynucleotides in a dialysis reactor)

— *sociobiological studies* of evolutionary stable traits of animal behavior (distributions of behavioral phenotypes in a given species)

— *population ecology* (densities of interacting species)

In population genetics, *Fisher-Wright-Haldane's model* regards the state $x \in S^n$ as the frequencies of alleles in a gene pool and the matrix $A := (a_{ij})_{i,j=1,\ldots,n}$ as the *fitness matrix*, where a_{ij} represents the fitness of the genotype (i,j). In this case, the matrix A is obviously symmetric and we denote by

$$\tilde{u}(x) :=< Ax, x > \quad \textit{the average fitness}$$

Since the growth rate can be derived from the potential $U(x) := \tilde{u}(x)/2$, we conclude that whenever A is positive-definite, the *average adaptability does not decrease*[13] *along viable solutions.*

[12]Observe that if for each i, all the a_{ij} are equal to b_i, we find 0-order replicator systems

[13]This property is known as the fundamental theorem of natural selection in population genetics.

In the theory of *prebiotic evolution*, the state represents the concentrations of polynucleotides. It is assumed in *Eigen-Schuster's "hypercycle"* that the growth rate of the i^{th}-polynucleotide is proportional to the concentration of the preceding one:

$$\forall\, i = 1, \ldots, n, \quad g_i(x) \;=\; c_i x_{i-1} \;\text{ where }\; x_{-1} := x_n$$

In other words, the growth of polynucleotide i is catalyzed by its predecessor by Michaelis-Menten type chemical reactions.

The feedback $\tilde{u}(x) = \sum_{i=1}^{n} c_i x_i x_{i-1}$ can be regarded as a selective pressure to maintain the concentration.

The equilibrium α of such a system is equal to

$$\forall\, i = 1, \ldots, n, \quad \alpha_i \;=\; \frac{1}{c_{i+1}} \left(\sum_{j=1}^{n} \frac{1}{c_j} \right)^{-1} \quad \text{where } \; c_{n+1} := c_1$$

First order replicator systems also offer a quite interesting model of *dynamic game* theory proposed in 1974 by J. Maynard-Smith to explain the evolution of genetically programmed behaviors of individuals of an animal species.

We denote by $i = 1, \ldots, n$ the n possible "strategies" used in interindividual competition in the species and denote by a_{ij} the "gain" when strategy i is played against strategy j. The state of the system is described by the "mixed strategies" $x \in S^n$, which are the probabilities with which the strategies are implemented. Hence the growth rate $g_i(x) := \sum_{j=1}^{n} a_{ij} x_j$ is the gain obtained by playing strategy i against the mixed strategy x and $\tilde{u}(x) := \sum_{i,j=1}^{n} a_{ij} x_i x_j$ can be interpreted as the *average gain* .

So the growth rate of the strategy i in the replicator system is equal to the difference between the gain of i and the average gain (a behavior which had been proposed in 1978 by Taylor and Jonker.)

We shall say that an equilibrium α is *evolutionary stable* if and only if the property

$$\exists\, \eta > 0 \quad \text{such that } \; x \neq \alpha, \; \sum_{i=1}^{n} g_i(x)(\alpha_i - x_i) > 0$$

holds true in a neighborhood of α.

This implies that

$$\frac{d}{dt}V_\alpha(x(t)) = -V_\alpha(x(t))\sum_{i=1}^{n}(x_i(t)-\alpha_i)(g_i(x(t))-g_i(\alpha)) > 0$$

in a neighborhood of α.

It is interesting to observe that first order replicator systems can be used at the two extremes of biological evolution, prebiotic evolution at the molecular level and behavioral evolution in ethology (animal behavior.)

In ecology, the main models are elaborations of the *Lotka-Volterra equations*

$$\forall\, i=1,\ldots,n,\ \ x_i'(t) = x_i(t)\left(a_{i0}+\sum_{j=1}^{n}a_{ij}x_j(t)\right)$$

where the growth rate of each species depend in an affine way upon the number of organisms of the other species. A very simple transformation replaces this system by a first order replicator system. We compactify $\mathbf{R}_+^n$ by introducing homogeneous coordinates. We set $x_0 := 1$ and we introduce the map

$$\forall\, i=0,\ldots,n,\ \ y_i := \frac{x_i}{\sum_{j=1}^{n}x_j}$$

from $\mathbf{R}_+^n$ onto S^{n+1}, the inverse of which is defined by $x_i := y_i/y_0$.

We set $a_{0j} = 0$ for all j, so that Lotka-Volterra's equation becomes

$$\forall\, i=1,\ldots,n,\ \ y_i' = \frac{y_i}{y_0}\left(\sum_{j=0}^{n}a_{ij}y_j - \sum_{k,l=1}^{n}a_{kl}y_ly_k\right)$$

because

$$y_i' = \frac{x_i'}{\sum x_j} - \frac{x_i\sum x_j'}{(\sum x_j)^2} = x_i\left(\sum_{j=0}^{n}a_{ij}x_j\right)y_0 - x_i\left(\sum_{k,l=0}^{n}a_{kl}x_lx_k\right)y_0^2$$

This is, up to the multiplication by $\frac{1}{y_0}$, i.e., up to a modification of the time scale, a $(n+1)$-dimensional first order replicator system.

So, first-order replicator systems appear as a common denominator underlying these four biological processes.

1.5 Stochastic Viability and Invariance

The aim of this section is to extend to the stochastic case Nagumo's Theorem on viability and/or invariance properties of closed subsets with respect to a differential equation.

1.5.1 Stochastic Tangent Sets

Let us consider a σ-complete probability space $(\Omega, \mathcal{F}, P)$, an increasing family of σ-sub-algebras $\mathcal{F}_t \subset \mathcal{F}$ and a finite dimensional vector-space $X := \mathbf{R}^n$.

The constraints are defined by closed subsets $K_\omega \subset X$, where the set-valued map

$$K : \omega \in \Omega \rightsquigarrow K_\omega \subset X$$

is assumed to be $\mathcal{F}_0$-measurable (which can be regarded as a *random set-valued variable.*)

We denote by $\mathcal{K}$ the subset

$$\mathcal{K} \ := \ \{u \in L^2(\Omega, \mathcal{F}, P) \mid \text{for almost all } \omega \in \Omega, \ \ u_\omega \in K_\omega\}$$

For simplicity, we restrict ourselves to scalar $\mathcal{F}_t$-Wiener processes $W(t)$.

Definition 1.5.1 (Stochastic Contingent Set) *Let us consider an $\mathcal{F}_t$-random variable $x \in K$ (i.e., an $\mathcal{F}_t$-measurable selection of K.)*

We define the stochastic contingent set $\mathcal{T}_K(t, x)$ *to K at x (with respect to $\mathcal{F}_t$) as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: There exist sequences of $h_n > 0$ converging to 0 and of $\mathcal{F}_{t+h_n}$-random variables a^n and b^n such that*

$$\left\{ \begin{array}{ll} i) & \mathbf{E}(\|a^n\|^2) \to 0 \\ ii) & \mathbf{E}(\|b^n\|^2) \to 0 \\ iii) & \mathbf{E}(b^n) \ = \ 0 \\ iv) & b^n \ \text{ is independent of } \ \mathcal{F}_t \end{array} \right. \tag{1.15}$$

and satisfying

$$\forall \, n \geq 0, \ \ x + v(W(t+h_n) - W(t)) + h_n\gamma + h_n a^n + \sqrt{h_n} b^n \ \in \ \mathcal{K} \tag{1.16}$$

1.5.2 Stochastic Viability

We consider the stochastic differential equation

$$d\xi \ = \ f(\xi(t))dt + g(\xi(t))dW(t) \tag{1.17}$$

where f and g are Lipschitz.

We say that a stochastic process $\xi(t)$ defined by

$$\xi(t) = \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s) \quad (1.18)$$

is a solution to the stochastic differential equation (1.17) if the functions f and g satisfy:

$$\text{for almost all } \omega \in \Omega, \quad f(\xi(\cdot)) \in L^1(0,T;X) \;\; \& \;\; g(\xi(\cdot)) \in L^2(0,T;X)$$

Definition 1.5.2 *We shall say that a stochastic process $x(\cdot)$ is* viable in *K if and only if*

$$\forall\, t \in [0,T], \;\; x(t) \in \mathcal{K} \quad (1.19)$$

i.e., if and only if

$$\forall\, t \in [0,T], \;\; \text{for almost all } \omega \in \Omega, \;\; \xi_\omega(t) \in K_\omega$$

The random set-valued variable K is said to be (stochastically) invariant *by the pair (f,g) if every solution ξ to the stochastic differential equation starting at a random variable $x \in K$ is* viable in *K.*

When K is a subset of X (i.e., a constant set-valued random variable) and when the maps (f,g) are defined on K, we shall say that K enjoys the (stochastic) viability property *with respect to the pair (f,g) if for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at a x which is* viable in *K.*

Since K_ω and $\xi_\omega(0)$ are $\mathcal{F}_0$ measurable, the projection $\Pi_{K_\omega}(\xi_\omega(0))$ is also a $\mathcal{F}_0$-measurable map (see Theorem 8.2.13, p. 317 of SET-VALUED ANALYSIS.) Then there exists a $\mathcal{F}_0$-measurable selection $y_\omega \in \Pi_{K_\omega}(\xi_\omega(0))$, which we call a *projection of the random variable $\xi(0)$ onto the random set-valued variable K.*

Theorem 1.5.3 (Stochastic Viability) *Let K be a closed convex subset of X. Then the following conditions are equivalent:*

1. — The subset K enjoys the stochastic viability property with respect to the pair (f,g)

2. — for every $\mathcal{F}_t$-random variable x viable in K,

$$(f(x), g(x)) \in \mathcal{T}_K(t,x) \quad (1.20)$$

We shall deduce this theorem from more general Theorems 1.5.4 and Theorems 1.5.5 below dealing with set-valued random variables instead of closed convex subsets.

1.5.3 Necessary Conditions

Let K be a set-valued random variable.

Theorem 1.5.4 *If the random set-valued variable K is invariant by the pair (f,g), then for every $\mathcal{F}_t$-random variable x viable in K,*

$$(f(x), g(x)) \in \mathcal{T}_K(t,x) \tag{1.21}$$

Proof — We consider the viable stochastic process $\xi(t)$

$$\xi(h) = x + \int_0^h f(\xi(s))ds + \int_0^h g(\xi(s))dW(s) \tag{1.22}$$

which is a solution to the stochastic differential equation (1.17) starting at x.

We can write it in the form

$$\xi(t) = \xi(0) + hf(\xi(0)) + g(\xi(0))W(h) + \int_0^h a(s)ds + \int_0^h b(s)dW(s)$$

where

$$\begin{cases} a(s) = f(\xi(s)) - f(\xi(0)) \\ b(s) = g(\xi(s)) - g(\xi(0)) \end{cases}$$

converge to 0 with s.

We set

$$a^h := \frac{1}{h}\int_t^{t+h} a(s)ds$$

and

$$b^h := \frac{1}{\sqrt{h}}\int_t^{t+h} b(s)dW(s)$$

and we observe that

$$\begin{cases} \mathbf{E}\left(\|a^h\|^2\right) = \frac{1}{h^2}\mathbf{E}\left(\left\|\int_t^{t+h} a(s)ds\right\|^2\right) \\ \leq \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|(a(s)\|^2)\right)ds \end{cases}$$

converges to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \leq t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

In the same way,

$$\begin{cases} \mathbf{E}\left(\|b^h\|^2\right) = \frac{1}{h}\mathbf{E}\left(\left\|\int_t^{t+h} b(s)dW(s)\right\|^2\right) \\ = \frac{1}{h}\int_t^{t+h}\mathbf{E}\left(\|b(s)\|^2\right)ds \end{cases}$$

also converges to 0 because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

The expectation of b^h is obviously equal to 0 and b^h is independent of $\mathcal{F}_t$. Since $\xi(h)_\omega$ belongs to K_ω for almost all ω, we deduce that the pair $(f(x), g(x))$ belongs to $\mathcal{T}_K(t,x)$. □

1.5.4 Sufficient Conditions for Stochastic Invariance

Theorem 1.5.5 (Stochastic Invariance) *Assume that the set-valued random variable K satisfies the following property: for every $\mathcal{F}_t$-random variable x, there exists an $\mathcal{F}_t$- measurable projection $y \in \Pi_K(x)$ such that*

$$(f(x), g(x)) \in \mathcal{T}_K(t,y) \tag{1.23}$$

Then the set-valued random variable K is invariant by (f,g)

Remark — Observe that the sufficient condition of invariance requires the verification of the "stochastic tangential condition" (1.23) for every stochastic process y, including stochastic processes which are not viable in K. □

In order to prove Theorem 1.5.5, we need the following:

Lemma 1.5.6 *Let K be a random set-valued variable, $\xi(0)$ a $\mathcal{F}_0$-adapted stochastic process.*

We define

$$\xi(t) = \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s)$$

and we choose a $\mathcal{F}_0$- measurable projection $y \in \Pi_K(\xi(0))$.

Then, for any pair of $\mathcal{F}_0$-random variable (γ, v) in the stochastic contingent set $\mathcal{T}_K(0,y)$, the following estimate

$$\begin{cases} \liminf_{t_n \to 0} \left(\mathbf{E}(d_K^2(\xi(t_n)) - \mathbf{E}(d_K^2(\xi(0)))\right)/t_n \\ \\ \leq 2\mathbf{E}\left(\langle \xi(0) - y, f(\xi(0)) - \gamma\rangle\right) + \mathbf{E}(\|g(\xi(0)) - v\|^2) \end{cases}$$

holds true.

Proof — Let us set $x = \xi(0)$, choose a projection $y \in \Pi_K(x)$ and take (γ, v) in the stochastic contingent set $\mathcal{T}_K(0,y)$. This means that there exist sequences $t_n > 0$ converging to 0 and $\mathcal{F}_{t_n}$-measurable a^n and b^n satisfying the assumptions (1.15) and

$$\forall\, n \geq 0, \text{ for almost all } \omega \in \Omega,\ y_\omega + v_\omega W_\omega(t_n) + \gamma_\omega t_n + t_n a_\omega^n + \sqrt{t_n} b_\omega^n \in K_\omega$$

Therefore

$$\begin{cases} d_K^2(\xi(t_n)) - d_K^2(\xi(0)) \leq \\ \left\| x + \int_0^{t_n} f(\xi(s))ds + \int_0^{t_n} g(\xi(s))dW(s) - y - vW(t_n) - \gamma t_n - t_n a^n - \sqrt{t_n} b^n \right\|^2 \\ - \|x-y\|^2 \\ = \left\| (x-y) + \int_0^{t_n} (f(\xi(s)) - \gamma)ds + \int_0^{t_n} (g(\xi(s)) - v)dW(s) - t_n a^n - \sqrt{t_n} b^n \right\|^2 \\ - \|x-y\|^2 \\ =: I \end{cases}$$

The latter term can be split in the following way:

$$\begin{cases} I \;=\; 2 < x-y, \int_0^{t_n} (g(\xi(s)) - v)dW(s) > & I_1 \\ + 2 < x-y, \int_0^{t_n} (f(\xi(s)) - \gamma)ds > & I_2 \\ + \| \int_0^{t_n} (g(\xi(s)) - v)dW(s)\|^2 & I_3 \\ + \| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2 & I_4 \\ +2 \left\langle \int_0^{t_n} (g(\xi(s)) - v)dW(s), \int_0^{t_n} (f(\xi(s)) - \gamma)ds \right\rangle & I_5 \\ -2 \left\langle x - y + \int_0^{t_n} (f(\xi(s)) - \gamma)ds + \int_0^{t_n} (g(\xi(s)) - v)dW(s), t_n a^n \right\rangle & I_6 \\ -2 \left\langle x - y + \int_0^{t_n} (f(\xi(s)) - \gamma)ds + \int_0^{t_n} (g(\xi(s)) - v)dW(s), \sqrt{t_n} b^n \right\rangle & I_7 \\ + \|t_n a^n + \sqrt{t_n} b^n\|^2 & I_8 \end{cases}$$

We take the expectation in both sides of this inequality and estimate each term of the right hand-side. First, we observe that

$$\mathbf{E}\left(\left\langle x-y, \int_0^{t_n} (g(\xi(s)) - v)dW(s)\right\rangle\right) \;=\; 0$$

so that the expectation of the first term I_1 of the right-hand side of the above inequality vanishes.

The second term I_2 is estimated by $2t_n\alpha_n$ where

$$\alpha_n \;:=\; \mathbf{E}\left(\left\langle x-y, \frac{1}{t_n}\int_0^{t_n} (f(\xi(s)) - \gamma)ds\right\rangle\right)$$

converges to

$$\alpha := \mathbf{E}\left(\langle x - y, f(\xi(0)) - \gamma\rangle\right)$$

The third term I_3 is estimated by $t_n\beta_n$ where

$$\begin{cases} \beta_n := \frac{1}{t_n}\mathbf{E}\left(\| \int_0^{t_n} (g(\xi(s)) - v)dW(s))\|^2\right) \\ = \frac{1}{t_n}\int_0^{t_n} \mathbf{E}\left(\|g(\xi(s)) - v\|^2\right) ds \end{cases}$$

converges to

$$\beta := \mathbf{E}(\|g(\xi(0)) - v\|^2)$$

because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)dW(s)\right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$. The fourth term I_4 is easily estimated by $t_n\delta_n$ where

$$\begin{cases} \delta_n := \frac{1}{t_n}\mathbf{E}\left(\| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2\right) \\ \leq \int_0^{t_n} \mathbf{E}\left(\|f(\xi(s)) - \gamma\|^2\right) ds \leq ct_n \end{cases}$$

because $\mathbf{E}\left(\left\|\int_0^t \varphi(s)ds\right\|^2\right) \leq t\int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$.

By the Cauchy-Schwarz inequality, the term I_5 is estimated by $2t_n\eta_n$ where

$$\begin{cases} \eta_n := \frac{1}{t_n}\mathbf{E}\left(\left\langle \int_0^{t_n} (g(\xi(s)) - v)dW(s), \int_0^{t_n} (f(\xi(s)) - \gamma)ds\right\rangle\right) \\ \leq \frac{1}{t_n}\mathbf{E}(\| \int_0^{t_n} (g(\xi(s)) - v)dW(s)\|^2)^{1/2}\mathbf{E}(\| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2)^{1/2} \\ = \frac{1}{t_n}\left(\int_0^{t_n} \mathbf{E}(\|g(\xi(s)) - v\|^2 ds)\right)^{1/2} (\mathbf{E}(\| \int_0^{t_n} (f(\xi(s)) - \gamma)ds\|^2)^{1/2} \\ \leq \sqrt{t_n}\left(\frac{1}{t_n}\int_0^{t_n} \mathbf{E}(\|g(\xi(s)) - v\|^2)\right)^{1/2} \left(t_n \int_0^{t_n} (\mathbf{E}(\|f(\xi(s)) - \gamma\|^2)\right)^{1/2} \\ \leq ct_n^{\frac{1}{2}} \end{cases}$$

We now estimate the three latter terms involving the errors a^n and b^n. We begin with I_6. First,

$$\mathbf{E}\left(\langle x - y, a^n\rangle\right) \leq \mathbf{E}\left(\|x - y\|^2\right)^{\frac{1}{2}} \left(\mathbf{E}(\|a^n\|^2)\right)^{\frac{1}{2}}$$

which converges to 0 by assumption (1.15)i).

Now, the Cauchy-Schwarz inequality implies that

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(f(\xi(s))-\gamma)ds, a^n\right\rangle\right) \\ \\ \leq \mathbf{E}\left(\left\|\int_0^{t_n}(f(\xi(s))-\gamma)ds\right\|^2\right)^{\frac{1}{2}} \mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

Finally, the stochastic term is estimated in the following way:

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(g(\xi(s))-v)dW(s), a^n\right\rangle\right) \\ \\ \leq \mathbf{E}\left(\left\|\int_0^{t_n}(g(\xi(s))-v)dW(s)\right\|^2\right)^{\frac{1}{2}} \mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \\ \\ = \left(\int_0^{t_n}\mathbf{E}(\|g(\xi(s))-v)\|^2 ds\right)^{\frac{1}{2}} \mathbf{E}\left(\|a^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

which obviously converges to 0.

We continue with I_7. We have

$$\mathbf{E}\left\langle x-y, \frac{1}{\sqrt{t_n}}b^n\right\rangle = 0$$

since b^n is independent of $x-y$ and $\mathbf{E}(b^n)=0$.

The Cauchy-Schwarz inequality implies that

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(f(\xi(s))-\gamma)ds, \frac{1}{\sqrt{t_n}}b^n\right\rangle\right) \\ \\ \leq \mathbf{E}\left(\left\|\int_0^{t_n}(f(\xi(s))-\gamma)ds\right\|^2\right)^{\frac{1}{2}} \mathbf{E}\left(\|\frac{1}{\sqrt{t_n}}b^n\|^2\right)^{\frac{1}{2}} \\ \\ = \frac{1}{\sqrt{t_n}}\left(t_n\int_0^{t_n}\mathbf{E}(\|(f(\xi(s))-\gamma)\|^2)ds\right)^{\frac{1}{2}} \mathbf{E}\left(\|b^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

Finally, the worst term of I_7 is estimated in the following way:

$$\left\{\begin{array}{l} \mathbf{E}\left(\left\langle \int_0^{t_n}(g(\xi(s))-v)dW(s), \frac{1}{\sqrt{t_n}}b^n\right\rangle\right) \\ \\ \leq \frac{1}{\sqrt{t_n}}\sqrt{\mathbf{E}\left(\left\|\int_0^{t_n}(g(\xi(s))-v)dW(s)\right\|^2\right)}\sqrt{\mathbf{E}\left(\|b^n\|^2\right)} \\ \\ = \left(\frac{1}{t_n}\int_0^{t_n}\mathbf{E}(\|g(\xi(s))-v)\|^2 ds\right)^{\frac{1}{2}} \mathbf{E}\left(\|b^n\|^2\right)^{\frac{1}{2}} \end{array}\right.$$

which converges to 0 by assumption (1.15)ii).

It remains to estimate the last term of I_8. There is no difficulty because

$$\frac{1}{t_n}\mathbf{E}\left(\left\|t_n a^n + \sqrt{t_n} b^n\right\|^2\right) = \mathbf{E}\left(\left\|\sqrt{t_n} a^n + b^n\right\|^2\right)$$

converges to 0.

Putting everything together, we deduce the inequality of the lemma. □

Proof of Theorem 1.5.5 Since the solution to the stochastic differential equation can be written for any $h \geq 0$

$$\xi(t+h) = \xi(t) + \int_t^{t+h} f(\xi(s))ds + \int_t^{t+h} g(\xi(s))dW(s)$$

we deduce from Lemma 1.5.6 that

$$\begin{cases} \liminf_{h\to 0+}\left(\mathbf{E}(d_K^2(\xi(t+h)) - \mathbf{E}(d_K^2(\xi(t)))\right)/h \\ \leq 2\mathbf{E}\left(\langle \xi(t) - y(t), g(\xi(t)) - \gamma\rangle\right) + \mathbf{E}(\|g(\xi(t)) - v\|^2) \end{cases}$$

for any $\mathcal{F}_t$-measurable selection $y(t)$ of $\Pi_K(\xi(t))$ and any $(v(t), \gamma(t)) \in \mathcal{T}_K(t, y(t))$.

Since there exists a selection $y(t)$ of $\Pi_K(\xi(t))$ such that we can take $v(t) := g(\xi(t))$ and $\gamma(t) := f(\xi(t))$ by assumption, we infer that setting

$$\varphi(t) := \mathbf{E}\left(d_K^2(\xi(t))\right)$$

the contingent epiderivative

$$D_\uparrow\varphi(t)(1) := \liminf_{h\to 0+} \frac{\varphi(t+h) - \varphi(t)}{h}$$

is non positive.

This implies that $\varphi(t) \leq 0$ for all $t \in [0, T]$. If not, there would exist $T > 0$ such that $\varphi(T) > 0$. Since φ is continuous, there exists $\eta \in]0, T[$ such that

$$\forall\, t \in]T-\eta, T],\;\; \varphi(t) > 0$$

Let us introduce the subset

$$A := \{s \in [0, T] \mid \forall\, t \in]s, t],\;\; \varphi(t) > 0\}$$

and $t_0 := \inf A$.

We observe that for any $t \in]t_0, T], \varphi(t) > 0$ and that $\varphi(t_0) = 0$. Indeed, if $\varphi(t_0) > 0$, there would exist $t_1 \in]t_1, t_0[$ such that $\varphi(t) > 0$ for all $t \in]t_1, t_0]$, i.e., $t_1 \in A$, so that t_0 would no be an infimum.

Therefore, $D_\uparrow\varphi(t)(1) \leq 0$ for any $t \in]t_0, T]$ and thus, we obtain the contradiction

$$0 < \varphi(T) = \varphi(T) - \varphi(t_0) \leq 0$$

Consequently, for every $t \in [0, T]$, we have

$$\mathbf{E}(d_K^2(\xi(t))) = \int_\Omega d_{K_\omega}^2(\xi_\omega(t))dP(\omega) = 0$$

Since the integrand is nonnegative, we infer that $d_{K_\omega}(\xi_\omega(t)) = 0$ almost surely, i.e., that the stochastic process ξ is viable in K. □

Proof of Theorem 1.5.3 The necessary condition following obviously from Theorem 1.5.4, it remains to prove that it is sufficient. For that purpose, we extend the maps f and g defined on K by the maps $\widetilde{f}$ and $\widetilde{g}$ defined on the whole space by

$$\widetilde{f}(x) := f(\pi_K(x)) \;\&\; \widetilde{g}(x) := g(\pi_K(x))$$

Then the pair $(\widetilde{f}, \widetilde{g})$ satisfies obviously condition

$$(\widetilde{f}(x), \widetilde{g}(x)) \in \mathcal{T}_K(t, \pi_K(x))$$

so that K is invariant by $(\widetilde{f}, \widetilde{g})$ thanks to Theorem 1.5.5. Since these maps do coincide on K, we infer that K is a viability domain of (f, g). □

Chapter 2

Set-Valued Maps

Introduction

We shall gather in this chapter some of the results dealing with set-valued maps that we shall need. Only the properties of upper semicontinuous set-valued maps and, among them, the Convergence Theorem 2.4.4, and some notions on the set-valued analogues of continuous linear operators, the *closed convex processes* are required in the short term. Hence, further results, in particular those dealing with *lower semicontinuous criteria* and selections of lower semicontinuous maps, are postponed to Chapter 6.

We refer to the monograph SET-VALUED ANALYSIS for an exhaustive presentation of continuity properties of set-valued maps. We only provide here for the convenience of the reader the main definitions and statements, as well as the proofs of the main results such as the convergence theorem which plays a central role in viability theory, in order to make this book as self-contained as possible.

After defining upper semicontinuous and lower semicontinuous set-valued maps, providing an example (parametrized set-valued map) and proving the classical Maximum Theorem, we shall devote our attention to convex-valued maps. For that purpose, we shall recall in Section 3 the statement of the Separation Theorem and the properties of support functions of convex subsets we need for defining the handy concept of *upper hemicontinuous set-valued maps*. We shall also state the useful Bipolar Theorem and Closed Range Theorem.

We next prove the fundamental Convergence Theorem, an adaptation of the Mazur Theorem to our purpose, which is needed to check that the limit of approximate solutions built according to Euler's precepts is a viable solution in the proof of the Viability Theorem (a step which was easy in the case of differential equations.)

We conclude this chapter with a section on the set-valued analogues of continuous linear operators and their transpose.

Since the graph of a continuous linear operator $A \in \mathcal{L}(X, Y)$ is a (closed) vector subspace of $X \times Y$, it is quite natural to regard set-valued maps, with closed convex cones as their graphs, as these set-valued analogues. Such set-valued maps are called *closed convex processes*[1].

The main class of examples of closed convex processes is provided by derivatives of set-valued maps which are introduced in Chapter 7.

We shall mention that closed convex processes enjoy (almost) all properties of continuous linear operators, including Banach's Open Mapping and Closed Graph Theorems, the Uniform Boundedness Theorem and a kind of Banach-Steinhauss Theorem which we shall use later in this book.

Like continuous linear operators, closed convex processes can be *transposed* and the Bipolar Theorem can be adapted to closed convex processes. They thus enjoy the benefits of a duality theory. For instance, a duality criterion of a viability domain of a closed convex process is given in Chapter 5.

2.1 Semicontinuous Set-Valued Maps

2.1.1 Definitions

We begin to recall in this section the basic definitions dealing with set-valued maps, also called *multifunctions*, *multivalued functions*, *point to set maps* or *correspondences*.

Definition 2.1.1 *Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its* graph *$Graph(F)$, the subset*

[1]The term "process" has been coined by R.T. Rockafellar for denoting maps the graph of which are cones in a study of economic "processes" (with constant return to scale.)

of the product space $X \times Y$ *defined by*

$$\text{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

We shall say that $F(x)$ *is the* image *or the* value *of* F *at* x. *A set-valued map is said to be* nontrivial *if its graph is not empty, i.e., if there exists at least one element* $x \in X$ *such that* $F(x)$ *is not empty.*

We say that F *is* strict *if all images* $F(x)$ *are not empty. The* domain *of* F *is the subset of elements* $x \in X$ *such that* $F(x)$ *is not empty:*

$$\text{Dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$$

The image *of* F *is the union of the images (or values)* $F(x)$ *when* x *ranges over* X:

$$\text{Im}(F) := \bigcup_{x \in X} F(x)$$

The inverse F^{-1} *of* F *is the set-valued map from* Y *to* X *defined by*

$$x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \text{Graph}(F)$$

The domain of F is thus the image of F^{-1} and coincides with the projection of the graph onto the space X and, in a symmetric way, the image of F is equal to the domain of F^{-1} and to the projection of the graph of F onto the space Y.

If K is a subset of X, we denote by $F|_K$ the *restriction* of F to K, defined by

$$F|_K(x) := \begin{cases} F(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

We shall write

$$F \subset G \iff \text{Graph}(F) \subset \text{Graph}(G)$$

and say that G is an *extension of* F.

2.1.2 Continuity Concepts

Let d denote the distance of the metric space X. When K is a subset of X, we denote by

$$d_K(x) := d(x, K) := \inf_{y \in K} d(x, y)$$

the *distance from* x *to* K, where we set $d(x, \emptyset) := +\infty$. The *ball of radius* $r > 0$ *around* K *in* X is denoted by

$$B_X(K, r) := \{x \in X \mid d(x, K) \leq r\}$$

When there is no ambiguity, we set

$$B(K, r) := B_X(K, r)$$

The balls $B(K, r)$ are neighborhoods of K. When K is compact, each neighborhood of K contains such a ball around K.

When X is a normed space whose *unit ball* is denoted by B (or B_X if the space must be mentioned), we observe that

$$B_X(K, r) = \overline{K + rB_X}$$

and we set

$$\|K\| := \sup_{x \in K} \quad \& \quad \mathrm{Diam}(K) = \|K - K\|$$

We first need to adapt to the set-valued case the concept of continuity. There are two equivalent definitions of a continuous map f at x, the "$\varepsilon - \eta$" definition and the fact that f maps every sequence x_n converging to x to a sequence $f(x_n)$ converging to $f(x)$. Unfortunately, the natural generalizations of these statements to set-valued maps are no longer equivalent.

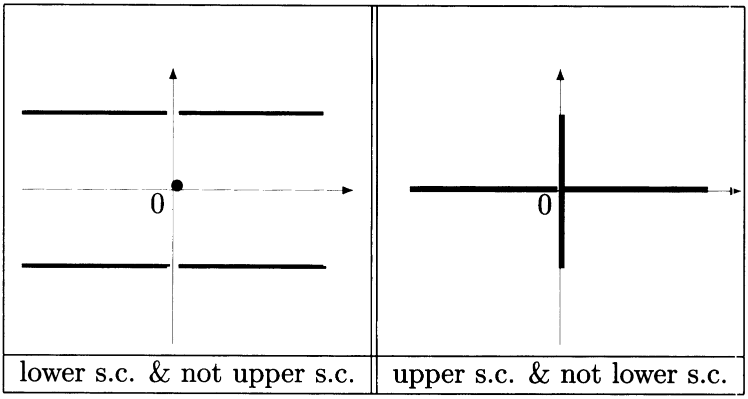
First, let us introduce these statements:

Definition 2.1.2 *A set-valued map* $F : X \rightsquigarrow Y$ *is called*

— upper semicontinuous *at* $x \in X$ *if and only if for any neighborhood* $\mathcal{U}$ *of* $F(x)$,

$$\exists\, \eta > 0 \quad \text{such that } \forall\, x' \in B_X(x, \eta), \quad F(x') \subset \mathcal{U}.$$

Figure 2.1: Semicontinuous and Noncontinuous Maps



It is said to be upper semicontinuous *if and only if it is upper semicontinuous at any point of* X.

— lower semicontinuous *at* $x \in \mathrm{Dom}(F)$ *if and only if for any* $y \in F(x)$ *and for any sequence of elements* $x_n \in \mathrm{Dom}(F)$ *converging to* x, *there exists a sequence of elements* $y_n \in F(x_n)$ *converging to* y. *It is said to be* lower semicontinuous *if it is lower semicontinuous at every point* $x \in \mathrm{Dom}(F)$.

— continuous *at* $x \in \mathrm{Dom}(F)$ *if it is* both *upper semicontinuous and lower semicontinuous at* x, *and that it is* continuous *if and only if it is continuous at every point of* $\mathrm{Dom}(F)$.

Indeed, there exist set-valued maps which enjoy one property without satisfying the other.

Examples — The set-valued map F_1 defined by

$$F_1(x) := \begin{cases} [-1,+1] & \text{if} \quad x \neq 0 \\ \{0\} & \text{if} \quad x = 0 \end{cases}$$

is lower semicontinuous at zero but not upper semicontinuous at zero.

The set-valued map $F_2 : \mathbf{R} \leadsto \mathbf{R}$ defined by

$$F_2(x) := \begin{cases} \{0\} & \text{if} \quad x \neq 0 \\ [-1,+1] & \text{if} \quad x = 0 \end{cases}$$

is upper semicontinuous at zero but not lower semicontinuous at zero. □

Remark — Let us point out that $F : X \rightsquigarrow Y$ *is upper semicontinuous if and only if* $\mathrm{Dom}(F)$ *is closed and if the restriction* $F : \mathrm{Dom}(F) \rightsquigarrow Y$ *is upper semicontinuous.*

Indeed, if F is upper semicontinuous and $F(x_0)$ is empty, we take two disjoint neighborhoods of $F(x_0)$, so that the upper semicontinuity of F at x_0 implies the existence of a neighborhood of x_0 which is mapped by F into this empty intersection of neighborhoods. This shows that the complement of the domain of F is open. The restriction of F to its domain is then obviously upper semicontinuous.

The converse statement is easy. □

We shall also need to adapt to the set-valued case the concept of Lipschitz applications.

Definition 2.1.3 *When* X *and* Y *are normed spaces, we shall say that* $F : X \rightsquigarrow Y$ *is* Lipschitz around $x \in X$ *if there exist a positive constant* λ *and a neighborhood* $\mathcal{U} \subset \mathrm{Dom}(F)$ *of* x *such that*

$$\forall\, x_1, x_2 \in \mathcal{U}, \quad F(x_1) \;\subset\; F(x_2) + \lambda \|x_1 - x_2\| B_Y$$

It is said to be locally Lipschitz *on an open subset* $\Omega \subset X$ *if it is Lipschitz around every point* $x \in \Omega$ *and* Lipschitz *on* Ω *if the constant* λ *is independent of* $x \in \Omega$*. In this case, if the images of* F *are closed, we denote by* $\|F\|_\Lambda$ *the smallest constant* λ*.*

If y *is given in* $F(x)$, F *is said to be* pseudo-Lipschitz *around* $(x, y) \in \mathrm{Graph}(F)$ *if there exist a positive constant* λ *and neighborhoods* $\mathcal{U} \subset \mathrm{Dom}(F)$ *of* x *and* $\mathcal{V}$ *of* y *such that*

$$\forall\, x_1, x_2 \in \mathcal{U}, \quad F(x_1) \cap \mathcal{V} \;\subset\; F(x_2) + \lambda \|x_1 - x_2\| B_Y$$

Example: Parametrized Set-Valued Maps The main class of examples that we shall use throughout this book is provided by parametrized set-valued maps.

Let us consider three metric spaces X, Y and Z, a set-valued map $U : X \rightsquigarrow Z$ and a single-valued map $f : \mathrm{Graph}(U) \mapsto Y$. We associate with these data the set-valued map $F : X \rightsquigarrow Y$ defined by

$$\forall\, x \in X, \quad F(x) := (f(x, u))_{u \in U(x)}$$

Proposition 2.1.4 *Assume that f is* continuous *from* Graph(U) *to Y.*

— *If U is lower semicontinuous, so is F.*

— *If U is upper semicontinuous with compact values, so is F.*

We leave the proof as an exercise[2]. □

The Maximum Theorem We also use the continuity properties of the marginal maps.

Definition 2.1.5 (Marginal Functions) *Consider a set-valued map $F : X \rightsquigarrow Y$ and a function f : Graph(F) $\mapsto$ R. We associate with them the* marginal function $g : X \mapsto \mathbf{R}$ *defined by*

$$g(x) := \sup_{y \in F(x)} f(x, y)$$

Theorem 2.1.6 (Maximum Theorem) *Let metric spaces X, Y, a set-valued map $F : X \rightsquigarrow Y$ and a function f : Graph(F) $\mapsto$ R be given.*

— *If f and F are* lower semicontinuous, *so is the marginal function.*

— *If f and F are* upper semicontinuous *and if the* values of F are compact, *so is the marginal function.*

The proof is an exercise of topology which is found in many books[3].

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[2]See Proposition 1.4.14 of SET-VALUED ANALYSIS for more details.

[3]We still provide a proof of the second statement for the convenience of the reader:

Pick $x \in X$ and fix $\varepsilon > 0$. Since f is upper semicontinuous, we can associate with any $y \in F(x)$ open neighborhoods $\mathcal{V}(y)$ of y and $\mathcal{U}_y(x)$ of x such that

$$\forall\, u \in \mathcal{U}_y(x) \text{ and } v \in \mathcal{V}(y), \;\; f(u, v) \leq f(x, y) + \varepsilon \tag{2.1}$$

Since $F(x)$ is compact, it can be covered by n neighborhoods $\mathcal{V}(y_i)$, $i = 1, \cdots, p$, the union of which makes up a neighborhood of $F(x)$. Then there exists a neighborhood $\mathcal{U}_0(x)$ such that

$$\forall\, x' \in \mathcal{U}_0(x), \;\; F(x') \subset \bigcup_{i=1}^{p} \mathcal{V}(y_i)$$

because F is upper semicontinuous. By taking u in the neighborhood

$$\mathcal{U}(x) := \mathcal{U}_0(x) \cap \bigcap_{i=1}^{p} \mathcal{U}_{y_i}(x)$$

we observe that

$$\forall\, u \in \mathcal{U}(x), \;\; \forall\, v \in F(u), \;\; f(u, v) \leq \sup_{i=1,\cdots,p} f(x, y_i) + \varepsilon \;\leq\; g(x) + \varepsilon$$

We will use the following corollary quite often:

Corollary 2.1.7 *If a set-valued map F is lower semicontinuous (*resp. *upper semicontinuous with compact values), then the function $(x, y) \mapsto d(y, F(x))$ is upper semicontinuous (*resp. *lower semicontinuous.)*

2.2 Closed Set-Valued Maps

We know that the graph of a continuous (single-valued) map is closed and that the converse is true under further assumptions (when we assume that the image of f is relatively compact, for instance.)

This result can be extended to upper semicontinuous set-valued maps. First, it is convenient to introduced *closed set-valued maps.*

Definition 2.2.1 (Closed Map) *Let us consider a set-valued map $F : X \rightsquigarrow Y$. We shall say that it is* closed *if and only if its graph is closed.*

2.2.1 Upper Semicontinuity of Closed Maps

Closed set-valued maps almost characterize upper semicontinuous set-valued maps, as the following result shows.

Proposition 2.2.2 *The graph of an upper semicontinuous set-valued map $F : X \rightsquigarrow Y$ with closed values is closed.*

The converse is true if we assume that the domain of F is closed and that Y is compact.

Remark — We shall state later (Theorem 2.5.5) that a closed convex process from a Banach space to another is also upper semicontinuous (and even, Lipschitz.) □

Proof — Let us consider a sequence of elements (x_n, y_n) of the graph of F converging to (x, y). Since F is upper semicontinuous, for all positive ε, there exists an integer $N(\varepsilon)$ such that, for all $n > N(\varepsilon)$, we have

$$y_n \in F(x_n) \subset F(x) + \varepsilon B$$

(thanks to (2.1)) and we deduce that

$$\forall\, u \in \mathcal{U}(x),\;\; g(u) \;\leq\; g(x) + \varepsilon \;\; \Box$$

We thus deduce that y belongs to the closure of $F(x)$, which coincides with $F(x)$. Then the limit (x, y) does belong to the graph of F.

Let us assume now that the graph of F is closed and that Y is compact. Let $x \in X$ and $\mathcal{V}$, an open neighborhood of $F(x)$, be given. Let us denote by M the complement of $\mathcal{V}$, which is compact and disjoint from $F(x)$. Since for all $y \in M$, the pair (x, y) does not belong to the graph of F, which is closed, there exist neighborhoods $\mathcal{W}_y(x)$ of x and $\mathcal{U}(y)$ of y such that

$$\text{Graph}(F) \cap ((\mathcal{W}_y(x)) \times (\mathcal{U}(y))) \quad = \quad \emptyset$$

The compact subset M being covered by n neighborhoods $\mathcal{U}(y_i)$, we consider the neighborhood $\mathcal{W}_0(x) := \bigcap_{i=1}^n \mathcal{W}_{y_i}(x)$. It is clear that

$$\forall\, x' \in \mathcal{W}_0(x), \quad F(x') \cap \left(\bigcup_{i=1}^{n} \mathcal{U}(y_i) \right) = \emptyset$$

Therefore, since $(Y \backslash \mathcal{V}) \subset \bigcup_{i=1}^n \mathcal{U}(y_i)$, we infer that $F(x') \subset \mathcal{V}$, i.e., that F is upper semicontinuous at x. □

This will be particularly useful since it provides an easy way to construct upper semicontinuous set-valued maps, by intersecting closed set-valued maps with closed balls, the radii of which are upper semicontinuous (real-valued) functions:

Corollary 2.2.3 *Let $F : X \rightsquigarrow Y$ be a* closed *set-valued map and $r : X \mapsto R$ be an upper semicontinuous function. If the* dimension of Y is finite , *then the cut set-valued map $F_r : X \rightsquigarrow Y$ defined by*

$$F_r(x) := F(x) \cap r(x)B \tag{2.2}$$

is upper semicontinuous.

It follows from Proposition 2.2.2 and the remark that the upper semicontinuity of $r : X \mapsto \mathbf{R}$ implies the upper semicontinuity of $x \rightsquigarrow r(x)B$.

2.2.2 Marchaud Maps

We denote by

$$\|F(x)\| := \sup_{y \in F(x)} \|y\|$$

and we say that F has *linear growth* if there exists a positive constant c such that

$$\forall\, x \in \mathrm{Dom}(F), \ \ \|F(x)\| \ \le \ c(\|x\|+1)$$

Definition 2.2.4 (Marchaud Map) *We shall say that F is a* Marchaud map *if it is nontrivial, upper semicontinuous, has compact convex images and linear growth.*

We deduce from Corollary 2.2.3 the following result:

Corollary 2.2.5 *If Y is a finite dimensional vector-space, to say that a nontrivial set-valued map F is a Marchaud map amounts to saying that*

$$\left\{\begin{array}{ll} i) & \textit{the graph and the domain of } F \textit{ are closed} \\ ii) & \textit{the values of } F \textit{ are convex} \\ iii) & \textit{the growth of } F \textit{ is linear} \end{array}\right.$$

More generally, let $U : X \rightsquigarrow Z$ be a nontrivial set-valued map and $f : \mathrm{Graph}(U) \mapsto Y$ be a single-valued map. Assume that

$$\left\{\begin{array}{ll} i) & \textit{the graph and the domain of } U \textit{ are closed} \\ ii) & \textit{the values of } U \textit{ are convex} \\ iii) & \textit{the growth of } U \textit{ is linear} \end{array}\right.$$

and that f is continuous, is affine with respect to the second variable and has linear growth. Then the parametrized map $x \rightsquigarrow f(x, U(x))$ is a Marchaud map.

2.3 Support Functions

When the values of a set-valued map are closed and convex, we can use the Hahn-Banach Separation Theorem to characterize them by their *support functions*. This is quite convenient for a lot of reasons, one of them being the possibility of replacing the continuity properties of set-valued maps by the more familiar continuity properties of real-valued functions.

Definition 2.3.1 *Let K be a nonempty subset of a Banach space X. We associate with any continuous linear form $p \in X^\star$*

$$\sigma_K(p) := \sigma(K,p) := \sup_{x\in K} < \langle p,x\rangle > \in \mathbf{R}\cup\{+\infty\}$$

The function $\sigma_K : X^\star \mapsto \mathbf{R}\cup\{+\infty\}$ is called the support function of K. *Its domain is a convex cone called the* barrier cone *denoted by*

$$b(K) := \mathrm{Dom}(\sigma_K) := \{p \in X^\star \mid \sigma_K(p) < \infty\} \qquad (2.3)$$

We say that the subsets of $X^\star$ defined by

$$\begin{cases} i) & K^\circ := \{p \in X^\star \mid \sigma_K(p) \le 1\} \\ ii) & K^- := \{p \in X^\star \mid \sigma_K(p) \le 0\} \\ iii) & K^+ := -K^- \\ iv) & K^\perp := \{p \in X^\star \mid \forall\, x \in K, \; < \langle p,x\rangle > = 0\} \end{cases}$$

are the polar set, (negative) polar cone, positive polar cone *and* orthogonal *of K respectively.*

When $L \subset X^\star$, we define the polar set $L^\circ \subset X$ as the subset of elements $x \in X$ (and not $X^{\star\star}$) satisfying $< \langle p,x\rangle > \le 1$ for all $p \in L$. The polar cone $L^- \subset X$ and the orthogonal $L^\perp \subset X$ of L are defined in the same way. The subsets

$$K^{\circ\circ} := (K^\circ)^\circ \subset X \;\&\; K^{--} := (K^-)^- \subset X$$

are called respectively the *bipolar set* and *bipolar cone* of a subset $K \subset X$ and the subspace $K^{\perp\perp} := (K^\perp)^\perp \subset X$ the *biorthogonal* of K.

It is clear that K° is a *closed convex subset containing* 0, that K^- is a *closed convex cone*, that $K^\perp$ is a *closed subspace* of $X^\star$ and that

$$K^\perp = K^- \cap K^+ \subset K^- \subset K^\circ \subset b(K)$$

Examples

- When $K = \{x\}$, then $\sigma_K(p) =< \langle p,x\rangle >$

- When $K = B_X$, then $\sigma_{B_X}(p) = \|p\|_\star$ where $\|\cdot\|_\star$ is the *dual norm* defined by

$$\|p\|_\star := \sup_{x \in X} \frac{\langle p, x \rangle}{\|x\|}$$

- If K is a *cone*, then

$$\sigma_K(p) = \begin{cases} 0 & \text{if} \quad p \in K^- \\ +\infty & \text{if} \quad p \notin K^- \end{cases} \quad \square$$

When $K = \emptyset$, we set $\sigma_\emptyset(p) = -\infty$ for every $p \in X^\star$.

The Separation Theorem can be stated in the following way:

Theorem 2.3.2 (Separation theorem) *Let K be a nonempty subset of a Banach space X. Its* closed convex hull *is characterized by linear constraint inequalities in the following way*

$$\overline{co}(K) = \{x \in X \mid \forall\, p \in X^\star, \;\; < \langle p, x \rangle > \leq \sigma_K(p)\}$$

Furthermore, there is a bijective correspondence between *nonempty closed convex subsets of X* and *nontrivial lower semicontinuous positively homogeneous convex functions on $X^\star$.*

Remark — The Separation Theorem holds true not only in Banach spaces, but in any Hausdorff locally convex topological vector-space. In particular, we can use it when X is supplied with the weakened topology. The geometrical interpretation can be stated as follows: the closed convex hull of a nonempty subset is the intersection of all closed half-spaces containing it. $\square$

We observe that a subset K is *bounded if and only if its support function is finite.*

We mention the following consequence, known as the *Bipolar theorem.*

Theorem 2.3.3 (Bipolar Theorem) *The* bipolar *cone K^{--} is the closed convex cone spanned by K.*

If $A \in \mathcal{L}(X,Y)$ is a continuous linear operator from X to Y and K is a subset of X, then

$$(A(K))^- = A^{\star^{-1}}(K^-)$$

and thus the closed cone spanned by $A(K)$ is equal to $(A^{\star^{-1}}(K^-))^-$.

where $A^\star$ denotes the transpose of A. We state now a simple criterion which implies that the image of a closed subset is closed[4].

Theorem 2.3.4 (Closed Range Theorem) *Let X be a Banach space, Y be a reflexive space, $K \subset X$ be a* weakly closed *subset and $A \in \mathcal{L}(X,Y)$ a continuous linear operator satisfying*

$$\mathrm{Im}(A^\star) + b(K) = X^\star \tag{2.4}$$

Then the image $A(K)$ is closed. In particular, if K is a closed convex cone *and if*

$$\mathrm{Im}(A^\star) + K^- = X^\star$$

then

$$A(K) = \left(A^{\star^{-1}}(K^-)\right)^-$$

For the convenience of the reader, we list in Table 2.1 some useful formulas of the calculus of support functions and barrier cones[5].

2.4 Convergence Theorem

We begin with the following consequence of the Maximum Theorem 2.1.6:

Corollary 2.4.1 *If a set-valued map F from a metric space X to a normed space Y is upper semicontinuous with compact values (*resp. *lower semicontinuous), then the function*

$$(x,q) \in X \times Y^\star \quad \mapsto \quad \sigma(F(x),q)$$

*is upper semicontinuous (*resp. *lower semicontinuous.)*

[4] See Theorem 2.4.4 of SET-VALUED ANALYSIS for a proof.

[5] See Chapter 3 of APPLIED NONLINEAR ANALYSIS for instance.

Table 2.1: **Properties of Support Functions.**

(1) ▷ If $K \subset L \subset X$, then $b(L) \subset b(K)$ and $\sigma_K \leq \sigma_L$

(2) ▷ If $K_i \subset X, \quad i \in I$, then
$b(\overline{co}(\bigcup_{i\in I} K_i)) \subset \bigcap_{i\in I} b(K_i)$
$\sigma(\overline{co}(\bigcup_{i\in I} K_i), p) = \sup_{i\in I} \sigma_{K_i}(p)$

(3) ▷ If $K_i \subset X_i, \quad (i = 1, \cdots, n)$, then
$b(\prod_{i=1}^n K_i) = \prod_{i=1}^n b(K_i)$
$\sigma(\prod_{i=1}^n K_i, (p_1, \cdots, p_n)) = \sum_{i=1}^n \sigma_{K_i}(p_i)$

(4)a) ▷ If $A \in \mathcal{L}(X, Y)$, then
$b(\overline{A(K)}) = A^{\star^{-1}} b(K)$
$\sigma_{\overline{A(K)}}(p) = \sigma_K(A^\star p)$

(4)b) ▷ If K_1 and K_2 are contained in X, then
$b(K_1 + K_2) = b(K_1) \cap b(K_2)$
$\sigma_{K_1+K_2}(p) = \sigma_{K_1}(p) + \sigma_{K_2}(p)$
In particular, if $K \subset X$ and P is a cone, then
$b(K + P) = b(K) \cap P^-$ and
$\sigma_{K+P}(p) = \sigma_K(p)$ if $p \in P^-$ and $+\infty$ if not

(5) ▷ If $L \subset X$ and $M \subset Y$ are *closed convex* subsets and $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that the *qualification constraint condition*
$0 \in \text{Int}(M - A(L))$ holds true, then
$b(L \cap A^{-1}(M)) = b(L) + A^\star b(M)$ and
$\forall\, p \in b(L \cap A^{-1}(M)),\ \exists\, \overline{q} \in Y^\star$ such that
$\sigma_{L\cap A^{-1}(M)}(p) = \sigma_L(p - A^\star \overline{q}) + \sigma_M(\overline{q})$
$= \inf_{q\in Y^\star}(\sigma_L(p - A^\star q) + \sigma_M(q))$

(5)a) ▷ If $M \subset Y$ is a *closed convex* subset and if $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that $0 \in \text{Int}(\text{Im}(A) - M)$, then $b(A^{-1}(M)) = A^\star b(M)$ and, for every $p \in b(A^{-1}(M))$, there exists $\overline{q} \in b(M)$ such that $\sigma_{A^{-1}(M)}(p) = \sigma_M(\overline{q}) = \inf_{A^\star q = p}(\sigma_M(q))$

(5)b) ▷ If K_1 and K_2 are closed convex subsets of X such that
$0 \in \text{Int}(K_1 - K_2)$, then $b(K_1 \cap K_2) = b(K_1) + b(K_2)$
and $\forall\, p \in b(K_1 \cap K_2),\ \exists\, \overline{q}_i \in X^\star, (i = 1, 2)$ such that
$\sigma_{K_1\cap K_2}(p) = \sigma_{K_1}(\bar{q}_1) + \sigma_{K_2}(\bar{q}_2)$
$= \inf_{p=p_1+p_2}(\sigma_{K_1}(p_1) + \sigma_{K_2}(p_2))$

Remark — This property still holds true when Y is supplied with the weak topology. One can prove that the converse is true when the values of F are convex and compact. □

Therefore, it is quite convenient to introduce the following definition.

Definition 2.4.2 (Upper Hemicontinuous Map) *We shall say that a set-valued map $F : X \rightsquigarrow Y$ is* upper hemicontinuous *at $x_0 \in X$ if and only if for every $p \in Y^\star$, the function $x \in X \mapsto \sigma(F(x), p) \in \overline{\mathbf{R}}$ is upper semicontinuous at x_0. It is said to be upper hemicontinuous if and only if it is upper hemicontinuous at every point of X.*

Proposition 2.4.3 *The graph of an upper hemicontinuous set-valued map with* closed convex *values is* closed.

Proof — Consider a sequence of elements (x_n, y_n) of Graph(F) converging to a pair (x, y). Then, for every $p \in Y^\star$,

$$< p, y >= \lim_{n\to\infty} < p, y_n >\leq \limsup_{n\to\infty} \sigma(F(x_n), p) \leq \sigma(F(x), p)$$

by the upper semicontinuity of $x \mapsto \sigma(F(x), p)$. This inequality implies that $y \in F(x)$ since these subsets are closed and convex, thanks to the Separation Theorem 2.3.2.

We thus have shown that (x, y) belongs to Graph(F), which ends the proof. □

When we shall adapt the Nagumo Theorem 1.2.1 to the set-valued case, the third step will naturally become much more difficult. This step, as well as many other properties of differential inclusions for Peano maps, will follow from the Convergence Theorem.

Let $a(\cdot)$ be a measurable strictly positive real-valued function from an interval $I \subset R$ to R_+. We denote by $L^1(I, Y; a)$ the space of classes of measurable functions from I to Y integrable for the measure $a(t)dt$.

Theorem 2.4.4 (Convergence Theorem) *Let X be a topological vector space, Y a Hilbert space and F be a nontrivial set-valued map* *from* X *to* Y. *We assume that* F is upper hemicontinuous with closed convex images.

Let I *be an interval of* R *and let us consider* measurable *functions* $x_m(\cdot)$ *and* $y_m(\cdot)$ *from* I *to* X *and* Y *respectively, satisfying:*

for almost all $t \in I$ *and for all neighborhood* $\mathcal{U}$ *of* 0 *in the product space* $X \times Y$, *there exists* $M := M(t, \mathcal{U})$ *such that*

$$\forall\, m > M,\ (x_m(t), y_m(t)) \in \text{Graph}(F) + \mathcal{U} \tag{2.5}$$

If we assume that

$$\begin{cases} i) & x_m(\cdot) \text{ converges almost everywhere to a function } x(\cdot) \\ ii) & y_m(\cdot) \in L^1(I, Y; a) \text{ converges } \textit{weakly} \text{ in } L^1(I, Y; a) \\ & \textit{to a function } y(\cdot) \in L^1(I, Y; a) \end{cases} \tag{2.6}$$

then

$$\text{for almost all } \ t \in I, \ \ y(t) \in F(x(t)) \tag{2.7}$$

Proof — Let us recall that in a Banach space ($L^1(I, Y; a)$, for instance), the closure (for the normed topology) of a set coincides with its weak closure (for the weakened topology[6]

$$\sigma(L^1(I, Y; a), L^\infty(I, Y^\star; a^{-1}))$$

We apply this result: for every m, the function $y(\cdot)$ belongs to the weak closure of the convex hull $\text{co}(\{y_p(\cdot)\}_{p \geq m}.)$ It coincides with the (strong) closure of $\text{co}(\{y_p(\cdot)\}_{p \geq m})$. Hence we can choose functions

$$v_m(\cdot) := \sum_{p=m}^{\infty} a_m^p y_p(\cdot) \in \text{co}(\{y_p(\cdot)\}_{p \geq m})$$

[6]By definition of the weakened topology, the continuous linear functionals and the weakly continuous linear functionals coincide. Therefore, the closed half-spaces and weakly closed half-spaces are the same. The Hahn-Banach Separation Theorem, which holds true in Hausdorff locally convex topological vector spaces, states that closed convex subsets are the intersection of the closed half-spaces containing them. Since the weakened topology is locally convex, we then deduce that closed convex subsets and weakly closed convex subsets do coincide. This result is known as *Mazur's theorem*.

(where the coefficients a_m^p are positive or equal to 0 but for a finite number of them, and where $\sum_{p=m}^{\infty} a_m^p = 1$) *which converge strongly* to $y(\cdot)$ in $L^1(I, Y; a)$. This implies that the sequence $a(\cdot)v_m(\cdot)$ converges strongly to the function $a(\cdot)y(\cdot)$ in $L^1(I, Y)$, since the operator of multiplication by $a(\cdot)$ is continuous from $L^1(I, Y; a)$ to $L^1(I, Y)$.

Thus, there exists another subsequence (again denoted by) $v_m(\cdot)$ such that[7]

$$\text{for almost all } t \in I, \quad a(t)v_m(t) \text{ converges to } a(t)y(t)$$

Since the function $a(\cdot)$ is strictly positive, we deduce that

$$\text{for almost all } t \in I, \quad v_m(t) \text{ converges to } y(t)$$

— Let $t \in I$ such that $x_m(t)$ converges to $x(t)$ in X and $v_m(t)$ converges to $y(t)$ in Y. Let $p \in Y^\star$ be such that $\sigma(F(x(t)), p) < +\infty$ and let us choose $\lambda > \sigma(F(x(t)), p)$. Since F is upper hemicontinuous, there exists a neighborhood $\mathcal{V}$ of 0 in X such that

$$\forall\, u \in x(t) + \mathcal{V}, \quad \text{then } \sigma(F(u), p) \le \lambda \tag{2.8}$$

Let N_1 be an integer such that

$$\forall\, q \ge N_1, \quad x_q \in x(t) + \frac{1}{2}\mathcal{V}$$

[7] *Strong convergence of a sequence in Lebesgue spaces L^p implies that some subsequence converges almost everywhere.* Let us consider indeed a sequence of functions f_n converging *strongly* to a function f in L^p. We can associate with it a subsequence f_{n_k} satisfying

$$\|f_{n_k} - f\|_{L^p} \le 2^{-k}; \quad \cdots < n_k < n_{k+1} < \cdots$$

Therefore, the series of integrals

$$\sum_{k=1}^{\infty} \int \|f_{n_k}(t) - f(t)\|_Y^p dt < +\infty$$

is convergent. The Monotone Convergence Theorem implies that the series

$$\sum_{k=1}^{\infty} \|f_{n_k}(t) - f(t)\|_Y^p$$

converges almost everywhere. For every t where this series converges, we infer that the general term converges to 0.

Let $\eta > 0$ be given. Assumption (2.5) of the theorem implies the existence of N_2 and of elements (u_q, v_q) of the graph of F such that

$$\forall\, q \geq N_2,\ \ u_q \in x_q(t) + \frac{1}{2}\mathcal{V},\ \ \|y_q(t) - v_q\| \leq \eta$$

Therefore u_q belongs to $x(t) + \mathcal{V}$ and we deduce from (2.8) that

$$\left\{ \begin{array}{lll} <p, y_q(t)> & \leq & <p, v_q> + \eta\|p\|_\star \\ & \leq & \sigma(F(u_q), p) + \eta\|p\|_\star \\ & \leq & \lambda + \eta\|p\|_\star \end{array} \right.$$

Let us fix $N \geq \max(N_1, N_2)$, multiply the above inequalities by the nonnegative a_m^q and add them up from $q = 1$ to ∞. We obtain :

$$<p, v_m(t)> \leq \lambda + \eta\|p\|_\star$$

By letting m go to infinity, it follows that

$$<p, y(t)> \leq \lambda + \eta\|p\|_\star$$

Letting now λ converge to $\sigma(F(x(t)), p)$ and η to 0, we obtain:

$$<p, y(t)> \leq \sigma(F(x(t)), p)$$

Since this inequality is automatically satisfied for those p such that $\sigma(F(x(t)), p) = +\infty$, it thus holds true for every $p \in Y^\star$. Hence, the images $F(x)$ being closed and convex, the Separation Theorem implies that $y(t)$ belongs to $F(x(t))$. The Convergence Theorem ensues. □

2.5 Closed Convex Processes

Let us introduce the set-valued analogues of continuous linear operators, which are the closed convex processes.

Definition 2.5.1 (Closed Convex Process) *Let $F : X \rightsquigarrow Y$ be a set-valued map from a normed space X to a normed space Y. We shall say that F is*

— convex *if its graph is convex*

— closed *if its graph is closed*

— *a* process *(or* positively homogeneous*) if its graph is a cone*

Hence a closed convex process *is a set-valued map whose graph is a closed convex cone.*

We shall see that most of the properties of continuous linear operators are enjoyed by closed convex processes.

Let us begin by the following obvious statements.

Lemma 2.5.2 *A set-valued map F is convex if and only if*

$$\begin{cases} \forall\, x_1,\ x_2 \in \mathrm{Dom}(F), \ \ \forall\, \lambda \in [0,1], \\ \lambda F(x_1) + (1-\lambda)F(x_2) \ \subset \ F(\lambda x_1 + (1-\lambda)x_2) \end{cases}$$

It is a process if and only if

$$\forall\, x \in X,\ \lambda > 0,\ \lambda F(x) \ = \ F(\lambda x) \ \text{ and } \ 0 \ \in \ F(0)$$

and a convex process if and only if it is a process satisfying

$$\forall\, x_1,\ x_2 \in X,\ \ F(x_1) + F(x_2) \ \subset \ F(x_1 + x_2)$$

We observe that the domain and the image of a closed convex process are convex cones (not necessarily closed.)

The main examples of closed processes will be provided by contingent derivatives of set-valued maps that we shall introduce in Chapter 7. □

We associate with a closed convex process its *norm* defined in the following way.

Definition 2.5.3 (Norm of a Closed Convex Process) *Let $F : X \rightsquigarrow Y$ be a closed convex process. Its* norm *$\|F\|$ is equal to*

$$\begin{cases} \|F\| & := \ \sup_{x\in\mathrm{Dom}(F)} d(0,F(x))/\|x\| \\ & = \ \sup_{x\in\mathrm{Dom}(F)} \inf_{v\in F(x)} \|v\|/\|x\| \\ & = \sup_{x\in\mathrm{Dom}(F)\cap B} \inf_{v\in F(x)} \|v\| \end{cases} \tag{2.9}$$

The Banach Open Mapping Theorem has been extended to closed convex processes by Robinson and Ursescu[8]:

Theorem 2.5.4 (Open Mapping) *Let X, Y be Banach spaces. Assume that a closed convex process $F : X \rightsquigarrow Y$ is* surjective *(in the sense that $\mathrm{Im}(F) = Y$.) Then F^{-1} is Lipschitz: There exists a constant $l > 0$ such that, for all $x_1 \in F^{-1}(y_1)$ and for any $y_2 \in Y$, we can find a solution $x_2 \in F^{-1}(y_2)$ satisfying:*

$$\|x_1 - x_2\| \ \le \ l\|y_1 - y_2\|$$

[8]See the original papers or Theorem 3.3.1 of APPLIED NONLINEAR ANALYSIS in the nonreflexive case and Theorem 2.2.1 of SET-VALUED ANALYSIS in the reflexive case where a much simpler proof is sufficient.

As in the case of continuous linear operators, the Open Mapping Theorem is equivalent to the Closed Graph Theorem, which can be stated as follows.

Theorem 2.5.5 (Closed Graph Theorem) *A closed convex process F from a Banach space X to another Y whose domain is the whole space is Lipschitz: there exists a (Lipschitz) constant $l > 0$ such that*

$$\forall\, x_1,\, x_2\, \in X,\;\; F(x_1)\; \subset\; F(x_2)\; +\; l\|x_1 - x_2\|B \qquad (2.10)$$

Thus, the norm of F is finite whenever $\mathrm{Dom}(F) = X$.

Proof — It is sufficient to apply the Open Mapping Theorem 2.5.4 to F^{-1}. □

One can now adapt the Uniform Boundedness Theorem to the case of closed convex processes[9].

Theorem 2.5.6 (Uniform Boundedness) *Let X and Y be Banach spaces and F_h be a family of closed convex processes from X to Y, "pointwise bounded" in the sense that*

$$\forall\, x \in X,\;\; \exists\, y_h\; \in\; F_h(x) \quad \textit{such that} \;\; \sup_h \|y_h\|\; <\; +\infty \qquad (2.11)$$

Then this family is "uniformly bounded" in the sense that

$$\sup_h \|F_h\|\; <\; +\infty$$

Hence we can speak of *bounded* families of closed convex processes, without specifying whether it is pointwise or uniform.

We shall need the following consequence of Uniform Boundedness Theorem 2.5.6 extends to closed convex processes the following useful convergence result.

Theorem 2.5.7 (Crossed Convergence) *Consider a metric space U, Banach spaces X and Y, and a set-valued map associating to each $u \in U$ a* closed convex process $F(u) : X \rightsquigarrow Y$. *Let us assume that the family of closed convex processes $F(u)$ is pointwise bounded.*

Then the following conditions are equivalent:

$$\begin{cases} i) & \textit{the map}\;\; u \rightsquigarrow \mathrm{Graph}(F(u))\;\; \textit{is lower semicontinuous} \\ ii) & \textit{the map}\;\; (u,x) \rightsquigarrow F(u)(x)\;\; \textit{is lower semicontinuous} \end{cases}$$

[9]See Theorem 2.3.1 of SET-VALUED ANALYSIS for instance.

Proof — For proving that i) implies ii), let us consider a sequence of elements (u_n, x_n) converging to (u, x) and an element $y \in F(u)(x)$. We have to approximate it by elements $y_n \in F(u_n)(x_n)$.

Since $u \rightsquigarrow \text{Graph}(F(u))$ is lower semicontinuous, we can approximate (x, y) by elements $(\widehat{x}_n, \widehat{y}_n) \in \text{Graph}(F(u_n))$. By the pointwise boundedness assumption and Theorem 2.5.6, there exist $l > 0$ and solutions $f_n \in F(u_n)(x_n - \widehat{x}_n)$ satisfying

$$\|f_n\| \leq l\|x_n - \widehat{x}_n\|$$

The right hand side of the above inequality converges to zero when n goes to infinity. Because $F(u_n)$ is a convex process, the element $y_n := \widehat{y}_n + f_n$ does belong to $F(u_n)(x_n)$. Consequently, using that y_n converge to y, we deduce that the set-valued map $(u, x) \rightsquigarrow F(u)(x)$ is lower semicontinuous at (u, x).

The converse is obviously true (even when the family $(F(u))_{u \in U}$ is unbounded.) □

Like continuous linear operators, closed convex processes can be *transposed*

Definition 2.5.8 (Transpose of a Process) *Let X, Y be Banach spaces, $F : X \rightsquigarrow Y$ be a process. Its* left-transpose *(in short, its* transpose*) $F^\star$ is the closed convex process from $Y^\star$ to $X^\star$ defined by*

$$\begin{cases} p \in F^\star(q) \quad \text{if and only if} \\ \forall\, x \in X, \;\; \forall\, y \in F(x), \;\; < p, x > \; \leq \; < q, y > \end{cases} \tag{2.12}$$

In particular, the transpose $F^\star$ of a linear process F is defined by

$$p \in F^\star(q) \;\text{ if and only if } \forall\, x \in X, \; \forall\, y \in F(x), \;\; < p, x > = < q, y >$$

The graph of the transpose $F^\star$ of F is related to the polar cone of the graph of F in the following way:

Lemma 2.5.9 (Graph of the Transpose) *Consider Banach spaces X, Y and let $F : X \rightsquigarrow Y$ be a process. Then*

$$(q, p) \in \text{Graph}(F^\star) \iff (p, -q) \in (\text{Graph}(F))^-$$

In the case of linear processes, we observe that $p \in F^\star(q)$ if and only if $(p, -q)$ belongs to $\text{Graph}(F)^\perp$ and we see at once that the *bitranspose* of a closed linear process F coincides with F.

The definition of a bitranspose of a convex process is not symmetric: If $G : Y^\star \rightsquigarrow X^\star$ is a convex process, we define its transpose $G^\star : X \rightsquigarrow Y$ by the formula

$$(x, -y) \in (\text{Graph}(G))^-$$

(instead of the formula $(-x, y) \in (\text{Graph}(G))^-$ obtained by exchanging the roles of X and $Y^\star$, Y and $X^\star$ respectively.)

With this definition, *the bitranspose of a closed convex process F coincides with F.*

We provide now a formula for transposing the product of closed convex processes[10].

Theorem 2.5.10 (Transpose of a Product) *Let W, X, Y, Z be Banach spaces, F be a closed convex process from X to Y, $A \in \mathcal{L}(W, X)$ and $B \in \mathcal{L}(Y, Z)$ be continuous linear operators. Assume that*

$$\text{Im}(A) - \text{Dom}(F) = X \tag{2.13}$$

Then the transpose of BFA is equal to:

$$(BFA)^\star = A^\star F^\star B^\star$$

One can adapt to the case of closed convex processes the Bipolar Theorem 2.3.3, which the source of most duality properties[11]:

Theorem 2.5.11 (Bipolar Theorem) *Consider Banach spaces X, Y and let $F : X \rightsquigarrow Y$ be a closed convex process, and $K \subset X$ be a cone satisfying* $\text{Dom}(F) - K = X$. *Then*

$$(F(K))^- = -F^{\star^{-1}}(K^+)$$

The above condition is obviously satisfied when the domain of F is the whole space. In this case we obtain

Corollary 2.5.12 *Let $F : X \rightsquigarrow Y$ be a strict closed convex process. Then Dom($F^\star$) $= F(0)^+$ and $F^\star$ is upper hemicontinuous (see Definition 2.4.2) with bounded closed convex images, mapping the unit ball to the ball of radius $\|F\|$. In particular, $F^\star(0) = \{0\}$.*

In particular, we obtain the following example of transpose:

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[10]See Theorem 2.5.3 of SET-VALUED ANALYSIS for instance.

[11]See Theorem 2.5.7 of SET-VALUED ANALYSIS for instance.

Proposition 2.5.13 *Let X, Y be Banach spaces, $G : X \rightsquigarrow Y$ be a closed convex process and $P \subset X$ and $Q \subset Y$ be closed convex cones. Let us consider the convex process F defined by*

$$F(x) := \begin{cases} G(x) + Q & \text{if } x \in P \\ \emptyset & \text{if } x \notin P \end{cases}$$

It is closed when we suppose that $\mathrm{Dom}(G^\star) + Q^- = Y^\star$. *If we assume that* $\mathrm{Dom}(G) - P = X$, *then its transpose is defined by*

$$F^\star(q) := \begin{cases} G^\star(q) + P^- & \text{if } q \in Q^+ \\ \emptyset & \text{if } q \notin Q^+ \end{cases}$$

Chapter 3

Viability Theorems for Differential Inclusions

Introduction

This is the basic chapter of this book, where the main viability theorems for differential inclusions in finite dimensional vector spaces are gathered and proved. (Invariance Theorems are the topic of Chapter 5.)

We must begin by defining the class of functions in which to seek solutions to differential inclusions. An adequate choice is a weighted Sobolev space, made of absolutely continuous functions. The first section is devoted to these spaces and the derivatives in the sense of distributions.

Viability domains K of a set-valued map F are presented and studied in the second section: They are defined by

$$\forall\, x \in K, \; F(x) \cap T_K(x) \neq \emptyset$$

or, equivalently, when K is closed and F is upper semicontinuous with convex compact values, by

$$\forall\, x \in K, \; F(x) \cap \overline{co}\,(T_K(x)) \neq \emptyset$$

or also, by a dual condition involving the polar cone of the contingent cone (called the *subnormal cone*).

Viability Theorems are stated in the third section. They claim that a subset K is viable under F (in the sense that for any initial state x_0, there exists one solution starting at x_0 which is viable in K) if and only if K is a *viability domain* of F.

We consider successively the cases when K is locally compact, open and closed. The proofs are gathered in the fourth section.

We then show in the fifth section that the *solution map* $\mathcal{S}$ associating with any initial state the (possibly empty) subset of solutions to the differential inclusion *is upper semicontinuous*.

We also prove Kurzhanski's Representation Theorem stating that the restriction of a set-valued map to a closed convex subset is a *countable intersection of unconstrained set-valued maps.* In the same way that Lagrange multipliers allows us to replace a constrained optimization problem with unconstrained problems by "adding the constraints to the functional", this representation theorem enables us to represent the set of viable solutions to a differential inclusion as a countable intersection of sets of solutions to unconstrained differential inclusions obtained by "adding the constraints " to the right-hand of the original differential inclusion.

We recall that the *upper limit* of a sequence of subsets K_n is the set of cluster points of sequences of elements $x_n \in K_n$. We then answer in the sixth section a natural *stability question*: does the upper limit of a sequence of viability domains remain a viability domain? We also extend this result to the case when the subsets K_n are viability domains of maps F_n. We define the *upper graphical limit* $F^\sharp$ of a sequence of set-valued maps F_n by saying that the graph of $F^\sharp$ is the upper limit of the graphs of F_n's. We then prove that the *upper limit of viability domains of set-valued maps F_n is a viability domain of the map $\overline{co}(F^\sharp)$.*

We proceed by giving examples of closed viability domains. In the seventh section, we show that the *limit sets* of solutions to a differential inclusion are closed viability domains. In particular, *trajectories of periodic solutions are closed viability domains* and thus, *limits of solutions when $t \to +\infty$, if they exist, are equilibria.* These limit sets are among the most interesting features of a dynamical system. They are naturally subsets of the largest closed viability domain contained in a closed set K, the existence of which is proved in Chapter 4. This set, which we call the *viability kernel* of K, plays such an important role that we devote the whole chapter 4 to some of its properties, which we shall use throughout this book.

This motivates a further study of existence theorems of an equilibrium. We begin by pointing out that an equilibrium does exist if there exists a solution $x(\cdot)$ viable in a compact subset such that a sequence of average velocities

$$v_n := \frac{1}{t_n} \int_0^{t_n} \|x'(\tau)\| d\tau$$

converges to 0.

The question arises as to whether a *closed viability domain* K *contains an equilibrium.* This is the case when K is compact and the range $F(K)$ is convex.

This is also the case when K is compact and convex. This striking statement, linking viability and nonlinear analysis, is actually equivalent to the Brouwer Fixed Point Theorem. In both cases, one can say that *viability implies stationarity.*

In Section 8, we adapt to the set-valued case an efficient result of D. Saari on the *chaotic behavior* of discrete systems. Assume that the domain of a differential inclusion is covered by a family of compact subsets K_a satisfying an adequate controllability property: Any point can be reached from any subset K_a. Take any arbitrary sequence $K_{a_0}, K_{a_1}, K_{a_2}, \ldots$ of such sets. Then there exist a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ and a sequence of instants t_j such that $x(t_j) \in K_{a_j}$ for all j.

Throughout this chapter, X denotes a finite dimensional vector-space so long as it is not explicitly mentioned that this is not the case.

3.1 Solution Class

We are going to extend Nagumo's Theorem 1.2.1 to the case of differential inclusions $x'(t) \in F(x(t))$. But we have first to agree on what we shall call a solution to such differential inclusions.

In the case of differential equations, there is no ambiguity since the derivative $x'(\cdot)$ of one solution $x(\cdot)$ to a differential equation

$x'(t) = f(t, x(t))$ inherits the properties of the map f and of the function $x(\cdot)$. It is continuous whenever f is continuous and measurable whenever f is continuous with respect to x and measurable with respect to t.

This is no longer the case with differential inclusions. We have to choose a space of functions or distributions in which we shall look for a solution.

We cannot hope to obtain without further restrictions a continuously differentiable, or even a plain differentiable solution. We shall be content to deal only with functions which are almost everywhere differentiable. Namely, we shall look for solutions among *absolutely continuous* functions, as it was proposed by T. Ważewski at the beginning of the sixties.

We denote by $L^1(0, \infty; X, e^{-bt}dt)$ the *weighted Lebesgue space* of (classes of) measurable functions $x(\cdot)$ from $[0, \infty[$ to X satisfying

$$\|x(\cdot)\| := \int_0^\infty e^{-bt}\|x(t)\|dt < +\infty$$

Definition 3.1.1 (Absolutely Continuous Functions) *A continuous function $x : [0, T] \mapsto X$ is said to be* absolutely continuous *if there exists a locally integrable function v such that*

$$\text{for all } t, s \in [0, T], \quad \int_t^s v(\tau)d\tau = x(s) - x(t)$$

In this case,

$$\text{for almost all } t \in [0, T], \quad x'(t) := v(t)$$

and we shall say that $x'(\cdot)$ is the weak derivative *of the function $x(\cdot)$.*

We shall denote by $W^{1,1}(0, \infty; X; e^{-bt}dt)$ (for some $b \geq 0$) the space of absolutely continuous functions defined by

$$\{x(\cdot) \in L^1(0, \infty; X, e^{-bt}dt) \mid x'(\cdot) \in L^1(0, \infty; X, e^{-bt}dt)\}$$

and, when $T < +\infty$, by $W^{1,1}(0, T; X)$ the space

$$\{x(\cdot) \in L^1(0, T; X) \mid x'(\cdot) \in L^1(0, T; X)\}$$

We shall supply them with the topology for which a sequence $x_n(\cdot)$ converges to $x(\cdot)$ if and only if

$$\begin{cases} i) & x_n(\cdot) \text{ converges } \textit{uniformly} \text{ to } x(\cdot) \\ & \text{(on compact intervals if } T = \infty) \\ \\ ii) & x_n'(\cdot) \text{ converges } \textit{weakly} \text{ to } x'(\cdot) \text{ in } L^1(0,T;X) \\ & \text{(in } L^1(0,\infty;X,e^{-bt}dt) \text{ if } T = +\infty) \end{cases}$$

Remark — The above spaces are *weighted Sobolev spaces*. To define them, it may be best to recall what *distributions* and *derivatives in the sense of distributions* are[1].

We denote by $\mathcal{D}(0,T;X)$ the space of indefinitely differentiable functions from $]0,T[$ to the finite dimensional vector space X *with compact support in* $]0,T[$. The choice of the simplest scalar product

$$< x, y > := \int_0^T x(t)y(t)dt$$

allows us to identify the space $\mathcal{D}(0,T;X)$ with a subspace of the dual $D^\star(0,T;X)$ of continuous linear functionals on $\mathcal{D}(0,T;X)$, called *distributions* since their discovery by Laurent Schwartz.

For that purpose, we identify a function $x(\cdot)$ with the continuous linear functional

$$y \mapsto \int_0^T x(t)y(t)dt$$

which belongs to the dual of $\mathcal{D}(0,T;X)$.

In other words, the fundamental idea is to regard the usual functions in a novel way: Instead of viewing them as maps from $]0,T[$ to X, we shall also regard them as continuous linear functionals on the infinite dimensional space $\mathcal{D}(0,T;X)$. In particular, *integrable functions* (actually, classes of measurable and integrable functions) are instances of distributions.

This very same scalar product defines the topology of quadratic convergence on $\mathcal{D}(0,T;X)$. Taking the completion of this space for this scalar product, we obtain the celebrated space $L^2(0,T;X)$. Since this scalar product was already used to identify $\mathcal{D}(0,T,X)$ with a subspace of its dual, it will also be used to identify $L^2(0,T,X)$ with its dual thanks to Riesz' Theorem. We thus obtain the inclusions:

$$\mathcal{D}(0,T;X) \subset L^2(0,T;X) = L^2(0,T;X)^\star \subset \mathcal{D}^\star(0,T;X)$$

[1]We refer to the text APPLIED FUNCTIONAL ANALYSIS by the author or any of the many books on distributions for more details.

The first (and most important) consequence of this concept is the possibility of differentiating integrable functions, and more generally, distributions.

Definition 3.1.2 (Distributional Derivative) *If $x(\cdot)$ is a measurable locally integrable function from $]0,T[$ to a finite dimensional vector space X, we shall say that the continuous linear functional $x' \in \mathcal{D}^\star(0,T;X)$ defined on the space $\mathcal{D}(0,T;X)$ by*

$$y(\cdot) \longmapsto -\int_0^T x(t)y'(t)dt$$

is the weak derivative *(or the* distributional derivative*) of $x(\cdot)$.*

A distributional derivative defined in such a way does not need to be a function, even measurable. In any case, *it is a distribution*. The weak derivative of a function of $\mathcal{D}(0,T,X)$ naturally coincides with the usual derivative.

Sobolev spaces are then defined in the following way:

Definition 3.1.3 *Let $a(\cdot)$ be a strictly positive measurable function. We denote by*

$$W^{1,p}(0,T;X;a) \ := \ \{x \in L^p(0,T;X;a) \ \mid x' \in L^p(0,T;X;a)\}$$

the weighted Sobolev space *of measurable p^{th}-integrable functions $x(\cdot)$ (for the measure $a(t)dt$) whose derivative $x'(\cdot)$ in the sense of distributions belongs to the space $L^p(0,T;X;a)$.*

If $a \equiv 1$, we set $W^{1,p}(0,T;X) := W^{1,p}(0,T;X;a)$ This is a Sobolev space. *If $p=2$, we often use the notation*

$$H^1(0,T;X) \ = \ W^{1,2}(0,T;X)$$

They are Banach spaces for the norm:

$$\|x\|_{1,p;a} \ := \ (\|x\|_{p,a}^p + \|x'\|_{p,a}^p)^{1/p}$$

For our study, we endowed $W^{1,1}(0,\infty;X;e^{-bt}dt)$ with a weaker topology, for reasons which will soon become clear. □

The generalization of the concept of derivative provided by the theory of distributions is not the only one we can conceive. This approach allows us to *keep the linearity properties of the differential operator $x \mapsto x'$*. Actually, one can show that the distribution x' is the limit in the space $\mathcal{D}^\star(0,T;X)$ of the differential quotients

$$\frac{x(\cdot+h)-x(\cdot)}{h}$$

The topology of $\mathcal{D}^\star(0,T;X)$ is so much weaker than the pointwise convergence topology that not only do differential quotients of any function converge, but also differential quotients of distributions. In this distributional sense, functions and distributions are indefinitely differentiable.

The price one pays to obtain this paradisiac situation is that the space of distributions may be too large, and that distributions are no longer functions.

We will propose in Chapter 9 another concept of derivative (contingent epiderivative) for studying Lyapunov functions: They are *lower epilimits of these difference quotients*, as we shall explain later, and are usual functions instead of distributions. But the contingent epiderivative of a function no longer depends linearly on this function.

3.2 Viability Domains

Let X be a finite dimensional vector-space. We describe the (nondeterministic) dynamics of the system by a set-valued map F from the finite dimensional vector-space X to itself.

The contingent cone was introduced by G. Bouligand[2] in the early thirties: When K is a subset of X and x belongs to K, we recall that the *contingent cone* $T_K(x)$ to K at x is the closed cone of elements

[2]who wrote: "... Nous poserons les définitions suivantes:

1. Une demi-droite OT, issue du point d'accumulation O de l'ensemble E, sera dite une *demi-tangente* au point O, à l'ensemble E, si tout cône droit à base circulaire, de sommet O et d'axe OT, contient (si faibles en soient la hauteur et l'angle au sommet) un point de l'ensemble E distinct du point O;

2. L'ensemble de toutes les demi-tangentes à l'ensemble E en un même point d'accumulation sera appelé, moyennant une désignation abrégée conforme à l'étymologie, le *contingent* de l'ensemble E au point O.

Le mot *contingent* a déjà été employé comme adjectif, en matière philosophique, ou comme substentif, au point de vue militaire. L'emploi nouveau que nous en faisons ne peut évidemment créer aucune équivoque."

v such that

$$\liminf_{h\to 0+} \frac{d(x+hv,K)}{h} = 0$$

(see Definition 1.1.3 and Section 5.1 below[3].)

3.2.1 Definition of Viability Domains

There are two ways to extend the concept of viability domain K to set-valued maps. The first one is to require that for any state x, there exists *at least* one velocity $v \in F(x)$ which is *contingent* to K at x. The second demands that *all* velocities $v \in F(x)$ are *contingent* to K at x.

Definition 3.2.1 (Viability Domain) *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map. We shall say that a subset $K \subset \mathrm{Dom}(F)$ is a* viability domain *of F if and only if*

$$\forall\, x \in K, \quad F(x) \cap T_K(x) \neq \emptyset$$

Since the contingent cone to a singleton is obviously reduced to 0, we observe that *a singleton $\{\overline{x}\}$ is a viability domain if and only if $\overline{x}$ is an equilibrium of F*, i.e., a stationary solution to the differential inclusion, which is a solution to the inclusion

$$0 \in F(\overline{x}) \tag{3.1}$$

In other words, the equilibria of a set-valued map provide the first examples of viability domains, actually, the *minimal viability domains.*

Remark — If K is a viability domain of a set-valued map F, the subset

$$D := \bigcap_{x\in K} (T_K(x) - F(x))$$

is the subset of *disturbances* of the system which do not destroy the fact that K is still a viability domain, because K remains a viability domain of any *perturbed set-valued map* $x \rightsquigarrow F(x) + G(x)$ where $x \mapsto G(x)$ maps K into D. □

[3] By using the concept of *upper limits* of sets introduced in Definition 3.6.1 of Section 3.3.6 below, we observe that the *contingent cone $T_K(x)$ is the upper limit of the differential quotients $\frac{K-x}{h}$ when $h \to 0+$.*

3.2.2 Subnormal Cones

In order to provide a dual characterization of viability domains, we need to introduce the dual concept of a contingent cone: the *subnormal cone*.

Definition 3.2.2 *Let x belong to $K \subset X$. We shall say that the (negative) polar cone*

$$N_K^0(x) := T_K(x)^- = \{p \in X^\star \mid \forall\, v \in T_K(x),\ < p, v > \leq 0\}$$

is the subnormal cone *to K at x.*

We see at once that

$$N_K^0(x) := (\overline{co}(T_K(x)))^-$$

The subnormal cone is equal to the whole space whenever the tangent cone $T_K(x)$ is reduced to 0.

Let us point out the following property:

Proposition 3.2.3 *Let K be a subset of a finite dimensional vector-space and Π_K denote its projector of best approximation. Then*

$$\forall\, y \notin K,\ \forall\, x \in \Pi_K(y),\ y - x \in N_K^0(x)$$

Proof — Let v belong to the contingent cone $T_K(x)$: there exists a sequence $h_n > 0$ converging to 0 and a sequence v_n converging to v such that $x + h_n v_n$ belongs to K for all n. Since $\|y - x\| \leq \|y - x - h_n v_n\|$, we deduce that $< x - y, v > \geq 0$ for all $v \in T_K(x)$. □

3.2.3 Dual Characterization of Viability Domains

We now prove a very important characterization of viability domains:

Theorem 3.2.4 *Assume that the set-valued map $F : K \rightsquigarrow X$ is upper semicontinuous with convex compact values. Then the three following properties are equivalent:*

$$\left\{\begin{array}{ll} i) & \forall\, x \in K,\ F(x) \cap T_K(x) \neq \emptyset \\ ii) & \forall\, x \in K,\ F(x) \cap \overline{co}\,(T_K(x)) \neq \emptyset \\ iii) & \forall\, x \in K,\ \forall\, p \in N_K^0(x),\ \sigma(F(x), -p) \geq 0 \end{array}\right. \quad (3.2)$$

Proof — Since property i) implies ii), assume that ii) holds true and fix $x \in K$. Let $u \in F(x)$ and $v \in T_K(x)$ achieve the distance between $F(x)$ and $T_K(x)$:

$$\|u - v\| = \inf_{y \in F(x),\ z \in T_K(x)} \|y - z\|$$

and set $w := \frac{u+v}{2}$. We have to prove that $u = v$. Assume the contrary.

Since v is contingent to K at x, there exist sequences $h_n > 0$ converging to 0 and v_n converging to v such that $x + h_n v_n$ belongs to K for every $n \geq 0$. We also introduce a projection of best approximation

$x_n \in \Pi_K(x + h_n w)$ of $x + h_n w$ onto K and we set $z_n := \dfrac{x_n - x}{h_n}$

so that, by Proposition 3.2.3, we know that

$$w - z_n \in N_K^0(x_n)$$

By assumption ii), there exists an element $y_n \in F(x_n) \cap \overline{co}\,(T_K(x_n))$. Consequently,

$$\langle w - z_n, y_n \rangle \leq 0 \tag{3.3}$$

Since x_n converges to x, the upper semicontinuity of F at x implies that for any $\varepsilon > 0$, there exists N_ε such that for $n \geq N_\varepsilon$, y_n belongs to the neighborhood $F(x) + \varepsilon B$, which is compact. Thus a subsequence (again denoted by) y_n converges to some element $y \in F(x)$.

We shall now prove that z_n converges to v. Indeed, the inequality

$$\begin{cases} \|w - z_n\| = \frac{1}{h_n} \|x + h_n w - x_n\| \\ \leq \frac{1}{h_n} \|x + h_n w - x - h_n v_n\| = \|w - v_n\| \end{cases}$$

implies that the sequence z_n has a cluster point and that every cluster point z of the sequence z_n belongs to $T_K(x)$, because $x + h_n z_n = x_n \in K$ for every $n \geq 0$. Furthermore, every such z satisfies $\|w - z\| \leq \|w - v\|$.

We now observe that v *is the unique best approximation of* w *by elements of* $T_K(x)$. If not, there would exist $p \in T_K(x)$ satisfying either $\|w-p\| < \|w-v\|$ or $p \neq v$ and $\|w-p\| = \|w-v\| = \|w-u\|$. In the latter case, we have $\langle u-w, w-p\rangle < \|u-w\|\|w-p\|$, since the equality holds true only for $p = v$. Each of these conditions together with the estimates

$$\begin{cases} \|u-p\|^2 & = \|u-w\|^2 + \|w-p\|^2 + 2\langle u-w, w-p\rangle \\ & \leq (\|u-w\| + \|w-p\|)^2 \leq \|u-v\|^2 \end{cases}$$

imply the strict inequality $\|u-p\| < \|u-v\|$, which is impossible since v is the projection of u onto $T_K(x)$. Hence $z = v$.

Consequently, all the cluster points being equal to v, and we conclude that z_n converges to v.

Therefore, we can pass to the limit in inequality (3.3) and obtain, observing that $w - v = (u-v)/2$,

$$\langle u-v, y\rangle = 2\langle w-v, y\rangle \leq 0 \text{ where } y \in F(x) \tag{3.4}$$

Since $F(x)$ is closed and convex and since $u \in F(x)$ is the projection of v onto $F(x)$, we infer that

$$\langle u-v, u-y\rangle \leq 0 \tag{3.5}$$

Finally, $T_K(x)$ being a cone and $v \in T_K(x)$ being the projection of u onto this cone, and in particular onto the half-line $v\mathbf{R}_+$, we deduce that

$$\langle u-v, v\rangle = 0 \tag{3.6}$$

Therefore, properties (3.4), (3.5) and (3.6) imply that

$$\|u-v\|^2 = \langle u-v, -v\rangle + \langle u-v, u-y\rangle + \langle u-v, y\rangle \leq 0$$

and thus, that $u = v$.

The equivalence between ii) and iii) follows from the Separation Theorem. Indeed, by ii), to saying that K is a viability domain amounts to say that for all $x \in K$, 0 belongs to $F(x) - \overline{co}(T_K(x))$, which is closed and convex whenever $F(x)$ is compact. Hence the Separation Theorem implies that this condition is equivalent to the one stated in the Theorem. □

We can deduce right away from Theorem 3.2.4 the following very useful fact:

Proposition 3.2.5 *Let us assume that two set-valued maps F_1 and F_2 are upper semicontinuous with compact convex images. If K is a viability domain of F_1 and F_2, it is still a viability domain of $\lambda_1 F_1 + \lambda_2 F_2$ (where $\lambda_1, \lambda_2 > 0$.)*

3.3 Statement of Viability Theorems

We now consider initial value problems (or Cauchy problems) associated with the differential inclusion

$$\text{for almost all } t \in [0, T], \quad x'(t) \in F(x(t)) \tag{3.7}$$

satisfying the initial condition $x(0) = x_0$.

Definition 3.3.1 (Viability and Invariance Properties) *Let K be a subset of the domain of F. A function $x(\cdot) : I \mapsto X$ is said to be* viable in K on the interval I *if and only if*

$$\forall t \in I, \quad x(t) \in K$$

We shall say that K is locally viable under F *(or enjoys the* local viability property for the set-valued map F*) if for any initial state x_0 in K,* there exist *$T > 0$ and a solution on $[0, T]$ to differential inclusion (3.7) starting at x_0 which is* viable *in K. It is said to be* (globally) viable under F *(or to enjoy the* (global) viability property*) if we can take $T = \infty$.*

The subset K is said to be locally invariant (respectively invariant) under F *if for any initial state x_0 of K,* all *solutions to differential inclusion (3.7) are* viable in K on some interval (respectively for all $t \geq 0$). *We also say that F enjoys the* local invariance *(respectively* invariance*) property.*

Remark — We should emphasize as we did for ordinary differential equations that the concept of invariance *depends upon the behavior of F on its domain outside of K.* □

We would naturally like to characterize closed subsets viable under F as closed viability domains. This is more or less the situation that we shall meet: The main viability theorems hold true for the class of *Marchaud maps, i.e., the nontrivial upper hemicontinuous* *set-valued maps with nonempty compact convex images and with linear growth* (or equivalently, in the case of finite dimensional state spaces, *closed set-valued maps with closed domain, convex values and linear growth.* (See Corollary 2.2.3).)

We observe that the only truly restrictive condition is the *convexity* of the images of these set-valued maps, since the continuity requirements are kept minimal. But we cannot dispense with it, as the following counter example shows.

Example — Let us consider $X := \mathbf{R}$, $K := [-1, +1]$ and the set-valued map $F : K \rightsquigarrow \mathbf{R}$ defined by

$$F(x) \quad := \quad \begin{cases} -1 & \text{if} \quad x > 0 \\ \{-1, +1\} & \text{if} \quad x = 0 \\ +1 & \text{if} \quad x < 0 \end{cases}$$

Obviously, no solution to the differential inclusion $x'(t) \in F(x(t))$ can start from 0, since 0 is not an equilibrium of this set-valued map!

We note however that

- The graph of F is closed
- F is bounded
- K is convex and compact
- K is a viability domain of F.

But the value $F(0)$ of F at 0 is not convex. Observe that if we had set $F(0) := [-1, +1]$, then 0 would have been an equilibrium.

This example shows that upper semicontinuity is not strong enough to compensate the lack of convexity. Stronger continuity or differentiability requirements allow us to relax this assumption.

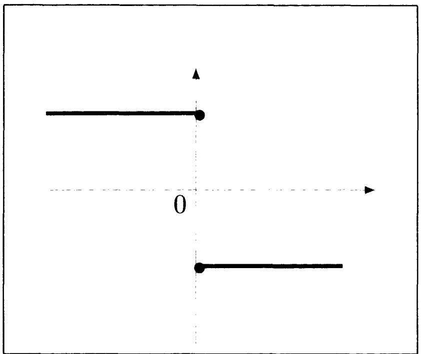
But we shall keep our continuity requirements minimal, and thus, be ready to pay the price of considering systems whose sets of velocities are convex. This is possible thanks to the extension of the Nagumo Theorem 1.2.1. □

Theorem 3.3.2 *Let us assume that*

$$\begin{cases} i) & F : X \rightsquigarrow X \textit{ is upper semicontinuous} \\ ii) & \textit{the images of } F \textit{ are convex and compact} \\ iii) & K \textit{ is locally compact} \end{cases}$$

Then K is locally viable under F if and only if K is a viability domain of F.

Figure 3.1: Example of a Map without Convex Values



There is no solution starting at 0

Since open subsets of finite dimensional vector spaces are locally compact viability domains of any set-valued map, we obtain the extension of Peano's Theorem 1.2.2 to differential inclusions due to Marchaud, Zaremba[4] and Ważewski[5]:

Theorem 3.3.3 *Let* Ω *be an open subset of a finite dimensional vector space* X *and* $F : \Omega \rightsquigarrow X$ *be a strict upper semicontinuous set-valued map with convex compact images.*

Then, for any $x_0 \in \Omega$*, there exists* $T > 0$ *such that differential inclusion (3.7) has a solution on the interval* $[0, T]$ *starting from* x_0*.*

[4]who proved independently in the thirties the existence of respectively *contingent* and *paratingent* solutions to differential inclusions (called *champs de demi-cônes* at the time.) The generalization of the concept of derivative to the notion of contingent derivative is due to B. Bouligand, who wrote: "... Nous ferons tout d'abord observer ... que la notion de contingent éclaire celle de différentielle".)

[5]who wrote: "... I learned the results of Zaremba's dissertation before the second world war, since I was a referee of that paper. Then a few years ago I came across with some results on optimal control and I have noticed a close connection between the optimal control problem and the theory of Marchaud-Zaremba." The author learned that this "coming across" happened during a seminar talk of C. Olech on a paper by LaSalle at Ważewski's seminar.

Ważewski proved that one can replace the contingent or paratingent derivatives of functions by derivatives of absolutely continuous functions defined almost everywhere in the definition of a solution to a differential inclusion, that he called *orientor field*.

The interesting case from the viability point of view is the one when the viability subset K is *closed*. In this case, we derive from Theorem 3.3.2 a more precise statement.

Theorem 3.3.4 (Local Viability Theorem) *Consider a* nontrivial upper semicontinuous set-valued map F with compact convex images *from* X *to* X *and a* closed *subset* $K \subset \mathrm{Dom}(F)$.

If K *is a* viability domain, *then for any initial state* $x_0 \in K$, *there exist a positive* T *and a solution on* $[0, T]$ *to differential inclusion (3.7) starting from* x_0, *viable in* K *and satisfying*

$$\begin{cases} \textit{either} & T = \infty \\ \quad \textit{or} & T < \infty \quad \textit{and} \quad \limsup_{t \to T-} \|x(t)\| = \infty \end{cases}$$

Further adequate information — a priori estimates on the growth of F — allow us to exclude the case when $\limsup_{t \to T-} \|x(t)\| = \infty$.

This is the case for instance when F is bounded on K, and, in particular, when K is bounded.

More generally, we can take $T = \infty$ when F enjoys linear growth:

Theorem 3.3.5 (Viability Theorem) *Consider a* Marchaud map $F : X \rightsquigarrow X$ *and a* closed *subset* $K \subset \mathrm{Dom}(F)$ *of a finite dimensional vector space* X.

If K *is a* viability domain, *then for any initial state* $x_0 \in K$, *there exists a* viable *solution on* $[0, \infty[$ *to differential inclusion (3.7.) More precisely, if we set*

$$c := \sup_{x \in \mathrm{Dom}(F)} \frac{\|F(x)\|}{\|x\| + 1}$$

then every solution $x(\cdot)$ *starting at* x_0 *satisfies the estimates*

$$\begin{cases} \forall\, t \geq 0, \;\; \|x(t)\| \;\leq\; (\|x_0\| + 1)e^{ct} \\ \text{and} \\ \text{for almost all } t \geq 0, \;\; \|x'(t)\| \;\leq\; c(\|x_0\| + 1)e^{ct} \end{cases}$$

and thus belongs to the space $W^{1,1}(0, \infty; X; e^{-bt}dt)$ *for* $b > c$.

Actually, we shall also use another more convenient formulation of this theorem. We agree for that purpose to set the distance $d(x, \emptyset)$ to the empty set equal to $+\infty$.

Theorem 3.3.6 (Second Viability Theorem) *Let us consider a* Marchaud map $F : X \rightsquigarrow X$ *and a* closed *subset* $K \subset \mathrm{Dom}(F)$ *of a finite dimensional vector space* X. *We assume that there exists a constant* $c > 0$ *such that*

$$\sup_{x \in K} \frac{d(0, F(x) \cap T_K(x))}{\|x\| + 1} \leq c < +\infty \tag{3.8}$$

Then for any initial state $x_0 \in K$, *there exists a* viable *solution on* $[0, \infty[$ *to differential inclusion (3.7) starting from* x_0, *which belongs to the space* $W^{1,1}(0, \infty; X; e^{-bt}dt)$ *for* $b > c$.

One can look right away at the *control version of the viability Theorems* in Section 6.1 in the framework of control systems and a very simple economic example in Section 6.2, in which other concepts such as viability kernels and heavy solutions are illustrated. Viability (and invariance) theorems for *linear differential inclusions* are presented in section 5.6 and can be checked over now.

3.4 Proofs of the Viability Theorems

We gather in this section the proofs of the theorems stated in the preceding one.

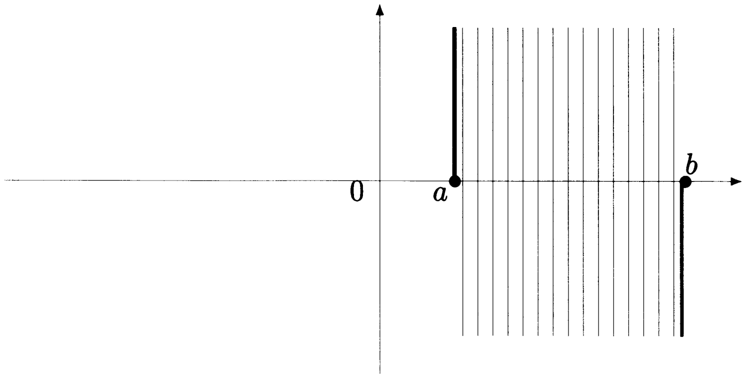
Since viable absolutely continuous functions $x(\cdot) : [0, T] \mapsto K$ satisfy $x'(t) \in T_K(x(t))$ for almost all $t \in [0, T]$, we could be tempted to derive viability theorems from existence theorems of solutions to differential inclusion $x'(t) \in R_K(x(t))$ where we set $R_K(x) := F(x) \cap T_K(x)$. Unfortunately, this is not possible because $T_K(\cdot)$ may be neither upper semicontinuous nor lower semicontinuous[6]. For instance, it is not upper semicontinuous as soon as inequality constraints are involved: take for example $K := [-1, +1]$. *The graph of* $T_K(\cdot)$, *equal to*

$$\{-1\} \times \mathbf{R}_+ \cup]-1, +1[\times \mathbf{R} \cup \{+1\} \times \mathbf{R}_-$$

is not closed, and not even locally compact: See figure 3.2.

[6] See Section 4.1., p. 178 of DIFFERENTIAL INCLUSIONS for an example of subset K such that $T_K(\cdot)$ is neither upper semicontinuous nor lower semicontinuous.

Figure 3.2: The Graph of $T_{[a,b]}(\cdot)$



So we have to devise a specific proof of Theorem 3.3.2, which consists in proving Propositions 3.4.1 and 3.4.2 below.

Proposition 3.4.1 (Necessary Condition) *Let us assume that*

$$\begin{cases} i) & F : X \rightsquigarrow X \text{ is upper hemicontinuous} \\ ii) & \text{the images of } F \text{ are convex and compact} \end{cases}$$

Let us consider a solution $x(\cdot)$ to differential inclusion (3.7) starting at x_0 and satisfying

$$\forall\, T > 0,\ \exists\, t \in]0, T] \text{ such that } x(t) \in K \tag{3.9}$$

(Naturally, viable solutions do satisfy this property.) Then

$$F(x_0) \cap T_K(x_0) \neq \emptyset$$

Proof — By assumption (3.9), there exists a sequence $t_n \to 0+$ such that $x(t_n) \in K$. Since F is upper hemicontinuous at x_0, we can associate with any $p \in X^\star$ and $\varepsilon > 0$ an $\eta_p > 0$ such that

$$\forall\, \tau \in [0, \eta_p],\ < p, x'(\tau) > \le \sigma(F(x(\tau)), p) \le \sigma(F(x_0), p) + \varepsilon \|p\|_\star$$

By integrating this inequality from 0 to t_n, setting $v_n := \frac{x(t_n)-x_0}{t_n}$ and dividing by $t_n > 0$, we obtain for n larger than some N_p

$$\forall\, p \in X^\star, \ \ \forall\, n \geq N_p, \ \ < p, v_n > \leq \ \sigma(F(x_0), p) + \varepsilon\|p\|_\star$$

Therefore, v_n lies in a bounded subset of a finite dimensional vector space, so that a subsequence (again denoted) v_n converges to some $v \in X$ satisfying

$$\forall\, p \in X^\star, \ \ < p, v > \leq \ \sigma(F(x_0), p) + \varepsilon\|p\|_\star$$

By letting ε converge to 0, we deduce that v belongs to the closed convex hull of $F(x_0)$.

On the other hand, since for any n, $x(t_n) = x_0 + t_n v_n$ belongs to K, we infer that v belongs to the contingent cone $T_K(x_0)$ since

$$\begin{cases} \liminf_{n\to\infty} d_K(x_0 + hv)/h \\ \leq \lim_{n\to\infty} \|x_0 + t_n v - x(t_n)\|/t_n \ = \ \lim_{n\to\infty} \|v_n - v\| \ = \ 0 \end{cases}$$

The intersection $F(x_0) \cap T_K(x_0)$ is then nonempty, so that the necessary condition ensues. □

Proposition 3.4.2 (Sufficient Condition) *Let us assume that*

$$\begin{cases} i) & F : X \rightsquigarrow X \text{ is upper semicontinuous} \\ ii) & \text{the images of } F \text{ are convex and compact} \end{cases}$$

Let K be a locally compact subset of the domain of F and $K_0 \subset K$ be a compact neighborhood of x_0 such that

$$\forall\, x \in K_0, \ \ F(x) \cap T_K(x) \ \neq \ \emptyset$$

Then there exist $T > 0$ and a solution to differential inclusion (3.7) starting at x_0 and viable in K on $[0, T]$.

Proof — We adapt the proof of Nagumo's Theorem 1.2.1 to the case of differential inclusion by following the same strategy: construct approximate solutions by modifying Euler's method to take into account the viability constraints, then deduce from available estimates that a subsequence of these solutions converges in some sense to a limit, and finally, check that this limit is a viable solution to differential inclusion (3.7). The two first steps are slight variations of the analogous steps of the proof of Nagumo's Theorem. The third step, which is specific to the set-valued case, uses the Convergence Theorem 2.4.4.

1. — Construction of Approximate Solutions

By assumption, there exists $r > 0$ such that the compact neighborhood $K_0 \subset K$ contains the ball $B_K(x_0, r) := K \cap (x_0 + rB)$. We set

$$C := F(K_0) + B, \quad T := r/\|C\|$$

We observe that C *is bounded.* We begin by proving

Lemma 3.4.3 *We posit the assumptions of Proposition 3.4.2. For any integer m, there exists $\theta_m \in]0, 1/m[$ such that for any $x \in K_0$, there exist $h \in [\theta_m, 1/m]$ and $u \in X$ satisfying*

$$\begin{cases} i) & u \in C \\ ii) & x + hu \in K \\ iii) & (x, u) \in B(\text{Graph}(F), 1/m) \end{cases}$$

Proof of Lemma 3.4.3 — By assumption, we know that for all $y \in K_0$, there exists an element $f(y) \in F(y) \cap T_K(y)$. By definition of the contingent cone, there exists $h_y \in]0, 1/m[$ such that

$$d_K(y + h_y f(y)) < h_y/2m$$

We introduce the subsets

$$N(y) := \{x \in K_0 \mid d_K(x + h_y f(y)) < h_y/2m\}$$

These subsets are obviously *open*. Since y belongs to $N(y)$, there exists $\eta_y \in]0, 1/m[$ such that $B(y, \eta_y) \subset N(y)$. The compactness of K_0 implies that it can be covered by q such balls $B(y_j, \eta_j)$, $j = 1, \dots, q$. We set

$$\theta_m := \min_{j=1,\dots,q} h_{y_j}$$

Let us choose any $x \in K_0$. Since it belongs to one of the balls $B(y_j, \eta_j) \subset N(y_j)$, there exists $z_j \in K$ such that

$$\begin{cases} \|x + h_{y_j} f(y_j) - z_j\|/h_{y_j} \\ \leq d_K(x + h_{y_j} f(y_j))/h_{y_j} + 1/2m \leq 1/m. \end{cases}$$

Let us set

$$u_j := \frac{z_j - x}{h_{y_j}}$$

We see that $\|x - y_j\| \leq \eta_j \leq 1/m$, that $x + h_{y_j} u_j = z_j \in K$ and that $\|u_j - f(y_j)\| \leq 1/m$. Hence,

$$(x, u_j) \in B((y_j, f(y_j)), 1/m) \subset B(\text{Graph}(F), 1/m)$$

and $u_j \in B(F(K_0), 1/m) \subset C$. Hence the Lemma ensues. □

We can now construct by induction a sequence of positive numbers $h_j \in]\theta_m, 1/m[$ and a sequence of elements $x_j \in K_0$ and $u_j \in C$ such that

$$\begin{cases} i) & x_{j+1} := x_j + h_j u_j \in K_0, \quad u_j \in C \\ ii) & (x_j, u_j) \in B(\text{Graph}(F), 1/m) \end{cases}$$

as long as $\sum_{i=0}^{j-1} h_i \leq T$.

Indeed, the elements x_j belong to K_0, since

$$\|x_j - x\| \leq \sum_{i=0}^{i=j-1} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{i=j-1} h_i \|C\| \leq T\|C\| = r$$

Since the h_j's are larger than or equal to $\theta_m > 0$, there exists J such that

$$h_1 + \cdots + h_{J-1} \leq T < h_1 + \cdots + h_J$$

We introduce the nodes $\tau_m^j := h_0 + \cdots + h_{j-1}$, $j = 1, \ldots, J+1$ and we interpolate the sequence of elements x_j at the nodes τ_m^j by the piecewise linear functions $x_m(t)$ defined on each interval $[\tau_m^j, \tau_m^{j+1}[$ by

$$\forall\, t \in [\tau_m^j, \tau_m^{j+1}[, \quad x_m(t) := x_j + (t - \tau_m^j) u_j$$

We observe that this sequence satisfies the following estimates

$$\begin{cases} i) & \forall\, t \in [0,T], \;\; x_m(t) \in \operatorname{co}(K_0) \\ ii) & \forall\, t \in [0,T], \;\; \|x'_m(t)\| \le \|C\| \end{cases} \tag{3.10}$$

Let us fix $t \in [\tau_m^j, \tau_m^{j+1}[$. Since $\|x_m(t) - x_m(\tau_m^j)\| \le h_j\|u_j\| \le \|C\|/m$, and since $(x_j, u_j) \in B(\text{Graph}(F), 1/m)$ by Lemma 3.4.3, we deduce that these functions are approximate solutions in the sense that

$$\begin{cases} i) & \forall\, t \in [0,T], \;\; x_m(t) \in B(K_0, \varepsilon_m) \\ ii) & \forall\, t \in [0,T], \;\; (x_m(t), x'_m(t)) \in B(\text{Graph}(F), \varepsilon_m) \end{cases} \tag{3.11}$$

where $\varepsilon_m := (\|C\| + 1)/m$ converges to 0.

2. — Convergence of the Approximate Solutions

Estimates (3.10) imply that for all $t \in [0,T]$, the sequence $x_m(t)$ remains in the compact subset $\operatorname{co}(K_0)$ and that the sequence $x_m(\cdot)$ is *equicontinuous*, because the derivatives $x'_m(\cdot)$ are bounded. We then deduce from Ascoli's Theorem that it remains in a compact subset of the Banach space $\mathcal{C}(0,T;X)$, and thus, that a subsequence (again denoted) $x_m(\cdot)$ converges uniformly to some function $x(\cdot)$.

Furthermore, the sequence $x'_m(\cdot)$ being bounded in the dual of the Banach space $L^1(0,T;X)$ which is equal to $L^\infty(0,T;X)$, it is weakly relatively compact thanks to Alaoglu's Theorem[7]. But since T is finite, the Banach space $L^\infty(0,T;X)$ is contained in $L^1(0,T;X)$ with a stronger topology[8]. The identity map being continuous for the

[7] Alaoglu's Theorem states that any *bounded subset of the dual of a Banach space is weakly compact.*

[8] Since the Lebesgue measure on $[0,T]$ is finite, we know that

$$L^\infty(0,T;X) \subset L^1(0,T;X)$$

with a stronger topology. The weak topology $\sigma(L^\infty(0,T;X),\; L^1(0,T;X))$ (weak-star topology) is stronger than the weakened topology $\sigma(L^1(0,T;X),\; L^\infty(0,T;X))$ since the canonical injection is continuous. Indeed, we observe that the seminorms of the weakened topology on $L^1(0,T;X)$, defined by finite sets of functions of $L^\infty(0,T;X)$, are seminorms for the weak-star topology on $L^\infty(0,T;X)$), since they are defined by finite sets of functions of $L^1(0,T;X)$.

norm topologies, is still continuous for the weak topologies. Hence the sequence $x'_m(\cdot)$ is weakly relatively compact in $L^1(0,T;X)$ and a subsequence (again denoted) $x'_m(\cdot)$ converges weakly to some function $v(\cdot)$ belonging to $L^1(0,T;X)$. Equations

$$x_m(t) - x_m(s) \quad = \quad \int_s^t x'_m(\tau)d\tau$$

imply that this limit $v(\cdot)$ is actually the weak derivative $x'(\cdot)$ of the limit $x(\cdot)$.

In summary, we have proved that

$$\begin{cases} i) & x_m(\cdot) \text{ converges uniformly to } x(\cdot) \\ ii) & x'_m(t) \text{ converges weakly to } \; x'(\cdot) \text{ in } L^1(0,T;X) \end{cases}$$

3. — The Limit is a Solution

Condition (3.11)i) implies that

$$\forall\, t \in [0,T], \;\; x(t) \;\in\; K_0$$

i.e., that $x(\cdot)$ is viable. The Convergence Theorem 2.4.4 and properties (3.11)ii) imply that

$$\text{for almost all } \; t \in [0,T], \;\; x'(t) \;\in\; F(x(t))$$

i.e., that $x(\cdot)$ is a solution to differential inclusion (3.7). □

Proof of Theorem 3.3.4 — First, K is locally compact since it is closed and the dimension of X is finite.

Second, we claim that starting from any x_0, there exists a maximal solution. Indeed, denote by $\mathcal{S}_{[0,T[}(x_0)$ the set of solutions to the differential inclusion defined on $[0,T[$.

We introduce the set of pairs $\{(T,x(\cdot))\}_{T>0,\, x(\cdot)\in\mathcal{S}_{[0,T[}(x_0)}$ on which we consider the order relation $\prec$ defined by

$$(T,x(\cdot)) \prec (S,y(\cdot)) \text{ if and only if } T \;\leq\; S \;\;\&\;\; \forall\, t \in [0,T[, \;\; x(t) = y(t)$$

Since every totally ordered subset has obviously a majorant, Zorn's Lemma implies that any solution $y(\cdot) \in \mathcal{S}_{[0,S[}(x_0)$ defined on some interval $[0, S[$ can be extended to a solution $x(\cdot) \in \mathcal{S}_{[0,T[}(x_0)$ defined on a maximal interval $[0, T[$.

Third, we have to prove that if T is finite, we cannot have

$$c \; := \; \limsup_{t \to T-} \|x(t)\| \; < \; +\infty$$

Indeed, if $c < +\infty$, there would exist a constant $\eta \in]0, T[$ such that

$$\forall \, t \in [T - \eta, T[, \quad \|x(t)\| \; \leq \; c + 1$$

Since F is upper semicontinuous with compact images on the compact subset $K \cap (c+1)B$, we infer that

$$\forall \, t \in [T - \eta, T[, \quad x'(t) \in F(K \cap (c+1)B), \quad \text{which is compact}$$

and thus bounded by a constant ρ. Therefore, for all $\tau, \sigma \in [T-\eta, T[$, we obtain:

$$\|x(\tau) - x(\sigma)\| \; \leq \; \int_{\sigma}^{\tau} \|x'(s)ds\| \; \leq \; \rho|\tau - \sigma|$$

Hence the Cauchy criterion implies that $x(t)$ has a limit when $t \to T-$. We denote by $x(T)$ this limit, which belongs to K because it is closed. Equalities

$$x(T_k) \; = \; x_0 + \int_0^{T_k} x'(\tau)d\tau$$

and Lebesgue's Theorem imply that by letting $k \to \infty$, we obtain:

$$x(T) \; = \; x_0 + \int_0^{T} x'(\tau)d\tau$$

This means that we can extend the solution up to T and even beyond, since Theorem 3.3.2 allows us to find a viable solution starting at $x(T)$ on some interval $[T, S]$ where $S > T$. Hence c cannot be finite. □

Proof of Theorem 3.3.5 — Since the growth of F is linear,

$$\exists \, c \geq 0, \quad \text{such that } \forall \, x \in \text{Dom}(F), \quad \|F(x)\| \leq c(\|x\| + 1)$$

Therefore, any solution to differential inclusion (3.7) satisfies the estimate:

$$\|x'(t)\| \leq c(\|x(t)\| + 1)$$

The function $t \to \|x(t)\|$ being locally Lipschitz, it is almost everywhere differentiable. Therefore, for any t where $x(t)$ is different from 0 and differentiable, we have

$$\frac{d}{dt}\|x(t)\| = \left\langle \frac{x(t)}{\|x(t)\|}, x'(t) \right\rangle \leq \|x'(t)\|$$

These two inequalities imply the estimates:

$$\|x(t)\| \leq (\|x_0\| + 1)e^{ct} \ \& \ \|x'(t)\| \leq c(\|x_0\| + 1)e^{ct} \tag{3.12}$$

Hence, for any $T > 0$, we infer that

$$\limsup_{t \to T-} \|x(t)\| < +\infty$$

Theorem 3.3.4 implies that we can extend the solution on the interval $[0, \infty[$.

Furthermore, estimates (3.12) imply that for $b > c$, the solution $x(\cdot)$ belongs to the weighted Sobolev space $W^{1,1}(0, \infty; X; e^{-bt}dt)$ since the multiplication by $e^{-(b-c)t}$ is continuous from $L^\infty(0, \infty; X)$ to $L^1(0, \infty; X)$. □

Proof of Theorem 3.3.6 — We introduce the set-valued map G defined on K by

$$G(x) := F(x) \cap c(\|x\| + 1)B$$

Corollary 2.2.3 implies that G is a Marchaud map. Assumption (3.8) implies that K is a viability domain of G. Therefore by Theorem 3.3.5, we know that for any $x_0 \in K$, there exists a viable solution to differential inclusion

$$x'(t) \in G(x(t))$$

on $[0, \infty[$, which is also a solution to differential inclusion (3.7) viable in K. □

3.5 Solution Map

We denote by $\mathcal{S}(x_0)$ or by $\mathcal{S}_F(x_0)$ the (possibly empty) set of solutions to differential inclusion (3.7.)

Definition 3.5.1 (Solution Map) *We shall say that the set-valued map $\mathcal{S}$ defined by* $\mathrm{Dom}(F) \ni x \mapsto \mathcal{S}(x)$ *is the* solution map *of F (or of differential inclusion (3.7).)*

When a closed subset K is viable under F, we denote

$$\mathcal{S}_F^K(x) \ := \ \mathcal{S}_F(x)(K) \cap \mathcal{K}$$

the set of solutions starting from $x \in K$ and viable in K.

We shall devote this section to the study of the solution map.

3.5.1 Upper Semicontinuity of Solution Maps

We recall that the space $W^{1,1}(0,\infty;X;e^{-bt}dt)$ is supplied with the topology for which a sequence $x_n(\cdot)$ converges to $x(\cdot)$ if and only if

$$\begin{cases} i) & x_n(\cdot) \text{ converges } \textit{uniformly} \text{ to } x(\cdot) \text{ on compact sets} \\ ii) & x_n'(\cdot) \text{ converges } \textit{weakly} \text{ to } x'(\cdot) \text{ in } L^1(0,T;X,e^{-bt}dt) \end{cases}$$

Theorem 3.5.2 (Continuity of the Solution Map) *Let us consider a finite dimensional vector space X and a* Marchaud map $F : X \rightsquigarrow X$. *We set*

$$c := \sup_{x \in \mathrm{Dom}(F)} \frac{\|F(x)\|}{\|x\|+1}$$

Then the solution map $\mathcal{S}$ is upper semicontinuous with compact images from its domain to the space $\mathcal{C}(0,\infty;X)$ supplied with the compact convergence topology.

Actually, for $b > c$, the solution map $\mathcal{S}$ is upper semicontinuous with compact images from its domain to the space $W^{1,1}(0,\infty;X;e^{-bt}dt)$.

Furthermore, the graph of the restriction of $\mathcal{S}|_L$ to any compact subset L of $\mathrm{Dom}(F)$ *is compact in $X \times W^{1,1}(0,\infty;X;e^{-bt}dt)$.*

Proof — We shall show that the graph of the restriction $\mathcal{S}|_L$ of the solution map $\mathcal{S}$ to a compact subset $L \subset \mathrm{Dom}(F)$ (assumed to be nontrivial) is compact.

Let us choose a sequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of the solution map $\mathcal{S}$. They satisfy:

$$x_n'(t) \in F(x_n(t)) \;\; \& \;\; x_n(0) = x_{0_n} \in L$$

A subsequence (again denoted) x_{0_n} converges to some $x_0 \in L$ because L is compact.

Then inequalities

$$\text{for almost all } t \geq 0, \quad \frac{d}{dt}\|x_n(t)\| \leq \|x_n'(t)\| \leq c(\|x_n(t)\| + 1)$$

imply that

$$\forall\, n \geq 0, \;\; \|x_n(t)\| \;\leq\; (\|x_{0_n}\| + 1)e^{ct} \;\; \& \;\; \|x_n'(t)\| \;\leq\; c(\|x_{0_n}\| + 1)e^{ct}$$

Therefore, by Ascoli's Theorem, the sequence $x_n(\cdot)$ is relatively compact in the Fréchet space $\mathcal{C}(0,\infty;X)$ and by Alaoglu's Theorem, the sequence $x_n'(\cdot)e^{-ct}$ is weakly relatively compact in $L^\infty(0,\infty;X)$.

Let us take $b > c$. Since the multiplication by $e^{-(b-c)t}$ is continuous from $L^\infty(0,\infty;X)$ to $L^1(0,\infty;X)$, it remains continuous when these spaces are supplied with weak topologies[9].

We have proved that the sequence $x_n'(\cdot)$ is *weakly relatively compact* in the weighted space $L^1(0,\infty;X;e^{-bt}dt)$.

We thus deduce that a subsequence (again denoted) x_n converges to x in the sense that:

$$\begin{cases} i) & x_n(\cdot) \text{ converges uniformly to } x(\cdot) \text{ on compact sets} \\ ii) & x_n'(\cdot) \text{ converges weakly to } x'(\cdot) \text{ in } L^1(0,\infty;X;e^{-bt}dt) \end{cases}$$

[9] If u_n converges weakly to u in $L^\infty(0,\infty;X)$, then $e^{-(b-c)t}u_n$ converges weakly to $e^{-(b-c)t}u$ in $L^1(0,\infty;X)$, because, for every $\varphi \in L^\infty(0,\infty;X) = L^1(0,\infty;X)^\star$, the values

$$< u_n, e^{-(b-c)t}\varphi > := \int_0^\infty e^{-(b-c)t}u_n(t)\varphi(t)dt$$

converge to

$$< u, e^{-(b-c)t}\varphi > := \int_0^\infty e^{-(b-c)t}u(t)\varphi(t)dt$$

since $e^{-(b-c)t}\varphi(\cdot)$ belongs to $L^1(0,\infty;X)$.

Inclusions

$$\forall\, n > 0, \quad (x_n(t), x_n'(t)) \in \text{Graph}(F)$$

imply that

$$\text{for almost all } t > 0, \quad x'(t) \in F(x(t))$$

thanks to the Convergence Theorem 2.4.4.

We thus have proved that a subsequence of elements $(x_{0_n}, x_n(\cdot))$ of the graph of $\mathcal{S}|_L$ converges to an element $(x_0, x(\cdot))$ of this graph. This shows that it is compact, and thus, that the solution map $\mathcal{S}$ is upper semicontinuous with compact images. □

Remark — We shall prove in Chapter 4 that the domain of the solution map $\mathcal{S}_F$ associated with a Marchaud map is a closed subset, called the *viability kernel* of Dom(F). Chapter 4 is devoted to the study of viability kernels. □

Remark — The "contingent derivative" of the solution map is contained in the solution map of the "variational inclusion", which is a "set-valued linearization" of the differential inclusion. (See Section 5, Chapter 10 of SET-VALUED ANALYSIS.) □

3.5.2 Closure of a Viability Domain

The first application of the upper semicontinuity of the solution map is that the closure of any subset viable under F is a viability domain:

Proposition 3.5.3 *Let us consider a* Marchaud map $F : X \rightsquigarrow X$ *and a subset* $\Omega \subset \text{Dom}(F)$ *viable under* F*. Then its closure* $\overline{\Omega}$ *is still viable under* F*.*

Proof — Indeed, let a sequence $x_n \in \Omega$ converge to x given in $\overline{\Omega}$. It remains in a compact subset L of the finite dimensional vector space X. Let us choose a sequence of solutions $x_n(\cdot) \in \mathcal{S}_F(x_n)$ viable in Ω, which exist by assumption.

Since the graph of the restriction $\mathcal{S}_F|_L$ of $\mathcal{S}_F$ to the compact subset L is compact, Theorem 3.5.2 implies that $(x_n, x_n(\cdot))$ belongs to the compact subset Graph($\mathcal{S}_F|_L$). Therefore a subsequence converges to some $(x, x(\cdot))$ of the graph of $\mathcal{S}_F|_L$, so that $x(\cdot)$ belongs to $\mathcal{S}_F(x)$. Since $x_n(t) \in \Omega$ for all $t \geq 0$, we infer the limit $x(\cdot)$ is viable in $\overline{\Omega}$. □

3.5.3 Reachable Map

We associate with the solution map $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ of the differential inclusion (3.7) the *reachable map*, (or flow, or set-valued semi-group) defined in the following way:

Definition 3.5.4 *For any $t \geq 0$, we denote by $R_F(t)(x) := (\mathcal{S}_F(x))(t)$ the set of states $x(t)$ reached from x through differential inclusion (3.7), by*

$$\begin{cases} R_F^K(t) := (\mathcal{S}_F(K))(t) \\ \\ Q_F^K(t) := (\mathcal{S}_F^K(K))(t) \end{cases}$$

the set of states $x(t)$ reached from K by solutions $x(\cdot) \in \mathcal{S}_F(x)$ and by solutions $x(\cdot) \in \mathcal{S}_F^K(x)$ viable in K respectively. They are called the reachable map *and* viable reachable map *respectively.*

The reachable map $R_F(t)(x)$ enjoys the *semigroup property*:

$$\forall\, t, s \;\geq\; 0, \;\; R_F(t+s)(x) \;=\; R_F(t)\left(R_F(s)(x)\right)$$

The maps $t \rightsquigarrow R_F^K(t)$ and $t \rightsquigarrow Q_F^K(t)$ are examples of *viability tubes* which shall be studied in Chapter 11. For the time, let us prove that these maps are closed:

Proposition 3.5.5 *Assume that $F : X \rightsquigarrow X$ is a Marchaud map and that a closed subset K is contained in the domain of $\mathcal{S}_F$. Then the graphs of the maps $t \rightsquigarrow R_F^K(t)$ and $t \rightsquigarrow Q_F^K(t)$ are closed.*

Proof — Let us consider a sequence (t_n, x_n) of the graph of $R_F^K(\cdot)$ converging to (t, x). By definition, there exist solutions $x_n(\cdot) \in \mathcal{S}_F(x_{0n})$ such that $x_{0n} \in K$ and $x_n(t_n) = x_n$. Since the sequence x_n is bounded, so that a subsequence converges to some $x \in K$, a slight modification of the proof of Theorem 3.5.2 obtained by writing that

$$x_n(t) \;=\; x_n + \int_{t_n}^{t} x_n'(\tau)d\tau$$

implies that a subsequence converges to some solution $x_\star(\cdot)$ such that $x_{n'} = x_{n'}(t_{n'})$ converges to $x = x_\star(t)$. Since a subsequence of x_{0n} converges to $x_\star(0)$, hence $x_\star(0)$ belongs to K and we deduce that $x \in R_F^K(t)$. □

The reachable maps play an important role in control theory. One can state that under adequate assumptions, $co(F)$ is its *infinitesimal generator*:

Theorem 3.5.6 (Frankowska) *If F is continuous with compact values, then*

$$\lim_{h \to 0+} \frac{R_F(h)(x) - x}{h} = co(F(x))$$

We shall see in Chapter 7 that the left-hand side of this formula is the *derivative* of the reachable map $R_F(\cdot)(x)$ at $(0, x)$, so that this theorem states that the when F is continuous, $co(F(x))$ is the derivative of the reachable map at $(0, x)$.

We refer to Hélène Frankowska's monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS for a proof of this basic theorem which plays a very important role for studying local controllability and value functions in optimal control.

3.5.4 Representation Property

When the viability subset is convex, we can represent the set of viable solutions in K as a countable intersection of solution sets to *unconstrained differential inclusions*, a property which is analogous to the duality property in convex minimization.

Theorem 3.5.7 (Kurzhanski) *Consider a set-valued map $F : X \rightsquigarrow X$ with nonempty compact values. Assume that $K := A^{-1}(M)$ is the inverse image of a closed convex subset $M \subset Y$ by a surjective linear operator $A \in \mathcal{L}(X, Y)$. Denote by $F|_K$ the restriction of F to K. Then, for any right-inverse $B \in \mathcal{L}(Y, X)$ of A,*

$$\forall\, x \in X, \quad F|_K(x) = \bigcap_{n \in \mathbf{Z}} (F(x) + nBAx - nB(M))$$

Consequently, for any $x \in K$, the set $\mathcal{S}|_K(x)$ of solutions to the differential inclusion $x'(t) \in F(x(t))$ viable in K is the intersection of *the sets of solutions $\mathcal{S}_n(x)$ to the unconstrained differential inclusions $x'(t) \in F(x(t)) + nBAx(t) - nB(M)$ when n ranges over* $\mathbf{Z}$.

Proof — Consider first the case when $x \in K$. Since $F(x) = F(x) + 0B(Ax - M)$, the intersection of the subsets $F(x) + nBAx - nB(M)$ is contained in $F(x)$. On the other hand, 0 belonging to $Ax - M$, we infer that $F(x) \subset F(x) + nB(Ax - M)$ for any $n \in \mathbf{Z}$, so that

$$F(x) \subset \bigcap_{n \in \mathbf{Z}} (F(x) + nBAx - nB(M))$$

Consider now the case when $x \notin K$ and let us show that

$$\bigcap_{n \in \mathbf{Z}} (F(x) + nBAx - nB(M)) = \emptyset$$

Since any right inverse B of A is injective, 0 does not belong to the closed convex subset $B(Ax - M)$, and thus can be separated from 0: There exist $p \in X^\star$ and $\varepsilon > 0$ such that

$$\sigma(B(Ax - M), p) = -\varepsilon < 0$$

Now, we observe that $F(x)$ being bounded, the support function $\sigma(F(x) - F(x)), p)$ is nonnegative and bounded. We claim that for any $n > (\sigma(F(x) - F(x)), p)/2\varepsilon$,

$$(F(x) - nB(Ax - M)) \cap (F(x) + nB(Ax - M)) = \emptyset$$

Otherwise, there would exist u_1 and u_2 in $F(x)$ such that $u_1 - u_2$ would belong both to $F(x) - F(x)$ and to $2nB(Ax - M)$, so that we would obtain the contradiction

$$-\sigma(F(x) - F(x), p) \leq < p, u_1 - u_2 > \leq -2n\varepsilon \quad \square$$

3.6 Stability of Viability Domains

Let us recall the definition of Painlevé-Kuratowski upper limit[10] of sets:

[10]The concepts of upper and lower limits of sets were introduced by Painlevé in 1902 and popularized by Kuratowski in his famous book TOPOLOGIE, to the point that they are often Christened *Kuratowski upper limits*. See the first chapter of

Definition 3.6.1 *Let K_n be a sequence of subsets of a metric space X. we say that*

$$K^\sharp := \text{Limsup}_{n\to\infty} K_n := \{y \in Y \mid \liminf_{n\to\infty} d(y, K_n) = 0\}$$

is its upper limit.

In other words, *it is the closed subset of cluster points of sequences of elements $x_n \in K_n$.*

We observe that the *contingent cone*

$$T_K(x) \;=\; \text{Limsup}_{h\to 0+} \frac{K - x}{h}$$

is the upper limit of the differential quotients $\frac{K-x}{h}$ when $h \to 0+$.

Let us consider now a sequence of closed subsets K_n viable under a set-valued map F. *Is the upper limit of these closed subsets still viable under F?* The answer is positive.

Theorem 3.6.2 *Let us consider a* Marchaud map $F : X \rightsquigarrow X$. *Then the upper limit of a sequence of closed subsets viable under F is still viable under F.*

In particular, the intersection of a decreasing family of closed viability domains is a closed viability domain.

Proof — We shall prove that the upper limit $K^\sharp$ of a sequence of subsets K_n viable under F is still viable under F.

Let x belong to $K^\sharp$. It is the limit of a subsequence $x_{n'} \in K_{n'}$. Since the subsets K_n are viable under F, there exist solutions $y_{n'}(\cdot)$ to differential inclusion $x' \in F(x)$ starting at $x_{n'}$ and viable in $K_{n'}$. The upper semicontinuity of the solution map implies that a subsequence (again denoted) $y_{n'}(\cdot)$ converges uniformly on compact intervals to a

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SET-VALUED ANALYSIS for an exhaustive study of these upper and lower limits of sequences of sets. Recall only that if the space X is *compact* , then the upper limit $K^\sharp$ enjoys

$$\text{for all neighborhood } \mathcal{U} \text{ of } K^\sharp, \;\; \exists N \;\; \text{such that } \forall n > N, \;\; K_n \subset \mathcal{U}$$

solution $y(\cdot)$ to differential inclusion $x' \in F(x)$ starting at x. Since $y_{n'}(t)$ belongs to $K_{n'}$ for all n', we deduce that $y(t)$ does belong to $K^\sharp$ for all $t > 0$.

When the sequence K_n is decreasing, we know that its upper limit is equal to the intersection of the K_n. □

What happens if we deal with the upper limit $K^\sharp$ of a sequence of closed viability domains K_n of set-valued maps F_n?

For that purpose, we introduce the concept of *graphical upper limit* of a sequence of set-valued maps F_n.

Definition 3.6.3 *We shall say that the set-valued maps* $\mathrm{Lim}^\sharp{}_{n\to\infty}F_n$ *from* X *to* X *defined by*

$$\mathrm{Graph}(\mathrm{Lim}^\sharp{}_{n\to\infty}F_n) \;:=\; \mathrm{Limsup}_{n\to\infty}\mathrm{Graph}(F_n)$$

is the graphical upper limit of the set-valued maps F_n.

For simplicity, we set $F^\sharp := \mathrm{Lim}^\sharp{}_{n\to\infty}F_n$. One can find more details on graphical limits in Chapter 7 of SET-VALUED ANALYSIS.

The question then arises whether the upper limit $K^\sharp$ of a sequence of closed subsets K_n viable under set-valued maps F_n is viable under the closed convex hull of the upper graphical limit $\overline{\mathrm{co}}F^\sharp$ of the set-valued maps F_n?

Theorem 3.6.4 (Stability of Solution Maps) *Let us consider a sequence of nontrivial set-valued maps* $F_n : X \rightsquigarrow X$ *satisfying a uniform linear growth: there exists* $c > 0$ *such that*

$$\forall\, x \in X,\;\; \|F_n(x)\| \;\le\; c(\|x\| + 1)$$

Then

1. — The upper limit of the solution maps $\mathcal{S}_{F_n}$ *is contained in the solution map* $\mathcal{S}_{\overline{\mathrm{co}}(F^\sharp)})$ *of the convex hull of the graphical upper limit of the set-valued maps* F_n

2. — If the subsets $K_n \subset \mathrm{Dom}(F_n)$ *are viable under the set-valued maps* F_n*, then the upper limit* $K^\sharp$ *is viable under* $\overline{\mathrm{co}}(F^\sharp)$.

It follows from the adaptation of the Convergence Theorem to limits of set-valued maps:

Theorem 3.6.5 *Let X be a topological vector space, Y be a finite dimensional vector space and F_n be a sequence of nontrivial set-valued maps from X to Y satisfying a uniform linear growth.*

Let us consider measurable *functions x_m and y_m from $[0,\infty[$ to X and Y respectively, satisfying:*

for almost all $t \in [0,\infty[$ and for all neighborhood $\mathcal{U}$ of 0 in the product space $X \times Y$, there exists $M := M(t,\mathcal{U})$ such that

$$\forall\, m > M,\ (x_m(t), y_m(t)) \in \text{Graph}(F_m) + \mathcal{U} \qquad (3.13)$$

If we assume that

$$\begin{cases} i) & x_m(\cdot) \text{ converges almost everywhere to a function } x(\cdot) \\ ii) & y_m(\cdot) \in L^1(0,\infty,Y;a) \text{ and converges } \textit{weakly} \text{ in } L^1(0,\infty,Y;a) \\ & \text{to a function } y(\cdot) \in L^1(0,\infty,Y;a) \end{cases}$$

then,

$$\text{for almost all } t \in [0,\infty[,\ \ y(t) \in \overline{co}(F^\sharp(x(t)))$$

We refer to Theorem 7.2.1 of SET-VALUED ANALYSIS for a proof.

3.7 ω-Limit Sets and Equilibria

3.7.1 ω-Limit Sets

The ω-limit sets of the solutions to differential inclusion

$$\text{for almost all } t \geq 0,\ \ x'(t) \in F(x(t)) \qquad (3.14)$$

provide examples of closed viability domains:

Definition 3.7.1 (ω-Limit set) *Let $x(\cdot)$ be a function from $[0,\infty[$ to X. We say that the subset*

$$\omega(x(\cdot)) := \bigcap_{T>0} cl(x([T,\infty[)) = \text{Limsup}_{t\to+\infty}\{x(t)\}$$

of its cluster points when $t \to \infty$ is the ω-limit set *of $x(\cdot)$.*

If F is a set-valued map, K a subset of $\text{Dom}(\mathcal{S}_F)$ *and $R_F^K(\cdot)$ the reachable map, we denote by*

$$\omega_F(K) := \text{Limsup}_{t\to+\infty} R_F^K(t)$$

the ω-limit set of the subset K. If K is a closed subset viable under F, the viable ω-limit set of K *is defined by*

$$\omega_F^K(K) := \text{Limsup}_{t\to+\infty} Q_F^K(t)$$

Being upper limits, *the ω-limit sets of solutions and sets are closed subsets.* They also are viable under F. We begin with the case of ω-limit sets of solutions:

Theorem 3.7.2 (ω-Limit sets are viability domains) *Let us consider a* Marchaud map $F : X \rightsquigarrow X$. *Then the ω-limit set of a solution to the differential inclusion (3.14) is a* closed viability domain[11].

In particular, the limits of solutions to the differential inclusion (3.14), when they exist, are equilibria *of F and the trajectories of periodic solutions to the differential inclusion (3.14) are also closed viability domains.*

If K is a viability domain of F, then the ω-limit sets of viable solutions are contained in K.

Proof — Let $\overline{x}$ belong to the ω-limit set of a solution $x(\cdot)$. It is the limit of a sequence of elements $x(t_n)$ when $t_n \to \infty$. We then introduce the functions $y_n(\cdot)$ defined by $y_n(t) := x(t+t_n)$. They are solutions to the differential inclusion (3.14) starting at $x(t_n)$. By Theorem 3.5.2 on the upper semicontinuity of the solution map, a subsequence (again denoted) $y_n(\cdot)$ converges uniformly on compact intervals to a solution $y(\cdot)$ to the differential inclusion (3.14) starting at $\overline{x}$. On the other hand, for all $t > 0$,

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} x(t+t_n) \in \omega(x(\cdot))$$

i.e., $y(\cdot)$ is *viable* in the ω-limit set $\omega(x(\cdot))$. Hence the ω-limit set is viable under F. The necessary condition of the Viability Theorem 3.3.2 implies that this ω-limit set is a viability domain.

[11] which is *connected when $\omega(x(\cdot))$ is compact*. If not, $\omega(x(\cdot))$ would be covered by two nonempty disjoint closed subsets K_1 and K_2. So, they can be separated by two disjoint open neighborhoods $U_1 \supset K_1$ and $U_2 \supset K_2$.

Since $U_1 \cup U_2$ is a neighborhood of the compact subset $\omega(x(\cdot))$, there exists T such that the subset $\Gamma := \{x(t)\}_{t\geq T}$ is contained in $U_1 \cup U_2$. This set is connected as the continuous image of $[T, \infty[$. We observe that the subsets $\Gamma_i := \Gamma \cap U_i$ are not empty, open, disjoint and cover Γ: this is a contradiction of the connectedness of Γ.

When a solution has a limit $\overline{x}$ when $t \rightarrow \infty$, the subset $\{\overline{x}\}$ is a viability domain, and thus, $\overline{x}$ is an equilibrium. $\square$

We consider now the case of ω-limit sets of closed subsets:

Proposition 3.7.3 *Let us consider a Marchaud map $F : X \rightsquigarrow X$ and a closed subset K of the domain of $\mathcal{S}_F$. Then the ω-limit set $\omega_F(K)$ is viable under F.*

If there exists $T \geq 0$ such that $\bigcup_{t \geq T} R_F^K(t)$ is bounded, then $\omega_F(K)$ is an universal attractor *in the sense that*

$$\forall\, x \in K,\ \forall\, x(\cdot) \in \mathcal{S}_F(x),\ \lim_{t \rightarrow \infty} d(x(t), \omega_F(K)) \ = \ 0$$

If K is viable under F, then the viable ω-limit set $\omega_F^K(K)$ is a closed viability domain contained in K.

If K is compact, it is an attractor *in the sense that*

$$\forall\, x \in K,\ \exists\ x(\cdot) \in \mathcal{S}_F^K(x) \quad \textit{such that} \ \lim_{t \rightarrow \infty} d(x(t), \omega_F^K(K)) \ = \ 0$$

Proof — The closed subset $\omega_F(K)$ is viable under F. Indeed, let ξ belong to $\omega_F(K)$. Then $\xi = \lim \xi_n$ where $\xi_n \ \in \ R_F^K(t_n)$. We associate with the solutions $x_n(\cdot)$ to the differential inclusion

$$x_n'(t) \ \in F(x_n(t)),\ \ x_n(t_n) \ = \ \xi_n$$

the functions $y_n(\cdot)$ defined by $y_n(t) := x_n(t + t_n)$ which are solutions to

$$y_n'(t) \ \in \ F(y_n(t)),\ \ y_n(0) \ = \ \xi_n$$

Theorem 3.5.2 implies that these solutions remain in a compact subset of $\mathcal{C}(0, \infty; X)$. Therefore, a subsequence (again denoted by) $y_n(\cdot)$ converges to $y(\cdot)$, which is a solution to

$$y'(t) \ \in \ F(y(t)),\ \ y(0) \ = \ \xi$$

Furthermore, this solution is viable in $\omega_F(K)$ since for all $t \geq 0$, $y(t)$ is the limit of a subsequence of $y_n(t) = x_n(t + t_n) \in R_F^K(t + t_n)$, and thus belongs to $\omega_F(K)$.

Let us prove now that $\omega_F(K)$ is an universal attractor. If not, there would exist $x_0 \in K$, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$, $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$\forall\, n \geq 0, \;\; d\,(x(t_n), \omega_F(K)) \;\geq\; \delta \;>\; 0$$

Since the closure of $\bigcup_{t \geq T} R_F^K(t)$ is compact by assumption, a subsequence (again denoted by) $x(t_n)$ converges to some $x_\star$ which belongs to the ω-limit set $\omega_F(K)$. We thus obtain a contradiction.

The proofs of the statements about $\omega_F^K(K)$ are analogous. □

We shall see in Chapter 11 that *upper limits of viability tubes* $t \mapsto P(t)$ *when* $t \to \infty$ *are closed subsets viable under* F *which are attractors* when $\bigcup_{t \geq T} P(t)$ is relatively compact. If we regard such ω-limit sets as *"asymptotic targets"* (because they are made of cluster points of solutions viable in such tubes), we must look for asymptotic targets among the closed subsets viable under F. □

3.7.2 Cesaro means of the velocities

The property of the Cesaro means described in the assumptions of the next theorem implies the existence of an equilibrium:

Theorem 3.7.4 *Let us assume that* F *is upper hemicontinuous with closed convex images and that* $K \subset \mathrm{Dom}(F)$ *is compact. If there exists a solution* $x(\cdot)$ *viable in* K *such that*

$$\inf_{t>0} \frac{1}{t} \int_0^t \|x'(\tau)\| d\tau \;=\; 0$$

then there exists a viable equilibrium $\bar{x}$*, i.e., a state* $\bar{x} \in K$ *solution to the inclusion* $0 \in F(\bar{x})$*.*

Proof — Let us assume that there is no viable equilibrium, i.e., that for any $x \in K$, 0 does not belong to $F(x)$. Since the images of F are closed and convex, the Separation Theorem implies that there exists $p \in \Sigma$, the unit sphere, and $\varepsilon_p > 0$ such that

$\sigma(F(x), -p) < -\varepsilon_p$. In other words, we can cover the compact subset K by the subsets

$$\mathcal{V}_p := \{ x \in K \mid \sigma(F(x), -p) < -\varepsilon_p \}$$

when p ranges over Σ. They are open thanks to the upper hemicontinuity of F, so that the compact subset K can be covered by q open subsets $\mathcal{V}_{p_j}$. Set $\varepsilon := \min_{i=1,\ldots,q} \varepsilon_{p_i} > 0$.

Consider now any viable solution to differential inclusion (3.14). Hence, for any $t \geq 0$, $x(t)$ belongs to some $\mathcal{V}_{p_j}$, so that

$$-\|x'(t)\| \leq \langle -p_j, x'(t) \rangle \leq \sigma(F(x(t)), -p_j) < -\varepsilon$$

and thus, by integrating from 0 to t, we have proved that there exists $\varepsilon > 0$ such that, for all $t > 0$,

$$\varepsilon < \frac{1}{t} \int_0^t \|x'(\tau)\| d\tau$$

a contradiction of the assumption of the theorem. □

3.7.3 Viability implies Stationarity

When K is a compact viability domain, then the convexity of either $F(K)$ or of K implies the existence of a viable equilibrium.

Theorem 3.7.5 *Let F be a Marchaud map. If $K \subset \mathrm{Dom}(F)$ is a compact viability domain and if $F(K)$ is convex, then there exists an equilibrium.*

Proof — Assume that there is no equilibrium. Hence, this means that 0 does not belong to the closed convex subset $F(K)$, so that the Separation Theorem implies the existence of some $p \in X^\star$ and $\varepsilon > 0$ such that

$$\sup_{x \in K, v \in F(x)} < v, -p > = \sigma(F(K), -p) < -\varepsilon$$

Hence, let us take any viable solution $x(\cdot)$ to differential inclusion (3.14), which exists by the Viability Theorem. We deduce that

$$\forall\, t \geq 0, \ < -p, x'(t) > \leq -\varepsilon$$

so that, integrating from 0 to t, we infer that

$$\varepsilon\, t \;\leq < p, x(t) - x(0) >$$

But K being bounded, we thus derive a contradiction. □

We shall state now that any convex compact viability domain contains an equilibrium.

Theorem 3.7.6 (Equilibrium Theorem) *Let X be a Banach space*[12] *and $F : X \leadsto X$ be an upper hemicontinuous set-valued map with closed convex images.*

If $K \subset X$ is a convex compact viability domain of F, then it contains an equilibrium of F.

This theorem is equivalent to the Kakutani and Brouwer Fixed Point Theorems; we shall not prove this equivalence here[13].

We show only that the Equilibrium Theorem 3.7.6 implies the Kakutani Fixed Point Theorem[14], which is the set-valued version of the Brouwer fixed Point Theorem.

Theorem 3.7.7 (Kakutani Fixed Point Theorem) *Let K be a convex compact subset of a Banach space X and $G : K \leadsto K$ be a strict upper hemicontinuous set-valued map with closed convex values. Then G has a fixed point*[15] *$\bar{x} \in K \cap G(\bar{x})$.*

Proof — We set $F(x) := G(x) - x$, which is also upper hemicontinuous with convex values. Since K is convex, then $K - x \subset T_K(x)$, and since $G(K) \subset K$, we deduce that K is a viability domain of F because $F(x) \subset T_K(x)$. Hence there exists a viable equilibrium $\bar{x} \in K$ of F, which is a fixed point of G. □

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[12] Actually, this theorem remains true for any Hausdorff locally convex topological vector space and in particular, for spaces endowed with weak topologies.

[13] See Appendix C of MATHEMATICAL METHODS OF GAME AND ECONOMIC THEORY for a proof of the Brouwer Fixed Point Theorem based on Sperner's Lemma and the second chapter of APPLIED NONLINEAR ANALYSIS for a proof based on differential geometry. We refer to these books or SET-VALUED ANALYSIS for a proof of the equivalence between these statements and the Ky Fan Inequality.

[14] called Ky Fan's Fixed Point Theorem in infinite dimensional spaces.

[15] which can be regarded as an equilibrium for the discrete set-valued dynamical system $x_{n+1} \in G(x_n)$.

Actually, Equilibrium Theorem 3.7.6 can be derived from the Brouwer Fixed-Point Theorem via the *Ky Fan Inequality.* We recall it below not only because we shall use it later, but because of its efficiency for proving many results of nonlinear analysis.

Theorem 3.7.8 (Ky Fan Inequality) *Let K be a compact convex subset of a Banach space and $\varphi : K \times K \mapsto \mathbf{R}$ be a function satisfying*

$$\left\{ \begin{array}{lll} i) & \forall\, y \in K, & x \mapsto \varphi(x,y) \text{ is lower semicontinuous} \\ ii) & \forall\, x \in K, & y \mapsto \varphi(x,y) \text{ is concave} \\ iii) & \forall\, y \in K, & \varphi(y,y) \leq 0 \end{array} \right. \tag{3.15}$$

Then, there exists $\bar{x} \in K$, a solution to

$$\forall\, y \in K, \;\; \varphi(\bar{x},y) \leq 0 \tag{3.16}$$

The Ky Fan inequality implies readily the von Neumann Minimax Theorem:

Theorem 3.7.9 (Minimax) *Let X and Y be Banach spaces*[16]*, $L \subset X$ and $M \subset Y$ be compact convex subsets and $f : L \times M \mapsto \mathbf{R}$ be a real valued function satisfying*

$$\left\{ \begin{array}{lll} i) & \forall\, y \in M, & x \mapsto f(x,y) \text{ is lower semicontinuous and convex} \\ & & \\ ii) & \forall\, x \in L, & y \mapsto f(x,y) \text{ is upper semicontinuous and concave} \end{array} \right.$$

Then there exists a saddle point $(\bar{x},\bar{y}) \in L \times M$ *of f:*

$$\forall\, (x,y) \in L \times M, \;\; f(\bar{x},y) \leq f(\bar{x},\bar{y}) \leq f(x,\bar{y})$$

Proof — We apply the Ky Fan Inequality with $K := L \times M$ and φ defined by

$$\varphi((\bar{x},\bar{y}),(x,y)) := f(\bar{x},y) - f(x,\bar{y}) \;\; \square$$

Actually, we often need a weaker version of the Minimax Theorem, called the Lop-Sided Minimax Theorem. We recall its statement:

[16] actually, Hausdorff locally convex topological vector spaces.

Theorem 3.7.10 (Lop-Sided Minimax Theorem) *Let X and Y be Banach spaces*[17]*, $L \subset X$ be a compact convex subset, $M \subset Y$ be a convex subset and $f : L \times M \mapsto \mathbf{R}$ be a real valued function satisfying*

$$\begin{cases} i) & \forall\, y \in M, \ \ x \mapsto f(x,y) \text{ is lower semicontinuous and convex} \\ ii) & \forall\, x \in L, \ \ y \mapsto f(x,y) \text{ is and concave} \end{cases}$$

Then there exists $\bar{x} \in L$ satisfying

$$\forall\, y \in M, \ \ f(\bar{x},y) \ \leq \ \inf_{x \in L} \sup_{y \in M} f(x,y) \ = \ \sup_{y \in M} \inf_{x \in L} f(x,y)$$

Proof — We refer to Theorem 6.2.7. of APPLIED NONLINEAR ANALYSIS for an instance of proof using only the Separation Theorem. □

Remark — A slight modification of the proof of the Equilibrium Theorem yields a whole family of sufficient conditions for the existence of zeros of a set-valued map from $K \subset X$ to another space Y. □

Theorem 3.7.11 *Let K be a convex compact subset of a Banach space X and F be a nontrivial upper hemicontinuous set-valued map with closed convex values from X to another Banach space Y.*

Let us consider also a continuous map $B : K \to \mathcal{L}(X,Y)$. If K, F and B are related by the condition

$$\forall\, x \in K, \ \ F(x) \cap \overline{B(x)T_K(x)} \ \neq \ \emptyset$$

then

$$\begin{cases} i) & \exists\, \bar{x} \in K \text{ such that } 0 \ \in \ F(\bar{x}) \\ ii) & \forall\, y \in K, \ \exists\, \hat{x} \in K \text{ such that } B(\hat{x})y \ \in \ B(\hat{x})\hat{x} - F(\hat{x}) \end{cases}$$

As an example, we derive the existence of a solution to the equation $f(\bar{x}) = 0$ where the solution $\bar{x}$ must belong to a compact convex subset K:

[17] or, more generally, an Hausdorff locally convex topological vector spaces.

Theorem 3.7.12 *Let X and Y be Banach spaces, $K \subset X$ be a compact convex subset, $\Omega \supset K$ be an open neighborhood of K and $f : \Omega \mapsto Y$ be a continuously differentiable single-valued map. Assume that*

$$\forall\, x \in K, \ \ -f(x) \ \in \ f'(x)T_K(x)$$

Then there exists a solution $\bar{x} \in K$ to the equation $f(\bar{x}) = 0$. In particular, when $x_0 \in K$ is given, there exists a sequence of elements $x_n \in K$ satisfying

$$\forall\, n \geq 0, \ \ f'(x_n)(x_n - x_{n-1}) \ = \ -f(x_n)$$

i.e., the implicit version of the Newton algorithm, studied in more details in Chapter 10.

Proof — We take $F(x) := \{f(x)\}$ and $B(x) = -f'(x)$ in Theorem 3.7.11. □

3.8 Chaotic Solutions to Differential Inclusions

Let $F : X \rightsquigarrow X$ be a Marchaud map, describing the dynamics of the differential inclusion

$$\text{for almost all } t \geq 0, \ \ x'(t) \ \in \ F(x(t)) \tag{3.17}$$

Theorem 3.8.1 (Chaotic Behavior) *Let us assume that a compact viability domain K of the Marchaud map F is covered by a family of closed subsets K_a $(a \in \mathcal{A})$ such that the following* controllability assumption *holds true: There exists $T < \infty$ such that*

$$\forall\, a \in \mathcal{A}, \forall\, y \in K, \ \ \exists\, x \in K_a, \ x(\cdot) \in \mathcal{S}(x) \ \&\ t \in [0,T[\text{ with } x(t) = y$$

Then, for any sequence $a_0, a_1, \ldots, a_n, \ldots$, there exists at least one solution $x(\cdot) \in \mathcal{S}(x)$ to differential inclusion (3.17) and a sequence of elements $t^j \geq 0$ such that $x(t^j) \ \in \ K_{a_j}$ for all $j \geq 0$.

Proof — Let $M \subset K$ be any closed subset. We associate with any solution $x(\cdot)$ starting at $x \in K$ and intersecting M at some time $t \in [0,T]$ the number $\tau_M := \inf(t \in [0,T] \mid x(t) \in M)$.

We associate with the sequence $a_0, a_1, \ldots$ the subsets $M_{a_0 a_1 \cdots a_n}$ defined by induction by $M_{a_n} := K_{a_n}$,

$$M_{a_{n-1}a_n} := \{x \in K_{a_{n-1}} \mid \exists\, x(\cdot) \in \mathcal{S}(x) \text{ such that } x(\tau_{M_{a_n}}) \in K_{a_n}\}$$

and, for $j = n-2, \ldots, 0$, by:

$$\begin{cases} M_{a_j a_{j+1} \cdots a_n} := \{x \in K_{a_j} \mid \exists\, x(\cdot) \in \mathcal{S}(x) \\ \text{such that } x(\tau_{M_{a_{j+1} \cdots a_n}}) \in M_{a_{j+1} \cdots a_n}\} \end{cases}$$

The controllability assumption implies that they are nonempty. They are closed thanks to Theorem 3.5.2. Since the family of subsets $M_{a_0 a_1 \cdots a_n}$ form a nonincreasing family and since K is compact, the intersection $K_\infty := \bigcap_{n=0}^{\infty} M_{a_0 a_1 \cdots a_n}$ is nonempty.

Let us take an initial state x in K_∞ and fix n. Hence there exists $x_n(\cdot) \in \mathcal{S}(x)$ and a sequence of $t_n^j \in [0, jT]$ such that

$$\forall\, j = 1, \ldots, n, \;\; x_n(t_n^j) \in M_{a_j \cdots a_n} \subset K_{a_j}$$

Indeed, there exist $y_1 \in \mathcal{S}(x)$ and $\tau_{M_{a_1 \cdots a_n}} \in [0, T]$ such that $y_1(\tau_{M_{a_1 \cdots a_n}})$ belongs to $M_{a_1 \cdots a_n}$. We set $t_n^1 := \tau_{M_{a_1 \cdots a_n}}$ and $x_n(t) := y_1(t)$ on $[0, t_n^1]$.

Assume that we have built $x_n(\cdot)$ on the interval $[0, t_n^k]$ such that $x_n(t_n^j) \in M_{a_j \cdots a_n} \subset K_{a_j}$ for $j = 1, \ldots, k$. Since $x_n(t_n^k)$ belongs to $M_{a_k \cdots a_n}$, there exist $y_{k+1} \in \mathcal{S}(x_n(t_n^k))$ and $\tau_{M_{a_{k+1} \cdots a_n}} \in [0, T]$ such that $y_{k+1}(\tau_{M_{a_{k+1} \cdots a_n}})$ belongs to $M_{a_{k+1} \cdots a_n}$. We set

$$t_n^{k+1} := t_n^k + \tau_{M_{a_{k+1} \cdots a_n}} \;\;\&\;\; x_n(t) := y_{k+1}(t + \tau_{M_{a_{k+1} \cdots a_n}})$$

on $[t_n^k, t_n^{k+1}]$.

Since for some $b > 0$, the sequence $x_n(\cdot) \in \mathcal{S}(x)$ is compact in the space $W^{1,1}(0, \infty; X; e^{-bt}dt)$, a subsequence (again denoted $x_n(\cdot)$) converges to some solution $x(\cdot) \in \mathcal{S}(x)$ to the differential inclusion. By extracting successive converging subsequences of $t_{n_1}^1, \ldots, t_{n_j}^j, \ldots$, we infer the existence of t_j's in $[0, jT]$ such that $x_{n_j}(t_{n_j}^j)$ converges to $x(t_j) \in K_{a_j}$, because the functions $x_n(\cdot)$ remain in an equicontinuous subset. □

Chapter 4

Viability Kernels and Exit Tubes

Introduction

If a closed subset K is not a viability domain, the question arises as to whether there are closed viability subsets of K viable under F and even, whether there exists a largest closed subset of K viable under F. The answer is positive for Marchaud maps, and we call *viability kernel* of a closed subset K *the largest closed subset viable under F contained in K*. Actually, we shall prove in the first section that it is equal to the set of initial states of K from which there exists at least one solution viable in K.

It is not only an attractive concept in the framework of viability theory, but also an efficient mathematical tool which we shall use very often. We illustrate this fact by characterizing the *permanence property* introduced by J. Hofbauer and K. Sigmund and the *fluctuation property* introduced by V. Krivan in terms of viability kernels.

If the initial state x_0 does not belong to the viability kernel of K, then every solution $x(\cdot) \in \mathcal{S}(x_0)$ must eventually leave K in finite time (in the sense that for some $T > 0$, $x(T) \notin K$) and never meets $\mathrm{Viab}(K)$ as long as it remains in K. This justifies the fact that a closed subset with empty viability kernel is called a *repeller*.

This property of the complement of the viability kernel (that we could call the *shadow* according to a poetic term coined by Henri

Poincaré) motivates the introduction in Section 2 of the *exit tube* of a closed subset, which associates with any $T > 0$ the subset of initial states $x \in K$ such that at least one solution $x(\cdot)$ to the differential inclusion starting at x remains in K for all $t \in [0, T]$. When $T = +\infty$, this is the viability kernel of K.

We continue in the third section with a more precise study of the boundary of a closed subset K of the domain of a Marchaud map F. We partition the boundary of K into three areas (the anatomy of K): The first one, the *inward area*, where all solutions to differential inclusion $x' \in F(x)$ starting from it enter the interior of K, the second one, *the outward area*, from which all solutions to the differential inclusion leave K, and the third one, which is a viability domain of the boundary of K.

This follows from the characterization of the contingent cone to the boundary of K as the intersection of the contingent cone to K and the contingent cone to its complement.

Due to the importance of the viability kernel, the question of finding an algorithm converging to the viability kernel arises. There exists a natural algorithm which was introduced in the linear case by Basile & Marro and Silverman for linear control systems (under the name of *structure algorithm*) and by Byrnes & Isidori in the case of smooth systems and sets defined by equality constraints (under the name of *zero dynamics algorithm*). Unfortunately, a simple counter-example shows that it does not converge when the closed subsets are defined by inequality constraints instead of equality constraints. The reason is that in general, the graph of the contingent cone map $T_K(\cdot)$ is not closed and that the contingent cone to an upper limit is not necessarily the upper limit of the contingent cones. In order to obtain these properties, we suggest introducing the subset $T_K^c(x)$ of directions $v \in T_K(x)$ such that $x(t) := x + tv + \int_0^t (t - \tau)x''(\tau)d\tau$ is viable in K and $x''(\cdot)$ is measurable and bounded by the constant c. These subsets, which can be interpreted as *global contingent sets*, enjoy properties that the contingent cones may lack, and which are useful in some questions such as the convergence of a modified version of the *zero dynamics algorithm* to a closed viability domain (instead of the viability kernel.)

We then present the fast viability kernel algorithm due to Frankowska and Quincampoix which converges to the viability kernel: We define it in this chapter and prove its convergence in Chapter 5.

The Viability Kernel Algorithm provides an upper estimate of the viability kernel $\text{Viab}_F(K)$ of a closed subset $K \subset \text{Dom}(F)$.

We can obtain lower estimates of the viability kernel by introducing finite-difference approximations of F. They are discrete dynamical systems, for which we can define analogous concepts to viable subsets under a discrete map, viability kernels, etc. We show in the fifth section that the upper limit of the viability kernels of a compact subset K for finite-difference approximations of F is contained in the viability kernel of K for F.

4.1 Viability Kernels

Let $F : X \rightsquigarrow X$ be a Marchaud map, describing the dynamics of the differential inclusion

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t)) \tag{4.1}$$

Consider a closed subset K of the domain of F. We shall prove the existence of the largest closed subset of K viable under F.

Definition 4.1.1 (Viability Kernel) *Let K be a subset of the domain of a set-valued map $F : X \rightsquigarrow X$. We shall say that the largest closed subset of K viable under F (which may be empty) is the* viability kernel of K for F *and denote it by* $\text{Viab}_F(K)$ *or, simply,* $\text{Viab}(K)$.

If the viability kernel of K is empty, we say that K is a repeller.

4.1.1 Existence of the Viability Kernel

We begin by proving that such a viability kernel does exist and by characterizing it.

Theorem 4.1.2 *Let us consider a Marchaud map $F : X \rightsquigarrow X$. Let $K \subset \text{Dom}(F)$ be closed. Then the viability kernel of K exists (possibly empty) and is equal to the subset of initial states such that at least one solution starting from them is viable in K.*

Proof — Let $\mathcal{K} \subset \mathcal{C}(0, \infty; X)$ denote the closed subset of functions viable in K. Set

$$\text{Viab}_F(K) := \{x \in K \mid \mathcal{S}_F(x) \cap \mathcal{K} \neq \emptyset\}$$

— *It is closed.* Indeed, let a sequence $x_n \in \text{Viab}_F(K)$ converge to x. It remains in a compact subset L of the finite dimensional vector space X. Let us choose a sequence of solutions $x_n(\cdot) \in \mathcal{S}_F(x_n) \cap \mathcal{K}$.

Since the graph of the restriction $\mathcal{S}_F|_L$ of $\mathcal{S}_F$ to the compact subset L is compact, Theorem 3.5.2 implies that $(x_n, x_n(\cdot))$ belongs to the compact subset $\text{Graph}(\mathcal{S}_F|_L)$. Therefore a subsequence converges to some $(x, x(\cdot))$ of the graph of $\mathcal{S}_F|_L$, so that $x(\cdot)$ belongs to both $\mathcal{S}_F(x)$ and $\mathcal{K}$, which is closed. Consequently, the limit x belongs to $\text{Viab}_F(K)$.

— *The subset* $\text{Viab}_F(K)$ *is also viable under* F. Indeed, for any element $x_0 \in \text{Viab}_F(K)$, there exists a viable solution $x(\cdot)$ to the differential inclusion starting from x_0. For all $t > 0$, the function $y(\cdot)$ defined by $y(\tau) := x(t+\tau)$ is also a viable solution to the differential inclusion, starting at $x(t)$. Hence $x(t) \in \text{Viab}_F(K)$.

— *It is the largest one.* Indeed, let us assume that $L \subset K$ is a closed viability domain of F. Then for all $x_0 \in L$, there exists a solution $x(\cdot)$ to differential inclusion (4.1) starting from x_0 which is viable in L, and thus, in K. □

In particular, the above proof implies the existence of a viability kernel of the domain of F.

Corollary 4.1.3 *Let us consider a Marchaud map* $F : X \rightsquigarrow X$. *Then the domain of the solution map* $\mathcal{S}_F$ *is the largest closed viability domain contained in the domain of* F.

The viability kernels may inherit properties of both F and K. For instance, *if the graph of* F *and the subset* K *are convex, so is the viability kernel of* K. *If* F *is a closed convex process and if* K *is a closed convex cone, the viability kernel is a closed convex cone.*

It may be useful to state the following consequence:

Proposition 4.1.4 *Let F be a Marchaud map and K be a closed subset of the domain of F. If x_0 belongs to $K \backslash \mathrm{Viab}_F(K)$, then every solution $x(\cdot) \in \mathcal{S}_F(x_0)$ must eventually leave K in finite time (in the sense that for some $T > 0$, $x(T) \notin K$) and never meets $\mathrm{Viab}_F(K)$ as long as it remains in K.*

In particular, if K is a repeller, every solution starting from K leaves it in finite time.

Following the terminology coined by Henri Poincaré in the case of differential equations, we could call the complement of the viability kernel of K its *shadow (ombre)*: it is the subset of K from which every solution leaves K in finite time.

Proof — The first statement follows from Theorem 4.1.2. If the second statement is false, then there would exist a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ which would be viable in K, so that x_0 would belong to the viability kernel of K, which is impossible. □

We shall prove in Section 4.3 that when $\mathrm{Viab}_F(K)$ is contained in the interior of K, then the boundary of the viability kernel is viable under F and even that, when the solution map is continuous, it is *semipermeable* in the sense that no solution starting from the boundary can enter the interior of the viability kernel (see Theorem 5.5.3.)

Example We shall compute explicitly the viability kernel of a simple system of differential inclusions in Section 6-2 and illustrate by computer experiments the semipermeable property of the boundary of this viability kernel. □

We deduce from Theorem 3.6.4 the following consequence:

Corollary 4.1.5 *Let us consider a sequence of nontrivial set-valued maps $F_n : X \rightsquigarrow X$ satisfying a uniform linear growth.*

Then the upper limit of the viability kernels of the set-valued maps F_n is contained in the viability kernel of $\overline{co}(F^\sharp)$:

$$\mathrm{Limsup}_{n\to\infty} \left(\mathrm{Viab}_{F_n}(K_n)\right) \subset \mathrm{Viab}_{\overline{co}(F^\sharp)} \left(\mathrm{Limsup}_{n\to\infty} K_n\right)$$

The following property is quite useful:

Proposition 4.1.6 *Let $F : X \leadsto X$ be a Marchaud map and $K \subset \mathrm{Dom}(F)$ be a compact subset viable under F. Then*

$$\omega_F^K(K) \subset \mathrm{Viab}_{-F}(K)$$

Proof — Let $x \in K$ be given, $x(\cdot) \in \mathcal{S}_F^K(x)$ be a viable solution and $x_\star := \lim_{n\to\infty} x(t_n)$ belong to $\omega_F^K(K)$. We associate with it the functions $y_n(\cdot)$ defined by $y_n(t) := x(t_n - t)$. They belong to $\mathcal{S}_{-F}(x(t_n))$, satisfy $y_n(t_n) = x$ and for every $t \leq t_n$, $y_n(t) \in K$.

By Theorem 3.5.2, a subsequence (again denoted by) $y_n(\cdot)$ converges to some solution $y_\star(\cdot) \in \mathcal{S}_{-F}(x_\star)$ which is viable in K. Indeed, for any $t \geq 0$,

$$y(t) \;=\; \lim_{n\to\infty} y_n(t) \;=\; \lim_{n\to\infty} x(t_n - t) \;\in\; K \;\square$$

Remark — Actually, we deduce that

$$K \;\subset \mathrm{Limsup}_{t\to\infty,\; x\to\omega_F^K(K)} R_{-F}(x)$$

because $x = y_n(t_n)$ belongs to the upper limit of $R_{-F}(t_n)(x(t_n))$ when $n \to \infty$. $\square$

Remark: Lyapunov Stability — The viability kernel plays an important role in this book. It underlies many classical results, since several concepts can be reformulated in terms of viability kernels. This is the case for instance of *Lyapunov stability of an equilibrium* $c \in F^{-1}(0)$. It means that for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}$ of c such that for every initial state $x \in \mathcal{U}$, there exists a solution $x(\cdot) \in \mathcal{S}_F(x)$ which remains in the ball $B(c, \varepsilon)$.

Proposition 4.1.7 *Let $K \subset X$ be a closed subset, $F : K \leadsto X$ be a strict Marchaud map and $c \in K$ be an equilibrium of F. It is* stable *if and only if*

$$c \;\in\; \bigcap_{\varepsilon>0} \mathrm{Int}(\mathrm{Viab}_F(B_K(c,\varepsilon)))$$

Remark: Zero Dynamics — The concept of *zero dynamics* introduced by Byrnes and Isidori in the framework of smooth control systems

$$x'(t) = f(x) + g(x)u$$

and smooth equality constraints $h(x) = 0$ is closely related to the notion of viability kernel: one can define the *zero dynamics of this system* as its restriction to the viability kernel of $K := h^{-1}(0)$. □

4.1.2 Permanence and Fluctuation

Let us begin by observing that a subset whose boundary is viable under F is itself viable.

Proposition 4.1.8 *Let F be a Marchaud map. If the boundary ∂K of a closed subset $K \subset \mathrm{Dom}(F)$ with nonempty interior is viable under F, so is K.*

Proof — Indeed, take x_0 in the interior of K. Thanks to the Marchaud Theorem 3.3.3, there exists a local solution $x(\cdot) \in \mathcal{S}_F(x_0)$ viable in the interior of K on some interval. By Theorem 3.3.5, it can be extended to a solution which is viable in the interior of K either for all positive t or for $t \in [0, T[$ where $x(T) \in \partial K$.

In the latter case, the boundary being assumed to be viable under F, there exists one solution $y(\cdot) \in \mathcal{S}_F(x(T))$ starting at $x(T)$ remaining in ∂K. Then the function $\hat{x}(\cdot)$ obtained by concatenating $x(\cdot)$ on $[0, T]$ and $y(t - T)$ on $[t, \infty[$ is a viable solution starting at x_0. This shows that K is viable under F. □

If $\partial K = \mathrm{Viab}_F(K)$, then $\partial K = K$, so that the interior of K is empty.

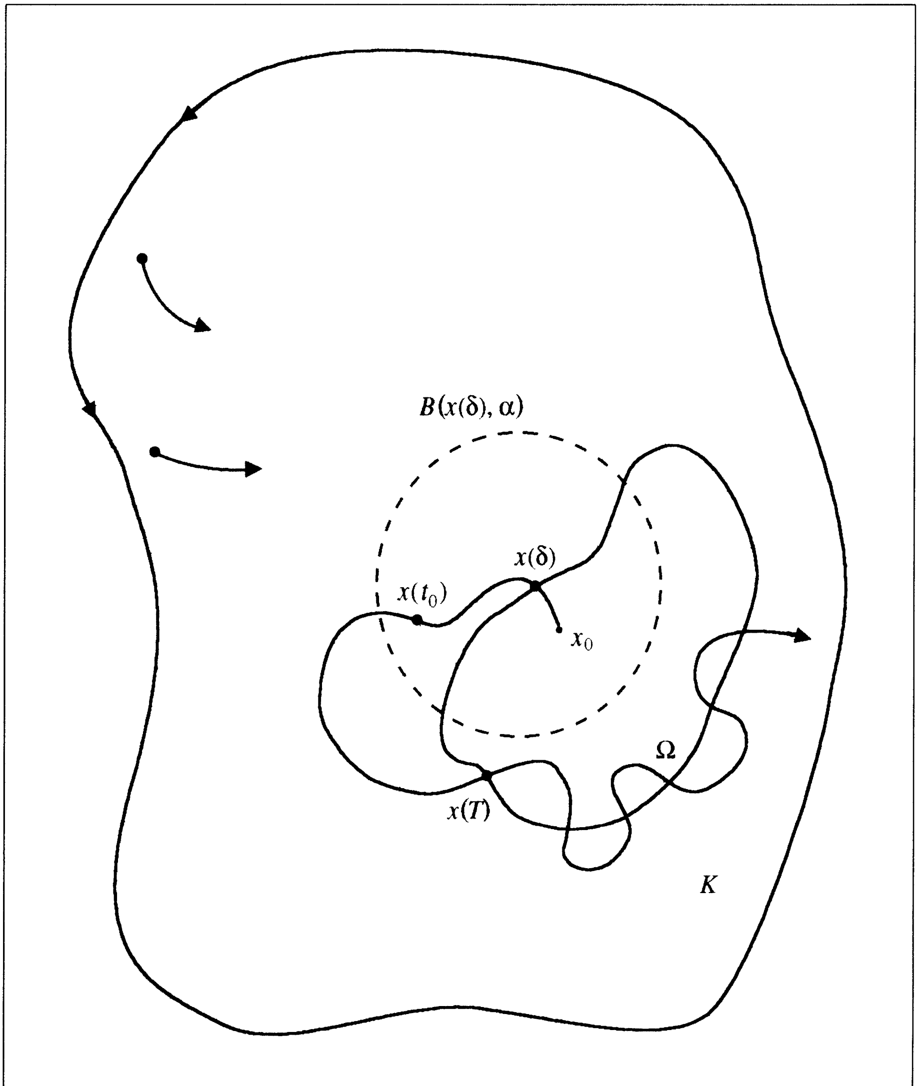
Assume now that the interior of K is not empty and that the boundary ∂K is the viability kernel of some closed subset $K \backslash \Omega$ where $\overline{\Omega} \subset \mathrm{Int}(K)$. We then obtain the following situation:

Theorem 4.1.9 *Let $F : X \rightsquigarrow X$ be a Marchaud map and K be a closed subset of its domain with a nonempty interior.*

Assume that there exists a nonempty open subset Ω such that $\overline{\Omega} \subset \mathrm{Int}(K)$ and that

$$\partial K = \mathrm{Viab}_F(K \backslash \Omega)$$

Figure 4.1: Fluctuation Property



When $\overline{\Omega}$ is a repeller, the solutions starting from the interior of K enter and leave alternatively Ω in finite time.

Then

1. *the boundary* ∂K *of* K *is* semipermeable *under* F *in the sense that for every* $x_0 \in \partial K$, any solution $x(\cdot) \in \mathcal{S}_F(x_0)$ viable in $K \backslash \Omega$ is also viable in ∂K.

2. *The interior* $\text{Int}(K)$ *of* K *is invariant under* F.

Actually, starting from $x_0 \in \text{Int}(K) \backslash \Omega$, *we can associate with any solution* $x(\cdot) \in \mathcal{S}_F(x_0)$ *an increasing sequence of instants* t_n *satisfying*

$$x(t_{2k+1}) \in \Omega \ \ \& \ \ x(t_{2k}) \in \text{Int}(K) \backslash \Omega$$

such that $t_{2k+1} > t_{2k}$ *is finite whenever* t_{2k} *is finite.*

If this sequence is finite, it stops at an odd instant $t_{2k_\infty - 1}$ *for the first index* k_∞ *such that*

$$\inf\{s \geq t_{2k_\infty - 1} \mid x(s) \notin \overline{\Omega}\} \ = \ \infty$$

Proof — We observe first that the boundary ∂K is a viability domain since it is a viability kernel.

For proving that it is semipermeable, fix $x_0 \in \partial K$. We observe that *any solution* $x(\cdot) \in \mathcal{S}_F(x_0)$ *viable in* $K \backslash \Omega$ *is actually viable in* ∂K*:* If not, there would exist $t_1 > 0$ such that $x(t_1) \in \text{Int}(K)$. Therefore, setting $y(t) := x(t + t_1)$, we observe that $y(\cdot) \in \mathcal{S}_F(x(t_1))$ and is viable in $K \backslash \Omega$. This means that $x(t_1)$ belongs to its viability kernel, which is the boundary ∂K. This is impossible.

We prove next that the interior of K is invariant.

Assume first that $x_0 \in \text{Int}(K) \backslash \Omega$ and consider any solution $x(\cdot) \in \mathcal{S}_F(x_0)$. We know that there exists some $T < +\infty$ such that $x(T) \notin K \backslash \Omega$. Let us introduce $\theta := \inf\{s \geq 0 \mid x(s) \in \Omega\}$ and $\tau := \inf\{s \geq 0 \mid x(s) \notin K\}$.

We claim that θ is finite and $\theta \leq \tau$. If not, we would have $x(\tau) \in \partial K$ and we know that there exists a solution $y(\cdot) \in \mathcal{S}_F(x(\tau))$ viable in K. Consequently, the function $z(\cdot)$ defined by

$$\forall\, t \geq 0, \ \ z(s) \ := \left\{ \begin{array}{lll} x(s) & \text{if} & s \in [0, \tau] \\ y(s - \tau) & \text{if} & s \in [\tau, \infty[\end{array} \right.$$

is a solution starting at x_0 and viable in $K\backslash\Omega$. This would imply that x_0 belongs to ∂K, which contradicts the assumption.

Assume that $x_0 \in \Omega$ and consider any solution $x(\cdot) \in \mathcal{S}_F(x_0)$. We introduce $\delta := \inf\{s \geq 0 \mid x(s) \notin \overline{\Omega}\} \in [0,\infty]$. Either δ is infinite, and the solution is viable in $\overline{\Omega}$ and thus, in the interior of K, or δ is finite. In this case, the function $x(\cdot)$ being continuous and $\overline{\Omega}$ being contained in the interior of K, we know that there exist α and η such that

$$x(t) \in B(x(\delta),\alpha) \subset \mathrm{Int}(K)$$

for every $t \in [\delta, \delta+\eta]$, and also that there exists $t_0 \in]\delta, \delta+\eta[$ such that $x(t_0) \in \mathrm{Int}(K)\backslash\Omega$. Setting $y(t) := x(t+t_0)$, we observe that $y(\cdot) \in \mathcal{S}_F(x(t_0))$, so that, thanks to what was proved above, there exists $t_1 \in [t_0,\infty[$ such that $y(t_1-t_0) = x(t_1) \in \Omega$ and $y(t) = x(t_0+t)$ belongs to $\mathrm{Int}(K)\backslash\Omega$ for $t \in [t_0, t_1-t_0[$. This means that

$$\exists\ T < +\infty \quad \text{such that } \forall\, t \in [0,T],\ \ x(t) \in \mathrm{Int}(K)\ \ \&\ \ x(T) \in \Omega$$

Putting these facts together, we have proved that starting from $x_0 \in \mathrm{Int}(K)$, every solution remains in the interior of K. Furthermore, there exists an increasing sequence of t_n such that $x(t_{2k}) \in \mathrm{Int}(K)\backslash\Omega$ and $x(t_{2k+1}) \in \Omega$, such that $t_{2k+1} > t_{2k}$ is finite whenever t_{2k} is finite. □

If $\overline{\Omega}$ is invariant under F, we obtain the *permanence property* introduced by J. Hofbauer and K. Sigmund:

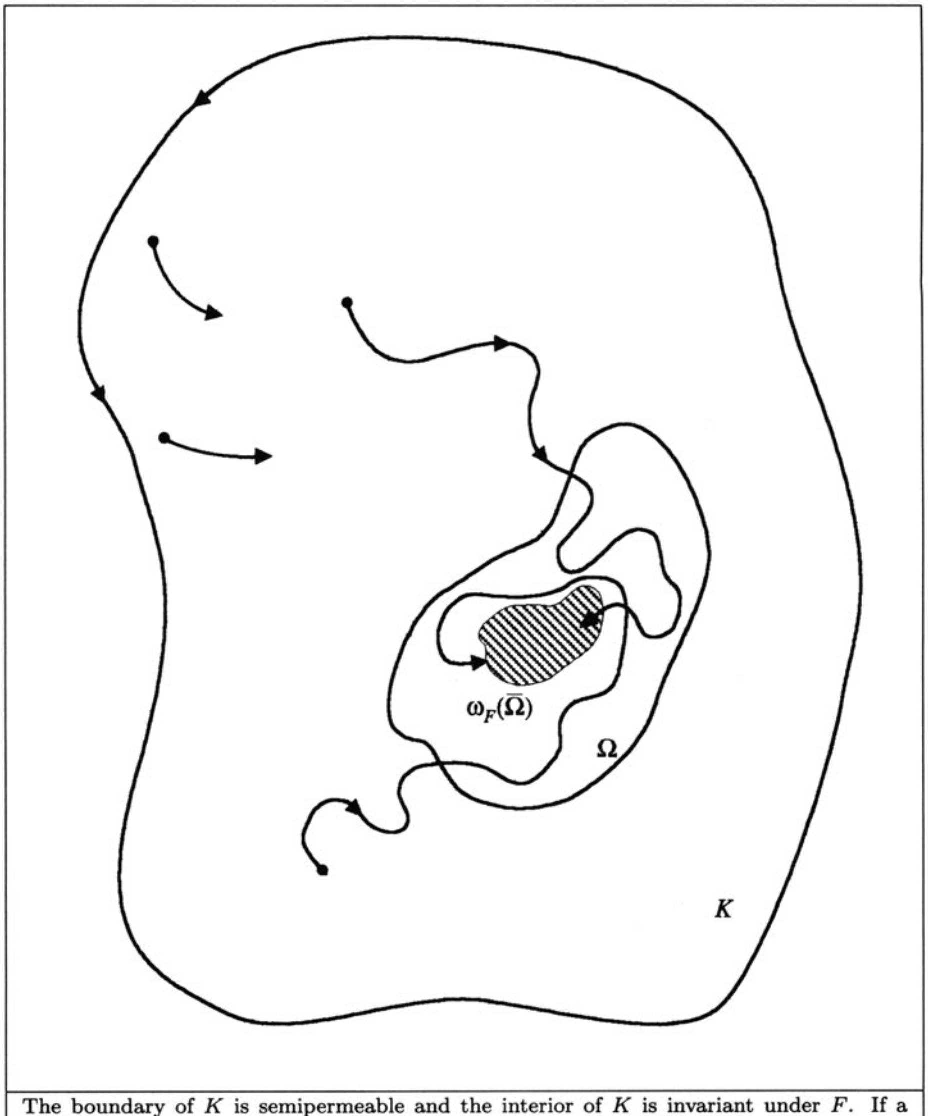
Theorem 4.1.10 *Let $F : X \rightsquigarrow X$ be a Marchaud map and K be a closed subset of its domain with a nonempty interior.*

Assume that there exists a nonempty open subset Ω such that $\overline{\Omega} \subset \mathrm{Int}(K)$ and that

$$\begin{cases} i) & \partial K = \mathrm{Viab}_F(K\backslash\Omega) \\ ii) & \overline{\Omega} \text{ is invariant under } F \end{cases}$$

Then the boundary is semipermeable, the interior of K is invariant, Ω is absorbing *in* $\mathrm{Int}(K)$ *in the sense that for any $x_0 \in \mathrm{Int}(K)$,*

Figure 4.2: Permanence Property



The boundary of K is semipermeable and the interior of K is invariant under F. If a solution leaves Ω, it must return to it.

all solutions $x(\cdot) \in \mathcal{S}_F(x_0)$ *enters* Ω *in finite time and remain in* $\overline{\Omega}$ *afterwards.*

Furthermore, if Ω *is relatively compact, the* ω*-limit set* $\omega_F(\overline{\Omega}) \subset \overline{\Omega}$ *is an universal attractor of the interior of* K*.*

Proof — Proposition 3.7.3 states that $\omega_F(\overline{\Omega})$ is an universal attractor of $\overline{\Omega}$, which is contained in $\overline{\Omega}$ since it is invariant. To prove that $\omega_F(\overline{\Omega})$ is an universal attractor of the interior of K, we then deduce from Theorem 4.1.9 that $\overline{\Omega}$ is absorbing since Ω is reached from the interior of K in finite time and since $\overline{\Omega}$ is assumed to be invariant. Hence from any point x_0 of the interior of K, all solutions $x(\cdot) \in \mathcal{S}(x_0)$ reach Ω in finite time, remain in $\overline{\Omega}$ and converge to $\omega_F(\overline{\Omega})$. □

If $\overline{\Omega}$ is a repeller, we obtain the *fluctuation property* introduced by V. Krivan:

Theorem 4.1.11 *Let* $F : X \rightsquigarrow X$ *be a Marchaud map and* K *be a closed subset of its domain with a nonempty interior.*

Assume that there exists a nonempty open subset Ω *such that* $\overline{\Omega} \subset \mathrm{Int}(K)$ *and that*

$$\begin{cases} i) & \partial K = \mathrm{Viab}_F(K \backslash \Omega) \\ ii) & \mathrm{Viab}_F(\overline{\Omega}) \neq \emptyset \end{cases}$$

Then the boundary is semipermeable, the interior of K *is invariant, and starting from* $x_0 \in \mathrm{Int}(K) \backslash \Omega$*, we can associate with any solution* $x(\cdot) \in \mathcal{S}_F(x_0)$ *an increasing sequence of finite instants* t_n *satisfying*

$$x(t_{2k+1}) \in \Omega \ \ \& \ \ x(t_{2k}) \in \mathrm{Int}(K) \backslash \Omega$$

4.1.3 Viability Envelopes

Proposition 4.1.12 *Let* $K \subset \mathrm{Dom}(F)$ *be a closed viability domain of a Marchaud map* F*.*

Then any closed subset $L \subset K$ *is contained into a minimal closed viability domain. These minimal viability domains containing* L *are called the* viability envelopes of L*.*

Proof — We apply Zorn's lemma for the inclusion order on the family of nonempty closed viability domains of F between L and K. For that purpose, consider any decreasing family of closed viability domains M_i and its intersection $M_\star := \bigcap M_i$. It is a closed viability domain thanks to the Stability Theorem 3.6.2. Therefore every subset $L \subset K$ is contained in a minimal element for this preorder. □

When $L = \emptyset$, we have to assume that K is compact to guarantee that the intersection of any decreasing family of nonempty closed viability domains is not empty. In this case, we obtain the following

Proposition 4.1.13 *Let K be a nonempty compact viability domain of a Marchaud map M. Then nonempty minimal viability domains M do exist and are made of ω-limit sets of viable solutions. Actually, they enjoy the following property:*

$$\forall\, x \in M, \ \ \exists\, x(\cdot) \in \mathcal{S}_F(x) \mid x \in M = \omega(x(\cdot))$$

Proof — Let $M \subset K$ be a minimal closed viability domain. Since it is a closed viability domain, we can associate with any $x \in M$ a viable solution $x(\cdot) \in \mathcal{S}_F(x)$ starting at x. Hence its limit set $\omega(x(\cdot))$ is contained in M. But limit sets being closed viability domains by Theorem 3.7.2 and M being minimal, it is equal to $\omega(x(\cdot))$, so that $x \in \omega(x(\cdot))$. □

4.2 Hitting and Exit Tubes

Let us consider a strict Marchaud map $F : X \rightsquigarrow X$ (i.e., a Marchaud map with nonempty values), the differential inclusion

$$x'(t) \ \in \ F(x(t)) \tag{4.2}$$

a closed subset $K \subset X$ and its boundary ∂K.

We introduce and study the properties of the functions which associate with any initial state $x \in K$ the first instant when a solution to the differential inclusion to (4.2) reaches the boundary ∂K (the *hitting time*) and the first instant when a solution leaves K (the *exit time*.)

4.2.1 Hitting and Exit Functionals

We begin by defining the hitting and exit of a continuous function $x(\cdot) \in \mathcal{C}(0, \infty; X)$.

Definition 4.2.1 *Let $K \subset X$ be a closed subset and $x(\cdot) \in \mathcal{C}(0, \infty; X)$ be a continuous function. We denote by*

$$\theta_K : \mathcal{C}(0, \infty; X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$$

the hitting functional *associating with $x(\cdot)$ its* hitting time $\theta_K(x(\cdot))$ *defined by*

$$\theta_K(x(\cdot)) \ := \ \inf \{t \in [0, +\infty[\ | \ x(t) \notin \mathrm{Int}(K)\}$$

In the same way, the functional $\tau_K : \mathcal{C}(0, \infty; X) \mapsto \mathbf{R}_+ \cup \{+\infty\}$ associating with $x(\cdot)$ its exit time $\tau_K(x(\cdot))$ *defined by*

$$\tau_K(x(\cdot)) \ := \ \inf \{t \in [0, \infty[\ | \ x(t) \notin K\}$$

is called the exit functional.

We observe that

$$\theta_K(x(\cdot)) \ \leq \ \tau_K(x(\cdot))$$

that

$$\forall\, t \in [0, \theta_K(x(\cdot))[, \ \ x(t) \in \mathrm{Int}(K) \ \ \& \ \ \forall\, t \in [0, \tau_K(x(\cdot))[, \ \ x(t) \in K$$

and that, when $\theta_K(x(\cdot))$ (respectively $\tau_K(x(\cdot))$) is finite,

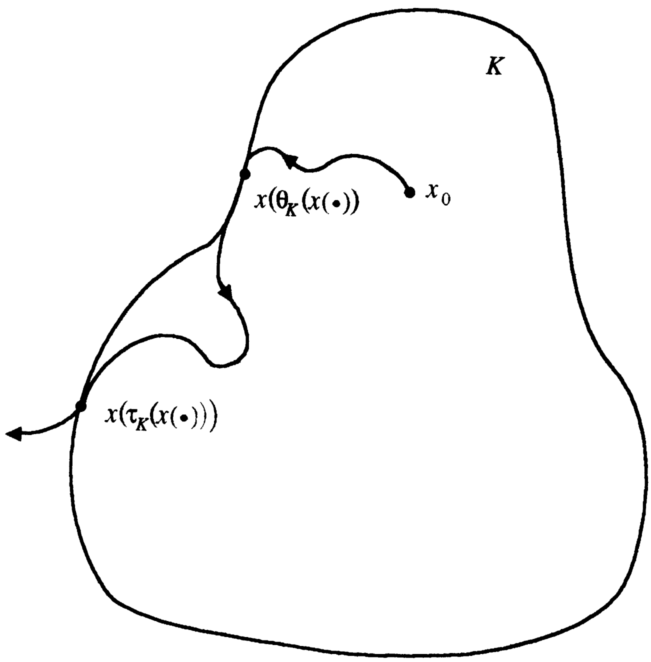
$$x(\theta_K(x(\cdot))) \in \partial K \ \ \& \ \ x(\tau_K(x(\cdot))) \in \partial K \ \ \text{respectively}$$

Remark also that $\theta_K(x(\cdot)) \equiv 0$ when the interior of K is empty.

Remark — Following the terminology coined by Henri Poincaré in the case of differential equations, we could call the first point $x(\theta_K(x(\cdot)))$ reaching the boundary the *consequent of* x_0. Always in the framework of differential equations, the hitting time is called by Ważewski *exit time* and our exit time called *strict exit time*. □

We continue to use the convention $\inf\{\emptyset\} := +\infty$, so that $\theta_K(x(\cdot))$ is infinite means that $x(t) \in \mathrm{Int}(K)$ for all $t \in [0, +\infty[$ and that $\tau_K(x(\cdot)) = +\infty$ means that $x(t) \in K$ for all $t \geq 0$.

Figure 4.3: Hitting and Exit Times



The first point $x(\theta_K(x(\cdot)))$ reaching the boundary is the *consequent of* x_0. The solution leaves K at the exit time $\tau_K(x(\cdot))$.

Lemma 4.2.2 *Let $K \subset X$ be a closed subset. The functional τ_K is upper semicontinuous when $\mathcal{C}(0,\infty;X)$ is supplied with the pointwise convergence topology and the functional θ_K is lower semicontinuous when $\mathcal{C}(0,\infty;X)$ is supplied with the compact convergence topology.*

Proof — By the Maximum Theorem 2.1.6, the upper semicontinuity of τ_K follows from the lower semicontinuity of the set-valued map $x(\cdot) \rightsquigarrow \Xi(x(\cdot)) \subset \mathbf{R}_+$ where

$$\Xi(x(\cdot)) \ := \ \{t \in [0,\infty[\mid x(t) \notin K\}$$

since $\tau_K(x(\cdot)) = \inf\{\Xi(x(\cdot))\}$.

Indeed, for any $t \in \Xi(x(\cdot))$ and any sequence $x_n(\cdot)$ converging pointwise to $x(\cdot)$, we see that $t \in \Xi(x_n(\cdot))$ for n large enough because $x_n(t)$ belongs to the open set $X \backslash K$ (since $x(t) \in X \backslash K$.)

Let us check now that the function θ_K is lower semicontinuous for the compact convergence topology: take any $T \geq 0$ and any sequence $x_n(\cdot)$ satisfying $\theta_K(x_n(\cdot)) \leq T$ converging to $x(\cdot)$ uniformly over compact subsets and show that $\theta_K(x(\cdot)) \leq T$. Let us introduce the subsets

$$\Theta_{T'}(x(\cdot)) \ := \ \{t \in [0,T'] \mid x(t) \notin \mathrm{Int}(K)\}$$

By construction, for any $T' > T$, the subsets $\Theta_{T'}(x_n(\cdot))$ are not empty. We also observe that the graph of the set-valued map $x(\cdot) \rightsquigarrow \Theta_{T'}(x(\cdot))$ is closed in the Banach space $\mathcal{C}(0,T';X) \times [0,T']$: Indeed, if $(x_n(\cdot), t_n) \in \mathrm{Graph}(\Theta_{T'})$ converges to $(x(\cdot), t)$, then $x_n(t_n) \in X \backslash \mathrm{Int}(K)$ converges to $x(t)$, which thus belongs to the closed subset $X \backslash \mathrm{Int}(K)$, so that $(x(\cdot), t) \in \mathrm{Graph}(\Theta_{T'})$. Taking its values in the compact interval $[0,T']$, the set-valued map $x(\cdot) \rightsquigarrow \Theta_{T'}(x(\cdot))$ is actually upper semicontinuous. Therefore, for any given $\varepsilon > 0$, $\Theta_{T'}(x_n(\cdot)) \subset \Theta_{T'}(x(\cdot)) + [-\varepsilon, +\varepsilon]$.

We thus infer that $\theta_K(x(\cdot)) \leq \theta_K(x_n(\cdot)) + \varepsilon \leq T + \varepsilon$ for every $\varepsilon > 0$. □

4.2.2 Hitting and Exit Functions

Consider now a strict Marchaud map $F : X \rightsquigarrow X$ and denote by $\mathcal{S}_F(x)$ the set of solutions $x(\cdot)$ to differential inclusion (4.2) starting at the initial state x.

Definition 4.2.3 *Let $K \subset X$ be a closed subset. The function $\theta_K^\flat : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\theta_K^\flat(x) := \inf_{x(\cdot)\in\mathcal{S}_F(x)} \theta_K(x(\cdot))$$

is called the hitting function *and the function $\tau_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by*

$$\tau_K^\sharp(x) := \sup_{x(\cdot)\in\mathcal{S}_F(x)} \tau_K(x(\cdot))$$

the exit function.

Lemma 4.2.2 and Theorem 3.5.2 on the upper semicontinuity of the solution map $x \in K \rightsquigarrow \mathcal{S}_F(x) \subset \mathcal{C}(0,\infty;X)$ (supplied with the compact convergence topology) imply

Proposition 4.2.4 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Then the hitting function $\theta_K^\flat$ is lower semicontinuous and the exit function $\tau_K^\sharp$ is upper semicontinuous.*

We are thus led to single out the following subsets:

Definition 4.2.5 *We associate with any $T \geq 0$ the subsets*

$$\left\{ \begin{array}{lll} i) & \mathrm{Hit}_F(K,T) := \left\{x \in K \mid \theta_K^\flat(x) \leq T\right\} \\ ii) & \mathrm{Exit}_F(K,T) := \left\{x \in K \mid \tau_K^\sharp(x) \geq T\right\} \end{array} \right. \tag{4.3}$$

We shall say that the set-valued map $T \rightsquigarrow \mathrm{Hit}_F(K,T)$ is the hitting tube *and that the set-valued map $T \rightsquigarrow \mathrm{Exit}_F(K,T)$ is the* exit tube.

Proposition 4.2.4 implies that the *graphs of the hitting and exit tubes are closed.*

Proposition 4.2.6 *Let F be a strict Marchaud map and $K \subset X$ be a closed subset.*

Then $\mathrm{Hit}_F(K,T)$ is the closed subset of initial states $x \in K$ such that the boundary ∂K is reached before T by one solution $x(\cdot)$ to differential inclusion (4.2) starting at x.

The closed subset $\text{Exit}_F(K,T)$ *is the subset of initial states* $x \in K$ *such that one solution* $x(\cdot)$ *to differential inclusion (4.2) starting at* x *remains in* K *for all* $t \in [0,T]$. *Actually, such a solution satisfies*

$$\forall\, t \in [0,T], \;\; x(t) \in \text{Exit}_F(K, T-t)$$

In particular, for $T = +\infty$,

$$\text{Viab}_F(K) \;=\; \text{Exit}_F(K,+\infty) \;=\; \bigcap_{T>0} \text{Exit}_F(K,T)$$

Proof — Since the subset of initial states $x \in K$ such that the boundary ∂K is reached before T by a solution $x(\cdot)$ to differential inclusion (4.2) starting at x is obviously contained in $\text{Hit}_F(K,T)$, consider an element $x \in \text{Hit}_F(K,T)$ and prove that it satisfies the above property.

By definition of the hitting function, we can associate with any $\varepsilon > 0$ a solution $x_\varepsilon(\cdot) \in \mathcal{S}_F(x)$ satisfying $\theta_K(x_\varepsilon(\cdot)) \le T + \varepsilon/2$, and, by definition of the hitting functional, a time $t_\varepsilon \le T + \varepsilon$ such that $x_\varepsilon(t_\varepsilon) \in X \backslash \text{Int}(K)$.

Since $\mathcal{S}_F(x)$ is compact in $\mathcal{C}(0,\infty;X)$ supplied with the compact convergence topology, subsequences (again denoted by) $x_\varepsilon(\cdot)$ and t_ε converge to $x(\cdot) \in \mathcal{S}_F(x)$ and $t \in [0, T+\varepsilon]$, so that the limit $x(t)$ of $x_\varepsilon(t_\varepsilon) \in X \backslash \text{Int}(K)$ belongs to the closed subset $X \backslash \text{Int}(K)$. This implies that $\theta_K(x(\cdot)) \le T + \varepsilon$ for every $\varepsilon > 0$.

In the same way, let $T \ge 0$ be finite or infinite. We observe that the subset of initial states $x \in K$ such that a solution $x(\cdot)$ to differential inclusion (4.2) starting at x remains in K for all $t \in [0, T[$ is contained in $\text{Exit}_F(K,T)$, so that it is enough to prove that for any $x \in \text{Exit}_F(K,T)$, there exists $x(\cdot) \in \mathcal{S}_F(x)$ satisfying the above property.

By definition of the exit function, we can associate with any sequence $t_n < T$ converging to T a solution $x_n(\cdot) \in \mathcal{S}_F(x)$ satisfying $t_n \le \tau_K(x_n(\cdot))$. By Theorem 3.5.2, a subsequence of solutions $x_n(\cdot)$ converges to a solution $x(\cdot) \in \mathcal{S}_F(x)$. Let $t \in [0, T[$ be given and choose n such that $t < t_n < T$. Observing that $x_n(t) \in K$ converges to $x(t)$, we infer that $x(t) \in K$ for any $t < T$. $\square$

Proposition 4.2.7 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset.*

The complement $K \backslash \mathrm{Exit}_F(K,T)$ is equal to the set

$$\{x \in K \mid \forall\, x(\cdot) \in \mathcal{S}_F(x),\ \ \exists\, t \in [0,T] \ \ \textit{such that}\ \ x(t) \notin K\}$$

of initial states x from which all solutions $x(\cdot) \in \mathcal{S}_F(\cdot)$ leave K at some $t \leq T$.

Consequently, if $M \subset K \backslash \mathrm{Viab}_F(K)$ is compact, there exists $T \geq 0$ such that, for every $x \in M$ and any solution $x(\cdot) \in \mathcal{S}_F(x)$, there exists $t \in [0,T]$ such that $x(t) \notin K$.

In particular, if K is a compact repeller, there exists $T < +\infty$ such that for every $x \in K$ and any solution $x(\cdot) \in \mathcal{S}_F(x)$, there exists $t \in [0,T]$ such that $x(t) \notin K$.

4.2.3 Exit Tubes

We observe that if $T_1 \leq T_2$,

$$\partial K \ = \ \mathrm{Hit}_F(K,0) \ \subset \ \mathrm{Hit}_F(K,T_1) \ \subset \ \mathrm{Hit}_F(K,T_2) \ \subset \ \ldots$$

and

$$\mathrm{Viab}_F(K) \subset \mathrm{Exit}_F(K,T_2) \subset \mathrm{Exit}_F(K,T_1) \subset \ldots \subset \mathrm{Exit}_F(K,0) = K$$

Proposition 4.2.8 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset.*

Let $0 < T_1 < T_2$ and $x \in \mathrm{Exit}_F(K,T_1) \backslash \mathrm{Exit}_F(K,T_2)$. Then all solutions $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0,T_1]$ remain in the complement of the exit tube:

$$\forall\, t \in [0,T_1],\ \ x(t) \in \mathrm{Exit}_F(K,T_1 - t) \backslash \mathrm{Exit}_F(K,T_2 - t)$$

In particular, for any $x \in \mathrm{Exit}_F(K,T) \backslash \mathrm{Viab}(K)$, all solutions $x(\cdot) \in \mathcal{S}_F(x)$ satisfy

$$\forall\, t \in [0,T],\ \ x(t) \ \in \ \mathrm{Exit}_F(K,T-t) \backslash \mathrm{Viab}(K)$$

Proof — Assume that $x \in \mathrm{Exit}_F(K,T_1) \backslash \mathrm{Exit}_F(K,T_2)$ and that for some solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0,T_1]$, there exists $\tau \in [0, T_1]$ such that $x(\tau) \in \text{Exit}_F(K, T_2 - \tau)$. Then there exists a solution $y(\cdot) \in \mathcal{S}_F(x(\tau))$ such that $y(0) = x(\tau)$ and $y(s) \in \text{Exit}_F(K, T_2 - \tau - s)$. Therefore, the concatenated function

$$\widehat{x}(t) := \begin{cases} x(t) & \text{if } t \in [0, \tau] \\ y(t-\tau) & \text{if } t \in [\tau, T_2] \end{cases}$$

is a solution to differential inclusion (4.2) starting at x which is viable in K on $[0, T_2]$ because $\widehat{x}(t) \in \text{Exit}_F(K, T_2 - t) \subset K$ for every $t \in [0, T_2]$. This means that the initial state x belongs to $\text{Exit}_F(K, T_2)$, a contradiction. □

Proposition 4.2.9 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. If $\text{Exit}_F(K, T)$ is contained in the interior of K, then its complement is invariant under F.*

In particular, if the viability kernel of K is contained in the interior of K, then the complement $X \backslash \text{Viab}_F(K)$ is invariant under F.

Proof — Assume that the conclusion is false: there exist $x \in X \backslash \text{Exit}_F(K, T)$, a solution $x(\cdot) \in \mathcal{S}_F(x)$ and τ such that $x(\tau) \in \text{Exit}_F(K, T)$. Since the solution is not entirely contained in the closed subset $\text{Exit}_F(K, T)$, then

$$t^\star := \sup \{ t \in [0, \tau] \mid x(t) \notin \text{Exit}_F(K, T) \}$$

is finite. Since the subset $\text{Exit}_F(K, T)$ is contained in the interior of K and since the function $x(\cdot)$ is continuous, we know that there exist α and η such that

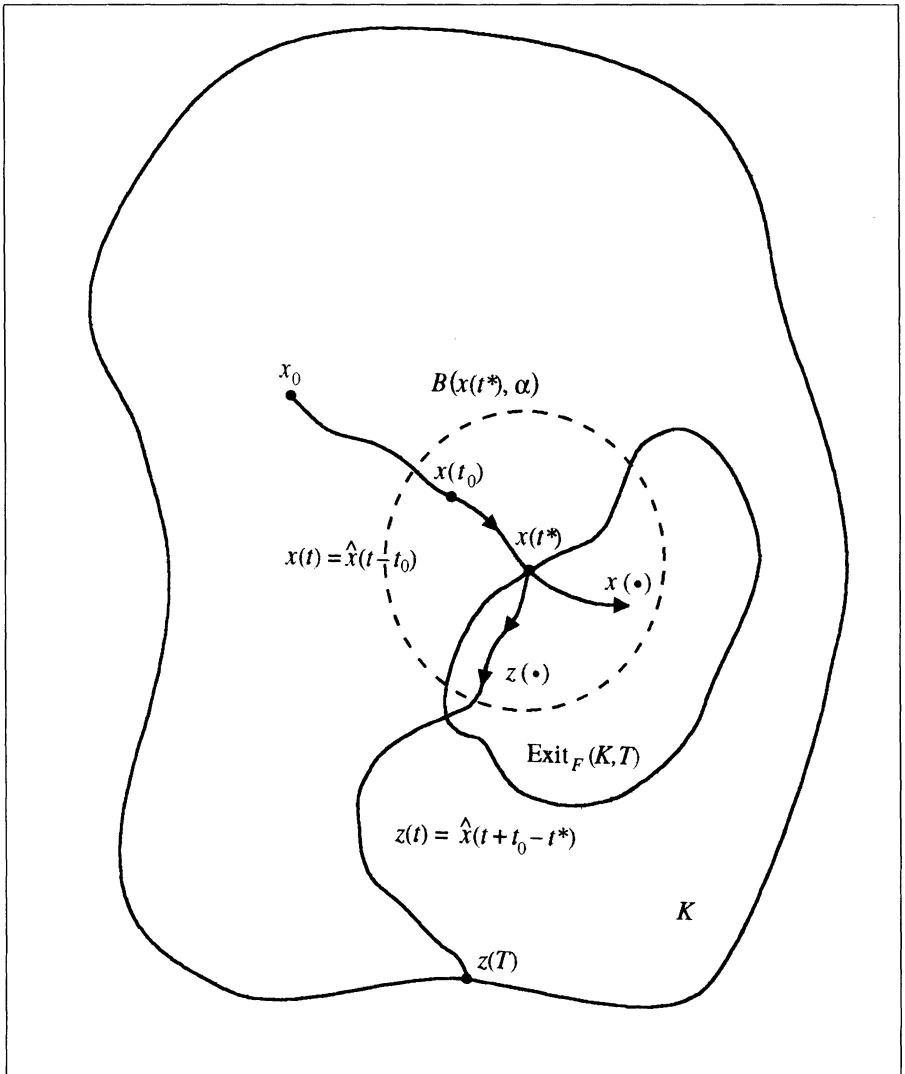
$$x(t) \in B(x(t^\star), \alpha) \subset \text{Int}(K)$$

for every $t \in [t^\star - \eta, t^\star]$, and also that there exists $t_0 \in]t^\star - \eta, t^\star[$ such that $x(t_0) \notin \text{Exit}_F(K, T)$. Therefore,

$$\forall\, t \in [t_0, t^\star], \;\; x(t) \in K$$

Since $x(t^\star) \in \text{Exit}_F(K, T)$, there exists $z(\cdot) \in \mathcal{S}_F(x(t^\star))$ such that $z(t) \in \text{Exit}_F(K, T - t)$. We introduce the solution $\widehat{x} \in \mathcal{S}_F(x)$ defined by $\widehat{x}(t) = x(t + t_0)$ when $t \in [0, t^\star - t_0]$ and $\widehat{x}(t) = z(t + t_0 - t^\star)$ when $t \in [t^\star - t_0, T + t^\star - t_0]$. The function $\widehat{x}(\cdot)$ is a solution to differential inclusion (4.2) starting at $x(t_0)$ which is viable in K on the interval $[0, T + t^\star - t_0]$. Hence $x(t_0) = y(0) \in \text{Exit}_F(K, T + t^\star - t_0) \subset \text{Exit}_F(K, T)$, a contradiction. □

Figure 4.4: Invariance of the Complement of a Viability Kernel



If $x_0 \notin \text{Exit}_F(K,T)$, every solution must leave K before hitting $\text{Exit}_F(K,T)$. When this set lies in $\text{Int}(K)$, there exists t_0 such that $x(t_0) \in \text{Int}(K)$.

4.3 Anatomy of a Set

4.3.1 Contingent Cone to the Boundary

Let us consider now any subset K. We denote by

$$\widehat{K} := X \backslash \mathrm{Int}(K) = \overline{X \backslash K}$$

the complement of the interior of K and by

$$\partial K := \overline{K} \cap \widehat{K}$$

the *boundary* of K. We observe that K is the closure of its interior if and only if $X \backslash K$ is the interior of $\widehat{K}$.

We introduce the Dubovitsky-Miliutin cone defined by

Definition 4.3.1 *The* Dubovitsky-Miliutin tangent cone $D_K(x)$ *to* K *is defined by:*

$$\begin{cases} v \in D_K(x) \text{ if and only if} \\ \exists\ \varepsilon > 0,\ \exists\ \alpha > 0 \text{ such that } x +]0, \alpha](v + \varepsilon B) \subset K \end{cases}$$

Lemma 4.3.2 *For any x in the boundary of K, the* Dubovitsky-Miliutin cone $D_K(x)$ *to K at x is the complement of the contingent cone $T_{X\backslash K}(x)$ to the complement $X\backslash K$ of K at $x \in \partial K$:*

$$\forall\, x \in \partial K,\ D_K(x) = X \setminus T_{X\backslash K}(x)$$

We need the following characterization of the contingent cone to the boundary:

Theorem 4.3.3 (Quincampoix) *Let K be a closed subset of a normed space and $\widehat{K}$ denote the closure of its complement. Then*

$$\forall\, x \in \partial K,\quad T_{\partial K}(x) = T_K(x) \cap T_{\widehat{K}}(x)$$

so that the whole space can be partitioned in the following way:

$$\forall\, x \in \partial K,\ D_{\mathrm{Int}(K)}(x) \cup D_{X\backslash K}(x) \cup T_{\partial K}(x) = X$$

In particular, any nonempty connected subset disjoint from the contingent cone $T_{\partial K}(x)$ to the boundary must be contained in either $D_{\mathrm{Int}(K)}(x)$ or $D_{X\backslash K}(x)$.

Proof — If the interior of K is empty, $\partial K = K$, so that the formula holds true. Assume that the interior of K is not empty and take any $x \in \partial K$. Since inclusion $T_{\partial K}(x) \subset T_K(x) \cap T_{\widehat{K}}(x)$ is obviously true, we have to prove that any u in the intersection $T_K(x) \cap T_{\widehat{K}}(x)$ is contingent to the boundary ∂K at x.

Indeed, there exist sequences $k_n > 0$ and $l_n > 0$ converging to $0+$ and sequences $v_n \in X$ and $w_n \in X$ converging to u such that

$$\forall\, n \geq 0, \quad x + k_n v_n \in K \ \ \& \ \ x + l_n w_n \in \widehat{K}$$

We shall prove that there exists $\lambda_n \in [0,1]$ such that, setting

$$h_n \ := \ \lambda_n k_n + (1-\lambda_n) l_n \in [\min(k_n, l_n), \max(k_n, l_n)]$$

and

$$u_n \ := \ \frac{\lambda_n k_n v_n + ((1-\lambda_n) l_n) w_n}{\lambda_n k_n + (1-\lambda_n) l_n}$$

we have

$$\forall\, n \geq 0, \quad x + h_n u_n \in \partial K$$

Indeed, we can take λ_n either 0 or 1 when either $x + k_n v_n$ or $x + l_n w_n$ belongs to the boundary. If not, $x + k_n v_n \in \text{Int}(K)$ and $x + l_n w_n \in X \backslash K$. Since the interval $[0,1]$ is connected, it cannot be covered by the two nonempty disjoint open subsets

$$\Omega_+ \ := \ \{\lambda \in [0,1] \mid x + \lambda k_n v_n + (1-\lambda) l_n w_n \ \in \text{Int}(K)\}$$

and

$$\Omega_- \ := \ \{\lambda \in [0,1] \mid x + \lambda k_n v_n + (1-\lambda) l_n w_n \ \in X \backslash K\}$$

Then there exists $\lambda_n \in [0,1] \backslash (\Omega_+ \cup \Omega_-)$ so that

$$x + \lambda_n k_n v_n + (1-\lambda_n) l_n w_n = x + h_n u_n \ \in \ \partial K$$

Since $h_n > 0$ converges to $0+$ and u_n converges to u, we infer that u belongs to the contingent cone to ∂K.

This formula and Lemma 4.3.2 imply the decomposition formula. Hence any nonempty connected subset U disjoint from $T_{\partial K}(x)$ cannot be partitioned by two nonempty open subsets $U \cap D_{\text{Int}(K)}(x)$ and $U \cap D_{X \backslash K}(x)$. □

Remark — Actually, the same proof implies that for two subsets K_1 and K_2, we have

$$\forall\, x \in K_1 \cap K_2,\;\; T_{K_1}(x) \cap T_{K_2}(x) \cap D_{K_1 \cup K_2} \;\subset\; T_{K_1 \cap K_2}(x) \quad \square$$

4.3.2 Strict Invariance

Let us consider a closed subset K of the domain of a set-valued map F and differential inclusion (4.2):

$$x'(t) \in F(x(t))$$

The Viability Theorem implies the following useful result:

Theorem 4.3.4 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images. Let $K \subset \mathrm{Dom}(F)$ be closed with nonempty interior and $x_0 \in \partial K$. Then each of the following conditions implies the next one:*

$$\left\{\begin{array}{ll} i) & F(x_0) \;\subset\; D_{\mathrm{Int}(K)}(x_0) \\ ii) & \text{for any solution starting from } x_0,\; \exists\; T > 0 \text{ such that} \\ & \forall\, t \in [0,T],\;\; x(t) \in \mathrm{Int}(K) \\ iii) & \exists \text{ a sequence } x_n \in \partial K \text{ converging to } x_0 \text{ such that} \\ & F(x_n) \;\subset\; D_{\mathrm{Int}(K)}(x_n) \end{array}\right.$$

Proof — We introduce the closure $\widehat{K}$ of $X \backslash K$, $x_0 \in \partial K$ and we observe that each of the following condition implies the next one:

$$\left\{\begin{array}{ll} i) & \exists\, r > 0 \text{ such that for all } x \in \partial K \cap (x_0 + rB), \text{ we have} \\ & F(x) \cap T_{\widehat{K}}(x) \;\neq\; \emptyset \\ ii) & \exists\, T > 0 \text{ and a solution starting at } x_0 \text{ viable in } \widehat{K} \text{ on } [0,T] \\ iii) & \exists \text{ a solution starting at } x_0 \text{ such that } \forall\, T > 0,\;\; \exists\, t \in]0,T] \\ & \text{such that } x(t) \in \widehat{K} \\ iv) & F(x_0) \cap T_{\widehat{K}}(x_0) \;\neq\; \emptyset \end{array}\right.$$

Indeed, the first implication follows from Proposition 3.4.2 applied to $\widehat{K}$, the second implication is obvious and the third one ensues from Proposition 3.4.1 still applied to $\widehat{K}$[1].

Proposition 4.3.4 follows from the negation of $i) \Longrightarrow iii) \Longrightarrow iv)$.
□

Remark — Actually, one can prove a stronger property of inwardness:

Proposition 4.3.5 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images bounded on a neighborhood of K and assume that the interior of K is not empty. If*

$$F(x) \subset D_K(x)$$

then there exist $\rho_x > 0$ and $T_x > 0$ such that

$$\forall\, x(\cdot) \in \mathcal{S}_F(x),\ \ \forall\, t \in [0, T_x],\ \ d(x(t), \partial K)\ \geq\ \rho_x t$$

Proof — Since $F(x) \subset D_K(x)$, we know that we can associate with any $v \in F(x)$ the element

$$\rho_v\ :=\ \liminf_{h \to 0+} \frac{d\left(x + hv, \widehat{K}\right)}{4h}\ >\ 0$$

This implies that there exists $\tau_v > 0$ such that

$$\forall\, h \in]0, \tau_v],\ \ d(x + hv, \widehat{K})\ \geq\ 3\rho_v h$$

Hence, for any $w \in v + \rho_v B$, we infer that

$$\forall\, h \in]0, \tau_v],\ \ d(x + hw, \widehat{K})\ \geq\ 2\rho_v h$$

[1] If in addition

$$x \in \partial K \rightsquigarrow R(x) := F(x) \cap T_{\widehat{K}}(x) \text{ is lower semicontinuous at } x_0 \in \partial K$$

then for any $v_0 \in R(x_0)$, there exists a neighborhood $\partial K \cap (x_0 + rB)$ of x_0 such that $(v_0 + B) \cap R(x) \neq \emptyset$ on this neighborhood. Hence pointwise viability property $iv)$ implies the local one, and thus, the existence of at least one local viable solution starting from 0.

The compact subset $F(x)$ can be covered by q balls $v_i + \rho_{v_i} B$. Set

$$\rho_x := \min_{1 \leq i \leq q} \rho_{v_i} > 0 \;\; \& \;\; t_x := \min_{1 \leq i \leq q} \tau_{v_i} > 0$$

We thus deduce that

$$\forall\, h \in]0, t_x], \;\; d(x + h(F(x) + \rho_x B), \widehat{K}) \;\geq\; \rho_x h$$

Let us consider now any solution $x(\cdot) \in \mathcal{S}_F(x)$. Since F is upper semicontinuous, we know that $F(z) \subset F(x) + \rho_x B$ whenever $\|z - x\| \leq \eta_x$ for some η_x. Since F is bounded by a constant $c > 0$ on the ball $B(x, \eta_x)$, we infer that

$$\|x(t) - x\| \;\leq\; \int_0^t \|F(x(s))\| ds \;\leq\; ct \leq \eta_x$$

when $t \leq T_x := \min\{t_x, \eta_x / c\}$. In this case, we observe that $x(t) - x \in t(F(x) + \rho_x B)$, so that for any $t \in]0, T_x]$,

$$d(x(t), \widehat{K}) \;=\; d(x + x(t) - x, \widehat{K}) \;\geq\; d(x + t(F(x) + \rho_x B), \widehat{K}) \;\geq\; \rho_x t \;\; \square$$

As a consequence, we obtain the

Theorem 4.3.6 (Strict Invariance Theorem) *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \rightsquigarrow X$ with compact convex images and assume that the interior of K is not empty. If*

$$\forall\, x \in \partial K, \;\; F(x) \;\subset\; D_{\mathrm{Int}(K)}(x)$$

then, for any initial state x_0 in the boundary ∂K of K, for any solution $x(\cdot)$ to differential inclusion (4.2) starting from x_0, there exists $T > 0$ such that $x(\cdot)$ remains in the interior of K on $]0, T]$.

4.3.3 Inward and Outward Areas

We then can split the boundary of ∂K into three areas depending on F:

$$\begin{cases} K_{\Leftarrow} & := \{ x \in \partial K \mid F(x) \subset D_{\mathrm{Int}(K)}(x) \} \\ & \text{the inward area} \\ K_{\Rightarrow} & := \{ x \in \partial K \mid F(x) \subset D_{X \backslash K}(x) \} \\ & \text{the outward area} \\ K_{\Leftrightarrow} & := \{ x \in \partial K \mid F(x) \cap T_{\partial K}(x) \neq \emptyset \} \end{cases}$$

Proposition 4.3.7 *Let us consider a nontrivial upper semicontinuous set-valued map $F : X \leadsto X$ with compact convex images and a closed subset K of its domain with a nonempty interior.*

1. — Whenever $x \in K_{\Leftarrow}$, all solutions starting at x must enter the interior of K on some open time interval $]0,T[$, and whenever $x \in K_{\Rightarrow}$, all solutions starting at x must leave the subset K on some $]0,T[$.

2. — If $\partial K \cap (x + rB) \subset K_{\Leftrightarrow}$ for some $r > 0$, then at least one solution starting at x remains in the boundary ∂K on some $[0,T]$.

We may also introduce the subsets

$$\begin{cases} \widetilde{K_{\Leftarrow}} & := \{ x \in \partial K \mid F(x) \cap T_K(x) \neq \emptyset \} = \partial K \backslash K_{\Rightarrow} \supset K_{\Leftarrow} \\ \widetilde{K_{\Rightarrow}} & := \{ x \in \partial K \mid F(x) \cap T_{\widehat{K}}(x) \neq \emptyset \} = \partial K \backslash K_{\Leftarrow} \supset K_{\Rightarrow} \end{cases}$$

and we observe that *if $\partial K \cap (x + rB) \subset \widetilde{K_{\Leftarrow}}$ for some $r > 0$, then at least one solution starting at x is viable in K on some $[0,T]$ and the analogous statement holds true for $\widetilde{K_{\Rightarrow}}$.*

Since the images $F(x)$ are convex, we deduce from the second statement of Theorem 4.3.3 that

$$K_{\Leftrightarrow} = \widetilde{K_{\Leftarrow}} \cap \widetilde{K_{\Rightarrow}}$$

The boundary of K can be partitioned into the three areas $K_{\Leftarrow}$, $K_{\Rightarrow}$ and $K_{\Leftrightarrow}$. From the inward area $K_{\Leftarrow}$, all solutions must enter K, from the outward area $K_{\Rightarrow}$, all solutions must leave K, from $K_{\Leftrightarrow}$, a solution can remain locally in the boundary ∂K, (respectively in K or in $\widehat{K}$) if there exists $c > 0$ such that $\partial K \cap (x + rB) \subset K_{\Leftrightarrow}$ (respectively $\partial K \cap (x + rB) \subset \widetilde{K_{\Leftarrow}}$ and $\partial K \cap (x + rB) \subset \widetilde{K_{\Rightarrow}}$). □

4.3.4 Boundary of Viability Kernels

The boundary of the viability kernel is viable in certain cases:

Theorem 4.3.8 (Saint-Pierre) *Let $F : X \rightsquigarrow X$ be a Marchaud map and K be a closed subset with nonempty interior. If the viability kernel* $\mathrm{Viab}_F(K)$ *of K is contained in the interior of K, then its boundary is also a viability domain.*

Proof — Set $K_\infty := \mathrm{Viab}_F(K)$. We shall actually prove that for any $x \in \partial K_\infty \cap \mathrm{Int}(K)$, we have $F(x) \cap T_{\partial K_\infty}(x) \neq \emptyset$.

Assume the contrary: Then, for some $x_0 \in \partial K_\infty$,

$$F(x_0) \cap T_{\partial K_\infty}(x_0) = \emptyset$$

Since $F(x_0)$ is convex and contained in the contingent cone to K_∞ at x_0, we deduce from the second statement of Theorem 4.3.3 that

$$F(x_0) \subset D_{K_\infty}(x_0)$$

We shall derive a contradiction by showing that

$$F(x_0) \cap D_{\widehat{K_\infty}} \neq \emptyset \text{ where } \widehat{K_\infty} := \overline{X \backslash K_\infty}$$

Indeed, x_0 can be approximated by elements $x_n \in \mathrm{Int}(K) \backslash K_\infty$. Since $F : X \rightsquigarrow X$ is a Marchaud map, there exist solutions $x_n(\cdot) \in \mathcal{S}_F(x_n)$. By Theorem 3.5.2, a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x_\star(\cdot) \in \mathcal{S}_F(x_0)$.

Since x_0 belongs to the interior of K, we know that its hitting $\theta_K(x_\star(\cdot))$ is strictly positive. Being lower semicontinuous thanks to Lemma 4.2.2, we deduce that for n large enough,

$$0 < T := \theta_K(x_\star(\cdot))/2 \leq \theta_K(x_n(\cdot))$$

Hence, for every $t \in]0,T]$ and every n large enough, we have $x_n(t) \in K$. By Proposition 4.1.4, this implies that $x_n(t)$ does not belong to the viability kernel K_∞. Consequently,

$$\forall\, t \in]0,T], \quad x(t) \notin K_\infty$$

Since F is a Marchaud map, we infer that $F(x_0) \cap D_{\widehat{K_\infty}} \neq \emptyset$ by the necessary condition of the Viability Theorem, i.e., the contradiction we were looking for. □

4.4 Viability Domain Algorithms

4.4.1 Viability Kernel Algorithm

This algorithm has been introduced by Basile & Marro and Silverman (under the name of *structure algorithm*) for linear control systems and by Byrnes & Isidori (under the name of *zero dynamics algorithm*) for smooth nonlinear control systems[2].

It has been shown to converge to the viability kernel of closed subsets defined by equality constraints, i.e., subsets of the form $K := h^{-1}(0)$ where h is a map from X to a finite dimensional vector-space Y.

Let us describe this algorithm in the general case: Consider a closed subset K of the domain of a set-valued map $F : X \rightsquigarrow X$.

This algorithm starts with $K_0 := K$. We then construct

$$K_1 := \mathrm{Dom}(R_{K_0}) \;\text{ where }\; R_{K_0}(x) \;:=\; F(x) \cap T_{K_0}(x)$$

Since the viability kernel $\mathrm{Viab}_F(K)$ is contained in K and thus, since $T_{\mathrm{Viab}_F(K)}(x) \subset T_K(x)$, we infer that $\mathrm{Viab}_F(K) \subset K_1$.

Assume that a decreasing sequence of subsets K_n satisfying

$$\mathrm{Viab}_F(K) \subset K_n \subset K_{n-1} \subset K$$

has been defined up to n. We then set $R_{K_n}(x) := F(x) \cap T_{K_n}(x)$, define $K_{n+1} := \mathrm{Dom}(R_{K_n})$ and we observe that $\mathrm{Viab}_F(K) \subset K_{n+1}$.

[2] See the forthcoming book NONLINEAR FEEDBACK DESIGN by Byrnes and Isidori for an exhaustive exposition of this topic.

Therefore

$$\mathrm{Viab}_F(K) \subset \bigcap_{n=0}^{\infty} K_n$$

The problem is to show that equality holds true. Several requirements have to be met to solve the problem.

— The first one is that the subsets K_n should be closed.

— The second one is that the upper limit of the contingent cones $T_{K_n}(x)$ is contained in the contingent cone to the upper limit of the subsets K_n (which, in this case, is the intersection of the decreasing sequence of the subsets K_n).

These conditions are missing for finding the viability kernel of $K := [0,1] \times \mathbf{R}$ for the system $F(x,v) := \{v\} \times cB$.

Indeed, $K_1 = \{0\} \times \mathbf{R}_+ \cup]0,1[\times \mathbf{R} \cup \{1\} \times \mathbf{R}_-$, and we observe that $K_2 = K_1$. Therefore, the algorithm stops at K_1. It is a viability domain which is not closed (and not even locally compact). Therefore, it is not the viability kernel. One can compute explicitly the viability kernel of K under this set-valued map F: We introduce the set-valued map T^c defined by

$$\forall\, x \in [0,1], \;\; T^c(x) = [-\sqrt{2cx}, \sqrt{2c(1-x)}]$$

We observe that the viability kernel of K under F is the graph of T^c: See figure 4.5

4.4.2 Global Contingent Sets

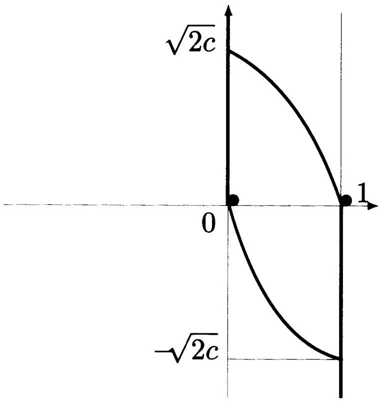
However, one can still recover some result by introducing closed subsets of the contingent cone (called *global contingent sets*) which enjoy the missing properties of the contingent cones.

Let $K \subset X$ be a closed subset. We shall define subsets $T_K^c(x)$ of the contingent cones $T_K(x)$ which have global properties instead of just local ones.

Definition 4.4.1 *Let $K \subset X$ be closed and $c > 0$ be a positive constant. Denote by $T_K^c(x)$ the subset of elements $v \in T_K(x)$ such that there exists a measurable function $\gamma(\cdot)$ bounded by c and satisfying*

$$\forall\, t \geq 0, \;\; x + tv + \int_0^t (t-\tau)\gamma(\tau)d\tau \;\in\; K$$

Figure 4.5: Viability Kernel of $[0,1] \times \mathbf{R}$ for $F(x,v) := \{v\} \times cB$



Naturally, the measurable function $\gamma(\cdot)$ is equal almost everywhere to the weak second derivative of $x(\cdot)$.

We observe that $0 \in T_K^c(x)$ for all $x \in K$.

Example — We can check easily that for $K := [0,1]$, the contingent cone $T_K(x)$ is defined by

$$T_K(x) = \begin{cases} \mathbf{R}_+ & \text{if } x = 0 \\ \mathbf{R} & \text{if } x \in]0,1[\\ \mathbf{R}_- & \text{if } x = 1 \end{cases}$$

and the global contingent set is equal to

$$\forall\, x \in [0,1],\ T_K^c(x) = \left[-\sqrt{2cx}, \sqrt{2c(1-x)}\right]$$

(See figure 4.5) □

We deduce from the properties of the viability kernels the following statements.

Proposition 4.4.2 *The graph of the set-valued map $x \rightsquigarrow T_K^c(x)$ is closed. Let $K^\sharp := \limsup_{n\to\infty} K_n$ denote the upper limit of a sequence of closed subsets K_n. Then the upper limit of the graphs of $T_{K_n}^c$ is contained in the graph of $T_{K^\sharp}^c$.*

Proof — We introduce the Marchaud map F from $X \times X$ to itself defined by $F(x,v) := \{v\} \times cB$. The map

$$t \mapsto x(t) \ := \ x_0 + tv_0 + \int_0^t (t-\tau)\gamma(\tau)d\tau$$

where $\|\gamma(\tau)\| \leq c$ is a solution to the differential inclusion $x'' \in cB$ and $(x(\cdot), x'(\cdot))$ is a solution to the differential inclusion

$$(x'(t), v'(t)) \ \in \ F(x(t), v(t)), \ \ x(0) \ = \ x_0, \ v(0) \ = \ v_0$$

We remark at once that $\text{Graph}(T_K^c(\cdot))$ is the viability kernel of the closure of the graph $\text{Graph}(T_K(\cdot))$ for the set-valued map $(x,v) \rightsquigarrow \{v\} \times cB$.

Then the proof follows from the fact that the viability kernel of a closed subset is closed and that the upper limit of a sequence of closed viability domains is a viability domain.

Let us consider any element (x,v) of the upper limit of the sequence of viability kernels $\text{Viab}(\text{Graph}(T_{K_n}))$. Then (x,v) is the limit of a subsequence (x_n, v_n) of elements of $\text{Viab}(\text{Graph}(T_{K_n}))$, so that there exist solutions $x_n(\cdot)$ to the differential inclusion $\|x''\| \leq c$ starting at (x_n, v_n) and converging to some function $x(\cdot)$ satisfying $x(0) = x$ and $x'(0) = v$. Since $x_n(t) \in K$ for all $t \geq 0$, then $x(t) \in K^\sharp$ for all $t \geq 0$. Therefore, $x'(t) \in T_{K^\sharp}(x(t))$. Hence, the pair $(x(t), x'(t))$ is a solution which is viable in $\text{Graph}(T_{K^\sharp})$ and consequently, $(x,v) \in \ \text{Viab}(\text{Graph}(T_{K^\sharp}))$. □

Obviously, if $c_1 \leq c_2$, then $T_K^{c_1} \subset T_K^{c_2}$.
We also observe that

$$\forall \, x \ \in \ K, \ \ \forall \, v \ \in \ T_K^c(x), \ \ DT_K^c(x,v)(v) \cap cB \ \neq \ \emptyset$$

Proposition 4.4.3 *Let Y be a finite dimensional space, $A \in \mathcal{L}(X,Y)$ be a linear operator and $K \subset X$, $M \subset Y$ be closed subsets. Then*

$$\forall \, x \ \in \ K, \ \ A\,(T_K^c(x)) \ \subset \ T_{\overline{A(K)}}^{c\|A\|}(Ax)$$

and thus for all $x \in A^{-1}(M)$, $T_{A^{-1}(M)}^c(x) \subset A^{-1}\left(T_M^{c\|A\|}(Ax)\right)$. Furthermore, if A is surjective, then there exists $\rho > 0$ such that

$$\forall \, x \ \in \ A^{-1}(M), \ \ T_M^c(Ax) \ \subset \ A\left(T_{A^{-1}(M)}^{c\rho}(x)\right)$$

Proof — Let $v \in T_K^c(x)$. Then there exists a solution $x(\cdot)$ to $x'' \in cB$ viable in K and satisfying $(x(0), x'(0)) = (x, v)$. So $y(t) := A(x(t))$ is a solution to the differential inclusion $y'' \in cA(B_X) \subset c\|A\|B_Y$, viable in $\overline{A(K)}$, such that $(y(0), y'(0)) = (Ax, Av)$.

The second statement follows by taking $K := A^{-1}(M)$.

To prove the last one, consider $w \in T_M(Ax)$ and a map

$$y(t) := y + tw + \int_0^t (t-\tau) y''(\tau) d\tau, \quad \|y''(\tau)\| \le c$$

viable in M. Since A is surjective, there exist a constant $\rho > 0$ and solutions x and v to the equations $Ax = y$ and $Av = w$ satisfying inequalities $\|x\| \le \rho\|y\|$ and $\|v\| \le \rho\|w\|$. Furthermore, by Theorem 8.2.9 of SET-VALUED ANALYSIS, there exists a measurable solution $z(\cdot)$ to the equation $Az(\tau) = y''(\tau)$ satisfying $\|z(\tau)\| \le \rho\|y''(\tau)\| \le \rho c$. Then

$$x(t) \ := \ x + tv + \int_0^t (t-\tau) z(\tau) d\tau$$

is a solution to the differential inclusion $\|x''\| \le \rho c$ which is viable in $A^{-1}(M)$. □

4.4.3 Viability Domain Algorithm

Thanks to Proposition 4.4.2, by replacing the contingent cones $T_K(x)$ by the subsets $T_K^c(x)$ in the *viability kernel algorithm*, we can prove that the modified version converges to a closed viability domain.

Let us set $K_0^c := K$. For defining $K_1^c \subset K_0^c$, we introduce the set-valued map R_0^c defined by $R_0^c(x) := F(x) \cap T_{K_0^c}^c(x)$ and set $K_1^c := \mathrm{Dom}(R_0^c)$.

If the subsets K_i^c have been defined up to n, we set $R_n^c(x) := F(x) \cap T_{K_n^c}^c(x)$ and we define $K_{n+1}^c := \mathrm{Dom}(R_n^c)$.

Proposition 4.4.4 *Assume that K is compact and that $F : K \rightsquigarrow X$ is upper semicontinuous with nonempty closed values. Then either K_i^c is empty for some step i or $K_\infty := \bigcap_{i=1}^\infty K_i^c$ is a nonempty closed viability domain of F: Actually,*

$$\forall\, x \in K_\infty, \ \ F(x) \cap T_{K_\infty}^c(x) \ \neq \ \emptyset$$

Proof — First, since the graph of R_i^c is the intersection of the graph of F and the graph of $T_{K_i^c}^c$ which are both closed, it is also closed. Furthermore, the subset K_i^c is closed since $F(K)$ is compact. Then the K_i^c's form a decreasing sequence of closed subsets of a compact subset. Either one of the K_i^c's is empty or the intersection K_∞ is not empty. In this case, let x be chosen in K_∞. For any n, there exists $v_n \in F(x) \cap T_{K_n^c}^c(x)$. The v_n's remaining in the compact subset $F(K)$, a subsequence (again denoted) v_n converges to some v. Since (x, v_n) belongs to the graph of $T_{K_n^c}^c$, we know that (x, v) belongs to the graph of $T_{K_\infty}^c$, since K_∞ is the upper limit of the decreasing sequence of the subsets K_i^c. Hence v belongs to $F(x) \cap T_{K_\infty}(x)$. □

4.4.4 Fast Viability Kernel Algorithm

Actually, the subsets K_n defined in the above algorithm are in general too large. Recalling the definition of the outward area $K_{\Rightarrow}$ of a closed subset, the subsets K_n are defined recursively by the formula

$$K_{n+1} \; = \; K_n \backslash (K_{n\,\Rightarrow})$$

so that $\text{Viab}_F(K) = \text{Viab}_F(K_n) \subset K_n$.

We propose now to take away not only the points of the outward area $K_{n\,\Rightarrow}$, but also all the balls $\overset{\circ}{B}(x_n, \varepsilon(x_n))$ centered at points $x_n \in K_{n\,\Rightarrow}$ such that

$$\overset{\circ}{B}(x_n, \varepsilon(x_n)) \cap \text{Viab}_F(K) \; = \; \emptyset$$

Hence, starting with $\widetilde{K}_0 := K$, we define the *fast viability kernel algorithm* by

$$\widetilde{K}_{n+1} \; := \; \widetilde{K}_n \backslash \bigcup_{x_n \in \widetilde{K}_{n\,\Rightarrow}} \overset{\circ}{B}(x_n, \varepsilon(x_n))$$

where $\overset{\circ}{B}(x_n, \varepsilon(x_n)) \cap \text{Viab}_F(K) \; = \; \emptyset$.

We observe that $\widetilde{K}_n \subset K_n$ and that $\text{Viab}_F(K) = \text{Viab}_F(\widetilde{K}_n) \subset \widetilde{K}_n$. This implies

Lemma 4.4.5 *Let K be a closed subset. Then*

$$\text{Viab}_F(K) \; \subset \; \bigcap_{n \geq 0} \widetilde{K}_n$$

We shall prove that equality holds true when F is both Marchaud and Lipschitz:

Theorem 4.4.6 (Frankowska-Quincampoix) *Assume that $F : X \rightsquigarrow X$ is both Marchaud and Lipschitz and that $K \subset X$ is closed. Then there exists a sequence of $\varepsilon(x_n) > 0$ (independent of the unknown viability kernel of K) such that the fast viability kernel algorithm converges to the viability kernel of K:*

$$\text{Viab}_F(K) = \bigcap_{n \geq 0} \widetilde{K_n}$$

We postpone the proof of this theorem to Chapter 5 since it requires Filippov's Theorem.

4.5 Finite-Difference Approximation of Viability Kernels

We begin by defining the discrete analogues of viability domains and viability kernels for discrete dynamical systems and prove that the discrete version of the Viability Kernel Algorithm provides the discrete viability kernel of a compact subset.

4.5.1 Viable Subsets under a Discrete System

Let X be a metric space and $G : X \rightsquigarrow X$ be a nontrivial set-valued map describing a discrete dynamical system

$$\forall\, n \geq 0, \;\; x_{n+1} \in G(x_n)$$

We denote $\vec{x} := (x_0, \ldots, x_n, \ldots)$ a solution to this system and by $\vec{S}(x_0)$ the set of solutions to this system starting at x_0.

We shall say that a subset $K \subset \text{Dom}(G)$ is *discretely viable under* G if for any initial state $x_0 \in K$, there exists at least one solution $\vec{x} \in \vec{S}(x_0)$ viable in K in the sense that

$$\forall\, n \geq 0, \;\; x_n \in K$$

A subset $K \subset \text{Dom}(G)$ is a *discrete viability domain* of G if

$$\forall\, x \in K, \;\; G(x) \cap K \neq \emptyset$$

It is obvious that a subset $K \subset X$ is a discrete viability domain of G if and only if it is discretely viable under G. If $K \subset \mathrm{Dom}(G)$ is closed, we define its *discrete viability kernel* as the largest closed viability domain contained in K. The discrete version of the viability kernel algorithm converges to the viability kernel:

Proposition 4.5.1 *Assume that $K \subset \mathrm{Dom}(G)$ is compact and that G is upper semicontinuous with closed images. We define recursively the closed subsets K_j by*

$$\begin{cases} i) & K_0 := K \\ ii) & K_j := \{x \in K_{j-1} \mid G(x) \cap K_{j-1} \neq \emptyset\} \end{cases} \qquad (4.4)$$

Then the discrete viability kernel of K exists and satisfies

$$\mathrm{Viab}_G(K) = \bigcap_{j=1}^{\infty} K_j$$

Proof — Inclusion

$$\mathrm{Viab}_G(K) \subset \bigcap_{j=1}^{\infty} K_j$$

is obvious. Since K is compact, either some K_j is empty or $\bigcap_{j=1}^{\infty} K_j$ is not empty. In the latter case, take $x \in \bigcap_{j=1}^{\infty} K_j$ and a sequence of elements $x_j \in G(x) \cap K_{j-1}$ which are not empty. A subsequence of elements $x_{j'}$ converges to some y. It belongs to both $G(x)$ and the upper limit of the K_j, which is equal to the intersection $\bigcap_{j=1}^{\infty} K_j$. Hence this intersection is a closed discrete viability domain, and thus contained in the discrete viability kernel of K. □

4.5.2 Finite-Difference Approximations

Let us consider a Marchaud map $F : X \rightsquigarrow X$, with which we associate discretizations $G_h : X \rightsquigarrow X$ satisfying

$$\forall\, \varepsilon > 0,\ \exists\, h_\varepsilon > 0 \mid \forall\, h \in]0, h_\varepsilon],\ \ \mathrm{Graph}\left(\frac{G_h - \mathbf{1}}{h}\right) \subset \mathrm{Graph}(F) + \varepsilon B$$

This is naturally the case if we take the explicit discretization $G_h^0 := \mathbf{1} + hF$.

Proposition 4.5.2 *Let us consider a sequence of discretizations G_h of a Marchaud map $F : X \rightsquigarrow X$. Then the upper limit* $\text{Limsup}_{n\to\infty}(K_h)$ *of a sequence of discrete viability domains K_h for G_h is a closed subset viable under F.*

In particular, if $K \subset \text{Dom}(F)$ is a closed subset,

$$\text{Limsup}_{h\to 0+}\left(\text{Viab}_{G_h}(K)\right) \subset \text{Viab}_F(K)$$

Proof — Indeed, consider a solution $\vec{x}_h := (x_{h_0}, \ldots, x_{h_n}, \ldots)$ to the discrete system G_h viable in K_h, defined by

$$\forall\, n \geq 0,\;\; x_{h_{n+1}} \in G_h(x_{h_n}) \cap K_h$$

We associate with it the functions $x_h(\cdot) \in \mathcal{C}(0,\infty;X)$ interpolating this sequence at the nodes nh:

$$\forall\, n \geq 0,\;\; \forall\, t \in [nh,(n+1)h[,\;\; x_h(t) := x_{h_n} + \frac{x_{h_{n+1}} - x_{h_n}}{h}(t - nh)$$

They satisfy for almost all $t \geq 0$

$$(x_h(t), x_h'(t)) \in \text{Graph}(F) + \varepsilon B$$

Also, by the Ascoli and Alaoglu Theorems, we derive as in the proof of the Viability Theorem, that a subsequence (again denoted by) $x_h(\cdot)$ satisfies

$$\begin{cases} i) & x_h(\cdot) \text{ converges uniformly to some } x(\cdot) \\ ii) & x_h'(t) \text{ converges weakly to } x'(\cdot) \text{ in } L^1(0,T;X) \end{cases}$$

Convergence Theorem 2.4.4 implies that this limit $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$. On the other hand, each $t \geq 0$ is the limit of nodes $n_t h$, so that $x(t)$ is the limit of $x_h(n_t h) \in K_h$. This implies that $x(t)$ belongs to the upper limit of the subsets K_h.
□

Chapter 5

Invariance Theorems for Differential Inclusions

Introduction

We devote this chapter to subsets *invariant* under a set-valued map, to invariance domains, kernels and envelopes, and to some of their properties.

Since the invariance property of a subset K involves the behavior of F outside of K, we need to extend the contingent cone to a subset K to the whole domain of F: we define for that purpose the concept of *external contingent cone* to K at any element $x \in X$.

Also, to proceed further, we need some regularity property of the subset, a kind of "$\mathcal{C}^1$-regularity", which here takes the following form: *the set-valued map $K \ni x \rightsquigarrow T_K(x)$ is lower semicontinuous.* Since this property will be used quite often, we give it a name: *sleekness.* We shall check that *the contingent cones to sleek subsets are convex. Convex subsets as well as smooth manifolds are sleek.*

Since we have seen the crucial role played by these contingent cones in viability theorems, we take this opportunity to study them further and to mention their calculus summarized in Table 5.2 for the convenience of the reader. Details are provided in chapter 4 of SET-VALUED ANALYSIS.

The second section is devoted to criteria for a subset to be *invariant under a set-valued map.* These criteria involve the concepts of external contingent cone introduced in the first section.

In the third section, we shall derive from Filippov's Theorem[1] the characterization of closed subsets K locally invariant under a Lipschitz set-valued map F as closed invariance domains.

We define in the fourth section the *invariance kernel* of a closed subset K, which is the largest closed subset of K invariant under F. We prove its existence when the solution map of the differential inclusion is lower semicontinuous. We also introduce the *invariance envelopes*, which are the smallest closed subsets containing K invariant under F, and relate them to the backward invariance kernel of the complement of K.

We study the stability of sequences of closed subsets invariant by set-valued maps F_n and invariance kernels, by showing for instance that the lower limit of invariance kernels of closed subsets K_n is contained in the invariance kernel of the lower limit.

We devote the fifth section to the study of semipermeability and viability properties of the boundaries of the viability and invariance kernels of a closed subset. We apply these results to define the defeat and victory domains of an open target and show that the boundary of the victory domain is a semipermeable barrier.

We illustrate in the sixth section the notions and results obtained so far with the example of *linear differential inclusions* $x' \in F(x)$, where the right-hand side F is a closed convex process. We mention in particular that in the case of linear differential inclusions, a closed convex cone is an invariance domain if and only if its polar cone is a viability domain of the transpose. In this sense, one can say that the concepts of viability and invariance are dual.

5.1 External Contingent Cones

5.1.1 External Contingent Cones

We begin by introducing the following notation:

$$D_{\uparrow}d_K(x)(v) \ := \ \liminf_{h\to 0+}(d_K(x+hv)-d_K(x))/h$$

[1]that we shall not prove here. We refer to Hélène Frankowska's monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS or to DIFFERENTIAL INCLUSIONS for an exposition of the fundamental Filippov's Theorem and its numerous applications.

which will be justified later[2]. We observe that when $x \in K$, a direction v is contingent to K at x if and only if $D_\uparrow d_K(x)(v) \leq 0$.

In order to study invariance properties of a subset K which involve the behavior of the set-valued map F outside of K, we need to extend our definition of the contingent cone to points outside of K:

Definition 5.1.1 *Let K be a subset of a finite dimensional vector-space X and x belong to X. We extend the notion of contingent cone to the one of* external contingent cone *to K at points outside of K in the following way:*

$$T_K(x) := \{v \mid D_\uparrow d_K(x)(v) \leq 0\}$$

We point out an easy but important relation between the external contingent cone at a point and the contingent cone at its projection:

Lemma 5.1.2 *Let K be a closed subset of a finite dimensional vector-space and $\Pi_K(y)$ be the set of projections of y onto K, i.e., the subset of $z \in K$ such that $\|y - z\| = d_K(y)$. Then the following inequalities:*

$$D_\uparrow d_K(y)(v) \quad \leq \quad d(v, T_K(\Pi_K(y)))$$

hold true. Therefore,

$$T_K(\Pi_K(y)) \subset T_K(y)$$

Proof — Choose $z \in \Pi_K(y)$ and $w \in T_K(z)$. Then

$$\left\{ \begin{array}{l} \frac{d_K(y+hv) - d_K(y)}{h} \leq \frac{\|y-z\| + d_K(z+hv) - d_K(y)}{h} \\ = \frac{d_K(z+hv)}{h} \leq \frac{d_K(z+hw)}{h} + \|v - w\| \end{array} \right.$$

Since z belongs to K and $w \in T_K(z)$, the above inequality implies that

$$D_\uparrow d_K(y)(v) \quad \leq \quad d(v, T_K(z)) \ \Box$$

[2]this is the contingent epiderivative of the distance functions d_K. (See Definition 9.1.2 of Chapter 9.)

5.1.2 Sleek Subsets

We define now the tangent cone $C_K(x)$ introduced in 1975 by F. H. Clarke.

Definition 5.1.3 *Let $K \subset X$ be a subset of a normed space X and $x \in \overline{K}$ belong to the closure of K. We define the* (Clarke) tangent cone *(or* circatangent cone*)* $C_K(x)$ *by*

$$C_K(x) := \{v \mid \lim_{h \to 0+, K \ni x' \to x} d_K(x' + hv)/h = 0\}$$

We see at once that $C_K(x) \subset T_K(x)$ and that if x belongs to Int(K), then $C_K(x) = X$.

It is very convenient to observe that when x belongs to $\overline{K}$,

$$\begin{cases} v \in C_K(x) \text{ if and only if } \forall\ h_n \to 0+,\ \ \forall\, K \ni x_n \to x, \\ \exists\ v_n \to v \text{ such that } \forall\, n,\ \ x_n + h_n v_n \in K \end{cases}$$

The charm of the tangent cone C_K at x is that *it is always convex*[3]. Unfortunately, *the price to pay for enjoying this convexity property of the Clarke tangent cones is that they may often be reduced to the trivial cone* $\{0\}$.

However, we shall show that the Clarke tangent cone and the contingent cone do coincide at those points x where the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous:

Definition 5.1.4 (Sleek Subsets) *We shall say that a subset K of X is* sleek *at $x \in K$ if the set-valued map*

$$K \ni x' \ \rightsquigarrow\ T_K(x') \text{ is lower semicontinuous at } x$$

and that it is sleek *if and only if it is sleek at every point of K.*

---

[3] Let v_1 and v_2 belong to $C_K(x)$. To prove that $v_1 + v_2$ belongs to this cone, let us choose any sequence $h_n > 0$ converging to 0 and any sequence of elements $x_n \in K$ converging to x. There exists a sequence of elements v_{1n} converging to v_1 such that the elements $x_{1n} := x_n + h_n v_{1n}$ do belong to K for all n. But since x_{1n} does also converge to x in K, there exists a sequence of elements v_{2n} converging to v_2 such that

$$\forall\, n,\ \ x_{1n} + h_n v_{2n} \ = \ x_n + h_n(v_{1n} + v_{2n}) \ \in \ K$$

This implies that $v_1 + v_2$ belongs to $C_K(x)$ because the sequence of elements $v_{1n} + v_{2n}$ converges to $v_1 + v_2$.

Theorem 5.1.5 *Let K be a closed subset of a finite dimensional vector-space X. Consider a set-valued map $F : K \rightsquigarrow X$ satisfying*

$$\begin{cases} i) & F \text{ is lower semicontinuous at } x \\ ii) & \exists\, \delta > 0 \quad \text{such that } \forall\, z \in B_K(x,\delta), \ \ F(z) \subset T_K(z) \end{cases}$$

Then $F(x) \subset C_K(x)$.

In particular, if K is sleek at $x \in K$, then $T_K(x) = C_K(x)$ is a closed convex cone.

Proof — Let us take $x \in K$ and $v \in F(x)$, assumed to be different from 0. Since F is lower semicontinuous at x, Corollary 2.1.7 implies that we can associate with any $\varepsilon > 0$ a number $\eta \in]0, \delta[$ such that $d(v, F(z)) \leq d(v, F(x)) + \varepsilon = \varepsilon$ for any $z \in B_K(x, \eta)$ (because $d(v, F(x)) = 0$). Therefore, for any $y \in B(x, \eta/4)$ and $\tau \leq \eta/4\|v\|$, the inequality

$$\forall\, z \in \Pi_K(y + \tau v), \ \|z - x\| \leq 2\|y + \tau v - x\| \leq 2\|x - y\| + 2\tau\|v\| \leq \eta$$

implies that

$$\begin{cases} d(v, T_K(\Pi_K(y + \tau v))) & \leq & d(v, F(\Pi_K(y + \tau v))) \\ & \leq & d(v, F(x)) + \varepsilon \ = \ \varepsilon \end{cases}$$

We set $g(\tau) := d_K(y + \tau v)$. By Lemma 5.1.2, we obtain

$$\begin{cases} \liminf_{h \to 0+} \left(g(\tau + h) - g(\tau)\right)/h \ = \ D_{\uparrow} d_K(y + \tau v)(v) \\ \leq \ d(v, T_K(\Pi_K(y + \tau v))) \ \leq \ \varepsilon \end{cases}$$

The function g being Lipschitz, it is almost everywhere differentiable, so that $g'(t) \leq \varepsilon$ for almost all t small enough. Integrating this inequality from 0 to h, we obtain

$$d_K(y + hv) \ = \ g(h) \ = \ g(h) - g(0) \ \leq \ \varepsilon h$$

for any $y \in B(x, \eta/4)$ and $\tau \leq \eta/4\|v\|$. This shows that v belongs to $C_K(x)$.

By taking $F(x) = T_K(x)$, we deduce that $T_K(x) \subset C_K(x)$ whenever K is sleek at $x \in K$, and thus, that they coincide. □

5.1.3 Tangent Cones to Convex Sets

For convex subsets K, the Clarke tangent cone and the contingent cone coincide with the closed cone spanned by $K - x$:

Proposition 5.1.6 (Tangent Cones to Convex Sets) *We denote by*

$$S_K(x) \;:=\; \bigcup_{h>0} \frac{K-x}{h}$$

the cone spanned by $K - x$. If K is convex, the contingent cone $T_K(x)$ to K at x is convex and

$$C_K(x) \;=\; T_K(x) \;=\; \overline{S_K(x)}$$

The subnormal cone is equal to

$$N_K(x) \;:=\; S_K(x)^- \;=\; \{\, p \in X^\star \;|\; \max_{y\in K} < p, y > = < p, x > \}$$

Furthermore, the normal cones $N_K(x)$ to a convex subset K are contained in the barrier cone of K: for every $x \in K$, $N_K(x) \;\subset\; b(K)$.

Remark — We shall denote by $T_K(x)$ the common value of these cones, and call it the *tangent cone* to the convex subset K at x. The subnormal cone coincides with the normal cone of K at x of convex analysis. □

Actually, closed convex subsets are sleek:

Theorem 5.1.7 *Any closed convex subset of a finite dimensional vector-space X is sleek.*

We refer to Theorem 4.2.2 of SET-VALUED ANALYSIS for the proof of this Theorem. □

It may be useful to recall the characterization of the interior of the tangent cone to a convex subset.

Proposition 5.1.8 (Interior of a Tangent Cone) *Assume that the interior of $K \subset X$ is not empty. Then*

$$\forall\, x \in K, \quad \mathrm{Int}(T_K(x)) \;=\; \bigcup_{h>0} \left(\frac{\mathrm{Int}(K) - x}{h} \right)$$

Furthermore, the graph of the set-valued map $K \ni x \rightsquigarrow \mathrm{Int}(T_K(x))$ *is open.*

For the convenience of the reader, we list in the Table 5.1 some useful formulas of the calculus of tangent cones to convex subsets (see Section 4.1. of APPLIED NONLINEAR ANALYSIS, in which the subsets K, K_i, L, M, ... are assumed to be convex.)

We shall need the following characterization of the normal cone to a convex cone:

Lemma 5.1.9 *Let* $K \subset X$ *be a convex cone of a normed space* X *and* $x \in K$. *Then*

$$p \in N_K(x) \Longleftrightarrow x \in K,\ p \in K^- \ \&\ < p, x > = 0 \Longleftrightarrow x \in N_{K^-}(p)$$

where $N_{K^-}(p) := \{x \in K \mid \forall\, q \in K^-,\ \ < q - p, x > \leq 0\}$.

Proof — To say that $p \in N_K(x)$ means that $< p, x >= \sigma_K(p)$, which is equal to 0 if and only if $p \in K^-$, and the first statement of the lemma follows. □

5.1.4 Calculus of Contingent Cones

We summarize in Table 5.2 the calculus of contingent cones. Formulas (1) to (4) are straightforward. The other properties are valid when K is sleek, and are a consequence of the Constrained Inverse Function Theorem, which we do not prove in this book[4].

See also Quincampoix's Theorem 4.3.3 and the remark following it for another set of sufficient conditions.

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[4]We refer to Chapter 4 of SET-VALUED ANALYSIS for the proofs of these formulas and more detailed results.

We mention also that transversality condition of formula (5) implies the constraint qualification assumption $0 \in \mathrm{Int}(f(L) - M)$ and that the stronger transversality condition

$$\exists\, c > 0 \mid \forall\, x \in K,\ \ B_Y \subset f'(x)(T_L(x) \cap cB_X) - T_M(Ax)$$

implies that if L and M are sleek and f is continuously differentiable, then K is also sleek.

Table 5.1: **Properties of Tangent Cones to Convex Sets.**

(1)	▷	If $x \in K \subset L \subset X$, then $T_K(x) \subset T_L(x)$ & $N_L(x) \subset N_K(x)$
(3)	▷	If $x_i \in K_i \subset X_i$, $(i = 1, \cdots, n)$, then $T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n T_{K_i}(x_i)$ $N_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) = \prod_{i=1}^n N_{K_i}(x_i)$
(4)a)	▷	If $A \in \mathcal{L}(X, Y)$ and $x \in K \subset X$, then $T_{A(K)}(Ax) = \overline{A(T_K(x))}$ $N_{A(K)}(Ax) = A^{\star^{-1}} N_K(x)$
(4)b)	▷	If K_1, $K_2 \subset X$, $x_i \in K_i$, $i = 1, 2$, then $T_{K_1+K_2}(x_1 + x_2) = \overline{T_{K_1}(x_1) + T_{K_2}(x_2)}$ $N_{K_1+K_2}(x_1 + x_2) = N_{K_1}(x_1) \cap N_{K_2}(x_2)$ In particular, if $x_1 \in K$ and x_2 belongs to a closed subspace P of X, then $T_{K+P}(x_1 + x_2) = \overline{T_{K_1}(x_1) + P}$ $N_{K+P}(x_1 + x_2) = N_K(x_1) \cap P^{\perp}$
(5)	▷	If $L \subset X$ and $M \subset Y$ are *closed convex* subsets and $A \in \mathcal{L}(X, Y)$ satisfies the *constraint qualification assumption* $0 \in \text{Int}(M - A(L))$, then, for every $x \in L \cap A^{-1}(M)$, $T_{L \cap A^{-1}(M)} = T_L(x) \cap A^{-1} T_M(Ax)$ $N_{L \cap A^{-1}(M)} = N_L(x) + A^{\star} N_M(Ax)$
(5)a)	▷	If $M \subset Y$ is *closed convex* and if $A \in \mathcal{L}(X, Y)$ satisfies $0 \in \text{Int}(\text{Im}(A) - M)$, then, for any $x \in A^{-1}(M)$, $T_{A^{-1}(M)}(x) = A^{-1} T_M(Ax)$ $N_{A^{-1}(M)}(x) = A^{\star} N_M(Ax)$
(5)b)	▷	If K_1, $K_2 \subset X$ are closed convex and satisfy $0 \in \text{Int}(K_1 - K_2)$, then, for any $x \in K_1 \cap K_2$ $T_{K_1 \cap K_2}(x) = T_{K_1}(x) \cap T_{K_2}(x)$ $N_{K_1 \cap K_2}(x) = N_{K_1}(x) + N_{K_2}(x)$
(5)c)	▷	If $K_i \subset X$, $(i = 1, \ldots, n)$, are closed and convex, $x \in \bigcap_{i=1}^n K_i$ and if there exists $\gamma > 0$ satisfying $\forall x_i$ such that $\|x_i\| \leq \gamma$, $\bigcap_{i=1}^n (K_i - x_i) \neq \emptyset$, then $T_{\bigcap_{i=1}^n K_i}(x) = \bigcap_{i=1}^n T_{K_i}(x)$ $N_{\bigcap_{i=1}^n K_i}(x) = \sum_{i=1}^n N_{K_i}(x)$

Table 5.2: **Properties of Contingent Cones.**

(1)	▷	If $K \subset L$ and $x \in \overline{K}$, then $T_K(x) \subset T_L(x)$
(2)	▷	If $K_i \subset X$, $(i = 1, \cdots, n)$ and $x \in \overline{\bigcup_i K_i}$, then $T_{\cup_{i=1}^n K_i}(x) \; = \; \cup_{i \in I(x)} T_{K_i}(x)$ where $I(x) := \{i \mid x \in \overline{K_i}\}$
(3)	▷	If $K_i \subset X_i$, $(i = 1, \cdots, n)$ and $x_i \in \overline{K_i}$, then $T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) \; \subset \; \prod_{i=1}^n T_{K_i}(x_i)$
(4)	▷	If $g \in \mathcal{C}^1(X, Y)$, if $K \subset X$, $x \in \overline{K}$ and $M \subset Y$, then $g'(x)(T_K(x)) \; \subset \; T_{g(K)}(g(x))$ $T_{g^{-1}(M)}(x) \; \subset \; g'(x)^{-1} T_M(g(x))$
(5)	▷	If $L \subset X$ and $M \subset Y$ are *closed sleek* subsets, $f \in \mathcal{C}^1(X, Y)$ is a continuously differentiable map and $x \in L \cap f^{-1}(M)$ satisfies the *transversality condition* $f'(x) T_L(x) - T_M(f(x)) = Y$, then $T_{L \cap f^{-1}(M)}(x) \; = \; T_L(x) \cap f'(x)^{-1} T_M(f(x))$
(5)a)	▷	If $M \subset Y$ is a *closed sleek* subset, $f \in \mathcal{C}^1(X, Y)$ is a continuously differentiable map and $x \in f^{-1}(M)$ satisfies $\mathrm{Im}(f'(x)) - T_M(f(x)) = Y$, then $T_{f^{-1}(M)}(x) \; = \; f'(x)^{-1} T_M(f(x))$
(5)b)	▷	If K_1 and K_2 are *closed sleek* subsets contained in X and $x \in K_1 \cap K_2$ satisfies $T_{K_1}(x) - T_{K_2}(x) = X$, then $T_{K_1 \cap K_2}(x) \; = \; T_{K_1}(x) \cap T_{K_2}(x)$
(5)c)	▷	If $K_i \subset X$, $(i = 1, \ldots, n)$, are *closed sleek* and $x \in \bigcap_i K_i$ satisfies $\forall v_i = 1, \ldots, n, \;\; \bigcap_{i=1}^n (T_{K_i}(x) - v_i) \neq \emptyset$ then, $T_{\bigcap_{i=1}^n K_i}(x) \; = \; \bigcap_{i=1}^n T_{K_i}(x)$

5.1.5 Inequality Constraints

We also state the following example of the contingent cone to a set defined by equality and inequality constraints[5]:

Theorem 5.1.10 *Let us consider a closed subset L of a finite dimensional vector-space X and two continuously differentiable maps $g := (g_1, \ldots, g_p) : X \mapsto \mathbf{R}^p$ and $h := (h_1, \ldots, h_q) : X \mapsto \mathbf{R}^q$ defined on an open neighborhood of L.*

Let K be the subset of L defined by the constraints

$$K := \{x \in L \mid g_i(x) \geq 0,\ i = 1, \ldots, p,\ \&\ h_j(x) = 0,\ j = 1, \ldots, q\}$$

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of active constraints.

We posit the following transversality condition at a given $x \in K$:

$$\begin{cases} i) & h'(x)C_L(x) = \mathbf{R}^q \\ ii) & \exists\, v_0 \in C_L(x) \quad \textit{such that } h'(x)v_0 = 0 \\ & \text{and } \forall\, i \in I(x), \ < g_i'(x), v_0 >> 0 \end{cases}$$

Then u belongs to the contingent cone to K at x if and only if u belongs to the contingent cone to L at x and satisfies the constraints

$$\forall\, i \in I(x),\ < g_i'(x), u > \geq 0 \ \&\ \forall\, j = 1, \ldots, q,\ h_j'(x)u = 0$$

Unfortunately, the graph of $T_K(\cdot)$ is not necessarily closed. However, there exists a closed set-valued map $T_K^{\diamond}(\cdot)$ contained in $T_K(\cdot)$ introduced by N. Maderner. Set

$$\gamma_K(x) := \min_{i \notin I(x)} \frac{g_i(x)}{\|g_i'(x)\|} \in]0, +\infty] \tag{5.1}$$

We observe that γ_K *is upper semicontinuous whenever the functions g_i are continuously differentiable*. Indeed, let $x_n \in K$ converge to x_0 and $a_n \leq \gamma_K(x_n)$ converge to a_0. Since $g_i(x_0) > 0$ whenever $i \notin I(x_0)$, we infer that $i \notin I(x_n)$ for n large enough. Hence inequalities

[5]See Proposition 4.3.6 of SET-VALUED ANALYSIS

$a_n\|g_i'(x_n)\| \leq g_i(x_n)$ hold true for any $i \notin I(x_0)$ and imply at the limit that $a_0 \leq \gamma_K(x_0)$.

The growth of the function γ_K *is linear whenever we assume that there exists a constant* $c > 0$ *such that*

$$\forall\, i = 1,\ldots,p, \quad \|g_i'(x)\| \;\geq\; c\,\frac{g_i(x)}{\|x\|+1}$$

Theorem 5.1.11 (Maderner) *We posit the assumptions of Theorem 5.1.10. Then the set-valued map* $T_K^{\diamond}(\cdot) : K \rightsquigarrow X$ *defined by:*
$u \in T_K^{\diamond}(x)$ *if and only if* $u \in T_L(x)$ *and*

$$\begin{cases} \forall\, i = 1,\ldots,p, & g_i(x)+ < g_i'(x), u > \;\geq\; 0 \\ \forall\, j = 1,\ldots,q, & h_j'(x)u \;=\; 0 \end{cases}$$

is contained in the contingent cone $T_K(x)$ *and satisfy*

$$T_K(x) \cap \gamma_K(x)B \;\subset\; T_K^{\diamond}(x)$$

Its graph is closed whenever the graph of $x \rightsquigarrow T_L(x)$ *is closed.*

Proof — Let u belong to $T_K^{\diamond}(x)$. Then, when $i \in I(x)$, we see that $\langle g_i'(x), u\rangle = g_i(x) + \langle g_i'(x), u\rangle \geq 0$, so that $u \in T_K(x)$.

Conversely, let us choose u in $T_K(x)$ satisfying $\|u\| \leq \gamma_K(x)$. Then either $i \in I(x)$ and $g_i(x)+\langle g_i'(x), u\rangle = \langle g_i'(x), u\rangle \geq 0$ or $g_i(x) > 0$ so that

$$i \notin I(x) \;\;\&\;\; g_i(x) + \langle g_i'(x), u\rangle \;\geq\; g_i(x) \;-\; \|g_i'(x)\|\|u\| \;\geq\; 0$$

because $\|u\| \leq \gamma_K(x) \leq g_i(x)/\|g_i'(x)\|$. Thus, in both cases, $g_i(x) + \langle g_i'(x), u\rangle \geq 0$, so that u belongs to $T_K^{\diamond}(x)$. □

5.2 Invariance Domains

Let us consider the differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t)) \tag{5.2}$$

We recall the definition of invariant subsets K under a set-valued map F: *A subset* K *is said to be* (locally) invariant under F *(or* *enjoys the* invariance property*) if for any initial state x_0 of K,* all *solutions to the differential inclusion (5.2) starting at x_0 are* viable (on some interval $[0, T]$).

We emphasize again that the concept of invariance *depends upon the behavior of F on the domain of F outside of K.* But we can tackle this issue since we have extended the concept of contingent cone to K at points outside of K (Definition 5.1.1). This enables us to provide an *Invariance Criterion* (by contrast with the Strict Invariance Theorem 4.3.6).

Theorem 5.2.1 *Let K be a subset of the domain of a nontrivial set-valued map F. If F is locally bounded and if*

$$\forall\, x \in \mathrm{Dom}(F), \quad F(x) \subset T_K(x)$$

then K is invariant under F.

Proof — Let $x(\cdot) \in \mathcal{S}(x_0)$ be any solution to the differential inclusion (5.2) defined on some interval $[0, T]$. Let us set $g(t) := d_K(x(t))$, which is absolutely continuous. Let t be a point where both $x'(t)$ and $g'(t)$ exist. Then there exists $\varepsilon(h)$ converging to 0 with h such that $x(t+h) = x(t) + hx'(t) + h\varepsilon(h)$ and

$$\begin{cases} g'(t) & = \lim_{h\to 0+} \frac{d_K(x(t)+hx'(t)+h\varepsilon(h)) - d_K(x(t))}{h} \\ & = D_\uparrow d_K(x(t))(x'(t)) \end{cases}$$

Since $x'(t) \in F(x(t)) \subset T_K(x(t))$ almost everywhere, we infer that $g'(t) \leq 0$ for almost all t. Therefore $x(\cdot)$ is viable whenever the initial state x_0 is in K. If not, there would exist $t > 0$ such that $x(t) \notin K$. But we derive a contradiction since:

$$0 < d_K(x(t)) = d_K(x(t)) - d_K(x(0)) = g(t) - g(0) = \int_0^t g'(\tau)d\tau \leq 0 \quad \square$$

We are tempted to call an invariance domain of F a subset $K \subset \mathrm{Dom}(F)$ satisfying the condition $F(x) \subset T_K(x)$ for all $x \in \mathrm{Dom}(F)$. But actually, we shall study the stronger property where the above condition holds true only for $x \in K$.

Definition 5.2.2 (Invariance Domain) *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map. We shall say that a subset $K \subset \mathrm{Dom}(F)$ is an* invariance domain *of F if*

$$\forall x \in K, \quad F(x) \subset T_K(x)$$

Since the contingent cone to a singleton is reduced to 0, we observe that a singleton $\{\overline{x}\}$ is an invariance domain if and only if $\overline{x}$ is a "stopping point" of F, i.e., a solution to the inclusion

$$F(\overline{x}) \;=\; \{0\}$$

(No velocity can take such a stopping point away.)

Corollary 5.2.3 *Let K be a subset of the domain of a nontrivial set-valued map F. Assume that F satisfies*

$$\forall\, x \in \mathrm{Dom}(F), \quad F(x) \subset F(\Pi_K(x))$$

If K is an invariance domain, then it is invariant under F.

Proof — It follows from Theorem 5.2.1, since $F(x) \subset F(\Pi_K(x)) \subset T_K(\Pi_K(x)) \subset T_K(x)$ thanks to Lemma 5.1.2. □

For instance, when K is a closed convex set, we can extend a set-valued map $F : K \rightsquigarrow X$ to a set-valued map $\widetilde{F} : X \rightsquigarrow X$ by setting

$$\forall\, x \in X, \quad \widetilde{F}(x) := F(\pi_K(x))$$

Corollary 5.2.4 *Let K be a closed convex subset and $F : K \rightsquigarrow X$ be a set-valued map satisfying*

$$\forall\, x \in K, \quad F(x) \subset T_K(x)$$

Then K is invariant under the extension $\widetilde{F}$ of F.

Corollary 5.2.5 *Let K be a closed subset of the domain of a nontrivial set-valued map F. If*

$$\forall\, x \in \mathrm{Dom}(F), \quad \forall\, v \in F(x), \quad \forall\, y \in \Pi_K(x), \quad < x - y, v > \;\leq\; 0$$

then K is invariant under F.

Proof — It follows trivially from Corollary 5.2.3 and Proposition 3.2.3. □

We can regard the next result as a *structural stability* property:

Proposition 5.2.6 *Let us assume that K is convex with nonempty interior. Assume that the graph of F is compact and that*

$$\forall\, x \in K, \quad F(x) \subset \mathrm{Int}(T_K(x))$$

Then there exists a neighborhood $\mathcal{U}$ of the graph of F such that the above condition is verified for all set-valued maps G whose graph is contained in $\mathcal{U}$.

Proof — Since the graph of F is compact and contained in the graph of $K \ni x \rightsquigarrow \mathrm{Int}\,(T_K(x))$ which is open by Proposition 5.1.8, the latter is such a neighborhood $\mathcal{U}$. □

5.3 Invariance Theorem

5.3.1 Filippov's Theorem

In order to characterize the local invariance property of a closed subset K, i.e., to prove that K is an invariance domain of F, we need to know that given any $x \in K$ and $v \in F(x)$, there exists a solution $x(\cdot)$ to differential inclusion (5.2) such that $x(0) = x$ and $x'(0) = v$.

This is the case when the right-hand side F is Lipschitz in a neighborhood of K, thanks to the Filippov Theorem[6]. Actually, Filippov's Theorem is much more than a mere existence theorem. It also provides an estimate of the distance between a function $y(\cdot)$ and the set $\mathcal{S}_F(x_0)$ of solutions starting at some initial state x_0.

Theorem 5.3.1 (Filippov) *Assume that $F : X \rightsquigarrow X$ is λ-Lipschitz with closed values on the interior of its domain. Let $y(\cdot)$ be a given*

[6] We do not provide the proof of the Filippov Theorem, but refer the reader to Corollary 2.4.1, p.121 of DIFFERENTIAL INCLUSIONS or to Hélène Frankowska's CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS.

absolutely continuous function such that $t \to d(y'(t), F(y(t)))$ is integrable(for the measure $e^{-\lambda s}ds$). We associate with a fixed x_0 the function η defined by

$$\eta(t) \;=\; e^{\lambda t}\left(\|x_0 - y(0)\| \;+\; \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right)$$

Let $T > 0$ be finite or infinite chosen such that the tube

$$\{y(t) + \eta(t)B\}_{t\in[0,T[}$$

is contained in the interior of the domain of F.

Then there exists a solution $x(\cdot)$ to differential inclusion (5.2) such that, for all $t \in [0,T[$,

$$\|x(t) - y(t)\| \leq e^{\lambda t}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \tag{5.3}$$

and for almost all $t \in [0,T[$,

$$\begin{cases} \|x'(t) \;-\; y'(t)\| \;\leq\; d(y'(t), F(y(t))) \\ \\ +\lambda e^{\lambda t}\left(\|x_0 - y(0)\| \;+\; \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \end{cases}$$

Proof — Filippov's Theorem yields an estimate on any finite interval $[0,T]$ such that the tube $\{y(t) + \eta(t)B\}_{t\in[0,T]}$ is contained in the interior of the domain of F.

Actually, we can extend it to the interval $[0,+\infty[$ if the tube

$$\{y(t) + \eta(t)B\}_{t\in[0,+\infty[}$$

is contained in the interior of the domain of F. Indeed, there exists a solution $x(\cdot)$ to differential inclusion (5.2) defined on $[0,T]$ starting at x_0 satisfying estimate (5.3) and in particular

$$\|x(T) - y(T)\| \;\leq\; e^{\lambda T}\left(\|x_0 - y(0)\| \;+\; \int_0^T d(y'(s), F(y(s)))e^{-\lambda s}ds\right)$$

There also exists a solution $z(\cdot)$ to differential inclusion (5.2) starting at $x(T)$ estimating the function $t \mapsto y(t+T)$ and satisfying

$$\begin{cases} \|z(t) - y(t+T)\| \\ \leq\; e^{\lambda t}\left(\|x(T) - y(T)\| \;+\; \int_0^t d(y'(s+T), F(y(s+T)))e^{-\lambda s}ds\right) \end{cases}$$

Hence we can extend $x(\cdot)$ on the interval $[0, 2T]$ by concatenating it with the function $t \mapsto x(t) := z(t - T)$ on the interval $[T, 2T]$ and we observe that the above estimates yield (5.3) for $t \in [0, 2T]$. We reiterate this process as long as the tube $\{y(t) + \eta(t)B\}_{t \in [0,nT]}$ is contained in the interior of the domain of F. □

It implies the existence of a solution:

Corollary 5.3.2 *Assume that F is Lipschitz on the interior of its domain. Then, for any $x_0 \in \text{Int}(\text{Dom}(F))$ and $v_0 \in F(x_0)$, there exist $T > 0$ and a solution $x(\cdot)$ to differential inclusion (5.2) defined on $[0, T]$ and satisfying $x(0) = x_0$ and $x'(0) = v_0$.*

Proof — We apply Filippov's Theorem with $y(t) := x_0 + tv_0$ and $x_0 := y(0)$. Then $d(y'(t), F(y(t))) \leq \lambda t \|v_0\|$ and

$$\eta(t) \leq e^{\lambda t} \int_0^t \lambda \tau \|v_0\| e^{-\lambda \tau} d\tau \leq \frac{\|v_0\|}{\lambda} \left(e^{\lambda t} - 1 - \lambda t\right)$$

Filippov's Theorem implies the existence of a solution $x(\cdot)$ to differential inclusion (5.2) starting at x_0 and satisfying

$$\|x(t) - x_0 - tv_0\| \leq \frac{\|v_0\|}{\lambda} \left(e^{\lambda t} - 1 - \lambda t\right)$$

Dividing by $t > 0$ and letting t converge to $0+$, we infer $x'(0) = v_0$.
□

It also implies the Lipschitz dependence of the solution map on the initial condition:

Corollary 5.3.3 *Let $y(\cdot) \in \mathcal{S}_F(y_0)$ and assume that F, $y(\cdot)$ satisfy the assumptions of Filippov's Theorem 5.3.1. Then*

$$d\left(y(t), \mathcal{S}_F(x_0)(t)\right) \leq \|x_0 - y_0\| \, e^{\lambda t}$$

so that the solution map $\mathcal{S}_F$ is lower semicontinuous.

Remark — Observe that if we set

$$\delta(t) := d\left(\text{Exit}_F(K, t), \partial K\right)$$

Filippov's Theorem 5.3.1 implies that for all $0 < T_0 < T$,

$$\forall\, x \in \text{Exit}_F(K, T), \;\; B\left(x, \frac{\delta(T - T_0)}{e^{\lambda T_0}}\right) \subset \text{Exit}_F(K, T_0) \;\; \square$$

5.3.2 Characterization of Local Invariance

We are ready to prove the characterization of invariant domains under a Lipschitz map:

Theorem 5.3.4 *Let us assume that F is Lipschitz on the interior of its domain and has compact values. Then a closed subset $K \subset$ $\mathrm{Int}(\mathrm{Dom}(F))$ is locally invariant under F if and only if K is an invariance domain.*

Proof — Let us assume that K is an invariance domain and let $x(\cdot)$ be any solution to differential inclusion (5.2) starting at x_0 and defined on some interval $[0,T]$. Let us set $g(t) := d_K(x(t))$, which is absolutely continuous on $[0,T]$.

Let t be a point such that both $x'(t)$ and $g'(t)$ exist and $x'(t)$ belongs to $F(x(t))$. Then there exists $\varepsilon(h)$ converging to 0 with h such that

$$x(t+h) \;=\; x(t) + hx'(t) + h\varepsilon(h)$$

and

$$\begin{cases} g'(t) \;=\; \lim_{h\to 0+}(d_K(x(t)+hx'(t)+h\varepsilon(h)) - d_K(x(t)))/h \\ \;=\; D_\uparrow d_K(x(t))(x'(t)) \end{cases}$$

Lemma 5.1.2 implies that

$$D_\uparrow d_K(x)(x'(t)) \;\leq\; d(x'(t), T_K(\Pi_K(x(t))))$$

Let us denote by $\lambda > 0$ the Lipschitz constant of F and choose any y in $\Pi_K(x(t))$. We deduce that:

$$d(x'(t), T_K(\Pi_K(x(t)))) \leq d(x'(t), T_K(y)) \leq d(x'(t), F(y))$$
(since K is an invariance domain)

$$\leq d(x'(t), F(x(t))) + \lambda\|y - x(t)\| \quad \text{(since } F \text{ is Lipschitz)}$$

$$= \; 0 + \lambda d_K(x(t)) \;=\; \lambda g(t)$$

Then g is a solution to

$$\text{for almost all } t \in [0,T], \;\; g'(t) \;\leq\; \lambda g(t) \;\;\&\;\; g(0) = 0$$

We deduce that $g(t) = 0$ for all $t \in [0, T]$, and therefore, that $x(t)$ is viable in K on $[0, T]$.

— Let us assume that K is locally invariant under F. Let $x_0 \in K$. We have to prove that any $u_0 \in F(x_0)$ is contingent to K at x_0. Corollary 5.3.2 implies that for all x_0 and $u_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to differential inclusion (5.2) satisfying $x(0) = x_0$ and $x'(0) = u_0$. Since K is locally invariant under F, it is viable on some interval $[0, T]$. We thus infer that u_0 belongs to $T_K(x_0)$. Hence $F(x_0)$ is contained in $T_K(x_0)$. □

5.3.3 Graphical Lower Limits of Solution Maps

Let us recall the concepts of *lower limits* of subsets and of *graphical lower limit* of set-valued maps.

Let K_n be a sequence of subsets of a metric space X. We say that

$$K^\flat := \mathrm{Liminf}_{n\to\infty} K_n := \{y \in X \mid \lim_{n\to\infty} d(y, K_n) = 0\}$$

is its *lower limit.* In other words, it is the closed subset of limits of sequences of elements $x_n \in K_n$.

We shall say that the set-valued map $\mathrm{Lim}^\flat{}_{n\to\infty} F_n$ from X to X defined by

$$\mathrm{Graph}(\mathrm{Lim}^\flat{}_{n\to\infty} F_n) := \mathrm{Liminf}_{n\to\infty} \mathrm{Graph}(F_n)$$

is the *graphical lower limit of the set-valued maps* F_n. For simplicity, we set $F^\flat := \mathrm{Lim}^\flat{}_{n\to\infty} F_n$.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L, M) := \sup_{y\in L} \inf_{z\in M} d(y, z) = \sup_{y\in L} d(y, M)$$

their *semi-Hausdorff distance*[7], and recall that $\Delta(L, M) = 0$ if and only if $L \subset M$. If Φ and Ψ are two set-valued maps, we set

$$\Delta(\Phi, \Psi)_\infty = \sup_{x\in X} \Delta(\Phi(x), \Psi(x))$$

[7]The Hausdorff distance between L and M is equal to $\max(\Delta(L, M), \Delta(M, L))$.

Filippov's Theorem provides an example of a situation where the solution map $\mathcal{S}_F$ is the graphical lower limit of a sequence of solution maps $\mathcal{S}_{F_n}$.

Theorem 5.3.5 *Let $F_n : X \rightsquigarrow X$ and $F : X \rightsquigarrow X$ be λ-Lipschitz set-valued maps with closed images and uniform linear growth: there exists $c > 0$ such that*

$$\forall\, n \geq 0, \ \ \forall\, x \in X, \ \ \|F_n(x)\| \ \leq \ c(\|x\| + 1)$$

Then

$$\Delta\left(\mathcal{S}_F(x_0), \mathcal{S}_{F_n}(x_{0n})\right)_\infty \ \leq \ e^{\lambda t}\|x_0 - x_{0n}\| + \frac{e^{\lambda t} - 1}{\lambda}\Delta(F, F_n)_\infty$$

and

$$\Delta\left(\mathcal{S}_F, \mathcal{S}_{F_n}\right)_\infty \ \leq \ \frac{e^{\lambda t} - 1}{\lambda}\Delta(F, F_n)_\infty$$

Consequently, if[8] $\lim_{n\to\infty} \Delta(F, F_n)_\infty = 0$*, then*

$$\mathcal{S}_F \ \subset \ \mathrm{Lim}^\flat{}_{n\to\infty}\left(\mathcal{S}_{F_n}\right)$$

Proof — Let us consider any solution $x(\cdot) \in \mathcal{S}_F(x_0)$ to differential inclusion (5.2). Therefore,

$$d(x'(t), F_n(x(t))) \ \leq \ \Delta(F(x(t)), F_n(x(t))) \ \leq \ \Delta(F, F_n)_\infty$$

By Filippov Theorem 5.3.1 applied to the map F_n, there exists a solution $x_n(\cdot) \in \mathcal{S}_{F_n}(x_{0n})$ such that

$$\begin{cases} \|x_n(t) - x(t)\| \ \leq \ e^{\lambda t}\|x_0 - x_{0n}\| + \displaystyle\int_0^t e^{\lambda(t-s)}\Delta(F, F_n)_\infty ds \\[2ex] = \ e^{\lambda t}\|x_0 - x_{0n}\| + \Delta(F, F_n)_\infty \dfrac{e^{\lambda t} - 1}{\lambda} \end{cases}$$

Then for any $t \geq 0$, $x(t)$ is the limit of $x_n(t)$, so that our claim is proved. □

[8] This implies that F is contained in the graphical lower limit $F^\flat$ of the set-valued maps F_n.

Remark — We can obtain other estimates. Set

$$\Delta(\Phi,\Psi)_1 = \sup_{x\in X} \frac{\Delta(\Phi(x),\Psi(x))}{\|x\|+1}$$

Let $F_n : X \rightsquigarrow X$ and $F : X \rightsquigarrow X$ be λ-Lipschitz set-valued maps with closed images and uniform linear growth. *Then, for any* $\lambda > c$,

$$\Delta\left(\mathcal{S}_F,\mathcal{S}_{F_n}\right)_1 \le \frac{e^{\lambda t}-e^{ct}}{\lambda - c}\Delta(F,F_n)_1$$

so that $\Delta\left(\mathcal{S}_F,\mathcal{S}_{F_n}\right)_1$ *converges to* 0 *and thus*

$$\mathcal{S}_F \subset \mathrm{Lim}^{\flat}{}_{n\to\infty}\left(\mathcal{S}_{F_n}\right)$$

when $\lim_{n\to\infty}\Delta(F,F_n)_1 = 0$.

Indeed, consider any solution $x(\cdot)\in\mathcal{S}_F(x_0)$ to differential inclusion (5.2). Since

$$\begin{cases} d(x'(t),F_n(x(t))) \le \Delta(F(x(t)),F_n(x(t))) \le \Delta(F,F_n)_1(\|x(t)\|+1) \\ \le \Delta(F,F_n)_1(\|x_0\|+1)e^{ct} \end{cases}$$

Filippov Theorem 5.3.1 applied to the map F_n implies that there exists a solution $x_n(\cdot)\in\mathcal{S}_{F_n}(x_0)$ such that

$$\begin{cases} \|x_n(t)-x(t)\| \le e^{\lambda t}\int_0^t \Delta(F,F_n)_1(\|x_0\|+1)e^{-(\lambda-c)s}ds \\ = \Delta(F,F_n)_1(\|x_0\|+1)\frac{e^{\lambda t}-e^{ct}}{\lambda-c} \end{cases} \quad \square$$

5.3.4 Accessibility Map

We recall that the *reachable map* R_F is defined by

$$R_F(t)x := (\mathcal{S}_F(x))(t)$$

(See Definition 3.5.4.)

Definition 5.3.6 *We shall denote by* $\mathcal{R}_F : X \rightsquigarrow X$ *the map defined by*

$$\mathcal{R}_F(x) := \bigcup_{T\ge 0} R_F(T)x$$

and call it the accessibility map.

Proposition 5.3.7 *Assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty closed values. Then $\mathcal{R}_F$ maps open subsets onto open subsets. If K is a closed subset satisfying $K = \overline{\mathrm{Int}(K)}$, then*

$$\overline{\mathrm{Int}\,(\mathcal{R}_F(K))} \;=\; \overline{\mathcal{R}_F\,(\mathrm{Int}(K))}$$

Proof — Let Ω be an open subset and fix any $y \in \mathcal{R}_F(x)$ where $x \in \Omega$: By definition, there exist $T > 0$ and a solution $x(\cdot)$ on $[0, T]$ to the differential inclusion (5.2) starting at x such that $x(T) = y$. Let $y(\cdot)$ be a solution to the backward inclusion $y' \in -F(y)$ starting at x and consider the solution $\widetilde{y}(\cdot) \in \mathcal{S}_{-F}(y)$ to the reverse differential inclusion defined by

$$\widetilde{y}(s) \;:=\; \begin{cases} x(T-s) & \text{if } 0 \;\le\; s \;\le\; T \\ y(s-T) & \text{if } T \;\le\; s \;<\; \infty \end{cases}$$

Since $-F$ is Lipschitz, Filippov's Theorem 5.3.1 implies that there exists a neighborhood $\mathcal{N}(y)$ of y such that, for every $z \in \mathcal{N}(y)$, one can find a solution $z(\cdot) \in \mathcal{S}_{-F}(z)$ satisfying $z(T) \in \Omega$. This means that z can be reached from Ω in finite time.

We thus deduce that $\mathcal{R}_F(\mathrm{Int}(K))$ is contained in $\mathrm{Int}(\mathcal{R}_F(K))$, so that the inclusion

$$\overline{\mathcal{R}_F(\mathrm{Int}(K))} \;\subset\; \overline{\mathrm{Int}\,(\mathcal{R}_F(K))}$$

holds true. It remains to prove the converse inclusion when we assume that $K = \overline{\mathrm{Int}(K)}$. We shall actually prove that any $y \in \mathcal{R}_F(K)$ belongs to $\overline{\mathcal{R}_F(\mathrm{Int}(K))}$. We know that there exist $x \in K$, $T > 0$ and a solution $x(\cdot)$ to differential inclusion (5.2) defined on $[0, T]$ starting at x such that $x(T) = y$. Take any $\varepsilon > 0$. Since $x \in \overline{\mathrm{Int}(K)}$, Filippov's Theorem 5.3.1 implies that there exists $\delta > 0$ such that for any $z \in B(x, \delta) \cap \mathrm{Int}(K)$, one can obtain a solution $z(\cdot)$ to differential inclusion (5.2) starting at z and satisfying $z(T) \in B(y, \varepsilon)$. Hence y can be approximated by elements $z(T) \in \mathcal{R}_F(\mathrm{Int}(K))$. □

5.3.5 Proof of Convergence of the Fast Viability Kernel Algorithm

Proposition 5.3.8 *Assume that F is both Marchaud and λ-Lipschitz. Let x belong to the outward area $K_{\Rightarrow}$ and set*

$$\delta_K(x) \;:=\; d(F(x), T_K(x))/2 \;>\; 0$$

We denote by $\theta_K(x) > 0$ the largest positive number θ such that

$$\forall\, h \in]0,\theta],\;\; d\left(x + h\left(F(x) + \delta_K(x)B\right), K\right) \;>\; 0$$

(which does exist). Let us fix $r > 0$ and set

$$\begin{cases} C_x \;:=\; F(B(x,r)),\;\; T := \min\{r/\|C_x\|, 1/\lambda\} \\ t_K(x) \;:=\; \min\{\delta_K(x)/2\lambda\|C_x\|, \theta_K(x), T\} \\ \varepsilon_K(x) := \delta_K(x)t_K(x)/2e^{\lambda t_K(x)} \end{cases}$$

Then $\varepsilon_K(x)$, which depends only upon x and K and does not involve $\mathrm{Viab}_F(K)$*, satisfies*

$$\overset{\circ}{B}\left(x, \varepsilon_K(x)\right) \cap \mathrm{Viab}_F(K) \;=\; \emptyset$$

Proof — The compactness of $F(x) + \delta_K(x)B$ and the very definition of the contingent cone imply that there exists a positive $\theta > 0$ such that

$$\forall\, h \in]0,\theta],\;\; d\left(x + h\left(F(x) + \delta_K(x)B\right), K\right) \;>\; 0$$

(See the proof of Proposition 4.3.5.) Therefore $\theta_K(x) > 0$ is positive and we observe that

$$\forall\, h \in]0,\theta_K(x)],\;\; d\left(x + h\left(F(x) + \frac{\delta_K(x)}{2}B\right), K\right) \;>\; \frac{\delta_K(x)h}{2}$$

Let us consider any solution $x(\cdot) \in \mathcal{S}(x)$ starting at x. Since $F(y) \subset C_x$ when y ranges over the ball $B(x,r)$, we first infer that

$$\forall\, t \le r/\|C_x\|,\;\; \|x(t) - x\| \;\le\; \int_0^t \|F(x(\tau))\|d\tau \;\le\; \|C_x\|t$$

Since F is λ-Lipschitz, we deduce that

$$\begin{cases} x(t) - x \ \in \ \int_0^t F(x(\tau)d\tau \ \subset \ \int_0^t (F(x) + \lambda\|x(\tau) - x\|B)d\tau \\ \subset \ t\left(F(x) + \frac{\delta_K(x)}{2}B\right) \end{cases}$$

whenever $t < t_K(x) := \min\{\delta_K(x)/2\lambda\|C_x\|, \theta_K(x), T\}$. Consequently, for every positive $t < t_K(x)$,

$$d(x(t), K) \ \geq \ d\left(x + t\left(F(x) + \frac{\delta_K(x)}{2}B\right), K\right) \ \geq \ \frac{\delta_K(x)}{2}t$$

Furthermore, by the Filippov Theorem 5.3.1, we know that for any $y(\cdot) \in \mathcal{S}(y)$, there exists a solution $x(\cdot) \in \mathcal{S}(x)$ such that

$$\|x(t) - y(t)\| \ \leq \ e^{\lambda t}\|x - y\|$$

We set $\varepsilon_K(x) \ := \ \delta_K(x)t_K(x)/2e^{\lambda t_K(x)}$. This implies that for any $y \in \overset{\circ}{B}(x, \varepsilon_K(x))$,

$$\begin{cases} d(y(t_K(x)), K) \geq d(x(t_K(x)), K) - \|x(t_K(x)) - y(t_K(x))\| \\ \geq \ \delta_K(x)t_K(x)/2 - e^{\lambda t_K(x)}\|x - y\| \ > \ 0 \end{cases}$$

This means that such initial states y do not belong to the viability kernel of K, because all solutions leave K in finite time. □

We shall need the following result.

Lemma 5.3.9 *Let P be a convex closed cone with compact sole*[9] *and M be a compact subset of X. Then there exists $y \in M$ such that:*

$$(y + P) \cap M \ = \ \{y\}$$

[9]Let P be a closed convex cone. We recall that the following conditions are equivalent:

$$\begin{cases} i) & \text{P is spanned by a convex compact set disjoint from 0} \\ ii) & \text{the interior of the polar cone } P^+ \text{ is not empty} \\ iii) & S := \{x \in P \mid < p_0, x >= 1\} \text{ where } p_0 \in \text{Int}(P^+) \text{ spans } P; \end{cases}$$

The compact convex subset S is called the *sole*, and such closed convex cones are called *cones with compact sole.*

Proof — The proof follows from Zorn's lemma. Let us define the following preorder relation on M:

$$a \leq b \iff b \in a + P$$

which is actually an order since P has a compact sole. We next prove that every subset L of M *which is totally ordered* has a majorant.

Clearly, for any $a \in L$, $(a + P) \cap M \neq \emptyset$. Since these sets are nonempty and compact and since $(b+P)\cap M \subset (a+P)\cap M$ whenever $a \leq b$, we deduce that:

$$\bigcap_{a\in L} ((a + P) \cap M) \neq \emptyset$$

Let b belong to $\bigcap_{a\in L}(a + P) \cap M$. Obviously, b is larger than any element of L for the order $\leq$. According to Zorn's lemma, there exists a maximal element $y \in M$: Namely, if $z \in M$ is different from y , then, $y \notin z + P$. Hence, $(y + P) \cap M = \{y\}$. □

Proof of Theorem 4.4.6 — By Lemma 4.4.5, we already know that

$$\text{Viab}_F(K) = \text{Viab}_F(\widetilde{K}_\infty) \subset \widetilde{K}_\infty$$

Assume that $\widetilde{K}_\infty$ is not a viability domain: there would exist $x \in \widetilde{K}_{\infty \Rightarrow}$. Set

$$\delta_\infty := \delta_{\widetilde{K}_\infty}(x) := \frac{d\left(F(x), T_{\widetilde{K}_\infty}(x)\right)}{2}$$

and

$$\theta_\infty := \theta_{\widetilde{K}_\infty}(x) > 0$$

We shall derive a contradiction by constructing a sequence of elements $x_n \in \widetilde{K}_{n \Rightarrow}$ converging to x such that $\varepsilon_{\widetilde{K}_n}(x_n)$ is bounded below by some $\varepsilon_\infty > 0$ that we shall define: In this case, we would have $\|x_n - x\| \geq \varepsilon_{\widetilde{K}_n}(x_n) \geq \varepsilon_\infty$ because

$$x \in \widetilde{K}_{n+1} \subset \widetilde{K}_n \backslash \overset{\circ}{B}(x_n, \varepsilon_{\widetilde{K}_n}(x_n))$$

by the very definition of the algorithm and thus, the contradiction ensues. We thus have to define this positive lower bound ε_∞.

Since the convex compact set $F(x) + \delta_\infty B$ does not contain 0, the cone P spanned by this set has a compact sole. Set

$$M_n := \widetilde{K}_n \cap (x + [0, \theta_\infty](F(x) + \delta_\infty B))$$

We can assert, thanks to Lemma 5.3.9, that:

$$\exists\, x_n \in M_n \text{ such that } (x_n + P) \cap M_n \;=\; \{x_n\}$$

On the other hand, by the very definition of $\widetilde{K}_\infty$ and the choice of x, the sequence x_n converges to x. Hence for all n large enough,

$$x_n \in\; x + \left[0, \frac{\theta_\infty}{2}\right] (F(x) + \delta_\infty B)$$

Thus,

$$\widetilde{K}_n \cap \left(x_n + \left[0, \tfrac{\theta_\infty}{2}\right] (F(x) + \delta_\infty B)\right) \;\subset$$

$$\widetilde{K}_n \cap (x + [0, \theta_\infty]\,(F(x) + \delta_\infty B)) \cap (x_n + P) = (x_n + P) \cap M_n = \{x_n\}$$

Since F is Lipschitz, we have for n large enough, $F(x_n) \subset F(x) + \delta_\infty B/2$, so that for any $t < \theta_\infty/2$,

$$d\left(x_n + t\,(F(x_n) + \delta_\infty B/2)\,, \widetilde{K}_n\right) \;\geq\; d\left(x_n + t\,(F(x) + \delta_\infty B)\,, \widetilde{K}_n\right) \;>\; 0$$

Thus $d(F(x_n), T_{\widetilde{K}_n}(x_n)) \geq \delta_\infty/2$, i.e., $\delta_{\widetilde{K}_n}(x_n) \geq \delta_\infty/4$. By Proposition 5.3.8, we deduce that $\theta_{\widetilde{K}_n}(x_n) \geq \theta_\infty/2$ and thus, setting

$$t_\infty \;:=\; \min\left\{\frac{\theta_\infty}{2}, \frac{\delta_\infty}{2\lambda\|C_x\|}, T\right\}$$

that $t_{\widetilde{K}_n}(x_n) \geq t_\infty/2$. Since $t \mapsto t/e^{\lambda t}$ is increasing for $0 \leq t \leq 1/\lambda$, we infer that

$$\varepsilon_\infty \;:=\; \frac{\delta_\infty t_\infty}{16 e^{\lambda t_\infty/2}} \;\leq\; \frac{\delta_{\widetilde{K}_n}(x_n) t_{\widetilde{K}_n}(x_n)}{2e^{\lambda t_{\widetilde{K}_n}(x_n)}} \;:=\; \varepsilon_{\widetilde{K}_n}(x_n)$$

We have thus constructed a lower bound ε_∞ of the radii $\varepsilon_{\widetilde{K}_n}(x_n)$ for n large enough which implies the contradiction we claimed at the beginning of the proof. □

5.4 Invariance Kernels

We now introduce the concepts of invariance kernel and envelope:

Definition 5.4.1 (Invariance Kernels and Envelopes) *Let K be a subset of the domain of a set-valued map $F : X \rightsquigarrow X$. The largest closed subset of K invariant under F, which we denote by* $\mathrm{Inv}_F(K)$ *or* $\mathrm{Inv}(K)$*, is called the* invariance kernel *of K. We shall say that the smallest closed subset invariant under F containing K is the* invariance envelope $\mathrm{Env}_F(K)$ *of K.*

Since the intersection of closed subsets invariant under F is still a closed subset invariant under F, the invariance envelope of a closed subset does exist.

5.4.1 Existence of the Invariance Kernel

We now prove the existence of the invariance kernel of a closed subset (possibly empty).

Recall that $\mathcal{S}_F$ denotes the solution map associating with any x_0 the set of solutions to differential inclusion $x' \in F(x)$ starting at x_0 and that it is lower semicontinuous when F is Lipschitz with closed values (see Corollary 5.3.3.) We shall set

$$\Omega \ := \ \mathrm{Dom}\,(\mathcal{S}_F)$$

Naturally, invariant subsets are necessarily contained in Ω. We supply the space $\mathcal{C}(0,\infty;X)$ with the topology of pointwise convergence.

Theorem 5.4.2 *Let us assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from Ω to $\mathcal{C}(0,\infty;X)$. Then, for any closed subset $K \subset \Omega$, there exists an invariance kernel (possibly empty) of K. It is the subset of initial points such that all solutions starting from them are viable in K.*

Proof — Let us denote by $\mathcal{K} \subset \mathcal{C}(0,+\infty;X)$ the subset of continuous functions $x(\cdot)$ which are viable in K and by Inv(K) the subset of initial state $x \in K$ such that $\mathcal{S}_F(x) \subset \mathcal{K}$, possibly empty.

Since the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ supplied with the topology of pointwise convergence and since $\mathcal{K}$ is closed, we deduce that Inv(K) is also a closed subset of K (See Proposition 1.4.4 of SET-VALUED ANALYSIS.)

It obviously contains any closed subset of K invariant under F.

It remains to be shown that it is also invariant under F. For that purpose, let us take $x \in$ Inv(F) and show that any solution $x(\cdot) \in \mathcal{S}_F(x)$ is viable in Inv(K) (by checking that for any $T > 0$, $x(T) \in$ Inv(K)). Let $y(\cdot)$ belongs to $\mathcal{S}_F(x(T))$. Hence the function $z(\cdot)$ defined by

$$z(t) \; := \; \begin{cases} x(t) & \text{if } t \in [0,T] \\ y(t-T) & \text{if } t \in [T,\infty[\end{cases}$$

is a solution to the differential inclusion (5.2) starting at x at time 0, and thus, is viable in K by the very definition of Inv(K). Hence for all $t \geq 0$, $y(t) = z(t+T)$ belongs to K, so that we have proved that $\mathcal{S}_F(x(T)) \subset \mathcal{K}$, i.e., $x(T) \in$ Inv(K). □

Remark — It is clear that

$$\text{Inv}(K_1 \cap K_2) \; = \; \text{Inv}(K_1) \cap \text{Inv}(K_2)$$

and more generally, that the invariance kernel of any intersection of closed subsets K_i $(i \in I)$ is the intersection of the invariance kernels of the K_i. □

5.4.2 Complement of the Invariance Kernel

Proposition 5.4.3 *Assume that $K \subset \Omega :=$ Dom$(\mathcal{S}_F)$ is compact with nonempty interior, that $F(K)$ is bounded and that its invariance kernel $\text{Inv}_F(K)$ is contained in the interior of K. Then the complement $\Omega \backslash \text{Inv}_F(K)$ of the invariance kernel is viable under F.*

Proof — Since we assume that the invariance kernel is compact, there exists $\eta > 0$ such that $\text{Inv}_F(K) + 2\eta B \subset K$.

We observe that property

$$\forall\, x(\cdot) \in \mathcal{S}_F(x),\; \exists\, t \leq \tau_K(x(\cdot)) \;\; \text{such that} \;\; x(t) \in \text{Inv}_F(K)$$

implies that x belongs to the invariance kernel of K.

Therefore, if $x_0 \in K \backslash \mathrm{Inv}_F(K)$, there exists a solution $x_1(\cdot) \in \mathcal{S}_F(x_0)$ such that $x_1(t) \notin \mathrm{Inv}_F(K)$ for every $t \in [0, \tau_K(x_1(\cdot))]$.

If $\tau_K(x_1(\cdot)) = +\infty$, we deduce that $x_1(\cdot)$ is viable in $K \backslash \mathrm{Inv}_F(K)$. If not, we set $t_1 := \tau_K(x_1(\cdot))$ and $x_1 := x_1(t_1) \in \partial K$.

Let $x_2(\cdot) \in \mathcal{S}_F(x_1)$ and define $\rho(x_2(\cdot)) := \inf\{t \geq 0 \mid x_2(t) \in \mathrm{Inv}_F(K) + \eta B\}$. Then either $\rho(x_2(\cdot)) = +\infty$ and the solution obtained in concatenating $x_1(\cdot)$ and $x_2(\cdot)$ is viable in $\Omega \backslash \mathrm{Inv}_F(K)$, or $t_2 := \rho(x_2(\cdot))$ is finite and $x_2 := x_2(t_2) \in \partial(\mathrm{Inv}_F(K) + \eta B)$.

We also check that $t_2 - t_1 \geq \eta / \|F(K)\|$ because $\|x_2 - x_1\| \leq (t_2 - t_1)\|F(K)\|$ and $\|x_2 - x_1\| \geq \eta$.

Now we iterate this procedure to construct a solution $x(\cdot)$ which is viable in $X \backslash \mathrm{Inv}_F(K)$. □

Let us point out this easy but useful remark:

Proposition 5.4.4 *If the boundary ∂K of a closed subset $K \subset \mathrm{Dom}(\mathcal{S}_F)$ is invariant under F, so is K.*

Proof — Indeed, take x_0 in the interior of K and any solution $x(\cdot) \in \mathcal{S}_F(x_0)$. If it is not viable in K, there would exist a finite exit time $T := \inf\{s \geq 0 \mid x(s) \notin K\}$, at which $x(T) \in \partial K$. Since the boundary is invariant, any solution starting at $x(T)$ remains in ∂K. This is the case of the solution $y(\cdot)$ defined by $y(t) := x(t + T)$, so that $x(t) \in \partial K$ for every $t \geq T$. This contradicts the assumption that $x(\cdot)$ is not viable in K. □

5.4.3 Stability of Invariance Domains

Let us consider now a sequence of closed subsets K_n invariant under a set-valued map F. *Is their lower limit still invariant under F?*

Proposition 5.4.5 *Let us assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from Ω to $\mathcal{C}(0, \infty; X)$. Then the lower limit of closed subsets $K_n \subset \Omega$ invariant under F is also invariant under F.*

In particular, the lower limit of the invariance kernels of a sequence of closed subsets $K_n \subset \Omega$ contains the invariance kernel of the lower limit of the sequence K_n:

$$\mathrm{Liminf}_{n\to\infty} (\mathrm{Inv}(K_n)) \supset \mathrm{Inv}\left(\mathrm{Liminf}_{n\to\infty} K_n\right)$$

Proof — Let the initial set $x_0 := \lim_{n\to\infty} x_{0n}$ belong to the lower limit $K^\flat$ of the sequence K_n and $x(\cdot) \in \mathcal{S}_F(x_0)$ be any solution to differential inclusion (5.2). Since the solution map is lower semicontinuous, there exist solutions $x_n(\cdot) \in \mathcal{S}_F(x_{0n})$ converging pointwise to $x(\cdot)$. The subsets K_n being invariant under F, we conclude that for any $t \geq 0$, $x_n(t) \in K_n$. This implies that $x(t) \in K^\flat$ for every $t \geq 0$. Hence $K^\flat$ is invariant under F. □

More generally, we can prove that the lower limit $K^\flat$ of a sequence of closed subsets K_n invariant under set-valued maps F_n are invariant under some set-valued map F.

Theorem 5.4.6 (Stability) *Let us consider set-valued maps $F_n : X \rightsquigarrow X$ and $F : X \rightsquigarrow X$ such that the solution map $\mathcal{S}_F$ is contained in the graphical lower limit of the solution maps $\mathcal{S}_{F_n}$. Then if the closed subsets $K_n \subset \mathrm{Dom}(\mathcal{S}_{F_n})$ are invariant under the set-valued maps F_n, their lower limit $K^\flat$ is invariant under F.*

In particular, the lower limit of the invariance kernels of closed subsets K_n for the set-valued maps F_n contains the invariance kernel of the lower limit $K^\flat$ for F:

$$\mathrm{Liminf}_{n\to\infty}\left(\mathrm{Inv}_{F_n}(K_n)\right) \supset \mathrm{Inv}_F\left(\mathrm{Liminf}_{n\to\infty} K_n\right)$$

5.4.4 Global Exit and Hitting Functions

When the solution map $\mathcal{S}_F$ is lower semicontinuous, we can deduce from Proposition 4.2.2 and the Maximum Theorem 2.1.6 that the function $\theta_K^\sharp : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\theta_K^\sharp(x) := \sup_{x(\cdot)\in\mathcal{S}_F(x)} \theta_K(x(\cdot))$$

is lower semicontinuous and that the function $\tau_K^\flat : K \mapsto \mathbf{R}_+ \cup \{+\infty\}$ defined by

$$\tau_K^\flat(x) := \inf_{x(\cdot)\in\mathcal{S}_F(x)} \tau_K(x(\cdot))$$

is upper semicontinuous.

Therefore the graphs of the "tubes" associating with $t \in [0, +\infty[$ the subsets

$$\left\{ \begin{array}{l} \left\{ x \in K \mid \theta_K^\sharp(x) \leq T \right\} \\ \\ \left\{ x \in K \mid \tau_K^\flat(x) \geq T \right\} \end{array} \right. \tag{5.4}$$

are closed.

The first subset is the subset of initial states $x \in K$ such that the boundary ∂K is reached before T by all solutions $x(\cdot)$ to the differential inclusion (5.2) starting at x.

The second subset is the subset of initial states $x \in K$ such that all solutions $x(\cdot)$ to the differential inclusion (5.2) starting at x remain in K for all $t \in [0, T]$.

We then observe that the invariance kernel is equal to

$$\mathrm{Inv}_F(K) = \bigcap_{T \geq 0} \left\{ x \in K \mid \tau_K^\flat(x) \geq T \right\}$$

5.4.5 Invariance Envelopes

One can relate invariance envelopes with the accessibility map:

Proposition 5.4.7 *Assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty closed values. Then the invariance envelope and the accessibility map are related by*

$$\mathrm{Env}_F(K) = \overline{\mathcal{R}_F(K)}$$

Proof — The subset $\mathcal{R}_F(K)$ is obviously contained in any closed invariant subset M containing K and in particular, in the invariance envelope of K.

Conversely, it is enough to prove that $\overline{\mathcal{R}_F(K)}$ is invariant. If not, there would exist $x_0 \in \overline{\mathcal{R}_F(K)}$, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ and $T > 0$ such that $x(T)$ does not belong to $\overline{\mathcal{R}_F(K)}$. Let $\varepsilon > 0$ be such that

$$B(x(T), \varepsilon) \cap \overline{\mathcal{R}_F(K)} = \emptyset$$

By Filippov's Theorem 5.3.1, there exists $\delta > 0$ such that for every $y_0 \in B(x_0, \delta)$, one can find a solution $y(\cdot) \in \mathcal{S}_F(y_0)$ starting from y_0 such that

$$y(T) \in B(x(T), \varepsilon) \subset X \setminus \overline{\mathcal{R}_F(K)}$$

Since x_0 belongs to the closure of $\mathcal{R}_F(K)$, one can choose such an initial state y_0 in $\mathcal{R}_F(K)$, so that there exists $z_0 \in K$, a solution $z(\cdot) \in \mathcal{S}_F(z_0)$ and $T_0 > 0$ satisfying $z(T_0) = y_0$. We then introduce the concatenation $\widetilde{y}(\cdot)$ defined by

$$\widetilde{y}(s) := \begin{cases} z(s) & \text{if } 0 \le s \le T_0 \\ y(s - T_0) & \text{if } T_0 \le s < \infty \end{cases}$$

Therefore $\widetilde{y}(\cdot) \in \mathcal{S}_F(z_0)$ is a solution starting from K such that $\widetilde{y}(T + T_0) = y(T)$, so that $y(T)$ belongs to $\mathcal{R}_F(K)$, a contradiction. □

Proposition 5.4.8 *Assume that $F : X \rightsquigarrow X$ is Lipschitz with nonempty closed values and that $K = \overline{\mathrm{Int}(K)}$. Then*

$$\mathrm{Env}_F(K) = \overline{X \setminus \mathrm{Inv}_{-F}(\widehat{K})} \text{ where } \widehat{K} := \overline{X \setminus K}$$

Proof — Since these two sets contain K, it is enough to prove the equality for the elements outside of K.

Let x_0 be outside of both K and $\mathrm{Inv}_{-F}(\widehat{K})$. We infer that there exists a solution $x(\cdot) \in \mathcal{S}_{-F}(x_0)$ and $T > 0$ such that $x(T) \in X \backslash \widehat{K} = \mathrm{Int}(K)$. Let us associate with a solution $y(\cdot) \in \mathcal{S}_F(x_0)$ the solution $\widetilde{y}(\cdot) \in \mathcal{S}_F(x(T))$ defined by

$$\widetilde{y}(s) := \begin{cases} x(T - s) & \text{if } 0 \le s \le T \\ y(s - T) & \text{if } T \le s < \infty \end{cases}$$

which thus satisfies $\widetilde{y}(T) = x_0 \in \mathcal{R}_F(\mathrm{Int}(K))$. Proposition 5.4.7 implies that the latter subset is contained in $\mathrm{Env}_F(K)$.

Conversely, let y belong to $\mathrm{Int}(\mathcal{R}_F(K)) \backslash K$. Since the interior of $\mathcal{R}_F(K)$ is equal to $\mathcal{R}_F(\mathrm{Int}(K))$ by Proposition 5.3.7, there exist $x_0 \in \mathrm{Int}(K)$, a solution $x(\cdot) \in \mathcal{S}_F(x_0)$ and $T > 0$ such that $y = x(T) \in X \backslash K$. We then associate with a solution $y(\cdot) \in \mathcal{S}_F(x_0)$ the solution $\widetilde{y}(\cdot) \in \mathcal{S}_F(y)$ defined by

$$\widetilde{y}(s) := \begin{cases} x(T - s) & \text{if } 0 \le s \le T \\ y(s - T) & \text{if } T \le s < \infty \end{cases}$$

which thus satisfies $\widetilde{y}(T) = x_0 \in \mathrm{Int}(K)$. Hence such a solution is not viable in $\widehat{K} = X\backslash\mathrm{Int}(K)$ and thus, $y = x(T)$ does not belong to the invariance kernel of $\widehat{K}$, so that we have proved that

$$\mathrm{Int}(\mathcal{R}_F(K)) \subset \overline{X \setminus \mathrm{Inv}_{-F}(\widehat{K})}$$

We conclude, thanks to Proposition 5.4.7. □

5.5 Boundaries of Viability and Invariance Kernels

5.5.1 Semipermeability of the Boundary of the Viability Kernel

We shall prove in this section that if the solution map is lower semicontinuous, then every viable solution starting on the boundary of the exit tube (respectively the viability kernel) remains on it.

Theorem 5.5.1 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$.*

Then, if

$$x \in \partial(\mathrm{Exit}_F(K,T)) \cap \mathrm{Limsup}_{t\to T-}\left(\mathrm{Exit}_F(K,t)\backslash\mathrm{Exit}_F(K,T)\right)$$

every solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0,T]$ remains on the boundary of the exit tube:

$$\forall\, t \in [0,T], \quad x(t) \in \partial\left(\mathrm{Exit}_F(K,T-t)\right)$$

Proof — Let $x(\cdot) \in \mathcal{S}_F(x)$ be a solution viable in K on $[0,T]$, which exists by assumption, and which thus satisfies

$$\forall\, t \in [0,T], \quad x(t) \in \mathrm{Exit}_F(K,T-t)$$

Also by assumption, there exists a sequence of $T_n < T$ converging to T and a sequence of elements $x_n \in \mathrm{Exit}_F(K,T_n)\backslash\mathrm{Exit}_F(K,T)$ converging to x.

Since the solution map is assumed to be lower semicontinuous, there exist solutions $x_n(\cdot)$ to the differential inclusion (5.2) starting at x_n defined on $[0,T]$ converging pointwise to $x(\cdot)$. On the other hand, by Proposition 4.2.8, we know that for any $t \in [0,T_n]$,

$$x_n(t) \in \mathrm{Exit}_F(K,T_n - t)\backslash\mathrm{Exit}_F(K,T-t)$$

Consequently, by passing to the limit, we obtain for all $t \in [0,T]$,

$$x(t) \in \mathrm{Exit}_F(K,T-t) \cap \overline{X\backslash\mathrm{Exit}_F(K,T-t)} = \partial\left(\mathrm{Exit}_F(K,T-t)\right)$$

i.e., the solution remains in the boundary of the exit tube. □

By using Proposition 4.2.9 instead of Proposition 4.2.8 in the proof of Theorem 5.5.1, we obtain the following statement:

Theorem 5.5.2 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ and that* $\mathrm{Exit}_F(K,T)$ *is contained in the interior of K. Then, if $x \in \partial(\mathrm{Exit}_F(K,T))$, every solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K on $[0,T]$ remains on the boundary of the exit tube:*

$$\forall\, t \in [0,T], \quad x(t) \in \partial\left(\mathrm{Exit}_F(K,T-t)\right)$$

For $T = +\infty$, we obtain the following consequence:

Theorem 5.5.3 (Quincampoix) *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a closed subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ and that the viability kernel of K is contained in the interior of K. Then the viability kernel is* semipermeable *in the sense that if $x \in \partial(\mathrm{Viab}_F(K))$, every solution $x(\cdot) \in \mathcal{S}_F(x)$ viable in K remains in the boundary of the viability kernel.*

In other words, this means that every solution starting from the boundary of the viability kernel can either remain in the boundary or leave the viability kernel, or equivalently, that no solution starting from outside of the viability kernel can cross its boundary: such solutions can only remain on the boundary of the viability kernel, or leave it.

5.5.2 Viability of the Boundary of the Invariance Kernel

In a symmetric way, we can prove that the boundary of the invariance kernel is viable:

Theorem 5.5.4 (Quincampoix) *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $K \subset X$ be a compact subset. Assume that the solution map $\mathcal{S}_F$ is lower semicontinuous from K to $\mathcal{C}(0,\infty;X)$ and that the invariance kernel of K is contained in the interior of K. Then, the boundary $\partial(\mathrm{Inv}_F(K))$ is viable under F.*

Proof — Let x_0 belong to $\partial(\mathrm{Inv}_F(K))$ and consider a sequence of elements $x_n \in K\backslash\mathrm{Inv}_F(K)$ converging to x_0.

By Proposition 5.4.3, we know that $X\backslash\mathrm{Inv}_F(K)$ is viable under F: there exist solutions $x_n(\cdot)$ to differential inclusion (5.2) starting at x_n which are viable in $X\backslash\mathrm{Inv}_F(K)$.

Since F is a Marchaud map, we infer from Theorem 3.5.2 that a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x(\cdot) \in \mathcal{S}_F(x_0)$ which is viable in the closure of the complement of $\mathrm{Inv}_F(K)$. Theorem 5.4.2 implies that this solution is also viable in the invariance kernel of K, and thus, that it is viable in the boundary of $\partial(\mathrm{Inv}_F(K))$. □

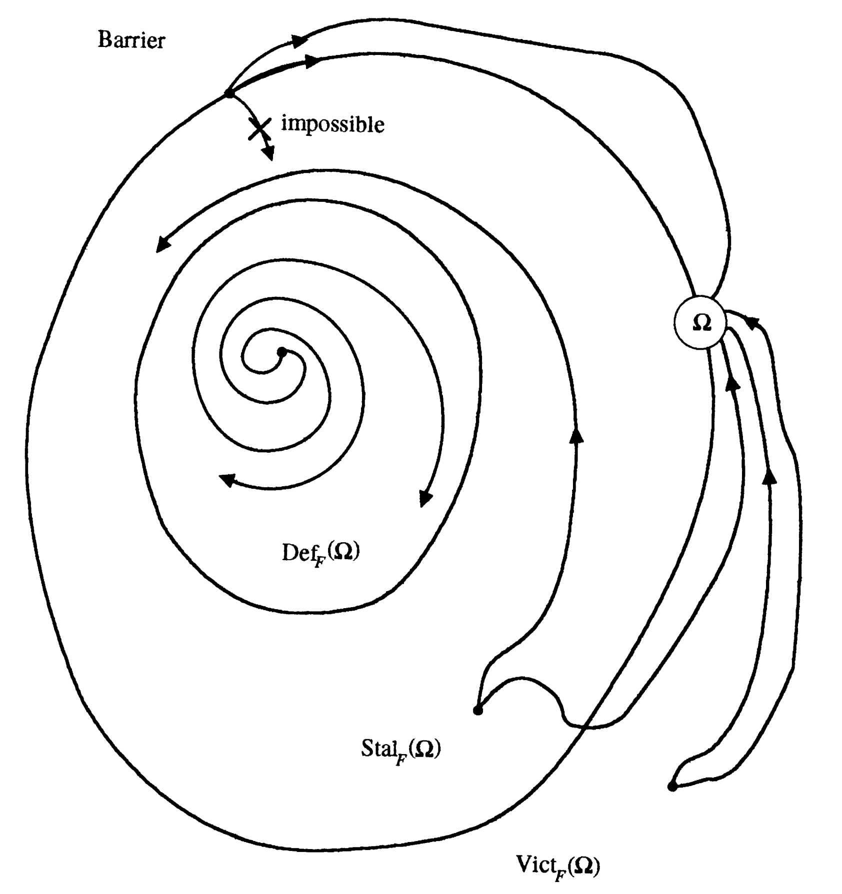
5.6 Defeat and Victory domains of a Target and its Barrier

We can apply the above theorems to the complement of an open target Ω. Let us introduce the following notations:

$$\begin{cases} i) & \mathrm{Defeat}_F(\Omega) := \mathrm{Inv}_F(X\backslash\Omega) \\ ii) & \mathrm{Stal}_F(\Omega) := \mathrm{Viab}_F(X\backslash\Omega) \setminus \mathrm{Inv}_F(X\backslash\Omega) \\ iii) & \mathrm{Vict}_F(\Omega) := X\backslash\mathrm{Viab}_F(X\backslash\Omega) \end{cases}$$

Theorem 5.6.1 (Quincampoix) *Let $F : X \rightsquigarrow X$ be a Marchaud and Lipschitz map. Consider an open target $\Omega \subset X$. Then*

Figure 5.1: Victory and Defeat Domains



The target Ω, the defeat domain, the barrier and the victory domains are symbolically represented.

1. $\text{Defeat}_F(\Omega)$ *is the* defeat domain*:* $\forall\ x_0 \in \text{Defeat}_F(\Omega)$*, every solution starting from* x_0 *never reaches the target* Ω

2. $\text{Vict}_F(\Omega)$ *is the* victory domain: $\forall\ x_0 \in \text{Vict}_F(\Omega)$*, every solution reaches the target* Ω *in finite time*

3. $\text{Stal}_F(\Omega)$ *is the* stalemate domain: $\forall\ x_0 \in \text{Stal}_F(\Omega)$

   - *there exists one solution which never reaches the target* Ω
   - *there exists one solution hitting the target* Ω

4. $\partial\,(\text{Viab}_F(X\backslash\Omega))$ *is the* barrier: $\forall\ x_0 \in \partial\,(\text{Viab}_F(X\backslash\Omega))$*, there exists a solution which is viable in the barrier as long as it does not reach the target* Ω*, and no solution enters the interior of* $\text{Stal}_F(\Omega)$

5. $\partial\,(\text{Defeat}_F(\Omega))$ *is viable under* F

We can also introduce

$$\text{Vict}_F(\Omega, T) \ := \ X\backslash\text{Exit}_F(X\backslash\Omega, T)$$

which is the open subset

$$\text{Vict}_F(\Omega, T) \ = \ \left\{x \quad \text{such that } \tau^{\sharp}_{X\backslash\Omega}(x) < T\right\}$$

of initial states from which all solutions reach the target Ω before T.

We deduce that the victory domain is equal to:

$$\text{Vict}_F(\Omega) \ = \ \bigcup_{T>0} \text{Vict}_F(\Omega, T)$$

The subset

$$\begin{cases} \text{vict}_F(\Omega, T) \ := \ \text{Hit}_F(X\backslash\Omega, T) \\ = \left\{x \notin \Omega \mid \exists\, x(\cdot) \in \mathcal{S}_F(x), \ \exists\, t \in [0, T] \ \text{ such that } x(t) \in \overline{\Omega}\right\} \end{cases}$$

is the set of initial states such that at least one solution to the differential inclusion reaches the closure of Ω at some $t \leq T$.

For compact targets C, we obtain the following characterization:

Proposition 5.6.2 *Let $F : X \rightsquigarrow X$ be a strict Marchaud map and $C \subset X$ be a nonempty compact subset. The set*

$$\{x \notin C \mid \exists\, x(\cdot) \in \mathcal{S}_F(x),\ \ \exists\, t \in [0,T] \quad \text{such that } x(t) \in C\}$$

of initial states such that at least one solution to the differential inclusion reaches the target C at some $t \leq T$ is equal to

$$\bigcup_{\eta>0} \bigcap_{0<\varepsilon\leq\eta} \mathrm{vict}_F(\overset{\circ}{B}(C,\varepsilon),T)$$

Proof — Let us choose $x \notin C$. Then we know that there exists $\eta > 0$ such that for any $\varepsilon \leq \eta$, there exists at least one solution $x_\varepsilon(\cdot) \in \mathcal{S}_F(x)$ to the differential inclusion reaching the ball $B(C,\varepsilon)$ at some $t_\varepsilon \leq T$, thanks to the above remark with $\Omega := \overset{\circ}{B}(C,\varepsilon)$. Since $\mathcal{S}_F(x)$ is compact in $\mathcal{C}(0,\infty;X)$ supplied with the compact convergence topology, subsequences (again denoted by) $x_\varepsilon(\cdot)$ and t_ε converge to $x(\cdot) \in \mathcal{S}_F(x)$ and $t \in [0,T]$ respectively, so that the limit $x(t)$ of $x_\varepsilon(t_\varepsilon) \in B(C,\varepsilon)$ belongs to the closed target C. □

5.7 Linear Differential Inclusions

5.7.1 Viability Cones

Let us consider the case when the right-hand side of the differential inclusion is a closed convex process. Since closed convex processes are set-valued analogues of continuous linear operators, it is legitimate to call such differential inclusions *linear differential inclusions*.

The domain of a closed convex process being a convex cone, it is quite natural to restrict the class of viability domains of closed convex processes to closed convex cones.

Theorem 5.7.1 (Linear Differential Inclusions) *Let X be a finite dimensional vector-space, $F : X \rightsquigarrow X$ be a closed convex process and $K \subset X$ be a closed convex cone. We posit the following assumptions:*

$$\begin{cases} i) & \forall\, x \in K,\ \ R(x) := F(x) \cap (\overline{K + \mathbf{R}x}) \neq \emptyset \\ ii) & \text{the norm (see Definition 2.5.3) of } \|R\| \text{ is finite} \end{cases}$$

Then, for any initial state $x_0 \in K$, there exists a solution $x(\cdot)$ to the linear differential inclusion

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t)) \tag{5.5}$$

starting at x_0 and viable in the cone K.

Proof — It is a direct consequence of the Second Viability Theorem 3.3.6 and formula

$$T_K(x) = \overline{K + \mathbf{R}x}$$

since, by the very definition of the norm of R, we have:

$$\forall\, x \in K, \quad d(0, F(x) \cap T_K(x)) \leq \|R\|\|x\| \quad \square$$

Hence it remains to prove the following

Lemma 5.7.2 *Let $K \subset X$ be a convex cone of a normed space X and $x \in K$. Then*[10] $T_K(x) = \overline{K + \mathbf{R}x}$.

The proof is left as an exercise (see also Lemma 4.2.5 of SET-VALUED ANALYSIS.) □

Example — Let $A \in \mathcal{L}(X, X)$ be a linear operator and $P \subset X$ and $Q \subset X$ be closed convex cones. Then the set-valued map F defined by

$$F(x) := Ax + Q \text{ if } x \in P \ \& \ \emptyset \text{ if not} \tag{5.6}$$

is a closed convex process. We then deduce a useful corollary for linear control systems with inequality constraints on both the state and the control variables:

Corollary 5.7.3 *Let X be a finite dimensional vector-space, $A \in \mathcal{L}(X, X)$ be a linear operator and $P \subset X$ and $Q \subset Y$ be closed convex cones. If*

$$\begin{cases} i) & \forall\, x \in P, \ Ax \in \overline{P + \mathbf{R}x} - Q \\ ii) & \exists\, c > 0 \ \text{ such that } \inf_{u \in \left(\overline{P + \mathbf{R}x}\right) \cap (Q + Ax)} \|u\| \leq c\|x\| \end{cases}$$

then, for any initial state $x_0 \in P$, there exists a solution to the differential equation $x'(t) = Ax(t) + u(t)$, where $u(t) \in Q$, which is viable in the closed convex cone P. □

[10] If we assume that $K^- + \{x\}^- = X^\star$ and that X is reflexive, then $T_K(x) = K - \mathbf{R}_+ x$ thanks to Closed Range Theorem 2.3.4.

5.7.2 Projection on the sphere

We shall "project" the solutions $x(\cdot)$ onto the unit sphere Σ. We shall show that the evolutions of these projections are governed by a differential inclusion the right-hand side of which is the "projection" of the linear differential inclusion onto the tangent space to this sphere defined in the following way: we associate with any $y \in \Sigma$ the orthogonal projector $\pi(y)$ onto the tangent space $T_\Sigma(y)$ to Σ at y defined by

$$\pi(y)z \;:=\; z - \langle y, z \rangle y$$

We observe the following property:

Lemma 5.7.4 *If K is a convex cone of a finite dimensional vector-space X, then, for any $y \in K \cap \Sigma$, $\pi(y)T_K(y) \subset T_{K\cap\Sigma}(y)$.*

Proof — Let $z \in T_K(y)$. We already know that $\pi(y)z$ belongs to $T_\Sigma(y)$. It belongs to $T_K(y)$ because

$$\pi(y)z \;=\; z - \langle y, z \rangle y \in \overline{K + \mathbf{R}y} + \mathbf{R}y \;\subset\; \overline{K + \mathbf{R}y}$$

Then it belongs to the intersection of $T_\Sigma(y) = \{y\}^\perp$ and $T_K(y)$. It is equal to $T_{K\cap\Sigma}(y)$ (see Table 5.2), because the transversality condition $T_\Sigma(y) - T_K(y) = X$ is satisfied since we can write

$$\forall\, z \in X, \;\; z \;=\; \pi(y)z + \langle y, z \rangle y \;\; \square$$

We now associate with a closed convex process $F : X \leadsto X$ its "projection" defined by

$$H(y) \;:=\; \pi(y)(F(y) \cap \|R\|B)$$

It is obviously a set-valued map with closed convex images contained in the ball $\|R\|B$, which is compact.

We deduce from the above lemma that *if a closed convex cone K is a viability domain of F, then $K \cap \Sigma$ is a viability domain of its projection H.* This implies the following consequence:

Proposition 5.7.5 *We posit the assumptions of Theorem 5.7.1.*

Then $x(\cdot)$ is a never vanishing viable solution to linear differential inclusion (5.5) if and only if $y(\cdot) := x(\cdot)/\|x(\cdot)\|$ is a solution to the projected differential inclusion

$$\text{for almost all } t \geq 0, \;\; y'(t) = \pi(y(t))z(t) \;\; \text{where} \;\; z(t) \in F(y(t))$$

viable in $K \cap \Sigma$ and we can write:

$$x(t) = y(t)\|x_0\|e^{\int_0^t \langle y(\tau), z(\tau) \rangle d\tau}$$

Proof — The proof follows easily from the relation

$$y(t) = \frac{x(t)}{\|x(t)\|} \ \& \ z(t) = \frac{x'(t)}{\|x(t)\|}$$

and the property

$$\frac{d}{dt}\|x(t)\| = \|x(t)\| < y(t), z(t) > \ \square$$

Remark — Let us introduce the constants

$$\lambda_- := \inf_{y\in\Sigma\cap K, v\in F(y)\cap\|R\|B} < v, y > \ \& \ \lambda_+ := \sup_{y\in\Sigma\cap K, v\in F(y)\cap\|R\|B} < v, y >$$

We deduce that the solutions $x(\cdot)$ obey the estimates

$$\|x_0\|e^{\lambda_- t} \leq \|x(t)\| \leq \|x_0\|e^{\lambda_+ t} \ \square$$

We deduce that if $\lambda_+ < 0$, then the origin is an *attractor* and that if $\lambda_- > 0$, the origin is a *source* of the system.

5.7.3 Projection on a compact sole

It may be advantageous to project a linear differential inclusion on the sole of a cone instead of the sphere, if one needs convexity, for instance. In particular, this allows us to prove that a closed convex process F does have an eigenvector in cones with compact soles.

We associate with the closed convex cone K and an element $p_0 \in \mathrm{Int}(K^+)$ the "compact sole"

$$S := \{x \in K \mid < p_0, x > = 1\}$$

We associate with any element $y \in S$ the projector $\varpi(y)$ onto the orthogonal hyperplane to p_0, defined by

$$\forall\, z \in X, \ \varpi(y)z := z- < p_0, z > y$$

We then remark that:

Lemma 5.7.6 *If K is a convex cone with compact sole of a finite dimensional vector-space X, then, for any $y \in S$, $\varpi(y)T_K(y) \subset T_S(y)$.*

Proof — The tangent cone to the sole S of K is equal to

$$T_S(x) = \{v \in T_K(x) \mid < p_0, v >= 0\} \qquad (5.7)$$

since S can be written in the form $K \cap p_0^{-1}(1)$. Indeed, the constraint qualification assumption $0 \in \text{Int}(p_0(K) - 1)$ is satisfied because $p_0(K)$ is a cone of $\mathbf{R}$ containing 1. We then deduce from Table 5.1 that $T_S(x) = T_K(x) \cap p_0^{-1} T_{\{1\}}(1)$, i.e., formula (5.7).

We now check that

$$\forall\, y \in S, \;\; \varpi(y) T_K(y) \;\subset\; T_S(y) \tag{5.8}$$

Indeed, Lemma 5.7.2 implies that if $u \in T_K(y)$, then

$$\varpi(y)u := u - < p_0, u > y \;\in\; \overline{(K + \mathbf{R}y)} + \mathbf{R}y \subset \overline{(K + \mathbf{R}y)} = T_K(y)$$

(because K is a closed convex cone) and

$$< p_0, \varpi(y)u > = < p_0, u > - < p_0, u > < p_0, y > = 0$$

(because $< p_0, y > = 1$). We deduce that $\varpi(y)u$ belongs to $T_S(y)$ thanks to (5.7). □

Let us project the closed convex process F to the set-valued map G defined on the compact sole S by

$$G(y) \;:=\; \varpi(y)(F(y) \cap \|R\|B)$$

which is naturally a set-valued map with closed convex images contained in the ball $\|R\|B$, which is compact. Since its graph is closed, we deduce that G is upper semicontinuous from S to X. By (5.8), S is a viability domain of the set-valued map G since the cone K is a viability domain of the closed convex process F, so that:

Proposition 5.7.7 *We posit the assumptions of Theorem 5.7.1. Therefore $x(\cdot)$ is a never vanishing viable solution to linear differential inclusion (5.5) if and only if*

$$y(\cdot) \;:=\; x(\cdot) / < p_0, x(\cdot) >$$

is a solution to the projected differential inclusion

$$\text{for almost all } t \ge 0, \;\; y'(t) = \varpi(y(t)) z(t) \;\; \text{where} \;\; z(t) \in F(y(t))$$

viable in the sole S and we can write:

$$x(t) \;=\; y(t) < p_0, x_0 > e^{\int_0^t < y(\tau), z(\tau) > d\tau}$$

Proof — The solutions $x(\cdot)$ and $y(\cdot)$ are related by

$$y(t) \;=\; \frac{x(t)}{< p_0, x(t) >} \;\; \& \;\; z(t) \;=\; \frac{x'(t)}{< p_0, x(t) >} \quad \square$$

Since the compact sole is a compact viability domain of the projection G, the Equilibrium Theorem 3.7.6 implies the existence of eigenvectors:

Theorem 5.7.8 (Eigenvector of a Closed Convex Process) *Let X be a finite dimensional vector-space and $F : X \leadsto X$ be a closed convex process. Assume that a closed convex cone $K \subset X$ enjoys the following properties.*

$$\begin{cases} i) & K \text{ has a compact sole} \\ ii) & K \text{ is a viability domain of } F \\ iii) & \text{the norm } \|R\| \text{ is finite} \end{cases}$$

Then there exists a nonzero eigenvector $\bar{x} \in K$ *of the closed convex process F associated with an* eigenvalue $\bar{\lambda}$, *i.e., a solution to the problem*

$$\bar{x} \in K, \;\; \bar{x} \neq 0, \;\; \bar{\lambda} \in \mathbf{R} \;\; \& \;\; \bar{\lambda}\bar{x} \in F(\bar{x}) \tag{5.9}$$

The eigenvalue $\bar{\lambda}$ is therefore nonnegative whenever $F(K) \subset K$.

Proof — Indeed, there exists an equilibrium $\bar{x} \in S$ of G, i.e., a solution to $0 \in G(\bar{x})$, in other words, a solution to

$$\bar{x} \in S, \;\; 0 \; = \; \varpi(y)(\bar{x})\bar{y} \; = \; \bar{y} - <p_0, \bar{y}> \bar{x} \;\text{ where }\; \bar{y} \in F(\bar{x})$$

By setting $\bar{\lambda} := <p_0, \bar{y}>$, we see that the pair $(\bar{\lambda}, \bar{x})$ is a solution to inclusion (5.9). □

5.7.4 Duality between Viability and Invariance

Let us consider the case when the right-hand side of the differential inclusion is a closed convex process F whose domain is the whole space.

Then we know that F is Lipschitz and that its transpose $F^\star$ is upper semicontinuous with compact images on its domain $F(0)^+$.

Theorem 5.7.9 (Polar of a Viability Domain) *Let X be a finite dimensional vector-space, $F : X \leadsto X$ be a strict closed convex process and K be a closed convex cone. Then K is an invariance domain of F if and only if K^+ is a viability domain of its transpose:*

$$\begin{cases} i) & \forall\, x \in K, \quad F(x) \subset T_K(x) \\ & \Updownarrow \\ ii) & \forall\, q \in K^+, \quad F^\star(q) \cap T_{K^+}(q) \neq \emptyset \end{cases}$$

where $K^+ := -K^- = \{p \in X^\star \mid \forall\, x \in K, \; <p, x> \geq \; 0\}$.

We refer to Section 4.2 (Theorem 4.2.6) of SET-VALUED ANALYSIS for the proof of this Theorem. □

Chapter 6

Regulation of Control Systems

Introduction

In this chapter, we interpret viability theorems in the framework of *control systems with a priori feedbacks*[1] *and state constraints*. The dynamics (U, f) of the control system are described by

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases}$$

Observe that solutions to a control system are solutions to the differential inclusion $x'(t) \in F(x(t))$ where, for each state x, $F(x) := f(x, U(x))$ is the subset of feasible velocities. Conversely, a differential inclusion is an example of a control system in which the controls are the velocities ($f(x, u) = u$ & $U(x) = F(x)$).

Observe also that *whenever the controls obey state-dependent constraints, the control system can no longer be regarded as a family of*

[1] i.e., with state-dependent constraints on the controls. If we regard differential equation (i) as an "input-output map" associating with an input-control an output-state, inclusion (ii), which associates input-controls with output-states, "feeds back" the system. The a priori feedback relation is set-valued, otherwise, we no longer deal with a control problem. *Regulating the system* means looking for a subset of controls which provide solutions satisfying a given property, either optimality (optimal feedbacks) as in optimal control theory, or viability, which is the issue tackled here.

differential equations parametrized by an open loop control $u(\cdot)$, *but as a differential inclusion.*

The "state constraints" are described by the viability[2] subset K of states which satisfy them.

We shall associate with each viability domain K the *regulation map* $R_K \subset U$ associating with every state $x \in K$ the set

$$R_K(x) := \{u \in U(x) \mid f(x,u) \in T_K(x)\}$$

of *viable controls.*

Under adequate assumptions on U and f, Viability Theorem 3.3.5 states in essence that K is a viability domain if and only if the images $R_K(x)$ of the regulation map are not empty for all $x \in K$ and that the "open-loop" controls $u(\cdot)$ which regulate viable solutions obey the *regulation law*

$$\text{for almost all } t \geq 0, \quad u(t) \in R_K(x(t))$$

Hence, a deep knowledge of the regulation map, its regularity properties and its calculus (derived from the calculus of contingent cones) is in order.

Time-dependent controls are called "open-loop controls". What we are aiming at is the construction of "closed-loop" controls[3], i.e., single-valued maps $\tilde{r}(\cdot)$ which are *selections* of the regulation map in the sense that $\tilde{r}(x) \in R_K(x)$ for all $x \in K$. Then the solutions to the differential equation

$$x'(t) = f(x(t), \tilde{r}(x(t)))$$

(when they exist) are viable since the implemented controls $u(t) := \tilde{r}(x(t))$ obey the regulation law by construction. This will be the case when $\tilde{r}(\cdot)$ is continuous (thanks to Nagumo's Theorem), but may still be the case for discontinuous, but explicit, closed-loop feedbacks.

[2]For linear and smooth nonlinear control systems, the viability property has been introduced under the name of "controlled invariance" (and the invariance property under the name of "conditional invariance").

[3]The terminology comes from systems theory. Controls given by the regulation law can be regarded as mixed open-loop and closed-loop, because they still depend upon the state, but in a set-valued way instead of a deterministic manner.

Hence, we have to carry out two tasks. First, find *selection procedures* of the regulation map which provide either continuous selections (Michaels' Continuous Selection Theorem 6.5.7) or discontinuous selections for which the above differential equation still has solutions (section 4.) We shall see that these selection procedures require that the *regulation map should be lower semicontinuous with convex values.* Providing sufficient conditions for the regulation map to be lower semicontinuous is thus the second preliminary task.

Observe that this is not at all desperate, since we know that the set-valued map $x \rightsquigarrow T_K(x)$ which is involved in the definition of the regulation map is lower semicontinuous with convex values whenever K is *sleek*, and thus, whenever K is convex or smooth. Then, if we add the assumption that $U(\cdot)$ is also lower semicontinuous, one can expect R_K to be lower semicontinuous as well. This statement is true under further adequate assumptions (constraint qualification or transversality), as we show in the lower semicontinuity criteria that we prove in the second section and apply to regulation maps in the third one.

Finally, we build closed-loop controls in the fifth section. Michael's Theorem naturally provides the existence of continuous closed-loop controls, but, being proved in a nonconstructive way, *does not furnish algorithmic ways to construct them.*

On the other hand, we can think of explicitly selecting some controls of the regulation map, for instance, the control $r^\circ(x) \in R_K(x)$ with minimal norm. Viable solutions obtained with this closed-loop control are called *slow viable solutions.* Unfortunately, lower semicontinuity of the regulation map is not sufficient for implying the continuity of this minimal norm closed-loop control, so Nagumo's Theorem 1.2.3 cannot be used. Still, one can prove that slow viable solutions do exist, as well as the ones obtained by selection procedures involving optimization or game theoretical mechanisms.

6.1 Regulation Map

We translate the viability theorems in the language of Control Theory and continue our investigations in this framework. From now on, we introduce two finite dimensional vector-spaces:

1. — the state space X
2. — the control space Z

and a *feedback set-valued map* $U : X \rightsquigarrow Z$ associating with any state x the (possibly empty) subset $U(x)$ of feasible controls when the state of the system is x. In other words, we assume that the available *controls of the system are required to obey constraints which may depend upon the state.* We shall investigate later the cases when the controls depend also upon the time and/or the history of the solution to the system.

The dynamics of the system are further described by a (single-valued) map $f : \text{Graph}(U) \mapsto X$ which assigns to each state-control pair $(x, u) \in \text{Graph}(U)$ the *velocity* $f(x, u)$ of the state.

Hence the set

$$F(x) \quad := \quad \{f(x, u)\}_{u \in U(x)}$$

is the set of available velocities to the system when its state is x.

Definition 6.1.1 (Control System) *A* control system *denoted by* (U, f) *is defined by*

— *a feedback set-valued map* $U : X \rightsquigarrow Z$

— *a map* $f : \text{Graph}(U) \mapsto X$ *describing the dynamics of the system.*

The evolution of the system (U, f) *is governed by the differential inclusion*

$$\begin{cases} i) & \text{for almost all } t, \ x'(t) \ = \ f(x(t), u(t)) \\ ii) & \text{where } u(t) \ \in \ U(x(t)) \end{cases} \tag{6.1}$$

Let us remark that when we take for controls the velocities, i.e., when we take $Z = X$, $U = F$ and $f(x, u) = u$, we find the usual differential inclusions again.

The case when the feedback map U is constant, i.e., when $\forall x \in \text{Dom}(U)$, $U(x) = U$, is the most often used in the literature.

Definition 6.1.2 (Regulation Map) *Consider a system* (U, f) *described by a feedback map* U *and dynamics* f. *We associate with any subset* $K \subset \text{Dom}(U)$ *the* regulation map $R_K : K \rightsquigarrow Z$ *defined by*

$$\forall x \in K, \ \ R_K(x) \ := \ \{u \in U(x) \mid f(x, u) \in T_K(x)\}$$

Controls u *belonging to* $R_K(x)$ *are called* viable.

We observe that K *is a viability domain if and only if the regulation map* R_K *is strict (has nonempty values).*

It is convenient to introduce the following definition:

Definition 6.1.3 *We shall say that the system* (U, f) *is a* Marchaud control system *if it satisfies the following conditions:*

$$\left\{\begin{array}{ll} i) & \text{Graph}(U) \textit{ is closed} \\ ii) & f \textit{ is continuous} \\ iii) & \textit{the velocity subsets } F(x) \textit{ are convex} \\ iv) & f \textit{ and } U \textit{ have linear growth} \end{array}\right. \tag{6.2}$$

and that it is an affine control system *if*

$$\left\{\begin{array}{ll} i) & \forall (x,u) \in \text{Graph}(U), \quad f(x,u) := c(x) + g(x)u \\ ii) & \text{Graph}(U) \textit{ is closed and the images of } U \textit{ are convex} \\ iii) & c : \text{Dom}(U) \mapsto X \textit{ is continuous} \\ iv) & g : \text{Dom}(U) \mapsto \mathcal{L}(Z,X) \textit{ is continuous and bounded} \\ v) & c \textit{ and } U \textit{ have linear growth} \end{array}\right. \tag{6.3}$$

Naturally, affine control systems are Marchaud systems. In this case, the regulation map R_K is defined by

$$R_K(x) \;:=\; \{u \in U(x) \mid g(x)u \in T_K(x) - c(x)\} \tag{6.4}$$

Hence Viability Theorem 3.3.5 can be restated in the following form:

Theorem 6.1.4 (Viability Theorem) *Let us consider a Marchaud control system* (U, f)*. Then a closed subset* $K \subset \text{Dom}(U)$ *is viable*[4] *under* F *(or is controlled invariant) if and only if it is a* viability domain.

Furthermore, any "open loop" *control* $u(\cdot)$ *regulating a viable solution* $x(\cdot)$ *in the sense that*

$$\text{for almost all } t, \;\; x'(t) = f(x(t), u(t))$$

[4]This means that for any initial state $x_0 \in K$, there exists a solution on $[0, \infty[$ to the control system (6.1) *viable in* K.

obeys the regulation law

$$\text{for almost all } t, \;\; u(t) \in R_K(x(t)) \tag{6.5}$$

Otherwise, if K is not a viability domain of the control system, there exists a viability kernel Viab(K) *of K.*

Remark — The Filippov Measurable Selection Theorem[5] actually allows us to choose open loop controls obeying the regulation law (6.5) which are *measurable.* We shall also provide in Chapter 7 conditions implying the existence of open loop controls belonging to the space $W^{1,1}(0,\infty; X; e^{-bt}dt)$. □

Remark — Naturally, we can replace assumption (6.2)iv) by the weaker assumption (which depends upon the viability domain K)

$$c_K \;:=\; \inf_{x\in K} \frac{d(0, R_K(x))}{\|x\|+1} \;<\; +\infty \;\square \tag{6.6}$$

Therefore, using viability theorems amounts to proving that the regulation maps R_K have nonempty values and then, to exploiting the regulation law. This will be possible when the regulation map is lower semicontinuous with convex values. This is the topic of the next section.

But before, we shall illustrate some of the concepts introduced so far and new ones by a very simple dynamical economic model (one commodity, one consumer.)

6.2 A Simple Economic Example.

Let $K := [0, b]$ the subset of a scarce commodity x. Assume that the consumption rate of a consumer is equal to $a > 0$, so that, without any further restriction, its exponential consumption will leave the viability subset $[0, b]$. Hence its consumption is slowed down by a price which is used as a control. In summary, the evolution of its consumption is governed by the control system

$$\text{for almost all } t \geq 0, \;\; x'(t) = ax(t) - u(t), \;\; \text{where } \; u(t) \geq 0$$

[5] See Theorem 8.2.10 of SET-VALUED ANALYSIS for instance.

subjected to the constraints

$$\forall\, t \geq 0, \quad x(t) \in [0, b]$$

(See figure 6.1)

The a priori feedback map U is defined by $U(x) := \mathbf{R}_+$. Hence the regulation map is given by the formula

$$R_K(0) \;=\; \{0\}, \;\; R_K(x) = \mathbf{R}_+ \;\text{ when }\; x \in]0, b[\;\;\&\;\; R_K(b) = [ab, +\infty[$$

Its graph is not closed, and its closure is the graph of U, equal to $[0, b] \times \mathbf{R}_+$

We see at once that the viable equilibria of the system range over the *equilibrium line* $u = ax$. Viability is guaranteed each time that the price $u(t)$ is chosen in $R(x(t))$, i.e., $u = 0$ when $x = 0$ (and thus, the system cannot leave the equilibrium because negative prices are not allowed "to start" the system) and $u \geq ab$ when $x = b$, so that the price is large enough to stop or decrease consumption.

Assume that the system obeys the inertia principle: *it keeps the price constant as long as it works*. Take for instance $x_0 > 0$ and $u_0 \in [0, ax_0[$. Then the consumption increases[6] and when it reaches the boundary b of the interval, the system has to switch very quickly, actually instantaneously, to a velocity large enough to slow down the consumption for the solution to remain in the interval $[0, b]$. This would require impulse controls.

But there is a bound to the growth of prices (and inflation rates), so that we should set a bound on price velocities: $|u'(t)| \leq c$. *We shall associate with such a bound a "last warning" threshold to modify the price*: there is a level of consumption after which it will be impossible to slow down the consumption with a velocity smaller than or equal to c to forbid it to increase beyond the boundary b.

We shall find this threshold by introducing *heavy solutions* (which will be studied in full generality later in Chapter 7) for building this regulation law. They are the one whose controls evolve with the "smallest velocity". It may be useful to be acquainted with

[6]it is equal to $\frac{e^{at}(ax_0 - u_0) + u_0}{a}$.

this concept on an example, and this one illustrates well how heavy solutions evolve.

We thus consider the solutions to the system

$$\begin{cases} i) & \text{for almost all } t \geq 0, \;\; x'(t) = ax(t) - u(t) \\ ii) & \text{and } \;\; -c \leq u'(t) \leq c \end{cases} \tag{6.7}$$

which are viable in Graph(U) (which is the closure of the graph of R_K).

We may call them the c-bounded state-control solutions.

We next introduce the functions $\rho_c^\sharp$ and $\rho_c^\flat$ defined on $[0, \infty[$ by

$$\begin{cases} i) & \rho_c^\flat(u) \;:=\; \frac{c}{a^2}(e^{-au/c} - 1 + \frac{a}{c}u) \approx \frac{u^2}{2c} \\ ii) & \rho_c^\sharp(u) \;:=\; -ce^{a(u-ab)/c}/a^2 + u/a + c/a^2 \end{cases}$$

and the functions $r_c^\sharp$ and $r_c^\flat$ defined on $[0, b]$ by

$$\begin{cases} i) & r_c^\flat(x) = u \text{ if and only if } x = \rho_c^\flat(u) \\ ii) & r_c^\sharp(x) = 0 \text{ if } x \in [0, \rho_c^\sharp(0)] \;\; (\rho_c^\sharp(0) = \frac{c}{a^2}(1 - e^{-a^2b/c})) \\ iii) & r_c^\sharp(x) = u \text{ if and only if } x = \rho_c^\sharp(u) \text{ when } x \in [\rho_c^\sharp(0), b] \end{cases}$$

Proposition 6.2.1 *The viability kernel of the graph of U under the system (6.7) is the graph of the regulation map R^c defined by*

$$\forall\, x \in [0, b], \;\; R^c(x) = [r_c^\sharp(x), r_c^\flat(x)] \tag{6.8}$$

Proof — Indeed, set $u^\sharp(t) := u_0 + ct$ and $u^\flat := u_0 - ct$ and denote by $x^\sharp(\cdot)$ and $x^\flat(\cdot)$ the solutions starting at x_0 to differential equations $x' = ax - u^\sharp(t)$ and $x' = ax - u^\flat(\cdot)$ respectively. Then any solution $(x(\cdot), u(\cdot))$ to the system (6.7) satisfies $u^\flat(\cdot) \leq u(\cdot) \leq u^\sharp(\cdot)$ and thus, $x^\sharp(\cdot) \leq x(\cdot) \leq x^\flat(\cdot)$ because

$$x(t) = e^{at}x_0 - \int_0^t e^{a(t-s)}u(s)ds$$

We also observe that the equations of the curves $t \mapsto (x^\sharp(\cdot), u^\sharp(\cdot))$ and $t \mapsto (x^\flat(\cdot), u^\flat(\cdot))$ passing through (x_0, u_0) are solutions to the differential equations

$$d\rho_c^\sharp = \frac{1}{c}(a\rho_c^\sharp - u)du \;\;\&\;\; d\rho_c^\flat = -\frac{1}{c}(a\rho_c^\flat - u)du$$

Figure 6.1: Evolution of a Heavy Solution

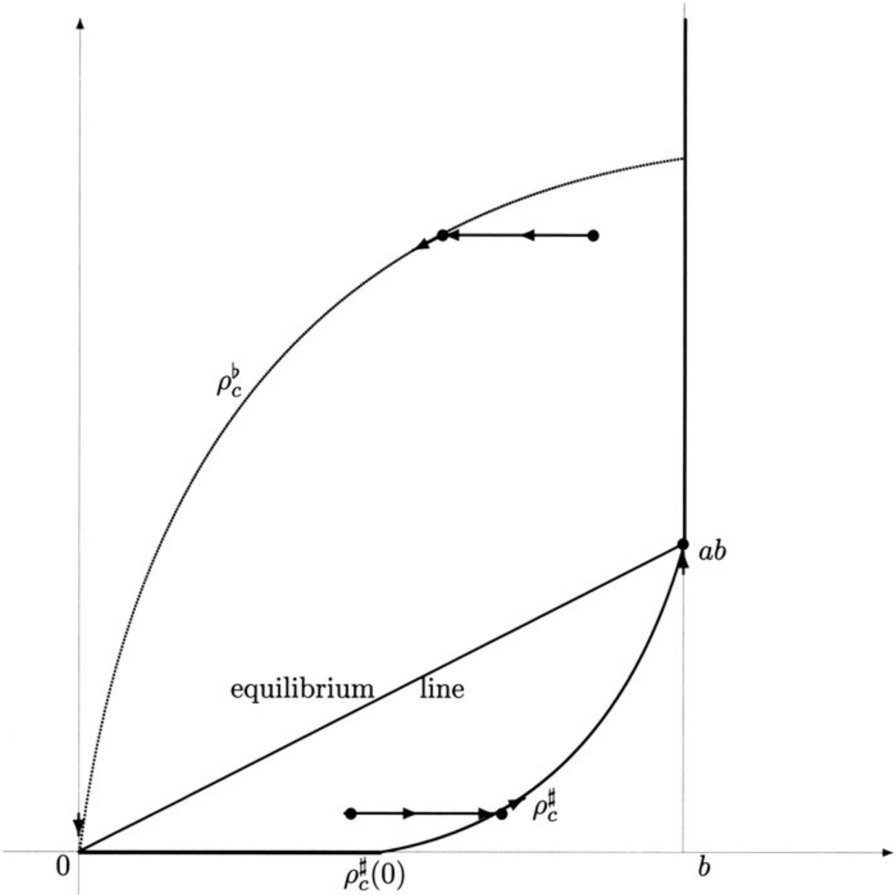
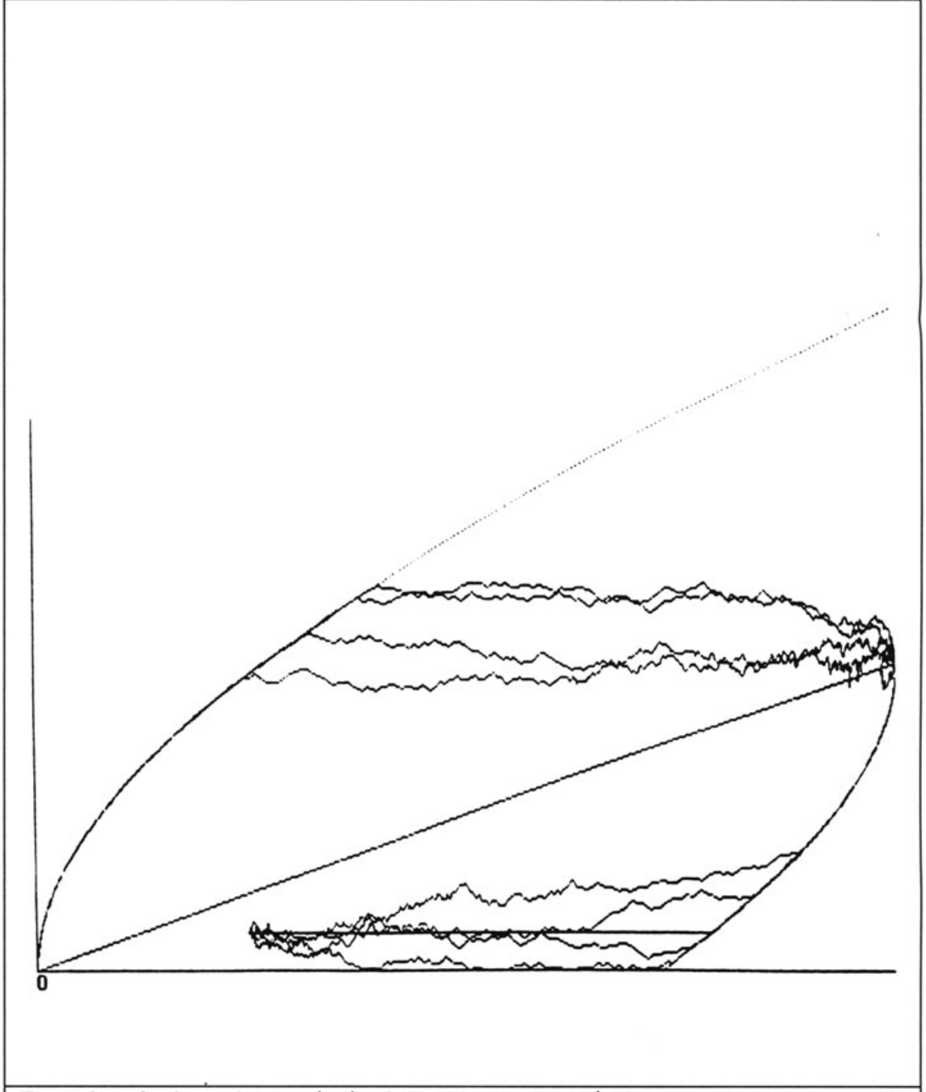


Figure 6.2: Other Solutions and Invariance of the Boundary



Examples of other solutions (6.7) where the velocities u' of the controls are randomly generated. This computer simulation (due to Morin & Vandanjon) illustrate the barrier properties of the part of the boundary of the viability kernel contained in the interior of Graph(U) (Theorem 5.5.3.)

the solutions of which are

$$\begin{cases} i) & \rho_c^\sharp(u) = e^{a(u-u_0)/c}(x_0 - u_0/a - c/a^2) + u/a + c/a^2 \\ ii) & \rho_c^\flat(u) = e^{a(u_0-u)/c}(x_0 - u_0/a + c/a^2) + u/a - c/a^2 \end{cases}$$

Let $\rho_c^\flat$ be the solution passing through $(0,0)$, which is equal to $\rho_c^\flat(u) = \frac{c}{a^2}(e^{-au/c} - 1 + \frac{a}{c}u)$ and $\rho_c^\sharp(u) = -ce^{a(u-ab)/c}/a^2 + u/a + c/a^2$ be the solution passing through the pair (ab, b).

— We check that the viability kernel is contained in the graph of R^c by contradiction.

If $u_0 > r_c^\flat(x_0)$, then any solution $(x(\cdot), u(\cdot))$ starting from (x_0, u_0) satisfies

$$x(t) \le x^\flat(t) = \rho_c^\flat(u^\flat(t)) \le \rho_c^\flat(u(t))$$

because $\rho_c^\flat(\cdot)$ is nondecreasing. Hence, when $x(t_1) = 0$, we deduce that $u(t_1) > 0$, so that such solution is not viable, and thus, (x_0, u_0) does not belong to the viability kernel.

If $0 \le u_0 < r_c^\sharp(x_0)$, any solution $(x(\cdot), y(\cdot))$ satisfies inequalities

$$x(t) \ge x^\sharp(t) = \rho_c^\sharp(u^\sharp(t)) \ge \rho_c^\sharp(u(t))$$

Therefore, when $x(t_1) = b$ for some time t_1, its velocity $x'(t_1) = ab - u(t_1)$ is positive, so that the solution is not viable.

— It remains to prove that the viability kernel is equal to the graph of R^c by constructing particular viable solutions starting from any point (x_0, u_0) of this graph. We choose the *heavy solutions*.

The equilibrium line $u = ax$ is contained in the viability kernel: if we start from an equilibrium, both the state and the controls can be kept constant.

We shall now investigate the cases when the initial control u_0 is below or above the equilibrium line.

Consider the case when $x_0 > 0$ and the price $u_0 \in [r_c^\sharp(x_0), ax_0[$. Since we want to choose the price velocity with minimal norm, we take $u'(t) = 0$[7] as long as the solution $x(\cdot)$ to the differential equation $x' = ax - u_0$ yields a consumption $x(t) < \rho_c^\sharp(u_0)$. When for some time t_1, the consumption $x(t_1) = \rho_c^\sharp(u_0)$, it has to be slowed down. Indeed,

[7] and realize in this case the dream of economists, who, despite the teachings of history, are looking for constant prices and commodities ...

otherwise $(x(t_1+\varepsilon), u_0)$ will be below the curve $\rho_c^\sharp$ and we saw that in this case, any solution starting from this situation will eventually cease to be viable. Therefore, prices should increase to slow down the consumption growth. The idea is to take the smallest velocity u' such that the vector $(x'(t_1), u')$ takes the state inside the graph of R^c: they are the velocities $u' \geq x'(t_1)/\rho_c^{\sharp'}(u_0)$. By construction, it is achieved by the velocity of $x^\sharp(\cdot)$, which is the highest one allowed to increase prices. Therefore, by taking

$$x(t) := x^\sharp(t) := e^{a(t-t_1)}(x(t_1)-u_0/a-c/a^2)+c(t-t_1)/a+u_0/a+c/a^2$$

and $u(t) := u_0 + c(t-t_1)$ for $t \in [t_1, t_1 + (ab - u_0)/c]$, we get a solution which ranges over the curve $x^\sharp(t) = \rho_c^\sharp(u^\sharp(t))$. This a heavy solution because, for the same reason as above, the smallest velocity of the price (which is unique along this curve) is chosen. According to the above differential equation, we see that $x(t)$ increases to b where it arrives with velocity 0 and the price increases linearly until it arrives at the equilibrium price ab. Since (b, ab) is an equilibrium, the heavy solution stays there: we take $x(t) \equiv b$ and $u(t) \equiv ab$ when $t \geq t_1+u_0/c$. So we have built a viable solution starting from (x_0, u_0). Therefore the region between the "curve $\rho_c^\sharp$" and the equilibrium line is contained in the viability kernel, i.e., the graph of R^c.

Consider now the case when $u_0 \in [ax_0, r_c^\flat(x_0)]$, where we follow the same construction of the heavy viable solution. We start by taking $u'(t) = 0$, and thus, $u(t) = u_0$, as long as the solution $x(\cdot)$ to the differential equation $x' = ax - u_0$, which decreases, satisfies $x(t) > \rho_c^\flat(u_0)$. Then, when $x(t_1) = \rho_c^\flat(u_0)$ for some t_1, we take

$$x(t) = x^\flat(t) := e^a(t-t_1)(x(t_1)-u_0/a+c/a^2)-c(t-t_1)/a+u_0/a-c/a^2$$

and $u(t) := u_0 - c(t-t_1)$ for $t \in [t_1, t_1+u_0/c]$ in order to avoid leaving the viability kernel. Finally, for $t \geq t_1 + u_0/c$, we take $x(t) \equiv 0$ and $u(t) \equiv 0$. This particular solution being viable, the pairs (x_0, u_0) where $u_0 \in [ax_0, r_c^\flat(x_0)]$ belong to the viability kernel. □

Remark — We observe that for any $x \in]0, b[$,

$$\lim_{c \to 0+} r_c^\flat(x) = \lim_{c \to 0+} r_c^\sharp(x) = ax, \quad \lim_{c \to \infty} r_c^\sharp(x) = 0 \ \& \ \lim_{c \to \infty} r_c^\flat(x) = +\infty$$

In other words, the graph of R^c starts from the equilibrium line when $c = 0$ and "converges" in some sense to the graph of U when $c \to +\infty$. □

Remark — One can also compute easily the regulation map based on Theorem 5.1.11. In this case, it is easy to observe that

$$R^{\diamond}(x) \;=\; [\max\{0, (1+a)x - b\}, (1+a)x]$$

We observe that $R^{\diamond} \subset R^c$ for any c satisfying

$$\rho_c^{\sharp}(0) \;\geq\; \frac{b}{a+1} \;\;\&\;\; r_c^{\flat}(b) \;\geq\; (a+1)b$$

6.3 Lower Semicontinuity Criteria

To proceed further, we need the regulation map to be lower semicontinuous with convex compact values.

We can always assume that U is lower semicontinuous and we know that the set-valued map $T_K(\cdot)$ is lower semicontinuous whenever the viability set is sleek (and, in particular, smooth or convex).

We thus need *lower semicontinuity criteria* to derive that the regulation map $R_K(\cdot)$ is lower semicontinuous. Therefore, we gather in this section the lower semicontinuity criteria which are useful for building *closed-loop* controls regulating viable solutions. We refer to SET-VALUED ANALYSIS for more details, although we provide most of the proofs for the convenience of the reader.

Proposition 6.3.1 *Consider a metric space X, two normed spaces Y and Z, two set-valued maps T and U from X to Y and Z respectively and a (single-valued) map f from $X \times Z$ to Y satisfying the following assumptions:*

$$\begin{cases} i) & T \text{ and } U \text{ are lower semicontinuous with convex values} \\ ii) & f \text{ is continuous} \\ iii) & \forall\, x,\;\; u \mapsto f(x,u) \;\text{ is affine} \end{cases}$$

We posit the following condition:

$\forall\, x \in X,\ \exists\, \gamma > 0,\ \delta > 0,\ c > 0,\ r > 0$ such that $\forall\, x' \in B(x, \delta)$ we have

$$\gamma B_Y \subset f(x', U(x') \cap rB_Z) - T(x')$$

Then the set-valued map $R : X \rightsquigarrow Z$ defined by

$$R(x) := \{u \in U(x) \mid f(x,u) \in T(x)\} \tag{6.9}$$

is lower semicontinuous with nonempty convex values.

Proof — Let us fix $u \in R(x)$ and a sequence x_n converging to x. Since U and T are lower semicontinuous and f is continuous, there exist sequences $u_n \in U(x_n)$ and $y_n \in T(x_n)$ converging to u and $f(x,u)$ respectively. Let us set $\varepsilon_n := \|f(x_n, u_n) - y_n\|$ and $\theta_n := \frac{\gamma}{\gamma + \varepsilon_n} \in]0,1[$. Then ε_n converges to 0. Since

$$\theta_n \varepsilon_n = (1 - \theta_n)\gamma$$

we deduce that

$$\begin{cases} \theta_n(f(x_n, u_n) - y_n) & \in & \theta_n \varepsilon_n B \\ & = & (1-\theta_n)\gamma B \\ & \subset & (1-\theta_n)(f(x_n, U(x_n) \cap rB_Z)) - T(x_n)) \end{cases}$$

Therefore, there exist $\widehat{u}_n \in U(x_n) \cap rB_Z$ and $\widehat{y}_n \in T(x_n)$ such that

$$f(x_n, \theta_n u_n + (1-\theta_n)\widehat{u}_n) = \theta_n y_n + (1-\theta_n)\widehat{y}_n$$

This implies that

$$v_n := \theta_n u_n + (1-\theta_n)\widehat{u}_n$$

belongs to $U(x_n)$ and that

$$u_n - v_n = (1-\theta_n)(u_n - \widehat{u}_n) \in (1-\theta_n)(r + \|u\| + 1)B$$

because $\|\widehat{u}_n\| \leq r$, $\|u_n\|$ is bounded, $U(x_n)$ and $T(x_n)$ are convex and f is affine with respect to u. Hence the elements $v_n \in R(x_n)$ converge to u given in $R(x)$. □

We state now another condition which is less symmetric.

Proposition 6.3.2 *Consider a metric space X, two normed spaces Y and Z, two set-valued maps T and U from X to Y and Z respectively and a (single-valued) map f from $X \times Z$ to Y such that*

$$\begin{cases} i) & U \text{ is lower semicontinuous with convex values} \\ ii) & f \text{ is continuous} \\ iii) & \forall\, x,\ \ u \mapsto f(x,u) \text{ is affine} \\ iv) & \forall\, x,\ \ T(x) \text{ is convex and its interior is nonempty} \\ v) & \text{the graph of the map } x \rightsquigarrow \operatorname{Int}(T(x)) \text{ is open} \end{cases}$$

We posit the following condition:

$$\forall\, x \in X,\ \exists\, u \in U(x) \text{ such that } f(x,u) \in \operatorname{Int}(T(x)) \tag{6.10}$$

Then the set-valued map R defined by (6.9) is lower semicontinuous with convex values.

Proof

1. — We introduce the set-valued map $S : X \rightsquigarrow Z$ defined by

$$S(x) \ := \ \{u \in U(x) \ \mid \ f(x,u) \in \operatorname{Int}(T(x))\} \subset R(x)$$

Assumption (6.10) implies that $S(x)$ is not empty. *We claim that S is lower semicontinuous.* Indeed, if $x_n \to x$ and if u belongs to $S(x) \subset U(x)$, there exists $u_n \in U(x_n)$ which converges to u because U is lower semicontinuous. Since

$$(x_n, f(x_n,u_n)) \text{ converges to } (x, f(x,u)) \in \operatorname{Graph}(\operatorname{Int}(T(\cdot)))$$

by continuity of f and since the graph of $\operatorname{Int}(T(\cdot))$ is open, the elements $f(x_n,u_n)$ belong to $\operatorname{Int}(T(x_n))$ for n large enough and thus, the elements u_n belong to $S(x_n)$ and converge to u.

2. — Convexity of $U(x)$ and $T(x)$ implies that $\overline{S(x)} = R(x)$. Indeed, let us fix $u \in R(x)$ and $u_0 \in S(x)$. Then $v_\theta := \theta u_0 + (1-\theta)u$ belongs to $S(x)$ when $\theta \in]0,1[$, because $T(x)$ is convex and $f(x,u_0)$ belongs to the interior of $T(x)$, so that for every $\theta \in]0,1[$,

$$f(x,u) + \theta(f(x,u_0) - f(x,u)) = f(x, u + \theta u_0 - \theta u) = f(x, v_\theta)$$

belongs to the interior of $T(x)$. Then u is the limit of v_θ when $\theta > 0$ converges to 0.

3. — The theorem ensues because the closure of any lower semicontinuous set-valued map is still lower semicontinuous. □

We now extend the lower semicontinuity criterion above to infinite intersection of set-valued maps.

Theorem 6.3.3 *Let us consider a metric space X, normed vector-spaces Y and Z and set-valued maps $F : X \times Y \rightsquigarrow Z$ and $H : X \rightsquigarrow Y$. We assume that*

$$\begin{cases} i) & F \text{ is lower semicontinuous with convex values} \\ ii) & H \text{ is upper semicontinuous with compact values} \end{cases}$$

and that there exist positive constants γ, δ, c such that for every single-valued map $e : Y \mapsto \gamma B$ we have

$$\forall\, x' \in B(x,\delta),\; cB \cap \bigcap_{y \in H(x')} (F(x',y) - e(y)) \neq \emptyset \tag{6.11}$$

Then the set-valued map $G : X \rightsquigarrow Z$ defined by

$$\forall\, x \in X,\; G(x) := \bigcap_{y \in H(x)} F(x,y)$$

is lower semicontinuous (with nonempty convex images).

Remark — When the set-valued map F is locally bounded (in the sense that it maps some neighborhood of each point to a bounded subset), we do not need the constant c and we can replace (6.11) by

$$\forall\, x' \in B(x,\delta),\; \bigcap_{y \in H(x')} (F(x',y) - e(y)) \neq \emptyset \;\square$$

Proof — Let us choose any sequence of elements $x_n \in \mathrm{Dom}(G)$ converging to x and $z \in G(x)$. We have to approximate z by elements $z_n \in G(x_n)$.

We introduce the following numbers:

$$e_n := \sup_{y \in H(x_n)} d(z, F(x_n,y))/2 \tag{6.12}$$

Now, let us choose for each $y \in H(x_n)$ an element $u_n(y) \in F(x_n,y)$ satisfying

$$\|z - u_n(y)\| \leq 2d(z, F(x_n,y)) \leq e_n$$

and set $\theta_n := \gamma/(\gamma + e_n)$. Consequently,

$$\theta_n(z - u_n(y)) \in \theta_n e_n B = (1-\theta_n)\gamma B$$

so that there exists $a_n(y) \in \gamma B$ such that

$$\theta_n(z - u_n(y)) = (1-\theta_n)a_n(y)$$

Therefore, assumption (6.11) implies the existence for all n large enough of elements $w_n \in cB$ and elements $v_n(y) \in F(x_n,y)$ such that $a_n(y) = v_n(y) - w_n$ for all $y \in H(x_n)$.

Hence we can write

$$\theta_n(z - u_n(y)) = (1-\theta_n)(v_n(y) - w_n)$$

So that the common value:

$$z_n := \theta_n z + (1-\theta_n)w_n = \theta_n u_n(y) + (1-\theta_n)v_n(y)$$

does not depend on y, belongs to all $F(x_n, y)$ (by convexity) and converges to z because

$$\|z - z_n\| \;=\; (1-\theta_n)\|z - w_n\| \;\leq\; (1-\theta_n)(\|z\| + c)$$

and because $1 - \theta_n = e_n/(\gamma + e_n)$ converges to 0 because e_n converges to 0 thanks to the following lemma. □

Lemma 6.3.4 *Let us assume that F is lower semicontinuous and that H is upper semicontinuous with compact images. Then the numbers e_n defined by (6.12) converge to* 0.

Proof — Since F is lower semicontinuous, Corollary 2.1.7 to the Maximum Theorem implies that the function

$$(x, y, z) \;\mapsto\; d(z, F(x,y))$$

is upper semicontinuous. Therefore, for any $\varepsilon > 0$ and any $y \in H(x)$, there exist an integer N_y and a neighborhood $\mathcal{V}_y$ of y such that

$$\forall\, y' \in \mathcal{V}_y,\; \forall\, n \geq N_y,\;\; d(z, F(x_n, y')) \;\leq\; \varepsilon \tag{6.13}$$

because $d(z, F(x,y)) = 0$. Hence the compact set $H(x)$ can be covered by p neighborhoods $\mathcal{V}_{y_i}$. Furthermore, H being upper semicontinuous, there exists an integer N_0 such that,

$$\forall\, n \geq N_0,\;\; H(x_n) \;\subset\; \bigcup_{i=1,\ldots,p} \mathcal{V}_{y_i}$$

Set $N := \max_{i=0,\ldots,p} N_{y_i}$. Then, for all $n \geq N$ and $y \in H(x_n)$, y belongs to some $\mathcal{V}_{y_i}$, so that, by (6.13), $d(z, F(x_n, y)) \leq \varepsilon$. Thus,

$$\forall\, n \geq N,\;\; e_n \;:=\; \sup_{y \in H(x_n)} d(z, F(x_n, y))/2 \;\leq\; \varepsilon/2$$

i.e., our lemma is proved. □

Remark — Theorem 6.3.1 can be extended to set-valued maps with nonconvex images: we state the following proved in Theorem 1.5.5 of SET-VALUED ANALYSIS:

Theorem 6.3.5 *Let $G : X \rightsquigarrow Z$ be a closed lower semicontinuous set-valued map from a metric space X to a Banach space Z and $f : X \times Y \mapsto Z$ a continuous (single-valued) map, where Y is another Banach space. Let us assume that f is differentiable with respect to y and that there exist constants $c > 0$ and $\eta > 0$ such that*

$$\left\{ \begin{array}{l} \forall\, x \in B(x_0, \eta),\ y \in B(y_0, \eta) \text{ and } z \in B(f(x_0, y_0), \eta) \cap G(x) \\ B_Z \subset c f'_y(x, y)(B_Y) - T_{G(x)}(z) \end{array} \right. \tag{6.14}$$

Then the set-valued map R defined by

$$R(x) := \{y \in Y \mid f(x, y) \in G(x)\}$$

is lower semicontinuous at x_0.

6.4 Lower Semicontinuity of the Regulation Map

6.4.1 General State Constraints

From now on, we shall assume that the viability domain K is sleek, i.e., that

$$\text{the set-valued map } x \rightsquigarrow T_K(x) \text{ is lower semicontinuous}$$

We recall that convex and smooth manifolds are sleek and that in this case *the contingent cones are convex.*

We shall derive from these properties that under adequate conditions, *the regulation map has convex images and is lower semicontinuous.* These further properties of the regulation map will allow us to provide sufficient conditions for checking that a closed subset K is a viability domain and to devise selection procedures of slow solutions as well as other kinds of selection procedures.

Theorem 6.4.1 *Assume that the control system is affine and that K is a closed* sleek *viability domain. Then the regulation map R_K has compact convex values.*

Let us assume furthermore that the set-valued map U is lower *semicontinuous and that*[8]

$$\begin{cases} \forall x \in K, \ \exists \gamma > 0, \delta > 0 \ \textit{such that} \ \forall x' \in B_K(x, \delta), \\ \gamma B \ \subset \ c(x') + g(x')(U(x') \cap c_K B) - T_K(x') \end{cases}$$

Then the regulation map is lower semicontinuous. *In this case, the support function of the regulation map is equal to:*

$$\sigma(R_K(x), p) = \inf_{q \in N_K^\circ(x)} (\sigma(U(x), p - g(x)^\star q) - < q, c(x) >)$$

6.4.2 Output Dependent Constraints

We consider the case when the viability domain $K := h^{-1}(M)$ is defined by more explicit constraints through a map h from X to an *output space* or *observation space* Y: we introduce three finite dimensional vector spaces:

1. — the state space X
2. — the output or observation space Y
3. — the control space Z

and we define the viability subset by the constraints

$$K := h^{-1}(M)$$

where we assume that the *observation map* h satisfies

$$\begin{cases} i) & h \text{ is a } \mathcal{C}^1\text{-map from } X \text{ to } Y \\ \\ ii) & \forall x \in K, \ \ Y = \operatorname{Im}(h'(x)) - T_M(h(x)) \end{cases}$$

Let us recall that in this case[9]:

$$T_K(x) \;\; = \;\; h(x)'^{-1} T_M(h(x))$$

[8]or that the interior of the contingent cones are not empty and

$$\forall x \in K, \ \exists u \in U(x) \cap c_K B \ \mid \ c(x) + g(x)u \in \operatorname{Int} T_K(x)$$

[9]If we assume furthermore that there exists a positive constant c such that

$$\forall \, x \in h^{-1}(M), \ \ B_Y \ \subset \ h'(x)(cB_X) - T_M(h(x))$$

then $h^{-1}(M)$ is also sleek.

The regulation map R_K can be written:

$$R_K(x) \quad := \quad \{u \in U(x) \mid h'(x)g(x)u \in T_M(h(x)) - h'(x)c(x)\}$$

By replacing K by M, $g(x)$ by $B(x) := h'(x)g(x)$ and $c(x)$ by $b(x) := h'(x)c(x)$, we obtain the following corollary:

Corollary 6.4.2 *Assume that the control system is affine and that the constraints satisfy*

$$\begin{cases} i) & M \text{ is a closed sleek subset of } Y \\ ii) & h \text{ is a } C^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K, \ \ Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \\ iv) & \forall x \in h^{-1}(M), \ \exists u \in U(x) \text{ such that} \\ & h'(x)g(x)u \in T_M(h(x)) - h'(x)c(x) \end{cases}$$

Then the regulation map R_K has compact nonempty convex values. Let us assume furthermore that the set-valued map U is lower semicontinuous and that[10]

$$\begin{cases} \forall x \in K, \ \ \exists \gamma > 0, \delta > 0 \text{ such that } \forall x' \in B_K(x,\delta), \\ \gamma B \subset h'(x')c(x') + h'(x')g(x')(U(x') \cap c_K B) - T_M(h(x')) \end{cases}$$

Then the regulation map is lower semicontinuous *and its support function is equal to:*

$$\sigma(R_K(x),p) = \inf_{q \in N_M^\circ(h(x))} (\sigma(U(x), p - g(x)^\star h'(x)^\star q) - < q, h'(x)c(x) >)$$

We also remark that checking whether $h^{-1}(M)$ is a viability domain amounts to solving for all $x \in K$ the inclusions

$$\text{find } u \in U(x) \text{ satisfying } 0 \in h'(x)c(x) + h'(x)g(x)u - T_M(h(x))$$

Hence we can use Theorem 3.7.6 to derive sufficient conditions for $h^{-1}(M)$ to be a viability domain.

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[10] or that the interior of the contingent cones are not empty and

$$\forall x \in K, \ \exists u \in U(x) \cap c_K B \text{ such that } h'(x)(c(x) + g(x)u) \in \mathrm{Int} T_M(h(x)) \,.$$

Proposition 6.4.3 *Let us assume that the control system is affine, that the values of the feedback map U are compact and that*

$$\begin{cases} i) & M \text{ is a closed sleek subset of } Y \\ ii) & h \text{ is a } \mathcal{C}^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K,\ Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \\ iv) & h^{-1}(M) \subset \mathrm{Dom}(U) \end{cases}$$

Assume furthermore that

$$\begin{cases} \text{there exists a continuous map } B : \mathrm{Graph}(U) \mapsto \mathcal{L}(Z,Y) \\ \text{such that } \forall x \in K,\ \ \forall u \in U(x), \\ h'(x)(c(x) + g(x)u) \in T_M(h(x)) + B(x,u)T_{U(x)}(u) \end{cases}$$

Then $K := h^{-1}(M)$ is a viability domain.

Let us emphasize the fact that in this statement, the map $B : \mathrm{Graph}(U) \to \mathcal{L}(Z,Y)$ is a parameter. It thus provides many possibilities for checking whether a given subset is a viability domain.

6.4.3 Output Regulation Map

Definition 6.4.4 *We shall say that the set-valued map $Q_M : Y \rightsquigarrow Z$ defined by*

$$\forall\, y \in M,\ \ Q_M(y) \ := \ \bigcap_{x \in h^{-1}(y)} R_K(x)$$

is the output regulation map *of the controlled system.*

We observe that

$$\forall\, x \in K,\ \ Q_M(h(x)) \ \subset \ R_K(x)$$

Therefore, if the output regulation map is strict (i.e., has nonempty images), the evolution of solutions viable in $h^{-1}(M)$, i.e., solutions satisfying

$$\forall\, t \geq 0,\ \ y(t) \ := \ h(x(t)) \ \in \ M$$

can be regulated by output-dependent controls

$$\text{for almost all } t \geq 0,\ \ u(t) \ \in \ Q_M(h(x(t)))$$

and not only by merely state-dependent controls.

Observe that Theorem 6.3.3 provides sufficient conditions for the output regulation map Q_M to be lower semicontinuous.

Observe also that the state regulation map R_K can be regarded as an output regulation map in the sense that

$$\forall\, x \in K,\ \ R_K(x) := Q_M(h(x))$$

whenever the following commutativity conditions hold true:

$$\begin{cases} i) & U(x) \ := \ V(h(x)) \text{ where } V : Y \leadsto Z \\ ii) & h'(x)g(x) \ := \ e(h(x)) \text{ where } e : Y \mapsto \mathcal{L}(Z,X) \\ iii) & h'(x)c(x) \ := \ d(h(x)) \quad d : Y \mapsto X \end{cases}$$

In this case

$$\forall\, y \in M,\ Q_M(y) \ := \ \{u \in V(y) \mid e(y)u \in T_M(y) - d(y)\}$$

6.4.4 Duality Criterion

We shall now characterize viability domains through a dual formulation. For that purpose, we associate with any subset $K \subset \mathrm{Dom}(U)$ the subnormal cone[11] $N_K^\circ(x)$ and the function β_K defined by:

$$\forall\ (x,p) \in \mathrm{Graph}(N_K^\circ),\ \ \beta_K(x,p) \ := \ \inf_{u \in U(x)} < p, g(x)u + c(x) >$$

We deduce from Theorem 3.2.4 the following:

Proposition 6.4.5 *Let us assume that the control system is affine and that the values of the feedback map U are compact. Then a closed subset K is a viability domain if and only if*

$$\forall (x,p) \in \mathrm{Graph}(N_K^\circ),\ \ \beta_K(x,p) \leq 0$$

If we assume in particular that

$$\begin{cases} i) & M \text{ is a closed subset of } Y \\ ii) & h \text{ is a } \mathcal{C}^1\text{-map from } X \textit{ to } Y \\ iii) & \forall x \in K,\ \ Y = \mathrm{Im}(h'(x)) - T_M(h(x)) \end{cases}$$

[11] The subnormal cone $N_K^\circ(x)$ to a subset K at a point $x \in K$ is the negative polar cone to the contingent cone $T_K(x)$.

then $K := h^{-1}(M)$ *is a viability domain if and only if*

$$\begin{cases} \forall\, x \in K,\ \forall\, q \in N_M^\circ(h(x)), \\ d_M(x,q) \ := \ \inf_{u\in U(x)} < q, h'(x)g(x)u + h'(x)c(x) > \ \leq 0 \end{cases}$$

For instance, this condition holds true when the following abstract Walras law *holds true:*

$$\begin{cases} i) & Z = Y, \quad U(x) \subset N_M^\circ(h(x)) \\ ii) & \forall q \in N_M^\circ(h(x)), \quad < q, h'(x)c(x) + h'(x)g(x)q > \leq 0 \end{cases}$$

6.4.5 Decoupling the Regulation Map

Finally, let us mention that the calculus of the contingent cones can be transferred to a calculus of regulation maps. For instance, a quite common type of viability constraints are of the form $K := L \cap h^{-1}(M)$ where we assume that

$$\begin{cases} i) & L \subset X \text{ and } M \subset Y \text{ are sleek} \\ ii) & h \text{ is a } \mathcal{C}^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K := L \cap h^{-1}(M), \ \ Y = h'(x)T_L(x) - T_M(h(x)) \end{cases}$$

Indeed, K is the inverse image of the product $L \times M$ by the map $\mathbf{1} \times h$ from X to $X \times Y$.

This a particular case of a more general situation when both X, Y and Z are product spaces. It may then be convenient to provide once and for all the explicit formulas of the regulation map when this is the case. Let us assume namely that

$$\begin{cases} i) & X := \prod_{i=1}^n X_i & & \\ ii) & Y := \prod_{j=1}^m Y_j, & M & := \prod_{j=1}^m M_j \\ iii) & Z := \prod_{k=1}^l Z_k, & U(x) & := \prod_{k=1}^l U_k(x) \end{cases} \tag{6.15}$$

and that

$$\begin{cases} i) & \forall x \in X,\ g(x)u := (g_1(x)u, \ldots, g_m(x)u) \ \&\ g_i(x)u := \sum_{k=1}^l g_i^k(x)u_k \\ ii) & c(x) \ := (c_1(x), \ldots, c_n(x)) \\ iii) & \forall x \in X,\ \ h(x) \ := \ (h_1(x), \ldots, h_m(x)) \ \text{ where} \\ & h_j(x) \ := \ \sum_{i=1}^n h_j^i(x_i) \end{cases} \tag{6.16}$$

Therefore, K is the intersection of the subsets K_j defined by:

$$K_j \ := \ \{x \in X \mid \sum_{i=1}^n h_j^i(x_i) \in M_j\} \tag{6.17}$$

Let us introduce the matrix $B(x) := h'(x)g(x)$ of operators

$$B_j^k(x) \;=\; \sum_{i=1}^{n} h_j^{i'}(x)g_i^k(x) \;\in\; \mathcal{L}(U_k, Y_j)$$

and the vector $b(x) := h'(x)c(x)$ of components

$$b_j(x) \;=\; \sum_{i=1}^{n} h_j^{i'}(x)c_i(x)$$

Corollary 6.4.6 *We posit the assumptions (6.15), (6.16) and (6.17). We assume also that*

$$\left\{\begin{array}{ll} i) & \forall k,\ \text{Graph}(U_k) \text{ is closed and the images of } U_k \\ & \text{are convex} \\ ii) & \forall i,\ c_i : \text{Dom}(U) \mapsto X_i \text{ is continuous} \\ iii) & \forall k, i,\ g_k^i : \text{Dom}(U) \mapsto \mathcal{L}(Z_k, X_i) \text{ is continuous} \\ iv) & \forall k, i,\ c_i \text{ and } g_k^i \text{ are bounded and } U_k \\ & \text{have linear growth} \end{array}\right.$$

and that

$$\left\{\begin{array}{ll} i) & \text{the subsets } M_j \text{ are closed and sleek} \\ ii) & \text{the maps } h_j^i \text{ are } \mathcal{C}^1 \\ iii) & \forall v_j \in Y_j\ (j = 1, \ldots, n),\ \exists u_i \in X_i \text{ such that} \\ & v_j \in \sum_{i=1}^n h_j^{i'}(x_i)u_j + T_M\left(\sum_{i=1}^n h_j^i(x_i)\right) \end{array}\right. \tag{6.18}$$

Then the regulation map R_K is defined by

$$\left\{\begin{array}{ll} i) & R_K(x) \;=\; \bigcap_{j=1}^m R_{K_j}(x) \text{ where} \\ ii) & R_{K_j}(x) = \{u = (u_1, \ldots, u_l) \in \prod_{k=1}^l U_k(x) \text{ such that} \\ & \sum_{k=1}^l B_j^k(x)u_k \;\in\; T_{M_j}(\sum_{i=1}^n h_j^i(x_i) - b_j(x)\} \end{array}\right. \tag{6.19}$$

and has compact values. If it is strict, then K is a viability domain of the system, and thus, for any initial state $x_0 \in K$, there exist viable *solutions $x_i(\cdot)$ on $[0, \infty[$ starting at x_0 to the system of differential equations*

$$\forall i = 1, \ldots, m,\ x_i'(t) \;=\; c_i(x(t)) + \sum_{k=1}^{l} g_i^k(x(t))u_k(t)$$

and open loop *controls regulating this viable solution $x(\cdot)$ in the sense that the* regulation laws

$$\forall j = 1, \ldots, m,\ \text{ for almost all } t,\ u(t) \in R_{K_j}(x(t))$$

are satisfied.

Proof — Assumptions (6.18) imply that the subsets

$$K_j \text{ and } K := \bigcap_{j=1}^{m} K_j \quad \text{are sleek}$$

and that

$$\begin{cases} i) & T_K(x) = \bigcap_{j=1}^{m} T_{K_j}(x), \\ ii) & T_{K_j}(x) = \\ & \{v \in X \mid \sum_{i=1}^{n} h_j^{i'}(x_i)(v_i) \in T_{M_j}(\sum_{i=1}^{n} h_j^i(x_i))\} \end{cases}$$

This implies obviously formulas (6.19). □

Definition 6.4.7 (Decoupled Regulation Map) *We posit the assumptions (6.15), (6.16) and (6.17). We shall say that the regulation map is* decoupled *if*

$$Z = Y \text{ and } \forall j \neq k, \ B_j^k(x) = 0$$

Corollary 6.4.8 *We posit the assumptions of Corollary 6.4.6. If the regulation map is decoupled, then each "partial" viability domain K_j is regulated by the i^{th} component of the control in the sense that*

$$R_{K_j}(x) = \{u_j \in U_j(x) \mid B_j^j(x)u_j \in T_{M_j}(\sum_{i=1}^{n} h_j^i(x_i) - b_j(x)\}$$

6.5 Selection Theorems

6.5.1 Minimal Selection

A quite natural question arises: if a set-valued map $F : X \rightsquigarrow Y$ is, say, upper semicontinuous, does there exist what is called a *continuous selection* f of F, i.e., a continuous single-valued map satisfying

$$\forall x \in X, \ f(x) \in F(x) \tag{6.20}$$

Furthermore, we need more than a mere existence theorem: we would like to explicitly construct such selections. If the values of a set-valued map F are closed and convex and if Y is a Hilbert space, we can take, for instance, the *minimal selection* defined by

$$\begin{aligned} F^\circ(x) & := m(F(x)) \\ & := \{u \in F(x) \mid \|u\| = \min_{y \in F(x)} \|y\|\} \end{aligned} \tag{6.21}$$

The upper semicontinuity of F, even when it is closed convex valued, is not strong enough to imply the continuity of the minimal selection[12].

However, we can still prove the following

Proposition 6.5.1 *Let us assume that $F : X \rightsquigarrow Y$ is closed and lower semicontinuous with convex values. Then the graph of the minimal selection is closed*[13].

Proof — The projection of 0 onto the closed convex set $F(x)$ is the element $u := m(F(x)) \in F(x)$ such that

$$\|u\|^2 + \sigma(-F(x), u) = \sup_{y \in F(x)} < u - 0, u - y > \leq 0 \tag{6.22}$$

(It is actually equal to 0). Let us introduce the set-valued map $S_F : X \rightsquigarrow Y$ defined by

$$u \in S_F(x) \text{ if and only if } \|u\|^2 + \sigma(-F(x), u) \leq 0 \tag{6.23}$$

Therefore, the graph of the minimal selection is equal to:

$$\text{Graph}(m(F)) = \text{Graph}(F) \cap \text{Graph}(S_F)$$

Since F is lower semicontinuous, the function $(x, u) \mapsto \sigma(-F(x), u)$ is lower semicontinuous, so that the graph of S_F, and thus, of $m(F(\cdot))$, is closed. □

6.5.2 Selection Procedures

This simple property of the minimal selection can be naturally extended to different selection procedures of elements of $F(x)$, by means of other minimization problems than the one of minimizing the norm or by problems such as game theoretical models. It may be useful to introduce the following definition:

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[12] Consider the set-valued map $F : \mathbf{R} \rightsquigarrow \mathbf{R}$ defined by

$$F(x) := \begin{cases} \{2\} & \text{if} \quad x \neq 0 \\ [1, 2] & \text{if} \quad x = 0 \end{cases}$$

It is upper semicontinuous with compact convex values and its minimal selection is obviously not continuous.

[13] If moreover F is upper hemicontinuous, then the minimal selection is continuous. See Theorem 9.3.4 of SET-VALUED ANALYSIS.

Definition 6.5.2 (Selection Procedure) *Let Y be a normed space. A* selection procedure *of a set-valued map $F : X \rightsquigarrow Y$ is a set-valued map $S_F : X \rightsquigarrow Y$ satisfying*

$$\begin{cases} i) & \forall x \in \mathrm{Dom}(F), \ \ S(F(x)) := S_F(x) \cap F(x) \neq \emptyset \\ ii) & \textit{the graph of } S_F \textit{ is closed} \end{cases}$$

The set-valued map $S(F) : x \rightsquigarrow S(F(x))$ is called the selection *of F.*

Then, obviously,

$$\begin{cases} \text{if the graph of } F \text{ is closed, so is the graph of the selection} \\ x \rightsquigarrow S(F(x)) \end{cases}$$

and the selection is a single-valued map denoted $s(F(\cdot))$ whenever

$$\forall x \in \mathrm{Dom}(F), \ \ s(F(x)) \ := \ S_F(x) \cap F(x) \ \text{ is a singleton}$$

The set-valued map defined by (6.22) is naturally a selection procedure of a set-valued map with closed convex values which provides the minimal selection. In the case of finite dimensional vector-spaces, we could also have used the selection procedure S_F° defined by

$$S_F^\circ(x) \ := \ \{y \in Y \mid \|y\| \ \leq \ d(0, F(x))\}$$

We can easily provide other examples of selection procedures through optimization thanks to the Maximum Theorem.

Proposition 6.5.3 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with compact values. Let V : $\mathrm{Graph}(F) \mapsto \mathbf{R}$ be continuous. Then the set-valued map S_F defined by:*

$$S_F(x) := \left\{ y \in Y \mid V(x,y) \leq \inf_{z \in F(x)} V(x,z) \right\}$$

is a selection procedure *of F which yields the selection $S(F)$ equal to:*

$$S(F(x)) = \left\{ y \in F(x) \mid V(x,y) \leq \inf_{z \in F(x)} V(x,z)) \right\}$$

Proof — Since F is lower semicontinuous, the function

$$(x, y) \mapsto V(x, y) + \sup_{z \in F(x)} (-V(x, z))$$

is lower semicontinuous thanks to the Maximum Theorem. Our proposition follows from :

$$\text{Graph}(S_F) = \\ \left\{(x, y) \mid V(x, y) + \sup_{z \in F(x)}(-V(x, z)) \le 0\right\} \quad \Box$$

Most selection procedures through game theoretical models or equilibria are instances of this general selection procedure based on Ky Fan's Inequality (Theorem 3.7.8).

Proposition 6.5.4 *Let us assume that a set-valued map $F : X \rightsquigarrow Y$ is lower semicontinuous with convex compact values. Let $\varphi : X \times Y \times Y \mapsto \mathbf{R}$ satisfy*

$$\begin{cases} i) & \varphi(x, y, z) \text{ is lower semicontinuous} \\ ii) & \forall (x, y) \in X \times Y, \quad z \mapsto \varphi(x, y, z) \text{ is concave} \\ iii) & \forall (x, y) \in X \times Y, \quad \varphi(x, y, y) \le 0 \end{cases}$$

Then the map S_F associated with φ by the relation

$$S_F(x) := \left\{ y \in Y \mid \sup_{z \in F(x)} \varphi(x, y, z) \le 0 \right\}$$

is a selection procedure of F yielding the selection map $x \mapsto S(F(x))$ defined by

$$S_F(x) := \left\{ y \in F(x) \mid \sup_{z \in F(x)} \varphi(x, y, z) \le 0 \right\}$$

Proof — Ky Fan's inequality states that the subsets $S_F(x)$ are not empty since the subsets $F(x)$ are convex and compact. The graph of S_F is closed thanks to the assumptions and the Maximum Theorem because it is equal to the lower section of a lower semicontinuous function:

$$\text{Graph}(S_F) = \left\{ (x, y) \mid \sup_{z \in F(x)} \varphi(x, y, z) \le 0 \right\} \quad \Box$$

Proposition 6.5.5 *Assume that* $Y = Y_1 \times Y_2$, *that a set-valued map* $F : X \rightsquigarrow Y$ *is lower semicontinuous with convex compact values and that* $a : X \times Y_1 \times Y_2 \to \mathbf{R}$ *satisfies*

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall\, (x, y_2) \in X \times Y_2, \;\; y_1 \mapsto a(x, y_1, y_2) \text{ is convex} \\ iii) & \forall\, (x, y_1) \in X \times Y_1, \;\; y_2 \mapsto a(x, y_1, y_2) \text{ is concave} \end{cases}$$

Then the set-valued map S_F *associating to any* $x \in X$ *the subset*

$$\begin{array}{l} S_F(x) \;\; := \;\; \{(y_1, y_2) \in Y_1 \times Y_2 \text{ such that} \\ \forall (z_1, z_2) \in F(x), \;\; a(x, y_1, z_2) \le a(x, z_1, y_2)\} \end{array}$$

is a selection procedure of F *(with convex values). The selection map* $S(F(\cdot))$ *associates with any* $x \in X$ *the subset*

$$\begin{array}{l} S(F)(x) \;\; := \;\; \{(y_1, y_2) \in F(x) \text{ such that} \\ \forall (z_1, z_2) \in F(x), \;\; a(x, y_1, z_2) \le a(x, y_1, y_2) \le a(x, z_1, y_2)\} \end{array}$$

of saddle-points *of* $a(x, \cdot, \cdot)$ *in* $F(x)$.

Proof — We take

$$\varphi(x, (y_1, y_2), (y_1', y_2')) \;\; := a(x, y_1, y_2') - a(x, y_1', y_2)$$

and we apply the above theorem. □

We derive from the Equilibrium Theorem 3.7.6 selection procedures yielding equilibria in $F(x)$.

Proposition 6.5.6 *Let us assume that a set-valued map* $F : X \rightsquigarrow Y$ *has nonempty compact convex values. Let us consider an upper semicontinuous set-valued map* E *with nonempty compact convex values from* Graph(F) *to* Y *satisfying:*

$$\forall (x, y) \in \text{Graph}(F), \;\; E(x, y) \cap T_{F(x)}(y) \neq \emptyset$$

Then the set-valued map S_F *defined by*

$$S_F(x) \;\; := \;\; \{y \in Y \mid 0 \in E(x, y)\}$$

is selection procedure of F. *The selection map* $S(F)$ *associates with any* $x \in$ Dom(F) *the set*

$$S(F)(x) \;\; := \;\; \{y \in F(x) \mid 0 \in E(x, y)\}$$

of equilibria of $E(x, \cdot)$ *in* $F(x)$.

6.5.3 Michael's Selection Theorem

We shall now state the celebrated Michael's theorem stating that lower semicontinuous convex-valued maps do have *continuous selections*.

Theorem 6.5.7 (Michael's Theorem) *Let F be a lower semicontinuous set-valued map with closed convex values from a compact metric space X to a Banach space Y. It does have a continuous selection.*

In particular, for every $\overline{y} \in F(\overline{x})$ there exists a continuous selection f of F such that $f(\overline{x}) = \overline{y}$.

We refer to Section 9.1 of SET-VALUED ANALYSIS for the proof of this Theorem. □

6.6 Closed-Loop Controls and Slow Solutions

6.6.1 Continuous Closed Loop Controls

Viable solutions to the control system (6.1) are regulated by the controls whose evolution is governed by the regulation law (6.1.4). Continuous single-valued selections r_K of the regulation map R_K are *viable closed loop controls*, since the Nagumo Theorem states that the differential equation

$$x'(t) \quad = \quad f(x(t), r_K(x(t)))$$

enjoys the viability property.

Indeed, by construction, K is a viability domain of the single-valued map $x \in K \mapsto f(x, r_K(x))$. Hence, when the regulation map is lower semicontinuous with convex values, we deduce from Michael's Theorem 6.5.7 the existence of viable continuous closed loop controls.

Proposition 6.6.1 *Consider a Marchaud control system (U, f). If its regulation map is lower semicontinuous with nonempty convex values, then the control system can regulate viable solutions in K by* continuous closed loop controls.

Remark — If $\tilde{r}$ is a closed-loop control, it is obvious that the viability kernel of K for $f(\cdot, \tilde{r}(\cdot))$ is contained in the viability kernel $\mathrm{Viab}(K)$ for the set-valued map F.

On the other hand, if $\tilde{r}$ is a closed-loop control regulating a viability domain K of F, i.e., if $\tilde{r}(x) \in R_K(x)$ for all $x \in K$, K is equal to its viability kernel for $f(\cdot, \tilde{r}(\cdot))$. □

6.6.2 Slow Viable Solutions

This result is not useful in practice, since Michael's Selection Theorem does not provide constructive ways to find those continuous closed loop controls.

Therefore, we are tempted to use explicit selections of the regulation map R_K, such as the minimal selection r_K° (see (6.21)). Unfortunately, since there is no hope of having continuous regulation maps R_K in general (as soon as we have inequalities constraints), this minimal selection is not continuous. But we can still prove that by taking the minimal selection r_K°, the differential equation

$$x'(t) = f(x(t), r_K^\circ(x(t))) \tag{6.24}$$

does enjoy the viability property.

Definition 6.6.2 *The solutions to differential equation (6.24) are called* slow *viable solutions to control system (6.1).*

We shall derive from Theorem 6.6.5 below the existence of slow viable solutions:

Theorem 6.6.3 *Consider a Marchaud control system (U, f). If the regulation map is lower semicontinuous with nonempty convex values, then the control system (6.1) has slow viable solutions.*

Furthermore, the solution map associating with any $x_0 \in K$ the set of slow viable solutions starting at x_0 is upper semicontinuous from K to $\mathcal{C}(0, \infty; X)$.

Example: Slow viable solutions on smooth subsets.
When $K := h^{-1}(0)$ is smooth, one can obtain explicit differential equations yielding slow viable solutions.

Corollary 6.6.4 *Let us assume that $h : X \mapsto Y$ is a continuously differentiable map and that the viability subset is $K := h^{-1}(0)$, that $U(x) \equiv Z$ is constant, and that the system is affine, so that*

$$\forall\, x \in K, \;\; R(x) := \{u \in Z \mid h'(x)f(x,u) = h'(x)(c(x)) + h'(x)g(x)u = 0\}$$

Then there exist slow solutions viable in K, which are the solutions to the system

$$\left\{ \begin{array}{l} x'(t) = -g(x(t))^\star h'(x(t))^\star \\ (h'(x(t))g(x(t))g(x(t))^\star h'(x(t))^\star)^{-1} h'(x(t))c(x(t)) \end{array} \right.$$

Proof — The element $u_0 \in R(x)$ of minimal norm is the solution of the quadratic minimization problem under equality constraints

$$h'(x)g(x)u = -h'(x)c(x)$$

and is given explicitly by the formula[14]

$$u_0 = -g(x)^\star h'(x)^\star (h'(x)g(x)g(x)^\star h'(x)^\star)^{-1} h'(x)c(x) \;\; \square$$

Slow viable solutions in affine spaces. Consider the case when $K := \{x \in X \mid Lx = y\}$ where $L \in \mathcal{L}(X,Y)$ is surjective. Then the differential equation yielding slow viable solutions is given by

$$x'(t) = -g(x(t))^\star L^\star (Lg(x(t))g(x(t))^\star L^\star)^{-1} Lc(x(t))$$

When $Y := \mathbf{R}$ and $K := \{x \in X \mid \; < p, x >= y\}$ is a hyperplane, the above equation becomes

$$x'(t) = -\frac{< p, c(x(t)) >}{\|g(x(t))^\star p\|^2} g(x(t))^\star p$$

Slow viable solutions in the sphere. Let $L \in \mathcal{L}(X,X)$ be a symmetric positive-definite linear operator, with which we associate the viability subset

$$K \;:=\; \{x \in X \mid \; < Lx, x >= 1\}$$

Then slow viable solutions are given by the differential equation

$$x'(t) = -\frac{< Lx(t), c(x(t)) >}{\|g(x(t))^\star L(x(t))\|^2} g(x(t))^\star Lx(t)$$

[14]Recall that the unique element which minimizes $x \mapsto \|x\|$ under the constraint $Bx = y$, where $B \in \mathcal{L}(X,Y)$ is surjective, is equal to $B^+ y$, where $B^+ = B^\star(BB^\star)^{-1}$ denotes the *orthogonal right-inverse of B*.

6.6.3 Other Selections of Viable Solutions

The reason why Theorem 6.6.3 on the existence of slow viable solutions holds true is that the minimal selection is obtained through the selection procedure defined in (6.22). It is this fact which matters. So, Theorem 6.6.3 can be extended to any selection procedure of the regulation map R_K yielding single-valued selections.

Theorem 6.6.5 *Consider a Marchaud control system (U, f) and suppose that K is a viability domain. Let S_{R_K} be a selection procedure of the regulation map R_K. Suppose that the values of S_{R_K} are convex and that the selection map*

$$s(R_K(\cdot)) := S_{R_K}(\cdot) \cap R_K(\cdot) \text{ is single-valued}$$

Then the selection $s(R_K)(\cdot)$ is a closed loop control *regulating viable solutions of the control system (6.1).*

Furthermore, the solution map associating with any $x_0 \in K$ the set of viable solutions to the differential equation

$$x'(t) = f(x(t), s(R_K)(x(t)))$$

starting at x_0 is upper semicontinuous from K to $\mathcal{C}(0, \infty; X)$.

Even if the selection is not single-valued, we may be still interested in regulating viable solutions by controls ranging over a set-valued selection $S(R_K)$ of the regulation map. In any case, Theorem 6.6.5 is a consequence of

Theorem 6.6.6 *Consider a Marchaud control system (U, f) and suppose that K is a viability domain. Let S_{R_K} be a selection of the regulation map R_K. Suppose that the values of S_{R_K} are convex. Then, for any initial state $x_0 \in K$, there exist a viable solution starting at x_0 and a viable control to control system (6.1) which are regulated by the selection $S(R_K)$ of the regulation map R_K, in the sense that*

$$\begin{cases} \text{for almost all } t \geq 0, \\ u(t) \in S(R_K)(x(t)) := R_K(x(t)) \cap S_{R_K}(x(t)) \end{cases}$$

Furthermore, the solution map associating with any $x_0 \in K$ the set of solutions to control system (6.1) starting at x_0 which are regulated by the selection $S(R_K)$ is upper semicontinuous from K to $\mathcal{C}(0,\infty;X)$.

Proof — Since the convex selection procedure S_{R_K} has a closed graph and convex values, we can replace the affine control system (6.1) by the control system

$$\begin{cases} i) & x'(t) = f(x(t),u(t)) \\ ii) & \text{for almost all } t, \ u(t) \in U(x(t)) \cap S_{R_K}(x(t)) \end{cases} \tag{6.25}$$

which satisfies the assumptions of the Viability Theorem 6.1.4. It remains to check that K is still a viability domain for this smaller system. But by construction, we know that for all $x \in K$, there exists $u \in S(R_K)(x)$, which belongs to the intersection $U(x) \cap S_{R_K}(x)$ and which is such that $f(x,u)$ belongs to $T_K(x)$.

Hence the new control system (6.25) enjoys the viability property, so that, for all initial states $x_0 \in K$, there exist a viable solution and a viable control to the control system (6.25) which, for almost all $t \geq 0$, are related by

$$\begin{cases} i) & u(t) \in U(x(t)) \cap S_{R_K}(x(t)) \\ ii) & f(x(t),u(t)) \in T_K(x(t)) \end{cases}$$

Therefore, for almost all $t \geq 0$, $u(t)$ belongs to the intersection of $R_K(x(t))$ and $S_{R_K}(x(t))$, i.e., to the selection $S(R_K)(x(t))$ of the regulation map R_K.

Since the solution map $\mathcal{T}$ of the system (6.25) is upper semicontinuous by Theorem 3.5.2 and since the set

$$\mathcal{K} := \{x(\cdot) \in \mathcal{C}(0,\infty;X) \mid \forall\, t \geq 0, \ \ x(t) \in K\}$$

is closed, the set-valued map $x \rightsquigarrow \mathcal{T}(x) \cap \mathcal{K}$ is also upper semicontinuous. It associates with any $x \in K$ the set of solutions to control system (6.1) starting at x_0 which are regulated by the selection $S(R_K)$. □

We can now multiply the possible corollaries, since we have given several instances of selection procedures of set-valued maps.

6.6.4 Examples of Selection of Viable Solutions

We shall just mention some of the examples. We begin by selecting viable solutions through minimization procedures:

Proposition 6.6.7 *Consider a Marchaud control system (U, f) and suppose that the regulation map is lower semicontinuous with nonempty convex images. Let*

$$V : (x, u) \in \text{Graph}(U) \mapsto V(x, u) \in \mathbf{R}$$

be continuous and convex with respect to u. Then, for any initial state $x_0 \in K$, there exists a viable solution $x(\cdot)$ to the control system (6.1) regulated by an open loop control $u(\cdot)$ satisfying for almost all $t \geq 0$,

$$u(t) \in R_K(x(t)) \;\&\; V(x(t), u(t)) = \inf_{v \in R_K(t)} V(x(t), v)$$

Proof — This is a consequence of Theorem 6.6.6 and Proposition 6.5.3. □

When the control space $Z := Z_1 \times Z_2$ is the product of two control spaces, viable controls can be required to be *saddle-points* of two-person games:

Proposition 6.6.8 *Consider a Marchaud control system (U, f), where $U(x) = U_1(x) \times U_2(x)$ is the product of two control sets and $f(x, u) := c(x) + g_1(x)u_1 + g_2(x)u_2$. Assume that the regulation map*

$$R_K(x) := \{(u_1, u_2) \in U_1(x) \times U_2(x) \mid g_1(x)u_1 + g_2(x)u_2 \in T_K(x) - c(x)\}$$

is lower semicontinuous with nonempty convex values. Let $a : X \times Z_1 \times Z_2 \to \mathbf{R}$ satisfy

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall (x, u_2) \in X \times Y_2, \;\; u_1 \mapsto a(x, u_1, u_2) \text{ is convex} \\ iii) & \forall (x, u_1) \in X \times Y_1, \;\; u_2 \mapsto a(x, u_1, u_2) \text{ is concave} \end{cases}$$

Then, for any initial state $x_0 \in K$, there exist a viable solution $x(\cdot)$ and open loop controls $u_1(\cdot)$ & $u_2(\cdot)$ satisfying for almost all $t \geq 0$,

$$\begin{cases} i) & x'(t) = c(x(t)) + g_1(x(t))u_1(t) + g_2(x(t))u_2(t) \\ ii) & u_1(t) \in U_1(x(t)) \;\&\; u_2(t) \in U_2(x(t)) \\ iii) & \forall (v_1, v_2) \in R_K(x), \\ & a(x(t), u_1(t), v_2) \leq a(x(t), u_1(t), u_2(t)) \leq a(x(t), v_1, u_2(t))\} \end{cases}$$

Proof — The proof follows from Theorem 6.6.6 and Proposition 6.5.7. □

We finally state a selection method of viable solutions regulated by viable controls satisfying a nonlinear equation.

Proposition 6.6.9 *Consider a Marchaud control system* (U, f) *and suppose that the regulation map is lower semicontinuous with nonempty convex images. Let us consider an upper semicontinuous set-valued map* E *with nonempty closed convex values from* Graph(U) *to* Z *satisfying:*

$$\forall (x,u) \in \text{Graph}(R_K), \quad E(x,u) \cap T_{R_K(x)}(u) \neq \emptyset$$

Then, for any initial state $x_0 \in K$*, there exists a viable solution* $x(\cdot)$ *to the control system (6.1) regulated by an open loop control* $u(\cdot)$ *satisfying for almost all* $t \geq 0$,

$$u(t) \in R_K(x(t)) \;\; \& \;\; 0 \in E(x(t), u(t))$$

Proof — We apply Theorem 6.6.6 and Proposition 6.5.6. □

Chapter 7

Smooth and Heavy Viable Solutions

Introduction

Let us still consider the problem of regulating a control system

$$(i) \quad \text{for almost all } t \geq 0, \; x'(t) = f(x(t), u(t)) \; \text{ where } \; u(t) \in U(x(t))$$

where $U : K \leadsto Z$ associates with each state x the set $U(x)$ of feasible controls (in general state-dependent) and $f : \text{Graph}(U) \mapsto X$ describes the dynamics of the system.

For simplicity, we take for viability subset the domain $K := \text{Dom}(U)$ of U[1]. We have seen in the preceding chapter that viable controls (which provide viable solutions $x(t) \in K := \text{Dom}(U)$) are the ones obeying the regulation law

$$\forall\, t \geq 0, \;\; u(t) \in R_K(t) \;\; (\text{or} \;\; (x(t), u(t)) \in \text{Graph}(R_K))$$

where

$$\forall\, x \in K, \;\; R_K(x) = \{\, u \in U(x) \mid f(x, u) \in T_K(x) \,\}$$

In this chapter, we are looking for a system of differential equations or of differential inclusions governing *the evolution of both viable states and controls*, so that we can look for

[1]or we replace U by its restriction to K. It is closed whenever $U : X \leadsto Z$ is upper semicontinuous.

— *heavy solutions*, which are evolutions where the controls evolve with minimal velocity

— *punctuated equilibria*, i.e., evolutions in which the control $\bar{u}$ remains constant whereas the state may evolve in the associated *viability cell*, which is the viability domain of $x \mapsto f(x, \bar{u})$,

— *regulation by ramp controls*, i.e., evolutions in which the open-control is linear, and more generally, polynomial open-loop controls

— and other related ideas.

The idea which allows us to achieve these aims is quite simple: *we differentiate the regulation law.*

This is possible whenever we know how to differentiate set-valued maps. Hence the first section is devoted to the definition and the elementary properties of the contingent derivative[2] $DF(x, y)$ of a set-valued map $F : X \rightsquigarrow Y$ at a point (x, y) of its graph: By definition, its graph is the contingent cone to the graph of F at (x, y). We refer to Chapter 5 of SET-VALUED ANALYSIS for further information on the differential calculus of set-valued maps.

In the second section, we differentiate the regulation law and deduce that

$$(ii) \quad \text{for almost all } t \geq 0, \quad u'(t) \in DR_K(x(t), u(t))(f(x(t), u(t)))$$

whenever the viable control $u(\cdot)$ is absolutely continuous,

This is the second half of the system of differential inclusions we are looking for.

Observe that this new differential inclusion has a meaning whenever the state-control pair $(x(\cdot), u(\cdot))$ remains viable in the graph of R_K.

Fortunately, by the very definition of the contingent derivative, the graph of R_K is a viability domain of the new system (i), (ii).

Unfortunately, as soon as viability constraints involve inequalities, there is no hope for the graph of the contingent cone, and thus, for the graph of the regulation map, to be closed, so that, the Viability Theorem cannot apply.

[2]We set $Df(x) := Df(x, f(x))$ whenever f is single-valued. When f is Fréchet differentiable at x, then $Df(x)(v) = f'(x)v$ is reduced to the usual directional derivative.

However, if the contingent derivative of U obeys a growth condition:

$$(\mathcal{G})\ \forall\,(x,u)\in \mathrm{Graph}(U),\quad \inf_{v\in DU(x,u)(f(x,u))}\|v\| \le c(\|u\|+\|x\|+1)$$

then there exists an absolutely continuous solution $(x(\cdot),u(\cdot))$ of (i) verifying

$$(iii)\ \text{for almost all } t\ \ge\ 0,\quad \|u'(t)\|\ \le\ c(\|u(t)\|+\|x(t)\|+1)$$

So, *a strategy to overcome the above difficulty is to introduce*[3] *the a priori growth condition (iii)* and to look for *graphs of closed set-valued maps R* contained in Graph(U) which are viable under this system of differential inclusions. We already illustrated that in the simple economic example of Section 6.2.

Such set-valued maps R are solutions to the *partial differential inclusion*

$$\forall\,x\in K,\ \ 0\in DR(x,u)(f(x,u))-c(\|x\|+\|u\|+1)B$$

satisfying the constraint

$$\forall\,(x,u)\in \mathrm{Graph}(R),\ \ R(x)\ \subset\ U(x)$$

Since we shall show that such closed set-valued maps R are all contained in the regulation map R_K, we call them *subregulation maps* associated with the system $i), iii)$. In particular, there exists a largest subregulation map denoted R^c.

In particular, any *single-valued* $r : K \mapsto Z$ with closed graph which is a solution to the *partial differential inclusion*

$$\forall\,x\in K,\ \ 0\in Dr(x)(f(x,r(x)))-c(\|x\|+\|r(x)\|+1)B$$

satisfying the constraint

$$\forall\,x\in K,\ \ r(x)\ \in\ U(x)$$

provides feedback controls regulating smooth solutions to the control system.

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[3]even if growth conditions on the contingent derivative of U are absent.

The set-valued and single-valued solutions to these partial differential inclusions are studied in Section 6 of Chapter 8.

Let us consider such a subregulation map R. Theorem 4.1.2 implies that *whenever the initial state* x_0 *is chosen in* $\mathrm{Dom}(R)$) *and the initial control* u_0 *in* $R(x_0)$, *there exists a solution to the system of differential inclusions* $i), iii)$ *viable in* $\mathrm{Graph}(R)$. The regulation law for the viable state-controls becomes

$$(iv) \quad u'(t) \in DR(x(t), u(t))(f(x(t), u(t))) \cap c(\|x(t)\| + \|u(t)\| + 1)B$$

We call it the *metaregulation law* associated with the subregulation map R.

This is how we can obtain *smooth viable state-control solutions* to our control problem by solving the system of differential inclusions $i), v)$.

Actually, the graphs of all such regulation maps are contained in the *viability kernel* of $\mathrm{Graph}(U)$ for the system of differential inclusions $i), iii)$. This viability kernel is then the graph of the largest subregulation map $R^c \subset U$.

We shall construct explicitly in the third section such a regulation map in the case of the simplest economic model we can think of.

To the extent where second order differential equations and inclusions are first-order systems in disguise, we devote section 7.4 to viability problems for second order differential inclusions. The situation is not as simple as in the first order case, because the viability constraint $x(t) \in K$ becomes $x'(t) \in T_K(x(t))$, or again, $(x(t), x'(t)) \in \mathrm{Graph}(T_K)$. It no longer defines closed (or even, locally compact) viability domains. So, here again, we shall overcome this type of difficulty by using the concept of viability kernel.

We can naturally follow the same route to obtain smoother open-loop controls by setting bounds on the m-th derivatives: for almost all $t \geq 0$,

$$(v) \quad \|u^{(m)}(t)\| \; \leq \; c(\|u^{(m-1)}(t)\| + \ldots + \|u(t)\| + \|x(t)\| + 1)$$

This is the topic of the fifth section.

We devote the sixth section to the particular case when $c = 0$. We observe that equation (iii) then yields constant controls u_0 and thus solutions $x(\cdot)$ to the problem $x'(t) = f(x(t), u_0)$ which are viable in the closed subset $U^{-1}(u_0)$ (whenever this subset is not empty.) If this is the case, we shall say that u_0 is a *punctuated equilibrium* and that $(R^0)^{-1}(u_0)$ is the associated *viability cell, the closed subset of states regulated by the constant control* u_0.

In the general case of smooth systems of order m, the 0-growth condition yields open-loop controls which are polynomial of degree m. In particular, for $m = 1$, first-degree polynomials open-loop controls are known under the more descriptive label of *ramp controls*.

The seventh section is devoted to selection procedures of *dynamical closed loops*, and, among them, of heavy viable solutions.

Instead of looking for closed loop control selections of the regulation map R_K as we did in Chapter 6, we now look for selections $g(\cdot,\cdot)$ of the metaregulation map

$$(x,u) \rightsquigarrow DR(x,u)(f(x,u)) \cap c(\|x\| + \|u\| + 1)B$$

called *dynamical closed-loops*.

Naturally, under adequate assumptions, Michael's Theorem implies the existence of a continuous dynamical closed loop. But under the same assumptions, we can take as dynamical closed-loop the minimal selection $g^\circ(\cdot,\cdot)$ defined by $\|g^\circ(x,u)\| = \min_{v \in DR(x,u)(f(x,u))} \|v\|$, which, in general, is not continuous.

However, we shall prove that this minimal dynamical feedback still yields smooth viable control-state solutions to the system of differential equations

$$x'(t) \ = \ f(x(t),u(t)) \quad \& \quad u'(t) \ = \ g^\circ(x(t),u(t))$$

called *heavy viable solutions*, (heavy in the sense of heavy trends.) They are the ones for which *the control evolves with minimal velocity*. In the case of the usual differential inclusion $x' \in F(x)$, where the controls are the velocities, they are the solutions with minimal acceleration (or maximal inertia.)

Heavy viable solutions obey the *inertia principle*: *"keep the controls constant as long as they provide viable solutions"*.

Indeed, if zero belongs to $DR(x(t_1),u(t_1))(f(x(t_1),u(t_1)))$, then the control will remain equal to $u(t_1)$ as long as for $t \geq t_1$, a solution $x(\cdot)$ to the differential equation $x'(t) = f(x(t),u(t_1))$ satisfies the condition $0 \in DR(x(t_1),u(t_1))(f(x(t_1),u(t_1)))$.

If at some time t_f, $u(t_f)$ is a punctuated equilibrium, then the solution enters the viability cell associated to this control and may remain in this viability cell forever[4] and the control will remain equal to this punctuated equilibrium.

The concept of a heavy viable solution will be extended to the m-th order, where we look for controls whose m-th derivative evolves as slowly as possible. They obey an *m-th order inertia principle: keep an m-degree polynomial open-loop control as long as the solution it regulates is viable.*

7.1 Contingent Derivatives

By coming back to the original point of view proposed by Fermat, we are able to geometrically define the derivatives of set-valued maps from the choice of tangent cones to the graphs, even though they yield very strange limits of differential quotients.

Definition 7.1.1 *Let $F : X \rightsquigarrow Y$ be a set-valued map from a normed space X to another normed space Y and $y \in F(x)$.*

The contingent derivative *$DF(x, y)$ of F at $(x, y) \in$ Graph(G) is the set-valued map from X to Y defined by*

$$\text{Graph}(DF(x,y)) := T_{\text{Graph}(F)}(x,y)$$

When $F := f$ is single-valued, we set $Df(x) := Df(x, f(x))$ and $Cf(x) := Cf(x, f(x))$.

We shall say that F is sleek *at $(x, y) \in$ Graph(F) if and only if the map*

$$(x', y') \in \text{Graph}(F) \rightsquigarrow \text{Graph}(DF(x', y'))$$

is lower semicontinuous at (x, y) (i.e., if the graph of F is sleek at (x, y).) The set-valued map F is sleek *if it is sleek at every point of its graph.*

Naturally, *when the map is sleek at (x, y), the contingent derivative $DF(x, y)$ is a closed convex process.*

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[4]as long as the viability domain does not change for external reasons which are not taken into account here.

We can easily compute the derivative of the inverse of a set-valued map F (or even of a noninjective single-valued map): *The contingent derivative of the inverse of a set-valued map F is the inverse of the contingent derivative*:

$$D(F^{-1})(y,x) \quad = \quad DF(x,y)^{-1}$$

If K is a subset of X and f is a single-valued map which is Fréchet differentiable around a point $x \in K$, then *the contingent derivative of the restriction of f to K is the restriction of the derivative to the contingent cone*:

$$D(f|_K)(x) = D(f|_K)(x, f(x)) = f'(x)|_{T_K(x)}$$

These contingent derivatives can be characterized by adequate limits of differential quotients[5]:

Proposition 7.1.2 *Let $(x,y) \in \mathrm{Graph}(F)$ belong to the graph of a set-valued map $F : X \leadsto Y$ from a normed space X to a normed space Y. Then*

$$\begin{cases} v \in DF(x,y)(u) & \text{if and only if} \\ \mathrm{liminf}_{h\to 0+, u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \end{cases}$$

If $x \in \mathrm{Int}(\mathrm{Dom}(F))$ and F is Lipschitz around x, then

$$v \in DF(x,y)(u) \;\; \textit{if and only if} \;\; \liminf_{h\to 0+} d\left(v, \frac{F(x+hu)-y}{h}\right) = 0$$

[5]We can reformulate Proposition 7.1.2 by saying that *the contingent derivative $DF(x,y)$ is the graphical upper limit (See Definition 3.6.3) of the differential quotients*

$$u \leadsto \nabla_h F(x,y)(u) \; := \; \frac{F(x+hu)-y}{h}$$

Indeed, we know that the contingent cone

$$T_{\mathrm{Graph}(F)}(x,y) \;=\; \mathrm{Limsup}_{h\to 0+} \frac{\mathrm{Graph}(F)-(x,y)}{h}$$

is the upper limit of the differential quotients $\frac{\mathrm{Graph}(F)-(x,y)}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathrm{Graph}(DF(x,y)) := T_{\mathrm{Graph}(F)}(x,y) \;\;\&\;\; \mathrm{Graph}(\nabla_h F(x,y)) = \frac{\mathrm{Graph}(F)-(x,y)}{h}$$

to conclude.

If moreover the dimension of Y *is finite, then*

$$\mathrm{Dom}(DF(x,y)) \;=\; X \text{ and } DF(x,y) \text{ is Lipschitz}$$

Proof — The first two statements being obvious, let us check the last one. Let u belong to X and l denote the Lipschitz constant of F on a neighborhood of x. Then, for all $h > 0$ small enough and $y \in F(x)$,

$$y \;\in\; F(x) \;\subset\; F(x+hu) \;+\; lh\|u\|B$$

Hence there exists $y_h \in F(x+hu)$ such that $v_h := (y_h - y)/h$ belongs to $l\|u\|B$, which is compact. Therefore the sequence v_h has a cluster point v, which belongs to $DF(x,y)(u)$. □

Remark — Lower Semicontinuously Differentiable Maps The lower semicontinuity of the set-valued map

$$(x,y,u) \in \mathrm{Graph}(F) \times X \leadsto DF(x,y)(u)$$

at some point (x_0, y_0, u_0) is often needed. Observe that it implies that F is sleek at (x_0, y_0). The converse needs further assumptions. We derive for instance from Theorem 2.5.7 the following criterion:

Proposition 7.1.3 *Assume that* X *and* Y *are Banach spaces and that* F *is sleek on some neighborhood* $\mathcal{U}$ *of* $(x_0, y_0) \in \mathrm{Graph}(F)$. *If the boundedness property*

$$\forall\, u \in X, \qquad \sup_{(x,y)\in\mathcal{U}\cap\mathrm{Graph}(F)} \; \inf_{v\in DF(x,y)(u)} \|v\| \;<\; +\infty$$

holds true, then the set-valued map

$$(x,y,u) \in \mathrm{Graph}(F) \times X \leadsto DF(x,y)(u)$$

is lower semicontinuous on $(\mathcal{U} \cap \mathrm{Graph}(F)) \times X$

7.2 Smooth Viable Solutions

7.2.1 Regularity Theorem

Let us consider a finite dimensional vector space Z and a control system (U, f) defined by a set-valued map $U : X \leadsto Z$ and a single-valued map $f : \mathrm{Graph}(U) \mapsto X$, where X is regarded as the state space, Z the control space, f as a description of the dynamics and U as the a priori feedback. The evolution of a state-control solution $(x(\cdot), u(\cdot))$ viable in Graph(U) is governed by

$$x'(t) = f(x(t), u(t)), \quad u(t) \in U(x(t)) \tag{7.1}$$

We shall look for viable solutions in $K := \mathrm{Dom}(U)$ which are smooth in the following sense:

Definition 7.2.1 (Smooth State-Control) *We say that the pair $(x(\cdot), u(\cdot))$ is* smooth *if both $x(\cdot)$ and $u(\cdot)$ are absolutely continuous and m-*smooth *if both $x(\cdot)$ and $u^{(m-1)}(\cdot)$ are absolutely continuous.*

*It is said to be φ-*smooth *(respectively φ-smooth of m-th order) if in addition for almost all $t \geq 0$, $\|u'(t)\| \leq \varphi(x(t), u(t))$ (respectively $\|u^{(m)}(t)\| \leq \varphi(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t))$), where $\varphi : X \times Z \mapsto \mathbf{R}_+$ (respectively $\varphi : X \times Z^m \mapsto \mathbf{R}_+$) is a given function.*

We obtain smooth viable solutions by setting a bound to the growth to the evolution of controls, as we did in the simple economic example of Section 6.2.

For that purpose, we associate to this control system and to any nonnegative continuous function $u \to \varphi(x, u)$ with linear growth[6] the system of differential inclusions

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) \in \varphi(x(t), u(t))B \end{cases} \tag{7.2}$$

Observe that any solution $(x(\cdot), u(\cdot))$ to (7.2) viable in Graph(U) is a φ-smooth solution to the control system (7.1).

We thus deduce from the Viability Theorem applied to the system (7.2) on the graph of U the following Regularity Theorem:

Theorem 7.2.2 *Assume that U is closed and f, φ are continuous with linear growth. Then the following two statements are equivalent:*

a) — For any initial state $x_0 \in \mathrm{Dom}(U)$ and control $u_0 \in U(x_0)$, there exists a φ-smooth state-control solution $(x(\cdot), u(\cdot))$ to the control system (7.1) starting at (x_0, u_0).

[6] which can be a constant ρ, or the function $(x, u) \to c\|u\|$, or the function $(x, u) \to c(\|u\| + \|x\| + 1)$. One could also take other dynamics $u' \in \Phi(x, u)$ where Φ is a Marchaud map.

b) — *The set-valued map U satisfies*

$$\forall\,(x,u) \in \text{Graph}(U),\;\; 0 \;\in\; DU(x,u)(f(x,u)) - \varphi(x,u)B \qquad (7.3)$$

Proof — The conclusion of the theorem amounts to saying that the closed subset Graph(U) enjoys the viability property. By Viability Theorem 3.3.5, which we can apply because $(x,u) \rightsquigarrow \{f(x,u)\} \times \varphi(x,u)B$ is a Marchaud map, this is the case if and only if it is a viability domain, i.e., if and only if

$$\forall\,(x,u) \in \text{Graph}(U), T_{\text{Graph}(U)}(x,u) \cap (\{f(x,u)\} \times \varphi(x,u)B) \neq \emptyset$$

By the very definition of the contingent derivative of U, this is the necessary and sufficient condition of the theorem. □

We know that whenever the right-hand side of an ordinary differential equation is differentiable, its solutions are twice differentiable. The extension of this property to the case of differential inclusions is just a consequence of the above theorem when we take $f(x,u) = u$:

Corollary 7.2.3 *Let $F : X \rightsquigarrow X$ be a closed set-valued map such that*

$$\forall\, x \in \text{Dom}(F),\; \forall\, v \in F(x),\; 0 \;\in\; DF(x,v)(v) - \varphi(x,v)B$$

where $(x,u) \rightarrow \varphi(x,u)$ is a nonnegative continuous function with linear growth.

Then, for any $x_0 \in \text{Dom}(F)$ and $v_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to the differential inclusion

$$x'(t) \;\in\; F(x(t)),\;\; x(0) \;=\; x_0 \;\&\; x'(0) \;=\; v_0$$

which belongs to the Sobolev space $W^{2,1}(0,\infty;X;e^{-bt}dt)$ (both $x(\cdot)$ and $x'(\cdot)$ are absolutely continuous.)

Remark — Naturally, we can consider other evolution laws of open-loop controls associated with the control system (U,f) which provide smooth open-loop controls yielding viable solutions.

First, we can introduce an observation space Y, replace the initial control space Z by another finite dimensional space Z_1, an observation map $\beta : X \mapsto Y$ and relate the new controls $v \in Z_1$ and the observation y to the former controls $u \in Z$ by a single-valued map of the form

$$u = \alpha(\beta(x), v)$$

where

$$\alpha : Y \times Z_1 \mapsto Z$$

We then define a new control system (g, V) defined by

$$\begin{cases} i) & g(x,v) := f(x, \alpha(\beta(x), v)) \\ ii) & V(x) := \{v \in Z_1 \mid \alpha(\beta(x), v) \in U(x)\} \end{cases}$$

Therefore the new control system governed by

$$\begin{cases} i) & x'(t) = g(x(t), v(t)) \\ ii) & v(t) \in V(x(t)) \end{cases} \tag{7.4}$$

provides the same dynamics of the state although through another parametrization.

This being done, we can propose any evolution law of the open-loop controls as long as they are compatible with the constraints $v(t) \in V(x(t))$ (or $u(t) \in U(x(t))$.)

For instance, if $A \in \mathcal{L}(Z_1, Z_1)$ and $\Phi : X \times Z_1 \rightsquigarrow Z_1$ and $\varphi : X \times Z_1 \rightsquigarrow Z_1$ is a Marchaud map, we can replace system (7.2) by the system of differential inclusions

$$\begin{cases} i) & x'(t) = g(x(t), v(t)) \\ ii) & v'(t) \in Av(t) + \Phi(x(t), v(t)) \end{cases} \tag{7.5}$$

(With an adequate choice of A, we are able to study the evolution of m time differentiable open-loop controls in next section.)

Then the Regularity Theorem becomes:

Theorem 7.2.4 *Assume that U is closed and sleek, that f, φ are continuous with linear growth, that the maps α and β are continuously differentiable with linear growth and that*

$$\forall\,(x,v) \in \mathrm{Graph}(V),\;\; \alpha'_v(\beta(x),v) \;\text{ is surjective}$$

Then the following two statements are equivalent:

a) — For any initial state $x_0 \in \mathrm{Dom}(V)$ and control $v_0 \in V(x_0)$, there exists a solution $(x(\cdot),v(\cdot))$ to the control system (7.5) starting at (x_0,v_0) (so that $x(\cdot)$ is still a solution to the control system (7.1)).

b) — The set-valued map V satisfies: for every $(x,v) \in \mathrm{Graph}(V)$,

$$\begin{array}{l} Av \in -\Phi(x,v) + \\ \alpha'_v(\beta(x),v)^{-1}\left[DU(x,\alpha(\beta(x),v))(g(x,v)) - \alpha'_y(\beta(x),v)\beta'(x)g(x,v)\right] \end{array}$$

Proof — By the Viability Theorem 3.3.5, we have to check that the graph of V is a viability domain for the set-valued map

$$(x,v) \rightsquigarrow \{g(x,v)\} \times (Av + \Phi(x,v))$$

Since the graph of V is the inverse image of the graph of U under the differentiable map $h : X \times Z_1 \mapsto X \in Z$ defined by

$$h(x,v) \;=\; (x, \alpha(\beta(x),v))$$

we can derive a formula to compute its contingent cone whenever U is sleek and the following transversality condition holds true:

$$\mathrm{Im}(h'(x,v)) - T_{\mathrm{Graph}(U)}(h(x,v)) \;=\; X \times Z$$

But the surjectivity of $\alpha'_v(\beta(x),v)$ implies obviously the surjectivity of $h'(x,v)$, so that this condition is satisfied. Hence, the contingent derivative of V is given by the formula

$$\left\{ \begin{array}{l} DV(x,v)(x') \;=\; \alpha'_v(\beta(x),v)^{-1}\,[\\ DU(x,\alpha(\beta(x),v))(x') - \alpha'_y(\beta(x),v)\beta'(x)x'\big] \end{array} \right.$$

Therefore, we observe that the second statement of the theorem states that the graph of V is a viability domain. □

7.2.2 Subregulation and Metaregulation Maps

The assumption of the above theorem is too strong, since it requires that property (7.3) is satisfied for all controls u of $U(x)$ (so that we have a solution for every initial control chosen in $U(x_0)$.) This means that, setting

$$R_K(x) := \{u \in U(x) \mid f(x,u) \in T_K(x)\}$$

we are in the situation where $R_K = U$.

We may very well be content with the existence of a smooth solution for only some initial control in a subset $R(x_0)$ of $U(x_0)$.

So, we can relax the problem by looking for closed set-valued feedback maps R contained in U in which we can find the initial state-controls yielding smooth viable solutions to the control system.

The Viability Theorem implies the following

Theorem 7.2.5 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \text{ is continuous and has linear growth} \end{cases} \tag{7.6}$$

Let $(x,u) \to \varphi(x,u)$ be a nonnegative continuous function with linear growth and $R : Z \rightsquigarrow X$ a closed set-valued map contained in U. Then the two following conditions are equivalent:

a) — R regulates φ-smooth viable solutions in the sense that for any initial state $x_0 \in \text{Dom}(R)$ and any initial control $u_0 \in R(x_0)$, there exists a φ-smooth state-control solution $(x(\cdot), u(\cdot))$ to the control system (7.1) starting at (x_0, u_0) and viable in the graph of R.

b) — R is a solution to the partial differential inclusion

$$\forall\ (x,u) \in \text{Graph}(R),\ \ 0 \in DR(x,u)(f(x,u)) - \varphi(x,u)B \tag{7.7}$$

satisfying the constraint: $\forall\ x \in K,\ \ R(x) \subset U(x)$.

In this case, such a map R is contained in the regulation map R_K, and is thus called a φ-subregulation map of U *or simply a* subregulation map . *The* metaregulation law *regulating the evolution of state-control solutions viable in the graph of R takes the form of the system of differential inclusions*

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) \in G_R(x(t), u(t)) \end{cases} \tag{7.8}$$

where the set-valued map G_R defined by

$$G_R(x,u) \;:=\; DR(x,u)(f(x,u)) \cap \varphi(x,u)B$$

is called the metaregulation map *associated with R.*

Furthermore, there exists a largest φ-subregulation map denoted R^{φ} contained in U.

Proof — Indeed, to say that R is a regulation map regulating φ-smooth solutions amounts to saying that its graph is viable under the system (7.2).

In this case, we deduce that for any $(x_0, u_0) \in \text{Graph}(R)$, there exists a solution $(x(\cdot), u(\cdot))$ viable in the graph of U, so that $x(\cdot)$ is in particular viable in K. Since $x'(t) = f(x(t), u(t))$ is absolutely continuous, we infer that $f(x_0, u_0)$ is contingent to K at x_0, i.e., that u_0 belongs to $R_K(x_0)$.

The regulation map for the system (7.2) associates with any $(x,u) \in \text{Graph}(R)$ the set of pairs $(x',u') \in \{f(x,u)\} \times \varphi(x,u)B$ such that (x',u') belongs to the contingent cone to the graph of R at (x,u), i.e., such that

$$u' \in DR(x,u)(f(x,u)) \cap \varphi(x,u)B \;=:\; C_R(x,u)$$

The graph of R^{φ} is the viability kernel of Graph(U) for the system of differential inclusions (7.2). □

Proposition 7.2.6 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} \text{i)} & U \text{ maps a neighborhood of every point to a compact subset} \\ \text{ii)} & \text{Graph}(U) \textit{ is upper semicontinuous with compact values} \\ \text{iii)} & f \textit{ is continuous and has linear growth} \end{cases}$$

Then the domain of every subregulation map is closed.

Proof — Let $x_n \in \text{Dom}(R)$ be a sequence converging to x_0 and let u_n belong to $R(x_n) \subset U(x_n)$. By assumption, the sequence u_n remains in a compact subset, so that a subsequence (again denoted by) u_n converges to some $u \in U(x)$. Since R is a subregulation map, there exist solutions $(x_n(\cdot), u_n(\cdot))$ to the system (7.2) of differential inclusions viable in the graph of R. Theorem 3.5.2 implies that a subsequence (again denoted by) $(x_n(\cdot), u_n(\cdot))$ converges to a solution $(x(\cdot), u(\cdot))$ starting at (x, u). Hence $u \in R(x)$ and thus, $x \in \text{Dom}(R)$. $\square$

We can be particularly interested in *single-valued regulation maps* $r : K \mapsto Z$, which are *closed-loop (feedback) controls regulating* φ-*smooth viable solutions*:

Proposition 7.2.7 *A closed single-valued continuous map* r *is a feedback control regulating* φ-*smooth viable solutions to the control problem if and only if* r *is a single-valued solution to the* inclusion

$$\forall\, x \in K,\;\; 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraint

$$\forall\, x \in K,\;\; r(x) \;\in\; U(x)$$

Then for any $x_0 \in K$, *there exists a solution to the differential equation* $x'(t) = f(x(t), r(x(t)))$ *starting at* x_0 *such that*

$$\forall\, t \geq 0,\; u(t) \;:=\; r(x(t)) \;\in\; U(x(t))$$

and

$$\text{for almost all } t \geq 0,\;\; \|u'(t)\| \leq \varphi(x(t), r(x(t)))$$

Remark — The study of set-valued and single-valued solutions to partial differential inclusion (7.7) will be carried over in Chapter 8 in the framework of the more general "tracking property". $\square$

Remark — We observe that any φ-subregulation map remains a ψ-subregulation map for $\psi \geq \varphi$ and in particular, that the largest subregulation maps R^{φ} are increasing with φ. $\square$

Example: Equality Constraints

Consider the case when $h : X \mapsto Y$ is a twice continuously differentiable map and when the viability domain is $K := h^{-1}(0)$.

Since $T_K(x) = \ker h'(x)$ when $h'(x)$ is surjective, we deduce that the regulation map is equal to

$$R_K(x) = \{u \in U(x) \mid h'(x)f(x,u) = 0\}$$

Proposition 7.2.8 *Assume that $h'(x) \in \mathcal{L}(X,Y)$ is surjective whenever $h(x) = 0$, that the graph of U is sleek and that for any $y \in Y$ and $v \in X$, the subsets*

$$DU(x,u)(v) \cap (h'(x)f'_u(x,u))^{-1}(y - h''(x)(f(x,u),v) - h'(x)f'_x(x,u)v)$$

are not empty. Then the contingent derivative $DR_K(x,u)(v)$ of the regulation map is equal to

$$DU(x,u)(v) \cap -(h'(x)f'_u(x,u))^{-1}(h''(x)(f(x,u),v) - h'(x)f'_x(x,u)v)$$

when $h'(x)v = 0$ and $DR_K(x,v) = \emptyset$ if not. In particular, if $U(x) \equiv Z$, then it is sufficient to assume that $h'(x)f'_u(x,u)$ is surjective and we have in this case

$$DR_K(x,u)(v) = -(h'(x)f'_u(x,u))^{-1}(h''(x)(f(x,u),v) - h'(x)f'_x(x,u)v)$$

when $h'(x)v = 0$ and $DR_K(x,v) = \emptyset$ if not.

Proof — The graph of R_K can be written as the subset of pairs $(x,u) \in \text{Graph}(U)$ such that $C(x,u) := (h(x), h'(x)f(x,u)) = 0$. Since the graph of U is closed and sleek, we know that the transversality condition

$$C'(x,u)T_{\text{Graph}(U)}(x,u) = C'(x,u)\text{Graph}(DU(x,u)) = Y \times Y$$

implies that the contingent cone to the graph of U is the set of elements $(v,w) \in \text{Graph}(DU(x,u))$ such that

$$\begin{cases} C'(x,u)(v,w) = \\ (h'(x)v, h'(x)f'_u(x,u)w + h'(x)f'_x(x,u)v + h''(x)(f(x,u),v)) = 0 \end{cases}$$

But the surjectivity of $h'(x)$ and the nonemptiness of the intersection imply this transversality condition. □

Therefore, the right-hand side of the metaregulation rule is equal to

$$\begin{cases} -(h'(x)f'_u(x,u))^{-1}(h''(x)(f(x,u),f(x,u)) - h'(x)f'_x(x,u)f(x,u)) \\ \cap DU(x,u)(f(x,u)) \cap \varphi(x,u)B \end{cases}$$

Example: Inequality Constraints

Consider the case when

$$K \ := \ \{x \in X \ : \ \forall\, i = 1, \ldots, p, \ \ g_i(x) \ \geq \ 0\}$$

is defined by inequality constraints (for simplicity, we do not include equality constraints.)

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of *active constraints* and we assume once and for all that for every $x \in K$,

$$\exists\, v_0 \in C_L(x) \ \text{ such that } \forall\, i \in I(x), \ < g_i'(x), v_0 >> 0$$

so that, by Theorem 5.1.10,

$$R_K(x) \ := \ \{u \in U(x) \mid \forall\, i \in I(x), \langle g_i'(x), f(x,u)\rangle \geq 0\}$$

We set $g(x) := (g_1(x), \ldots, g_p(x))$.

We have seen that the graph of the set-valued map $x \rightsquigarrow R_K(x)$ is not necessarily closed. However, we can find explicit subregulation maps by using Theorem 5.1.11. We thus introduce the set-valued map $R_K^\diamond : X \rightsquigarrow Z$ defined by

$$R_K^\diamond(x) \ := \ \{u \in U(x) \mid g(x) + g'(x)f(x,u) \geq 0\} \ \subset \ R_K(x)$$

We can regulate solutions viable in K by smooth open-loop controls by looking for solutions to the system of differential inclusions (7.2) which are viable in the graph of $R_K^\diamond$.

We thus need to compute the derivative of $R_K^\diamond$ in order to characterize the associated metaregulation map:

Proposition 7.2.9 *Assume that the stronger viability condition*[7]

$$\forall\, x \in K, \ \ R_K^\diamond(x) \ \neq \ \emptyset$$

[7] which holds true whenever K is a viability domain for the control system and

$$\forall\, x \in K, \ \exists\, u \in U(x) \ \text{ such that } \ \|f(x,u)\| \ \leq \ \gamma_K(x)$$

where the function γ_K is defined by (5.1) in Section 5.1. See Theorem 5.1.11.

is satisfied. We set

$$I(x,u) := \{i = 1,\ldots,p \mid g_i(x) + \langle g_i'(x), f(x,u)\rangle = 0\}$$

Assume that U *is sleek and closed and that for every* $(x,u) \in \text{Graph}(R_K^\diamond)$, *there exists* $u_0' \in DU(x,u)(x_0')$ *satisfying*

$$\forall\, i \in I(x,u),\ \langle g_i'(x), x_0' + f_x'(x,u)x_0' + f_u'(x,u)u_0'\rangle + g_i''(x)(f(x,u),x_0') \geq 0$$

Then the contingent derivative $DR_K^\diamond(x,u)(v)$ *of the subregulation map* $R_K^\diamond$ *is defined by:* $u' \in DR_K^\diamond(x,u)(x')$ *if and only if* $u' \in DU(x,u)(x')$ *and*

$$\forall\, i \in I(x,u),\ \langle g_i'(x), x' + f_x'(x,u)x' + f_u'(x,u)u'\rangle + g_i''(x)(f(x,u),x') \geq 0$$

If $U(x) \equiv Z$, *then it is sufficient to assume that* $g'(x)f_u'(x,u)$ *is surjective. We then have in this particular case*

$$\begin{cases} DR_K^\diamond(x,u)(x') := \{u' \in Z \mid \forall\, i \in I(x,u), \\ \langle g_i'(x), f_u'(x,u)u'\rangle \geq -\langle g_i'(x), x' + f_x'(x,u)x'\rangle - g_i''(x)(f(x,u),x')\} \end{cases}$$

Proof — By Theorem 5.1.10 applied to $L := \text{Graph}(U)$ and to the constraints defined by $\tilde{g}_i(x,u) := g_i(x) + \langle g_i'(x), f(x,u)\rangle$, we deduce that $u' \in DR_K^\diamond(x,u)(x')$ if and only if $u' \in DU(x,u)(x')$ and

$$\forall\, i \in I(x,u),\ \langle g_i'(x), x' + f_x'(x,u)x' + f_u'(x,u)u'\rangle + g_i''(x)(f(x,u),x') \geq 0 \quad \square$$

We then deduce from the above Proposition and the Regularity Theorem the following consequence:

Proposition 7.2.10 *We posit the assumptions of Proposition 7.2.9. If for any* $(x,u) \in \text{Graph}(R_K^\diamond)$, *there exists* u' *such that* $\|u'\| \leq \varphi(x,u)$, *then for any initial state* x_0 *and any* $u_0 \in R_K^\diamond(x_0)$, *there exists a solution* $(x(\cdot),u(\cdot))$ *to the control system (7.2) such that* $x(\cdot)$ *is viable in the set* K *defined by inequality constraints. The metaregulation law can then be written*

$$\begin{cases} i) & x'(t) = f(x(t),u(t)) \\ ii) & u'(t) \in G(x(t),u(t)) \end{cases} \tag{7.9}$$

where the metaregulation map G associated to $R_K^\diamond$

$$G(x,u) \;:=\; DR_K^\diamond(x,u)(f(x,u)) \cap \varphi(x,u)B$$

defined by:
$w \in G(x,u)$ *if and only if* $w \in DU(x,u)(f(x,u)) \cap \varphi(x,u)B$ *and*

$$\left\{\begin{array}{l} \forall\, i \in I(x,u),\;\; \langle g_i'(x), f_u'(x,u)u'\rangle \\ \geq\; -\langle g_i'(x), f(x,u) + f_x'(x,u)f(x,u)\rangle - g_i''(x)(f(x,u),f(x,u)) \end{array}\right.$$

Naturally, the graph of the metaregulation map G is not necessarily closed. However, we can still use Theorem 5.1.11 to obtain a "submetaregulation map" of this system of differential inclusions. We introduce the set-valued map $G^\diamond$ defined by: $u' \in G^\diamond(x,u)$ if and only if $\|u'\| \leq \varphi(x,u)$ and

$$\left\{\begin{array}{l} \forall\, i = 1,\ldots,p,\;\; \langle g_i'(x), f_u'(x,u)u'\rangle \\ \geq\; -g_i(x) - \langle g_i'(x), 2f(x,u) + f_x'(x,u)f(x,u)\rangle - g_i''(x)(f(x,u),f(x,u)) \end{array}\right.$$

Hence the system of differential inclusions

$$\left\{\begin{array}{lrcl} i) & x'(t) & = & f(x(t),u(t)) \\ \\ ii) & u'(t) & \in & G^\diamond(x(t),u(t)) \cap \varphi(x(t),u(t))B \end{array}\right. \tag{7.10}$$

regulates φ-smooth solutions which are viable in K.

7.3 Second Order Differential Inclusions

Viability problems for second order differential inclusions also require the use of viability kernels.

Let us consider a set-valued map $F : X \times X \rightsquigarrow X$ and the second order differential inclusion

$$\text{for almost all } t \geq 0,\;\; x''(t) \in F(x(t),x'(t)) \tag{7.11}$$

If we are looking for differentiable solutions $x(\cdot)$ which are viable in K, we know that $\forall\, t \geq 0,\;\; x'(t) \in T_K(x(t))$, i.e., $(x(t),x'(t)) \in \text{Graph}(T_K)$. So the viability condition $x(t) \in K$ involves the underlying viability condition $x'(t) \in T_K(x(t))$. Hence, a necessary condition for having viable solutions is that the closure of the graph of T_K is contained in the domain of F.

As usual, we regard the second order differential inclusion as the system of first order differential inclusions

$$\begin{cases} i) & \text{for almost all } t \geq 0, \quad x'(t) = u(t) \\ ii) & \text{and } \; u'(t) \in F(x(t), u(t)) \end{cases}$$

and the viability condition $x(t) \in K$ as the first order viability constraint

$$\forall\, t \geq 0, \quad (x(t), x'(t)) \in \text{Graph}(T_K)$$

So, by the very definition of contingent derivatives, the necessary condition of viability can be expressed in the form

$$\forall\, (x, u) \in \text{Graph}(T_K), \quad F(x, u) \cap DT_K(x, u)(u) \neq \emptyset \tag{7.12}$$

Viability Theorem 3.3.5 implies the following result:

Proposition 7.3.1 *Assume that the graph of the contingent cone $T_K(\cdot)$ is closed and contained in the domain of a Marchaud map F.*

Then the necessary and sufficient condition for the second order differential inclusion (7.11) to have viable solutions starting from any initial state $x_0 \in K$ and any initial velocity $u_0 \in T_K(x_0)$ is that condition (7.12) *is satisfied.*

This condition is satisfied whenever K is a smooth subset of the form $h^{-1}(0)$:

Corollary 7.3.2 *Let $h : X \mapsto Y$ be a twice continuously differentiable map such that $h'(x) \in \mathcal{L}(X, Y)$ is surjective whenever $h(x) = 0$ and $K := h^{-1}(0)$. Then differential inclusion (7.11) has a viable solution starting from any initial state $x_0 \in K$ and any initial velocity u_0 satisfying $h'(x_0)u_0 = 0$ if and only if*

$$\forall\, x \in K, \; \forall\, u \in \ker h'(x), \quad -h'(x)F(x, u) \cap h''(x)(u, u) \neq \emptyset$$

Proof — We already know that $T_K(x) = \ker h'(x)$ because $h'(x)$ is surjective, so that the transversality condition is satisfied.

Since the graph of T_K can be described by the equation $B(x,u) = 0$ where

$$B(x,u) \;:=\; (h(x), h'(x)v)$$

Its derivative $B'(x,u) \in \mathcal{L}(X \cap X, X \cap X)$ is equal to

$$B'(x,u)(v,w) = (h'(x)v, h''(x)(u,v) + h'(x)w)$$

and is surjective thanks to the surjectivity of $h'(x)$. Therefore, the contingent cone to the graph of the set-valued map $T_K(\cdot)$ is the subset of elements (v,w) such that $B'(x,u)(v,w) = 0$, i.e., the subset of elements $v \in T_K(x)$ and $w \in -h'(x)^{-1}h''(x)(u,v)$. In other words,

$$DT_K(x,u)(v) = \begin{cases} -h'(x)^{-1}h''(x)(u,v) & \text{if } v \in T_K(x) \\ \emptyset & \text{if } v \notin T_K(x) \end{cases}$$

Hence tangential condition (7.12) is equivalent to the condition of the corollary. □

Unfortunately, the graph of the contingent cone is not closed, nor even locally compact, as soon as the viability constraints involve inequality constraints. In this case, this condition is no longer sufficient, as the following example shows.

Example Take $X := \mathbf{R}$ and $K := \mathbf{R}_+$ and the differential inclusion $x''(t) = x(t)+1$. We see easily that the tangential condition (7.12) is satisfied. However, there is no solution to this second order differential equation starting from $(0,0)$. □

If the graph of $T_K(\cdot)$ is not closed, we can look for explicit closed set-valued maps contained in $T_K(\cdot)$, such as the maps $T_K^c(\cdot)$ (see Definition 4.4.1), or the maps $T_K^\diamond(\cdot)$ introduced by N. Maderner in the case of inequality constraints (see Theorem 5.1.11).

In the general case, we can regard the viability kernel of its closure as the graph of a closed set-valued map (possibly empty) R. Theorem 4.1.2 implies the following consequence:

Theorem 7.3.3 *Assume that $F : X \times X \rightsquigarrow X$ is a Marchaud map. Let K be a subset such that $\overline{\text{Graph}(T_K)} \subset \text{Dom}(F)$.*

Then there exists a largest closed set-valued map $R : X \rightsquigarrow X$ such that second order differential inclusion (7.11) has a viable solution for any initial state $x_0 \in \mathrm{Dom}(R)$ and initial velocity $u_0 \in R(x_0)$.

If we are not interested by global properties, but are satisfied with local properties, we can look for locally compact viability domains of $u \times F(x,u)$ comprised between the graph of R (the largest closed viability domain) and the graph of T_K (a viability domain which may not be locally compact), because Viability Theorem 3.3.2 requires only local compactness for having local viable solutions.

This happens whenever the graph of the interior of the contingent cone $\mathrm{Int}(T_K)$ is open (this is the case when the interior of a closed convex subset K is not empty, for instance.) Then, by taking initial velocities $u_0 \in \mathrm{Int}(T_K(x_0))$, we deduce from Theorem 3.3.2 the existence of a viable solution $x(\cdot)$ on some $[0,T]$.

In the nonconvex case, one can take initial velocities u_0 in the Dubovitsky-Miliutin cone $D_K(x_0)$ (see Definition 4.3.1.)

7.4 Metaregulation Map of High Order

The above results can naturally be extended to the regulation of control systems by smooth controls of order $m > 1$.

We introduce a set-valued map $U_m : X \times Z^{m-2} \rightsquigarrow Z$ satisfying

$$\text{if } \exists\, u_0, \ldots, u_{m-1} \mid u_{m-1} \in U_m(x, u_0, \ldots, u_{m-2}), \text{ then } u_0 \in U(x)$$

We can take for instance $\mathrm{Graph}(U_m) := \mathrm{Graph}(U) \times Z^{m-2}$, but we shall propose later other choices of closed maps U_m.

Let us consider a nonnegative continuous function

$$(x, u_0, \ldots, u_{m-1}) \in \mathrm{Graph}(U_m) \to \varphi(x, u_0, \ldots, u_{m-1}) \in \mathbf{R}_+$$

with linear growth.

We obtain smooth viable solutions of order m by setting a bound to the m-th derivative of the control. For that purpose, we associate with this control system and φ the system of differential inclusions

$$\begin{cases} i) \quad x'(t) = f(x(t), u(t)) \\ ii) \quad u^{(m)}(t) \in \varphi(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t))B \end{cases} \tag{7.13}$$

Let us consider a closed set-valued map $R_m : X \times Z^{m-1} \rightsquigarrow Z$. We also regard the graph of R_m as the graph of the set-valued map $N_m : Z^m \rightsquigarrow X$ defined by

$$x \in N_m(u_0, \ldots, u_{m-1}) \text{ if and only if } u_{m-1} \in R_m(x, u_0, \ldots, u_{m-2})$$

and $K_m := \operatorname{Im}(N_m)$ its image.

Theorem 7.4.1 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & \operatorname{Graph}(U_m) \text{ is closed} \\ ii) & f \text{ is continuous and has linear growth} \end{cases}$$

Let us consider a closed set-valued map $R_m : X \times Z^{m-2} \rightsquigarrow Z$ contained in U_m. Then the two following conditions are equivalent:

a) — R_m regulates φ-smooth viable solutions of order m in the sense that for any initial $(x_0, u_0, u_1, \ldots, u_{m-1}) \in \operatorname{Graph}(R_m)$, there exists a solution $x(\cdot) \in W^{1,1}(0,\infty; X, e^{bt})$ and a control $u(\cdot) \in W^{m,1}(0,\infty; Z, e^{bt})$ to the control system (7.1) satisfying the initial conditions

$$x(0) = x_0, \;\; u(0) = u_0, \;\; u'(0) = u_1 \;\; \ldots, \;\; u^{(m-1)}(0) = u_{m-1}$$

the growth condition

$$\left\| u^{(m)}(t) \right\| \leq \varphi(x(t), \ldots, u^{(m-1)}(t))$$

and the constraints[8]

$$\forall\, t \geq 0, \;\; x(t) \in N_m(u(t), u'(t), \ldots, u^{(m-2)}(t))$$

b) — R_m is a solution to the partial differential inclusion[9]

$$\begin{cases} \forall\, (x, u_0, \ldots, u_{m-1}) \in \operatorname{Graph}(R_m), \\ 0 \in DR_m(x, u_0, \ldots, u_{m-1})(f(x, u_0), u_1, \ldots, u_{m-1}) \\ -\varphi(x, u_0, \ldots, u_{m-1})B \end{cases}$$

[8]which can also be written in the form

$$\forall\, t \geq 0, \;\; u^{(m-1)}(t) \in R_m(x(t), u(t), u'(t), \ldots, u^{(m-2)}(t))$$

[9]or N_m is a solution to the partial differential inclusion

$$0 \in DN_m(u_0, \ldots, u_{m-1})(u_1, \ldots, u_{m-1}, \varphi(x, \ldots, u_{m-1})) - f(x, u_0)$$

satisfying the constraint: $R_m(x, u_0, \ldots, u_{m-2}) \subset U_m(x, u_0, \ldots, u_{m-2})$.

In this case, such a map R_m *is called a* φ-growth subregulation map of order m of U *or simply a* subregulation map of order m .

The metaregulation law of order m *regulating the evolution of state-control solutions viable in the graph of R takes the form of the system of differential inclusions*

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u^{(m)}(t) \in G_{R_m}(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t)) \end{cases} \tag{7.14}$$

where the metaregulation map G_{R_m} of order m *is defined by*

$$\begin{cases} G_{R_m}(x, u_0, \ldots, u_{m-1}) := \\ DR_m(x, u_0, \ldots, u_{m-1})(f(x, u_0), \ldots, u_{m-1}) \cap \varphi(x, u_0, \ldots, u_{m-1})B \end{cases}$$

There exists a largest φ*-growth subregulation map denoted* R_m^{φ} *contained in* U_m.

Proof — We introduce the differential inclusion

$$\begin{cases} x'(t) = f(x(t), u_0(t)) \\ u_0'(t) = u_1(t) \\ \cdots \\ u_{m-2}'(t) = u_{m-1}(t) \\ u_{m-1}'(t) \in \varphi(x(t), u_0(t), \ldots, u_{m-1}(t))B \end{cases} \tag{7.15}$$

where the state space is $X \times Z^m$ and the set of constraints is $\text{Graph}(U_m) \subset X \times Z^m$.

To say that R_m is a subregulation map regulating smooth solutions of order m amounts to saying that its closed graph is viable under the above system (7.15).

The metaregulation map of order m, which is the regulation map of the system (7.15) yielding viable solutions in the graph of R_m, is the set of velocities

$$(f(x, u_0), u_1, \ldots, u_{m-1}, u')$$

where $u' \in \varphi(x, u_0, \ldots, u_{m-1})B$ which are contingent to the graph of R_m at $(x, u_0, \ldots, u_m)$, i.e., which satisfy

$$u' \in DR_m(x, u_0, u_1, \ldots, u_{m-1})(f(x, u_0), u_1, \ldots, u_{m-1})$$

The graph of the largest subregulation map R_m^{φ} of order m is the viability kernel of $\text{Graph}(U_m)$ for this system of differential inclusions. □

7.5 Punctuated Equilibria, Ramp Controls and Polynomial Open-Loop Controls

The case when the growth φ is equal to 0 is particularly interesting, because the inverse N^0 of the 0-growth regulation map R^0 determines the areas $N^0(u)$ regulated by constant control u.

One could call $N^0(u)$ the *viability cell or niche* of u. A control u is called a *punctuated equilibrium* if and only if its viability cell is not empty. Naturally, *when the viability cell of a punctuated equilibrium is reduced to a point, this point is an equilibrium.*

So, punctuated equilibria are constant controls which regulate the control systems (in its viability cell):

Proposition 7.5.1 *The viability cell of a control u is the viability kernel of $U^{-1}(u)$ for the differential equation $x'(t) = f(x(t), u)$ parametrized by the constant control u.*

Proof — Indeed, viability cells describe the regions of $\text{Dom}(U)$ which are controlled by the constant control u because for any initial state x_0 given in $N^0(u)$, there exists a viable solution $x(\cdot)$ to the differential inclusion

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) = 0 \end{cases}$$

starting at (x_0, u), i.e., of the differential equation $x'(t) = f(x(t), u)$ which is *viable in the viability cell* $N^0(u)$ because $u \in R^0(x(t))$ for every $t \geq 0$. □

One can ask more generally whether linear open-loop controls $u(t) := u_0 + tu_1$ can regulate viable solutions to the control systems, and what are the largest areas of the viability domain which can be regulated by linear controls. Such controls are called *ramp controls* .

The advantage is that in such areas, finding the ramp controls amounts to looking only for two elements u_0 and u_1 in the finite dimensional space Z^2 rather than a general function $u(\cdot)$ in an infinite-dimensional space $W^{1,1}(0,\infty; Z, e^{bt})$.

Pursuing this point of view, the problem arises of *regulating viable solutions to a control system by polynomial open-loop controls of degree m*. For $m = 0$, we find the punctuated equilibria, for $m = 1$ the ramp controls, and so on.

We consider the graph $\mathrm{Graph}(R_m^0)$ of the largest m-smooth 0-growth regulation map of the system (7.15) and we denote by $K_m^0 := \mathrm{Im}(N_m^0)$.

Proposition 7.5.2 *We posit the assumptions of Theorem 7.4.1. Then $K_m^0 \subset K$ is the largest subset of initial states from which there exist viable solutions regulated by m-degree polynomial open-loop controls.*

Controlling the system from $x_0 \in K_m^0$ amounts to choosing initial controls $(u_0, u_1, \ldots, u_m) \in (N_m^0)^{-1}(x_0) \subset Z^{m+1}$. In this case, there exists a viable solution $x(\cdot)$ to the control system

$$x'(t) \;=\; f\left(x(t), u_0 + u_1 t + \ldots + u_{m-1}\frac{t^{m-1}}{(m-1)!}\right)$$

satisfying

$$x(0) \;=\; x_0,\;\; u(0) \;=\; u_0,\;\; u'(0) \;=\; u_1,\;\; \ldots,\;\; u^{(m-1)}(0) \;=\; u_{m-1}$$

and the regulation law written in the form

$$\forall\, t \geq 0,\;\; x(t) \;\in\; N_m^0\left(\sum_{j=0}^{m-1} u_j\frac{t^j}{j!}, \ldots, \sum_{j=0}^{m-k-1} u_{j+k}\frac{t^j}{j!}, \ldots, u_{m-1}\right)$$

We naturally obtain

$$K^0 \;:=\; K_1^0 \;\subset\; K_1^0 \;\subset\; \ldots \;\subset K_m^0 \;\subset\; \ldots \;\; K := \mathrm{Dom}(U)$$

and, for $k \leq m$,

$$N_k^0(u_0, \ldots, u_{k-1}) = N_m^0(u_0, \ldots, u_{k-1}, 0, \ldots, 0)$$

Remark — In the case of the general evolution of open-loop controls, the regulation maps are solutions to the partial differential inclusion

$$\forall (x, v) \in \text{Graph}(V), \quad Av \in DR(x, v)(g(x, v)) - \Phi(x, v)$$

subject to the constraint

$$\forall x \in X, \quad R(x) \subset V(x)$$

In particular, for $\Phi = 0$, we obtain the subset of initial states x_0 from which there exist viable solutions to the control system

$$x'(t) = f(x(t), e^{At}v_0)$$

regulated by open-loop controls

$$v(t) = e^{At}v_0$$

which are solutions to the system of differential equations

$$v'(t) = Av(t), \quad v(0) = v_0$$

7.6 Heavy Viable Solutions

7.6.1 Dynamical Closed Loops

Let us consider a control system (U, f), a regulation map $R \subset U$ which is a solution to the partial differential inclusion (7.7) and the metaregulation map

$$(x, u) \rightsquigarrow G_R(x, u) := DR(x, u)(f(x, u)) \cap \varphi(x, u)B$$

regulating smooth state-control solutions viable in the graph of R through the system (7.8) of differential inclusions.

The question arises as to whether we can construct selection procedures of the control component of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 7.6.1 (Dynamical Closed Loops) *Let R be a φ-growth subregulation map of U. We shall say that a selection g of the contingent derivative of the metaregulation map G_R associated with R mapping every $(x,u) \in \mathrm{Graph}(R)$ to*

$$g(x,u) \in G_R(x,u) := DR(x,u)(f(x,u)) \cap \varphi(x,u)B \tag{7.16}$$

is a dynamical closed loop *of R.*

The system of differential equations

$$\begin{cases} i) & x'(t) = f(x(t),u(t)) \\ ii) & u'(t) = g(x(t),u(t)) \end{cases} \tag{7.17}$$

is called the associated closed loop differential system.

Clearly every solution to (7.17) is also a solution to (7.8). Therefore, a dynamical closed loop being given, solutions to the system of ordinary differential equations (7.17) (if any) are smooth state-control solutions of the initial control problem (7.1).

Such solutions do exist when g is continuous (and if such is the case, they will be continuously differentiable.) But they also may exist when g is no longer continuous, as we saw when we built closed loop controls in Chapter 6. This is the case for instance when $g(x,u)$ is the element of minimal norm in $G_R(x,u)$.

In both cases, we need to assume that the metaregulation map G_R associated with R is lower semicontinuous with closed convex images. By Proposition 7.1.3, it will be sufficient to assume that:

$$\begin{cases} i) & R \text{ is sleek} \\ ii) & \sup_{(x,u)\in\mathrm{Graph}(R)} \|DR(x,u)\| < +\infty \end{cases} \tag{7.18}$$

Indeed, assumptions (7.18)i) and ii) imply that the set-valued map $(x,u,v) \rightsquigarrow DR(x,u,v)$ is lower semicontinuous. Since φ is continuous, we infer from Proposition 6.3.2 that the metaregulation map G_R is also lower semicontinuous.

We thus begin by deducing from Michael's Theorem 6.5.7 the existence of continuously differentiable viable state-control solutions.

Theorem 7.6.2 *Assume that U is closed and that f, φ are continuous and have linear growth. Let $R(\cdot) \subset U(\cdot)$ be a φ-growth subregulation map satisfying assumption (7.18). Then there exists a continuous dynamical closed loop g associated with R. The associated closed loop differential system (7.17) regulates continuously differentiable state-control solutions to (7.1) defined on* $[0, \infty[$.

7.6.2 Heavy Viable Solutions

Since we do not know constructive ways to build continuous dynamical closed loops, we shall investigate whether some explicit dynamical closed loop provides closed loop differential systems which do possess solutions.

The simplest example of dynamical closed loop control is the minimal selection of the metaregulation map G_R, which in this case is equal to the map g_R° associating with each state-control pair (x, u) the element $g_R^\circ(x, u)$ of minimal norm of $DR(x, u)(f(x, u))$ because for all (x, u), $\|g_R^\circ(x, u)\| \le \varphi(x, u)$ whenever $G_R(x, u) \neq \emptyset$.

Definition 7.6.3 (Heavy Viable Solutions) *Denote by $g_R^\circ(x, u)$ the element of minimal norm of $DR(x, u)(f(x, u))$. We shall say that the solutions to the associated closed loop differential system*

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) = g_R^\circ(x(t), u(t)) \end{cases}$$

are heavy viable solutions *to the control system (U, f) associated with R.*

Theorem 7.6.4 (Heavy Viable Solutions) *Let us assume that U is closed and that f, φ are continuous and have linear growth. Let $R(\cdot) \subset U(\cdot)$ be a φ-growth subregulation map satisfying assumption (7.18). Then for any initial state-control pair (x_0, u_0) in Graph(R), there exists a heavy viable solution to the control system (7.1).*

Remark — *Any heavy viable solution $(x(\cdot), u(\cdot))$ to the control system (7.1) satisfies the inertia principle*: Indeed, we observe that if for some t_1, the solution enters the subset $C_R(u(t_1))$ where we set

$$C_R(u) := \{x \in K \mid 0 \in DR(x, u)(f(x, u))\}$$

the control $u(t)$ remains equal to $u(t_1)$ as long as $x(t)$ remains in $C_R(u(t_1))$. Since such a subset is not necessarily a viability domain, the solution may leave it.

If for some $t_f > 0$, $u(t_f)$ is a punctuated equilibrium, then $u(t) = u_{t_f}$ for all $t \geq t_f$ and thus, $x(t)$ *remains in the viability cell* $N_1^0(u(t_f))$ *for all* $t \geq t_f$. □

The reason why this theorem holds true is that the minimal selection is obtained through the selection procedure defined by

$$S^{\circ}_{G_R}(x,u) \;:=\; \|g^{\circ}_R(x,u)\|\, B \tag{7.19}$$

It is this fact which matters. So, Theorem 7.6.4 can be extended to any selection procedure of the metaregulation map $G_R(x,u)$ defined in Chapter 6 (See Definition 6.5.2).

Theorem 7.6.5 *Let us assume that the control system (7.1) satisfies*

$$\begin{cases} i) & \text{Graph}(U) \;\textit{is closed} \\ ii) & f \;\textit{is continuous and has linear growth} \end{cases} \tag{7.20}$$

Let $(x,u) \to \varphi(x,u)$ *be a nonnegative continuous function with linear growth and* $R : Z \rightsquigarrow X$ *a closed set-valued map contained in* U.

Let S_{G_R} : Graph$(R) \rightsquigarrow X$ *be a selection procedure with convex values of the metaregulation map* G_R. *Then, for any initial state* $(x_0, u_0) \in$ Graph(R), *there exists a state-control solution to the associated closed loop system*

$$x' = f(x,u), \quad u' \in G_R(x,u) \cap S_{G_R}(x,u) \tag{7.21}$$

defined on $[0,\infty[$ *and starting at* (x_0,u_0). *In particular, if for any* $(x,u) \in$ Graph(R), *the intersection*

$$G_R(x,u) \cap S_{G_R}(x,u) \;=\; \{s\,(G_R(x,u))\}$$

is a singleton, then there exists a state-control solution defined on $[0,\infty[$ *and starting at* (x_0,u_0) *to the associated closed loop system*

$$x'(t) \;=\; f(x(t),u(t)), \;\; u'(t) \;=\; s\,(G_R(x(t),u(t)))$$

Proof — Consider the system of differential inclusions

$$x' = f(x,u), \quad u' \in S_{G_R}(x,u) \cap \varphi(x,u)B \tag{7.22}$$

subject to the constraints

$$\forall\, t \geq 0, \quad (x(t), u(t)) \in \mathrm{Graph}(R)$$

Since the selection procedure S_{G_R} has a closed graph and convex values, the right-hand side is an upper semicontinuous set-valued map with nonempty compact convex images and with linear growth. On the other hand Graph(R) is a viability domain of the map $\{f(x,u)\} \times (S_{G_R}(x,u) \times \varphi(x,u)B)$. Therefore, the Viability Theorem can be applied. For any initial state-control $(x_0, u_0) \in$ Graph(R), there exists a solution $(x(\cdot), u(\cdot))$ to (7.22) which is viable in Graph(R). Consequently, for almost all $t \geq 0$, the pair $(x'(t), u'(t))$ belongs to the contingent cone to the graph of R at $(x(t), u(t))$, which is the graph of the contingent derivative $DR(x(t), u(t))$. In other words, for almost all $t \geq 0$, $u'(t) \in DR(x(t), u(t))(f(x(t), u(t)))$. Since $\|u'(t)\| \leq \varphi(x(t), u(t))$, we deduce that $u'(t) \in G_R(x(t), u(t))$ for almost all $t \geq 0$. Hence, the state-control pair $(x(\cdot), u(\cdot))$ is a solution to (7.21). □

Proof of Theorem 7.6.4 — By the Maximum Theorem 2.1.6 the map $(x,u) \mapsto \|g_R^\circ(x,u)\|$ is upper semicontinuous. It has a linear growth on Graph(R). Thus the set-valued map $(x,u) \rightsquigarrow \|g_R^\circ(x,u)\|\, B$ is a selection procedure satisfying the assumptions of Theorem 7.6.5. □

Since we know many examples of selection procedures, it is possible to multiply examples of dynamical closed-loops as we did for usual closed loops. We shall see some examples in the framework of differential games in Chapter 14.

7.6.3 Heavy Viable Solutions under Equality Constraints

Consider the case when $h : X \mapsto Y$ is a twice continuously differentiable map, when the viability domain is $K := h^{-1}(0)$ and when there are no constraints on the controls ($U(x) = Z$ for all $x \in K$). We derive from Proposition 7.2.8 the following explicit formulas for the dynamical closed loop yielding heavy solutions.

Proposition 7.6.6 *We posit assumptions of Theorem 7.2.8. Assume further that* $U(x) \equiv Z$*, that the regulation map*

$$R(x) \; := \; \{u \in Z \mid h'(x)f(x,u) = 0\}$$

has nonempty values, that $h(x)$ *is surjective whenever* $x \in K$ *and that* $h'(x)f'_u(x,u) \in \mathcal{L}(Z,Y)$ *is surjective whenever* $u \in R(x)$.

Then there exist heavy solutions viable in K*, which are the solutions to the system of differential equations*

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -f'_u(x,u)^\star h'(x)^\star p(x,u) \text{ where} \\ & p(x,u) := (h'(x)f'_u(x,u)f'_u(x,u)^\star h'(x)^\star)^{-1}h'(x)f'_x(x,u)f(x,u) \end{cases}$$

Proof — The element $g(x,u) \in G(x,u)$ of minimal norm, being a solution to the quadratic minimization problem with equality constraints

$$h'(x)f'_u(x,u)w = -h'(x)f'_x(x,u)f(x,u) - h''(x)(f(x,u),f(x,u))$$

is equal to

$$\begin{array}{l} g(x,u) = -f'_u(x,u)^\star h'(x)^\star (h'(x)f'_u(x,u)f'_u(x,u)^\star h'(x)^\star)^{-1} \\ (h'(x)f'_x(x,u)f(x,u) + h''(x)(f(x,u),f(x,u))) \end{array}$$

because the linear operator $B := h'(x)f'_u(x,u) \in \mathcal{L}(Z,Y)$ is surjective.

Example: Heavy solutions viable in affine spaces. Consider the case when $K := \{x \in X \mid Lx = y\}$ where $L \in \mathcal{L}(X,Y)$ is surjective.

Let us assume that

$$\begin{cases} i) & \forall\, x \in K,\; R(x) := \{u \in Z \;\text{ such that }\; Lf(x,u) = 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\; \forall\, u \in R(x),\;\; Lf'_u(x,u) \;\text{ is surjective} \end{cases}$$

Then, for any initial state $x_0 \in K$ and initial velocity u_0 satisfying $Lf(x_0,u_0) = 0$, there exists a heavy viable solution given by the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -f'_u(x,u)^\star L^\star (Lf'_u(x,u)f'_u(x,u)^\star L^\star)^{-1} Lf'_x(x,u)f(x,u) \end{cases}$$

When $Y := \mathbf{R}$ and $K := \{x \in X \mid \; < p,x >= y\}$ is an hyperplane, the above assumption becomes

$$\begin{cases} i) & \forall\, x \in K,\; R(x) := \{u \in Z \;\text{ such that }\; < p, f(x,u) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\; \forall\, u \in R(x),\;\; f'_u(x,u)^\star p \neq 0 \end{cases}$$

and heavy viable solutions are solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -\frac{<p, f'_x(x,u), f(x,u)>}{\|f'_u(x,u)^\star p\|^2} f'_u(x,u)^\star p \end{cases}$$

Example: Heavy solutions viable in the sphere.

Let $L \in \mathcal{L}(X,X)$ be a symmetric positive-definite linear operator, with which we associate the viability subset

$$K := \{x \in X \mid < Lx, x >= 1\}$$

We assume that

$$\begin{cases} i) & \forall\, x \in K,\ R(x) := \{u \in Z \ \text{ such that } \ < Lx, f(x,u) >= 0\} \neq \emptyset \\ ii) & \forall\, x \in K,\ \forall\, u \in R(x),\ \ f'_u(x,u)^\star Lx \neq 0 \end{cases}$$

Then there exist heavy viable solutions in the sphere, which are solutions to the system of differential equations

$$\begin{cases} i) & x' = f(x,u) \\ ii) & u' = -\frac{f'_u(x,u)^\star Lx}{\|f'_u(x,u)^\star Lx\|^2} p(x,u) \ \text{ where} \\ & p(x,u) := < Lf(x,u), f(x,u) > + < Lx, f'_x(x,u) f(x,u) > \end{cases}$$

7.6.4 Heavy Viable Solutions of High Order

We shall extend the concept of heavy viable solutions to higher order.

For simplicity, we explain what happens for the first order, in the case when we want to satisfy both the inertia principle and a first-order inertia principle: keep a ramp control as long as it regulates a viable smooth solution.

We begin with the control system (U, f), we set $U_1 := U$ and $\varphi_1 := \varphi$, we choose a φ_1-growth subregulation map $R_1(\cdot) := R(\cdot) \subset U(\cdot)$ and we denote by

$$G_1(x,u) := DR_1(x,u)(f(x,u)) \cap \varphi_1(x,u)B$$

the metaregulation rule associated with R_1.

Since we know that the evolution of heavy viable solutions is governed by the differential equation

$$u_0'(t) = u_1(t) = g_1^\circ(x(t), u_0(t))$$

where g_1° is the minimal selection of G_1, the instinctive idea which comes to mind is to take for set-valued map U_1 the (single-valued) map g_1°. Unfortunately, its graph is not closed.

Since the minimal selection g_1° is obtained through the selection procedure defined by (7.19), another idea is to use any selection procedure S_{G_1} of the set-valued map G_1 and in particular the one defined by (7.19):

$$S_{G_1}^\circ(x, u) := \|g_1^\circ(x, u)\| B$$

We then define U_2 by

$$\text{Graph}(U_2) := (\text{Graph}(R_1) \times Z) \cap \text{Graph}(S_{G_1})$$

and we introduce a continuous function $\varphi_2 : \text{Graph}(U_2) \mapsto \mathbf{R}_+$ with linear growth.

The graph of U_2 is closed. This choice being made, we associate a φ_2-growth subregulation map $R_2 \subset U_2$ (for instance, the viability kernel of the graph of U_2.) We know that the evolution of the second derivative of the control is governed by the metaregulation law

$$u''(t) \in G_2(x(t), u(t), u'(t))$$

where we denote by

$$G_2(x, u_0, u_1) := DR_2(x, u_0, u_1)(f(x, u_0), u_1) \cap \varphi_2(x, u_0, u_1)B$$

the metaregulation map associated with R_2. We propose to govern the evolution of the second derivative of the control by selections of the map G_2, and in particular, by its selection of minimal norm g_2°, which then yields a second-order heavy viable solution.

Theorem 7.6.7 (Second-Order Heavy Viable Solutions) *Let us assume that U_1 is closed and that f, φ_1, φ_2 are continuous and have linear growth, that conditions (7.18) and*

$$\begin{cases} i) & \text{the subregulation map } R_2 \text{ is sleek} \\ ii) & \sup_{(x,u_0,u_1)\in\text{Graph}(R_2)} \|DR_2(x, u_0, u_1)\| < +\infty \end{cases} \quad (7.23)$$

hold true. Then for any initial data $u_1 \in R_2(x_0, u_0)$, there exists a second-order heavy viable solution to the control system (7.1), i.e., a solution to the system

$$\begin{cases} x'(t) &= f(x(t), u(t)) \\ u'(t) &= g_1^\circ(x(t), u(t)) \\ u''(t) &= g_2^\circ(x(t), u(t), u'(t)) \end{cases}$$

Remark — *Any second-order heavy viable solution satisfies the first-order inertia principle.*

For explaining why , let us introduce the subsets

$$\begin{cases} C_k(u_0, \ldots, u_{k-1}) \\ := \{x \in K | 0 \in DR_k(x, u_0, \ldots, u_{k-1})(f(x, u_0), \ldots, u_{k-1})\} \end{cases}$$

for $k = 1,\ 2$.

If for some t_0, the solution enters the subset $C_1(u(t_0))$, then the open-loop control $u(t)$ becomes constant as long as $x(t)$ remains in $C_1(u(t_0))$.

If for some t_1, the solution enters the subset $C_2(u(t_1), u'(t_1))$, then the open-loop control $u(t)$ becomes a ramp control as long as $x(t)$ remains in $C_2(u(t), u'(t))$. In this case, it is regulated by

$$x(t) \in N_2^0\left(u_{t_1} + (t - t_1)u'(t_1), u'(t_1)\right)$$

Since such a subset is not necessarily a viability domain, the solution may leave it.

If for some $t_r > 0$, the solution $x(t)$ enters the subset K_2^0, then it will be regulated by a ramp control, until some time[10] $t_f \in [t_r, \infty[$ where $x(t_f) \in K_1^0$. Then $u_{t_f} \in R_1(x(t_f))$ is a punctuated equilibrium, and $u(t) = u_{t_f}$ for all $t \geq t_f$, so that $x(t)$ *remains in the viability cell* $N_1^0(u(t_f))$ *for all* $t \geq t_f$. □

Naturally, as for heavy viable solutions, this theorem follows from:

[10] which may never be reached

Theorem 7.6.8 *Let us assume that the control system (7.1) and the functions* φ_1, φ_2 *satisfy*

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \;\&\; \varphi_i \text{ are continuous and have linear growth } (i = 1, 2) \end{cases}$$

Let $S_{G_1} : \text{Graph}(R_1) \rightsquigarrow X$ *be a selection procedure of the metaregulation map* G_1*,* U_2 *be defined by*

$$\text{Graph}(U_2) := (\text{Graph}(R_1) \times Z) \cap \text{Graph}(S_{G_1})$$

$R_2 \subset U_2$ *be a subregulation map and* $S_{G_2} : \text{Graph}(R_2) \rightsquigarrow Z$ *be a selection procedure of the metaregulation map* G_2 *with convex values. Set*

$$\begin{cases} S(G_1)(x, u) & := & G_1(x, u) \cap S_{G_1}(x, u) \\ S(G_2)(x, u, u') & := & G_2(x, u, u') \cap S_{G_2}(x, u, u') \end{cases}$$

Then, for any initial state $(x_0, u_0, u_1) \in \text{Graph}(R_2)$*, there exists a solution to the system*

$$\begin{cases} x'(t) & = & f(x(t), u(t)) \\ u'(t) & \in & S(G_1)(x(t), u(t)) \\ u''(t) & \in & S(G_2)(x(t), u(t), u'(t)) \end{cases} \tag{7.24}$$

defined on $[0, \infty[$ *and starting at* (x_0, u_0, u_1)*.*

In particular, if for any $(x, u, u') \in \text{Graph}(R_2)$*, the intersections*

$$S(G_1)(x, u) \;\&\; S(G_2)(x, u, u')$$

are singleta $\{s(G_1)(x, u)\}$ *and* $\{s(G_2)(x, u, u')\}$*, then there exists a state-control solution defined on* $[0, \infty[$ *and starting at* (x_0, u_0) *to the associated closed loop system*

$$\begin{cases} x'(t) & = & f(x(t), u(t)) \\ u'(t) & = & s(G_1)(x(t), u(t)) \\ u''(t) & = & s(G_2)(x(t), u(t), u'(t)) \end{cases}$$

Proof — We consider the system

$$\begin{cases} i) & x'(t) = f(x(t), u_0(t)) \\ ii) & u_0'(t) = u_1(t) \\ iii) & u_1'(t) \in S_{G_2}(x(t), u_0(t), u_1(t)) \cap \varphi_2(x(t), u_0(t), u_1(t))B \end{cases} \tag{7.25}$$

Since the selection procedure S_{G_2} has a closed graph and convex values, the right-hand side of this system of differential inclusions is a Marchaud map.

The closed subset $\text{Graph}(R_2)$ is a viability domain. Indeed, we know that there exists an element w in the selection $S(G_2)(x, u_0, u_1)$. Since $w \in G_2(x, u_0, u_1)) \subset DR_2(x, u_0, u_1)(f(x, u_0), u_1))$, we infer that

$$(f(x, u_0), u_1, w) \in \text{Graph}(DR_2(x, u_0, u_1)) := T_{\text{Graph}(R_2)}(x, u_0, u_1)$$

Hence $(f(x, u_0), u_1, w)$ is a velocity which is contingent to the graph of R_2.

Therefore the Viability Theorem implies the existence of a solution $(x(\cdot), u_0(\cdot), u_1(\cdot))$ to the system of differential inclusions (7.25) viable in the graph of R_2. This implies that for almost all $t \geq 0$, setting $u(\cdot) := u_0(\cdot)$,

$$u''(t) = u_1'(t) \in DR_2(x(t), u(t), u'(t))(f(x(t), u(t)), u'(t))$$

This, together with (7.25)iii), implies that for almost all $t \geq 0$,

$$u''(t) \in G_2(x(t), u(t), u'(t)) \cap S_{G_2}(x(t), u(t), u'(t))$$

Furthermore, since $\text{Graph}(R_2)$ is contained in $\text{Graph}(R_1) \times Z$, we deduce that

$$\forall\, t \geq 0, \;\; u(t) := u_0(t) \in R_1(x(t))$$

so that

$$\forall\, t \geq 0, \;\; u'(t) \in DR_1(x(t), u(t))(f(x(t)), u(t)) \subset G_1(x(t), u(t))$$

On the other hand, by the very choice of U_2, we know that

$$\forall\, t \geq 0,\;\; u'(t) \;:=\; u_1(t) \;\in\; R_2(x(t), u(t)) \;\subset\; S_{G_1}(x(t), u(t))$$

Hence we have proved the existence of a solution to the second-order system of partial differential inclusions (7.24) with a right-hand side which is not a Marchaud map. □

Naturally, we can extend this theorem up to the order m, by recursively choosing the map U_m by formula

$$\mathrm{Graph}(U_m) \;:=\; (\mathrm{Graph}(R_{m-1}) \times Z) \cap \mathrm{Graph}(S_{G_{m-1}})$$

and by taking a subregulation map $R_m \subset U_m$ (for instance, the map whose graph is a viability kernel for the system (7.15).)

In the case of the minimal selection, we take as selection procedure

$$S^{\circ}_{G_m}(x, u_0, \ldots, u_{m-1}) \;:=\; \|g^{\circ}_m(x, u_0, \ldots, u_{m-1})\|\, B$$

where actually, $g^{\circ}_m(x, u_0, \ldots, u_{m-1})$ is the element of minimal norm of

$$DR_m(x, u_0, \ldots, u_{m-1})(f(x, u_0), \ldots, u_{m-1})$$

Theorem 7.6.9 (m-th Order Heavy Viable Solutions) *Assume that U is closed and that f, φ_k are continuous and have linear growth for $0 \leq k \leq m$. We assume further that for $0 \leq k \leq m$,*

$$\begin{cases} i) & \text{the subregulation map } R_k \subset U_k \text{ is sleek} \\ ii) & \sup_{(x,u_0,\ldots,u_{k-1}) \in \mathrm{Graph}(R_k)} \|DR_k(x, u_0, \ldots, u_{k-1})\| < +\infty \end{cases}$$

Then for any initial data $u_{m-1} \in R_m(x_0, u_0, \ldots, u_{m-2})$, there exists an m-th order heavy viable solution to the control system

$$\begin{cases} x'(t) \;=\; f(x(t), u(t)) \\ \\ u'(t) \;=\; g^{\circ}_0(x(t), u(t)) \\ \\ \cdots \\ \\ u^{(m)}(t) \;=\; g^{\circ}_m(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t)) \end{cases}$$

It obeys an *m-th order inertia principle: keep an m-degree polynomial open-loop control as long as the solution it regulates is viable.*

This theorem follows from the more general

Theorem 7.6.10 *Let us assume that the control system (7.1) and the functions φ_k satisfy for $0 \leq k \leq m$*

$$\begin{cases} i) & \text{Graph}(U) \text{ is closed} \\ ii) & f \ \& \ \varphi_k \text{ are continuous and have linear growth} \end{cases}$$

Let S_{G_k} : $\text{Graph}(R_k) \rightsquigarrow X$ be selection procedures with convex values of the set-valued maps G_k. Set

$$S\left(G_k\right)\left(x, u_0, \ldots, u_{k-1}\right) := S_{G_k}(x, u_0, \ldots, u_{k-1}) \cap G_k(x, u_0, \ldots, u_{k-1})$$

Then, for any initial state $(x_0, u_0, u_1, \ldots, u_{m-1}) \in \text{Graph}(R_m)$, there exists a solution to the system

$$\begin{cases} x'(t) &= f(x(t), u(t)) \\ u'(t) &\in S(G_1)(x(t), u(t)) \\ \ldots & \\ u^{(m)}(t) &\in S(G_m)(x(t), u(t), u'(t), \ldots, u^{(m-1)}(t)) \end{cases}$$

defined on $[0, \infty[$ and starting at $(x_0, u_0, u_1, \ldots u_{m-1})$.

Chapter 8

Partial Differential Inclusions of Tracking Problems

Introduction

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \rightsquigarrow X$, $G : X \times Y \rightsquigarrow Y$ and the *system of differential inclusions*

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases}$$

We further introduce a set-valued map $H : X \rightsquigarrow Y$, regarded as an *observation map*.

We devote this chapter to several issues related to the following *tracking property*: for every $x_0 \in \mathrm{Dom}(H)$ and every $y_0 \in H(x_0)$, there exist solutions $(x(\cdot), y(\cdot))$ to the system of differential inclusions such that

$$\forall\, t \geq 0,\ \ y(t) \in H(x(t))$$

This is a *viability problem*, since we actually look for a solution $(x(\cdot), y(\cdot))$ which remains viable in the graph of the observation map H. So, if the set-valued maps F and G are Marchaud maps and if the graph of H is closed, the Viability Theorem states that the tracking property is equivalent to the fact that the graph of H is a viability domain of $(x, y) \leadsto F(x, y) \times G(x, y)$.

Recalling that the graph of the contingent derivative $DH(x, y)$ of H at a point (x, y) of its graph is the contingent cone to the graph of H at (x, y), the tracking property is then equivalent to saying that H is a (set-valued contingent) solution to the *system of partial differential inclusions*

$$\forall\, (x, y) \in \mathrm{Graph}(H), \quad 0 \in DH(x, y)(F(x, y)) - G(x, y)$$

We observe that when F and G are single-valued maps f and g and H is a differentiable single-valued map h, the partial differential inclusion boils down to the more familiar *system of first-order partial differential equations*

$$\forall\, j = 1, \ldots, m, \quad \sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i(x, h(x)) - g_j(x, h(x)) \;=\; 0$$

For special types of systems of differential equations, the graph of such a map h (satisfying additional properties) is called a *center manifold*. Theorems providing the existence of local center manifolds have been widely used for the study of stability near an equilibrium and in control theory.

Since the partial differential inclusion links the three data F, G and H, we can use it in three different ways:

1. — Knowing F and H, find G or selections g of G such that the tracking property holds (observation problem)

2. — Knowing G (regarded as an *exosystem*, following Byrnes & Isidori's terminology) and H, find F or selections f of F such that the tracking property holds (tracking problem)

3. — Knowing F and G, find observation maps H satisfying the tracking property, i.e., solve the above partial differential inclusion.

Furthermore, we can address other specific questions such as:

a) — Find the largest set-valued contingent solution to the partial differential inclusion contained in a given set-valued map (which then, contains all the other ones if any)

b) — Find single-valued contingent solutions h to the partial differential inclusion which then becomes

$$\forall\, x \in K, \quad 0 \in Dh(x)(F(x, h(x))) - G(x, h(x))$$

In this case, the tracking property states that there exists a solution to the *"reduced" differential inclusion*

$$x'(t) \in F(x(t), h(x(t)))$$

so that $(x(\cdot), y(\cdot) := h(x(\cdot)))$ is a solution to the initial system of differential inclusions starting at $(x_0, h(x_0))$. Knowing h allows us to divide the system by half, so to speak.

It may seem strange to accept set-valued maps as solutions to a system of first-order partial differential inclusions. But this may offer a way to describe shocks by the set-valued character of the solution (which may happen even for maps with smooth graphs, but whose projection leads to set-valued maps.)

Set-valued solutions provide a convenient way to treat hyperbolic problems.

Indeed, looking for "weak" solutions to this partial differential inclusion in Sobolev spaces or other spaces of distributions does not help here since we require *solutions h to be defined through their graph, and thus, solutions which are defined everywhere.*

However, derivatives in the sense of distributions do not offer the unique way to describe weak or generalized solutions.

The contingent derivative $v \mapsto Du(x)(v)$ of single-valued map u at x is obtained by taking *upper graphical limits* when $h \to 0$ of the difference-quotients $v \mapsto \frac{u(x+hv)-u(x)}{h}$ whereas the distributional derivatives are limits of the difference-quotients $x \mapsto \frac{u(x+hv)-u(x)}{h}$ in the space of distributions. In both cases, we use convergences weaker than the pointwise convergence for increasing the possibility for the difference-quotients to converge, and, in doing so, we may lose some properties by passing to these weaker limits. In the first case, the contingent derivative is no longer necessarily a single-valued map, but may be set-valued, whereas in the second case, the derivative may be a distribution. Further, each one of these weaker convergence allows us to differentiate set-valued maps U at (x, y) since one can check that the contingent derivative is the graphical upper limits of the difference quotients $v \rightsquigarrow \frac{U(x+hv)-y}{h}$ and to differentiate a distribution T by taking distributional limits of the difference quotients $\frac{\tau_v^h T - T}{h}$.

We devote the first section to general properties of set-valued contingent solutions to these partial differential inclusions.

We begin by deriving the existence of the largest closed solution contained in a given observation map and by providing a very useful stability theorem stating that graphical upper limits of solutions is still a solution.

The observation and tracking problems are two sides of the same coin because the set-valued map H and its inverse play symmetric roles. This is one of the reasons why we regard a single-valued map as a set-valued map characterized by its graph.

Consider then the observation problem: the idea is to observe solutions of a system $x' \in F(x,y)$ by a system $y' \in G(y)$ where $G : Y \leadsto Y$ provides simpler dynamics: equilibria, uniform movement, exponential growth, periodic solutions, etc. This allows us to observe complex systems[1] $x' \in F(x)$ in high dimensional spaces X by simpler systems $y' \in G(y)$ — or even better, $y' = g(y)$ — in lower dimensional spaces. We can think of H as an observation map, made of a small number of *sensors* taking into account uncertainty or lack of precision.

For instance, when $G \equiv 0$, we obtain constant observations. Observation maps H such that $F(x) \cap DH(x,y)^{-1}(0) \neq \emptyset$ for all $y \in H(x)$ provide solutions satisfying

$$\forall\, t \geq 0,\ \ x(t) \in H^{-1}(y_0) \ \text{ where } \ y_0 \in H(x_0)$$

In other words, inverse images $H^{-1}(y_0)$ are closed viability domains of F. *Viewed through such an observation map, the system appears to be in equilibrium.*

More generally, if there exists a linear operator $A \in \mathcal{L}(Y,Y)$ such that

$$\forall\, y \in \operatorname{Im}(H),\ \forall\, x \in H^{-1}(y),\ \ F(x) \cap DH(x,y)^{-1}(Ay) \neq \emptyset$$

[1]We can use this tracking property as a *mathematical metaphor* to model the concept of .... metaphors in epistemology. The simpler system (the model) $y' \in G(y)$ is designed to provide *explanations* of the evolution of the unknown system $x' \in F(x)$ and the tracking property means that the *metaphor* H is valid (*non falsifiable.*) Evolution of knowledge amounts to "increasing" the observation space Y and to *modifying* the system G (replacing the model) and/or the observation map H (obtain more experimental data), checking that the tracking property (the validity or the consistency of the metaphor) is maintained.

then we obtain solutions $x(\cdot)$ satisfying the time-dependent viability condition

$$\forall\, t \geq 0, \;\; x(t) \in H^{-1}(e^{At}y_0) \;\text{ where }\; y_0 \in H(x_0)$$

so that we can use the exhaustive knowledge of linear differential equations to derive behavioral properties of the solutions to the original system.

But instead of checking whether such given dynamics G satisfy the tracking property, we can look for systematic ways of finding them. For that purpose, it is natural to appeal to the selection procedures studied in section 4 of Chapter 6.

For instance, the most attractive idea is to choose the minimal selection $(x,y) \mapsto g^\circ(x,y)$ of the set-valued map

$$(x,y) \rightsquigarrow DH(x,y)(F(x,y))$$

which, by construction, satisfies the partial differential inclusion. We shall prove that under adequate assumptions, the system

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;=\; g^\circ(x(t),y(t)) \end{cases}$$

has solutions (satisfying automatically the tracking property) even though the minimal selection g° is not necessarily continuous.

The drawback of the minimal selection and the other ones of the same family is that g° depends upon x. We would like to obtain single-valued dynamics g independent of x. They are selections of the set-valued map G_H defined by

$$G_H(y) \;:=\; \bigcap_{x \in H^{-1}(y)} DH(x,y)(F(x,y))$$

We must appeal to Michael's Continuous Selection Theorem to find continuous selections g of this map, so that the system

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;=\; g(y(t)) \end{cases}$$

has solutions satisfying the tracking property.

When $F : X \rightsquigarrow X$ does not depend upon y, the size of the set-valued map G_H *measures in some sense a degree of inadequacy of the observation of the system* $x' \in F(x)$ through H, because the larger the images of G_H, the more dynamics g tracking an evolution of the differential inclusion.

Tracking problems, which are the topic of the second section, are intimately related to the observation problem: Here, the system $y' \in G(y)$, called the *exosystem*, is given. The problem is to *regulate the system* $x'(t) \in F(x(t), y(t))$ for finding solutions $x(\cdot)$ *that match the solutions to the exosystem* $y'(t) \in G(y(t))$ in the sense that $y(t) \in H(x(t))$, or, more to the point, $x(t) \in H^{-1}(y(t))$.

Decentralization of control systems and *decoupling properties* are instances of this problem.

An instance of decentralization can be described as follows: We take $X := Y^n$, $F(x) := \prod_{i=1}^n F_i(x_i)$, and the viability subset is given in the form

$$K := \{(x_1, \ldots, x_n) \mid \sum_{i=1}^n x_i \in M\}$$

so that we observe the individual evolutions $x_i'(t) \in F_i(x_i(t))$ through their sum $y(t) := \sum_{i=1}^n x_i(t)$. Decentralizing the system means solving

— first a differential inclusion $y'(t) \in G(y(t))$ providing a solution $y(\cdot)$ viable in the viability subset $M \subset Y$, and

— second, find solutions to the differential inclusions $x_i'(t) \in F_i(x_i(t))$ satisfying the (time-dependent) viability condition

$$\sum_{i=1}^n x_i(t) = y(t)$$

a condition which does not depend any more on M.

Hierarchical decomposition happens whenever the observation map is a composition product of several maps determining the *successive levels of the hierarchy.* The evolution at each level is linked to the state of the lower level and is regulated by controls depending upon the evolution of state-control of the lower level.

The third section is devoted to existence and comparison theorems of invariant manifolds.

We extend Hadamard's formula of solutions to linear hyperbolic differential equations to the set-valued case. We shall prove the existence of one set-valued contingent solutions $H_\star$ to the *decomposable system*

$$\forall\,(x,y) \in \text{Graph}(H_\star),\;\; Ay \in DH_\star(x,y)(\Phi(x)) - \Psi(x)$$

where $\Phi : X \rightsquigarrow X$ and $\Psi : X \rightsquigarrow Y$ are two Marchaud maps. If we denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x, then the set-valued map $H_\star : X \rightsquigarrow Y$ defined by

$$\forall\, x \in X,\;\; H_\star(x) \;:=\; -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

is the largest contingent solution with linear growth to this partial differential inclusion when $\lambda := \inf_{\|x\|=1}\langle Ax, x\rangle > 0$ is large enough. We also show that it is Lipschitz whenever Φ and Ψ are Lipschitz and compare the solutions associated with maps Φ_i and Ψ_i $(i = 1,2)$.

We then turn our attention to partial differential inclusions of the form

$$\forall\, x \in X,\;\; Ah(x) \;\in\; Dh(x)(f(x,h(x))) - G(x,h(x))$$

when $\lambda > 0$ is large enough, $f : X \times Y \mapsto X$ is Lipschitz, $G : X \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and satisfies the growth condition

$$\forall\, x,y,\;\; \|G(x,y)\| \leq c(1 + \|y\|)$$

When G is single-valued, we obtain a global *Center Manifold Theorem*, stating the existence and uniqueness of an invariant manifold for systems of differential equations with Lipschitz right-hand sides.

We end this section with comparison theorems between single-valued and set-valued solutions to such partial differential inclusions.

We characterize in the fourth section the single-valued contingent solutions to partial differential inclusions as minimizers of a functional. i.e., we provide a *variational principle*.

We apply in the sixth section some of the results obtained so far to the existence of closed-loop controls regulating smooth viable solutions to a control system. In chapter Chapter 7, we saw that closed-loop controls $r : K \mapsto Z$ regulating smooth solutions to a control system (U, f):

$$\begin{cases} i) & \text{for almost all } t, \; x'(t) = f(x(t), u(t)) \\ ii) & \text{where } u(t) \in U(x(t)) \end{cases}$$

in the sense that *for any* $x_0 \in K$, *there exists a solution to the differential equation* $x'(t) = f(x(t), r(x(t)))$ *starting at* x_0 *such that* $u(t) := r(x(t)) \in U(x(t))$ *is absolutely continuous and satisfies the growth condition*

$$\|u'(t)\| \leq \varphi(x(t), u(t))$$

for almost all t.

They are solutions to the following partial differential inclusion

$$\forall\, x \in K, \;\; 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraints

$$\forall\, x \in K, \;\; r(x) \in U(x)$$

This is a tracking problem, where the closed loop control is regarded as an observation map of a system where $F(x, u) := f(x, u)$ and $G(x, u) := \varphi(x, u)B$.

8.1 The Tracking Property

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \leadsto X$, $G : X \times Y \leadsto Y$ and a set-valued map $H : X \leadsto Y$, regarded as (and often called) the *observation map.*

Definition 8.1.1 *We shall say that* F, G *and* H *satisfy the* tracking property *if for any initial state* $(x_0, y_0) \in \text{Graph}(H)$, *there exists at least one solution* $(x(\cdot), y(\cdot))$ *to the* system of differential inclusions

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in G(x(t), y(t)) \end{cases} \tag{8.1}$$

starting at (x_0, y_0)*, defined on* $[0,\infty[$ *and satisfying*

$$\forall\, t \geq 0,\;\; y(t) \;\in\; H(x(t))$$

8.1.1 Characterization of the Tracking Property

We now consider the first-order system of the *partial differential inclusion*

$$\forall\,(x,y)\;\in\;\text{Graph}(H),\;\; 0 \in DH(x,y)(F(x,y)) - G(x,y) \tag{8.2}$$

Definition 8.1.2 *We shall say that a set-valued map* $H : X \rightsquigarrow Y$ *satisfying (8.2) is a solution to the* partial differential inclusion *if its graph is a* closed *subset of* $\text{Dom}(F) \cap \text{Dom}(G)$.

When $H = h : \text{Dom}(h) \mapsto Y$ *is a single-valued map with closed graph contained in* $\text{Dom}(F) \cap \text{Dom}(G)$*, the partial differential inclusion (8.2) becomes*

$$\forall\, x \in \text{Dom}(h),\;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) \tag{8.3}$$

We deduce at once from the Viability Theorem 3.3.5 and Theorem 4.1.2 the following:

Theorem 8.1.3 *Let us assume that* $F : X\times Y \rightsquigarrow X$*,* $G : X\times Y \rightsquigarrow Y$ *are Marchaud maps and that the graph of the set-valued map* H *is a closed subset of* $\text{Dom}(F) \cap \text{Dom}(G)$.

1. — The triple (F, G, H) *enjoys the tracking property if and only if* H *is a solution to the* partial differential inclusion *(8.2).*

2. — There exists a largest *solution* $H_\star$ *to the partial differential inclusion (8.2) contained in* H*. It enjoys the following property: whenever an initial state* $y_0 \in H(x_0)$ *does not belong to* $H_\star(x_0)$*, then all solutions* $(x(\cdot), y(\cdot))$ *to the system of differential inclusions (8.1) satisfy*

$$\begin{cases} i) & \forall\, t \geq 0, \quad y(t) \;\notin\; H_\star(x(t)) \text{ as long as } y(t) \;\in\; H(x(t)) \\ ii) & \exists\, T > 0 \quad \textit{such that } y(T) \;\notin\; H(x(T)) \end{cases} \tag{8.4}$$

3. — Any closed set-valued map $L \subset H_\star$ *is contained in a minimal set-valued map satisfying the tracking property.*

Naturally, the graph of $H_\star$ is the viability kernel of the graph of H.

We now translate in this framework the useful Stability Theorem 3.6.5. We recall that the graph of the *graphical upper limit* $H^\sharp$ of a sequence of set-valued maps $H_n : X \rightsquigarrow Y$ is by definition the graph of the upper limit of the graphs of the maps H_n. (See Chapter 7 of SET-VALUED ANALYSIS.)

Theorem 8.1.4 (Stability) *Let us consider a sequence of Marchaud maps $F_n : X \times Y \rightsquigarrow X$, $G_n : X \times Y \rightsquigarrow Y$ with uniform linear growth*[2] *and their graphical upper limit $F^\sharp$ and $G^\sharp$.*

Consider also a sequence of set-valued maps $H_n : X \rightsquigarrow Y$, solutions to the partial differential inclusions

$$\forall\,(x,y) \in \text{Graph}(H_n),\;\; 0 \in DH_n(x,y)(F_n(x,y)) - G_n(x,y) \quad (8.5)$$

Then the graphical upper limit $H^\sharp$ of the solutions H_n is a solution to

$$\forall\,(x,y) \in \text{Graph}(H^\sharp),\;\; 0 \in DH^\sharp(x,y)(\overline{co}F^\sharp(x,y)) - \overline{co}(G^\sharp(x,y)) \quad (8.6)$$

In particular, if the set-valued maps F_n and G_n converge graphically to maps F and G respectively, then the graphical upper limit $H^\sharp$ of the solutions H_n is a solution of (8.2).

It is an obvious consequence of Theorem 3.6.5.

We recall that graphical convergence of single-valued maps is weaker than pointwise convergence. This is why graphical limits of single-valued maps which are converging pointwise may well be set-valued.

Therefore, for single-valued solutions, the stability property implies the following statement: *Let h_n be single-valued solutions to the*

[2] In the sense that there exists a constant $c > 0$ such that

$$\sup_{n\geq 0} \max\left(\|F_n(x,y)\|\,,\|G_n(x,y)\|\right) \;\leq\; c(\|x\| + \|y\| + 1)$$

contingent partial differential inclusion (8.5). Then their graphical upper limit $h^\sharp$ *is a* (possibly set-valued) solution *to (8.6).*

Although set-valued solutions to first-order systems of partial differential inclusions make sense to describe shock and other phenomena, we may still need sufficient conditions for an upper graphical limit of single-valued maps to remain single-valued. This is the case for instance when *a sequence of continuous solutions* h_n *to the partial differential inclusion (8.5) is equicontinuous and converges pointwise to a function h. Then*[3] *h is a single-valued solution to (8.6).*

Remark — We could also introduce two other kinds of *partial differential inclusions*:

$$\forall\, (x,y) \in \text{Graph}(H),\;\; DH(x,y)(F(x,y)) \;\subset\; G(x,y)$$

and

$$\forall\, (x,y) \in \text{Graph}(H),\;\; G(x,y) \;\subset\; \bigcap_{u\in F(x,y)} DH(x,y)(u)$$

The first inclusion implies obviously that any solution $(x(\cdot), y(\cdot))$ to the viability problem

$$x'(t) \;\in\; F(x(t), y(t)) \;\;\&\;\; x(t) \;\in\; H^{-1}(y(t))$$

parametrized by the absolutely continuous function $y(\cdot)$ is a solution to the differential inclusion

$$y'(t) \;\in\; G(x(t), y(t))$$

The second inclusion states that the graph of H is an invariance domain of the set-valued map $F \times G$. Assume that F and G are Lipschitz with compact values on a neighborhood of the graph of F. By the Invariance Theorem 5.3.4, the second inclusion is equivalent to the following strong tracking property:

For any initial state $(x_0, y_0) \in \text{Graph}(H)$, *every solution* $(x(\cdot), y(\cdot))$ to the system of differential inclusions (8.1) starting at (x_0, y_0) satisfies $y(t) \in H(x(t))$ for all $t \geq 0$. □

[3]Indeed, a pointwise limit h of single-valued maps h_n is a selection of the graphical upper limit of the h_n. The latter is equal to h when h_n remain in an equicontinuous subset.

8.1.2 Construction of trackers

Any selection of the map Φ defined by

$$\forall\,(x,y) \in \text{Graph}(H), \quad \Phi(x,y) \ := \ DH(x,y)(F(x,y))$$

provides dynamics that satisfy the tracking property, provided that the system has solutions.

Naturally, we can obtain such selections by using selection procedures $G := S_\Phi$ of Φ (see Definition 6.5.2) that have convex values and linear growth, since the solutions to the system

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ \in \ S_\Phi(x(t),y(t)) \end{cases}$$

satisfying the tracking property (which exist by Theorem 8.1.3) are solutions to the system

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ \in \ S(\Phi)(x(t),y(t)) := \Phi(x(t),y(t)) \cap S_\Phi(x(t),y(t)) \end{cases}$$

Let us mention only the case of the minimal selection g° of Φ defined by

$$\begin{cases} i) & g^\circ(x,y) \ \in \ DH(x,y)(F(x,y)) \\ ii) & \|g^\circ(x,y)\| \ = \ \inf_{v\in DH(x,y)(F(x,y))} \|v\| \end{cases}$$

Theorem 8.1.5 *Assume that the Marchaud map F is continuous and that H is a sleek closed set-valued map satisfying, for some constant $c > 0$,*

$$\forall\,(x,y) \in \text{Graph}(H), \quad \|DH(x,y)\| \ \leq c$$

where $\|DH(x,y)\| := \sup_{\|u\|\leq 1} \inf_{v\in DH(x,y)(u)} \|v\|$ denotes the norm of the closed convex process $DH(x,y)$. Then the system observed by the minimal selection g° of $DH(\cdot,\cdot)(F(\cdot,\cdot))$

$$\begin{cases} i) & x'(t) \ \in \ F(x(t),y(t)) \\ ii) & y'(t) \ = \ g^\circ(x(t),y(t)) \end{cases}$$

has solutions enjoying the tracking property.

Proof — By Theorem 7.1.3, the set-valued map $(x, y, u) \rightsquigarrow DH(x, y)(u)$ is lower semicontinuous. We deduce then from the lower semicontinuity of F that the set-valued map Φ is also lower semicontinuous. Since $DH(x, y)$ is a convex process, it maps the convex subset $F(x, y)$ to the convex subset $\Phi(x, y)$. Therefore, Φ being lower semicontinuous with closed convex images, its minimal selection S_Φ° defined by

$$S_\Phi^\circ(x, y) \ := \ \{v \in Y \mid \|v\| \le d(0, \Phi(x, y))\}$$

is closed with convex values. Furthermore,

$$\|g^\circ(x, y)\| \ \le \ c\|F(x, y)\| \ \le \ c'(\|x\| + \|y\| + 1)$$

since $\|DH(x, y)\| \le c$ and the growth of F is linear. Then the system

$$\begin{cases} i) \quad x'(t) \ \in \ F(x(t), y(t)) \\ \\ ii) \quad y'(t) \ \in \ S_\Phi^\circ(x(t), y(t)) \cap c'(\|x(t)\| + \|y(t)\| + 1)B \end{cases}$$

has solutions enjoying the tracking property by Theorem 8.1.3. Therefore for almost all $t \ge 0$,

$$y'(t) \ \in \ \Phi(x(t), y(t)) \cap S_\Phi^\circ(x(t), y(t)) \ = \ g^\circ(x(t), y(t)) \ \square$$

8.1.3 The Observation Problem

We consider the important case when $G : Y \rightsquigarrow Y$ does not depend upon x. Hence the partial differential inclusion becomes

$$\forall\, x \in \text{Dom}(H), \forall\ y \in H(x), \quad G(y) \cap DH(x, y)(F(x, y)) \ne \emptyset$$

The behavior of observations of some solutions to the differential inclusion $x' \in F(x, y)$ may be given as the prescribed behavior of solutions to differential equations $y' = g(y)$, where g is a selection of

$$g(y) \in \bigcap_{x \in H^{-1}(y)} DH(x, y)(DF(x, y))$$

In the case when the differential equation $y' = g(y)$ has a unique solution $r(t)y_0$ starting from y_0, the solution $x(\cdot)$ satisfies the condition

$$\forall\, t \ge 0, \ \ x(t) \in H^{-1}(r(t)y(0)), \ \ x(0) \in H^{-1}(y(0))$$

When g is a linear operator $G \in \mathcal{L}(Y, Y)$, it can be written

$$\forall\, t \geq 0,\;\; x(t) \in H^{-1}(e^{Gt}y(0)),\;\; x(0) \in H^{-1}(y(0))$$

When $H \equiv h$ is a single-valued differentiable map, then the map G_H can be written

$$G_H(y) \;:=\; \bigcap_{h(x)=y} h'(x)F(x,y)$$

and a single-valued map g is a selection of G_H if and only if

$$\forall\, x \in \mathrm{Dom}(H),\;\; g(h(x)) \;\in\; h'(x)F(x,h(x))$$

The problem arises as to how to construct such maps g. But before studying it in the next subsection, we consider the particular case when $G \equiv 0$. Therefore, if F is a Marchaud map, H enjoys the tracking property if and only if it is a solution to

$$\forall\, (x,y) \in \mathrm{Graph}(H),\;\; 0 \in DH(x,y)(F(x,y)) \tag{8.7}$$

Since the tracking property of H amounts to saying that each subset $H^{-1}(y)$ enjoys the viability property for $F(\cdot, y)$, we observe that this condition is also equivalent to condition

$$\forall\, y \in \mathrm{Im}(H),\, \forall\, x \in H^{-1}(y),\;\; F(x,y) \cap T_{H^{-1}(y)}(x) \;\neq\; \emptyset$$

We may say that such a set-valued map H is an *energy map* of F. □

In the general case, the evolution with respect to a parameter y of the viability kernels of the closed subsets $H^{-1}(y)$ under the set-valued map $F(\cdot, y)$ is described in terms of $H_\star$:

Proposition 8.1.6 *Let $F : X \times Y \rightsquigarrow X$ be a Marchaud map and $H : X \rightsquigarrow Y$ be a closed set-valued map. Then there exists a largest solution $H_\star : X \rightsquigarrow Y$ contained in H to (8.7).*

The inverse images $H_\star^{-1}(y)$ are the viability kernels of the subsets $H^{-1}(y)$ under the maps $F(\cdot, y)$:

$$\mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y)) \;=\; H_\star^{-1}(y)$$

The graphical upper limit of energy maps is still an energy map.

Therefore *the graph of the map* $y \rightsquigarrow \mathrm{Viab}_{F(\cdot,y)}(H^{-1}(y))$ *is closed, and thus upper semicontinuous whenever the domain of* H *is bounded.*

When the observation map H is a single-valued map h, the partial differential inclusion becomes[4]:

$$\forall\, x,\ \exists\, u \in F(x,h(x)) \ \text{ such that }\ 0 \ \in\ Dh(x)(u)$$

The largest closed energy map $h_\star$ contained in h is necessarily the restriction of h to a closed subset $K_\star$ of the domain of h. Therefore, for all $y \in \mathrm{Im}(h)$, $K_\star \cap h^{-1}(y)$ *is the viability kernel of* $h^{-1}(y)$. The restriction of the differential inclusion $x'(t) \in F(x(t),y)$ to the viability kernel of $h^{-1}(y)$ is what Byrnes and Isidori call *zero dynamics of* F (in the framework of smooth nonlinear control systems.)

Remark — The Equilibrium Map. We associate with each parameter y the set

$$E(y) \ :=\ \{x \in H^{-1}(y) \mid 0 \in F(x,y)\}$$

of the equilibria of $F(\cdot,y)$ viable in $H^{-1}(y)$. We say that $E : Y \rightsquigarrow X$ is the *equilibrium map*.

We can derive some information on this equilibrium map from its derivative, which we can compute easily:

Theorem 8.1.7 *Assume that both* $H : X \rightsquigarrow Y$ *and* $F : X \times Y \rightsquigarrow X$ *are closed and sleek and that*

$$\begin{cases} \forall\,(x,y) \in \mathrm{Graph}(H),\ \ \forall\,(u,v,w)\ \in X \times Y \times X, \\ \exists\, v_1 \in DH(x,y)(u_1)\ \ \textit{such that}\ \ w \in DF(x,y,0)(u+u_1, v+v_1) \end{cases}$$

[4]When $h : X \mapsto \mathbf{R}$ is a continuous real function, we shall see in Chapter 9, Proposition 9.1.4 below, that the values

$$Df(x)(u) = [D_\uparrow f(x)(u), D_\downarrow f(x)(u)]$$

of the contingent derivative are intervals bounded by the epi and hypo contingent derivatives, so that the previous equation becomes a system of two contingent inequalities:

$$\forall\, x,\ \exists\, u \in F(x,h(x)) \ \text{ such that }\ D_\uparrow f(x)(u)\ \le\ 0\ \le\ D_\downarrow f(x)(u)$$

See Hélène Frankowska's CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS for an exhaustive study of contingent inequalities in the framework of Hamilton-Jacobi equations.

Then the contingent derivative of the equilibrium map is the equilibrium map of the contingent derivative of F:

$$u \in DE(y,x)(v) \iff u \in DH(x,y)^{-1}(v) \ \& \ 0 \in DF(x,y,0)(u,v)$$

Proof — We observe that by setting $\pi(x,y) := (x,y,0)$, the graph of E^{-1} can be written:

$$\text{Graph}(E^{-1}) \ := \ \text{Graph}(H) \cap \pi^{-1}(\text{Graph}(F))$$

and we apply formula (5) of Table 5.2, which states that if the transversality condition: for all $(x,y) \in \text{Graph}(E^{-1})$,

$$\pi\left(T_{\text{Graph}(H)}(x,y)\right) - T_{\text{Graph}(F)}(\pi(x,y)) \ = \ X \times Y \times X$$

holds true, then

$$T_{\text{Graph}(E^{-1})}(x,y) \ := \ T_{\text{Graph}(H)}(x,y) \cap \pi^{-1}\left(T_{\text{Graph}(F)}(\pi(x,y))\right)$$

Recalling that the contingent cone to the graph of a set-valued map is the graph of its contingent derivative, the assumption of our proposition implies the transversality condition. We then observe that the latter equality yields the conclusion of the proposition. □

Using the inverse function and the localization theorems presented in section 5.4 of SET-VALUED ANALYSIS, we can derive the following information. For instance, set

$$Q(y,x) \ := \ \left\{u \in DH(x,y)^{-1}(0) \mid 0 \ \in \ DF(x,y,0)(u,0)\right\}$$

Then, for any equilibrium $x \in E(y)$ and any closed cone P satisfying $P \cap Q(y,x) = \{0\}$, there exists $\varepsilon > 0$ such that

$$E(y) \cap (x + \varepsilon(P \cap B)) \ = \ \{x\}$$

where B denotes the unit ball. In particular, *an equilibrium $x \in E(y)$ is locally unique whenever $0 \in DH(x,y)^{-1}(0)$ is the unique equilibrium of $DF(x,y,0)(\cdot,0)$.*

Furthermore, if the set $E(y)$ of equilibria is convex, then

$$E(y) \ \subset \ x + Q(y,x) \ \square$$

8.1.4 Construction of Observers

These maps g are selections of the map $G_H : Y \rightsquigarrow Y$ defined by

$$G_H(y) := \bigcap_{x \in H^{-1}(y)} (DH(x,y)(F(x,y)))$$

The set-valued map G_H *measures so to speak a degree of disorder of the system* $x' \in F(x,y)$, because the larger the images of G_H, the more observed dynamics g tracking an evolution of the differential inclusion.

By using Michael's Continuous Selection Theorem, we obtain the following

Theorem 8.1.8 *Assume that the set-valued map F is continuous with convex compact images and linear growth, that H is a sleek closed set-valued map the domain of which is bounded and that there exists a constant $c > 0$ such that*

$$\forall \, (x,y) \in \text{Graph}(H), \ \ \|DH(x,y)\| \ \le c$$

Assume also that there exist constants $\delta > 0$ and $\gamma > 0$ such that, for any map $x \mapsto e(x) \in \gamma B$,

$$\delta B \cap \bigcap_{x \in H^{-1}(y)} (DH(x,y)(F(x,y)) - e(x)) \ \neq \ \emptyset$$

Then there exists a continuous map g such that the solutions of

$$\begin{cases} i) & x'(t) \ \in \ F(x(t), y(t)) \\ ii) & y'(t) \ = \ g(y(t)) \end{cases}$$

enjoy the tracking property for any initial state $(x_0, y_0) \in \text{Graph}(H)$.

Proof — The proof of Theorem 8.1.5 showed that the set-valued map Φ is lower semicontinuous with compact convex images. Furthermore, the set-valued map H^{-1} is upper semicontinuous with compact images since we assumed the domain of H bounded. Then the lower semicontinuity criterion Theorem 6.3.3 implies that the set-valued map G_H is also lower semicontinuous with compact convex images. Therefore there exists a continuous selection g of G_H, so that the above system does have solutions viable in the graph of H. □

8.2 The Tracking Problem

8.2.1 Tracking Control Systems

Let $H : X \rightsquigarrow Y$ be an observation map. We consider two control systems

$$\left\{ \begin{array}{lll} i) & \text{for almost all } t \geq 0, \ x'(t) & = & f(x(t), u(t)) \\ ii) & \text{where } u(t) & \in & U(x(t)) \end{array} \right. \tag{8.8}$$

and

$$\left\{ \begin{array}{ll} i) & \text{for almost all } t \geq 0, \ y'(t) = g(y(t), v(t)) \\ ii) & \text{where } v(t) \in V(y(t)) \end{array} \right. \tag{8.9}$$

on the state and observation spaces respectively, where $U : X \rightsquigarrow Z_X$ and $V : Y \rightsquigarrow Z_X$ map X and Y to the control spaces Z_X and Z_Y and where $f : \text{Graph}(U) \mapsto X$ and $g : \text{Graph}(V) \mapsto Y$.

We introduce the set-valued maps $R_H(x, y) : Z_Y \rightsquigarrow Z_X$ defined by

$$R_H(x, y; v) := \{u \in U(x) \mid f(x, u) \in DH(x, y)^{-1}(g(y, v))\}$$

if $v \in V(y)$ and $R_H(x, y; v) := \emptyset$ if $v \notin V(y)$.

Corollary 8.2.1 *Assume that the set-valued maps U and V are Marchaud maps and that the maps f and g are continuous, affine with respect to the controls and with linear growth. The set-valued map H enjoys the tracking property if and only if*

$$\forall\, (x, y) \in \text{Graph}(H), \ \ \text{Graph}(R_H(x, y)) \neq \emptyset$$

Then the system is regulated by the regulation law

$$\text{for almost all } t \geq 0, \ \ u(t) \in R_H(x(t), y(t); v(t))$$

When $H \equiv h$ is single-valued and differentiable and when we set $f(x, u) := c(x) + g(x)u$ and $g(y, v) := d(y) + e(y)v$ where $g(x)\cdot$ and $e(y)\cdot$ are linear operators, we obtain the formula

$$R_h(x; v) := U(x) \cap (h'(x)g(x))^{-1}(d(h(x)) - h'(x)c(x) + e(h(x)v))$$

8.2.2 Decentralization of a control system

We assume that the viability set of the control system (8.8) is defined by constraints of the form $K := L \cap h^{-1}(M)$ where

$$\left\{\begin{array}{ll} i) & L \subset X \text{ and } M \subset Y \text{ are closed and sleek} \\ ii) & h \text{ is a } \mathcal{C}^1\text{-map from } X \text{ to } Y \\ iii) & \forall x \in K := L \cap h^{-1}(M), \;\; Y = h'(x)T_L(x) - T_M(h(x)) \end{array}\right. \tag{8.10}$$

We associate with the two systems (8.8), (8.9) the *decoupled viability constraints*

$$\left\{\begin{array}{lll} i) & \forall t \geq 0, & x(t) \in L \\ ii) & \forall t \geq 0, & h(x(t)) = y(t) \\ iii) & \forall t \geq 0, & y(t) \in M \end{array}\right. \tag{8.11}$$

It is obvious that the *first component* $x(\cdot)$ of any pair of solutions $(x(\cdot), y(\cdot))$ to the system ((8.8),(8.9)) satisfying viability constraints (8.11) is a solution to the initial control system (8.8) viable in the set K defined by (8.10)iii).

On the other hand, solutions to the system (8.8) viable in K can be obtained in two steps:

— First, find a solution $y(\cdot)$ to the control system (8.9) *viable in M*

and then,

— second, find a solution $x(\cdot)$ the control system (8.8) satisfying the viability constraints

$$\left\{\begin{array}{lll} i) & \forall t \geq 0, & x(t) \in L \\ ii) & \forall t \geq 0, & h(x(t)) = y(t) \end{array}\right. \tag{8.12}$$

which does no longer involve the subset $M \subset Y$ of constraints.

This decentralization problem is a particular case of the observation problem for the set-valued map H defined by

$$H(x) := \left\{\begin{array}{ll} h(x) & \text{if } x \in L \;\&\; h(x) \in M \\ \emptyset & \text{if not} \end{array}\right.$$

whose contingent derivative is equal under assumptions (8.10) to

$$DH(x)(u) := \begin{cases} h'(x)u & \text{if } u \in T_L(x) \ \& \ h'(x)u \in T_M(h(x)) \\ \emptyset & \text{if not} \end{cases}$$

We know that the regulation map of the initial system (8.8), (8.9) on the subset K defined by (8.10) is equal to

$$R_K(x) = \{u \in U(x) \cap T_L(x) \mid h'(x)f(x,u) \in T_M(h(x))\}$$

The regulation map of the projected control system (8.9) on the subset M is defined by

$$R_M(y) = \{v \in V(y) \mid g(y,v) \in T_M(y)\}$$

We introduce now the set-valued map R_H which is equal to

$$R_H(x,y;v) := \{u \in U(x) \cap T_L(x) \mid h'(x)f(x,u) = g(y,v)\}$$

We observe that

$$\forall x \in K, \quad R_H(x,h(x);R_M(h(x))) \subset R_K(x)$$

The regulation map regulating solutions to the system ((8.8),(8.9)) satisfying viability conditions (8.11) is equal to

$$x \rightsquigarrow R_H(x,h(x);R_M(h(x)))$$

Therefore, the regulation law feeding back the controls from the solutions are given by: for almost all $t \geq 0$

$$\begin{cases} i) & v(t) \in R_M(y(t)) \\ ii) & u(t) \in R_H(x(t);v(t)) \end{cases}$$

The first law regulates the solutions to the control system (8.9) viable in M and the second regulates the solutions to the control system (8.8) satisfying the viability constraints (8.12).

Remark — The reason why this property is called decentralization lies in the particular case when $X := Y^n$, when $h(x) := \sum_{i=1}^n x_i$ and when the control system (8.8) is

$$\forall\, i = 1,\ldots,n, \quad x_i'(t) = f_i(x_i(t),u(t)) \ \text{ where } \ u(t) \in U_i(x_i(t))$$

constrained by

$$\forall\, i = 1, \ldots, n, \quad x_i(t) \in L_i \;\;\&\;\; \sum_{i=1}^{n} x_i(t) \in M$$

We introduce the regulation map R_H defined by

$$\begin{array}{l} R_H(x_1, \ldots, x_n, y; v) \\ := \; \{u \in \bigcap_{i=1}^{n} (U_i(x_i) \cap T_{L_i}(x_i)) \mid \sum_{i=1}^{n} f_i(x_i, u) = g(y, v)\} \end{array}$$

This system can be decentralized first by solving the viability problem for system (8.9) in the viability set M through the regulation law $v(t) \in R_M(y(t))$.

This being done, the state-control $(y(\cdot), v(\cdot))$ being known, it remains in a second step to study the evolution of the n control systems

$$\forall\, i = 1, \ldots, n, \quad x_i'(t) \;=\; f_i(x_i, u(t))$$

through the regulation law

$$u(t) \;\in\; R_H(x_1(t), \ldots, x_n(t), \sum_{i=1}^{n} x_i(t); v(t)) \;\; \square$$

Economic Interpretation — We can illustrate this problem with an economic interpretation: the state $x := (x_1, \ldots, x_n)$ describes an allocation of a commodity $y \in M$ among n consumers. The subsets L_i represent the consumptions sets of each consumer and the subset M the set of available commodities. The control u plays the role of the price system of the consumptions goods and v the price of the production goods. Differential equations $x_i' = f_i(x_i, u)$ represent the behavior of each consumer in terms of the consumption price and $y' = g(y, v)$ the evolution of the production process.

The decentralization process allows us to decouple the production problem and the consumption problem. $\square$

8.2.3 Hierarchical Decomposition Property

For simplicity, we restrict ourself here to the case when *the observation map* $H \equiv h := h_2 \circ h_1$ *is the product of two differentiable single-valued maps* $h_1 : X \mapsto Y_1$ and $h_2 : Y_1 \mapsto Y_2$.

We address the following issue: Can we observe the evolution of a solution to a control problem (8.8) through $h_2 \circ h_1$ by observing it

— first through h_1 by a control system

$$\begin{cases} i) & \text{for almost all } t \geq 0, \quad y_1'(t) = g_1(y_1(t), v_1(t)) \\ ii) & \text{where } v_1(t) \in V_1(y_1(t)) \end{cases} \tag{8.13}$$

and then,

— second, observing this system through h_2.

We introduce the maps R_h, R_{h_1} and R_{h_2} defined respectively by

$$\begin{cases} R_h(x; v) & := \{u \in U(x) \mid h'(x)f(x,u) = g(h(x), v) \\ & \quad \text{if } v \in V(h(x))\} \\ R_{h_1}(x; v_1) & = \{u \in U(x) \mid h_1'(x)f(x,u) = g_1(h_1(x), v_1) \\ & \quad \text{if } v_1 \in V(h_1(x))\} \\ R_{h_2}(x_1; v) & = \{v_1 \in V_1(x_1) \mid h_2'(x_1)g_1(x_1, v_1) = g(h_2(x_1), v) \\ & \quad \text{if } v \in V(h_2(x_1))\} \end{cases}$$

and we see at once that

$$R_{h_1}(x; R_{h_2}(h_1(x); v)) \subset R_h(x; v)$$

Therefore, if the graph of $v \rightsquigarrow R_{h_1}(x; R_{h_2}(h_1(x); v))$ is not empty, we can recover from the evolution of a solution $y(\cdot)$ to the control system (8.9) a solution $y_1(\cdot)$ to the control system (8.13) by the tracking law

$$\text{for almost all } t, \quad v_1(t) \in R_{h_2}(y_1(t), v(t))$$

and then, a solution $x(\cdot)$ to the control system (8.8) by the tracking law

$$\text{for almost all } t, \quad u(t) \in R_{h_1}(x(t), v_1(t))$$

This can illustrate hierarchical organization *which is found in the evolution of so many macrosystems. The decomposition of the observation map as a product of several maps determines the successive levels of the hierarchy. The evolution at each level obeys the constraint binding its state to the state of the lower level. It is regulated by controls determined (in a set-valued way) by the evolution of the state-control of the lower level.*

8.3 Partial Differential Inclusions

We shall begin by the decomposable case (or the set-valued linear systems) for which we have explicit formulas, that we next use to solve the general problem of finding a contingent solution to the system of partial differential inclusions

$$\forall\, x \in X,\;\; Ah(x) \;\in\; Dh(x)(f(x,h(x))) - G(x,h(x))$$

(where $A \in \mathcal{L}(Y,Y)$) whose graph is a viable manifold.

If $h : X \mapsto Y$, we set

$$\|h\|_\infty \;:=\; \sup_{x\in X}\|h(x)\| \;\;\&\;\; \|h\|_\Lambda \;:=\; \sup_{x\neq y}\frac{\|h(x)-h(y)\|}{\|x-y\|}$$

When G is Lipschitz with nonempty closed images, we denote by $\|G\|_\Lambda$ its Lipschitz constant, the smallest of the constants l satisfying

$$\forall\, z_1, z_2,\;\; G(z_1) \;\subset\; G(z_2) + l\,\|z_1 - z_2\|\,B$$

where B is the unit ball.

8.3.1 Decomposable Case

Let $K \subset X$, $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ be set-valued maps and $A \in \mathcal{L}(Y,Y)$. We set

$$\lambda \;:=\; \inf_{\|x\|=1}\langle Ax, x\rangle$$

and we recall that[5]

$$\forall\, y \in Y,\;\; \left\|e^{-At}y\right\| \;\leq\; e^{-\lambda t}\|y\|$$

[5]Indeed, $y(t) := e^{-At}y$ being a solution to the differential equation $y'(t) = -Ay(t)$ starting at y, we infer that

$$\frac{d}{dt}\|y(t)\|^2 \;=\; 2\langle y(t), -Ay(t)\rangle \;\leq\; -2\lambda\|y(t)\|^2$$

so that $\|y(t)\| \leq e^{-\lambda t}\|y\|$.

Consider the *decomposable* system of differential inclusions

$$\begin{cases} x'(t) \in \Phi(x(t)) \\ y'(t) \in Ay(t) + \Psi(x(t)) \end{cases} \tag{8.14}$$

which extends to the set-valued case the characteristic system of linear hyperbolic systems

$$\forall\, (x,y) \in \mathrm{Graph}(H_\star),\quad Ay \in DH_\star(x,y)(\Phi(x)) - \Psi(x) \tag{8.15}$$

the solutions of which are the maps satisfying the tracking property.

We denote by $\mathcal{S}_\Phi(x,\cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and viable in K.

We define the set-valued map $H_\star : K \rightsquigarrow Y$ by[6]

$$\forall\, x \in K,\quad H_\star(x) \ := \ -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt \tag{8.16}$$

Theorem 8.3.1 *Assume that $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ are Marchaud maps and that K is a closed viability domain[7] of Φ. If λ is large enough, then $H_\star : K \rightsquigarrow Y$ defined by (8.16) is the* largest solution with linear growth *to inclusion (8.15) and is bounded whenever Ψ is bounded.*

More precisely, if there exist positive constants α, β and γ such that

$$\forall\, x \in K,\quad \|\Phi(x)\| \ \le \ \alpha(\|x\|+1) \ \ \& \ \ \|\Psi(x)\| \ \le \ \beta + \gamma\|x\|$$

and if $\lambda > \alpha$, then

$$\forall\, x \in K,\quad \|H_\star(x)\| \ \le \ \frac{\beta}{\lambda} + \frac{\gamma}{\lambda-\alpha}(\|x\|+1) \tag{8.17}$$

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[6]By definition of the integral of a set-valued map (see Chapter 8 of SET-VALUED ANALYSIS for instance), this means that for every $y \in H_\star(x)$, there exists a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and $z(t) \in \Psi(x(t))$ such that

$$y \ := \ -\int_0^\infty e^{-At}z(t)dt \ \in \ H_\star(x)$$

[7]If K is closed, then $H_\star$ is defined on the viability kernel $\mathrm{Viab}_\Phi(K)$.

Moreover, if $K := X$ and Φ, Ψ are Lipschitz, then $H_\star : X \rightsquigarrow Y$ is also Lipschitz (with nonempty values) whenever λ is large enough:

$$\text{If } \lambda > \|\Phi\|_\Lambda, \; H_\star(x_1) \subset H_\star(x_2) + \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda} \|x_1 - x_2\| \, B$$

Formula (8.16) shows also that the graph of $H_\star$ is convex (respectively $H_\star$ is a closed convex process) whenever the graphs of the set-valued maps Φ and Ψ are convex (respectively Φ and Ψ are closed convex processes).

Proof

1. — We prove first that the graph of $H_\star$ satisfies contingent inclusion (8.15).

Indeed, choose an element y in $H_\star(x)$. By definition of the integral of a set-valued map, this means that there exist a solution $x(\cdot) \in \mathcal{S}_\Phi(x, \cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at x and viable in K and $z(t) \in \Psi(x(t))$ such that

$$y := -\int_0^\infty e^{-At} z(t)dt \in H_\star(x)$$

We check that for every $\tau > 0$

$$-\int_0^\infty e^{-At} z(t+\tau)dt \in H_\star(x(\tau)) = H_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right)$$

By observing that

$$\begin{cases} \frac{1}{\tau}\int_0^\infty e^{-At}\left(z(t) - z(t+\tau)\right)dt \\ = -\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At}z(t)dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At}z(t)dt \end{cases}$$

we deduce that

$$\begin{cases} y + \tau\left(-\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At}z(t)dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At}z(t)dt\right) \\ \in H_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right) \end{cases}$$

Since Φ is upper semicontinuous, we know that for any $\varepsilon > 0$ and t small enough, $\Phi(x(t)) \subset \Phi(x) + \varepsilon B$, so that $x'(t) \in \Phi(x) + \varepsilon B$ for almost all small t. Therefore, $\Phi(x)$ being closed and convex, we infer that for $\tau > 0$ small enough, $\frac{1}{\tau}\int_0^\tau x'(t)dt \in \Phi(x) + \varepsilon B$ thanks to the Mean-Value Theorem. This latter set being compact, there exists a sequence of $\tau_n > 0$ converging to 0 such that $\frac{1}{\tau_n}\int_0^{\tau_n} x'(t)dt$ converges to some $u \in \Phi(x)$.

In the same way, Ψ being upper semicontinuous, $\Psi(x(t)) \subset \Psi(x) + \varepsilon B$ for any $\varepsilon > 0$ and t small enough, so that $z(t) \in \Psi(x) + \varepsilon B$ for almost all small t. The Mean-Value Theorem implies that

$$\forall\, n > 0, \quad z_n := \frac{1}{\tau_n}\int_0^{\tau_n} z(t)dt \;\in\; \Psi(x) + \varepsilon B$$

since this set is compact and convex. Furthermore, there exists a subsequence of z_n converging to some $z_0 \in \Psi(x)$. Hence, since

$$\frac{1}{\tau_n}\int_0^{\tau_n} \left(e^{-At} - 1\right) z(t)dt \;\to\; 0$$

we infer that

$$Ay + z_0 \;\in\; DH_\star(x,y)(u)$$

so that $Ay \;\in\; DH_\star(x,y)(\Phi(x)) - \Psi(x)$.

2. — Let us prove now that the graph of $H_\star$ is closed when λ is large enough. Consider for that purpose a sequence of elements (x_n, y_n) of the graph of $H_\star$ converging to (x,y). There exist solutions $x_n(\cdot) \in \mathcal{S}_\Phi(x_n,\cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at x_n and viable in K and measurable selections $z_n(t) \in \Psi(x_n(t))$ such that

$$y_n := -\int_0^\infty e^{-At} z_n(t)dt \in H_\star(x_n)$$

The growth of Φ being linear, there exists $\alpha > 0$ such that the solutions $x_n(\cdot)$ obey the estimate

$$\|x_n(t)\| \le (\|x_n\| + 1)e^{\alpha t} \;\;\&\;\; \|x_n'(t)\| \le \alpha(\|x_n\| + 1)e^{\alpha t}$$

By Theorem 3.5.1, we know that there exists a subsequence (again denoted by) $x_n(\cdot)$ converging uniformly on compact intervals to a solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$.

The growth of Ψ also being linear, we deduce that, setting $u_n(t) := e^{-At}z_n(t)$,

$$\|z_n(t)\| \leq \beta+\gamma(\|x_n\|+1)e^{\alpha t} \ \& \ \|u_n(t)\| \leq \beta e^{-\lambda t}+\gamma(\|x_n\|+1)e^{-(\lambda-\alpha)t}$$

When $\lambda > \alpha$, Dunford-Pettis' Theorem implies that a subsequence (again denoted by) $u_n(\cdot)$ converges weakly to some function $u(\cdot)$ in $L^1(0,\infty;Y)$. This implies that $z_n(\cdot)$ converges weakly to some function $z(\cdot)$ in $L^1(0,\infty;Y;e^{-\lambda t}dt)$. The Convergence Theorem 2.4.4 states that $z(t) \in \Psi(x(t))$ for almost every t. Since the integrals y_n converge to $-\int_0^\infty e^{-At}z(t)dt$, we have proved that

$$y \ = \ -\int_0^\infty e^{-At}z(t)dt \ \in \ H_\star(x)$$

3. — Estimate (8.17) is obvious since any solution $x(\cdot) \in \mathcal{S}_\Phi(x,\cdot)$ satisfies

$$\forall\, t \geq 0, \ \ \|x(t)\| \ \leq \ (\|x\|+1)e^{\alpha t}$$

so that, if $\lambda > \alpha$,

$$\|H_\star(x)\| \ \leq \ \int_0^\infty e^{-\lambda t}\left(\beta+\gamma(\|x\|+1)e^{\alpha t}\right)dt \ = \ \frac{\beta}{\lambda}+\frac{\gamma}{\lambda-\alpha}(\|x\|+1)$$

Assume now that $M : K \rightsquigarrow Y$ is any set-valued contingent solution to inclusion (8.15) with linear growth: there exists $\delta > 0$ such that for all $x \in X$, $\|M(x)\| \leq \delta(\|x\|+1)$. Since M enjoys the tracking property, we know that for any $(x,y) \in \text{Graph}(M)$, there exists a solution $(x(\cdot),y(\cdot))$ to the system of differential inclusions

$$\begin{cases} i) & x'(t) \ \in \ \Phi(x(t)) \\ ii) & y'(t)-Ay(t) \ \in \ \Psi(x(t)) \end{cases} \tag{8.18}$$

starting at (x,y) such that $y(t) \in M(x(t))$ for all $t \geq 0$. We also know that $\|x(t)\| \leq (\|x\|+1)e^{\alpha t}$ so that $\|y(t)\| \leq \delta(1+(\|x\|+1)e^{\alpha t})$. The second differential inclusion of the above system implies that $z(t) := y'(t)-Ay(t)$ is a measurable selection of $\Psi(x(t))$ satisfying the growth condition

$$\|z(t)\| \ \leq \ \beta+\gamma(\|x\|+1)e^{\alpha t}$$

Therefore, if $\lambda > \alpha$, the function $e^{-At}z(t)$ is integrable. On the other hand, integrating by parts $e^{-At}z(t) := e^{-At}y'(t) - e^{-At}Ay(t)$, we obtain

$$e^{-AT}y(T) - y = \int_0^T e^{-At}z(t)dt$$

which implies that

$$y = -\int_0^\infty e^{-At}z(t)dt \in H_\star(x)$$

by letting $T \mapsto \infty$. Hence we have proved that[8] $M(x) \subset H_\star(x)$.

4. — Assume now that Φ and Ψ are Lipschitz, take any pair of elements x_1 and x_2 and choose $y_1 = -\int_0^\infty e^{-At}z_1(t)dt \in H_\star(x_1)$, where

$$x_1(\cdot) \in \mathcal{S}_\Phi(x_1,\cdot) \;\&\; z_1(t) \in \Psi(x_1(t))$$

By the Filippov Theorem 5.3.1, there exists a solution $x_2(\cdot) \in \mathcal{S}_\Phi(x_2,\cdot)$ such that

$$\forall\, t \geq 0,\;\; \|x_1(t) - x_2(t)\| \leq e^{\|\Phi\|_\Lambda t}\|x_1 - x_2\|$$

We denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi(x_2(t))$, which is measurable thanks to Corollary 8.2.13 of SET-VALUED ANALYSIS and which satisfies

$$\forall\, t \geq 0,\; \|z_1(t)-z_2(t)\| \leq \|\Psi\|_\Lambda\|x_1(t)-x_2(t)\| \leq \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda t}\|x_1-x_2\|$$

Therefore, if $\lambda > \|\Phi\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At}z_2(t)dt$ belongs to $H_\star(x_2)$ and satisfies

$$\|y_1-y_2\| \leq \int_0^\infty \|\Psi\|_\Lambda e^{-t(\lambda-\|\Phi\|_\Lambda)}\|x_1-x_2\|dt \leq \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda}\|x_1-x_2\| \;\square$$

[8]This proof actually implies that any set-valued contingent solution M with polynomial growth in the sense that for some $\rho \geq 0$,

$$\forall\, x \in X,\;\; \|M(x)\| \leq \delta(\|x\|^\rho + 1)$$

is contained in $H_\star$ if $\lambda > \alpha\rho$, i.e., that there is no contingent solution with polynomial growth other than with linear growth (and bounded when $\gamma = 0$).

We prove now a comparison result between solutions to two decomposable partial differential inclusions.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L,M) := \sup_{y\in L}\inf_{z\in M} d(y,z) = \sup_{y\in L} d(y,M)$$

their *semi-Hausdorff distance*[9], and recall that $\Delta(L,M)=0$ if and only if $L\subset M$. If Φ and Ψ are two set-valued maps, we set

$$\Delta(\Phi,\Psi)_\infty = \sup_{x\in X}\Delta(\Phi(x),\Psi(x)) := \sup_{x\in X}\sup_{y\in\Phi(x)} d(y,\Psi(x))$$

Theorem 8.3.2 *Consider now two pairs (Φ_1,Ψ_1) and (Φ_2,Ψ_2) of Marchaud maps defined on X and their associated solutions*

$$\forall\, x\in X,\quad H_{\star i}(x) := -\int_0^\infty e^{-At}\Psi_i(\mathcal{S}_{\Phi_i}(x,t))dt \quad (i=1,2)$$

If the set-valued maps Φ_2 and Ψ_2 are Lipschitz, and if $\lambda > \|\Phi_2\|_\Lambda$, then

$$\Delta(H_{\star_1},H_{\star_2})_\infty \le \frac{1}{\lambda}\Delta(\Psi_1,\Psi_2)_\infty + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda-\|\Phi_2\|_\Lambda)}\Delta(\Phi_1,\Phi_2)_\infty$$

Proof — Consider the two pairs (Φ_1,Ψ_1) and (Φ_2,Ψ_2) of set-valued maps and choose $y_1 = -\int_0^\infty e^{-At}z_1(t)dt \in H_{\star 1}(x)$ where

$$x_1(\cdot) \in \mathcal{S}_{\Phi_1}(x,\cdot) \;\&\; z_1(t) \in \Psi_1(x_1(t))$$

In order to compare $x_1(\cdot)$ with the solution-set $\mathcal{S}_{\Phi_2}(x,\cdot)$ via the Filippov Theorem, we use the estimate

$$d(x_1'(t),\Phi_2(x_1(t))) \le \sup_{z\in\Phi_1(x_1(t))} d(z,\Phi_2(x_1(t)))) \le \Delta(\Phi_1,\Phi_2)_\infty$$

Therefore, there exists a solution $x_2(\cdot)\in\mathcal{S}_{\Phi_2}(x,\cdot)$ such that

$$\forall\, t\ge 0,\; \|x_1(t)-x_2(t)\| \le \Delta(\Phi_1,\Phi_2)_\infty\frac{e^{t\|\Phi_2\|_\Lambda}-1}{\|\Phi_2\|_\Lambda}$$

[9]The *Hausdorff distance* between L and M is $\max(\Delta(L,M),\Delta(M,L))$, which may be equal to ∞.

by Filippov's Theorem. As before, we denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi_2(x_2(t))$, which is measurable and satisfies

$$\begin{cases} \forall\, t \geq 0, \ \ \|z_1(t) - z_2(t)\| \ \leq \ \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \|x_1(t) - x_2(t)\| \\ \leq \ \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \frac{e^{t\|\Phi_2\|_\Lambda - 1}}{\|\Phi_2\|_\Lambda} \end{cases}$$

Therefore $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $H_{\star 2}(x)$ and satisfies

$$\begin{cases} \|y_1 - y_2\| \\ \leq \ \int_0^\infty e^{-\lambda t} \Delta(\Psi_1, \Psi_2)_\infty dt + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \int_0^\infty \frac{e^{t\|\Phi_2\|_\Lambda - 1}}{\|\Phi_2\|_\Lambda} e^{-\lambda t} dt \\ \leq \ \frac{\Delta(\Psi_1, \Psi_2)_\infty}{\lambda} + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)} \Delta(\Phi_1, \Phi_2)_\infty \ \ \square \end{cases}$$

When Φ, Ψ are single-valued, we obtain:

Proposition 8.3.3 *Assume that φ and ψ are Lipschitz and that ψ is bounded. Then if $\lambda > 0$, the map $h := \Gamma(\varphi, \psi)$ defined by*

$$h(x) \ = - \int_0^\infty e^{-At} \psi(S_\varphi(x,t))dt$$

is the unique bounded single-valued solution to the contingent inclusion

$$Ah(x) \ \in \ Dh(x)(\varphi(x)) - \psi(x) \tag{8.19}$$

and satisfies

$$\|h\|_\infty \ \leq \ \frac{\|\psi\|_\infty}{\lambda} \ \ \& \ \forall\, \lambda > \|\varphi\|_\Lambda, \ \ \|h\|_\Lambda \ \leq \ \frac{\|\psi\|_\Lambda}{\lambda - \|\varphi\|_\Lambda} \tag{8.20}$$

The map $(\varphi, \psi) \mapsto \Gamma(\varphi, \psi)$ is continuous from $\mathcal{C}(X, X) \times \mathcal{C}(X, Y)$ to $\mathcal{C}(X, Y)$:

$$\|\Gamma(\varphi_1, \psi_1) - \Gamma(\varphi_2, \psi_2)\|_\infty \ \leq \ \frac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \frac{\|\psi_2\|_\Lambda}{\lambda(\lambda - \|\varphi_2\|_\Lambda)} \|\varphi_1 - \varphi_2\|_\infty$$

The proof follows Theorems 8.3.1 and 8.3.2.

8.3.2 Existence of a Lipschitz Contingent Solution

We shall now prove the existence of a contingent single-valued solution to inclusion

$$\forall\, x \in X, \;\; Ah(x) \;\in\; Dh(x)(f(x,h(x))) - G(x,h(x)) \qquad (8.21)$$

Theorem 8.3.4 *Assume that the map $f : X \times Y \mapsto X$ is Lipschitz, that $G : X \times Y \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and that*

$$\forall\, x,\; y,\;\; \|G(x,y)\| \le c(1+\|y\|)$$

Let $A \in \mathcal{L}(Y,Y)$ such that $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ (where ν is the dimension of X). Then there exists a bounded Lipschitz contingent solution to the partial differential inclusion (8.21).

Proof — Since for every Lipschitz single-valued map $s(\cdot)$, $x \rightsquigarrow G(x,s(x))$ is Lipschitz (with constant $\|G\|_\Lambda(1+\|s\|)_\Lambda)$ and has convex compact values, Theorem 9.4.3 of SET-VALUED ANALYSIS implies that the subset G_s of Lipschitz selections ψ of the set-valued map $x \rightsquigarrow G(x,s(x))$ with Lipschitz constant less than $\nu\,\|G\|_\Lambda(1+\|s\|_\Lambda)$ is not empty (where ν denotes the dimension of X.) We denote by φ_s the Lipschitz map defined by $\varphi_s(x) := f(x,s(x))$, with Lipschitz constant equal to $\|f\|_\Lambda(1+\|s\|_\Lambda)$.

The solutions h to inclusion (8.21) are the fixed points to the set-valued map $\mathcal{R} : \mathcal{C}(X,Y) \rightsquigarrow \mathcal{C}(X,Y)$ defined by

$$\mathcal{R}(s) \;:=\; \{\Gamma(\varphi_s,\psi)\}_{\psi\in G_s} \qquad (8.22)$$

Indeed, if $h \in \mathcal{R}(h)$, there exists a selection $\psi \in G_h$ such that

$$Ah(x) \;\in\; Dh(x)(f(x,h(x))) - \psi(x) \;\subset\; Dh(x)(f(x,h(x))) - G(x,h(x))$$

Since $\|G(x,y)\| \le c(1+\|y\|)$, we deduce that any selection $\psi \in G_s$ satisfies

$$\|\psi\|_\infty \;\le\; c(1+\|s\|_\infty)$$

Therefore, Proposition 8.3.3 implies that

$$\forall\, h \in \mathcal{R}(s),\;\; \|h\|_\infty \;\le\; \frac{c}{\lambda}(1+\|s\|_\infty) \;\;\&\;\; \|h\|_\Lambda \;\le\; \frac{\nu\|G\|_\Lambda(1+\|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1+\|s\|_\Lambda)}$$

We first observe that when $\lambda > c$,

$$\forall\, s \in \mathcal{C}(X,Y) \;\text{ such that } \|s\|_\infty \leq \frac{c}{\lambda - c},\; \forall\, h \in \mathcal{R}(s),\; \|h\|_\infty \leq \frac{c}{\lambda - c}$$

When $\lambda > 4\nu\|f\|_\Lambda \|G\|_\Lambda$, we denote by

$$\rho_\lambda := \frac{\lambda - \|f\|_\Lambda - \nu\|G\|_\Lambda - \sqrt{\lambda^2 - 2\lambda(\|f\|_\Lambda + \nu\|G\|_\Lambda) + (\|f\|_\Lambda - \nu\|G\|_\Lambda)^2}}{2\|f\|_\Lambda}$$

the smallest root of the equation

$$\lambda\rho \;=\; \|f\|_\Lambda\rho^2 + (\|f\|_\Lambda + \nu\|G\|_\Lambda)\rho + \nu\|G\|_\Lambda$$

which is positive. We observe that

$$\lim_{\lambda\to+\infty} \lambda\rho_\lambda \;=\; \nu\|G\|_\Lambda$$

and infer that

$$\forall\, s \in \mathcal{C}(X,Y) \;\text{ such that } \|s\|_\Lambda \leq \rho_\lambda,\; \forall\, h \in \mathcal{R}(s),\; \|h\|_\Lambda \;\leq\; \rho_\lambda$$

because h being of the form $\Gamma(\varphi_s, \psi_s)$, satisfies by Proposition 8.3.3:

$$\|h\|_\Lambda \leq \frac{\|\psi_s\|_\Lambda}{\lambda - \|\varphi_s\|_\Lambda} \leq \frac{\nu\,\|G\|_\Lambda\,(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)} \leq \frac{\nu\,\|G\|_\Lambda\,(1 + \rho_\lambda)}{\lambda - \|f\|_\Lambda(1 + \rho_\lambda)} = \rho_\lambda$$

Let us denote by $B^1_\infty(\lambda)$ the subset defined by

$$B^1_\infty(\lambda) \;:=\; \left\{ h \in \mathcal{C}(X,Y) \;\mid\; \|h\|_\infty \leq \frac{c}{\lambda - c} \;\;\&\;\; \|h\|_\Lambda \leq \rho_\lambda \right\}$$

which is compact (for the compact convergence topology) thanks to Ascoli's Theorem.

We have therefore proved that when $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$, the set-valued map $\mathcal{R}$ sends the compact subset $B^1_\infty(\lambda)$ to itself.

It is obvious that the values of $\mathcal{R}$ are convex. Kakutani's Fixed-Point Theorem implies the existence of a fixed point $h \in \mathcal{R}(h)$ if we prove that the graph of $\mathcal{R}$ is closed.

Actually, the graph of $\mathcal{R}$ is compact. Indeed, let us consider any sequence $(s_n, h_n) \in \text{Graph}(\mathcal{R})$. Since $B^1_\infty(\lambda)$ is compact, a subsequence (again denoted by) (s_n, h_n) converges to some function

$$(s,h) \;\in\; B^1_\infty(\lambda) \times B^1_\infty(\lambda)$$

But there exist bounded Lipschitz selections $\psi_n \in G_{s_n}$ with Lipschitz constant $\nu\|G\|_\Lambda(1+\rho_\lambda)$ such that

$$\forall\, n \geq 0, \;\; h_n \; = \; \Gamma(\varphi_{s_n}, \psi_n)$$

Therefore a subsequence (again denoted by) ψ_n converges to some function $\psi \in G_s$. Since φ_{s_n} converges obviously to φ_s, we infer that h_n converges to $\Gamma(\varphi_s, \psi)$ where $\psi \in G_s$, i.e., that $h \in \mathcal{R}(s)$, since Γ is continuous by Proposition 8.3.3. □

8.3.3 Comparison Results

The point of this section is to compare two solutions to inclusion (8.21), or even, a single-valued solution and a contingent set-valued solution $M : X \rightsquigarrow Y$.

We first deduce from Theorem 8.3.2 the following "localization property":

Theorem 8.3.5 *We posit the assumptions of Theorem 8.3.4, with $A \in \mathcal{L}(Y,Y)$ such that $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ (where ν is the dimension of X). Let $\Phi : X \rightsquigarrow X$ and $\Psi : X \rightsquigarrow Y$ be two Lipschitz and Marchaud maps with which we associate the set-valued map $H_\star$ defined by*

$$\forall\, x \in X, \;\; H_\star(x) \; := \; -\int_0^\infty e^{-At}\Psi(\mathcal{S}_\Phi(x,t))dt$$

Then any bounded single-valued contingent solution $h(\cdot)$ to inclusion (8.21) satisfies the following estimate

$$\begin{cases} \forall\, x \in X, \;\; d(h(x), H_\star(x)) \; \leq \; \frac{1}{\lambda}\sup_{x\in X}\Delta(G(x,h(x)), \Psi(x)) \\ +\frac{\|\Psi\|_\Lambda}{\lambda(\lambda-\|\Phi\|_\Lambda)}\sup_{x\in X} d(f(x,h(x)), \Phi(x)) \end{cases}$$

In particular, if we assume that

$$\forall\, y \in Y, \;\; f(x,y) \; \in \; \Phi(x) \;\; \& \;\; G(x,y) \; \subset \; \Psi(x)$$

then all bounded single-valued contingent solutions $h(\cdot)$ to inclusion (8.21) are selections of $H_\star$.

Proof — Let h be any bounded single-valued contingent solution to inclusion (8.21). One can show that h can be written in the form

$$h(x) = -\int_0^\infty e^{-At} z(t)dt \text{ where } z(t) \in G(x(t), h(x(t)))$$

by using the same arguments as in the first part of the proof of Theorem 8.3.1.

We also adapt the proof of Theorem 8.3.2 with $\Phi_1 := f(x, h(x))$, $z_1(t) := z(t)$, $\Phi_2 := \Phi$ and $\Psi_2 := \Psi$, to show that the estimates stated in the theorem hold true. □

8.4 The Variational Principle

We characterize in this section solutions to the partial differential inclusion (8.3) through a *variational principle*. For that purpose, we recall that

$$\sigma(M,p) := \sup_{z\in M} < p, z > \quad \& \quad \sigma^\flat(M,p) := \inf_{z\in M} < p, z >$$

denote the support functions of $M \subset X$ and $B_\star$ the unit ball of $Y^\star$. We also need the following

Definition 8.4.1 *Let $H : X \rightsquigarrow Y$ be a set-valued map and (x,y) belong to its graph. We shall say that the transpose $DH(x,y)^\star : Y^\star \rightsquigarrow X^\star$ of the contingent derivative $DH(x,y)$ is the* codifferential of H at (x,y). *When $H := h$ is single-valued, we set $Dh(x)^\star := Dh(x, h(x))^\star$.*

8.4.1 Definition of the Functional

Consider a closed subset $K \subset X$. We introduce the nonnegative functional Φ defined on the space $\mathcal{C}(K,Y)$ of continuous maps by

$$\Phi(h) := \sup_{q\in B_\star} \sup_{x\in K} \sup_{p\in Dh(x)^\star(q)} \Big(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q)\Big)$$

Theorem 8.4.2 (Variational Principle) *Let the set-valued maps F and G be upper semicontinuous with convex and compact values.*

Let $c > 0$. Then a single-valued map $h : K \mapsto Y$ is a solution to the partial differential inclusion

$$\forall\, x \in K,\;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

if and only if $\Phi(h) \leq c$.

Consequently, h is a solution to the partial differential inclusion (8.3) if and only if $\Phi(h) = 0$.

Proof — The first inclusion is easy: let $u \in F(x,h(x))$, $v \in G(x,h(x))$ and $e \in cB$ be such that $v - e \in Dh(x)(u)$. Then, for any $q \in B_\star$ and $p \in Dh(x)^\star(q)$, we know that

$$< p,u > - < q,v-e > \leq 0$$

so that

$$\begin{cases} \sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \\ \\ \leq < p,u > - < q,v > \leq < q,e > \leq c \end{cases}$$

By taking the supremum with respect to $x \in K$, $q \in B_\star$ and $p \in Dh(x)^\star(q)$, we infer that $\Phi(h) \leq c$.

Conversely, we can write inequality $\Phi(h) \leq c$ in the form of the minimax inequality: for any $x \in K$, $q \in Y^\star$,

$$\sup_{p \in Dh(x)^\star(q)} \inf_{u \in F(x,h(x))} \inf_{v \in G(x,h(x))} (< p,u > - < q,v >) \leq c\|q\|$$

Noticing that $c\|q\| = \sigma(cB,q)$ and setting

$$\beta(p,q;u,v,e) := < p,u > - < q,v-e >$$

this inequality can be written in the form: for every $x \in K$,

$$\sup_{(p,-q) \in \text{Graph}(Dh(x))^-} \inf_{(u,v,e) \in F(x,h(x)) \times G(x,h(x)) \times cB} \beta(p,q;u,v,e) \leq 0$$

Since the set $F(x,h(x)) \times G(x,h(x)) \times cB$ is convex compact and since the negative polar cone to the graph of $Dh(x)$ is convex, the

Lop-Sided Minimax Theorem 3.7.10 implies the existence of $u_0 \in F(x,h(x))$, $v_0 \in G(x,h(x))$ and $e_0 \in cB$ such that

$$\sup_{(p,-q)\in \mathrm{Graph}(Dh(x))^-}(<p,u_0> - <q,v_0-e_0>) =$$

$$\sup_{(p,-q)\in \mathrm{Graph}(Dh(x))^-}\inf_{(u,v,e)\in F(x,h(x))\times G(x,h(x))\times cB}\beta(p,q;u,v,e)$$

$$\leq 0$$

This means that $(u_0, v_0 - e_0)$ belongs to the bipolar of the graph of $Dh(x)$, i.e., its closed convex hull $\overline{co}(\mathrm{Graph}(Dh(x)))$. In other words, we have proved that

$$(F(x,h(x)) \times (G(x,h(x)) + cB)) \cap \overline{co}\left(T_{\mathrm{Graph}(h)}(x,h(x))\right) \neq \emptyset$$

But by Theorem 3.2.4, this is equivalent to the condition

$$(F(x,h(x)) \times (G(x,h(x)) + cB)) \cap T_{\mathrm{Graph}(h)}(x,h(x)) \neq \emptyset$$

i.e., h is a solution to the partial differential inclusion. □

Theorem 8.4.3 *Assume that the set-valued maps F and G are upper semicontinuous with nonempty convex compact images. Let $\mathcal{H} \subset \mathcal{C}(K,Y)$ be a compact subset for the compact convergence topology.*

Assume that $c := \inf_{h\in\mathcal{H}} \Phi(h) < +\infty$. Then there exists a solution $h \in \mathcal{H}$ to the partial differential inclusion

$$0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

Since $\mathcal{H}$ is a compact subset for the compact convergence topology, it is sufficient to prove that the functional Φ is lower semicontinuous on the space $\mathcal{C}(K,Y)$ for this topology: If it is proper (i.e., different from the constant $+\infty$), it achieves its minimum at some $h \in \mathcal{H}$, which is a solution to the above partial differential inclusion thanks to Theorem 8.4.2. So Theorem 8.4.3 follows from Proposition 8.4.4 below:

Proposition 8.4.4 *Assume that the set-valued maps F and G are upper semicontinuous with nonempty convex compact images. Then the functional Φ is lower semicontinuous on equicontinuous subsets of the space $\mathcal{C}(K,Y)$ for the compact convergence topology.*

To prove this result, we need more information about the convergence properties of the codifferentials.

8.4.2 Convergence Properties of the Codifferentials

Proposition 8.4.5 *Let X, Y be finite dimensional vector-spaces and $K \subset X$ be a closed subset. Assume that h is the pointwise limit of an equicontinuous family of maps $h_n : K \mapsto Y$. Let $x \in K$ and $p \in Dh(x)^\star(q)$ be fixed. Then there exist subsequences of elements $x_{n_k} \in K$ converging to x, q_{n_k} converging to q and $p_{n_k} \in Dh_{n_k}(x_{n_k})^\star(q_{n_k})$ converging to p.*

If the functions h_n are differentiable, we deduce that there exist subsequences of elements $x_{n_k} \in K$ converging to x and q_{n_k} converging to q such that $h'_{n_k}(x_{n_k})^\star(q_{n_k})$ converges to p.

Proof — We can reformulate the statement in the following way: we observe that $p \in Dh(x)^\star(q)$ if and only if

$$(p, -q) \in \left(T_{\mathrm{Graph}(h)}(x, h(x))\right)^-$$

so that we have to prove that there exist subsequences $x_{n_k} \in K$ and

$$(p_{n_k}, -q_{n_k}) \in \left(T_{\mathrm{Graph}(h_{n_k})}(x_{n_k}, h_{n_k}(x_{n_k}))\right)^-$$

converging to x and $(p, -q)$ respectively. Therefore the proposition follows from

Theorem 8.4.6 (Frankowska) *Let us consider a sequence of closed subsets K_n and an element $x \in \mathrm{Liminf}_{n\to\infty} K_n$ (assumed to be nonempty.) Set $K^\sharp := \mathrm{Limsup}_{n\to\infty} K_n$.*

Then, for any $p \in (T_{K^\sharp}(x))^-$, there exist subsequences of elements $x_{n_k} \in K_{n_k}$ and $p_{n_k} \in \left(T_{K_{n_k}}(x_{n_k})\right)^-$ converging to p and x respectively:

$$(T_{K^\sharp}(x))^- \subset \mathrm{Limsup}_{n\to\infty,\, x_n \to_{K_n} x} (T_{K_n}(x_n))^-$$

Proof — First, it is sufficient to consider the case when x belongs to the intersection $\bigcap_{n=1}^\infty K_n$ of the subsets K_n. If not, we set $\widehat{K_n} := K_n + x - u_n$ where $u_n \in K_n$ converges to x. We observe that $x \in \bigcap_{n=1}^\infty \widehat{K_n}$ and that $T_{\widehat{K_n}}(x_n) = T_{K_n}(x_n - x + u_n)$.

Let $p \in (T_{K^\sharp}(x))^-$ be given with norm 1. We associate with any positive λ the projection x_n^λ of $x + \lambda p$ onto K_n:

$$\left\| x + \lambda p - x_n^\lambda \right\| = \min_{x_n \in K_n} \| x + \lambda p - x_n \| \tag{8.23}$$

and set

$$v_n^\lambda := \frac{x_n^\lambda - x}{\lambda} \quad \& \quad p_n^\lambda := p - v_n^\lambda \in \left(T_{K_n}(x_n^\lambda) \right)^-$$

because $x + \lambda p - x_n^\lambda = \lambda(p - v_n^\lambda)$ belongs to the polar cone $\left(T_{K_n}(x_n^\lambda)\right)^-$ to the contingent cone $T_{K_n}(x_n^\lambda)$ by Proposition 3.2.3.

Let us fix for the time $\lambda > 0$. By taking $x_n = x \in K_n$ in (8.23), we infer that $\|v_n^\lambda\| \leq 2$. Therefore, the sequences x_n^λ and v_n^λ being bounded, some subsequences $x_{n'}^\lambda$ and $v_{n'}^\lambda$ converge to elements $x^\lambda \in K^\sharp$ and $v^\lambda = \frac{x^\lambda - x}{\lambda}$ respectively.

Furthermore, there exists a sequence $\lambda_k \to 0+$ such that v^{λ_k} converge to some $v \in T_{K^\sharp}(x)$ because $\|v^\lambda\| \leq 2$ and because for every λ,

$$x^\lambda = x + \lambda v^\lambda \in K^\sharp$$

Therefore $\langle p, v\rangle \leq 0$ since $p \in (T_{K^\sharp}(x))^-$.

On the other hand, we deduce from (8.23) the inequalities

$$\left\| p - v_n^\lambda \right\|^2 = \|p\|^2 + \|v_n^\lambda\|^2 - 2\langle p, v_n^\lambda\rangle \leq \|p\|^2$$

which imply, by passing to the limit, that $\|v\|^2 \leq 2\langle p, v\rangle \leq 0$.

We have proved that a subsequence v^{λ_k} converges to 0, and thus, that a subsequence $v_{n_k}^{\lambda_k} = p - p_{n_k}^{\lambda_k}$ converges also to 0. The lemma ensues. □

Proof of Proposition 8.4.4 — Assume that Φ is proper. Let h_n be a sequence of Φ satisfying for any n, $\Phi(h_n) \leq c$ and converging to some map h. We have to check that $\Phi(h) \leq c$. Indeed, fix $x \in K$, $q \in B_\star$ and $p \in Dh(x)^\star(q)$. By Proposition 8.4.5, there exist subsequences (again denoted by) $x_n \in K$ converging to x, q_n converging to q and $p_n \in Dh_n(x_n)^\star(q_n)$ converging to p such that $h_n(x_n)$ converges to $h(x)$.

We can always assume that $\|q_n\| \leq 1$. If not, we replace q_n by $\widehat{q}_n := \frac{\|q\|}{\|q_n\|} q_n$ and p_n by

$$\widehat{p}_n := \frac{\|q\|}{\|q_n\|} p_n \in Dh_n(x_n)^\star(\widehat{q}_n)$$

Since F and G are upper semicontinuous with compact values, we know that for any (p, q) and $\varepsilon > 0$, we have

$$\begin{cases} \sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q) \\ \leq \sigma^\flat(F(x_n, h_n(x_n)), p_n) - \sigma(G(x_n, h_n(x_n)), q) + \varepsilon \leq \Phi(h_n) + \varepsilon \end{cases}$$

for n large enough. Hence, by letting n go to ∞, we infer that for any $\varepsilon > 0$,

$$\sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q) \leq c + \varepsilon$$

Letting ε converge to 0 and taking the supremum on $q \in B_\star$, $x \in K$ and $p \in Dh(x)^\star(q)$, we infer that $\Phi(h) \leq c$. □

8.5 Feedback Controls Regulating Smooth Evolutions

Consider a control system (U, f):

$$\begin{cases} i) & \text{for almost all } t, \ x'(t) = f(x(t), u(t)) \\ ii) & \text{where } u(t) \in U(x(t)) \end{cases} \tag{8.24}$$

Let $(x, u) \to \varphi(x, u)$ be a nonnegative continuous function with linear growth.

We have proved in Chapter 7 that there exists a closed regulation map $R^\varphi \subset U$ larger than any closed regulation map $R : K \rightsquigarrow Z$ contained in U and enjoying the following viability property: *For any initial state $x_0 \in \mathrm{Dom}(R)$ and any initial control $u_0 \in R(x_0)$, there exists a solution $(x(\cdot), u(\cdot))$ to the control system (8.24) starting at (x_0, u_0) such that*

$$\forall\, t \geq 0, \ u(t) \in R(x(t))$$

and

$$\text{for almost all } t \geq 0, \ \|u'(t)\| \ \leq \ \varphi(x(t), u(t))$$

Let $K \subset \mathrm{Dom}(U)$ be a closed subset. We also recall that *a closed set-valued map $R : K \rightsquigarrow Z$ is a feedback control regulating viable solutions to the control problem satisfying the above growth condition if and only if R is a solution to the* partial differential inclusion

$$\forall\, x \in K, \ \ 0 \in DR(x,u)(f(x,u)) - \varphi(x,u)B$$

satisfying the constraint

$$\forall\, x \in K, \ \ R(x) \ \subset \ U(x)$$

In particular, a *closed graph single-valued regulation map* $r : K \mapsto Z$ is a solution to the *partial differential inclusion*

$$\forall\, x \in K, \ \ 0 \in Dr(x)(f(x,r(x))) - \varphi(x,r(x))B \qquad (8.25)$$

satisfying the constraint

$$\forall\, x \in K, \ \ r(x) \ \in \ U(x)$$

Such a solution can be obtained by a variational principle:
We introduce the functional Φ defined by

$$\Phi(r) \ := \ \sup_{q \in B_\star} \sup_{x \in K} \sup_{p \in Dr(x)^\star(q)} (< p, f(x,r(x)) > -\varphi(x,r(x))\|q\|)$$

Theorem 8.5.1 *Let $\mathcal{R} \subset \mathcal{C}(K,Y)$ be a nonempty compact subset of selections of the set-valued map U (for the compact convergence topology.)*

Suppose that the functions f and φ are continuous and that

$$c \ := \ \inf_{r \in \mathcal{R}} \Phi(r) \ < \ +\infty$$

Then there exists a solution $r(\cdot)$ to the partial differential inclusion

$$\forall\, x \in K, \ \ 0 \in Dr(x)(f(x,r(x))) - (\varphi(x,r(x)) + c)B$$

Chapter 9

Lyapunov Functions

Introduction

Consider a differential inclusion $x' \in F(x)$, a function $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ and a real-valued function $w(\cdot)$.

The function V is said *to enjoy the Lyapunov property* if and only if for any initial state x_0, there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq 0, \;\; V(x(t)) \;\leq\; w(t)$$

Such inequalities allow us to deduce many properties on the asymptotic behavior of V along the solutions to the differential inclusion (in numerous instances, $w(t)$ goes to 0 when $t \to +\infty$, so that $V(x(t))$ converges also to 0).

Recall that the *epigraph* of V is defined by

$$\mathcal{E}p(V) \;:=\; \{(x,\lambda) \in X \times \mathbf{R} \mid V(x) \leq \lambda\}$$

We see right away that when $w(\cdot)$ is a solution to a differential equation $w' = -\varphi(w)$, we have actually a viability problem in the epigraph of V because the Lyapunov property can be written: For any initial state x_0, there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq 0, \;\; (x(t), w(t)) \;\in\; \mathcal{E}p(V)$$

So that we can apply viability theorems whenever the epigraph of V is closed, i.e., whenever V is lower semicontinuous: V enjoys the

Lyapunov property if and only if its epigraph is a viability domain of the map $(x, w) \rightsquigarrow F(x) \times \{-\varphi(w)\}$.

Therefore, our first task is to study the contingent cone to the epigraph of an extended function V at some point $(x, V(x))$: it is the epigraph of a function denoted $D_\uparrow V(x)$ and called the *contingent epiderivative* of V at x. It is an extension of the concept of directional derivative: If V is Fréchet differentiable at x, then

$$\forall\, u \in X,\ D_\uparrow V(x)(u) \;=\; \langle V'(x), u\rangle$$

It is also an extension of the *lower Dini derivative* when V is locally Lipschitz around x and an extension of the derivative from the right of a convex function. We devote the first section to a minimal presentation of these contingent epiderivatives, which are studied thoroughly in Chapter 6 of SET-VALUED ANALYSIS.

Hence it is no surprise that lower semicontinuous extended functions V which satisfy the Lyapunov property are solutions to the *contingent Hamilton-Jacobi inequality*

$$\forall\, x \in \text{Dom}(V), \quad \inf_{v \in F(x)} D_\uparrow V(x)(v) + \varphi(V(x)) \leq 0$$

We call them *Lyapunov functions (with respect to φ)* because, when V is differentiable and $F \equiv f$ is single-valued, we recognize the classical definition of a Lyapunov function, solution to

$$< V'(x), f(x) > +\varphi(V(x)) \leq 0$$

Therefore, the use of contingent epiderivatives allows lower semicontinuous extended functions to rank among candidates to be solutions to such a contingent Hamilton-Jacobi inequality.

This is of particular importance whenever state constraints are involved, because the restriction of a smooth function to a closed subset is no longer smooth[1].

We prove and exploit these facts in the second section.

The main question we face is *how to construct Lyapunov functions*. Ever since Lyapunov proposed a century ago his second method

[1]By the way, we observe that the indicator function ψ_K of a closed subset K is a Lyapunov function (for $\varphi \equiv 0$) if and only if K is a viability domain.

for studying the behavior of a solution around an equilibrium, finding Lyapunov functions for such and such differential equation (or partial differential equation) has been a source of numerous problems requiring most often many clever tricks. The same difficulty is found here.

However, using the concept of viability kernel, we are able to assert in section 9.3 the existence of a *smallest lower semicontinuous Lyapunov function* $U_\star^\varphi$ *larger than or equal to a given function* U. Hence, starting with any lower semicontinuous function U, we know that there exists a lower semicontinuous Lyapunov function $U_\star^\varphi$ (may be identically equal to $+\infty$) such that

$$\forall\, t \geq 0, \;\; U(x(t)) \leq U_\star^\varphi(x(t)) \leq w(t)$$

whenever the initial state is in the domain of $U_\star^\varphi$.

This may be quite useful when U is the distance function $d_M(\cdot)$ to a subset. For instance, in the case when $\varphi(w) = aw$, the domain of this Lyapunov function $d_{M\star}^a$ provides the set of states (the basin) from which a solution to the differential inclusion converges exponentially to M because

$$\forall\, x_0 \in \mathrm{Dom}(D_{M\star}^a), \;\; d_M(x(t)) \leq d_{M\star}^a(x(t)) \leq d_{M\star}^a(x_0)e^{-at}$$

The results about Lyapunov functions are generalized in the section 9.4 to obtain inequalities of the type

$$\forall\, t \geq s \geq 0, \;\; V(x(t)) - V(x(s)) + \int_s^t W(x(\tau), x'(\tau)d\tau) \leq 0$$

which are very useful for studying the *asymptotic behavior of solutions to differential inclusions and for sufficient conditions for optimality in optimal control.* These important issues are not treated here: we refer to the monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène Frankowska for an exhaustive study of generalized solutions (both contingent and viscosity) to Hamilton-Jacobi equations.

We also show as an example that *gradient inclusions* $x' \in -\partial V(x)$ (where $\partial V(x)$ denotes the generalized gradient) have slow solutions along which V does not increase when V is locally Lipschitz. We refer to Section 3.4 of DIFFERENTIAL INCLUSIONS for the case of lower semicontinuous convex extended functions.

A real-valued function defines the preorder $\succeq$ by

$$x \succeq y \text{ if and only if } V(x) \leq V(y)$$

Since different functions can yield the same preorder, since some (total) preorders cannot be derived from a cost function and since it is needed to consider also any preorder, total or not, in such fields as economics, we address the problem of characterizing preorders satisfying the *Lyapunov property*: for any initial state x_0, there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq s \geq 0, \;\; x(t) \succeq x(s)$$

This problem and the comparison of solutions to two differential inclusions are the topics of section 9.5.

As an application, we touch upon the asymptotic observability problem for differential inclusions in the section 9.6. Here is the problem (for differential equations). We *do not know* the solution $x(\cdot)$ to a differential equation $x' = f(x)$, i.e., its initial value which would allow us to reconstruct it, *but only its observation* $y(t) = h(x(t))$ where $h : X \mapsto Y$ is an observation map.

How can we reconstruct the solution $x(\cdot)$ knowing only $y(\cdot)$? We investigated this *tracking problem* in Chapter 8.

Here, we address a less demanding problem: we only wish to approximate the solution $x(t)$ for large t's. In other words, we would like to build a differential equation $z'(t) = g(z(t), y(t))$ which yields a solution $z(\cdot)$ such that

$$U(x(t) - z(t)) \leq w(t)$$

where U measures some kind of distance and $w(t)$ goes to 0. This problem is known under the name of *asymptotic observability*.

9.1 Contingent Epiderivatives

9.1.1 Extended Functions and their Epigraphs

A function $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ is called an *extended (real-valued) function*. Its *domain* is the set of points at which V is finite:

$$\mathrm{Dom}(V) := \{x \in X \mid V(x) \neq \pm\infty\}$$

A function is said to be *nontrivial*[2] if its domain is not empty. Any function V defined on a subset $K \subset X$ can be regarded as the extended function V_K equal to V on K and to $+\infty$ outside of K, whose domain is K.

Since the order relation on the real numbers is involved in the definition of the Lyapunov property (as well as in minimization problems), we no longer characterize a real-valued function by its graph, but rather by its *epigraph*

$$\mathcal{E}p(V) := \{(x, \lambda) \in X \times \mathbf{R} \mid V(x) \leq \lambda\}$$

or by its *hypograph* defined in a symmetric way by

$$\mathcal{H}p(V) := \{(x, \lambda) \in X \times \mathbf{R} \mid V(x) \geq \lambda\} = -\mathcal{E}p(-V)$$

The graph of a function is then the intersection of its epigraph and its hypograph.

We also remark that some properties of a function are actually properties of their epigraphs. For instance, *an extended function V is convex (resp. positively homogeneous) if and only if its epigraph is convex (resp. a cone).* The epigraph of V is closed if and only if

$$\forall\, x \in X, \;\; V(x) = \liminf_{y \to x} V(y)$$

For extended functions V which never take the value $-\infty$, this is equivalent to the lower semicontinuity of V. We also observe that any positively homogeneous extended function is non trivial whenever $V(0) \neq -\infty$. In this case, $V(0) = 0$.

[2]Such a function is said to be *proper* in convex and non smooth analysis. We chose this terminology for avoiding confusion with proper maps.

Indicators ψ_K *of subsets* K defined by

$$\psi_K(x) := 0 \text{ if } x \in K \text{ and } +\infty \text{ if not}$$

which characterize subsets (as *characteristic functions* do for other purposes), provide important examples of extended functions.

The indicator ψ_K is lower semicontinuous if and only if K is closed and ψ_K is convex if and only if K is convex. One can regard the sum $V + \psi_K$ as the restriction of V to K.

We recall the convention $\inf(\emptyset) := +\infty$.

9.1.2 Contingent Epiderivatives

Before defining the contingent epiderivatives of a function by taking the contingent cones to its epigraph, we need to prove the following statement:

Proposition 9.1.1 *Let* $V : X \mapsto \mathbf{R}\cup\{\pm\infty\}$ *be a nontrivial extended function and* x *belong to its domain.*

Then the contingent cone to the epigraph of V *at* $(x, V(x))$ *is the epigraph of an extended function denoted* $D_\uparrow V(x)$*:*

$$\mathcal{E}p(D_\uparrow V(x)) = T_{\mathcal{E}p(V)}(x, V(x))$$

equal to[3]*:*

$$\forall\, u \in X,\ \ D_\uparrow V(x)(u) \ = \ \liminf_{h \to 0+, u' \to u} (V(x + hu') - V(x))/h$$

[3]We can reformulate this formula below by saying that *the contingent epiderivative* $D_\uparrow V(x)$ *is the lower epilimit (See Definition 9.2.4) of the differential quotients*

$$u \rightsquigarrow \nabla_h V(x)(u) \ := \ \frac{V(x + hu) - V(x)}{h}$$

Indeed, we know that the contingent cone

$$T_{\mathcal{E}p(V)}(x, V(x)) \ = \ \text{Limsup}_{h \to 0+} \frac{\mathcal{E}p(V) - (x, V(x))}{h}$$

is the upper limit of the differential quotients $\frac{\mathcal{E}p(V) - (x, V(x))}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathcal{E}p(D_\uparrow V(x)) := T_{\mathcal{E}p(V)}(x, y) \ \ \& \ \ \mathcal{E}p(\nabla_h F(x, y)) = \frac{\mathcal{E}p(V) - (x, V(x))}{h}$$

to conclude.

Proof — Indeed, to say that

$$(u,v) \in T_{\mathrm{Ep}(V)}(x, V(x))$$

amounts to saying that there exist sequences $h_n > 0$ converging to $0+$ and (u_n, v_n) converging to (u,v) satisfying

$$\forall\, n \geq 0, \quad \frac{V(x+h_n u_n) - V(x)}{h_n} \leq v_n$$

This is equivalent to saying that

$$\forall\, u \in X, \quad \liminf_{h \to 0+, u' \to u} (V(x+hu') - V(x))/h \leq v \;\square$$

Definition 9.1.2 *Let $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be an extended function and $x \in \mathrm{Dom}(V)$. We shall say that the function $D_\uparrow V(x)$ is the* contingent epiderivative *of V at x and that the function V is* contingently epidifferentiable *at x if for any $u \in X$, $D_\uparrow V(x)(u) > -\infty$ (or, equivalently, if $D_\uparrow V(x)(0) = 0$).*

A function V is episleek *(at x) if its epigraph is episleek (at $(x, V(x))$).*

Consequently, *the epigraph of the contingent epiderivative at x is a closed cone. It is then lower semicontinuous and positively homogeneous whenever V is contingently epidifferentiable at x.*

We shall need also the contingent cone to the epigraph of V at points (x,w) where $w > V(x)$:

Proposition 9.1.3 *Let $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and x belong to its domain. For all $w \geq V(x)$,*

$$T_{\mathcal{E}p(V)}(x,w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

and for all $w > V(x)$,

$$\mathrm{Dom}(D_\uparrow V(x)) \times \mathbf{R} \subset T_{\mathcal{E}p(V)}(x,w)$$

If the restriction of V to its domain is upper semicontinuous, then, for all $w > V(x)$,

$$T_{\mathcal{E}p(V)}(x,w) = T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$$

Proof

1. — Fix $w \geq V(x)$. Let us assume that (u,v) belongs to $T_{\mathcal{E}p(V)}(x,w)$. We infer that there exist sequences u_n, v_n and $h_n > 0$ converging to u, v and 0 such that

$$w + h_n v_n \;\geq\; V(x + h_n u_n)$$

We thus deduce that u belongs to the contingent cone to the domain of V at x, and thus, that $T_{\mathcal{E}p(V)}(x,w) \subset T_{\mathrm{Dom}(V)}(x) \times \mathbf{R}$.

2. — If u belongs to the domain of the contingent epiderivative of V at x, if $w > V(x)$ and if v is any real number, we check that (u,v) belongs to $T_{\mathcal{E}p(V)}(x,w)$.

Indeed, there exist sequences of elements $h_n > 0$, u_n and v_n converging to 0, u and $D_{\uparrow}V(x)(u)$ respectively such that

$$(x + h_n u_n, V(x) + h_n v_n) \;\in\; \mathcal{E}p(V)$$

But we can write

$$(x+h_n u_n, w+h_n v) = (x+h_n u_n, V(x)+h_n v_n)+(0, w-V(x)+h_n(v-v_n))$$

Since $w - V(x) + h_n(v - v_n)$ is strictly positive when h_n is small enough, we infer that $(x + h_n u_n, w + h_n v)$ belongs to the epigraph of V, i.e., that (u,v) belongs to the cone $T_{\mathcal{E}p(V)}(x,w)$.

3. — Let w be strictly larger than $V(x)$ and u belong to $T_{\mathrm{Dom}(V)}(x)$. Then there exist sequences u_n and $h_n > 0$ converging to u and 0 such that $V(x + h_n u_n) < +\infty$ for all n.

When V is upper semicontinuous on its domain, for all $\varepsilon \in]0, \frac{w-V(x)}{2}[$, there exists $\eta > 0$ such that, for all $h_n\|u_n\| < \eta$, we obtain

$$V(x + h_n u_n) \;\leq\; V(x) + \varepsilon \;<\; w - \varepsilon$$

Let v be given arbitrarily in $\mathbf{R}$. Then, for any $h_n > 0$ when $v \geq 0$ or for any $h_n \in]0, \frac{\varepsilon}{-v}[$ when $v < 0$, inequality $w - \varepsilon \leq w + h_n v$ implies that $V(x + h_n u_n) \leq w + h_n v$, i.e., that the pair (u,v) belongs to $T_{\mathcal{E}p(V)}(x,w)$. □

We then have to compare contingent derivatives with the contingent epiderivatives and hypoderivatives, defined in a analogous way:

the hypograph of the contingent hypoderivative $D_\downarrow V(x)$ *of* V *at* x *is the contingent cone to the hypograph of* V *at* $(x, V(x))$:

$$\mathcal{E}p(D_\downarrow V(x)) = T_{\mathcal{H}p(V)}(x, V(x))$$

It is equal to

$$\forall\, u \in X, \;\; D_\downarrow V(x)(u) \;=\; \limsup_{h \to 0+, u' \to u} (V(x+hu') - V(x))/h$$

Proposition 9.1.4 *Let* $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be an extended function and* x *belong to its domain. Take any* $u \in \mathrm{Dom}(D_\uparrow V(x)) \cap \mathrm{Dom}(D_\downarrow V(x))$. *Then*

$$\{D_\uparrow V(x)(u), D_\downarrow V(x)(u)\} \;\subset\; D\mathbf{V}(x)(u) \;\subset\; [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$$

Equality

$$D\mathbf{V}(x)(u) \;=\; [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$$

hods true either when V *is continuous on a neighborhood of* x *or when* V *is episleek at* x.

Proof — Since the contingent epiderivative of V at x in the direction u is equal to

$$D_\uparrow V(x)(u) := \liminf_{h \to 0+, u' \to u} \frac{V(x+hu') - V(x)}{h}$$

we see that $D_\uparrow V(x)(u)$ is the limit of a subsequence of $\frac{V(x+hu')-V(x)}{h}$, and thus, that $D_\uparrow V(x)(u) \in D\mathbf{V}(x)(u)$. The same is true with the contingent hypoderivative.

Since $\mathrm{Graph}(V) = \mathcal{E}p(V) \cap \mathcal{H}p(V)$, we deduce that the inclusions

$$T_{\mathrm{Graph}(V)}(x, V(x)) \subset T_{\mathcal{E}p(V)}(x, V(x)) \cap T_{\mathcal{H}p(V)}(x, V(x))$$

can be translated into

$$\mathrm{Graph}(D\mathbf{V}(x)) \subset \mathcal{E}p(D_\uparrow V(x)) \cap \mathcal{H}p(D_\downarrow V(x))$$

from which the inclusion $D\mathbf{V}(x)(u) \subset [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$ follows.

The image $DV(x)(u)$ being convex (and thus, an interval) when V is episleek at x, we infer that $[D_\uparrow V(x)(u), D_\downarrow V(x)(u)] \subset D\mathbf{V}(x)(u)$.

Assume now that V is continuous on a neighborhood of x. Then, on a neighborhood of $(x, V(x))$, the graph of V is the boundary of both the epigraph and the hypograph of V, so that Theorem 4.3.3 implies that

$$T_{\mathrm{Graph}(V)}(x, V(x)) \subset T_{\mathcal{E}p(V)}(x, V(x)) \cap T_{\mathcal{H}p(V)}(x, V(x))$$

and thus, that $D\mathbf{V}(x)(u) = [D_\uparrow V(x)(u), D_\downarrow V(x)(u)]$. □

The contingent epiderivative coincides with the directional derivative $< V'(x), u >$ when V is Fréchet differentiable.

If V is Fréchet differentiable around a point $x \in K$, then the *contingent epiderivative of the restriction is the restriction of the derivative to the contingent cone*:

$$D_\uparrow(V|_K)(x)(u) := \begin{cases} < V'(x), u > & \text{if } \ u \in T_K(x) \\ +\infty & \text{if } \ \text{not} \end{cases}$$

The formulas become much more simple when V is Lipschitz: the contingent epiderivative coincides with the *lower Dini derivative* :

Proposition 9.1.5 *Let us assume that $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ is Lipschitz around a point x of its domain. Then*

$$D_\uparrow V(x)(u) = \liminf_{h \to 0+} (V(x+hu) - V(x))/h \ \ \text{(the lower Dini derivative)}$$

and satisfies for some $l > 0$:

$$\forall\, u \in X, \ \ |D_\uparrow V(x)(u)| \ \leq \ l\|u\|$$

Remark — There are other intimate connections between contingent cones and contingent epiderivatives.

Let ψ_K be the *indicator* of a subset K. Then it is easy to check that

$$D_\uparrow(\psi_K)(x) = \psi_{T_K(x)}$$

Therefore we can either derive properties of the epiderivatives from properties of the tangent cones through epigraphs or take the opposite approach by using the above formula.

Recall that there is also an obvious link between the contingent cone and the contingent epiderivative of the distance function to K since we can write for every $x \in K$:

$$T_K(x) = \{v \in X \mid D_\uparrow d_K(x)(v) = 0\}$$

and that we used this formula to extend contingent cones to the whole space in Section 5.1. □

9.1.3 Epidifferential Calculus

We begin by computing epiderivatives of the sum and the composition product of functions:

Theorem 9.1.6 *Let us consider two finite dimensional vector-spaces X and Y, a continuous single-valued map $f : X \mapsto Y$, and two extended functions V and W from X and Y to $\mathbf{R} \cup \{+\infty\}$ respectively. Let x_0 belong to the domain of the functions $U := V + W \circ f$. We assume that f is continuously differentiable around x_0, that V and W are contingently epidifferentiable at x_0 and $f(x_0)$ respectively. Inequality*

$$D_\uparrow U(x_0)(u) \geq D_\uparrow V(x_0)(u) + D_\uparrow W(f(x_0))(f'(x_0)u)$$

is always true. If V is episleek at x_0 or W is episleek at $f(x_0)$ and the following transversality condition:

$$\mathrm{Dom}(D_\uparrow W(f(x_0))) - f'(x_0)\mathrm{Dom}(D_\uparrow V(x_0)) = Y$$

holds true, then

$$D_\uparrow U(x_0)(u) = D_\uparrow V(x_0)(u) + D_\uparrow W(f(x_0))(f'(x_0)u)$$

In particular, if K is a closed subset and V is a lower semicontinuous function, if $x_0 \in K \cap \mathrm{Dom}(V)$, if K is sleek at x_0 and V is episleek at x_0 and if the transversality condition

$$\mathrm{Dom}(D_\uparrow V)(x_0) - T_K(x_0) = X$$

holds true, then the contingent epiderivative of the restriction is the restriction of the contingent epiderivative to the contingent cone:

$$\forall\, u \in T_K(x_0), \;\; D_\uparrow V|_K(x_0)(u) = D_\uparrow V(x_0)(u)$$

Let us consider now a finite family of functions $V_i : X \mapsto \mathbf{R} \cup \{\pm\infty\}$, $(i \in I)$ with which we associate the function U defined by

$$U(x) \ := \ \max_{i \in I} V_i(x)$$

We set $I(x) := \{i \in I \mid V_i(x) = U(x)\}$. The following estimates are always true:

$$\forall\, u \in X, \ \ \max_{i \in I(x_0)} D_\uparrow V_i(x_0)(u) \ \leq \ D_\uparrow U(x_0)(u)$$

Equality holds true under transversality conditions:

Proposition 9.1.7 *Let us consider n extended lower semicontinuous functions $V_i : X \mapsto \mathbf{R} \cup \{+\infty\}$. If the dimension of X is finite, if the functions U_i are episleek at x_0 and if we posit the transversality assumption at $x_0 \in \mathrm{Dom}(U)$*

$$\forall\, u_i \in X, \ \ \bigcap_{i=1}^{n} (\mathrm{Dom}(D_\uparrow V_i(x_0)) - u_i) \neq \emptyset$$

then

$$\begin{cases} \forall\, u \in \bigcap_{i=1}^{n} \mathrm{Dom}(D_\uparrow V_i(x_0)), \\ DU(x_0)(u) \ = \ \max_{i \in I(x_0)} D_\uparrow V_i(x_0)(u) \end{cases}$$

Consider finally two normed vector spaces X and Y and an extended function $U : X \times Y \mapsto \mathbf{R} \cup \{\pm\infty\}$, with which we associate the marginal function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ defined by

$$V(x) \ := \ \inf_{y \in Y} U(x, y)$$

Proposition 9.1.8 *Let us consider two normed vector spaces X and Y, an extended function $U : X \times Y \mapsto \mathbf{R} \cup \{\pm\infty\}$, and its marginal function V. Suppose that there exists $y_0 \in Y$ which achieves the minimum of $U(x_0, \cdot)$ on Y:*

$$V(x_0) \ = U(x_0, y_0)$$

Then

$$\forall\, u \in X, \ \ D_\uparrow V(x_0)(u) = \liminf_{u' \to u} \left(\inf_{v \in Y} D_\uparrow U(x_0, y_0)(u', v) \right)$$

Equality holds true if U is convex.

9.2 Lyapunov Functions

9.2.1 The Characterization Theorem

We consider a differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\; \in \;\; F(x(t)) \tag{9.1}$$

and a time-dependent function $w(\cdot)$ defined as a solution to the differential equation

$$w'(t) = -\varphi(w(t)) \tag{9.2}$$

where $\varphi : \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth. This function φ is used as a parameter in what follows. (The main instance of such a function φ is the affine function $\varphi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a})$.

Our problem is to characterize either functions enjoying the φ -Lyapunov property, i.e., nonnegative extended functions $V : X \to \mathbf{R}_+ \cup \{+\infty\}$ (such that $\mathrm{Dom}(V) \subset \mathrm{Dom}(F)$) satisfying

$$\forall\, t \geq 0, \;\; V(x(t)) \;\; \leq \;\; w(t), \quad w(0) = V(x(0)) \tag{9.3}$$

along at least one solution $x(\cdot)$ to differential inclusion (9.1) and a solution $w(\cdot)$ to differential equation (9.2).

Definition 9.2.1 (Lyapunov Functions) *We shall say that a nonnegative contingently epidifferentiable*[4] *extended function V is a* Lyapunov function *of F associated with a function $\varphi(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}$ if and only if V is a solution to the* contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_\uparrow V(x)(v) + \varphi(V(x)) \;\; \leq \;\; 0 \tag{9.4}$$

Theorem 9.2.2 *Let V be a nonnegative contingently epidifferentiable lower semicontinuous extended function and $F : X \rightsquigarrow X$ be a Marchaud map. Then V is a Lyapunov function of F associated with $\varphi(\cdot)$ if and only if for any initial state $x_0 \in \mathrm{Dom}(V)$,* there exist solutions *$x(\cdot)$ to (9.1) and $w(\cdot)$ to (9.2) satisfying property (9.3).*

[4]We recall that this means that for all $x \in \mathrm{Dom}(V)$, $\forall\, v \in X$, $D_\uparrow V(x)(v) > -\infty$ and that $D_\uparrow V(x)(v) < \infty$ for at least a $v \in X$.

Proof — We set $G(x,w) := F(x) \times \{-\varphi(w)\}$. Obviously, the system (9.1), (9.2) has a solution satisfying (9.3) if and only if the system of differential inclusions

$$(x'(t), w'(t)) \in G(x(t), w(t)) \tag{9.5}$$

has a solution starting at $(x_0, V(x_0))$ viable in $\mathcal{K} := \mathcal{E}p(V)$. We first observe that $\mathcal{K}$ is a viability domain for G if and only if V is a Lyapunov function for F with respect to φ: If $\mathcal{K}$ is a viability domain of G, by taking $z = (x, V(x))$, we infer that

$$(v, -\varphi(V(x))) \in T_{\mathcal{K}}(x, V(x)) = \mathcal{E}p(D_\uparrow V(x))$$

for some $v \in F(x)$, hence (9.4).

Conversely, since $F(x)$ is compact and $v \mapsto D_\uparrow V(x)(v)$ is lower semicontinuous, (9.4) implies that there exists $v \in F(x)$ such that the pair $(v, -\varphi(V(x)))$ belongs to $T_{\mathcal{E}p(V)}(x, V(x))$. Hence

$$(x + h_n v_n, V(x) + h_n s_n) \in \mathcal{K}$$

with $h_n \to 0+$, $v_n \to v$ and $s_n \to -\varphi(V(x))$. If $w > V(x)$, this implies that for large n

$$\begin{cases} (x + h_n v_n, w - h_n\varphi(w)) = (x + h_n v_n, V(x) + h_n s_n) \\ +(0, w - V(x) - h_n(s_n + \varphi(w))) \in \mathcal{K} + \{0\} \times \mathbf{R}_+ = \mathcal{K} \end{cases}$$

so that $(v, -\varphi(w)) \in T_{\mathcal{K}}(x, w)$.

Remark — We can reformulate the viability theorem in the following way:

Corollary 9.2.3 *Let $F : X \rightsquigarrow X$ be a Marchaud map. A closed subset K enjoys the viability property if and only if its indicator ψ_K is a solution to the contingent equation*

$$\inf_{v \in F(x)} D_\uparrow \psi_K(x)(v) = 0$$

Remark — With an extended nonnegative function V, we can associate affine functions $w \to aw - b$ for which V is a solution to the contingent Hamilton-Jacobi inequalities (9.4).

For that purpose, we consider the convex function b defined by

$$b(a) := \sup_{x \in \mathrm{Dom}(F)} \left(\inf_{v \in F(x)} D_{\uparrow}V(x)(v) + aV(x) \right)$$

Then it is clear that V is a solution to the contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(F),\ \inf_{v \in F(x)} D_{\uparrow}V(x)(v) + aV(x) - b(a) \leq 0$$

Therefore, we deduce that there exists a solution to the differential inclusion satisfying

$$\forall\, t \geq 0,\ V(x(t)) \leq (V(x_0) - \frac{b(a)}{a})e^{-at} + \frac{b(a)}{a}$$

A reasonable choice of a is the largest of the minimizers of $a \in]0, \infty[\rightarrow \max(0, b(a)/a)$, for which $V(x(t))$ decreases as fast as possible to the smallest level set $V^{-1}(]0, \frac{b(a)}{a}])$ of V. $\square$

9.2.2 Stability Theorems

We address now a stability question: *Is the limit of a sequence of Lyapunov functions still a Lyapunov function?*

It depends on what we understand as "limit": the appropriate concept is the one of *lower epilimit* defined in the following way:

Definition 9.2.4 *The epigraph of the lower epilimit*

$$\lim{}^{\sharp}_{\uparrow n \to \infty} V_n$$

of a sequence of extended functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ is the upper limit of the epigraphs:

$$\mathcal{E}p(\lim{}^{\sharp}_{\uparrow n \to \infty} V_n) := \mathrm{Limsup}_{n \to \infty} \mathcal{E}p(V_n)$$

One can check that

$$\lim{}^{\sharp}_{\uparrow n \to \infty} V_n(x_0) = \liminf_{n \to \infty, x \to x_0} V_n(x)$$

We refer to Chapter 7 of SET-VALUED ANALYSIS for further details on *epigraphical convergence*.

Meanwhile, we deduce from Theorem 3.6.2 that

Theorem 9.2.5 *Let F be a Marchaud map. Then the lower epilimit of a sequence of Lyapunov functions V_n associated with a function φ is still a Lyapunov function of F associated with φ.*

We now consider the case when the functions V_n are Lyapunov functions of maps F_n:

Theorem 9.2.6 (Stability) *Let us consider a sequence of Marchaud maps $F_n : X \times Y \rightsquigarrow X$ with uniform linear growth and their graphical upper limit $F^\sharp$. Then the lower epilimit of a sequence of Lyapunov functions V_n of F_n associated with a function φ is a Lyapunov function of $\overline{co}F^\sharp$ associated with φ.*

It is an obvious consequence of Theorem 3.6.5.

9.2.3 W-Monotone Set-Valued Maps

Let $W : X \to \mathbf{R}_+ \cup \{+\infty\}$ be a nonnegative extended function. We say that a set-valued map F is W-monotone (with respect to φ) if

$$\forall\, x, y,\ \forall\, u \in F(x), v \in F(y),\ D_\uparrow W(x-y)(v-u) + \varphi(W(x-y)) \le 0 \tag{9.6}$$

We obtain for instance the following consequence:

Corollary 9.2.7 *Let W be a nonnegative contingently epidifferentiable extended lower semicontinuous function and $F : X \rightsquigarrow X$ be a nontrivial Marchaud map such that $-F$ is W-monotone with respect to some φ. Let $\bar{x}$ be an equilibrium of F (i.e., a solution to $0 \in F(\bar{x})$). Then, for any initial state x_0, there exist solutions $x(\cdot)$ and $w(\cdot)$ satisfying*

$$\forall\, t \ge 0,\ \ W(x(t) - \bar{x}) \ \le\ w(t)$$

In particular, for $W(z) := \frac{1}{2}\|z\|^2$, we find the usual concept of monotonicity (with respect to φ):

$$\forall\, x, y,\ \ \forall\, u \in F(x), v \in F(y),\ \ < u-v, x-y > \ \ge\ \varphi\left(\frac{1}{2}\|x-y\|^2\right) \quad \square$$

9.2.4 Attractors

Using distance functions as Lyapunov functions, we can study attractors:

Definition 9.2.8 *We shall say that a closed subset K is an* attractor *of order $\alpha \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(F)$, there exists at least one solution $x(\cdot)$ to differential inclusion (9.1) such that*

$$\forall\, t \geq 0,\ \ d_K(x(t)) \ \leq\ d_K(x_0)e^{-\alpha t}$$

We can recognize attractors by checking whether the distance function to K is a Lyapunov function:

Corollary 9.2.9 *Assume that F is a nontrivial Marchaud map. Then a closed subset $K \subset \mathrm{Dom}(F)$ is an attractor if and only if the function $d_K(\cdot)$ is a solution to the contingent inequalities:*

$$\forall\, x \in \mathrm{Dom}(F),\ \ \inf_{v \in F(x)} D_{\uparrow} d_K(x)(v) + \alpha d_K(x) \ \leq\ 0$$

Example Let us consider a function V defined through a nonnegative function $U : X \times Y \to \mathbf{R}_+ \cup \{+\infty\}$ in the following way:

$$V(x) := \inf_{y \in Y} U(x, y)$$

When we assume that the infimum is achieved at a point y_x, we recall that

$$D_{\uparrow} V(x)(u) \ \leq\ \inf_{v \in Y} D_{\uparrow} U(x, y_x)(u, v)$$

Hence, under the assumptions of Theorem 9.2.2, we deduce that assumption

$$\forall\, x \in \mathrm{Dom}(V),\ \ \inf_{u \in F(x), v \in Y} D_{\uparrow} U(x, y_x)(u, v) + \varphi(U(x, y_x)) \ \leq\ 0$$

implies that there exists a solution $x(\cdot)$ satisfying

$$\forall\, t \geq 0,\ \ \inf_{y \in Y} U(x(t), y) \ \leq\ w(t)$$

We can derive from this inequality and the calculus of contingent epiderivatives many consequences.

9.2.5 Universal Lyapunov Functions

We shall characterize the φ- universal Lyapunov property, for which property (9.3) is satisfied along *all* solutions to (9.1) and all solutions $w(\cdot)$ to (9.2).

We say that V is a *universal Lyapunov function* of F associated with a function φ if and only if V is a solution to the upper contingent Hamilton-Jacobi inequalities

$$\forall\, x \in \mathrm{Dom}(V), \quad \sup_{v\in F(x)} D_\uparrow V(x)(v) + \varphi(V(x)) \;\leq\; 0 \tag{9.7}$$

In the same way as in Theorem 9.2.2, one can check that the closed subset $\mathcal{E}pV$ is an invariance domain of the set-valued map G if and only if V is a universal Lyapunov function. Then the Invariance Theorem 5.3.4 implies:

Theorem 9.2.10 *Let V be a nonnegative contingently epidifferentiable lower semicontinuous extended function. If F is Lipschitz on the interior of its domain with compact values and*

$$\mathrm{Dom}(V) \subset \mathrm{Int}(\mathrm{Dom}(F))$$

then V is a universal Lyapunov function associated with φ if and only if for any initial state $x_0 \in$ Dom(V), all solutions $x(\cdot)$ *to (9.1) and* $w(\cdot)$ *to (9.2) do satisfy this property (9.3).*

If F is Lipschitz on the interior of its domain with compact values and φ is Lipschitz, then a subset $K \subset \mathrm{Dom}(F)$ is invariant under F if and only if its indicator ψ_K is a solution to the contingent equation

$$\sup_{v\in F(x)} D_\uparrow \psi_K(x)(v) \;=\; 0$$

We say that a subset $M \subset \mathrm{Dom}(F)$ is a *universal attractor* of order $\alpha \geq 0$ if and only if for any $x_0 \in \mathrm{Dom}(F)$, all solutions $x(\cdot)$ to differential inclusion (9.1) satisfy property.

We deduce that if F is Lipschitz with compact images, then K is a universal attractor if and only if

$$\forall\, x \in \mathrm{Dom}(F), \quad \sup_{v\in F(x)} D_\uparrow d_K(x)(v) + \alpha d_K(x) \;\leq\; 0$$

9.3 Optimal Lyapunov Functions

9.3.1 Smallest Lyapunov Functions

The functions φ and $U : X \to \mathbf{R}_+ \cup \{+\infty\}$ being given, we shall construct the smallest lower semicontinuous Lyapunov function larger than or equal to U, i.e., the smallest nonnegative lower semicontinuous solution $U_\star^\varphi$ to the contingent Hamilton-Jacobi inequalities (9.4) larger than or equal to U.

Theorem 9.3.1 *Let us consider a Marchaud map $F : X \rightsquigarrow X$, a continuous function $\varphi : \mathbf{R}_+ \to \mathbf{R}$ with linear growth and a proper nonnegative extended function U such that $\mathrm{Dom}(U) \subset \mathrm{Dom}(F)$.*

Then there exists a smallest nonnegative lower semicontinuous solution $U_\star^\varphi : \mathrm{Dom}(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the contingent Hamilton-Jacobi inequalities (9.4) larger than or equal to U (which can be the constant $+\infty$), which thus enjoys the property:

$$\forall\, x_0 \in \mathrm{Dom}(U_\star^\varphi),\ \textit{there exist solutions to (9.1) and (9.2)}$$
$$\text{satisfying } \forall\, t \geq 0,\ \ U(x(t)) \ \leq\ U_\star^\varphi(x(t)) \ \leq\ w(t)$$

Consequently, if $U(x_0) < U_\star^\varphi(x_0)$, all solutions $x(\cdot)$ to differential inclusion (9.1) and all solutions $w(\cdot)$ to differential equation (9.2) starting at $(x_0, U(x_0))$ satisfy

$$\begin{cases} i) & \forall\, t \geq 0,\ U_\star^\varphi(x(t)) \ >\ w(t) \text{ as long as } U(x(t)) \ \leq\ w(t) \\ & \\ ii) & \exists\, T > 0 \ \textit{ such that } \ U(x(T)) \ >\ w(T) \end{cases}$$

This happens for any solution $w(\cdot)$ whenever the initial state x_0 does not belong to the domain of $U_\star^\varphi$.

Proof — By Theorem 4.1.2, we know that there exists a largest closed viability domain $\mathcal{K} \subset \mathcal{E}p(U)$ (the viability kernel of the epigraph of U) of the set-valued map $(x,w) \rightsquigarrow G(x,w) := F(x) \times \{-\varphi(w)\}$. If it is empty, it is the epigraph of the constant function equal to $+\infty$.

If not, we have to prove that it is the epigraph of the nonnegative lower semicontinuous function $U_\star^\varphi$ defined by

$$U_\star^\varphi(x) \ :=\ \inf_{(x,\lambda)\in\mathcal{K}} \lambda$$

we are looking for. Indeed, the epigraph of any solution $V \geq U$ to the contingent inequalities (9.4) being a closed viability domain of the set-valued map G, is contained in the epigraph of $U_\star^\varphi$, so that $U_\star^\varphi$ is smaller than the lower semicontinuous solutions to (9.4) larger than U. Since

$$\mathcal{E}p(U_\star^\varphi) = \text{Graph}(U_\star^\varphi) + \{0\} \times \mathbf{R}_+ \subset \mathcal{K} + \{0\} \times \mathbf{R}_+$$

it is therefore enough to show that $\mathcal{K} + \{0\} \times \mathbf{R}_+ \subset \mathcal{K}$. In fact, we prove *if $\mathcal{M} \subset Dom(F) \times \mathbf{R}_+$ is a closed viability domain of G,, then so is the subset*

$$\mathcal{M}_0 := \mathcal{M} + \{0\} \times \mathbf{R}_+$$

Obviously, $\mathcal{M}_0$ is closed. To see that $G(x,w) \cap T_{\mathcal{M}_0}(x,w) \neq \emptyset$, let

$$U_{\mathcal{M}}(x) \ := \ \inf_{(x,\lambda)\in\mathcal{M}} \lambda, \ \ d := -\varphi(U_{\mathcal{M}}(x))$$

By assumption, there exists $v \in F(x)$ such that (v,d) belongs to the contingent cone to $\mathcal{M}$ at the point $(x, U_{\mathcal{M}}(x)) \in \mathcal{M}$. Hence, there exist sequences $h_n > 0$ converging to 0, v_n converging to v and d_n converging to d such that

$$\forall\, n \geq 0, \ \ (x + h_n v_n, U_{\mathcal{M}}(x) + h_n d_n) \ \in \ \mathcal{M}$$

This proves the claim when $w = U_{\mathcal{M}}(x)$ and the case $w > U_{\mathcal{M}}(x)$ follows as in the proof of Theorem 9.2.2. □

Corollary 9.3.2 *We posit the assumptions of Theorem 9.3.1.*

— *The indicator $\psi_{\text{Viab}(K)}$ of the viability kernel* Viab(K) *of a closed subset K is the smallest nonnegative lower semicontinuous solution to*

$$\forall\, x \in \text{Dom}(V), \ \ \inf_{v\in F(x)} D_\uparrow V(x)(v) \ \leq \ 0 \tag{9.8}$$

larger than or equal to ψ_K.

— *For all $a \geq 0$, there exists a smallest lower semicontinuous function $d_{M\star}^a : X \to \mathbf{R} \cup \{+\infty\}$ larger than or equal to d_M such that*

$$\forall\, x_0 \in \text{Dom}(d_{M\star}^a), \ \ \text{there exists a solution } x(\cdot) \text{ to (9.1) such that} \\ d_M(x(t)) \leq d_{M\star}^a(x_0)e^{-at}$$

We can regard the subsets $\mathrm{Dom}(d^a_{M\star})$ as the *basins of exponential attraction of* M.

Proof — Let us check that the smallest lower semicontinuous solution U_0 larger than or equal to $U \equiv 0$ is equal to the indicator of $\mathrm{Viab}(K)$. Since it is clear that it is a solution to the above contingent inequalities (9.8), then

$$\forall\, x \in \mathrm{Viab}(K),\ \ U_0(x) \ \le \psi_{\mathrm{Viab}(K)}(x)$$

Let x_0 belong to the domain of U_0. Then there exists a solution $x(\cdot)$ to the system of differential inclusions (9.5) starting at $(x_0, U_0(x_0))$ satisfying $U_0(x(t)) \le U_0(x_0)$ since $w(t) \equiv U_0(x_0)$. Therefore x_0 belongs to the largest closed viability domain $\mathrm{Viab}(K)$. Hence $U_0(x_0) \le \psi_{K_0}(x_0) = 0$.

The proof of the second statement is easy. □

Proposition 9.3.3 *We posit the assumptions of Theorem 9.3.1. Assume furthermore that* φ *vanishes at* 0. *Then if* U *vanishes at an equilibrium* $\bar{x}$ *of* F*, so does the function* $U^{\varphi}_{\star}$ *associated with* φ.

Let L *be the set-valued map associating to any solution* $x(\cdot)$ *to the differential inclusion (9.1) its limit set and* $\mathcal{S}$ *be the solution map. If* φ *is asymptotically stable, then for any* $x_0 \in \mathrm{Dom}(U^{\varphi}_{\star})$*, there exists a solution* $x(\cdot) \in \mathcal{S}(x_0)$ *such that* $L(x(\cdot)) \subset U^{-1}(0)$.

Proof

— If $\bar{x}$ is an equilibrium of F such that $U(\bar{x}) = 0$, then $(\bar{x}, 0)$ is an equilibrium of G restricted to the epigraph of U (because $\varphi(0) = 0$), so that the singleton $(\bar{x}, 0)$, being a viability domain, is contained in viability kernel of $\mathcal{E}p(U)$, which is the epigraph of $U^{\varphi}_{\star}$. Hence $0 \le U(\bar{x}) \le U^{\varphi}_{\star}(\bar{x}) \le 0$.

— If φ is asymptotically stable, then the solutions $w(\cdot)$ to the differential equation $w'(t) = -\varphi(w(t))$ do converge to 0 when $t \to +\infty$. Let x_0 belong to the domain of $U^{\varphi}_{\star}$ and $x(\cdot)$ be a solution satisfying

$$U(x(t)) \ \le\ U^{\varphi}_{\star}(x(t)) \ \le\ w(t)$$

Hence any cluster point ξ of $L(x(\cdot))$, which is the limit of a subsequence $x(t_n)$, belongs to ${U^{\varphi}_{\star}}^{-1}(0)$, because the limit $(\xi, 0)$ of the sequence of elements $(x(t_n), w(t_n))$ of the epigraph of $U_\star^\varphi$ belongs to it, for it is closed. Hence $0 \leq U(\xi) \leq U_\star^\varphi(\xi) \leq 0$. □

9.3.2 Smallest Universal Lyapunov Functions

Using the concept of invariance kernels, we can adapt the above results to optimal universal Lyapunov functions:

Theorem 9.3.4 *If F is Lipschitz on the interior of its domain with compact values and φ is Lipschitz, then there exists a smallest nonnegative lower semicontinuous solution $U_\triangleleft^\varphi : \mathrm{Dom}(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the upper contingent Hamilton-Jacobi inequalities (9.7) larger than or equal to U (which can be the constant $+\infty$), which enjoys the property:*

$$\begin{array}{l} \forall\, x_0 \in \mathrm{Dom}(U_\triangleleft^\varphi), \quad \textit{all solutions} \text{ to (9.1) and (9.2) satisfy} \\ \qquad \forall\, t \geq 0, \;\; U(x(t)) \;\leq\; U_\triangleleft^\varphi(x(t)) \;\leq\; w(t) \end{array}$$

Proof — The proof is analogous to the one of Theorem 9.3.1: When F and φ are Lipschitz, Theorem 5.4.2 implies that there exists a largest closed invariance domain $\widetilde{\mathcal{K}}$ contained in the epigraph of U. We prove that it is the epigraph of the smallest lower semicontinuous solution

$$U_\triangleleft^\varphi \;=\; \inf_{(x,\lambda)\in\widetilde{\mathcal{K}}} \lambda$$

to (9.7) we are looking for. This can be checked by showing that *if $\mathcal{M} \subset$ Dom(F) $\times \mathbf{R}_+$ is a closed invariance domain of the set-valued map G, then so is the subset $\mathcal{M} + \{0\} \times \mathbf{R}_+$.* □

We quote the following consequence:

Corollary 9.3.5 *Assume that F is Lipschitz on the interior of its domain with compact values.*

— *The indicator $\psi_{\mathrm{Inv}(K)}$ of the invariant kernel* Inv(K) *of a closed subset K (i.e., the largest closed invariance domain of F contained in K) is the smallest nonnegative lower semicontinuous solution to*

$$\forall\, x \in \mathrm{Dom}(V), \quad \sup_{v\in F(x)} D_\uparrow V(x)(v) \;\leq\; 0 \tag{9.9}$$

larger than or equal to ψ_K.

— *For all $a \geq 0$, there exists a smallest lower semicontinuous function $d_{M\triangleleft}^a : X \to \mathbf{R} \cup \{+\infty\}$ larger than or equal to d_M such that*

$$\begin{array}{l} \forall\, x_0 \in \mathrm{Dom}(d_{M\triangleleft}^a), \;\; \text{any solution } x(\cdot) \text{ to (9.1) satisfies} \\ \qquad d_M(x(t)) \leq d_{M\triangleleft}^a(x_0)e^{-at} \end{array}$$

We can regard the subsets $\mathrm{Dom}(d_{M\triangleleft}^a)$ as the basins of universal exponential attraction of M.

9.4 Other Monotonicity Properties

9.4.1 Monotone Solutions

We extend the Lyapunov property to more sophisticated inequalities:

Theorem 9.4.1 *Let $F : X \rightsquigarrow X$ be a Marchaud map,*

$$W : (x,v) \in \text{Graph}(F) \mapsto W(x,v) \in \mathbf{R}$$

a lower semicontinuous function convex with respect to v and $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ a nonnegative extended lower semicontinuous function whose domain is contained in the domain of F.

We assume that there exists a positive constant c such that

$$\begin{cases} \forall\, x \in \text{Dom}(V), \ \inf_{v \in F(x)} D_\uparrow V(x)(v) \ \geq \ -c(\|x\|+1) \\ \\ \forall\, (x,v) \in \text{Graph}(F), \ W(x,v) \ \geq \ -c(\|x\|+1) \end{cases} \tag{9.10}$$

*and that V is a W-*Lyapunov function *in the sense that it is a solution to the contingent Hamilton-Jacobi inequality*

$$\forall\, x \in \text{Dom}(V), \quad \inf_{v \in F(x)} D_\uparrow V(x)(v) + W(x,v) \leq 0 \tag{9.11}$$

Then, for any initial state $x_0 \in \text{Dom}(V)$, there exists a solution to differential inclusion (9.1) satisfying

$$\forall\, t \geq 0, \ V(x(t)) - V(x_0) + \int_0^t W(x(\tau), x'(\tau))d\tau \ \leq \ 0 \tag{9.12}$$

Proof — We introduce the set-valued map $G : X \times \mathbf{R} \rightsquigarrow X \times \mathbf{R}$ defined by

$$G(x,w) := \{(v,\lambda) \mid v \in F(x) \ \& \ \lambda \in [-c(\|x\|+1), -W(x,v)]\}$$

It is clear that the graph of G is closed and its values are convex and nonempty by definition (9.10) of c. Its growth is linear by construction. Furthermore, the epigraph of V is a closed viability domain of G: take $v \in F(x)$ achieving the minimum of the lower semicontinuous function $D_\uparrow V(x)(\cdot) + W(x,\cdot)$ on the compact subset $F(x)$. It satisfies $D_\uparrow V(x)(v) + W(x,v) \leq 0$ by assumption (9.11), so that the pair $(v, -W(x,v))$ belongs to the contingent cone to the epigraph of V at (x,w). This follows from the very definition of the epiderivative when $w := V(x)$ and from Proposition 9.1.3 when $w > V(x)$.

Hence $\mathcal{E}p(V)$ being a closed viability domain of $G(\cdot,\cdot)$, there exists a solution $(x(\cdot), w(\cdot))$ to differential inclusion

$$\text{for almost all } t \geq 0, \ \ (x'(t), w'(t)) \in G(x(t), w(t))$$

starting from $(x_0, V(x_0))$ and viable in the epigraph of V. Inequalities

$$w'(\tau) \leq -W(x(\tau), x'(\tau)) \ \& \ V(x(t)) \leq w(t)$$

for almost all $\tau \geq 0$ and all $t \geq 0$ imply by integration from 0 to t inequality (9.12). □

As a consequence, we deduce the following monotonicity theorem:

Theorem 9.4.2 *Let $F : X \rightsquigarrow X$ be a Marchaud map,*

$$W : (x,v) \in \text{Graph}(F) \mapsto W(x,v) \in \mathbf{R}_+$$

a nonnegative continuous function convex with respect to v and $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ a nonnegative extended lower semicontinuous function, continuous on its domain (assumed to be contained in the domain of F). We posit assumptions (9.10) and (9.11).

Then, for any initial sate $x_0 \in \text{Dom}(V)$, there exists a solution to differential inclusion (9.1) satisfying

$$\forall\, t \geq s \geq 0, \ \ V(x(t)) - V(x(s)) + \int_s^t W(x(\tau), x'(\tau))d\tau \ \leq \ 0 \quad (9.13)$$

Proof — We associate with $h \to 0+$ the grid jh, $(j = 1, \ldots)$ and we build a solution $x_h(\cdot) \in \mathcal{S}(x_0)$ to differential inclusion (9.4.1) by using Theorem 9.4.1 iteratively: for $j = 0$, we take $x_h(\cdot)$ on the interval $[0,h]$ satisfying (9.12), then we take $x_h(\cdot)$ on $[h, 2h]$ to be a solution starting at $x_h(h)$ and satisfying $V(x_h(t)) - V(x_h(h)) + \int_h^t W(x(\tau), x'(\tau))d\tau \leq 0$, etc.

Since the image $\mathcal{S}(x_0)$ is compact, a subsequence (again denoted) $x_h(\cdot)$ converges to some solution $x(\cdot) \in \mathcal{S}(x_0)$ in the Sobolev space

$W^{1,1}(0, \infty; X; e^{-bt}dt)$. Continuity of W and Proposition 6.3.4 of DIFFERENTIAL INCLUSIONS implies that the functional

$$x(\cdot) \mapsto \int_0^\infty W(x(\tau), x'(\tau))d\tau$$

is lower semicontinuous on $W^{1,1}(0, \infty; X; e^{-bt}dt)$. Hence

$$\int_0^\infty W(x(\tau), x'(\tau))d\tau \le \liminf_{h \to 0+} \int_0^\infty W(x_h(\tau), x'_h(\tau))d\tau$$

Let $t > s$ be approximated by $j_h h \geq k_h h$ so that

$$V(x_h(j_h h)) - V(x_h(k_h h)) + \int_{k_h h}^{j_h h} W(x_h(\tau), x'_h(\tau))d\tau \le 0$$

The function V being continuous on its domain, inequality (9.13) ensues. □

Remark — We refer to Chapter 6 of DIFFERENTIAL INCLUSIONS and above all, to CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène Frankowska for an exposition of the consequences of such an inequality and of generalized solutions (both contingent and viscosity) to Hamilton-Jacobi-Bellman equations.

Let us just mention that F and W being given and satisfying the assumptions of Theorem 9.4.2, the (extended) function V_F defined by

$$V_F(x) := \inf_{x(\cdot) \in \mathcal{S}(x)} \int_0^\infty W(x(\tau), x'(\tau))d\tau$$

is the smallest of the nonnegative solutions to the contingent inequality (9.11). Furthermore, a solution $\widehat{x}(\cdot) \in \mathcal{S}(x_0)$ satisfies inequality (9.13) for V_F if and only if it is a minimal solution to the optimal control problem

$$\int_0^\infty W(\widehat{x}(\tau), \widehat{x}'(\tau))d\tau = \inf_{x(\cdot) \in \mathcal{S}(x)} \int_0^\infty W(x(\tau), x'(\tau))d\tau$$

In this case, it obeys the "optimality principle"

$$\forall\, t \geq 0, \;\; V_F(\widehat{x}(t)) = \int_t^\infty W(\widehat{x}(\tau), \widehat{x}'(\tau))d\tau \;\; \square$$

For $W \equiv 0$, we obtain the following consequence:

Corollary 9.4.3 *Let $F : X \leadsto X$ be a Marchaud map and V be a nonnegative lower semicontinuous function satisfying*

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\geq\; -c(\|x\|+1)$$

Then V is a Lyapunov function of F if and only if

$$\inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\leq\; 0$$

Furthermore, if V is continuous on its domain, then, for any initial state $x_0 \in \mathrm{Dom}(V)$, V does not increase along at least one solution $x(\cdot)$ to differential inclusion (9.1).

9.4.2 LaSalle's Theorem

One can find attractors using Lyapunov functions by adapting to the set-valued case a classical result due to Lassale:

Theorem 9.4.4 *Assume that $F : X \leadsto X$ is a Marchaud map and that V is a nonnegative lower semicontinuous Lyapunov function continuous on its domain and satisfying*

$$\forall\, x \in \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_{\uparrow}V(x)(v) \;\geq\; -c(\|x\|+1)$$

We denote by

$$K \;:=\; \left\{ x \in \mathrm{Dom}(F) \mid \sup_{u \in F(x)} D_{\downarrow}V(x)(u) \;\geq\; 0 \right\}$$

If K is closed[5]*, then for any $x_0 \in \mathrm{Dom}(V)$, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ such that its ω-limit set*[6] *is contained in* $\mathrm{Viab}(K)$*:*

$$\omega(x(\cdot)) \;\subset\; \mathrm{Viab}(K)$$

[5]This happens whenever F is upper semicontinuous with compact values and $(x,v) \mapsto D_{\uparrow}V(x)(u)$ is upper semicontinuous.

[6]See Definition 3.7.1:

$$\omega(x(\cdot)) \;:=\; \bigcap_{T>0} cl(x([T,\infty[))$$

It is not empty if we assume that V is *inf-compact (or lower semicompact)* (this means that the lower sections $\{x \in X \mid V(x) \leq \lambda\}$ are relatively compact).

Proof — By Corollary 9.4.3, we know that for any $x_0 \in \mathrm{Dom}(V)$, there exists a solution $x(\cdot) \in \mathcal{S}(x_0)$ such that $t \mapsto V(x(t))$ is nonincreasing and converges to some $a \geq 0$. Let $x_\star \in \omega(x(\cdot))$ be a cluster point of the solution $x(\cdot)$ when $t \to \infty$. There exists a sequence $t_n \to \infty$ such that $x(t_n)$ converges to some $x_\star \in V^{-1}(a)$. The functions $y_n(\cdot)$ defined by $y_n(t) := x(t+t_n)$ belong to $\mathcal{S}(x(t_n))$, so that Theorem 3.5.2 implies that a subsequence (again denoted by) $y_n(\cdot)$ converges to a solution $y(\cdot) \in \mathcal{S}(x_\star)$.

The function V being continuous on its domain, inequalities

$$a \ \leq \ V(y_n(t)) \ = \ V(x(t+t_n)) \ \leq \ V(x(t_n))$$

imply by going to the limit that

$$\forall\, t \geq 0, \ \ V(y(t)) \ = \ a \ \text{ or, equivalently, } \ (y(t), a) \ \in \ \mathrm{Graph}(V)$$

Hence $y(\cdot)$ is viable in $V^{-1}(a)$. The necessary condition of the Viability Theorem implies that

$$\forall\, t \geq 0, \ \ 0 \ \in \ DV(y(t))(F(y(t)))$$

By Proposition 9.1.4, we infer that

$$\forall\, t \geq 0, \quad \sup_{u \in F(y(t))} D_\downarrow V(y(t))(u) \ \geq \ 0$$

i.e., that $y(\cdot)$ is viable in K. Hence $x_\star = y(0)$ belongs to the viability kernel of K. □

9.4.3 Example: Gradient Inclusions

Consider a locally Lipschitz sleek real-valued function $V : X \mapsto \mathbf{R}$. Since the contingent epiderivative $D_\uparrow V(x)$ is positively homogeneous, convex and lower semicontinuous, it is the support function of the bounded closed convex subset

$$\partial V(x) \ := \ \{p \in X^\star \mid \forall\, v \in X, \ \ <p, v> \leq D_\uparrow V(x)(v)\}$$

called the *generalized gradient* $\partial V(x)$. A *gradient inclusion* is the differential inclusion

$$\text{for almost all } t \geq 0, \ \ -x'(t) \ \in \ \partial V(x(t))$$

We shall show that a gradient inclusion does have a *slow solution*, i.e., a solution to the differential equation

$$\text{for almost all } t \geq 0, \quad -x'(t) = (\partial V)^\circ(x(t)) \subset \partial V(x(t)) \tag{9.14}$$

(where $\|(\partial V)^\circ(x)\| = \min_{v \in \partial V(x)} \|v\|$) along which the function V decreases.

Theorem 9.4.5 *Let us assume that* $V : X \mapsto \mathbf{R}$ *is a locally Lipschitz episleek real-valued function. Then there exists a slow solution* $x(\cdot)$ *to the gradient inclusion (9.14) satisfying*

$$\text{for almost all } t \geq 0, \quad D_\uparrow V(x(t))(x'(t)) + \|x'(t)\|^2 = 0 \tag{9.15}$$

Proof — We apply Theorem 9.4.2 above with $F(x) := -\partial V(x)$ and $W(x,v) := \|v\|^2$. Since V is locally Lipschitz, its generalized gradient $\partial V(x)$ is convex and compact. Being episleek, one can prove that the function

$$(x,u) \mapsto D_\uparrow V(x)(u) \text{ is upper semicontinuous}$$

Since $D_\uparrow V(x)$ is the support function of $\partial V(x)$, we infer that $\partial V(\cdot)$ is upper hemicontinuous. The solution $v \in -\partial V(x)$ to the equation $D_\uparrow V(x)(v) + \|v\|^2 \leq 0$ exists and is unique: it is the projection of 0 onto the closed convex $-\partial V(x)$. Therefore, there exists a solution to the gradient inclusion satisfying (9.15), i.e., such that for almost all $t \geq 0$, $-x'(t)$ is the projection of 0 onto $-\partial V(x(t))$. This is a slow solution. We also know that for all $t \geq s \geq 0$,

$$V(t) - V(s) = -\int_s^t \|x'(\tau)\|^2 d\tau$$

and thus, that $V(x(t))$ decreases whenever $x(\cdot)$ is not an equilibrium. □

9.4.4 Feedbacks Regulating Monotone Solutions

The regulation map R_V^W which provides solutions satisfying property (9.13) is defined by

$$R_V^W(x) := \{v \in F(x) \mid D_\uparrow V(x)(v) + W(x,v) \leq 0\}$$

Finding closed loop controls, slow solutions, etc., requires that the regulation map is lower semicontinuous with convex values. The following supplies a sufficient condition for this purpose.

Corollary 9.4.6 *We posit the assumptions of Theorem 9.4.2. If F is lower semicontinuous, if $(x,v) \mapsto D_\uparrow V(x,v)$ is upper semicontinuous and if*

$$\forall\, x \in \mathrm{Dom}(V),\ \exists\, \bar{v} \in F(x) \mid D_\uparrow V(x)(\bar{v}) + W(x,\bar{v}) < 0 \qquad (9.16)$$

then the regulation map is lower semicontinuous and there exists a continuous selection $\tilde{r}$ of R_V^W such that the solutions of differential equation $x'(t) = \tilde{r}(x(t))$ are solutions to differential inclusion (9.1) satisfying property (9.13).

Proof — It is analogous to the proof of Theorem 6.3.2. We first observe that the graph of the set-valued map S defined by $S(x) := \{v \mid D_\uparrow V(x)(v) + W(x,v)\}$ is open, then that $x \rightsquigarrow F(x) \cap S(x)$ is lower semicontinuous thanks to the lower semicontinuity of F and thus, that R_V^W is also lower semicontinuous because $R_V^W(x) = \overline{F(x) \cap S(x)}$ and because $F(x) \cap S(x)$ is convex.

Hence the assumptions of Michael's Theorem 6.5.7 are satisfied and there exits a continuous selection of R_V^W. □

Remark — Assumption (9.16) is satisfied for instance when V is both episleek and locally Lipschitz. When it is not satisfied, we can still derive the lower semicontinuity of the regulation map by using Theorem 6.3.1 and the lower semicontinuity of the set-valued map $x \rightsquigarrow T_V^W(x)$ defined by:

$$T_V^W(x) \ := \ \{\, v \in X \mid D_\uparrow V(x)(v) + W(x,v) \le 0 \,\}$$

Proposition 9.4.7 *Let us assume that V is episleek, that the restriction of V to its domain is continuous, that $W(\cdot,\cdot)$ is continuous and convex with respect to the second argument and that F is lower semicontinuous with closed convex values. If for any x, there exists $\bar{v} \in F(x)$ such that*

$$D_\uparrow V(x)(\bar{v}) + W(x,\bar{v}) \ < \ 0$$

then $x \rightsquigarrow T_V^W(x)$ is lower semicontinuous at x.

Proof — Let v belong to $T_V^W(x)$ be chosen and a sequence $x_n \in \text{Dom}(D_\uparrow(V))$ converge to x. Since the set-valued map $\mathcal{E}p(D_\uparrow V(\cdot))$ is lower semicontinuous, and since $(v, -W(x,v))$ belongs to $\mathcal{E}p(D_\uparrow V(x))$, there exist a subsequence (again denoted x_n), a sequence v_n converging to v and a sequence $\varepsilon_n \geq 0$ converging to 0 such that

$$(v_n, -W(x_n, v_n) + \varepsilon_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

Let us set $a_0 := -W(x,\bar{v}) - D_\uparrow V(x)(\bar{v}) > 0$. Since by assumption the pair $(\bar{v}, -W(x,\bar{v}) - a_0)$ belongs also to $\mathcal{E}p(D_\uparrow V(x))$, we deduce that there exist sequences $\bar{v}_n$ converging to $\bar{v}$ and $a_n > 0$ converging to a_0 such that

$$(\bar{v}_n, -W(x_n, \bar{v}_n) - a_n) \in \mathcal{E}p(D_\uparrow V(x_n))$$

We introduce now $\theta_n := \frac{\varepsilon_n}{2(\varepsilon_n + a_n)} \in [0,1]$ converging to 0, $u_n := (1-\theta_n)v_n + \theta_n \bar{v}_n$ converging to v and $\alpha_n := (1-\theta_n)W_n, v_n) + \theta_n W(x_n, \bar{v}_n) - W(x_n, u_n) \geq 0$ (thanks to the convexity of $W(x_n, \cdot)$). The lower semicontinuity of the contingent cone to the epigraph of V, which is the epigraph of $D_\uparrow V(\cdot)$, implies that these cones are convex. Hence

$$\begin{cases} (u_n, -W(x_n, u_n) - \varepsilon_n/2 - \alpha_n) \\ = (1-\theta_n)(v_n, -W(x_n, v_n) + \varepsilon_n) + \theta_n(\bar{v}_n, -W(x_n, \bar{v}_n) - a_n) \end{cases}$$

belongs to $\mathcal{E}p(D_\uparrow V(x_n))$. This can be written

$$D_\uparrow V(x_n)(u_n) \leq -W(x_n, u_n) - \varepsilon_n - \alpha_n/2 < -W(x_n, u_n)$$

Hence u_n belongs to $T_V^W(x_n)$ and converges to v. $\square$

9.5 Lyapunov Preorders

A given function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ defines the preorder

$$x \succeq y \iff V(x) \leq V(y)$$

i.e., a reflexive ($x \succeq x$ for every x) and transitive ($x \succeq y$ and $y \succeq z$ imply $x \succeq z$) binary relation.

Let us consider more generally a *preorder* $\succeq$ and look for solutions $x(\cdot)$ of differential inclusion (9.1) which do not decrease in the sense that

$$\forall\, t \geq s \geq 0, \;\; x(t) \succeq x(s)$$

For that purpose, it is useful to characterize a preorder by the set-valued map P defined[7] by

$$\forall\, x, \;\; P(x) := \; \{y \mid y \succeq x\}$$

the graph of which is the graph of the preorder.

Conversely, any set-valued map P *reflexive* (in the sense that $x \in P(x)$ for every x) and *transitive* (in the sense that $P(y) \subset P(x)$ for every $y \in P(x)$) defines the preorder $\succeq$ defined by $x \succeq y$ if and only if $x \in P(y)$.

Hence, from now on, we shall represent a preorder by a reflexive and transitive set-valued map.

9.5.1 Monotone solutions with respect to a preorder

Corollary 9.4.3 can be extended to general closed preorders.

Proposition 9.5.1 *Let F be a Marchaud map and P be a preorder with closed graph whose domain is contained in the domain of F.*

The following statements are equivalent:

$$\left\{\begin{array}{ll} i) & \forall\, x \in \mathrm{Dom}(P), \;\; F(x) \cap T_{P(x)}(x) \neq \emptyset \\ ii) & \forall\, (x,y) \in \mathrm{Graph}(P), \;\; F(y) \cap DP(x,y)(0) \neq \emptyset \\ iii) & \forall\, x_0 \in \mathrm{Dom}(P), \; \exists\, x(\cdot) \in \mathcal{S}(x_0) \;\; \textit{such that} \\ & \forall\, t \geq s \geq 0, \;\; x(t) \succeq x(s) \end{array}\right. \quad (9.17)$$

Proof

— Condition (9.17)i) implies (9.17)ii) because, for any $y \in P(x)$, there exists $v \in F(y) \cap T_{P(y)}(y)$, i.e., such that $y + h_n v_n \in P(y) \subset P(x)$ for some sequences $h_n \to 0+$ and $v_n \to v$. Hence,

[7] When the (total) preorder is defined by a function V, the set-valued map P associates with any x the subset $P(x) := \{y \mid V(y) \leq V(x)\}$. Its graph is closed if and only if V is continuous on its domain.

the pair $(x + h_n 0, y + h_n v_n)$ belongs to the graph of P, i.e., $v \in DP(x, y)(0)$.

— Condition (9.17)ii) implies (9.17)iii). First, observing that condition (9.17)ii) means that the graph of P is a closed viability domain of the set-valued map $(x, y) \rightsquigarrow \{0\} \times F(y)$, we infer that for any $(x_0, x_0) \in \mathrm{Graph}(P)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions $x' = 0$ and $y' \in F(y)$ which is viable in $\mathrm{Graph}(P)$, i.e., a solution $y(\cdot) \in \mathcal{S}(x_0)$ such that

$$\forall\, t \geq 0,\;\; y(t) \in P(x_0) \tag{9.18}$$

We associate now with $h \to 0+$ the grid jh, $(j = 1, \ldots)$ and we build a solution $x_h(\cdot) \in \mathcal{S}(x_0)$ to differential inclusion (9.1) iteratively: for $j = 0$, we take $x_h(\cdot) = y(\cdot)$ on the interval $[0, h]$ satisfying (9.18), then we take $x_h(\cdot)$ on $[h, 2h]$ to be a solution starting at $x_h(h)$ and satisfying $x_h(t) \in P(x_h(h))$, etc.

Since the image $\mathcal{S}(x_0)$ is compact, a subsequence (again denoted by) x_h converges to some solution $x(\cdot) \in \mathcal{S}(x_0)$ in the Sobolev space $W^{1,1}(0, \infty; X; e^{-bt}dt)$. Let $t > s$ be approximated by $j_h h \geq k_h h$ so that

$$x_h(jh) \;\in\; P(x_h(kh)) \;\text{ or }\; (x_h(kh), x_h(jh) \;\in\; \mathrm{Graph}(P))$$

The graph of P being closed, we infer that $(x(s), x(t)) \in \mathrm{Graph}(P)$, i.e., that $x(t) \in P(x(s))$.

— Condition (9.17)iii) implies (9.17)i) exactly as in the proof of the necessary condition of Haddad's Viability Theorem. □

9.5.2 Comparison of solutions

The same type of proofs yields results dealing with the comparison of solutions to two differential inclusions:

Proposition 9.5.2 *Let $F : X \rightsquigarrow X$ and $G : X \rightsquigarrow X$ be two Marchaud maps and a preorder P with closed graph whose graph is contained in* $\mathrm{Dom}(F) \times \mathrm{Dom}(G)$.

Then the following statements are equivalent:

$$\left\{\begin{array}{ll} i) & \forall\,(x,y)\in \mathrm{Graph}(P),\;\; G(y)\cap DP(x,y)(F(x))\neq\emptyset \\ \\ ii) & \forall\, x_0\in\mathrm{Dom}(P),\;\exists\, x(\cdot)\in\mathcal{S}_F(x_0)\;\;\&\;\; y(\cdot)\in\mathcal{S}_G(x_0)\;\;\textit{such that} \\ & \forall\, t\geq 0,\;\; y(t)\succeq x(t) \end{array}\right. \tag{9.19}$$

Proof — Condition (9.19)i) states that the graph of the preorder P is a closed viability domain of the set-valued map

$$(x,y)\in\mathrm{Graph}(P)\rightsquigarrow F(x)\times G(y)$$

and condition (9.19)ii) that it enjoys the viability property. We then apply Viability Theorem 3.3.5. □

Corollary 9.5.3 *Let $F: X\rightsquigarrow X$ and $G: X\rightsquigarrow X$ be two Marchaud maps, $K\subset \mathrm{Dom}(F)\cap\mathrm{Dom}(G)$ be a closed sleek subset and $Q\subset X$ be a closed convex cone*[8] *defining an order relation on X. We assume the transversality condition*

$$\forall\,(x,y)\in K\times K\;\;\textit{such that}\; y-x\in Q,\;\; T_K(y)-T_K(x)-T_Q(y-x)\;=\;X$$

Then the following statements are equivalent:

$$\left\{\begin{array}{ll} i) & \forall\,(x,y)\in K\times K\quad\textit{such that}\;\; y-x\in Q, \\ & 0\in G(y)-F(x)-T_Q(y-x) \\ \\ ii) & \forall\,(x_0,y_0)\in K\times K\quad\textit{such that}\;\; y_0-x_0\in Q, \\ & \exists\, x(\cdot)\in\mathcal{S}_F(x_0)\;\;\&\;\; y(\cdot)\in\mathcal{S}_G(y_0)\quad\textit{such that} \\ & \forall\, t\geq 0,\; y(t)-x(t)\in Q \end{array}\right.$$

Proof — We define the set-valued map P by

$$\mathrm{Graph}(P) := \{(x,y)\in K\times K\mid y-x\in Q\}$$

Since K is sleek, as well as Q which is convex, we infer from the transversality condition that the contingent derivative of P at (x,y) in the direction u is equal to

$$DP(x,y)(u) := \{v\in T_K(x)\mid v-u\in T_Q(y-x)\}\;\text{ if }\; u\in T_K(x)$$

We then apply Proposition 9.5.2 above. □

[8] We recall that the contingent cone $T_Q(z)$ to Q at z is equal to $\overline{Q+\mathbf{R}z}$.

9.6 Asymptotic Observability of Differential Inclusions

Let us consider a set-valued map F from a finite dimensional vector-space $X := \mathbf{R}^n$ to X and an observation map h from X to another finite dimensional vector-space $Y := \mathbf{R}^p$. We "observe" the evolution

$$\forall\, t \geq 0, \;\; y(t) \;\; := h(x(t))$$

of an unknown solution $x(\cdot)$ to the differential equation (9.1).

The problem is to "simulate asymptotically" at least an unknown state $x(\cdot)$ by a solution $z(\cdot)$ to a control system where the control is the observation of the state

$$z'(t) \;\; = g(z(t), y(t)) \tag{9.20}$$

We shall measure the asymptotic behavior of the error $x(\cdot) - z(\cdot)$ through a nonnegative lower semicontinuous extended function $U : X \mapsto \mathbf{R} \cup \{+\infty\}$ and through a function $w(\cdot)$ from $[0, +\infty]$ to $\mathbf{R}_+$ by inequalities

$$\forall\, t \geq 0, \;\; U(x(t) - z(t)) \;\; \leq \;\; w(t) \tag{9.21}$$

Typically, we would like that $w(t)$ converges to 0 when t goes to $+\infty$ (for instance, $w(t) = ce^{-at}$) and that $U^{-1}(0) = \{0\}$ (for instance, $U(x) := \|x\|^\alpha$) so that we deduce that the error $z(t) - x(t)$ between the observed state $z(t)$ and the unknown state $x(t)$ converges to 0. The bound $w(t)$ which sets an estimate of the measure of the error will be provided by a differential equation

$$w'(t) \;\; = \;\; -\varphi(w(t)), \;\; w(0) \;\; = \;\; U(x(0) - z(0)) \tag{9.22}$$

where $\varphi : [0, +\infty] \;\; \mapsto \;\; \mathbf{R}$ (such as $\varphi(w) = aw$ to obtain exponential decay).

Definition 9.6.1 *Let F, h, φ and U be given. We say that the dynamical system F observed through h is* stabilizable *by g with respect to U and φ if*

$$\forall\, x, z, \quad \inf_{v \in F(x)} D_\uparrow U(x - z)(v - g(z, h(x))) \;\; \leq -\varphi(U(x - z))$$

Proposition 9.6.2 *We assume that F is a Marchaud map, that g, h and φ are continuous with linear growth and that $U : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ is contingently epidifferentiable, lower semicontinuous and episleek. If the dynamical system F observed through h is stabilizable by g, then for any initial state x_0 and z_0, there exist solutions $x(\cdot)$ to (9.1), $z(\cdot)$ to (9.20) and $w(\cdot)$ to (9.22) starting at x_0, z_0 and $U(x_0 - z_0)$ respectively and satisfying inequalities (9.21).*

Proof — The conclusion of the theorem amounts to saying that the function $(x, z) \mapsto V(x, z) := U(x - z)$ enjoys the Lyapunov property with respect to φ for the system of differential inclusions

$$\begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & z'(t) = g(z(t), h(x(t))) \end{cases}$$

because, U being episleek, we infer that $D_\uparrow V(x, z)(x', z') = D_\uparrow U(x - z)(x' - z')$. We then apply Theorem 9.2.2. □

We now have to construct stabilizing maps g in various situations.

We begin by providing a first class of examples using (U, φ)-monotone maps. We derive from the definition of U-monotone maps with respect to φ the following obvious statement.

Proposition 9.6.3 *Let us assume that U, φ, f and h being given, we can find a continuous map $c : Y \mapsto X$ such that*

$$\text{the map } x \mapsto c(h(x)) - F(x) \text{ is } (U, \varphi)\text{-monotone}$$

Then for any continuous selection f of F, the single-valued map

$$g(z, y) := f(z) - c(h(z)) + c(y)$$

stabilizes F through h with respect to U and φ.

The problem now is to recognize whether there exist functions U and φ and a map c which make the set-valued map $c \circ h - F$ to be (U, φ)-monotone.

More generally, let us introduce the set-valued map H defined by

$$H(z, x) := \{v \mid \inf_{u \in F(x)} D_\uparrow U(x - z)(u - v) + \varphi(U(x - z)) \leq 0\}$$

The general problem of stabilizing F through h amounts to finding selections g of the set-valued map G defined by

$$\forall\,(z,y),\quad G(z,y) \;=\; \bigcap_{h(x)=y} H(z,x)$$

since by construction, such selections are stabilizing f through h. When G is lower semicontinuous with closed convex values, Michael's Theorem guarantees the existence of a continuous selection. Hence, in this case, we can stabilize F, at least in theory, since Michael's Theorem is not constructive.

Chapter 10

Miscellaneous Viability Issues

Introduction

This chapter gathers several topics which can be studied from the viability point of view:

1. — How to correct a differential inclusion to make a given closed subset a viability domain (variational differential inequalities)
2. — How to describe situations where some evolutions are more likely to be implemented than others (fuzzy viability)
3. — How to approximate viable solutions to differential inclusions (finite-difference schemes) and their equilibria (Newton's method).

1. — **Differential Variational Inequalities.**

In Chapter 4, we have studied the "anatomy" of a closed subset K which is not a viability domain of a set-valued map F and introduced its "viability kernel", which is the largest closed viability domain contained in it.

Instead of changing the viability constraints by replacing the original viability set K by its viability kernel, we can keep the same viability set and change the dynamics to make K a viability domain of the new system.

There is an obvious strategy which comes immediately to the mind: project the images $F(x)$ onto the contingent cone $T_K(x)$: unfortunately, we lose by doing so two nice features which are basic in the proof of the viability theorems: convexity of the images and upper semicontinuity of the new map.

But it happens that solutions to the "projected subset" are the solutions to another differential inclusion, called *variational differential inequalities* when K is convex. They have been around for a long time in mechanics (problems with unilateral constraints) and in optimization under constraints. It amounts to replacing the set-valued map F by the set-valued map $x \rightsquigarrow F(x) - N_K(x)$, where $N_K(x)$ denotes the normal cone to K at x.

Since K is a viability domain for this new problem (when K is convex or smooth, and more generally, when K is *sleek*), we have two equivalent ways to correct a dynamical system on the boundary of a viability set to make it a viability domain. Furthermore, under additional assumptions, we shall prove that the slow solutions of the two corrected problems coincide and exist.

2. — Fuzzy Viability.

Using differential inclusions for representing uncertainty(or contingent chance) can be criticized on the ground that it gives velocities of the system at state x the same "likelihood" to be chosen. Is there a possibility to discriminate among velocities and to choose among the viable ones those which are somewhat better?

To answer this problem, we suggest replacing the usual subset of velocities in the right hand side of the differential inclusion by a "fuzzy set" of velocities.

Fuzzy sets are usually represented by "membership functions" χ taking their values in the interval $[0, 1]$, the membership functions of usual subsets being their characteristic functions, taking their values in $\{0, 1\}$. Here, we rather characterize subsets by their indicators ψ_K taking their values in $\{0, +\infty\}$, so that we shall replace the classical membership functions of "fuzzy subsets" by extended functions $V : X \mapsto \mathbf{R} \cup \{+\infty\}$, which measure in some sense the membership cost to the fuzzy subset. Hence, a fuzzy differential inclusion is a differential inclusion in which the subsets of velocities are fuzzy subsets $U(x, \cdot)$ depending upon the state x, i.e.,

$$\text{for almost all } t \geq 0, \ \ U(x(t), x'(t)) < \infty$$

We are not only interested in characterizing viability domains of fuzzy differential inclusions (and even, fuzzy viability domains), but also in selecting the viable velocities which minimize the cost of belonging to the fuzzy subset of velocities.

3. — **Finite-Difference Schemes**

We address in the last section elementary numerical aspects of differential inclusions, by studying the convergence of implicit and projected explicit finite-difference schemes.

How can we find an equilibrium of a set-valued map? We shall adapt for that purpose the Continuous Newton's Method to the case of set-valued maps.

10.1 Variational Differential Inequalities

We shall assume here that $K \subset X$ is a closed sleek[1] subset of $X := \mathbf{R}^n$.

We know that K enjoys the viability property for the differential inclusion

$$\text{for almost all } t \geq 0, \ \ x'(t) \ \in \ F(x(t)) \tag{10.1}$$

if and only if K is a viability domain.

When K is not a viability domain, can we "correct" F on the boundary ∂K in order that K becomes a viability domain for the modified dynamics $\widehat{F}$.

An obvious choice[2] of a modified dynamics is the "projection of $F(x)$ onto the contingent cone $T_K(x)$" defined by

$$\forall\, x \in K, \ \ \widehat{F}(x) \ := \ \pi_{T_K(x)} F(x)$$

[1]This means that the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous. This implies that the contingent cone is convex. Let $N_K(x) := (T_K(x))^-$, the polar cone to the contingent cone, denote the *subnormal cone* to K at x, also called the *normal cone* in the case of sleek subsets. When K is sleek, the graph of the set-valued map $x \rightsquigarrow N_K(x)$ is closed. These are the two properties we shall use.

In particular, closed convex subsets are sleek.

[2]We saw in section 4 of Chapter 1 another example of correction procedure which "corrects" a given system by a replicator system which is viable in the simplex.

where π_T denotes the projector of best approximation onto T. (We observe that F is not modified in the interior of K). Hence we replace differential inclusion (10.1) by the *projected differential inclusion*

$$\text{for almost all } t \geq 0, \quad x'(t) \in \pi_{T_K(x(t))}F(x(t)) \tag{10.2}$$

But, by doing so, we may destroy both the convexity of the images and the upper semicontinuity, so that we cannot apply directly the viability theorem to deduce the existence of a solution to the projected differential inclusion.

10.1.1 The Equivalence Theorem

But we shall overcome this difficulty by observing that the viable solutions to the projected differential inclusion (10.2) are the viable solutions to

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t)) - N_K(x(t)) \tag{10.3}$$

Theorem 10.1.1 *The sets of viable solutions to differential inclusions (10.2) and (10.3) starting in K do coincide.*

Furthermore, if F is a Marchaud map, for any $x_0 \in K$, there exists a viable solution starting from x_0 to either projected differential inclusion (10.2) or differential inclusion (10.3).

Proof

— Let $x(\cdot)$ be a solution to the projected differential inclusion (10.2). Since $\mathbf{1} - \pi_{T_K}(x)$ is the orthogonal projector to its polar cone, the normal cone $N_K(x)$, we infer that

$$\pi_{T_K(x)}F(x) \subset F(x) - N_K(x)$$

so that $x(\cdot)$ is a solution to (10.3).

The converse statement follows from the fact that for almost all $t \geq 0$, the derivative $x'(t)$ of a viable solution $x(\cdot)$ to (10.3) belongs to $T_K(x(t)) \cap -T_K(x(t)) = (N_K(x(t)))^\perp$. Hence, if we write $x'(t) = f - p$ where $f \in F(x(t))$ and $p \in N_K(x(t))$, we deduce that for almost all $t \geq 0$, $< x'(t) - f, x'(t) >= - < p, x'(t) >= 0$. The

Projection Theorem[3] implies actually that $x'(t)$ is the projection of f onto $T_K(x(t))$.

— Let us prove now the existence of a solution to differential inclusion (10.3). We introduce the set-valued map G defined by

$$G(x) \ := \ F(x) - N_K(x) \cap c(\|x\| + 1)B$$

Since the graph of $N_K(\cdot)$ is closed, then $x \rightsquigarrow N_K(x) \cap c(\|x\| + 1)B$ is a Marchaud map, so is also G.

The closed subset K is a viability domain of G: indeed, for any $x \in K$ and $f \in F(x)$, $g := f - \pi_{N_K(x)} f$ belongs both to $F(x) - N_K(x)$ and to $T_K(x)$ (because $\mathbf{1} - \pi_{N_K(x)} = \pi_{T_K(x)}$). It also belongs to $G(x)$ because $\|\pi_{N_K(x)} f\| \le \|f\| \le c(\|x\| + 1)$, so that $\pi_{N_K(x)}$ belongs to $N_K(x) \cap c(\|x\| + 1)B$.

Hence the Viability Theorem implies the existence of a viable solution $x(\cdot)$ to the differential inclusion

$$\text{for almost all } t \ge 0, \ \ x'(t) \ \in \ G(x(t))$$

which is obviously a viable solution to (10.3). □

Remark — Let us consider the special case when K is a closed convex subset. We observe first that the characterization[4] of the normal cone to a convex subset implies

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[3]When T is a closed convex cone, the projector π_T of best approximation by elements of T is characterized by

$$< \pi_T(x) - x, \pi_T(x) > = \ 0 \ \ \& \ \ \forall\, v \in T, \ < \pi_T(x) - x, v > \le \ 0$$

and satisfies

$$\begin{cases} i) & \forall\, \lambda > 0, \ \ \pi_T(\lambda x) = \lambda \pi_T(x) \\ ii) & \|x\|^2 = \|\pi_T(x)\|^2 + \|(\mathbf{1} - \pi_T)(x)\|^2 \\ iii) & \|\pi_T(x)\| \le \|x\| \ \ \& \ \ |(\mathbf{1} - \pi_T)(x)\| \le \|x\| \end{cases}$$

Furthermore, if $N := T^-$ denotes the polar cone, $\mathbf{1} - \pi_T =: \pi_N$ is the projector on N and we have

$$T \ = \ \{ \, v \mid \pi_N(v) = 0 \, \} \ \ \& \ \ N \ = \ \{ \, v \mid \pi_T(v) = 0 \, \}$$

Maps π_T and π_N are called *orthogonal projectors* onto T and N respectively.

[4]If K is closed and convex, its normal cone is equal to

$$N_K(x) \ = \ \{ \, p \in X^\star \ \ \text{such that } \forall\, y \in K, \ < p, y - x > \le 0 \, \}$$

Proposition 10.1.2 *Let us assume that K is closed and convex. Then the solutions to (10.3) are the solutions to the "variational differential inequalities"*

$$\begin{cases} i) & \forall\, t \geq 0, \;\; x(t) \;\in\; K \\ ii) & \text{for almost all } t \geq 0, \;\; \exists\, v(t) \in F(x(t)) \quad \textit{such that} \\ & \forall\, y \in K, \;\; < x'(t) - v(t), x(t) - y > \leq 0 \end{cases}$$

When K is convex and compact, we obtain the existence of a solution to the associated variational inequalities[5]:

Proposition 10.1.3 *Let F be upper semicontinuous with compact convex images and $K \subset \mathrm{Dom}(F)$ be a compact convex subset. Then there exists a solution $\bar{x}$ to the* variational inequalities

$$\begin{cases} i) & \bar{x} \;\in\; K \\ ii) & \exists\, \bar{v} \in F(\bar{x}) \quad \textit{such that } \forall\, y \in K, \;\; < \bar{v}, y - \bar{x} > \leq 0 \end{cases} \;\square$$

Proof — We apply Ky Fan Inequality (Theorem 3.7.8) to the function φ defined on $K \times K$ by

$$\varphi(x, y) \;:=\; \inf_{v \in F(x)} < v, y - x >$$

which is obviously concave with respect to y, lower semicontinuous with respect to x thanks to the Maximum Theorem 2.1.6 and satisfy $\varphi(y, y) = 0$. Ky Fan's Theorem implies the existence of $\bar{x} \in K$ such that

$$\forall\, y \in K, \;\; \sup_{y \in K} \inf_{v \in F(\bar{x})} < v, \bar{x} - y > \;\leq\; 0$$

We conclude the proof by applying the lop-sided minimax theorem. □

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[5]Convex minimization problems with constraints lead to such variational inequalities thanks to the "Fermat Rule" which states that, $\bar{x} \in K$ minimizes a lower semicontinuous convex extended function V on K if and only if $0 \in \partial V(\bar{x}) + N_K(\bar{x})$ provided that the weak constraint qualification assumption $0 \in \mathrm{Int}(K - \mathrm{Dom}(V))$ holds true. Therefore, differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \;\in\; -\partial V(x(t)) - N_K(x(t))$$

is the continuous version of the gradient method for constrained minimization problems and one can show that viable solutions $x(t)$ converge to minimizers of V over K when $t \to \infty$. See Section 3.5. of DIFFERENTIAL INCLUSIONS.

Remark — V. Krivan suggested generalizing this procedure by replacing the normal cone $N_K(x)$ by any closed convex cone $M(x)$ such that the condition

$$\forall\, x \in K,\ \ (F(x) - M(x)) \cap T_K(x)\ \neq\ \emptyset$$

whenever the graph of the set-valued map $M : K \rightsquigarrow X$ is closed. For any $f \in X$, the element $u \in (f - M(x)) \cap T_K(x)$ of minimal norm is called the M-projection. Most of the results concerning variational differential inequalities are still valid when we replace the normal cone by such a closed set-valued map M the values of which are closed convex cones satisfying the above condition. In many concrete problems, it may be possible to find a "natural" map M which is simpler to handle than the normal cone map N_K. □

10.1.2 Slow Solutions

Actually, we shall prove that *slow viable solutions* to the projected differential inclusion (10.2) and to differential inclusion (10.3) do coincide and, when the set-valued map F is continuous, do exist.

Proposition 10.1.4 *Slow viable solutions to the projected differential inclusion (10.2) and to differential inclusion (10.3) do coincide and are solutions to the differential equation*

$$\begin{cases} \text{for almost all } t \geq 0, & x'(t) = m(F(x(t)) - N_K(x(t))) \\ & = m(\pi_{T_K(x(t))}F(x(t))) \end{cases} \tag{10.4}$$

The velocities satisfy the equation

$$\text{for almost all } t \geq 0,\ \ \|x'(t)\|^2 + \sigma(-F(x(t)), x'(t))\ =\ 0$$

The proof of this proposition follows from the following

Lemma 10.1.5 *Let $F \subset X$ be a convex compact subset, $T \subset X$ be a closed convex cone and $N := T^-$ be its polar cone. Then*

$$m(\pi_T(F))\ =\ m(F - N)$$

and this element is a solution to

$$\begin{cases} i) & u\ \in\ (F - N) \cap T \\ ii) & \|u\|^2 + \sigma(-F, u) = 0 \end{cases}$$

Furthermore, if $m(F)$ belongs to T, then $m(F) = m(\pi_T(F))$.

Proof — By writing $m(F-N) =: x_0 - y_0$ where $x_0 \in F$ and $y_0 \in N$, we obtain

$$\begin{cases} \|x_0 - y_0\| = \inf_{x\in F, y\in N} \|x-y\| = \inf_{x\in F} \|x - \pi_N(x)\| \\ = \inf_{x\in F} \|\pi_T(x)\| = \inf_{y\in N} \|x_0 - y\| = \|\pi_T(x_0)\| \end{cases}$$

Hence $\pi_T(x_0) = m(\pi_T(F)) = m(F-N)$.

The element $u \in F - N$ is the projection of 0 if and only if

$$\begin{cases} 0 = \sup_{y\in F, z\in N} < u, u-(y-z) > \\ = \|u\|^2 + \sigma(-F,u) + \sigma(N,u) = \|u\|^2 + \sigma(-F,u) \end{cases}$$

because, N being a cone, $\sigma(N,u)$ is either equal to 0 or to $+\infty$, and thus, being finite, is equal to 0 whenever $u \in N^- = T$.

Finally, when $m(F) = \pi_F(0)$ belongs to T, it satisfies

$$\begin{array}{l} \forall\, y \in F,\ \ \|m(F)\|^2 \ \le < m(F), y > \\ =< m(F), \pi_T(y) > + < m(F), \pi_N(y) > = < m(F), \pi_T(y) > \end{array}$$

because $< m(F), \pi_N(y) >= 0$. Hence we deduce inequalities

$$\forall\, y \in F,\ \ \|m(F)\|^2 \ \le < m(F), \pi_T(y) > \le \ \|m(F)\|\ \|\pi_T(y)\|$$

which imply that the norm of $m(F) \in \pi_T(F)$ is minimal. □

Theorem 10.1.6 *Let F be* continuous *with closed convex images and linear growth. Then, for any $x_0 \in K$, there exists a viable (slow) solution to differential equation (10.4) starting from x_0.*

Proof — We introduce the set-valued map H defined by

$$H(x) \ := \ \{\, u \in X \mid \|u\|^2 + \sigma(-F(x), u) \le 0 \,\}$$

Since F is lower semicontinuous, then $(x,u) \to \sigma(-F(x),u)$ is lower semicontinuous, so that the graph of H is closed. Hence the graph of the set-valued map G_1 defined by

$$\forall\, x \in K,\ \ G_1(x) := (F(x) - (N_K(x) \cap c(\|x\|+1)B \cap H(x)$$

is closed and its growth is linear. So G_1 is a Marchaud map. We observe that K is a viability domain of G_1: indeed, $v := m(F(x) -$

$N_K(x))$ belongs to both $G_1(x)$ and $T_K(x)$. Hence, for any $x_0 \in K$, there exists a viable solution to

$$\text{for almost all } t \geq 0, \ \ x'(t) \ \in \ G_1(x(t))$$

Being viable, then, for almost all $t \geq 0$, the velocity $x'(t)$ belongs to $(F(x(t)) - N_K(x(t))) \cap T_K(x(t))$. Since it also belongs to $H(x(t))$, we deduce from Lemma 10.1.5 that $x'(t) = m(F(x(t)) - N_K(x(t)))$. □

10.1.3 Projected differential inclusions onto smooth subsets

Let us consider the case when K is a smooth manifold described by

$$K \ := \ \{ \, x \in X \quad \text{such that} \ \ h(x) = 0 \, \}$$

where $g : X \to Y$ is a $\mathcal{C}^1$-map such that

$$\forall \, x \in K, \ \ h'(x) \ \text{ is surjective}$$

The orthogonal projection of $f(x)$ onto $T_K(x) := \text{Ker} h'(x)$ is equal to

$$h(x) := f(x) - h'(x)^\star (h'(x)h'(x)^\star)^{-1} h'(x) f(x) = f(x) - h'(x)^\star u(x)$$

where $u(x) := (h'(x)h'(x)^\star)^{-1}h'(x)f(x)$ can be regarded as a feedback rule[6].

Hence the slow viable solutions to the projected equation of $x' = f(x)$ are solutions to the differential equation

$$x'(t) \ = \ f(x(t)) - h'(x(t))^\star (h'(x(t))h'(x(t))^\star)^{-1} h'(x(t)) f(x(t))$$

In the case of a differential inclusion $x' \in F(x)$, the slow viable solutions to its projected inclusion are solutions to the differential equation

$$x'(t) = \bar{f}(x(t)) - h'(x(t))^\star (h'(x(t))h'(x(t))^\star)^{-1} h'(x(t)) \bar{f}(x(t))$$

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[6]Observe that $h'(x)^+ := h'(x)^\star (h'(x)h'(x)^\star)^{-1}$ is the *orthogonal right-inverse* of $h'(x)$, so that we can also write $h(x) = (\mathbf{1} - h'(x)^+ h'(x)) f(x)$.

where $\bar{f}(x) \in F(x)$ minimizes over $F(x)$ the function

$$f \to \|f - (h'(x)h'(x)^\star)^{-1}h'(x)f\|$$

— **Case of affine subspaces**

This is the case when $h(x) := Ax - b$, where $A \in \mathcal{L}(X,Y)$ is surjective and $b \in Y$. Then the slow solutions to the projected differential equation are the solutions to the equation

$$x'(t) = f(x(t)) - A^\star(AA^\star)^{-1}Af(x(t))$$

and, in the case of a differential inclusion $x' \in F(x)$, the slow viable solutions are solutions to the differential equation

$$x'(t) = \bar{f}(x(t)) - A^\star(AA^\star)^{-1}A\bar{f}(x(t))$$

where $\bar{f}(x) \in F(x)$ minimizes over $F(x)$ the function $f \to \|f - (AA^\star)^{-1}Af\|$.

When $Y = \mathbf{R}$ and $Ax :=< p, x >$, the differential equation becomes:

$$x'(t) = f(x(t)) - \frac{< p, f(x(t)) >}{\|p\|^2}p$$

— **Case of balls**

This is the case when $h(x) := \|x\|^2 - b :=< Jx, x > -b$ where $J \in \mathcal{L}(X,X)$ is symmetric and positive-definite. Then the slow solutions to the projected differential equation are the solutions to the equation

$$x'(t) = f(x(t)) - \frac{< Jx(t), f(x(t)) >}{\|Jx(t)\|^2}Jx(t)$$

and, in the case of a differential inclusion $x' \in F(x)$, the slow viable solutions are solutions to the differential equation

$$x'(t) = \bar{f}(x(t)) - \frac{< Jx(t), \bar{f}(x(t)) >}{\|Jx(t)\|^2}Jx(t)$$

where $\bar{f}(x) \in F(x)$ minimizes over $F(x)$ the function

$$f \to \left\|f - \frac{< Jx, f >}{\|Jx\|^2}Jx\right\|$$

10.2 Fuzzy Viability

10.2.1 Fuzzy Sets

We recall that any subset $K \subset X$ can be characterized by its *"indicator"* ψ_K, which is the nonnegative extended function defined by:

$$\psi_K(x) := \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases}$$

It can be regarded as a "cost function" or a "penalty function", assigning to any element $x \in X$ an infinite cost when x is outside K, and no cost at all when x belongs to K.

We also recall that K is closed (respectively convex) if and only if its indicator is lower semicontinuous (respectively convex).

We are led to regard any non negative extended function U from X to $\mathbf{R}_+ \cup \{+\infty\}$ as another implementation of the idea underlying "fuzzy sets", in which indicators replace characteristic functions. Instead of using membership functions taking values in the interval $[0, 1]$, we shall deal with *membership cost functions* taking their values anywhere between 0 and $+\infty$.

Definition 10.2.1 *We shall regard an extended nonnegative function* $U : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *as a* fuzzy set. *Its* domain *is the domain of* U, *i.e., the set of elements* x *such that* $U(x)$ *is finite, and the* core *of* U *is the set of elements* x *such that* $U(x) = 0$. *The* complement *of the fuzzy set* U *is the complement of its domain and the complement of its core is called the* fuzzy boundary.

We shall say that the fuzzy set U *is* closed *(respectively* convex*) if the extended function* U *is lower semicontinuous (respectively convex).*

Hence the membership function of the empty set is the constant function equal to $+\infty$.

Definition 10.2.2 *We shall say that a set-valued map* $\mathbf{U} : X \rightsquigarrow Y$ *associating with any* $x \in X$ *a fuzzy subset* $U(x)$ *of* Y *is a* fuzzy set-valued map. *Its* graph *is the fuzzy subset of* $X \times Y$ *associated with the extended nonnegative function* $(x, y) \mapsto U(x, y) := U(x)(y)$ *and its* domain *is*

$$\mathrm{Dom}(\mathbf{U}) := \{x \in X \mid U(x, y) < +\infty \text{ for some } y\}$$

A fuzzy set-valued map $\mathbf{U}$ *is said to be* closed *if and only if its graph is closed, i.e., if its membership function is lower semicontinuous. Its values are closed (respectively convex) if and only if the fuzzy subset* $U(x)$ *are closed (respectively convex). It has linear growth if and only if, for some positive constant c,*

$$U(x,v) < +\infty \implies \|v\| \le c(\|x\| + 1)$$

A nontrivial closed fuzzy set-valued map $\mathbf{U}$ *with convex images and linear growth is called a* Marchaud fuzzy set-valued map.

10.2.2 Fuzzy Differential Inclusions

By using indicators, we can reformulate the differential inclusion

$$\text{for almost all } t, \;\; x'(t) \;\in\; F(x(t))$$

as

$$\text{for almost all } t, \;\; \psi_{F(x(t))}(x'(t)) \;<\; +\infty$$

Then we are led to define "fuzzy dynamics" of a system by a fuzzy set-valued map $\mathbf{U}$ associating to any $x \in X$ a fuzzy set $U(x)$ of velocities $\{v \mid U(x,v) < +\infty\}$. In this case, we can write the associated *fuzzy differential inclusion* in the form

$$\text{for almost all } t \ge 0, \;\; U(x(t), x'(t)) < +\infty \tag{10.5}$$

or, equivalently, in the form

$$\text{for almost all } t \ge 0, \;\; (x(t), x'(t)) \in \text{Graph}(\mathbf{U})$$

which is a fuzzy subset instead of a usual subset.

We begin by characterizing usual subsets K enjoying the viability property for fuzzy differential inclusion: for any initial state $x_0 \in K$, there exists a solution $x(\cdot)$ to the fuzzy differential inclusion (10.5) which is viable in K.

Definition 10.2.3 *We shall say that a subset* $K \subset Dom(\mathbf{U})$ *is a viability domain of the fuzzy set-valued map* $\mathbf{U}$ *if and only if*

$$\forall\, x \in K, \;\; \mathbf{U}(x) \cap T_K(x) \;\neq\; \emptyset$$

i.e., if and only if

$$\forall\, x \in K, \;\; \exists\, v \in T_K(x) \;\text{ such that }\; U(x,v) \;<\; +\infty$$

We begin by proving an extension of the Viability Theorem 3.3.5 to fuzzy differential inclusions.

Theorem 10.2.4 (Fuzzy Viability) *Let us consider a Marchaud fuzzy set-valued map* **U** *from a finite dimensional vector-space* X *to itself. Any closed subset* $K \subset$ *Dom*(**U**) *enjoying the viability property with respect to* U *is a viability domain and the converse holds true if*

$$\forall\, x \in \mathrm{Dom}(\mathbf{U}),\ \ \lambda(x) \ := \inf_{v \in T_K(x)} U(x,v) \ \leq \ \mu(x) \ < \ +\infty$$

where μ *is upper semicontinuous.*

Proof — Let us introduce the set-valued map $F : K \rightsquigarrow X$ defined by

$$F(x) \ := \ \{\, v \in X \mid U(x,v) \leq \mu(x) \,\} \tag{10.6}$$

The subset K enjoys the viability property (is a viability domain) for the fuzzy differential inclusion (10.5) if and only if it does so for this set-valued map F. The set-valued map satisfies the assumptions of the Viability Theorem 3.3.5, because the graph of F is closed, its images are convex and its growth is linear. Then we infer that K enjoys the viability property if and only if it is a viability domain of F, and thus, of **U**. □

When the fuzzy set-valued map **U** is continuous, we can select a viable solution to the fuzzy differential inclusion (10.5) which is *sharpest*, in the sense that the cost of its velocity's membership is minimal:

$$\text{for almost all } t,\ \ U(x(t),x'(t)) = \inf_{v \in T_K(x(t))} U(x(t),v) \tag{10.7}$$

Theorem 10.2.5 *We posit the assumptions of Theorem 10.2.4. We assume moreover that the restriction of the membership function* U *to its domain (the graph of* **U***) is continuous and that the viability domain* K *is sleek.*

Then there exists a sharpest viable solution to the differential inclusion (10.5) (i.e., which satisfies condition (10.7)).

Proof — We introduce the function λ defined by

$$\lambda(x) := \inf_{v \in T_K(x)} U(x,v)$$

Since saying that K is sleek amounts to saying that the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous, the Maximum Theorem 2.1.6 implies that the function λ is upper semicontinuous, because we have assumed that U is upper semicontinuous.

We then introduce the set-valued map G defined by

$$G(x) := \{ v \in X \mid U(x,v) \le \lambda(x) \}$$

Then G has a closed graph, and the other assumptions of the Viability Theorem 3.3.5 are satisfied. There exists a viable solution to differential inclusion $x'(t) \in G(x(t))$, which is a sharpest viable solution to fuzzy differential inclusion (10.5.) □

10.2.3 Fuzzy Viability Domains

Is it possible to speak of fuzzy subsets having the viability property?

A way to capture this idea is to introduce a continuous function[7] $\varphi : \mathbf{R}_+ \to \mathbf{R}$ with linear growth (which is used as a parameter in what follows) and the associated differential equation

$$w'(t) = -\varphi(w(t)), \;\; w(0) = V(x_0) \tag{10.8}$$

whose solutions $w(\cdot)$ set an upper bound to the membership of a fuzzy subset when time elapses.

We shall say that a fuzzy set $V \subset \mathrm{Dom}(\mathbf{U})$ enjoys the "fuzzy viability property" (with respect to φ) if and only if for all initial state $x_0 \in \mathrm{Dom}(V)$, there exist solutions to the fuzzy differential inclusion (10.5) and to the differential equation (10.8) which are "fuzzily" viable in the sense that

$$\forall\, t \ge 0, \;\; V(x(t)) \le w(t), \;\; w(0) = V(x_0)$$

i.e., if and only if the function V satisfies the Lyapunov property (with respect to φ).

[7]The main instance of such a function φ is the affine function $\varphi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a}$.

We introduce now the "contingent set" $T_V^{\varphi}(x)$ (also denoted $T_V(x)$), the closed subset defined by:

$$T_V^{\varphi}(x) := \{ v \in X \mid D_{\uparrow}V(x)(v) + \varphi(V(x)) \leq 0 \}$$

Definition 10.2.6 (Fuzzy Viability Domain) *Let the continuous function φ with linear growth be given. We shall say that a fuzzy subset V is a fuzzy viability domain of a fuzzy set-valued map* **U** *(with respect to φ) if and only if*

$$\forall\, x \in \mathrm{Dom}(V), \quad \mathbf{U}(x) \cap T_V^{\varphi}(x) \neq \emptyset$$

i.e.,, if and only if

$$\forall\, x \in \mathrm{Dom}(V), \quad \exists\, v \in T_V^{\varphi}(x) \quad \textit{such that } U(x,v) < +\infty$$

Theorem 10.2.4 can be extended to fuzzy viability domains:

Theorem 10.2.7 *The fuzzy set-valued map* **U** *satisfies the assumptions of Theorem 10.2.4. We assume that $V \subset \mathrm{Dom}(\mathbf{U})$ is a closed fuzzy subset which is contingently epidifferentiable*[8]*. If a closed fuzzy subset V enjoys the viability property, then it is a closed fuzzy viability domain of* **U** *and the converse holds true if*

$$\forall\, x \in \mathrm{Dom}(\mathbf{U}), \quad \lambda(x) := \inf_{v \in T_V^{\varphi}(x)} U(x,v) \leq \mu(x) < +\infty$$

where μ is upper semicontinuous.

Proof — We apply the characterization theorem of Lyapunov functions (see Theorem 9.2.2) to the set-valued map F defined by (10.6). □

We proceed by extending Theorem 10.2.5 on selection of fuzzy viable solutions to fuzzy differential inclusions which are sharpest, in the sense that

$$\text{for almost all } t, \quad U(x(t), x'(t)) = \inf_{v \in T_V^{\varphi}(x(t))} U(x(t), v) \qquad (10.9)$$

[8]This means that for all $x \in \mathrm{Dom}(V)$, $\forall\, v \in X$, $D_{\uparrow}V(x)(v) > -\infty$ and that $D_{\uparrow}V(x)(v) < \infty$ for at least a $v \in X$.

Theorem 10.2.8 *We posit the assumptions of Theorem 10.2.4. We assume moreover that the restriction of the membership function U to its domain (the graph of* **U***) is continuous and that the fuzzy viability domain V satisfies*[9]

$$x \rightsquigarrow T_V^{\varphi}(x) \text{ is lower semicontinuous}$$

Then there exists a sharpest viable solution to the differential inclusion (10.5) (which satisfies condition (10.9)).

Proof — The proof is the same as the one of Theorem 10.2.5, where the function λ is now defined by

$$\lambda(x) \;:=\; \inf_{v \in T_V^{\varphi}(x)} U(x,v) \;\square$$

Let us consider now any closed fuzzy subset of the domain of **U**, which is not necessarily a fuzzy viability domain. The functions φ being given, we shall construct the largest closed fuzzy viability domain V_{φ} contained in V.

Theorem 10.2.9 *The fuzzy set-valued map satisfies the assumptions of Theorem 10.2.4. We assume that* $V \subset \mathrm{Dom}(\mathbf{U})$ *is a closed fuzzy subset which is contingently epidifferentiable.*

Then for any $\lambda > 0$*, there exists a largest closed fuzzy viability domain* V_{φ} *contained in V (for the fuzzy differential inclusion), which enjoys furthermore the property:*

$$\text{for almost all } t \geq 0, \;\; U(x(t), x'(t)) \leq \lambda$$

Proof — We apply the theorem on the existence of a smallest lower semicontinuous Lyapunov function V_{φ} larger than or equal to V (see Theorem 9.3.1) to the set-valued map F defined by (10.6). □

[9]See section 9.4.3. (with $W(x,v) \equiv \varphi(V(x))$) for criteria of lower semicontinuity.

10.3 Finite-Difference Schemes

We address now the problem of approximating a viable solution to differential inclusion

$$\text{for almost all } t \geq 0, \quad x'(t) \in F(x(t)), \quad x(0) = x_0 \tag{10.10}$$

by solutions to finite difference schemes, either *implicit*

$$\forall j \geq 0, \quad x_{j+1} \in (x_j + hF(x_{j+1})) \cap K, \quad x(0) = x_0 \tag{10.11}$$

or *explicit*:

$$\forall j \geq 0, \quad x_{j+1} \in \Gamma_K(x_j + hF(x_j)), \quad x(0) = x_0 \tag{10.12}$$

where Γ_K is a quasi-projector[10] from X onto K. We say that such solutions converge when $h \to 0+$ if a subsequence of the piecewise linear functions x_h which interpolates the x_j's on the nodes jh converges to a solution $x(\cdot)$ uniformly over compact intervals.

10.3.1 Implicit Finite-Difference Scheme

Implicit finite difference schemes require the knowledge at each step of a solution to an inclusion, whereas explicit schemes demand only the possibility of projecting a point to the viability domain (which is still a difficult problem).

Theorem 10.3.1 *Assume that K is a convex compact viability domain of a Marchaud map F. Then there exist solutions to the implicit finite-difference scheme (10.11) which converges to a solution to differential inclusion (10.10) when $h \to 0+$.*

Proof — We observe that K remains a viability domain of the set-valued map $x \rightsquigarrow x_j - x + hF(x)$ since, K being convex, $x_j - x \in T_K(x)$. Hence the assumptions of Equilibrium Theorem 3.7.6

[10] satisfying

$$\exists \lambda > 0 \quad \text{such that } \forall x \in K, \ \forall y \in X, \ \ \|\Gamma_K(y) - x\| \ \leq \ \lambda \|x - y\|$$

(See Definition 1.3.1).

are satisfied, so that it has a viable equilibrium $x_{j+1} \in K$, which is a solution to the implicit scheme.

The approximate solutions $x_h(\cdot)$ defined on each interval by

$$\forall\, t \in [jh, (j+1)h], \;\; x_h(t) \;:=\; x_j + (t - jh)v_j$$

obviously satisfy the estimates (3.10) of the proof of Viability Theorem 3.3.2, so that a subsequence (again denoted by) $x_h(\cdot)$ converges to some $x(\cdot)$ uniformly on compact intervals and a subsequence (again denoted by) $x'_h(\cdot)$ converges weakly to $x'(\cdot)$ in L^1, which is a solution to the problem thanks to Convergence Theorem 2.4.4. □

10.3.2 Explicit Finite-Difference Scheme

We extend now Theorem 1.3.3 to the case of differential inclusions.

Theorem 10.3.2 *Let us consider a continuous set-valued map with compact values and a compact subset $K \subset \mathrm{Dom}(F)$ such that*

$$\forall\, x \in K, \;\; F(x) \subset T_K(x)$$

Let Γ_K be a quasi-projector from X onto K. Let f be any selection of F and Φ_f the smallest convex valued upper semicontinuous set-valued map containing f, defined by

$$\Phi_f(x) \;:=\; \bigcap_{\eta>0} \overline{co} f(B(x,\eta))$$

Then, starting from $x_0 \in K$, the solutions to the projected explicit finite-difference scheme (10.12) converge to a solution to the differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \in \Phi_f(x(t))$$

Proof — Since $F(x) \subset T_K(x)$ for all $x \in K$, Lemma 5.1.2 implies that

$$\begin{cases} \forall\, (x,v) \in \mathrm{Graph}(F), \; \forall\, \varepsilon > 0, \; \exists\, \eta(x,v) \;\; \text{such that} \\ d_K(y + hu) \leq \varepsilon \;\text{ whenever }\; \max(\|x-y\|, \|v-u\|, h) \leq \eta(x,v) \end{cases}$$

The graph of F being compact, it can be covered by p balls $B((x_i, v_i), \eta(x_i, v_i))$. By setting $\eta := \min_{j=1,\dots,p} \eta(x_i, v_i)$, we deduce that $\forall\, \varepsilon > 0$,

$$\exists\, \eta > 0 \quad \text{such that} \quad \forall\, h \in]0, \eta],\ \forall\, (x, v) \in \text{Graph}(F),\ d_K(x+hv) \leq \varepsilon h$$

Starting from $(x_0, f(x_0)) \in \text{Graph}(F)$, we associate by induction the sequences

$$x_{j+1} := \Gamma_K(x_j + hf(x_j)) \ \& \ v_{j+1} := \frac{x_{j+1} - x_j}{h}$$

and the approximate solutions $x_h(\cdot)$ defined on each interval by

$$\forall\, t \in [jh, (j+1)h],\ \ x_h(t) \ := \ x_j + (t - jh)v_j$$

They obviously satisfy estimates (3.10) of the proof of Viability Theorem 3.3.2 so that a subsequence (again denoted by) $x_h(\cdot)$ converges to some $x(\cdot)$ uniformly over compact intervals and a subsequence (again denoted by) $x'_h(\cdot)$ converges weakly to $x'(\cdot)$ in L^1.

Furthermore, for all $t \geq 0$, we have

$$\begin{cases} (x_h(t), x'_h(t)) = \\ (x_j, f(x_j)) + ((t - jh)v_{j+1}, (x_{j+1} - x_j - hf(x_j))/h) \\ \in \ \text{Graph}(f) + (h\|C\| \times (d_K(x_j + hf(x_j))/h)B \\ \subset \ \text{Graph}(f) + \varepsilon(B \times B) \end{cases}$$

for h small enough. Since the set-valued map Φ_f is upper semicontinuous with closed convex images, we infer from the convergence theorem that the limit $x(\cdot)$ is a solution to the differential inclusion

$$x'(t) \ \in \ \Phi_f(x(t)) \quad \square$$

Remark — When f is not continuous, Cellina's Approximate Selection Theorem[11] allows us to approximate Φ_f by continuous functions f_h in the sense that, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\forall\, h \in]0, \eta],\ \ \text{Graph}(f_h) \ \subset \ \text{Graph}(\Phi_f) + \varepsilon(B \times B)$$

Then Theorem 10.3.2 still holds true when we use the approximate projected explicit finite-difference method

$$x_{j+1} \ = \ \Gamma_K(x_j + hf_h(x_j)) \ \square$$

[11] See Theorem 9.2.1 of SET-VALUED ANALYSIS for instance.

Remark — When f is not continuous, the problem is then to associate *explicitly* with any x_j an element y_j close to x_j such that

$$d_K(x_j + hf(y_j)) \leq h\varepsilon$$

instead of implicitly as in the proof of the Viability Theorem. □

10.3.3 Approximation of an Equilibrium

We can show that under suitable assumptions, the limit when $j \to \infty$ of a discrete solution $(x_j)_{j\geq 0}$ of the projected explicit scheme (if it exists) is an equilibrium of F:

Theorem 10.3.3 *Assume that K is a sleek invariance*[12] *domain of F and that $F(K)$ is bounded.*

If for h small enough, a sequence x_j of the projected finite difference scheme

$$\forall\, j \geq 0,\;\; x_{j+1} \in \Pi_K(x_j + hF(x_j)),\;\;\; x(0) = x_0$$

converges to some $\bar{x}$, then $\bar{x}$ is a (viable) equilibrium of F.

Proof — Indeed, let $u_j \in F(x_j)$ satisfy $x_{j+1} \in \Pi_K(x_j + hu_j)$. Since $F(K)$ is bounded, the sequence u_j is bounded. Let $u \in F^{-1}(\bar{x})$ be any cluster point. Since $(x_j + hu_j, x_{j+1})$ belongs to the graph of Π_K, which is closed, we infer that $\bar{x} \in \Pi_K(\bar{x} + hu)$, and, by Proposition 3.2.3, that $hu = \bar{x} + hu - \bar{x}$ belongs to the polar cone $T_K(\bar{x})^-$ of $T_K(\bar{x})$.

It also belongs to $T_K(\bar{x})$ because $F(\bar{x}) \subset T_K(\bar{x})$. Since this cone is convex because K is sleek, we deduce that $u = 0$. Hence, all cluster points of u_j being equal to 0, we infer that the sequence u_j converges to 0, and therefore, that $\bar{x}$ is an equilibrium. □

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[12]Recall that this means that

$$\forall\, x \in \mathrm{Dom}(F),\;\; F(x) \;\subset\; T_K(x)$$

10.4 Newton's Method

The classical Newton's Method for solving equation $f(x) = 0$ is defined by the algorithm $f'(x_n)(x_{n+1} - x_n) = -hf(x_n)$ and is known to converge to a solution $\bar{x}$ of this equation when f is invertible and when the initial point x_0 is close to this solution.

The continuous version of Newton's Method is given by the differential equation

$$f'(x(t))x'(t) = -f(x(t)), \quad x(0) = x_0$$

which makes sense when $f'(\cdot)$ is invertible. We observe that $y(t) := f(x(t))$ is a solution to the differential equation $y'(t) = -y(t)$ and thus, that it is equal to $y_0 e^{-t}$, so that the cluster points of $x(t)$ when $t \to \infty$ are equilibria of f. Using this remark, we can write the above differential equation in the form of the system of differential inclusions

$$\begin{cases} i) & x'(t) \in (f'(x(t)))^{-1}(-y(t)) \\ ii) & y'(t) = -y(t) \end{cases}$$

for which the graph of f is a viability domain.

We see at once how we can generalize this idea to any closed set-valued map F: Whenever we build pairs $(x(\cdot), y(\cdot))$ of functions such that

1. — the function $(x(\cdot), y(\cdot))$ is viable in the graph of F
2. — the limit when $t \to \infty$ of $y(t)$ is equal to 0

we obtain a continuous version of Newton's algorithm, in the sense that the ω-limit set of $x(\cdot)$, if it is not empty, is contained in the set $F^{-1}(0)$ of equilibria of F.

A way to guarantee that $\lim_{t\to\infty} y(t)$ is equal to 0 is to obtain this function as a solution to a differential equation

$$y'(t) + \psi(y(t)) = 0 \tag{10.13}$$

for which 0 is an asymptotically stable equilibrium: the simplest candidate is $\psi(y) := ay$ for some positive a, but it costs nothing to leave the choice of ψ open.

Therefore, a necessary condition for the function $(x(\cdot), y(\cdot))$ to be viable in the graph of F is that

$$\text{for almost all } t \geq 0, \quad (x'(t), y'(t)) \in T_{\text{Graph}(F)}(x(t), y(t))$$

or, equivalently, taking into account differential equation (10.13) and the definition of contingent derivatives[13], that it is a solution to the system of differential inclusions

$$\begin{cases} i) & x'(t) \in DF(x(t), y(t))^{-1}(\psi(y(t))) \\ ii) & y'(t) = -\psi(y(t) \end{cases}$$

(which are automatically viable in the graph of F).

Therefore, building a continuous Newton's method amounts to

1. — choose a "nice" single-valued Lipschitz map $\psi : Y \mapsto Y$ for which 0 is asymptotically stable

2. — choose a selection φ of the set-valued map

$$(x, y) \rightsquigarrow \Phi(x, y) := DF(x, y)^{-1}(-\psi(y))$$

such that the system of differential equations

$$\begin{cases} i) & x'(t) = \varphi(x(t), y(t)) \\ ii) & y'(t) = \psi(y(t)) \end{cases}$$

has solutions for initial states $(x_0, y_0) \in \text{Graph}(F)$.

We know how to answer these questions by using either Michael's Selection Theorem or any selection procedure introduced in section 4 of Chapter 6.

From now on, a "nice" single-valued Lipschitz map $\psi : Y \mapsto Y$ for which 0 is asymptotically stable is chosen once and for all. We set $\Phi(x, y) := DF(x, y)^{-1}(-\psi(y))$

Theorem 10.4.1 *Let $F : X \rightsquigarrow Y$ be a closed set-valued map. Let S_Φ be a selection procedure of the set-valued map Φ with convex values and linear growth. Then, for any initial state $(x_0, y_0) \in \text{Graph}(F)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions*

$$\begin{cases} i) & x'(t) \in S(\Phi)(x(t), y(t)) := \Phi(x(t), y(t)) \cap S_\Phi(x(t), y(t)) \\ ii) & y'(t) = -\psi(y(t)) \end{cases}$$

[13] which states that

$$T_{\text{Graph}(F)}(x(t), y(t)) =: \text{Graph}(DF(x(t), y(t)))$$

starting at (x_0, y_0)*, satisfying* $y(t) \in F(x(t))$ *for all* $t \geq 0$ *and such that the* ω*-limit set* $\omega(x(\cdot))$ *is contained in the set* $F^{-1}(0)$ *of equilibria.*

Proof — We apply the Viability Theorem 3.3.5 to the system of differential inclusions

$$\begin{cases} i) & x'(t) \in S_\Phi(x(t), y(t)) \\ ii) & y'(t) = -\psi(y(t)) \end{cases}$$

The set-valued map $(x, y) \rightsquigarrow S_\Phi(x, y) \times \{-\psi(y)\}$ being a Marchaud map, it remains to observe that by construction, $(u, -\psi(y))$ belongs to the contingent cone $T_{\mathrm{Graph}(F)}(x, y)$ whenever $u \in S(\Phi)(x, y)$. □

Naturally, this suggests checking that the *minimal selection* φ° defined by

$$\varphi^0(x, y) \in \Phi(x, y) \ \& \ \|\varphi^\circ(x, y)\| = \inf_{v \in \Phi(x,y)} \|v\|$$

can be obtained as an example of the above general method.

Theorem 10.4.2 *Assume that the graph of the set-valued map* $F : X \rightsquigarrow Y$ *is closed and that there exists a positive constant* c *such that*

$$\forall\ (x, y) \in \mathrm{Graph}(F), \quad \inf_{u \in DF(x,y)^{-1}(v)} \|u\| \leq c\|v\|$$

If F *is sleek, then for any initial state* $(x_0, y_0) \in \mathrm{Graph}(F)$*, there exists a solution* $(x(\cdot), y(\cdot))$ *to the system of differential inclusions*

$$\begin{cases} i) & x'(t) = \varphi^0(x(t), y(t)) \\ \\ ii) & y'(t) = -\psi(y(t)) \end{cases}$$

starting at (x_0, y_0)*, satisfying* $y(t) \in F(x(t))$ *for all* $t \geq 0$ *and such that the* ω*-limit set* $\omega(x(\cdot))$ *is contained in the set* $F^{-1}(0)$ *of equilibria.*

Proof — Since the set-valued map F is sleek, the contingent derivatives are closed convex processes. Proposition 7.1.3 applied to

F^{-1} and the assumption of our theorem imply that the set-valued map $(x, y, u) \rightsquigarrow DF(x, y)^{-1}(u)$ is lower semicontinuous with convex values. Therefore Φ is also lower semicontinuous with convex values. Furthermore,

$$\|\varphi^{\circ}(x, y)\| \leq c\|\psi(y)\|$$

so that the growth of the minimal selection is linear. Let S_{Φ}° be the minimal selection procedure defined by

$$S_{\Phi}^{\circ}(x, y) := \{u \in X \mid \|u\| \leq \inf_{v \in \Phi(x,y)} \|v\|\}$$

We apply the Viability Theorem to the system of differential inclusions

$$\begin{cases} i) & x'(t) \in S_{\Phi}^{\circ}(x(t), y(t)) \cap c\|\psi(y(t))\|B \\ ii) & y'(t) = -\psi(y(t)) \end{cases}$$

since the map

$$(x, y) \rightsquigarrow (S_{\Phi}^{\circ}(x, y) \cap c\|\psi(y)\|B) \times \{-\psi(y)\}$$

is a Marchaud map for which the graph of F is a viability domain. □

Corollary 10.4.3 *Let K be a closed sleek subset of a finite dimensional vector-space X and f a C^1-function from a neighborhood of K to a finite dimensional vector-space Y. Assume that*

$$\forall\, x \in K, \;\; f'(x)T_K(x) = Y$$

and that there exists a constant $c > 0$ such that, for all $x \in K$ and $v \in Y$, we can find a solution $u_x \in T_K(x)$ to the equation $f'(x)u = v$ such that $\sup_{x \in K} \|u_x\| < +\infty$.

Hence, there exists a solution $x(\cdot)$ to the differential equation

$$x'(t) = m\left(T_K(x(t)) \cap f'(x(t))^{-1}(-f(x(t)))\right) \tag{10.14}$$

whose ω-limit set is contained in the set $f^{-1}(0) \cap K$ of viable equilibria.

Remark — Recall that when K is compact and convex, Theorem 3.7.12 states that assumption (10.14) implies the existence of both a viable equilibrium of f and of a viable solution to the *implicit discrete scheme*

$$f'(x_n)(x_n - x_{n-1}) = -f(x_n), \quad x_0 \text{ being given in } K \quad \square$$

Chapter 11

Viability Tubes

Introduction

Let X be a finite dimensional vector space and $F : [0, \infty[\times X \rightsquigarrow X$ a set-valued map which associates with any state $x \in X$ and any time t the subset $F(t, x)$ of velocities of the system. The evolution of the system is governed by differential inclusion

$$x'(t) \in F(t, x(t))$$

We consider now *tubes*, i.e., set-valued maps $t \rightsquigarrow P(t)$ from $[0, \infty[$ to X. We say that a solution $t \mapsto x(t) \in X$ is *viable* (in the tube P) if

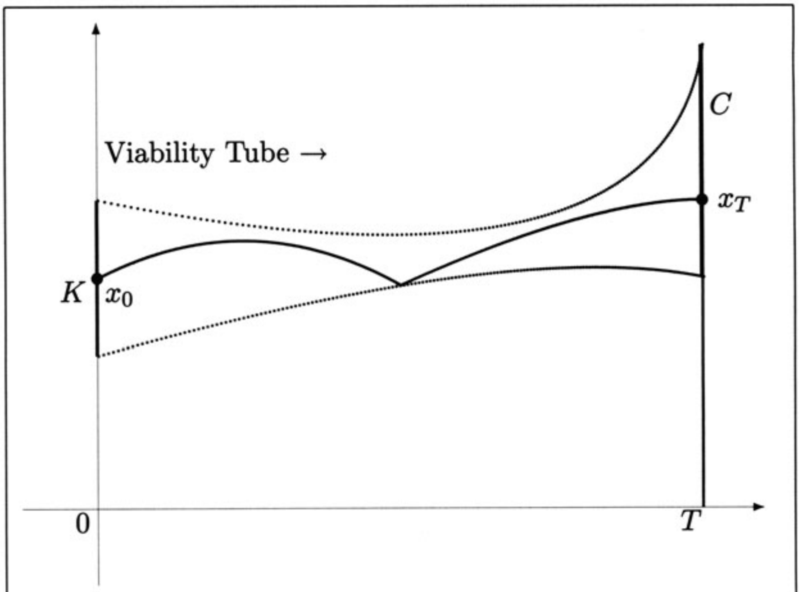
$$\forall\, t \geq 0, \;\; x(t) \in P(t)$$

A tube P is *viable* under F (or enjoys the *viability property*) if and only if, for all $t_0 \geq 0$ and $x_0 \in P(t_0)$, there exists at least one solution $x(\cdot)$ to the differential inclusion starting at x_0 which is viable in the tube P.

A single-valued tube $t \rightsquigarrow \{x(t)\}$ enjoys the viability property if and only if $x(\cdot)$ is a solution to the differential inclusion. Actually, this was the approach used in the first studies of differential inclusions by Marchaud and Zaremba in the 1930's.

The knowledge of a tube enjoying the viability property allows us to infer some information about the asymptotic behavior of some solutions to the differential inclusion. Therefore, they share with

Figure 11.1: Viability Tube



The $K - C$ Problem
Domain K & Target C

Lyapunov functions their usefulness as tools to address asymptotic stability problems.

We shall begin by characterizing tubes enjoying the viability property as *viability tubes*, which are those satisfying

$$\forall\, t \geq 0, \;\; \forall\, x \in P(t), \quad F(t,x) \cap DP(t,x)(1) \;\neq\; \emptyset$$

where $DP(t,x)(1)$ is the contingent derivative[1] of P at (t,x) in the "forward" direction 1.

We can also characterize viability tubes $P(t)$ by the indicator functions ψ_P of their graphs: P is a viability tube if and only if ψ_P is a solution to the *contingent Hamilton-Jacobi equation*[2]

$$\inf_{v \in F(t,x)} D_\uparrow \psi(t,x)(1,v) \;=\; 0$$

We prove in the second section that the reachable tube[3] $R_F^K(t)$ from a closed subset K is a closed viability tube satisfying $R_F^K(0) = K$. We prove that it contains *minimal* closed viability tubes $P(\cdot)$ satisfying $P(0) = K$. We interpret these minimal viability tubes as solutions to the Cauchy problem for viability tubes.

We show in the third section that the upper limit when $t \to \infty$ of a viability tube $P(t)$ is a viability domain and, under compactness assumptions, an attractor of the tube. If we regard such upper limits of viability tubes of a differential inclusion as *asymptotic targets*, we

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[1]See chapter 7 above and Chapter 5 of SET-VALUED ANALYSIS. The contingent derivative is defined by

$$DP(t,x)(1) \;=\; \left\{ v \in X \mid \liminf_{h \to 0+} d\left(v, \frac{P(t+h)-x}{h}\right) = 0 \right\}$$

[2]See chapter 9 above, Chapter 6 of SET-VALUED ANALYSIS and CONTROL THEORY OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène Frankowska. The contingent epiderivative is defined by

$$D_\uparrow \psi(t,x)(1,v) \;:=\; \liminf_{h \to 0+, v' \to v} \frac{\psi(t+h, x+hv') - \psi(t,x)}{h}$$

is the "contingent epiderivative" of ψ at (t,x) in the direction $(1,v)$.

[3]*Reachable tubes* or *funnels* are also solutions to *funnel equations* which are extensively studied in the Soviet literature.

then derive that such targets are necessarily viability domains of a differential inclusion.

We then address the following controllability problem: Given a subset K and a target C, can we reach some or every element $y \in C$ from an initial point $x \in K$ following a solution to the differential inclusion? One way to achieve this purpose is to build viability tubes "going from K to C". Examples are provided in Section 4, where we study tubes of the form $P(t) := \varphi(t, K, C)$ where $\varphi(0, K, C) = K$ and $\varphi(T, K, C) = C$, which *carry* a subset K to a subset C. We also provide in Section 5 a surjectivity criterion which may be useful for solving such problems.

We then investigate in Section 6 tubes enjoying the *invariance property:* for all $t_0 \geq 0$ and $x_0 \in P(t_0)$, *all* solutions to the differential inclusion are viable: we shall characterize them as *invariance tubes*, solutions to

$$\forall\, t \geq 0,\ \forall\, x \in P(t),\ \ F(t,x) \ \subset\ DP(t,x)(1)$$

We can also look for *Marchaud tubes* satisfying the opposite inclusion:

$$\forall\, t \geq 0,\ \forall\, x \in P(t),\ \ DP(t,x)(1) \ \subset\ F(t,x)$$

We see at once that any absolutely continuous selection $x(t) \in P(t)$ of a Marchaud tube is a solution to the differential inclusion.

Hence we have three ways to regard a tube as a *"multivalued solution"* to the differential inclusion, according whether for any $(t, x) \in \mathrm{Graph}(P)$, we have

- $DP(t,x) \subset F(t,x)$ (Marchaud tubes),
- $DP(t,x) \cap F(t,x) \neq \emptyset$ (viability tubes),
- $F(t,x) \subset DP(t,x)$ (invariance tubes).

In many time-dependent problems, the set-valued maps $t \rightsquigarrow F(t,x)$ are only measurable, and no longer upper semicontinuous. This is in particular the case of *variational inclusions* obtained by *linearization of a differential inclusion*

$$x'(t) \ \in\ F(x(t))$$

along a solution $x(\cdot)$:

$$w'(t) \in DF(x(t), x'(t))(w(t))$$

(See Chapter 10 of SET-VALUED ANALYSIS for the connections between the solution map of the variational inclusion and the contingent derivative of the solution map of the original differential inclusion, and SET-VALUED ANALYSIS AND CONTROL THEORY for its applications to local controllability and observability.)

We shall adapt the viability theorem to the case of set-valued map measurable with respect to the time in section 7.

11.1 Viability Tubes

Let X be a finite dimensional vector space. We consider a set-valued map $F : [0,\infty[\times X \rightsquigarrow X$ which associates with every (t,x) the subset $F(t,x)$ of velocities of the system at time t when its state is $x \in X$. We shall study the *differential inclusion*

$$\text{for almost all } t \in [0,T], \;\; x'(t) \in F(t,x(t)) \tag{11.1}$$

From now on, we regard a set-valued map P from $[0,\infty[$ to X as a "tube".

Definition 11.1.1 *We say that a tube P is* viable under F *(or enjoys the* viability property*) if for any initial time $t_0 \in [0,\infty[$ and any initial state $x_0 \in P(t_0)$, there exists a solution $x(\cdot)$ to (11.1) starting from x_0 at time t_0 which is "viable" in the sense that*

$$\forall\, t \in [t_0,\infty[, \;\; x(t) \in P(t)$$

In the finite horizon *case where $F : [0,T]\times X \rightsquigarrow X$ and $P : [0,T] \rightsquigarrow X$, we say that the tube P is* viable under F *on $[0,T]$ if for any initial time $t_0 \in [0,\infty[$ and any initial state $x_0 \in P(t_0)$, there exists a solution $x(\cdot)$ to (11.1) starting from x_0 at time t_0 which is "viable" in the sense that*

$$\forall\, t \in [t_0,T[, \;\; x(t) \in P(t)$$

and for $t \in [T,\infty[$, a solution $x(\cdot)$ to differential inclusion $x'(t) \in F(T,x(t))$ starting from $x(T)$ at time T which is "viable" in $P(T)$:

$$\forall\, t \geq T, \;\; x(t) \in P(T)$$

Remark that a subset K is viable under a time independent set-valued map $F : X \rightsquigarrow X$ if and only if the "constant tube" $t \rightsquigarrow P(t) := K$ is viable under F.

Our first task is to characterize tubes enjoying the viability property thanks to its "contingent derivative".

We observe that it is enough to know this contingent derivative only in the directions $1, 0$ and -1 because $\tau \rightsquigarrow DP(t,x)(\tau)$ is positively homogeneous. In particular, we note that

$$\begin{cases} DP(t,x)(1) & = & \left\{v \in X \mid \liminf_{h \to 0+} d\left(v, \frac{P(t+h)-x}{h}\right) = 0\right\} \\ T_{P(t)}(x) & \subset & DP(t,x)(0) \end{cases} \tag{11.2}$$

Definition 11.1.2 *A tube* $P : [0,\infty[\rightsquigarrow X$ *is called a* viability tube *of a set-valued map* $F : [0,\infty[\times X \rightsquigarrow X$ *if its graph is contained in the domain of* F *and if*

$$\forall\, t \in [0,T[, \forall\, x \in P(t), \quad F(t,x) \cap DP(t,x)(1) \neq \emptyset$$

In the finite horizon *case where* $F : [0,T] \times X \rightsquigarrow X$ *and* $P : [0,T] \rightsquigarrow X$*, we say that* P *is* a viability tube of F *if*

$$\begin{cases} i) & \forall\, t \in [0,T[, & \forall\, x \in P(t), & F(t,x) \cap DP(t,x)(1) & \neq & \emptyset \\ ii) & \text{if} \quad t = T, & \forall\, x \in P(T), & F(T,x) \cap DP(T,x)(0) & \neq & \emptyset \end{cases} \tag{11.3}$$

A tube is said to be closed *if and only if its graph is closed.*

Viability Theorem 3.3.5 for autonomous (i.e., time independent) systems and other results of Chapter 3 can be easily translated in the time dependent case and yield the following summary:

Theorem 11.1.3 *Assume that the* $F : [0,\infty[\times X \rightsquigarrow X$ *is a Marchaud map. Then*

1. — A necessary and sufficient condition for a closed tube to be viable under F *is that it is a viability tube.*

2. — There exists a largest closed viability tube $P_\star$ *contained in any given closed tube* P*, called the* viability kernel *of the tube* P.

3. — If P_n is a sequence of closed viability tubes, then the graphical upper limit P, which is the tube defined by the

$$\text{Graph}(P) := \limsup_{n\to\infty} \text{Graph}(P_n)$$

is also a (closed) viability tube.

4. — Any closed tube $Q \subset P_\star$ is actually contained in a minimal viability tube, called a viability envelope *of Q.*

Proof — We introduce the set-valued map G from Graph(P) to $\mathbf{R}_+ \times \mathbf{R}^n$ defined by

$$G(s,x) := \{1\} \times F(s,x)$$

in the infinite horizon case and by

$$G(s,x) := \begin{cases} \{1\} \times F(s,x) & \text{if} \quad s \in [0,T[\\ [0,1] \times F(T,x) & \text{if} \quad s = T \\ \{0\} \times F(T,x) & \text{if} \quad s > T \end{cases}$$

in the finite horizon case. We observe that $(s(\cdot), x(\cdot))$ is a solution to the differential inclusion

$$\text{for almost all } t, \quad (s'(t), y(t)) \in G(s(t), y(t))$$

starting at $(s(0), x(0)) = (t_0, x_0)$ if and only if the function $x(\cdot)$ defined by $x(t) := y(t - t_0)$ is a solution to differential inclusion (11.1) starting at x_0 at time t_0. We also note that the tube P is viable under F if and only if its graph is viable under G and that P is a viability tube if and only if its graph is a viability domain of G. It thus remains to translate the time independent results. □

We shall denote by $\mathcal{S}_F(t_0, x_0)$ the set of solutions $x(\cdot)$ to the differential inclusion (11.1) and by Graph(P_F) its closed domain, which is the graph of the largest viability tube of F.

We deduce from Theorem 3.5.2 that

Theorem 11.1.4 *Let us consider a finite dimensional vector space X and a* Marchaud map *$F : [0,\infty[\times X \rightsquigarrow X$. Then the solution map $\mathcal{S}_F$ is upper semicontinuous with compact images from* Graph(P_F) *to the space $\mathcal{C}(0,\infty;X)$ supplied with the compact convergence topology.*

Remark — We deduce from the properties of the viability kernel that if an initial state $x_s \in P(s)$ at time s does not belong to $P_\star(s)$, then any solution $x(\cdot)$ to the differential inclusion starting at x_s satisfies

$$\begin{cases} i) & \forall\, t \geq s,\ \ x(t) \notin P_\star(t) \ \text{ as long as } \ x(t) \in P(x(t)) \\ ii) & \exists\, T \geq s \ \text{ such that } \ x(T) \notin P(T) \quad \square \end{cases}$$

We may regard condition (11.3)i) involved in the definition of viability tubes as a "set-valued differential inclusion", the solutions to which are "viability tubes" and condition (11.3)ii) as a "final" condition.

Remark — Actually, conditions (11.3) defining "viability tubes" is a multivalued version of the Hamilton-Jacobi equation in the following sense.

We characterize a tube P by the indicator function ψ_P of its graph defined by

$$\psi_P(t,x) \ := \ \begin{cases} 0 & \text{if } \ x \in P(t) \\ +\infty & \text{if } \ x \notin P(t) \end{cases}$$

Hence, conditions (11.3) can be translated in the following way:

Proposition 11.1.5 *A tube P is a viability tube if and only if the indicator function ψ_P of its graph is a solution to the* contingent Hamilton-Jacobi *equation:*

$$\inf_{v \in F(t,x)} D_\uparrow \psi(t,x)(1,v) \ = \ 0 \tag{11.4}$$

satisfying the final condition (when $T < \infty$):

$$\inf_{v \in F(T,x)} D_\uparrow \psi(T,x)(0,v) \ = \ 0 \tag{11.5}$$

When the function ψ is differentiable, equation (11.4) can be written in the form

$$\frac{\partial \psi}{\partial t} + \inf_{v \in F(t,x)} \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(t,x) v_i \ = \ 0$$

We recognize the classical Hamilton-Jacobi equation (see Chapter 9.)

11.2 Cauchy Problem for Viability Tubes

We consider now a Marchaud map $F : [0, \infty[\times X \leadsto X$ and its largest closed viability tube $P_F : [0, \infty[\leadsto X$ (which is the viability kernel of the domain of F).

Let $K \subset P_F(0)$ be a closed subset. We are looking for viability tubes which satisfy the initial condition $P(0) = K$, which is the Cauchy problem for tubes.

The main example of a closed viability tube satisfying this initial condition is the *reachable tube* defined by $R_F(t)(K)$ defined by

$$R_F^K(t) \ := \ \{x(t)\}_{x(\cdot)\in\mathcal{S}_F(0,K)}$$

Proposition 3.5.6 can be adapted to the time-dependent case:

Proposition 11.2.1 *Assume that $F : [0, \infty[\times X \leadsto X$ is a Marchaud map and that a closed subset K is contained $P_F(0)$. Then the reachable tube $R_F^K(t)$ is a closed viability tube satisfying $R_F^K(0) = K$ and*

$$R_F^K(t) \ = \ \{x \in P_F(t) \mid R_{-F}(t)(x) \cap K \neq \emptyset\}$$

Furthermore, any closed viability tubes $P(\cdot) \subset R_F^K(\cdot)$ satisfying $P(0) = K$ contains a minimal closed viability tube satisfying $P(0) = K$.

Proof — The last statement is a consequence of Zorn's Lemma: Indeed, consider the family of closed viability tubes $P(\cdot) \subset R_F^K(\cdot)$ and a decreasing family $\{P_i\}_{i\in I}$ of such tubes. It is clear that the intersection $P(\cdot) := \bigcap_{i\in I} P_i(\cdot)$ is a closed viability tube satisfying $P(0) = K$. Hence each closed viability tube contains a minimal viability tube. □

Definition 11.2.2 *We shall say that the minimal closed viability tubes $P(\cdot) \subset R_F^K(\cdot)$ of F satisfying the initial condition $P(0) = K$ are the solutions to the Cauchy problem for the differential inclusion (11.1).*

Theorem 11.2.3 *Consider a Marchaud map* $F : [0, \infty[\times X \rightsquigarrow X$ *satisfying the uniqueness property*[4]*:* $\forall\, x \in P_F(0)$*, there exists a unique solution to the differential inclusion (11.1) starting at* x *at time* 0.

Let $K \subset P_F(0)$ *be a nonempty closed subset. Then the reachable tube* R_F^K *is the unique solution to the Cauchy problem for viability tubes.*

Proof of Theorem 11.2.3 — Let us consider a closed viability tube $P \subset R_F^K$ satisfying $P(0) = K$. Then $P = R_F^K$. Otherwise, there would exist $x_s \in R_K(s)$ such that $x_s \notin P(s)$. By assumption, there exists a solution $x(\cdot)$ to the differential inclusion (11.1) starting from $x(0) \in K$ such that $x(s) = x_s$. But starting from $x(0)$, the solution is unique, and is viable in the tube P since it is a viability tube satisfying $P(0) = K$. Therefore $x(s)$ belongs to $P(s)$, a contradiction. $\square$

11.3 Asymptotic Target

We shall now study the behavior of viability tubes of time independent maps $F : X \rightsquigarrow X$ when $t \to \infty$.

Theorem 11.3.1 *Consider a Marchaud map* F *from* X *to* X *and a closed viability tube* $P : [0, \infty[\rightsquigarrow X$*. Then the upper limit*

$$C \;:=\; \text{Limsup}_{t \to \infty} P(t)$$

is a viability domain of F.

If there exists $T \geq 0$ *such that* $\bigcup_{t \geq T} P(t)$ *is bounded, then it is an* attractor *in the sense that*

$$\forall\, t_0 \geq 0,\; x_0 \in P(t_0),\; \exists\; x(\cdot) \in \mathcal{S}_F(t_0, x_0) \;\mid\; \lim_{t \to \infty} d(x(t), C) \;=\; 0$$

[4]This happens whenever F enjoys a *monotonicity property* of the form: there exists a real constant c such that for every $t \geq 0$, for every pair x_i, $u_i \in F(t, x_i)$ $(i = 1,\ 2)$, we have

$$\langle u_1 - u_2, x_1 - x_2 \rangle \;\leq\; c\|x_1 - x_2\|^2$$

Proof — We shall prove that C enjoys the viability property. Let ξ belong to C. Then $\xi = \lim \xi_n$ where $\xi_n \in P(t_n)$. We consider the solutions $x_n(\cdot)$ to the differential inclusion

$$x_n'(t) \in F(x_n(t)), \; x_n(t_n) = \xi_n$$

which are viable in the sense that $\forall\, t \geq t_n, \quad x_n(t) \in P(t)$. The function $y_n(\cdot)$ defined by $y_n(t) := x_n(t + t_n)$ are solutions to

$$y_n'(t) \in F(y_n(t)), \; y_n(0) = \xi_n$$

Theorem 3.5.2 implies that these solutions remain in a compact subset of $\mathcal{C}(0, \infty; X)$. Therefore, a subsequence (again denoted by) $y_n(\cdot)$ converges to $y(\cdot)$, which is a solution to

$$y'(t) \in F(y(t)), \; y(0) = \xi$$

Furthermore, this solution is viable in C since for all $t \geq 0, \; y(t)$ is the limit of a subsequence of $y_n(t) = x_n(t+t_n) \in P(t+t_n)$, and thus belongs to C.

Let us prove now that C is an attractor. If not, there would exist $x_0 \in P(t_0)$ such that for all solutions $x(\cdot) \in \mathcal{S}_F(t_0, x_0)$, there exist $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$\forall\, n \geq 0, \; d(x(t_n), C) \geq \delta > 0$$

There is at least one such solution $x_\star(\cdot)$ which is viable in the tube $P(\cdot)$. Since the closure of $\bigcup_{t \geq T} P(t)$ is compact by assumption, a subsequence (again denoted by) $x_\star(t_n)$ converges to some $x_\star$ which belongs to the C. We thus obtain a contradiction. □

11.4 Examples of Viability Tubes

Let us consider two closed subsets K and C of X and a differentiable map Φ from a neighborhood of $[0, T] \times K \times C$ to X.

We consider tubes of the form

$$P(t) := \Phi(t, K, C) \tag{11.6}$$

Proposition 11.4.1 *Let us assume that for all $t \leq T$, for all $x \in P(t)$, there exists $(y, z) \in K \times C$ satisfying $\Phi(t, y, z) = x$ and there exists $(u, v) \in T_{K\times C}(y, z)$ such that*

$$\begin{cases} i) \quad \text{if } t < T, \ \ \Phi'_y(t, y, z)u + \Phi'_z(t, y, z)v \ \in \ F(t, x) - \Phi'_t(t, y, z) \\ ii) \quad \text{if } t = T, \ \ \Phi'_y(T, y, z) + \Phi'_z(T, y, z)v \ \in \ F(T, x) \end{cases} \tag{11.7}$$

Then the set-valued map P defined by (11.6) is a viability tube of F on $[0, T]$.

Proof — We observe that Graph(P) is the image of $[0, T] \times K \times C$ under the map Ψ defined by

$$\Psi(t, y, z) \ = \ (t, \Phi(t, y, z))$$

By formula (4) of Table 5.2,

$$\Psi'(t, y, z)T_{[0,T]\times K\times C}(t, y, z) \ \subset \ T_{\text{Graph}(P)}(\Psi(t, y, z))$$

Hence assumptions (11.7) imply that P is a viability tube. □

We can also characterize viability tubes of the form (11.6) through dual conditions (involving the subnormal cones) thanks to Theorem 3.2.4.

Proposition 11.4.2 *Let us assume that F is upper semicontinuous with compact convex values and that the subsets K and C are closed. If for any $t \in [0, T]$, $\forall\, x \in P(t)$, there exists $(y, z) \in K \times C$ satisfying $\Phi(t, y, z) = x$ such that for all*

$$p \text{ satisfying } (\Phi'_y(t, y, z)^\star p, \Phi'_z(t, y, z)^\star p) \ \in \ N^\circ_{K\times C}(y, z)$$

we have

$$\begin{cases} i) \quad \forall\, t < T, \ \ < p, \Phi'_t(t, y, z) > +\sigma(F(t, \Phi(t, y, z)), -p) \ \geq \ 0 \\ ii) \quad \text{for } t = T, \ \ \sigma(F(T, \Phi(T, y, z)), -p) \ \geq \ 0 \end{cases} \tag{11.8}$$

then the set-valued map P defined by (11.6) is a viability tube of F on $[0, T]$.

Let us consider now a tube of the form

$$P(t) \ := \ K + \varphi(t)C$$

where K and C are closed.

Corollary 11.4.3 *Let us assume that K and C are closed subsets and that F is upper semicontinuous with compact convex values. Let $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ be a differentiable function satisfying either one of the following equivalent conditions: For any $t \geq 0$, $\forall \, x \in P(t)$, there exist $y \in K, z \in C$ such that $x = y + \varphi(t)z$ and either*

$$\begin{cases} i) & (F(t, y + \varphi(t)z) - \varphi'(t)z) \cap (T_K(y) + T_C(z)) \neq \emptyset \ \text{ if } t < T \\ ii) & (F(T, y + \varphi(T)z) \cap (T_K(y) + T_C(z)) \neq \emptyset \ \text{ if } t = T \end{cases}$$

or, $\forall \, p$ satisfying $(p, \varphi'(t)p) \in N^{\circ}_{K \times C}(y, z)$,

$$\begin{cases} i) & \sigma_C(p) + \sigma(F(t, y + \varphi(t)z, -p) \ \geq \ 0 \text{ if } t < T \\ ii) & \sigma(F(T, y + \varphi(T)z, -p) \ \geq \ 0 \text{ if } t = T \end{cases} \tag{11.9}$$

Then the set-valued map P defined by

$$P(t) \ := \ K + \varphi(t)C$$

is a viability tube of F on $[0, T]$.

Let us consider the instance when $K = \{c\}$ and when 0 belongs to the interior of the closed convex subset C.

We introduce the function a_0 defined by

$$\begin{cases} a_0(t, w) \ := \\ \sup_{z \in C} \sup_{p \in N_C(x), \, \sigma_C(p) = 1} \inf_{v \in F(t, c + wz)} < p, v > \\ = \sup_{z \in C} \inf_{v \in F(t, c + wz)} \sup_{p \in N_C(x), \, \sigma_C(p) = 1} < p, v > \end{cases} \tag{11.10}$$

(The latter equation follows from the Lopsided Minimax Theorem).

Let us assume that there exists a continuous function $a : \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}$ satisfying $a(t, 0) = 0$ for all $t \geq 0$ and

$$\forall\ (t,w) \in \mathbf{R}_+ \times \mathbf{R}_+,\ \ a(t,w)\ \geq\ a_0(t,w)$$

Let φ be a solution to the differential equation

$$\varphi'(t)\ =\ a(t,\varphi(t))\ \&\ \varphi(0)\ =\ \varphi_0\ \text{ given} \tag{11.11}$$

satisfying

$$a(T,\varphi(T)) = 0$$

Since $\sigma_C(p) > 0$ for all $p \neq 0$, we deduce that for all $z \in C$ and all $p \in N_C(z)$,

$$\left\{\begin{array}{l} \varphi'(t)\sigma_C(p)\ \geq\ a(t,\varphi(t))\sigma_C(p)\ \geq\ a_0(t,\varphi(t))\sigma_C(p) \\ \\ \geq\ \sigma_C(p)\sup_{v\in F(t,c+\varphi(t)z)} < -\frac{p}{\sigma_C(p)}, v > \\ \\ =\ -\sigma(F(t,c+\varphi(t)z),-p) \end{array}\right.$$

Hence, condition (11.9)i) is satisfied. We also check that

$$0\ =\ a(T,\varphi(T))\ \ \geq\ a_0(T,\varphi(T))\ \geq\ \frac{-1}{\sigma_C(p)}\sigma(F(T,c+\varphi(T)z),-p)$$

Then the tube defined by $P(t)\ :=\ c+\varphi(t)C$ is a viability tube of F. □

For instance, if $C := B$ is the unit ball, then $\sigma_B(p)\ =\ \|p\|$ and $N_B(z)\ =\ \lambda z$ for all $z \in S := \{x \| \|x\|\ =\ 1\}$. Hence, in this case we have

$$a_0(t,w)\ \ :=\ \ \sup_{\|z\|=1}\ \inf_{v\in F(t,c+wz)}\ < v,z >$$

In other words, the function a_0 defined by (11.10) conceals all the information needed to check whether a given subset C can generate a tube P. □

Remark — When a is nonpositive and satisfies $a(t,0) = 0$ for all $t \geq 0$, then there exists a non-negative non-increasing solution $\varphi(\cdot)$ to the differential equation (11.11).

When $T = \infty$, we infer that $\int_0^\infty a(\tau, \varphi(\tau))d\tau$ is finite. Let us assume that for all $w_\star \in \mathbf{R}_+$,

$$\lim_{t\to\infty, w\to w_\star} a(t,w) \;=\; a_\star(w_\star)$$

Then the limit $\varphi_\star$ of $\varphi(t)$ when $t \to \infty$ satisfies the equation

$$a_\star(\varphi_\star) = 0$$

Otherwise, there would exist $\varepsilon > 0$ and T such that $a_\star(\varphi_\star) + \varepsilon < 0$ and for all $t > T, a(t, \varphi(t)) \le a_\star(\varphi_\star) + \varepsilon$ by definition of $a_\star$.

We deduce the contradiction

$$\varphi(t) - \varphi(T) \;=\; \int_T^t a(\tau, \varphi(\tau))d\tau \;\le\; (t - T)(a_\star(\varphi_\star) + \varepsilon)$$

when t is large enough.

Example — Let us consider the case when F does not depend upon t. We set

$$\rho_0 \;:=\; \sup_{\lambda\in\mathbf{R}} \inf_{w>0} (\lambda w - a_0(w))$$

Assume also that $\lambda_0 \in \mathbf{R}$ achieves the supremum. We can take $\psi(w) := \lambda_0 w - \rho_0$.

If $\rho_0 > 0$, the function

$$\varphi_T(t) \;:=\; \begin{cases} \frac{\rho_0}{\lambda_0}\left(1 - e^{\lambda_0(t-T)}\right) & \text{if} \quad \lambda_0 \ne 0 \\ -\rho_0(t-T) & \text{if} \quad \lambda_0 = 0 \end{cases}$$

is such that $P(t) := c + \varphi_T(t)C$ is a tube of F such that $P(T) = \{c\}$.

If $\rho_0 \le 0$ and $\lambda_0 < 0$, then the functions

$$\varphi(t) \;:=\; \frac{1}{\lambda_0}(\rho_0 - e^{\lambda_0 t})$$

are such that $P(t) := c + \varphi(t)C$ defines a tube of F on $[0, \infty[$ such that $P(t)$ decreases to the set $P_\infty := c + \frac{\rho_0}{\lambda_0}C$. □

11.5 An Abstract K − C Problem

We propose now a criterion which allows us to decide whether a compact convex subset C lies in the image $R(K)$ of a subset K of a

Hilbert space X by a set-valued map R (the reachable map in our framework) from X to another Hilbert space Y.

We want to solve the following problem (the *K-C problem*):

For every y in C, find x in K such that y belongs to $R(x)$

(i.e. we can reach any element of the target C from K). Assume that we know how to solve this problem for a "nicer" set-valued map Q from K to Y (say, a map with compact convex graph).

For every y in C, find x in K such that y belongs to $Q(x)$

The next theorem states how a relation linking R and Q (R is *outward with respect to Q*) allows us to deduce the surjectivity of R from the surjectivity of Q.

Theorem 11.5.1 *We assume that the graph of Q is convex and compact and that R is upper semicontinuous with convex values. We set*

$$K := \mathrm{Dom}(Q), \;\; C := \mathrm{Im}(Q)$$

If

$$\forall\, x \in K, \;\; \forall\, y \in Q(x), \;\; y \;\in\; R(x) + T_C(y),$$

(outwardness condition), then $C \subset R(K)$.

Proof — It is a simple consequence of Theorem 3.7.11. We replace X by $X \times Y$, K by Graph(Q) (which is convex compact), A by the projection π_Y from $X \times Y$ to Y and R by the set-valued map G from $X \times Y$ to Y defined by:

$$G(x,y) := R(x) - y_0 \;\text{ where } y_0 \text{ is given in } C$$

The outwardness condition implies that the tangential condition :

$$0 \;\in\; -y + R(x) + T_C(y)$$

if satisfied. Since $y_0 - y$ belongs to $T_C(y)$ (because $y_0 \in C$), then

$$0 \;\in\; -y_0 + R(x) + T_C(y) \;=\; G(x,y) + T_C(y)$$

We observe that

$$\begin{cases} T_C(y) = T_{\mathrm{Im}(Q)}(y) = T_{\pi_Y(\mathrm{Graph}(Q))}(\pi_Y(x,y)) \\ = \pi_Y(T_{\mathrm{Graph}(Q)}(x,y)) \end{cases}$$

so that

$$0 \in G(x,y) + \overline{\pi_Y}(T_{\mathrm{Graph}(Q)}(x,y))$$

Theorem 3.7.11 implies the existence of a solution $(\bar{x},\bar{y})$ in the graph of Q to the inclusion $0 \in G(\bar{x},\bar{y})$, i.e., to the inclusion $y_0 \in R(\bar{x})$.

Remark — The dual version of the *outwardness condition* is the following:

$$\forall\, q \in N_C(y),\ \ \forall\, x \in A^{-1}(y),\ \ < q, y > \leq \sigma(R(x), q)$$

where $N_C(y)$ denotes the normal cone to the convex set C at y and

$$\sigma(R(x), q) := \sup_{y \in R(x)} < q, y >$$

is the support function of $R(x)$.

11.6 Invariant Tubes

We distinguish between viability tubes and invariant tubes in the same way as viability domains and invariant domains.

Definition 11.6.1 *We say that a tube P is* invariant under F *(or enjoys the* invariance property*) if and only if for all t_0 and $x_0 \in P(t_0)$,* all the solutions *to differential inclusion (11.3.1) are viable in the tube P.*

We say that P is an invariant tube *if*

$$\begin{cases} i) & \forall\, t \in [0,T[,\ \forall\, x \in P(t), & F(t,x) \subset DP(t,x)(1) \\ ii) & \text{if } T < +\infty,\ \forall\, x \in P(T), & F(t,x) \subset DP(t,x)(0) \end{cases}$$

We obtain the following theorem.

Theorem 11.6.2 *Assume that $F : [0,T[\times\Omega \to X$ is Lipschitz with respect to x in the sense that*

$$\exists\, k(\cdot) \in L^1(0,T) \mid F(t,x) \subset F(t,y) + k(t)\|x-y\|B$$

(B is a unit ball). Let $t \rightsquigarrow P(t) \subset \Omega$ be a closed tube: If P is an invariant tube, then it is invariant under F.

The theorem follows from the following extension of Lemma 5.1.2:

Lemma 11.6.3 *Let P be a closed tube and $\pi_{P(t)}(y)$ denote the set of best approximations of y by elements of $P(t)$.*

$$\begin{cases} \liminf_{h\to 0+} \frac{d(y+hv,P(t+h))-d(y,P(t))}{h} \\ \leq \inf_{x\in\pi_{P(t)}(y)} d(v, DP(t,x)(1)) \end{cases}$$

Proof of the theorem — Let us associate with any solution to the differential inclusion $x'(t) \in F(t,x(t))$ the function $g(t) := d(x(t),P(t))$. Let us choose $y(t) \in \pi_{P(t)}(x(t))$. Inequalities

$$\begin{cases} \liminf_{h\to 0+}(g(t+h)-g(t))/h \\ = \liminf_{h\to 0+} (d(x(t)+hx'(t)+ho(h),P(t+h))-d(x(t),P(t)))/h \\ \leq \|o(h)\| + \liminf_{h\to 0+}(d(x(t)+hx'(t),P(t+h))-d(x(t),P(t)))/h \\ \leq d(x'(t),DP(t,y(t)(1))) \\ \leq d(x'(t),F(t,y(t))) \leq \sup_{v\in F(t,x(t))} d(v,F(t,y(t))) \\ \leq k(t)\|y(t)-x(t)\| = k(t)d(x(t),P(t)) = k(t)g(t) \end{cases}$$

imply that g(t) is a solution to the differential inequality

$$D_\uparrow g(t)(1) \leq k(t)g(t) \ \& \ g(t_0) = d(x_0,P(t_0)) = 0$$

Hence $d(x(t),P(t)) = g(t) = 0$ for all $t \in [t_0,T[$. □

Remark — If we assume that the condition

$$\forall (t,y) \in \mathrm{Dom}(F), \ \exists x \in \pi_{P(t)}(y) \text{ such that } F(t,y) \subset DP(t,x)(1)$$

holds true, then the tube P is invariant by F: this knowledge of the behavior of F outside the graph of the tube P allows us to dispose of the Lipschitz assumption. □

We can characterize the indicator functions of the graphs of invariant tubes in the following way:

Proposition 11.6.4 *A tube P is invariant by F if and only if the indicator function ψ_P of its graph is a solution to the equation*

$$\sup_{v\in F(t,x)} D_\uparrow\psi(t,x)(1,v) = 0$$

satisfying the final condition

$$\text{If } T < +\infty, \quad \sup_{v \in F(t,x)} D_{\uparrow}\psi(T,x)(0,v) \; = \; 0$$

11.7 Measurable Time Dependence

In this section, we consider the case of a differential inclusion

$$\text{for almost all } t \geq 0, \;\; x'(t) \; \in \; F(t, x(t)) \tag{11.12}$$

where, for all $x \in X$, the set-valued map $t \rightsquigarrow F(t,x)$ is *measurable* with respect to $t \in [0, \infty[$.

This means that the inverse images of any open subset of X by the set-valued map $t \rightsquigarrow F(t,x)$ is (Lebesgue) measurable. (See Chapter 8 of SET-VALUED ANALYSIS for a presentation of measurable set-valued maps for instance).

We are looking for solutions to this differential inclusion which are viable in a closed subset K.

Theorem 11.7.1 (Tallos) *Let X be a finite dimensional vector-space X and $F : \mathbf{R}_+ \times K \rightsquigarrow X$ be a nontrivial set-valued map satisfying*

$$\begin{cases} i) & \forall\, x \in K, \;\; t \rightsquigarrow F(t,x) \;\text{ is measurable} \\ ii) & \forall\, t \geq 0, \;\; x \rightsquigarrow F(t,x) \\ & \text{is upper semicontinuous with compact convex values} \\ iii) & \exists\, c(\cdot) \in L^1(0,\infty;\mathbf{R}_+) \;\; \textit{such that } \|F(t,x)\| \; \leq \; c(t)(\|x\| + 1) \end{cases}$$

If K is a viability domain in the sense that

$$\text{for almost all } t \geq 0, \;\; F(t,x) \cap T_K(x) \; \neq \; \emptyset$$

it is viable under F: for any initial state $x_o \in K$, there exists a solution to the differential inclusion (11.12) starting at x_0 which is viable in K.

Proof — The idea is

— first, to approximate the set-valued map F by Marchaud maps F_h defined by

$$\forall\, x \in K,\ \forall\, t \geq 0,\ \ F_h(t,x) \ := \ \frac{1}{h}\int_t^{t+h} F(s,x)ds$$

— and then, to show that the solutions x_h to approximate differential inclusions

$$\text{for almost all } t \geq 0,\ \ x_h'(t) \ \in \ F_h(t, x_h(t))$$

viable in K converge to a solution to the differential inclusion (11.12) viable in K.

• — For that purpose, we need first to recall the definition of the integral of a set-valued map $\Phi : \mathbf{R}_+ \rightsquigarrow X$. We denote by $\boldsymbol{\Phi}$ *the set of all integrable selections of* Φ:

$$\boldsymbol{\Phi} \ = \ \{\, f \in L^1(0,\infty;X) \mid f(t) \in \Phi(t) \ \text{ almost everywhere in } \ \mathbf{R}_+ \,\}$$

The integral of Φ on $\mathbf{R}_+$ is the set of integrals of integrable selections of Φ:

$$\int_0^\infty \Phi(t)dt \ := \ \left\{ \int_0^\infty f(t)dt \mid f \in \boldsymbol{\Phi} \right\}$$

The integral is convex whenever Φ has convex images (and even when the images are not convex, as Aumann's theorem states.)

We recall (see for instance Theorem 8.6.2 of SET-VALUED ANALYSIS) that the integral of a support function is the support function of the integral:

$$\frac{1}{h}\int_t^{t+h} \sigma(\Phi(s),p)ds \ = \ \sigma\left(\frac{1}{h}\int_t^{t+h} \Phi(s)ds, p\right)$$

and that

$$\text{for almost all } t \geq 0,\ \ \lim_{h\to 0+} \sigma\left(\frac{1}{h}\int_t^{t+h} \Phi(s)ds, p\right) \ = \ \sigma(\Phi(t),p)$$

• — *The set-valued maps F_h are upper semicontinuous.* Let $(t,x) \in \mathbf{R}_+ \times K$ and $\varepsilon > 0$ be given. We associate with any $\gamma > 0$

$$\eta(s,x,\gamma) \ := \ \sup_{y \in B(x,\gamma)} \ \sup_{v \in F(s,y)} \ d(v, F(s,x))$$

The function η is measurable with respect to s (see section 8.2. of SET-VALUED ANALYSIS), is bounded by $2c(s)(\|x\|+1)$ and

$$\lim_{\gamma\to 0+} \eta(s,x,\gamma) \;=\; 0$$

since $y \rightsquigarrow F(s,y)$ is upper semicontinuous at x.

Therefore, the Lebesgue Dominated Convergence Theorem implies that

$$\lim_{\gamma\to 0+}\left(\frac{1}{h}\int_t^{t+h}\eta(s,x,\gamma)ds\right) \;=\; 0$$

Choose now $\delta\in]0,\varepsilon h/4[$ such that

$$\frac{1}{h}\int_t^{t+h}\eta(s,x,\gamma)ds \;\leq\; \frac{\varepsilon}{2}$$

holds true.

Take $(r,y)\in\mathbf{R}_+\times K$ such that $|r-s|<\delta$ and $\|x-y\|<\delta$ and $w\in F_h(r,y)$. By definition of the integral of a set-valued map, there exists an integrable selection $g(s)\in F_h(s,y)$ satisfying

$$w \;=\; \frac{1}{h}\int_r^{r+h} g(s)ds$$

We introduce the set-valued map Φ defined by

$$\Phi(s) \;:=\; F(s,x)\cap\{u\in X \mid \|u-g(s)\|\leq\eta(s,x,\delta)\}$$

Then Φ is obviously measurable with nonempty closed values and thus, measurable. Choose a selection $\varphi(\cdot)$ of Φ and set

$$v \;:=\; \frac{1}{h}\int_t^{t+h}\varphi(s)ds$$

which belongs to $F_h(t,x)$ and which satisfies

$$\begin{cases} \|v-w\| \;=\; \dfrac{1}{h}\left\|\displaystyle\int_t^{t+h}\varphi(s)ds-\int_r^{r+h}g(s)ds\right\| \\ \leq\; \dfrac{1}{h}\left|\displaystyle\int_t^r\|g(s)\|ds\right|+\dfrac{1}{h}\displaystyle\int_t^{t+h}\|\varphi(s)-g(s)\|ds+\dfrac{1}{h}\left|\displaystyle\int_{t+h}^{r+h}\|g(s)\|ds\right| \\ \leq\; \dfrac{\varepsilon}{4}+\dfrac{1}{h}\displaystyle\int_t^{t+h}\eta(s,x,\delta)ds+\dfrac{\varepsilon}{4} \;\leq\; \varepsilon \end{cases}$$

by the choice of δ. We have proved that F_h is upper semicontinuous. We recall also that the values of F_h are closed and convex. If we set

$$c_h(t) \;:=\; \frac{1}{h}\int_t^{t+h} c(s)ds$$

we see that

$$c \;:=\; \int_0^{\infty} c(s)ds \;=\; \int_0^{\infty} c_h(t)dt$$

In summary, we have proved that F_h are Marchaud maps.

• — We prove now that K is a viability domain of the set-valued maps F_h:

$$\forall\, t \geq 0,\ \forall\, x \in K,\ \ F_h(t,x) \cap T_K(x) \;\neq\; \emptyset$$

By Theorem 3.2.4, this is equivalent to prove that

$$\forall\, t \geq 0,\ \forall\, x \in K,\ \ F_h(t,x) \cap \overline{co}\,(T_K(x)) \;\neq\; \emptyset$$

Since the set-valued map $t \rightsquigarrow F(t,x) \cap T_K(x)$ is measurable, the Measurable Selection Theorem (see Theorem 8.1.3 of SET-VALUED ANALYSIS for instance) implies the existence of a measurable selection $f(\cdot)$ satisfying

$$\text{for almost all } t \geq 0,\ \ f(t) \;\in\; F(t,x) \cap T_K(x)$$

Therefore

$$v_h := \frac{1}{h}\int_t^{t+h} f(s)ds \;\in\; F_h(t,x) \cap \overline{co}\,(T_K(x))$$

• — Since for every $h > 0$, F_h is a Marchaud map, we deduce from Theorem 11.1.3 the existence of a solution $x_h(\cdot)$ to the approximate differential inclusion viable in K starting at any initial state $x_0 \in K$.

Estimates

$$\text{for almost all } t \geq 0,\ \ \|x_h'(t)\| \;\leq\; c_h(t)(\|x\| + 1)$$

imply that a subsequence denoted by $x_n(\cdot) := x_{h_n}(\cdot)$ converges to some function $x(\cdot)$ uniformly over compact intervals and that $x_n'(\cdot)$

converges *weakly* in $L^1(0,\infty;X)$ to the distributional derivative $x'(\cdot)$ of $x(\cdot)$.

• — It remains to check that $x(\cdot)$ is a viable solution to differential inclusion (11.12).

Set $F_n := F_{h_n}$. Since $x_n(t)$ converges pointwise to $x(t)$, we deduce first that $x(t) \in K$ for any $t \geq 0$ and second, that $\|x_n(t) - x(t)\| \leq \delta$ for n large enough. Hence the inclusion

$$F(s, x_n(t)) \subset F(s, x(t)) + \eta(s, x(t), \delta)B$$

implies that for any $p \in X^\star$,

$$\sigma(F(s, x_n(t)), p) \leq \sigma(F(s, x(t)), p) + \eta(s, x(t), \delta)\|p\|$$

We then deduce by integrating the above inequality that

$$\sigma(F_n(t, x_n(t)), p) \leq \sigma(F_n(t, x(t)), p) + \frac{1}{h_n}\int_t^{t+h_n} \eta(s, x(t), \delta)\|p\| ds$$

In summary, we proved that for any $\varepsilon > 0$, there exists N large enough for

$$\forall\, n \geq N, \ < p, x_n'(t) > \leq \sigma(F_n(t, x_n(t)), p) \leq \sigma(F(t, x(t)), p) + \varepsilon$$

We borrow now from Olech the following adaptation of the proof of the Convergence Theorem 2.4.4. By the Mazur Theorem, there exists a sequence of convex combinations $v_m(\cdot)$ of elements $x_n'(\cdot)$ $(n \geq m)$ converging strongly to $x'(\cdot)$ in $L^1(0,\infty;X)$. It satisfies

$$\forall\, n \geq 0, \ < p, v_n(t) > \leq \sigma(F(t, x(t)), p) + \varepsilon$$

Since a subsequence of elements (again denoted by) $v_n(\cdot)$ converges almost everywhere to $x'(\cdot)$, then $< p, x'(t) > \leq \sigma(F(t, x(t)), p)$ for almost all $t \geq 0$. The values $F(t, x(t))$ being convex and compact, we have proved that $x'(t) \in F(t, x(t))$ for almost all $t \geq 0$. □

Chapter 12

Functional Viability

Introduction

Differential equations and inclusions describe the evolution of systems where, at each instant, the velocity of the state depends upon the value of the state at this very instant (in a single or multivalued way).

Differential inclusions with memory, also called *functional differential inclusions*, express that at each instant, the velocity of the state depends upon the history of its evolution up to this instant.

By *functional viability*, we mean viability constraints which also depend upon the history of the evolution of the state of the system, or even, when the constraints act not only on the state of the system, but on its past evolution.

This allows us to take into account delays, anticipations, cumulated consequences of the past, etc., in both the dynamics of the system and the viability constraints.

We shall adapt the techniques devised for the usual viability problems for differential inclusions to functional viability problems for differential inclusions with memory.

This will leas to a characterization of the functional viability property by a "functional tangential condition" stating that for any past evolution, there exists at least a velocity "tangent" to the set of past evolutions satisfying the functional viability constraints.

This characterization does not solve completely the problem, since, for concrete examples, we have to prove that it is satisfied. It is well known that invariance problems for differential equations with delays are difficult to solve.

But as in the case of differential equations and inclusions, the characterization of functional viability by functional tangential conditions offers easier routes to solve the problem since these conditions do not require the resolution of the functional differential inclusion.

The first section is devoted to the definitions and the presentation of the main classes of examples (differential inclusion with delays, Volterra type differential inclusions, etc.) and Haddad's functional viability theorem is proved in the third section.

We treat in the third section the particular cases of functional viability constraints of the form

$$\forall\, t \geq 0,\ \ x(t)\ \in\ M\left(\int_{-\infty}^{t} A(t-s)x(s)d\mu(s)\right)$$

and sufficient conditions involving the derivative of the set-valued map M are presented.

We end this chapter by adapting to the functional viability case the concepts of viability kernels and viability tubes.

12.1 Definitions and Examples

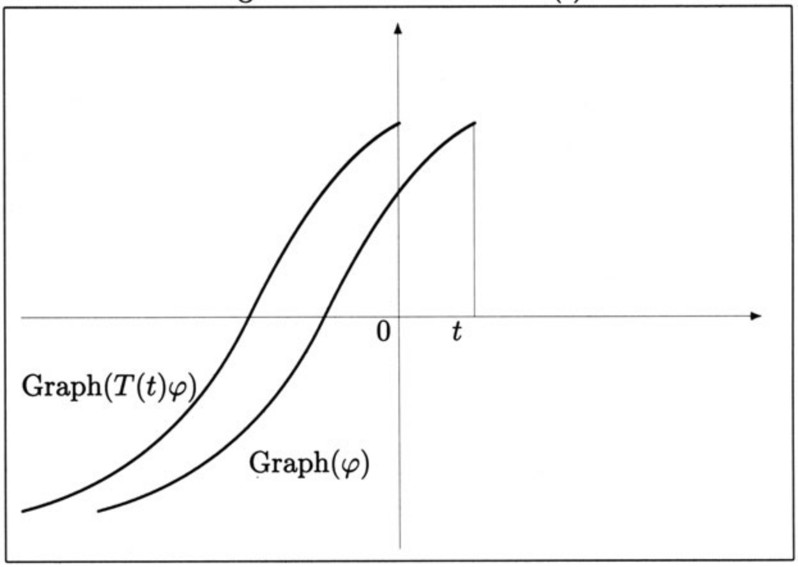
Our first task is to translate the concept of history of the evolution of the state up to the instant $t > 0$. We achieve this purpose by using the operator $T(t)$ from the Fréchet space $\mathcal{C}(-\infty,+\infty;X)$ to $\mathcal{C} := \mathcal{C}(-\infty,0;X)$ which associates with any continuous function $x(\cdot)$ its *history* $T(t)x$ *up to time* t defined by:

$$\forall\, \tau \in]-\infty, 0],\ \ T(t)x(\tau)\ :=\ x(t+\tau)$$

A differential inclusion with memory describes in the following way the link between the velocity $x'(t)$ and the history $T(t)x$ up to time t through a set-valued map $\mathcal{F}$ from $\mathcal{C}$ to X in the following manner:

$$\text{for almost all } t \in [0,\infty[,\ \ x'(t) \in\ \mathcal{F}(T(t)x) \qquad (12.1)$$

Figure 12.1: Translation $T(t)$



Initial conditions express that the history of the evolution of the state up to the initial state 0 is known: it is a function $\varphi \in \mathcal{C}$. Hence the initial condition is written in the form:

$$T(0)x \;=\; \varphi \tag{12.2}$$

The viability constraints bear not only on the state of the system, but on its evolution, by requiring that at each instant,

$$\forall\, t \in [0, \infty[,\;\; T(t)x \;\in\; \mathcal{K} \tag{12.3}$$

where $\mathcal{K} \subset \mathcal{C}$ is a given closed subset of state evolutions.

Definition 12.1.1 *We shall say that a subset $\mathcal{K} \subset \mathcal{C}$ is* viable under *$\mathcal{F}$ (or enjoys the* viability property *for $\mathcal{F} : \mathcal{C} \rightsquigarrow X$) if and only if for any initial evolution $\varphi \in \mathcal{K}$, there exists a solution $x(\cdot)$ to (12.1) starting at φ (in the sense of (12.2)) and viable in $\mathcal{K}$ (in the sense of (12.3))*

We first observe that by taking

$$\begin{cases} i) & \mathcal{F}(\varphi) := F(\varphi(0)) \\ ii) & \mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in K \} \end{cases}$$

where $K \subset X$ and $F : X \rightsquigarrow X$, usual viability problems are particular cases of functional viability problems because

$$\begin{cases} i) & x'(t) \in \mathcal{F}(T(t)x) = F((T(t)x)(0)) = F(x(t)) \\ ii) & x(t) = (T(t)x)(0) \in K \iff T(t)x \in \mathcal{K} \end{cases}$$

We can also extend this time-independent functional viability problem to the time-dependent case. We introduce for that purpose

$$\begin{cases} i) & \text{a set-valued map } \mathcal{P} : \mathbf{R} \rightsquigarrow \mathcal{C} \\ ii) & \text{a set-valued map } \mathcal{F} : \text{Graph}(\mathcal{P}) \rightsquigarrow X \end{cases}$$

We thus say that $\mathcal{P}$ enjoys the functional viability property if and only if for any t_0 and $\varphi \in \mathcal{P}_{t_0}$, there exists a solution $x(\cdot)$ to

$$\text{for almost all } t \geq t_0, \quad x'(t) \in \mathcal{F}(t, T(t)x) \tag{12.4}$$

satisfying the initial condition $T(t_0)x = \varphi$, and which is viable in the sense that:

$$\forall\, t \geq t_0, \quad T(t)x \in \mathcal{P}(t)$$

Before characterizing closed subsets $\mathcal{K}$ enjoying the viability property, we show that this class of viability problems covers many examples.

Example 1. Viability problems with delays. We consider p delay functions r_i from $[0,\infty[$ to $[0,\infty[$. A *differential inclusion with delays* is described by a set-valued map $F : X^p \rightsquigarrow X$ in the following way:

$$x'(t) \in F(x(t - r_1(t)), \ldots, x(t - r_p(t)))$$

In the same way, viability constraints with delays are described by q delay functions s_i from $[0,\infty[$ to $[0,\infty[$ and a set-valued map $D : X^q \rightsquigarrow X$:

$$\forall\, t \in [0,\infty[, \quad x(t) \in D(x(t - s_1(t)), \ldots, x(t - s_q(t)))$$

This viability problem fits the general framework of functional viability by taking

$$\begin{cases} i) & \mathcal{F}(\varphi) := F(\varphi(-r_1(t)), \ldots, \varphi(-r_p(t))) \\ ii) & \mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in D(\varphi(-s_1(t)), \ldots, \varphi(-s_q(t))) \} \end{cases} \square$$

Example 2. Volterra viability problems. We consider a function $k : \mathbf{R} \times \mathbf{R} \times X \to Y$ (called a *kernel*) which expresses the cumulated consequences $\int_{-\infty}^{t} k(t, s, x(s))ds$ in Y of the evolution of the solution up to t.

A *Volterra differential inclusion* is described by a set-valued map $F : Y \rightsquigarrow X$ in the following way:

$$\text{for almost all } t \in [0, \infty[, \;\; x'(t) \in F\left(\int_{-\infty}^{t} k(t, s, x(s))ds\right)$$

In the same way, Volterra viability constraints are described by a kernel $l : \mathbf{R} \times \mathbf{R} \times X \to Z$ and a set-valued map $D : Z \rightsquigarrow X$ through

$$\forall\, t \in [0, \infty[, \;\; x(t) \in D\left(\int_{-\infty}^{t} l(t, s, x(s))ds\right)$$

Volterra viability problems are particular cases of functional viability problems when we take

$$\begin{cases} i) & \mathcal{F}(\varphi) := F(\int_{-\infty}^{0} k(t, t+s, \varphi(s))ds) \\ ii) & \mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in \int_{-\infty}^{0} l(t, t+s, \varphi(s))ds \} \end{cases} \square$$

12.2 Functional Viability Theorem

To proceed, we have to adapt to functional viability problems the concept of viability domains;

Definition 12.2.1 (Functional Viability Domains) *Let φ be given in a subset $\mathcal{K} \subset \mathcal{C}$. We denote by $\mathcal{D}_{\mathcal{K}}(\varphi) \subset X$ the subset of* *elements* $v \in X$ *such that, for any* $\varepsilon > 0$, *there exist* $h \in]0, \varepsilon]$ *and* $\varphi_h \in \mathcal{C}(-\infty, h; X)$ *satisfying*

$$\begin{cases} i) & T(0)\varphi_h = \varphi, \quad T(h)\varphi_h \in \mathcal{K} \\ ii) & (\varphi_h(h) - \varphi_h(0))/h \in v + \varepsilon B \end{cases} \tag{12.5}$$

Let $\mathcal{F} : \mathcal{C} \rightsquigarrow X$ *be a set-valued map. We shall say that* $\mathcal{K} \subset \mathrm{Dom}(\mathcal{F})$ *is a* functional viability domain *of* $\mathcal{F}$ *if and only if*

$$\forall \varphi \in \mathcal{K}, \quad \mathcal{F}(\varphi) \cap \mathcal{D}_{\mathcal{K}}(\varphi) \neq \emptyset \tag{12.6}$$

We denote by $\mathcal{C}_\lambda$ the closed convex subset of λ-Lipschitz functions from $]-\infty, 0]$ to X. Ascoli's Theorem states that a *closed subset* $\mathcal{K} \subset \mathcal{C}_\lambda$ *is compact if and only if* $\mathcal{K}(0) := \{\varphi(0)\}_{\varphi \in \mathcal{K}}$ *is bounded*, because it is closed and equicontinuous (by assumption) and pointwise bounded because, for all $\psi \in \mathcal{K}$ and $\tau \leq 0$,

$$\|\psi(\tau)\| \leq \|\psi(\tau) - \psi(0)\| + \|\psi(0)\| \leq \lambda|\tau| + \|\mathcal{K}(0)\|$$

Theorem 12.2.2 *[Haddad] Let* $\mathcal{F} : \mathcal{C}_\lambda \rightsquigarrow X$ *be a Marchaud map and* $\mathcal{K} \subset \mathrm{Dom}(\mathcal{F})$ *a closed subset of* $\mathcal{C}_\lambda$.

Then $\mathcal{K}$ *enjoys the functional viability property if and only if it is a functional viability domain.*

Remark — We observe that when $\mathcal{K} := \{ \varphi \in \mathcal{C} \mid \varphi(0) \in K \}$, then

$$\mathcal{D}_{\mathcal{K}}(\varphi) = T_K(\varphi(0))$$

and that when $\mathcal{F}(\varphi) := F(\varphi(0))$, $\mathcal{K}$ is a functional viability domain of $\mathcal{F}$ if and only if K is a viability domain of F. Hence the Viability Theorem for differential inclusions is a consequence of Theorem 12.2.2. □

Proof of the necessary condition — Assume that a solution $x(\cdot)$ to the functional differential inclusion (12.1) satisfies: there exists a sequence t_n converging to 0 such that $T(t_n)x \in \mathcal{K}$.

Since $\mathcal{F}$ is upper hemicontinuous at φ, we can associate with any $p \in X^\star$ and $\varepsilon > 0$ a neighborhood $\mathcal{V}$ of 0 in $\mathcal{C}$ such that

$$\forall \psi \in \varphi + \mathcal{V}, \quad \sigma(\mathcal{F}(\psi), p) \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

Since $T(0)x = \varphi$, there exists $\eta > 0$ such that $T(\tau)x - \varphi \in \mathcal{V}$ for $|\tau| \leq \eta$. Hence, integrating inequalities

$$< x'(\tau), p > \leq \sigma(\mathcal{F}(T(\tau)x), p) \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

from 0 to t_n, we obtain

$$\forall\, p \in X^\star, \;\; < p, \frac{x(t_n) - x_0}{t_n} > \leq \; \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

This implies that the sequence $v_n := \frac{x(t_n)-x_0}{t_n}$ is relatively compact in X. Therefore, a subsequence (again denoted by) v_n converges to some $v \in X$. Since for all $p \in X^\star$, for n large enough,

$$< p, v_n > \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

we deduce that the limit v satisfies

$$\forall\, p \in X^\star, \; < p, v > \leq \sigma(\mathcal{F}(\varphi), p) + \varepsilon$$

Letting ε converge to 0, we obtain

$$\forall\, p \in X^\star, \; < p, v > \leq \sigma(\mathcal{F}(\varphi), p)$$

so that v belongs to the closed convex hull of $\mathcal{F}(\varphi)$, which is equal to $\mathcal{F}(\varphi)$ because it is closed and convex.

It remains to show that v belongs to $\mathcal{D}_{\mathcal{K}}(\varphi)$. Indeed, $T(t_n)x \in \mathcal{K}$ by assumption, $T(0)x = \varphi$, so that condition (12.5) is satisfied with $\varphi_h := x(\cdot)$.

Proof of the sufficient condition — Let us consider an initial evolution φ and choose $T := 1$. We shall construct a viable solution to (12.1) on $[0, 1]$, so that it will be possible to extend it on $[0, \infty[$. Let us set

$$\mathcal{K}_0 \; := \; \{\, \psi \in \mathcal{K} \mid \|\psi(0) - \varphi(0)\| \leq 2\lambda \,\}$$

Since $\mathcal{K} \subset \mathcal{C}_\lambda$ and $\mathcal{K}_0(0)$ is bounded, we deduce that this subset $\mathcal{K}_0$ is compact thanks to Ascoli's Theorem. Since $\mathcal{F}$ is upper semicontinuous with compact images, we know that $\mathcal{F}(\mathcal{K}_0)$ is bounded. We set $C \; := \; \mathcal{F}(\mathcal{K}_0) + B$ which is bounded.

For any integer m, we denote by $\mathcal{V}_m^a$ the neighborhood of $\mathcal{C}$ defined by

$$\mathcal{V}_m^a := \{ \psi \in \mathcal{C} \mid \sup_{\tau \in [-m,0]} \|\psi(\tau)\| \leq a \}$$

We shall construct a sequence of approximate solutions in a first step, show that this sequence converges to some limit in a second step and prove that this limit is a viable solution in the third one.

— **Construction of approximate solutions.**

We begin by proving

Lemma 12.2.3 *There exists $\theta_m \in]0, \frac{1}{m}[$ such that, for any $\chi \in \mathcal{K}_0$, we can find $h \in [\theta_m, \frac{1}{m}]$, $\psi \in \mathcal{C}(-\infty, +\infty; X)$ and $v \in \mathcal{F}(\mathcal{K}_0)$ satisfying*

$$\begin{cases} i) & T(0)\psi \in \mathcal{K},\ \ T(h)\psi \in \mathcal{K},\ \ (\psi(h) - \psi(0))/h \in v + \frac{1}{m}B \\ ii) & T(0)\psi \in \chi + \mathcal{V}_m^{h/m} \\ iii) & (T(0)\psi, v) \ \in \ \mathrm{Graph}(\mathcal{F}) \end{cases} \tag{12.7}$$

Proof — Condition (12.6) allows us to associate with any $\psi \in \mathcal{K}_0$ elements $v \in \mathcal{F}(\psi)$, $h_\psi \in]0, \frac{1}{m}[$ and $\psi_h \in \mathcal{C}$ such that

$$T(0)\psi_h = \psi,\ \ T(h_\psi)\psi_h \in \mathcal{K} \ \ \& \ \ \frac{\psi_h(h_\psi) - \psi(0)}{h_\psi} \in v + \frac{1}{m}B$$

We point out that the Lipschitz constant of ψ_h on the interval $]-\infty, h_\psi]$ is less than or equal to λ because $T(h_\psi)\psi_h$ belongs to $\mathcal{K} \subset \mathcal{C}_\lambda$.

Since $\mathcal{K}_0$ is compact, it can be covered by q neighborhoods $\psi_j + \mathcal{V}_m^{h_{\psi_j}/m}$. We set $\theta_m := \min_{1 \leq i \leq q} h_{\psi_i} \in]0, \frac{1}{m}]$.

Let us take any $\chi \in \mathcal{K}_0$. It belongs to one of these neighborhoods: then there exist elements ψ_i, $h_i := h_{\psi_i} > 0$, ψ_{h_i} and $v_i \in F(\psi_{h_i})$ satisfying properties (12.7). Hence the lemma ensues with $h := h_i$, $\psi := \psi_{h_i}$ and $v := v_i$. □

We take $m \geq 1/\lambda$. We thus deduce

Lemma 12.2.4 *There exist a finite sequence of $h_j \in [\theta_m, \frac{1}{m}]$, of functions $\psi_j \in \mathcal{C}(-\infty, +\infty; X)$ and elements $v_j \in \mathcal{F}(\mathcal{K}_0)$ such that $\psi_0 = \varphi, h_0 = 0$ and*

$$\begin{cases} i) & T(0)\psi_j \in \mathcal{K}_0, \;\; T(h_j)\psi_j \in \mathcal{K}_0, \\ & (\psi(h_j) - \psi_j(0))/h_j \in v_j + \frac{1}{m}B \\ ii) & T(0)\psi_j \in T(h_{j-1})\psi_{j-1} + \mathcal{V}_m^{h_j/m} \\ & (\text{i.e.}, \sup_{-m\leq\tau\leq 0} \|\psi_j(\tau) - \psi_{j-1}(\tau + h_{j-1})\| \leq h_j/m) \\ iii) & (T(0)\psi_j, v_j) \;\in\; \mathrm{Graph}(\mathcal{F}) \end{cases} \tag{12.8}$$

Proof — We proceed by induction. By Lemma 12.2.3, starting with $\psi_0 := \varphi$, there exist $h_1 \in [\theta_m, \frac{1}{m}]$, ψ_1 and $v_1 \in C$ such that the above properties (12.7) hold true. It remains to check that $T(h_1)\psi_1$ belongs to $\mathcal{K}_0$ to deduce that properties (12.8) are also satisfied, i.e., that $\|T(h_1)\psi_1(0) - \varphi(0)\| \leq 2\lambda$. This follows from the fact that

$$\begin{cases} \|T(h_1)\psi_1(0) - \varphi(0)\| \;\leq\; \|\psi_1(h_1) - \psi_1(0)\| + \|\psi_1(0) - \varphi(0)\| \\ \leq \; \lambda h_1 + \frac{1}{m}h_1 \;\leq\; 2\lambda h_1 \end{cases}$$

We apply Lemma 12.2.3 to the function $\chi := T(h_1)\psi_1$ and infer the existence of $h_2 \in [\theta_m, \frac{1}{m}]$, ψ_2 and v_2 satisfying properties (12.7) and we verify that

$$\|T(h_2)\psi_2(0) - \varphi(0)\| \leq \lambda h_2 + \frac{1}{m}h_2 + (\lambda + \frac{1}{m})h_1 \leq (\lambda + \frac{1}{m})(h_1 + h_2)$$

We proceed until the index J such that

$$(\lambda + \frac{1}{m})(h_1 + h_2 + \cdots + h_{J-1}) \;\leq 2\lambda \;<\; (\lambda + \frac{1}{m})(h_1 + h_2 + \cdots + h_J) \;\; \square$$

We set $\tau_m^0 := 0, \tau_m^1 = h_1, \ldots, \tau_m^J := \sum_{j=1}^J h_j$ so that $\tau_m^J \geq 1$. We define the functions $y_m(\cdot)$ on $]-\infty, \tau_m^p[$ by

$$\begin{cases} i) & y_m(t) := \varphi(t) \;\text{ if }\; t \leq 0 \\ ii) & y_m(t) := \psi_{j+1}(t - \tau_m^j) + \sum_{i=0}^j (\psi_i(h_i) - \psi_{i+1}(0)) \\ & \text{if } \; t \in [\tau_m^j, \tau_m^{j+1}[\end{cases}$$

and their values

$$x_j := y_m(\tau_m^j) = \psi_{j+1}(0) + \sum_{i=0}^{j}(\psi_i(h_i) - \psi_{i+1}(0))$$

We interpolate this sequence by piecewise linear functions defined on each interval $[\tau_m^j, \tau_m^{j+1}[$ by

$$x_m(t) := x_j + (t - \tau_m^j)(x_{j+1} - x_j)/h_{j+1}$$

and we set $x_m(\tau) := y_m(\tau) := \varphi(\tau)$ when $\tau \leq 0$.

Properties of the functions $x_m(\cdot)$ are summarized in the following

Lemma 12.2.5 *The functions $x_m(\cdot) :] - \infty, 1]$ are λ-Lipschitz, satisfy*

$$\begin{cases} i) & \forall\, t \in [0,1], \;\; x_m'(t) \in C \\ ii) & \forall\, t \in [0,1], \;\; \|x_m(t) - \varphi(0)\| \leq \lambda t \end{cases} \tag{12.9}$$

and

$$\begin{cases} i) & T(0)x_m \;=\; \varphi \\ ii) & \forall\, t \in] - \infty, 1], \;\; (T(t)x_m, x_m'(t)) \in \mathrm{Graph}(\mathcal{F}) + \left(\mathcal{V}_m^{\varepsilon_m} \times \frac{1}{m}B\right) \\ iii) & \forall\, t \in] - \infty, 1], \;\; T(t)x_m \in \mathcal{K} + \mathcal{V}_m^{\varepsilon_m} \end{cases} \tag{12.10}$$

where ε_m converges to 0.

Proof — The functions ψ_j being λ-Lipschitz, as translations of functions of $\mathcal{K}$, so are the functions $y_m(\cdot)$ and $x_m(\cdot)$.

The velocities of the approximate solutions belong to C because

$$\begin{cases} x_m'(t) = \frac{x_{j+1} - x_j}{h_{j+1}} = \frac{y_m(\tau_m^{j+1}) - y_m(\tau_m^j)}{h_{j+1}} \\ = \frac{\psi_{j+1}(h_j) - \psi_{j+1}(0)}{h_{j+1}} \in v_{j+1} + \frac{1}{m}B \end{cases}$$

On the other hand, since $T(0)x_m = \varphi$, we deduce that

$$\|x_m(t) - \varphi(0)\| = \|x_m(t) - x_m(0)\| \leq \lambda t$$

It remains to prove properties (12.10).

For that purpose, we shall prove by induction that for any $j = 0, \ldots, J-1$, we have

$$\sup_{-m \leq \tau \leq 0} \|T(\tau_m^{j+1}) y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \leq \frac{\tau_m^{j+1}}{m} \tag{12.11}$$

For $j = 0$ and $\tau \in [-h_1, 0]$, we obtain

$$\begin{cases} \|T(h_1)y_m(\tau) - T(h_1)\psi_1(\tau)\| = \|y_m(\tau + h_1) - \psi_1(\tau + h_1)\| \\ = \|\psi_1(\tau + h_1 + 0) + \psi_0(0) - \psi_1(0) - \psi_1(\tau + h_1)\| = \|\psi_1(0) - \psi_0(0)\| \\ = \|\psi_1(0) - \varphi(0)\| \leq \frac{h_1}{m} \end{cases}$$

When $\tau \in [-m, -h_1]$, then

$$y_m(\tau + h_1) - \psi_1(\tau + h_1) = \varphi(\tau + h_1) - \psi_1(\tau + h_1)$$

By (12.8)ii) with $j = 1$, we know that $T(h_1)\psi_1 \in \varphi + \mathcal{V}_m^{h_1/m}$. Then property (12.11) is satisfied for $j = 0$. Assume that it is satisfied for $j - 1$ and prove that it holds true for j.

First, when $\tau \in [-h_{j+1}, 0]$, we get

$$\begin{cases} \|T(\tau_m^{j+1})y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \\ = \|\psi_{j+1}(\tau + \tau_m^{j+1} - \tau_m^j) - \psi_{j+1}(\tau + h_{j+1}) + \sum_{i=0}^j (\psi_i(h_i) - \psi_{i+1}(0))\| \\ \leq \sum_{i=0}^j \|\psi_i(h_i) - \psi_{i+1}(0)\| \leq \sum_{i=0}^j \frac{h_i}{m} = \frac{\tau_m^j}{m} \leq \frac{\tau_m^{j+1}}{m} \end{cases}$$

When $\tau \in [-m, -h_{j+1}]$, we obtain

$$\begin{cases} \|T(\tau_m^{j+1})y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \\ \leq \|T(\tau_m^j)y_m(\tau + h_{j+1}) - T(h_j)\psi_j(\tau + h_{j+1})\| \\ + \|T(h_j)\psi_j(\tau + h_{j+1}) - \psi_{j+1}(\tau + h_{j+1})\| \quad \square \end{cases}$$

Induction hypothesis (12.11) and condition (12.8)iii) imply that for all $\tau \in [-m, -h_{j+1}]$,

$$\|T(\tau_m^{j+1})y_m(\tau) - T(h_{j+1})\psi_{j+1}(\tau)\| \leq \frac{\tau_m^j}{m} + \frac{h_{j+1}}{m} = \frac{\tau_m^{j+1}}{m}$$

Hence property (12.11) is established, from which we have to deduce properties (12.10).

We observe that

$$\sup_{t\leq\tau_m^J} \|x_m(t) - y_m(t)\| \leq \frac{2\lambda}{m} \tag{12.12}$$

This is obvious when $t \leq 0$ because these functions are equal in this case. Otherwise, when $t \in [\tau_m^j, \tau_m^{j+1}[$, we obtain

$$\|x_m(t) - y_m(t)\| \leq \|x_m(t) - x_j\| + \|x_j - y_m(t)\| \leq \frac{2\lambda}{m}$$

Therefore, inequalities (12.12) and (12.11) imply that

$$\|T(\tau_m^{j+1})x_m - T(h_{j+1})\psi_{j+1}\| \leq (2\lambda + \tau_m^{j+1})/m$$

and thus, that for all $t \in [\tau_m^j, \tau_m^{j+1}]$,

$$\begin{cases} \|T(t)x_m - T(h_{j+1})\psi_{j+1}\| \\ \leq \|T(t)x_m - T(\tau_m^{j+1})x_m\| + \|T(\tau_m^{j+1})x_m - T(h_{j+1})\psi_{j+1}\| \\ \leq \lambda h_{j+1} + (2\lambda + \tau_m^{j+1})/m \leq 3(\lambda+1)/m =: \varepsilon_m \end{cases}$$

Consequently, when $t \in [\tau_m^j, \tau_m^{j+1}]$,

$$T(t)x_m \in T(h_{j+1})\psi_{j+1} + \mathcal{V}_m^{\varepsilon_m} \subset \mathcal{K} + \mathcal{V}_m^{\varepsilon_m}$$

and

$$\begin{cases} (T(t)x_m, x_m'(t)) \in (T(h_{j+1})\psi_{j+1}, v_{j+1}) + \mathcal{V}_m^{\varepsilon_m} \times \frac{1}{m}B \\ \subset \mathrm{Graph}(\mathcal{F}) + \mathcal{V}_m^{\varepsilon_m} \times \frac{1}{m}B \ \square \end{cases}$$

— **Convergence of approximate solutions**

Conditions (12.9) of Lemma 12.2.5 allow us to apply Ascoli's Theorem. Hence a subsequence (again denoted by) x_m converges uniformly on every compact interval to a continuous function $x(\cdot) :]-\infty, 1] \to X$, so that for all $t \geq 0$, $T(t)x_m$ converges to $T(t)x$

in $\mathcal{C}$. Condition (12.9)i) and Alaoglu's Theorem imply also that a subsequence (again denoted by) $x'_m(\cdot)$ converges weakly to $x'(\cdot)$ in $L^1(0,1;X)$ for some positive constant b.

— **The limit is a solution**

Conditions (12.10) of Lemma 12.2.5 allow us to apply the Convergence Theorem 2.4.4, where $\mathcal{C}$ plays the role of X, X the role of Y, $T(t)x_m$ the role of $x_m(t)$ and $x'_m(\cdot)$ the role of $y_m(\cdot)$. Hence the limit $x(\cdot)$ is a solution to the functional differential inclusion (12.1), which is viable since $\mathcal{K}$ is closed. The proof of the Functional Viability Theorem is completed. □

12.3 History-dependent Viability Constraints

We consider the case when

$$\mathcal{K} := \{\varphi \in \mathcal{C} \text{ such that } \varphi(0) \in M(U(\varphi))\} \tag{12.13}$$

where $U \in \mathcal{L}(\mathcal{C}, Y)$ is a continuous linear operator and where $M : Y \leadsto X$ is a closed set-valued map.

We introduce the affine subspace $\Gamma(x) \subset \mathcal{C}(0,1;X)$ of functions $\psi \in \mathcal{C}_\lambda(0,1;X)$ satisfying $\psi(0) = x$. With any $\varphi \in \mathcal{C}_\lambda$ and $\psi \in \Gamma(\varphi(0))$ we associate the concatenated function $\varphi \vee \psi \in \mathcal{C}(-\infty,1;X)$ equal to φ on $]-\infty,0]$ and to ψ on $[0,1]$.

We denote by $\Lambda : \mathcal{C} \leadsto \mathcal{C}$ the set-valued map[1] associating with any $\varphi \in \mathcal{C}$ the subset $\Lambda\varphi$ of functions $\nu \in \mathcal{C}$ such that there exist sequences of $h_n > 0$ converging to $0+$, of ν_n converging to ν in $\mathcal{C}$ and of functions $\varphi_n \in \Gamma(\varphi(0))$ such that

$$\forall\, n \geq 0,\;\; T(h_n)(\varphi \vee \varphi_n) \;=\; \varphi + h_n\nu_n$$

Observe that if φ is the restriction to $]-\infty,0]$ of a differentiable function $\widetilde{\varphi}$ defined on $]-\infty,1]$, then $\Lambda\varphi := \widetilde{\varphi}'$ and that $\Lambda\varphi \neq \emptyset$ whenever φ is the restriction to $]-\infty,0]$ of a Lipschitz function $\widetilde{\varphi}$ defined on $]-\infty,1]$. In this case, every selection $\nu \in \Lambda\varphi$ is almost everywhere equal to φ':

$$\text{for almost all } t \geq 0,\;\; \nu(t) \;=\; \varphi'(t)$$

[1]We can regard Λ as a *contingent infinitesimal generator of the semi-group* $T(t)$.

We introduce now the *adjacent derivative* $D^\flat M(y,x)$ of M at (y,x) defined[2] in the following way: $u \in D^\flat M(y,x)(v)$ if and only if for all sequences $h_n > 0$ converging to 0, there exist sequences u_n converging to u and v_n converging to v such that

$$\forall\, n \geq 0,\;\; x + h_n u_n \;\in\; M(y + h_n v_n)$$

If M is sleek at (y,x), then both contingent and adjacent derivatives at (y,x) coincide. This is then the case when the graph of M is either convex or a smooth manifold. See Chapter 5 of SET-VALUED ANALYSIS for further details on adjacent derivatives of set-valued maps.

We provide more and more general sufficient conditions for subsets $\mathcal{K}$ defined by (12.13) to be viability domains.

Theorem 12.3.1 *We posit the following "surjectivity condition" on U: there exists a constant $c > 0$ such that for all $h > 0$,*

$$\begin{cases} \forall\,(v,u) \in Y \times X,\;\; \exists\, \psi_h \in \mathcal{C}(0,1;X) \quad \textit{such that} \\ \psi_h(0) = 0,\;\; \psi_h(h) = u,\;\; UT(h)(\varphi \vee \psi_h) = v \\ \text{and satisfying} \\ \|\psi_h\|_{\mathcal{C}(0,1;X)} \;\leq\; c(\|u\| + \|v\|) \end{cases}$$

Therefore

$$D^\flat M(U\varphi, \varphi(0))(U(\Lambda\varphi)) \;\subset\; \mathcal{D}_{\mathcal{K}}(\varphi)$$

The next statement trades surjectivity condition on U with restrictions on the size of the norm of U and the norm $\|D^\flat M(y,x)\|$

[2]Recall that the *adjacent tangent cone* $T_K^\flat(z)$ to a subset K at $z \in K$ is defined by

$$T_K^\flat(z) \;:=\; \text{Liminf}_{h\to 0+}\left(\frac{K-z}{h}\right) \;=\; \left\{v \mid \lim \frac{d_K(z+hv)}{h} \;=\; 0\right\}$$

Then

$$\text{Graph}(D^\flat M(y,x)) \;:=\; T^\flat_{\text{Graph}(M)}(y,x)$$

defined by

$$\left\|D^\flat M(y,x)\right\| \; := \; \inf_{u \in D^\flat M(y,x)(v)} \frac{\|u\|}{\|v\|}$$

Theorem 12.3.2 *Assume that for any* $(y,x) \in \text{Graph}(M)$*, the domain* $\text{Dom}(D^\flat M(y,x)) = Y$ *and that*

$$\forall\,(y,x) \in \text{Graph}(M), \;\; \left\|D^\flat M(y,x)\right\| \; \leq \; \beta \; < \; +\infty$$

and that there exists $\gamma > 0$ *such that*

$$\|UT(h)(0 \vee \psi)\| \; \leq \; \gamma h \|\psi\|$$

Therefore

$$D^\flat M(U\varphi, \varphi(0))(U(\Lambda\varphi)) \; \subset \; \mathcal{D}_K(\varphi)$$

These results follow from the more general sufficient condition, which looks quite involved, but which is flexible enough to cover a wide variety of examples.

Theorem 12.3.3 *We posit that the following "stability condition" linking* U *and* M*: there exist constants* $c, l > 0$ *and* $\alpha \in]0,1[$ *such that for all* $h > 0$*,*

$$\left\{\begin{array}{l} \forall\,(y,x) \in \text{Graph}(M), \; \forall\,(v,u) \in Y \times X, \\ \exists\,\psi_h \in \mathcal{C}(0,1;X), \; u_\alpha \in X, \; v_\alpha \in Y \quad \textit{such that } \psi_h(0) = 0 \;\; \& \\ \\ \psi_h(h) \in D^\flat M(U\varphi, \varphi(0))(UT(h)(0 \vee \psi_h) - v - v_\alpha) + u + u_\alpha \\ \\ \text{and satisfying} \\ \\ \|\psi_h\|_{\mathcal{C}(0,1;X)} \; \leq \; c(\|u\| + \|v\|), \;\; \|u_\alpha\| + \|v_\alpha\| \leq \; \alpha(\|u\| + \|v\|) \end{array}\right.$$

Therefore

$$D^\flat M(U\varphi, \varphi(0))(U(\Lambda\varphi)) \; \subset \; \mathcal{D}_K(\varphi)$$

Proof — Let us pick $\nu \in \Lambda\varphi$ and $u \in D^\flat M(U\varphi, \varphi(0))(U\nu)$ and check that u belongs to $\mathcal{D}_K(\varphi)$.

We know that there exist sequences $h_n > 0$ converging to $0+$, u_n converging to v and v_n converging to $U\nu$ such that

$$\forall\, n \geq 0, \;\; \varphi(0) + h_n u_n \; \in \; M(U\varphi + h_n v_n)$$

But we also know by definition of Λ that there exist sequences ν_n converging to ν and $\varphi_n \in \Gamma(\varphi(0))$ such that

$$\forall\, n \geq 0, \;\; T(h_n)(\varphi \vee \varphi_n) \;=\; \varphi + h_n \nu_n$$

Denote by A_n the Fréchet differentiable operator from $\mathcal{C}(0,1;X) \times Y \times X$ to $Y \times X$ defined by

$$A_n(\psi, y, x) \;:=\; (UT(h_n)(\varphi \vee \psi) - y, \psi(h_n) - x)$$

We observe that

$$A_n(\varphi_n, U\varphi + h_n v_n, \varphi(0) + h_n u_n) \;=\; h_n \left(U\nu_n - v_n, \nu_n(0) - u_n\right)$$

and that

$$A_n'(\psi, y, x)(\xi, v, u) \;=\; (UT(h_n)(0 \vee \xi) - y, \xi(h_n) - x)$$

We the apply Theorem 3.4.5 of SET-VALUED ANALYSIS which we now recall:

Theorem 12.3.4 *Let X be a Banach space, $L \subset X$ be a closed subset and Y a normed space. Consider a sequence of Fréchet differentiable operators A_n from X to Y and elements $x_{0n} \in L$ such that x_{0n} converges to $x_0 \in L$ and $A_n(x_{0n})$ to y_0.*

We assume that A_n verify the following stability assumption*: there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$\forall\, x \;\in\; L \cap B(x_0, \eta), \;\; B_Y \;\subset\; A_n'(x)(T_L(x) \cap cB_X) + \alpha B_Y \quad (12.14)$$

Then there exist $l > 0$ and $\gamma > 0$ such that

$$\forall\, y_n \;\in\; B(y_0, \gamma), \;\; d\left(x_{0n}, A_n^{-1}(y_n) \cap L\right) \;\leq\; l\|y_n - A_n(x_{0n})\|$$

We apply this theorem with

$$\begin{cases} X := \mathcal{C}(0,1;X) \times Y \times X, \;\; Y := Y \times X \\ L := \Gamma(\varphi(0)) \times \mathrm{Graph}(M) \\ y_n := 0 \;\; \& \;\; x_{0n} := (\varphi_n, U\varphi + h_n v_n, \varphi(0) + h_n u_n) \end{cases}$$

We have seen that $A_n(x_{0n})$ converges to $y_0 := 0$.

By the stability assumption, there exist $c > 0$ and $\alpha \in]0,1[$ such that for any (ψ, y, x) and any (v, u), there exists a solution

$$\begin{cases} (\psi_h, y_h, x_h) \in \\ \Gamma(0) \times T_{\mathrm{Graph}(M)}(U\varphi, \varphi(0)) \; = \; T_{\Gamma(\varphi(0)) \times \mathrm{Graph}(M)}(\varphi, U\varphi, \varphi(0)) \end{cases}$$

to the equation $A'_h(\psi, y, x)((\psi_h, y_h, x_h)) = (v, u)$ satisfying the above estimates.

Then we can apply Theorem 12.3.4: for each n, there exists a solution (ψ_n, y_n, x_n) to the equation $A_n(\mu_n, y_n, x_n) = 0$ belonging to

$$(\mu_n, y_n, x_n) \; \in \; \Gamma(\varphi(0)) \times \mathrm{Graph}(M)$$

and satisfying the inequalities

$$\begin{cases} \|\mu_n - \varphi_n\|_{\mathcal{C}(0,1;X)} + \|y_n - U\varphi - h_n v_n\| + \|x_n - \varphi(0) - h_n u_n\| \\ \le \; h_n(\|U\nu_n - v_n\| + \|\nu_n(0) - u_n\|) \end{cases}$$

This implies in particular that there exists a sequence e_n converging to 0 such that

$$\mu_n(h_n) = x_n = \varphi(0) + h_n u_n + h_n e_n \; \in \; M(UT(h_n)(\varphi \vee \mu_n))$$

and such that

$$\frac{\mu_n(h_n) - \varphi(0)}{h_n} \; = \; u_n + e_n \; \text{ converges to } \; u$$

Since μ_n belongs to $\Gamma(\varphi(0))$, we infer that the function $\varphi_n := \varphi \vee \mu_n$ satisfies the properties

$$T(h_n)\varphi_n \; \in \; \mathcal{K} \; \& \; \varphi_n(h_n) \; = \; \varphi(0) + h_n(u_n + e_n)$$

where $u_n + e_n$ converges to 0. We thus conclude that u belongs to $\mathcal{D}_{\mathcal{K}}(\varphi)$. $\square$

Proof of Theorem 12.3.2 — We have to prove that the stability condition of Theorem 12.3.3 holds true. We take $\psi_h \in \mathcal{C}(0,1;X)$ defined by $\psi_h(t) := tu/h$ if $t \in [0,h]$ and $\psi_h(t) := (1 -$

$t)u/((1-h))$ if $t \in [h,1]$, so that $\|\psi_h\| \leq \|u\|$ satisfies $\psi_h(0) = 0$ and $\psi_h(h) = u$. Let us set $\rho := \frac{\beta}{1+\beta}$. We then take

$$v_\alpha := UT(h)(0 \vee \psi_h) - \rho v$$

and

$$u_\alpha \in (1-\rho)D^\flat M(y,x)(-v)$$

with minimal norm. We thus see that

$$\psi_h(h) \in D^\flat M(y,x)(UT(h)(0 \vee \psi_h) - v - v_\alpha) + u - u_\alpha$$

and that

$$\|v_\alpha\| \leq \gamma h \|\psi_h\| + \rho\|v\| \ \ \& \ \ \|u_\alpha\| \leq (1-\rho)\beta\|v\| = \rho\|v\| \quad (12.15)$$

Therefore the stability assumption is satisfied with $\alpha \in]\rho, 1[$ for h small enough.

Proof of Theorem 12.3.1 — The stability assumption is obviously satisfied with $u_\alpha = 0$ and $v_\alpha = 0$. □

12.3.1 Viability constraints with delays

Here, we take $Y := X^p$ and $U(\varphi) := (\varphi(-\theta_1), \ldots, \varphi(-\theta_p))$. Then the surjectivity assumption of Theorem 12.3.1 is satisfied, so that we obtain the following consequence:

Corollary 12.3.5 *Let us consider p positive delays $\theta_1, \ldots, \theta_p$. Assume that*

$$\mathcal{K} := \{\varphi \in \mathcal{C} \ \ \textit{such that} \ \varphi(0) \in M(\varphi(-\theta_1), \ldots, \varphi(-\theta_p))\}$$

Then

$$D^\flat M(\varphi(-\theta_1), \ldots, \varphi(-\theta_p), \varphi(0))((\Lambda\varphi)(-\theta_1), \ldots, (\Lambda\varphi)(-\theta_p)) \subset \mathcal{D}_\mathcal{K}(\varphi)$$

12.3.2 Volterra Viability constraints

Let us consider a finite dimensional space Y, a set-valued map $M : Y \rightsquigarrow X$ and $A \in L^1(0,\infty;\mathcal{L}(X,Y))$. We consider the case when U is defined by

$$U\varphi := \int_{-\infty}^{0} A(-s)\varphi(s)ds$$

Corollary 12.3.6 *Let us consider $A \in L^1(0,\infty;\mathcal{L}(X,Y))$ satisfying*

$$\sup_{t\in[0,1]} \|A(t)\|_{\mathcal{L}(Y,X)} \leq \gamma < +\infty$$

Assume that for all $(y,x) \in \mathrm{Graph}(M)$, $\mathrm{Dom}(D^\flat M(y,x)) = Y$ and that

$$\sup_{(y,x)\in\mathrm{Graph}(M)} \|D^\flat M(y,x)\| \leq \beta < +\infty$$

Let $\mathcal{K}$ be the subset defined by

$$\mathcal{K} := \left\{\varphi \in \mathcal{C} \quad \text{such that } \varphi(0) \in M\left(\int_{-\infty}^{0} A(-s)\varphi(s)ds\right)\right\}$$

Then

$$D^\flat M\left(\int_{-\infty}^{0} A(-s)\varphi(s)ds, \varphi(0)\right)\left(\int_{-\infty}^{0} A(-s)\varphi'(s)ds\right) \subset \mathcal{D}_{\mathcal{K}}(\varphi)$$

Proof — It follows from Theorem 12.3.2. □

12.4 Functional Viability Kernel

The proof of Theorem 12.2.1 shows also that the solution map is upper semicontinuous and that there exist functional viability kernels of closed subsets $\mathcal{K} \subset \mathcal{C}_\lambda$.

We denote by $\mathcal{S}(\varphi)$ or by $\mathcal{S}_{\mathcal{F}}(\varphi)$ the (possibly empty) set of solutions to differential inclusion (12.1) starting from the initial evolution φ. We shall say that the set-valued map $\mathcal{S}$ defined by

$$\mathrm{Dom}(\mathcal{F}) \ni \varphi \longmapsto \mathcal{S}(\varphi)$$

is the *solution map* of $\mathcal{F}$ (or of functional differential inclusion (12.1).)

We shall say that $\mathcal{F}$ is a *Marchaud map* if it is a nontrivial upper hemicontinuous map with nonempty compact convex images and with linear growth in the sense that there exists $c > 0$ such that

$$\forall \varphi \in \mathcal{C}, \ \ \|\mathcal{F}(\varphi)\| \ \leq \ c\,(\|\varphi(0)\| + 1)$$

Theorem 12.4.1 (Continuity of the Solution Map) *Let us consider a Marchaud map $\mathcal{F} : \mathcal{C}_\lambda \rightsquigarrow X$.*

The solution map $\mathcal{S}$ is upper semicontinuous with compact images from its domain to the space $\mathcal{C}(-\infty, +\infty; X)$.

Actually, the graph of the restriction of $\mathcal{S}$ to any compact subset $\mathcal{K}$ of $\mathcal{C}_\lambda$ is compact.

Proof — We shall show that for all $\varphi \in \text{Dom}(\mathcal{F})$ and for all $\eta > 0$, the restriction to a compact subset $\mathcal{K} \subset \text{Dom}(\mathcal{F})$ of the set-valued map $\mathcal{S}$ is compact.

Let us choose a sequence of elements $(\varphi_n, x_n(\cdot))$ of the graph of the solution map $\mathcal{S}$. They satisfy:

$$\begin{cases} i) & x_n'(t) \ \in \ \mathcal{F}(T(t)x_n) \\ ii) & T(0)x_n \ = \ \varphi_n \end{cases}$$

The linear growth of $\mathcal{F}$ implies that

$$\|x_n'(t)\| \ \leq \ c(\|x_n(t)\| + 1)$$

and thus, that

$$\forall\, n \geq 0, \ \ \|x_n(t)\| \ \leq \ (\|\varphi_n(0)\|+1)e^{ct} \ \ \& \ \ \|x_n'(t)\| \ \leq \ c(\|\varphi_n(0)\|+1)e^{ct}$$

Therefore, since the sequence of $\varphi_n(0)$ is bounded, the sequence $x_n(\cdot)$ is relatively compact in the Fréchet space $\mathcal{C}(0, \infty; X)$ by Ascoli's Theorem, and the sequence $x_n'(\cdot)e^{-ct}$ is weakly relatively compact in $L^\infty(0, \infty; X)$ by Alaoglu's Theorem. Let us take $b > c$.

Hence a subsequence (again denoted by) x_n converges to x in the sense that:

$$\begin{cases} i) & x_n(\cdot) \text{ converges to } x(\cdot) \text{ uniformly on compact intervals} \\ ii) & x_n'(\cdot) \text{ converges to } x'(\cdot) \text{ weakly in } L^1(0, \infty; X; e^{-b\cdot}) \end{cases}$$

Inclusions

$$\forall n > 0, \quad (T(t)x_n, x_n'(t)) \in \text{Graph}(\mathcal{F})$$

imply that

$$\text{for almost all } t > 0, \;\; x'(t) \; \in \; \mathcal{F}(T(t)x)$$

thanks to the Convergence Theorem 2.4.4.

We thus have proved that a subsequence of the elements $(\varphi_n, x_n(\cdot))$ of the graph of $\mathcal{S}$ restricted to $\mathcal{K}$ converges to an element $(\varphi, x(\cdot))$ of this graph. This shows that it is compact, and thus, that the solution map $\mathcal{S}$ is upper semicontinuous with compact images. □

Definition 12.4.2 (Functional Viability Kernels) *Let $\mathcal{K} \subset \mathcal{C}_\lambda$ be a subset of the domain of a set-valued map $\mathcal{F} : \mathcal{C} \rightsquigarrow X$. We shall say that the largest closed functional viability domain contained in $\mathcal{K}$ (which may be empty) is the* viability kernel of $\mathcal{K}$ *and denote it by* $\text{Viab}_{\mathcal{F}}(\mathcal{K})$ *or, simply,* $\text{Viab}(\mathcal{K})$.

We can adapt to the functional case the existence theorem of a viability kernel.

Theorem 12.4.3 *Let us consider a Marchaud map $\mathcal{F} : \mathcal{C} \rightsquigarrow X$ with compact convex images. Then the viability kernel of $\mathcal{K}$ does exist and is the subset of initial evolutions $\varphi \in \mathcal{K}$ such that at least one solution starting from φ is viable in $\mathcal{K}$.*

12.5 Functional Viability Tubes

We can now extend this time-independent functional viability theorem to the time-dependent case. We consider

$$\begin{cases} i) & \text{a set-valued map } \mathcal{P} : \mathbf{R} \rightsquigarrow \mathcal{C}_\lambda \\ ii) & \text{a set-valued map } \mathcal{F} : \text{Graph}(\mathcal{P}) \rightsquigarrow X \end{cases}$$

Definition 12.5.1 *For any $\varphi \in \mathcal{P}(t)$, we denote by $D\mathcal{P}(t, \varphi)(1) \subset X$ the subset of elements $v \in X$ such that, for any $\varepsilon > 0$, there exist*

$h \in]0, \varepsilon]$ and $\varphi_h \in \mathcal{C}(-\infty, t+h)$ satisfying

$$\begin{cases} i) & T(t)\varphi_h = \varphi \\ ii) & T(t+h)\varphi_h \in \mathcal{P}(t+h) \\ iii) & (\varphi_h(t+h) - \varphi_h(t))/h \in v + \varepsilon B \end{cases} \quad (12.16)$$

We shall say that the set-valued map $\mathcal{P}$ is a functional viability tube *if and only if*

$$\forall\, t,\ \varphi \in \mathcal{P}(t),\ \ F(t,\varphi) \cap \mathcal{DP}(t,\varphi)(1) \neq \emptyset$$

Theorem 12.5.2 *Assume that the set-valued map $\mathcal{P} : \mathbf{R} \rightsquigarrow \mathcal{C}_\lambda$ takes its values into λ-Lipschitz functions and that its graph is closed.*

Assume also that $\mathcal{F}$ is a Marchaud map. Then $\mathcal{P}$ enjoys the functional viability property: for any t_0 and $\varphi \in \mathcal{P}_{t_0}$, there exists a solution $x(\cdot)$ to

$$\text{for almost all } t \geq t_0,\ \ x'(t) \in \mathcal{F}(t, T(t)x) \quad (12.17)$$

satisfying the initial condition $T(t_0)x = \varphi$ which is viable in the sense that:

$$\forall\, t \geq t_0,\ \ T(t)x \in \mathcal{P}(t)$$

if and only if is a functional viability tube.

Proof — The proof of the necessary condition is fully analogous to the time-independent case. We deduce the sufficient condition from the time-independent case by observing that the functional viability property for the new system

$$\begin{cases} i) & (s'(t), x'(t)) \in \{1\} \times \mathcal{F}((T(t)s)(0), T(t)x) \\ ii) & T(t_0)(s,x) = (t_0, \varphi) \end{cases}$$

and the closed subset $\mathcal{L}$ defined by

$$\mathcal{L} = \{\, \mathcal{C}(-\infty, 0; \mathbf{R} \times X)_{1,\lambda} \mid \varphi \in \mathcal{P}(s(0)) \,\}$$

is equivalent to the functional viability property of the time-dependent system (12.4).

The assumptions of the Functional Viability Theorem 12.2.2 are satisfied since the set-valued map $\mathcal{G}$ defined by $\mathcal{G}(s,\varphi) := \mathcal{F}(s(0),\varphi)$ is upper semicontinuous with compact convex images, taking its values in the subset of $\max(1,\lambda)$-Lipschitz functions.

It remains to check that $\mathcal{L}$ is a functional viability domain of $\mathcal{G}$ if and only if $\mathcal{P}$ is a functional viability tube of $\mathcal{F}$.

Indeed, take $\varepsilon > 0$ and $v \in \mathcal{F}(t,\varphi) \cap \mathcal{DP}(t,\varphi)(1)$ and prove that $1 \times v$ belongs to the intersection of $1 \times \mathcal{G}(s,\varphi)$ and $\mathcal{D}_{\mathcal{L}}(s,\varphi)$ for any function $s(\cdot)$ such that $s(0) = t$. Then $(s,\varphi) \in \mathcal{L}$ since $\varphi \in \mathcal{P}(s(0)) = \mathcal{P}(t)$.

We know that there exist $h \in]0,\varepsilon]$ and $\varphi_h \in \mathcal{C}(-\infty, t+h)$ such that properties (12.16) are satisfied. Let us define the functions s_h and ψ_h on $]-\infty,h]$ by

$$\begin{cases} i) & s_h(\tau) = s(\tau) \quad \text{if } \tau \leq 0 \;\text{ and }\; s_h(\tau) = t+\tau \;\text{ if } \tau \in [0,h] \\ ii) & \psi_h(\tau) = \varphi_h(\tau + t) \end{cases}$$

Then, properties

$$T(0)(s_h,\psi_h) \;=\; (s,\varphi) \;\&\; T(h)(s_h,\psi_h) \;\in\; \mathcal{L}$$

(because $T(h)\psi_h = T(t+h)\varphi_h \in \mathcal{P}(t+h) = \mathcal{P}((T(h)s_h)(0)))$ and

$$\frac{s_h(h) - s_h(0)}{h} = 1 \;\&\; \frac{\psi_h(h) - \psi_h(0)}{h} = \frac{\varphi_h(t+h) - \varphi_h(t)}{h} \in v + \varepsilon B$$

imply that $1 \times v$ belongs to $\mathcal{D}_{\mathcal{L}}(s,\varphi)$ □

Chapter 13

Viability Theorems for Partial Differential Inclusions

Introduction

We extend the viability theorems to the case of elliptic and parabolic differential equations and inclusions and consider the regulation of viable solutions to distributed control problems governed by a parabolic partial differential equation of the type:

$$\begin{cases} i) & \frac{\partial}{\partial t}x(t,\omega) - \Delta x(t,\omega) = f(x(t,\omega), u(t,\omega)) \\ & \text{(the state equation)} \\ ii) & \text{for almost all } t,\omega, \quad u(t,\omega) \in U(\omega, x(t,\omega)) \\ & \text{(state-dependent feedback controls)} \\ iii) & \forall\, t \in [0,T], \quad x(t,\omega)|_{\partial\Omega} = 0 \\ & \text{(Dirichlet boundary conditions)} \\ iv) & \text{for almost all } \omega \in \Omega, \quad x(0,\omega) = x_0(\omega) \\ & \text{(initial condition)} \end{cases}$$

where ω ranges over an open subset $\Omega \subset \mathbf{R}^n$ and where

$$\begin{cases} i) & f : \mathbf{R} \times \mathcal{U} \to \mathbf{R} \text{ is a single-valued map} \\ ii) & U : \Omega \times \mathbf{R} \rightsquigarrow \mathcal{U} \text{ is a set-valued map} \\ iii) & \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial \omega_i^2} \text{ is the Laplacian} \end{cases}$$

In this case, the viability sets are closed subsets K of (space dependent) functions of $L^2(\Omega)$. The viability property states that for any initial state $x_0 \in K$, there exists a solution to the above distributed control problem such that

$$\forall\, t \geq 0, \;\; \omega \;\to\; x(t, \omega) \text{ belongs to } \;K$$

As for ordinary differential inclusions, we associate with the viability subset K the regulation map R_K defined by

$$R_K(x) := \{\, u \in U(x) \;\;\text{ such that }\; \Delta x + f(x, u) \in T_K(x) \,\}$$

where the concept of contingent cone is adequately adapted to the infinite-dimensional case. We then shall prove that K is a viability domain if and only if $R_K(x) \neq \emptyset$ for all $x \in K$.

Here again, under adequate assumption, we shall prove that *the distributed control system enjoys the viability property if and only if it is a viability domain.*

Naturally, the Laplacian being just an example of an *unbounded operator* on the Hilbert space $H := L^2(\Omega)$, the viability theorems we want to prove shall hold true for any unbounded operator. We recall the properties we need in the first section and we prove the main viability theorem in the second section. The third section is devoted to the case when the unbounded operator is an *elliptic* operator, which is the case for most applications. The regulation of distributed control problems is tackled in the fourth section and we adapt to the case of partial differential inclusions the theorems on Lyapunov functions in the last section.

We shall assume through this chapter that *the Hilbert spaces are separable.*

We begin by recalling in the first section[1] what we need about unbounded operators on Hilbert spaces to proceed. Next, the second

[1] We refer to the text APPLIED FUNCTIONAL ANALYSIS by the author for further details on unbounded operators, their adjoint and the infinitesimal generators of semi-groups.

section is devoted to the viability theorem for operational differential inclusions. These results are applied to elliptic and parabolic differential inclusions in the third section and to distributed control problems in the fourth section. The results on Lyapunov functions are extended to the case of parabolic inclusions in the fifth section.

13.1 Unbounded operators

Let us consider two Hilbert spaces Y and Z such that Y *is embedded in* Z in the sense that

$$Y \subset Z; \text{ the canonical injection is continuous with dense image}$$

We are allowed to identify[2] the dual $Z^\star$ of Z to a dense subset of the dual $Y^\star$ of Y.

We introduce also a Hilbert space H in which Y is dense with a stronger topology and we choose to identify the dual $H^\star$ with H (we say that H is a *pivot space*) by identifying[3] the scalar product $<, \cdot, >$ on $H \times H$ with the duality pairing on $H^\star \times H$.

We summarize this situation by writing:

$$Y \subset Z \subset H = H^\star \subset Z^\star \subset Y^\star$$

the canonical injections from one space to a larger one being continuous and dense.

An *unbounded operator* on a pivot space H is a pair $(D(A), A)$ where $D(A) \subset H$ is a subspace of H (called the *domain* of A) and A is a linear operator from $D(A)$ to H. In order to make continuous

[2]The canonical injection $j : Y \to Z$ being injective with dense image, its transpose $j^\star : Z^\star \to Y^\star$ has a dense image and is injective. We then may identify $j^\star$ with the identity map, and thus, infer that the duality products

$$\forall\, x \in Y,\ y \in Z^\star,\ < x, y >_{Z \times Z^\star} := < jx, y >_{Z \times Z^\star} = < x, j^\star y >_{Y \times Y^\star} =: < x, y >_{Y \times Y^\star}$$

coincide on $Y \times Z^\star$, which is dense in both $Y \times Y^\star$ and $Z \times Z^\star$.

[3]This amounts to regarding an element $x \in H$ both as an element of H and as a continuous linear functional $y \to < x, y >$ on H. This point of view is at the very basis of the theory of distributions, when $H = L^2(\Omega)$, as we saw in Section 3.1..

both the map A and the injection from $D(A)$ to H, the domain $D(A)$ is supplied with the "graph scalar product"

$$((x,y))_{D(A)} := < x,y > + < Ax, Ay >$$

We observe that for all $\lambda \in \mathbf{R}$, the domains $D(A)$ and $D(A+\lambda)$ of A and $A+\lambda := A+\lambda\mathbf{I}$ do coincide (as Hilbert spaces).

The unbounded operator $(D(A), A)$ is said to be *closed* if and only if $D(A)$ is a Hilbert space (i.e., complete) for the scalar product and *densely defined* (or *with dense domain*) if the domain $D(A)$ is dense in H.

From now on, *we shall assume that the unbounded operators we are using are closed and densely defined*, because in this case, we can define the concept of an *adjoint* of an unbounded operator, which is also closed and densely defined.

Here is how we proceed to define such an adjoint: since $D(A)$ is dense in H and H is identified with $H^\star$, one can identify H to a dense subspace of the dual $D(A)^\star$. The transpose $A^\star$ of A is then a continuous linear operator from H to $D(A)^\star$. We define the domain of the adjoint by

$$D(A^\star) := \{ p \in H \mid A^\star p \in H \}$$

and we take the restriction of $A^\star$ to this domain to obtain the unbounded operator $(D(A^\star), A^\star)$, which is called the *adjoint of the unbounded operator* $(D(A), A)$.

One can prove that *whenever* $(D(A), A)$ *is closed and densely defined, so is its adjoint.*

Hence, a first consequence of this property is the possibility to *extend* A *to* H in the following manner.

Since the domain $D(A^\star)$ of $A^\star$ is dense in H, then we can identify $H = H^\star$ to a dense subspace of the dual $D(A^\star)^\star$ of $A^\star$, and the transpose $A^{\star\star} \in \mathcal{L}(H, D(A^\star)^\star)$ is the *unique extension*[4] of $A \in \mathcal{L}(D(A), H)$ to $A \in \mathcal{L}(H, D(A^\star)^\star)$.

[4]Indeed, when x belongs to $D(A) \subset H$, we deduce that for all y belonging to $D(A^\star) \subset H$,

$$\begin{array}{l} < Ax, y >_{H\times H} = < x, A^\star y >_{D(A)\times D(A^\star)} = < x, A^\star y >_{H\times H} \\ = < A^{\star\star}x, y >_{D(A^\star)^\star \times D(A^\star)} = < A^{\star\star}x, y >_{H\times H} \end{array}$$

so that A and $A^{\star\star}$ coincide on the dense subspace $D(A)$ of H.

In summary, when both the domains $D(A)$ and $D(A^\star)$ are dense in H, then

$$\begin{cases} i) & A \in \mathcal{L}(D(A), H) \cap \mathcal{L}(H, D(A^\star)^\star) \\ ii) & A^\star \in \mathcal{L}(D(A^\star), H) \cap \mathcal{L}(H, D(A)^\star) \end{cases}$$

Let $Z \supset Y$ be another Hilbert space supplied with a weaker topology. We denote by

$$W^p(0,T;Y,Z) := \{ x(\cdot) \in L^p(0,T;Y) \mid x'(\cdot) \in L^p(0,T;Z) \}$$

supplied with the norm

$$\begin{cases} \|x\|_{W^p}^p := \|x\|_{L^p(Y)}^p + \|x'\|_{L^p(Z)}^p & \text{if } p < +\infty \\ \\ \|x\|_{W^\infty} := \max(\|x\|_{L^\infty(Y)}, \|x'\|_{L^\infty(Z)}) & \text{if } p = +\infty \end{cases}$$

Lemma 13.1.1 (Compactness Lemma) *Let us consider a Hilbert space Y such that*

$$D(A^\star) \subset Y \subset H$$

satisfying

$$\exists\, \rho > 0,\ \theta \in [0,1] \mid \|x\|_{Y^\star} \le \rho^{1+\theta} \|x\|_H^{1-\theta}\ \|x\|_{D(A^\star)^\star}^\theta \tag{13.1}$$

Then

$$W^\infty(0,T;Y,D(A^\star)^\star) \subset \mathcal{C}(0,T;H)$$

and the canonical injection is compact whenever the injection from Y to H is compact (when $T = +\infty$, the space $\mathcal{C}(0,\infty;H)$ is supplied with the topology of compact convergence).

Proof — Inequality $\|x\|_H^2 \le \|x\|_Y\ \|x\|_{Y^\star}$ and inequality (13.1) imply

$$\|x\|_H \le \rho \|x\|_Y^{\frac{1}{1+\theta}}\ \|x\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}}$$

Therefore, if $x(\cdot)$ belongs to $W^\infty(0,T;Y,D(A^\star)^\star)$,

$$\|x(t) - x(s)\|_H \le \rho \|x(t) - x(s)\|_Y^{\frac{1}{1+\theta}} \|x(t) - x(s)\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}}$$

Hence $x(\cdot)$ is continuous from $[0,T]$ to H.

Assume now that the injection from Y to H (and thus, to $D(A^\star)^\star$) is compact and take a bounded sequence $x_n(\cdot)$ in $W^\infty(0,T;Y,D(A^\star)^\star)$.

Hence $x_m(t)$ being bounded in Y, is relatively compact in $D(A^\star)^\star$. The sequence $x_m(\cdot)$ is also *equicontinuous* in $D(A^\star)^\star$, because the derivatives $x'_m(\cdot)$ are bounded in $D(A^\star)^\star$.

We then deduce from Ascoli's Theorem that it remains in a compact subset of the Banach space $\mathcal{C}(0,T;D(A^\star)^\star)$ (and in the Fréchet space $\mathcal{C}(0,\infty;H)$ when $T = +\infty$), and thus, that a subsequence (again denoted by) $x_m(\cdot)$ converges uniformly (over compact intervals) in $D(A^\star)^\star$ to some function $x(\cdot)$.

Actually, it is converging uniformly in H (on compact intervals) because

$$\begin{cases} \|x_m(t) - x_p(t)\|_H \leq \rho \|x_m(t) - x_p(t)\|_Y^{\frac{1}{1+\theta}} \|x_m(t) - x_p(t)\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}} \\ \\ \leq 2\|K\|_Y^{\frac{1}{1+\theta}} \|x_m(t) - x_p(t)\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}} \quad \square \end{cases}$$

13.2 Operational Differential Inclusions

We consider an unbounded operator $(D(A), A)$ of a Hilbert space H (identified with its dual), a set-valued map $F : H \rightsquigarrow D(A^\star)^\star$ and the Cauchy problem for the operational differential inclusion: find $x(\cdot) \in W^\infty(0,T;H,D(A^\star)^\star)$ such that

$$\text{for almost all } t \in [0,T], \;\; x'(t) + Ax(t) \in F(x(t)) \tag{13.2}$$

The initial conditions x_0 are given in $D(A)$.

Actually, we would like to take initial conditions in larger spaces $Y \subset H$, in order to cover less regular cases needed in further examples. We thus shall assume that there exist Hilbert spaces Y and Z such that

$$D(A) \subset Y \subset H \;\&\; D(A^\star) \subset Z \subset H$$

(the injections being continuous and dense) in such a way that A maps Y to $Z^\star$. The main example is $Y := D(A)$ and $Z := H$, but there are many other choices[5] which make sense in applications, so that it costs nothing to use as parameters these Hilbert spaces Y and Z such that $A \in \mathcal{L}(Y, Z^\star)$.

Let K be a subset of Y. We denote by

$$d^{D(A^\star)^\star}(x; K) := \inf_{y \in K} \|x - y\|_{D(A^\star)^\star}$$

the distance in $D(A^\star)^\star$ to a subset K and $T_K^{D(A^\star)^\star}(x)$ the $D(A^\star)^\star$-*contingent cone to K at $x \in K$* in $D(A^\star)^\star$, which is the subset of elements $v \in D(A^\star)^\star$ satisfying

$$\liminf_{h \to 0+} d^{D(A^\star)^\star}(x + hv; K)/h = 0$$

Definition 13.2.1 *We shall say that a subset $K \subset Y$ is* viable under $F - A$ *(or enjoys the* viability property for $F - A$*) if for any $x_0 \in K$, there exists one solution $x(\cdot) \in W^\infty(0, T; Y, D(A^\star)^\star)$ to (13.2) starting at x_0 and viable in K.*

We shall say that K is a viability domain of $F - A$ *if and only if*

$$\forall\, x \in K, \quad Ax \in F(x) - T_K^{D(A^\star)^\star}(x) \tag{13.3}$$

Theorem 13.2.2 (Operational Differential Inclusions) *Let $(D(A), A)$ be a closed densely defined unbounded operator on H. We introduce a pair of Hilbert spaces Y and Z satisfying the following assumptions:*

$$\left\{\begin{array}{ll} i) & A \in \mathcal{L}(Y, Z^\star) \\ ii) & \text{the injection from } Y \text{ to } H \text{ is compact and} \\ & \exists\, \rho > 0,\ \theta \in [0,1] \mid \|x\|_{Y^\star} \leq \rho^{1+\theta}\|x\|_H^{1-\theta}\ \|x\|_{D(A^\star)^\star}^\theta \\ iii) & \text{the injection from } D(A^\star) \text{ to } Z \text{ is compact} \\ iv) & F : H \rightsquigarrow Z^\star \text{ is upper hemicontinuous} \\ & \text{with bounded closed convex images} \\ v) & K \subset Y \text{ is bounded in } Y \text{ and closed in } H \end{array}\right. \tag{13.4}$$

[5]The other extreme case is $Y = H$ and $Z := D(A^\star)^\star$, but this one will not work! The larger Y, the less K needs to be bounded.

Then K is viable under $F - A$ if and only if it is a viability domain. In this case, viable solutions belong to $W^{\infty}(0,T;Y;Z^{\star})$ and can be extended on $[0,\infty[$.

Proof of the necessary condition — Let us consider a solution $x(\cdot)$ to the operational differential inclusion (13.2) satisfying

$$\forall\, T > 0,\ \exists\, t \in]0,T] \text{ such that } x(t) \in K$$

or, equivalently: there exists a sequence $t_n \to 0+$ such that $x(t_n) \in K$.

Since F is upper hemicontinuous at x_0, we can associate with any $p \in B_Z$ and $\varepsilon > 0$ an $\eta > 0$ such that

$$\forall \tau \in [0,\eta],\ \ \sigma(F(x(\tau)),p) \leq \sigma(F(x_0),p) + \varepsilon$$

Hence, integrating inequalities

$$\begin{cases} \langle x'(\tau),p\rangle \leq \sigma(F(x(\tau)),p) - \langle Ax(\tau),p\rangle \\ \leq \sigma(F(x_0),p) + \|A\|_{\mathcal{L}(Y,Z^{\star})}\|K\|_Y + \varepsilon =: \delta(p) \end{cases}$$

from 0 to t_n, we obtain

$$\forall\, p \in B_Z,\ \left\langle p, \frac{x(t_n) - x_0}{t_n}\right\rangle \ \leq\ \delta(p)$$

The uniform boundedness theorem implies that the sequence

$$v_n \ := \ \frac{x(t_n) - x_0}{t_n}$$

is relatively weakly compact in $Z^{\star}$, and compact[6] in $D(A^{\star})^{\star}$. Therefore, a subsequence (again denoted by) v_n converges weakly in $Z^{\star}$ to some $v \in Z^{\star}$. On the other hand, $x(\cdot)$ belonging to $W^{\infty}(0,T;Y,D(A^{\star})^{\star})$, it is a continuous function from $[0,T]$ to H by the Compactness Lemma 13.1.1. Therefore

$$\frac{1}{t_n}\int_0^{t_n} x(\tau)d\tau \text{ converges to } x_0 \text{ in } H$$

[6]Indeed, the injection from $D(A^{\star})$ to Z being compact by assumption, so is, by transposition, the injection from $Z^{\star}$ to $D(A^{\star})^{\star}$.

Consequently, for all $p \in D(A^\star) \subset Z$, inequalities

$$\langle p, v_n \rangle \leq \sigma(F(x_0), p) + \varepsilon - \left\langle \frac{1}{t_n} \int_0^{t_n} x(\tau) d\tau, A^\star p \right\rangle$$

imply that the limit v satisfies

$$\forall\, p \in D(A^\star),\ \langle p, v \rangle \leq \sigma(F(x_0), p) + \varepsilon - \langle p, Ax_0 \rangle$$

Letting ε converge to 0, we obtain

$$\forall\, p \in D(A^\star),\ \langle p, v + Ax_0 \rangle \leq \sigma(F(x_0, p))$$

so that $v + Ax_0$ belongs to the closed convex hull of $F(x_0)$ in $D(A^\star)^\star$. Since $F(x_0)$ is closed, convex and bounded in $Z^\star$, hence weakly compact, it remains weakly compact in $D(A^\star)^\star$, so that $Ax_0 \in F(x_0) - v$. On the other hand, since

$$\forall\, n, \quad x(t_n) \;=\; x_0 + t_n v_n \;\in\; K$$

we infer that v belongs to the $D(A^\star)^\star$-contingent cone $T_K^{D(A^\star)^\star}(x_0)$.

Proof of the sufficient condition — We shall construct approximate solutions by modifying Euler's method to take into account the viability constraints, then deduce from available estimates that a subsequence of these solutions converges in some sense to a limit, and finally, check that this limit is a viable solution to the operational differential inclusion (13.2).

1 — **Construction of Approximate Solutions**

We observe first that K is compact in the Hilbert space H, since the injection from Y to H is compact. Since $F(K)$ is bounded because F is upper hemicontinuous, we deduce that $C := F(K) - A(K) + B_{D(A^\star)^\star}$ is bounded in $D(A^\star)^\star$. Next, we claim that the following is true:

Lemma 13.2.3 *For any integer m, there exist $\theta_m \in]0, 1/m[$ such that for all $x \in K$, there exist $h \in [\theta_m, 1/m]$ and $u \in D(A^\star)^\star$ satisfying*

$$\begin{cases} i) & u \in C \\ ii) & x + hu \in K \\ iii) & (x, u) \in \text{Graph}(F - A) + \frac{1}{m}(B_H \times B_{D(A^\star)^\star}) \end{cases}$$

Proof of Lemma — Since K is a *viability domain*, we know that for all $y \in K$, there exists an element $f(y) \in F(y)$ such that $f(y) - Ay \in T_K^{D(A^\star)^\star}(y)$. By the very definition of the contingent cone $T_K^{D(A^\star)^\star}(y)$, there exists $h_y \in]0, 1/m[$ such that

$$d^{D(A^\star)^\star}(y + h_y(f(y) - Ay); K) \ < \ h_y/2m$$

We introduce the subsets

$$N(y) \ := \ \left\{x \in K \mid d^{D(A^\star)^\star}(x + h_y(f(y) - Ay); K) < h_y/2m\right\}$$

These subsets are obviously not empty (they contain y), *open* in $D(A^\star)^\star$, and thus, in H. Since y belongs to $N(y)$, there exists $\eta_y \in]0, 1/m[$ such that $B_H(y, \eta_y) \subset N(y)$. The compactness of K in H implies that it can be covered by q such balls $B_H(y_j, \eta_j)$, $j = 1, \ldots, q$.

We set

$$\theta_m \ := \ \min_{j=1,\ldots,q} h_{y_j} \ > \ 0$$

Let us choose any $x \in K$. Since it belongs to one of the balls $B_H(y_j, \eta_j) \subset N(y_j)$, there exists $z_j \in K$ such that

$$\begin{array}{l} \|x + h_{y_j}(f(y_j) - Ay_j) - z_j\|_{D(A^\star)^\star}/h_{y_j} \\ \le \ d^{D(A^\star)^\star}(x + h_{y_j}(f(y_j) - Ay_j); K)/h_{y_j} + 1/2m \le 1/m \end{array}$$

Let us set $u_j := \frac{z_j - x}{h_{y_j}}$. We see that $\|x - y_j\|_H \le \eta_j \le 1/m$, that $x + h_{y_j}u_j = z_j \in K$ and that $\|u_j + Ay_j - f(y_j)\|_{D(A^\star)^\star} \le 1/m$. Hence,

$$\begin{array}{l} (x, u_j) \in (y_j, f(y_j) - Ay_j) + \frac{1}{m}(B_H \times B_{D(A^\star)^\star}) \\ \subset \text{Graph}(F - A) + \frac{1}{m}(B_H \times B_{D(A^\star)^\star}) \end{array}$$

and $u_j \in B_{D(A^\star)^\star}(F(K) - A(K), 1/m) \subset C$. So the proof of the Lemma is completed. □

We can now construct by induction a sequence of positive numbers $h_j \in]\theta_m, 1/m[$ and a sequence of elements $x_j \in K$ and $u_j \in C$

such that

$$\begin{cases} i) & x_{j+1} := x_j + h_j u_j \in K, \;\; u_j \in C \\ ii) & (x_j, u_j) \in \text{Graph}(F - A) + \frac{1}{m}(B_H \times B_{D(A^\star)^\star}) \end{cases}$$

We introduce the nodes $\tau_m^j := h_0 + \cdots + h_{j-1}$ and we interpolate the sequence of elements x_j at the nodes τ_m^j by the piecewise linear functions $x_m(t)$ defined on each interval $[\tau_m^j, \tau_m^{j+1}[$ by

$$\forall t \in [\tau_m^j, \tau_m^{j+1}[, \;\; x_m(t) := x_j + (t - \tau_m^j) u_j$$

We observe that this sequence satisfies the following estimates

$$\begin{cases} i) & \forall t \in [0,T], \;\; x_m(t) \in \overline{co}(K) \\ & \text{which is bounded in } Y \text{ and compact in } H \\ ii) & \forall t \in [0,T], \;\; \|x_m'(t)\|_{D(A^\star)^\star} \leq \|C\|_{D(A^\star)^\star} \end{cases} \tag{13.5}$$

Let us fix $t \in [\tau_m^j, \tau_m^{j+1}[$. We deduce from assumption (13.4)ii) that

$$\begin{cases} \|x_m(t) - x_m(\tau_m^j)\|_H \leq \rho \|x_m(t) - x_m(\tau_m^j)\|_Y^{\frac{1}{1+\theta}} \|x_m(t) - x_m(\tau_m^j) x\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}} \\ \leq 2\rho \|co(K)\|_Y^{\frac{1}{1+\theta}} h_j^{\frac{\theta}{1+\theta}} \|u_j\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}} \\ \leq \frac{2}{m} \|co(K)\|_Y^{\frac{1}{1+\theta}} h_j^{\frac{\theta}{1+\theta}} \|C\|_{D(A^\star)^\star}^{\frac{\theta}{1+\theta}} =: \varepsilon_m \end{cases}$$

Since $(x_j, u_j) \in \text{Graph}(F - A) + 1/m(B_H \times B_{D(A^\star)^\star})$, we deduce that the functions are approximate solutions in the sense that $\forall t \in [0,T]$,

$$\begin{cases} i) & x_m(t) \in B_{D(A^\star)^\star}(K, \varepsilon_m) \\ ii) & (x_m(t), x_m'(t)) \in \text{Graph}(F - A) + (\varepsilon_m + \frac{1}{m})(B_H \times B_{D(A^\star)^\star}) \end{cases} \tag{13.6}$$

where ε_m converges to 0.

— **Convergence of the Approximate Solutions**

Estimates (13.5) imply that for all $t \in [0,T]$, the sequence $x_m(t)$ remains in the subset $\overline{co}K$ which is bounded in Y and that the derivatives remain bounded in $D(A^\star)^\star$. Hence the sequence $x_m(t)$ is a bounded subset of $W^\infty(0,T;Y,D(A^\star)^\star)$, which is relatively compact in $\mathcal{C}(0,T;H)$ by Compactness Lemma 13.1.1, so that a subsequence (again denoted by) $x_m(\cdot)$ converges uniformly in H.

Furthermore, the sequences $x_m(\cdot)$ and $x'_m(\cdot)$ being bounded in $L^\infty(0,T;Y)$ and $L^\infty(0,T;D(A^\star))$ respectively (which are the dual of $L^1(0,T;Y^\star)$ and $L^1(0,T;D(A)))$, they are weakly relatively compact thanks to Alaoglu's Theorem. The identity map being continuous for the norm topologies, is still continuous for the weak topologies. Hence the sequences $x_m(\cdot)$ and $x'_m(\cdot)$ are weakly relatively compact in $L^1(0,T;Y)$ and $L^1(0,T;D(A^\star)^\star)$, so that subsequences (again denoted by) $x_m(\cdot)$ and $x'_m(\cdot)$ converge weakly $x(\cdot) \in L^1(0,T;Y)$ and $x'(\cdot) \in L^1(0,T;D(A^\star)^\star)$.

In summary, we have proved that

$$\begin{cases} i) & x_m(\cdot) \text{ converges to } x(\cdot) \text{ weakly in } L^1(0,T;Y) \\ & \text{and uniformly in } H \\ ii) & x'_m(t) \text{ converges weakly to } x'(\cdot) \text{ in } L^1(0,T;D(A^\star)^\star) \end{cases}$$

— **The Limit is a Solution**

We regard now $(F - A)$ as a set-valued map from H to $D(A^\star)^\star$. Hence the Convergence Theorem 2.4.4 and properties (13.6)ii) imply that

$$\text{for almost all } t \in [0,T], \;\; x'(t) \in F(x(t)) - Ax(t)$$

i.e., that $x(\cdot)$ is a solution to the operational differential inclusion (13.2).

Since K is compact in H, and thus, in $D(A^\star)^\star$, condition (13.6)i) implies that

$$\forall t \in [0,T], \;\; x(t) \in cl_{D(A^\star)^\star}(K) = K$$

i.e., that $x(\cdot)$ is viable.

Finally, since T was chosen arbitrarily, we can extend the solution to the whole half-line $[0, \infty[$. The proof of the theorem is completed. □

Remark — The proof of this theorem does not work for the other limit case when $Y := H$ and $Z := D(A^\star)$, because of the lack of compactness necessary to pass to the limit. □

When K is convex, we obtain under the assumptions of Theorem 13.2.2 the existence of a viable equilibrium $\overline{x} \in K$, which is a solution to the inclusion

$$Ax \in F(x) \tag{13.7}$$

Actually,

Theorem 13.2.4 *Let $(D(A), A)$ be a closed densely defined unbounded operator on H. We posit the assumptions*

$$\begin{cases} i) & F : H \rightsquigarrow D(A^\star)^\star \text{ is upper hemicontinuous} \\ & \text{with closed convex images} \\ & \\ ii) & K \subset H \text{ is a compact convex viability domain of } F - A \end{cases}$$

Then there exists a viable equilibrium, a solution $\overline{x} \in K$ to (13.7).

Proof — This is the direct consequence of Theorem 3.7.11 where $X = H$, $Y = D(A^\star)^\star$, F is replaced by $x \rightsquigarrow F(x) - Ax$ and $B(x)$ is replaced by the canonical injection from H to $D(A^\star)^\star$. □

Remark — The proof of the necessary condition shows that the viability property for solutions $x(\cdot) \in W^\infty(0, T; Y, Z^\star)$ implies that the tangential condition (13.3) is satisfied in a weak sense with $D(A^\star)^\star$ replaced by $Z^\star$. When K is convex, we can dispense with the assumption that the injection from $D(A^\star)$ to Z, is compact, thanks to the following

Lemma 13.2.5 *If $K \subset Y \subset Z$ is convex, then $T_K^Z(x)$ is the (weak or strong) closure in Z of $S_K(x) := \bigcup_{h>0} \frac{1}{h}(K - x)$ and v belongs to $T_K^Z(x)$ if and only if v is a weak limit of elements v_n such that $x + h_n v_n \in K$ for all $n \geq 0$. The normal cone is equal to*

$$N_K^Z(x) = N_K(x) \cap Z^\star$$

Proof — Since the cone $S_K(x)$ is convex whenever K is convex, its weak or strong closures in Z do coincide. Therefore, if v is a weak limit of elements v_n such that $x + h_n v_n \in K$ for all $n \geq 0$, then v belongs to the closure in Z of $S_K(x)$. Conversely, if v belongs to this closure in Z, then it is the limit (in Z) of elements v_n such that $x + k_n v_n$ belongs to K for some positive k_n. Since K is convex, $x + h_n v_n \in K$ for the sequence $h_n := \min(k_n, \frac{1}{n})$ which converges to 0. Hence v belongs to $T_K^Z(x)$.

If j denotes the canonical injection from Y to Z, we can write the above statement as $T_K^Z(x) = \overline{jT_K(x)}$, so that,

$$N_K^Z(x) = (jT_K(x))^- = j^{\star^{-1}} N_K(x) = N_K(x) \cap Z^\star$$

since $j^\star$ is the canonical injection from $Z^\star$ to $Y^\star$. □

We conclude with the following example:

Lemma 13.2.6 *Let $Y \subset Z$ be two Hilbert spaces, the injection being continuous and dense, and J the duality mapping from Y onto $Y^\star$. Then*

$$T_{\alpha B_Y}^Z(x) = \begin{cases} Z \text{ if } \|x\|_Y < \alpha \text{ or if } \|x\|_Y = \alpha \ \& \ Jx \notin Z^\star \\ \\ \{v \mid \langle Jx, v\rangle \leq 0\} \text{ if } \|x\|_Y = \alpha \ \& \ Jx \in Z^\star \end{cases}$$

Proof — Indeed, if $\|x\|_Y < 1$, then $T_{\alpha B_Y}^Z(x) \supset T_{\alpha B_Y}(x) = Y$. Since Y is dense in Z, we deduce that $T_{\alpha B_Y}^Z(x) = Z$. Assume now that $\|x\|_Y = \alpha$. We have seen in Lemma 13.2.5 that $N_{\alpha B_Y}^Z(x) = N_{\alpha B_Y}(x) \cap Z^\star$.

Since $N_{B_Y}(x) = \{\lambda Jx\}_{\lambda \geq 0}$, we infer that when $Jx \notin Z^\star$, then $N_{B_Y}^Z(x) = 0$ and that when $Jx \in Z^\star$, $N_{B_Y}^Z(x) = \{\lambda Jx\}_{\lambda \geq 0}$. The claim follows by polarity. □

13.3 Elliptic & Parabolic Inclusions

We suppose from now on that the unbounded operator is derived from an "elliptic" operator A, in which case the operational differential inclusion is called "parabolic", and we shall derive existence and regularity theorems for parabolic differential inclusions.

First, we need to recall some facts about elliptic operators.

Let us consider for that purpose a Hilbert space $X \subset H$ dense in H with a stronger topology, so that we have the following identifications:

$$X \subset H = H^\star \subset X^\star$$

We introduce a continuous linear operator $A \in \mathcal{L}(X, X^\star)$, and we observe that its transpose $A^\star$ also belongs to $\mathcal{L}(X, X^\star)$. We can associate with A and $A^\star$ the *unbounded operators* $(D(A), A)$ and $(D(A^\star), A^\star)$ on H, the domains of which are defined by:

$$D(A) := \{x \in X \mid Ax \in H\}, \quad D(A^\star) := \{x \in X \mid A^\star x \in H\}$$

These domains are Hilbert spaces[7] when they are supplied with the graph norms:

$$\|x\|^2_{D(A)} := \|x\|^2_X + \|Ax\|^2_H, \quad \|x\|^2_{D(A^\star)} := \|x\|^2_X + \|A^\star x\|^2_H$$

so that $(D(A), A)$ and $(D(A^\star), A^\star)$ are closed unbounded operators.

If we suppose that the domain $D(A^\star)$ of $A^\star$ is dense in X, and thus, in H, then one can also check that $(D(A^\star), A^\star)$ is the adjoint of $(D(A), A)$ (in the sense of unbounded operators) and the transpose $A^{\star\star} \in \mathcal{L}(H, D(A^\star)^\star)$ as the *unique extension* of $A \in \mathcal{L}(X, X^\star)$ to H. Since we can also identify the dual $X^\star$ of X to a dense subspace of the dual $D(A^\star)^\star$ of $A^\star$, we obtain the "embeddings"

$$D(A) \subset X \subset H = H^\star \subset X^\star \subset D(A^\star)^\star$$

[7]Let us consider a Cauchy sequence $x_n \in D(A)$. Then x_n is a Cauchy sequence in X and Ax_n is a Cauchy sequence in H. Hence they converge to x and p respectively. On the other hand, for all $y \in X$, the relations

$$< Ax_n, y > = < x_n, Ay >$$

imply by going to the limit that

$$< p, y > = < x, A^\star y > = < Ax, y >$$

Hence $p = Ax$ (in $X^\star$) and, since $p \in H$, then x belongs to $D(A)$ and x_n converges to x in $D(A)$.

In summary, when both the domains $D(A)$ and $D(A^\star)$ are dense in X, then

$$\begin{cases} i) & A \in \mathcal{L}(D(A),H) \cap \mathcal{L}(X,X^\star) \cap \mathcal{L}(H,D(A^\star)^\star) \\ ii) & A^\star \in \mathcal{L}(D(A^\star),H) \cap \mathcal{L}(X,X^\star) \cap \mathcal{L}(H,D(A)^\star) \end{cases}$$

This is the case when for some $\lambda \in \mathbf{R}$ the operator $A + \lambda$ is an isomorphism[8] from X to $X^\star$. If the canonical injection from X to H is compact, then so are the injections from $D(A)$ to X, from H to $X^\star$ and from $X^\star$ to $D(A^\star)^\star$.

We shall say that A *is X-elliptic* if and only if there exists a positive constant c such that

$$\forall\, x \in X, \;\; < Ax, x > \geq \; c\|x\|_X^2$$

and that A *is X-coercive* if and only if $A + \lambda$ is X-elliptic for some $\lambda \in \mathbf{R}$.

The Lax-Milgram Theorem[9] states that *X-elliptic operators are isomorphisms.*

Assumption (13.4) with $\theta = 1/2$ holds true when A is X-elliptic because its inverse A^{-1} is $X^\star$-elliptic:

$$< p, A^{-1}p >=< AA^{-1}p, A^{-1}p >\geq \|A^{-1}p\|_X^2 \geq c'\|p\|_{X^\star}^2$$

so that

$$c'\|p\|_{X^\star}^2 \leq< p, A^{-1}p >\leq \|p\|_H\|A^{-1}p\|_H \leq c''\|p\|_H\|p\|_{D(A^\star)^\star}$$

Consequently:

[8] If $x \in X$, then $Ax + \lambda x \in X^\star$ can be approximated by a sequence of elements $p_n \in H$, so that x can be approximated by the elements $x_n := (A+\lambda)^{-1}p_n \in D(A)$ in X. If the injection from X to H is compact, so is the injection from $D(A)$ to H, because, if x_n converges weakly to x in $D(A)$, $Ax_n + \lambda x_n$ converges weakly to $Ax + \lambda x$ in H and strongly if $X^\star$ (since the transpose of a compact operator is still compact), so that $x_n = (A + \lambda)^{-1}(Ax_n + \lambda x_n)$ converges strongly to x in X.

[9] which is the infinite-dimensional extension of the theorem stating that definite positive matrices are invertible, since in finite dimension, definite positive and X-elliptic matrices coincide.

Lemma 13.3.1 *If $A \in \mathcal{L}(X, X^\star)$ is X-coercive, then the domains $D(A)$ and $D(A^\star)$ are dense in X and H, the canonical injections being compact whenever the injection from X to H is compact.*

Furthermore, the functions $x(\cdot) \in W^\infty(0,T;X,D(A^\star)^\star)$ are continuous from $[0,T]$ to H and the injection from $W^\infty(0,T;X,D(A^\star)^\star)$ to $\mathcal{C}(0,T;H)$ is compact whenever the injection from X to H is compact.

We then obtain the following consequence of the Viability Theorem 13.2.2 for operational differential inclusions:

Corollary 13.3.2 *Let us assume that $A \in \mathcal{L}(X, X^\star)$ is X-coercive and that*

$$\begin{cases} i) & \text{the injection from } X \text{ to } H \text{ is compact} \\ ii) & F : H \rightsquigarrow X^\star \text{ is upper hemicontinuous} \\ & \text{with bounded closed convex images} \\ iii) & K \subset X \text{ is bounded in } X \text{ and closed in } H \end{cases}$$

Then K is viable under $F - A$ if and only if it is a viability domain and the viable solutions belong to $W^\infty(0,T;X,X^\star)$.

If K is convex, there exists a viable equilibrium $\overline{x} \in K$, a solution to

$$\overline{x} \in X \quad \& \quad A\overline{x} \in F(\overline{x})$$

Proof — Lemma 13.3.1 implies that the assumptions of Theorem 13.2.2 with $Y := Z := X$ hold true when A is X-coercive. □

We shall now see that *the ellipticity condition replaces in some cases the requirement that K is bounded in X.* This is the case for instance when we take $K = H$, the whole space, so that we derive an existence theorem for parabolic equations:

Theorem 13.3.3 (Shi Shuzhong) *Let us assume that*

$$\left\{\begin{array}{ll} i) & \text{the injection from } X \text{ to } H \text{ is compact} \\ ii) & A \text{ is } X\text{-elliptic and symmetric} \\ iii) & F : H \leadsto H \text{ is bounded and upper hemicontinuous} \\ & \text{with closed convex images} \end{array}\right. \tag{13.8}$$

Then

1 — There exists an equilibrium

2 — For any initial state $x_0 \in X$, there exists a solution $x(\cdot)$ to the operational differential inclusion (13.2).

Proof — It follows from Corollary 13.3.2 and the following

Lemma 13.3.4 *If $A \in \mathcal{L}(X, X^\star)$ is X-elliptic and symmetric and F is bounded, then, there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$ the ball of radius α in X is a viability domain of $F - A$. Actually, we have*

$$F(x) \subset Ax + T^H_{\alpha B_X}(x)$$

Proof — Since A is X-elliptic and symmetric, we can renorm X by taking $\|x\|_X := \sqrt{< Ax, x >}$, for which A becomes the duality mapping from X onto $X^\star$. The injection from $D(A)$ to X being continuous, there exists $\rho > 0$ such that $\|x\|_X \leq \rho \|Ax\|_H$ for every $x \in D(A)$. Then we take

$$\alpha \geq \alpha_0 := \rho \|F\|_\infty \text{ where } \|F\|_\infty := \sup_{x \in X} \|F(x)\|$$

For any x such that $\|x\|_X = \alpha$, inequalities

$$\left\{\begin{array}{l} < Ax, f > \leq \|Ax\|_H \|F\|_\infty \leq \frac{\alpha\|Ax\|_H}{\rho} \\ = \|Ax\|_H \frac{\|x\|_X}{\rho} \leq \|Ax\|^2_H = < Ax, Ax > \end{array}\right.$$

imply that

$$\forall \|x\|_X = \alpha, \quad < Ax, f - Ax > \leq 0$$

By Lemma 13.2.6, this implies that $f - Ax$ belongs to $T^H_{\alpha B_X}(x)$. □

Remark — We can relax the boundedness condition on F and replace it by the growth condition

$$l := \limsup_{\|x\|_X \to +\infty} \frac{d^H(0, F(x))}{\|Ax\|_H} < 1$$

Indeed, it implies that by taking $\varepsilon < 1 - l$, there exists α_0 such that, for any $\|x\|_X \geq \alpha_0$, there exists $f \in F(x)$ such that

$$\|f\|_H = d^H(0, F(x)) \leq (l + \varepsilon)\|Ax\|_H \leq \|Ax\|_H$$

We deduce from the above inequality that there exists $f \in F(x)$ such that

$$\forall \ \|x\|_X = \alpha, \quad < Ax, f - Ax > \leq 0$$

because

$$< Ax, f > \leq \ \|Ax\|_H \ \|f\|_H \ \leq \ \|Ax\|_H^2 \ = < Ax, Ax >$$

Hence $Ax \in F(x) - T^H_{\alpha B_X}(x)$. □

When we replace H by X in the above growth condition, we obtain the existence of a solution which takes its values in $D(A)$.

Theorem 13.3.5 *We posit the assumptions of Theorem 13.3.3 where the boundedness condition is replaced by the growth condition*

$$l_X := \limsup_{\|Ax\|_H \to +\infty} \frac{d^X(0, F(x))}{\|Ax\|_X} < 1$$

Then

1 — There exists an equilibrium $\overline{x}$, a solution to

$$\overline{x} \in D(A), \ A\overline{x} \in X \quad \& \quad A\overline{x} \in F(\overline{x})$$

2 — For any initial state $x_0 \in X$, there exists a solution $x(\cdot)$ in $W^\infty(0, T; D(A), H)$ to the operational differential inclusion (13.2).

Proof — Since A is an isomorphism from $D(A)$ to H, we can supply the domain with the norm $\|x\|_{D(A)} := \|Ax\|_H$, so that the duality mapping from $D(A)$ onto its dual is $A^\star A = A^2$.

We know that there exists α_0 such that, for any $\|Ax\|_H \geq \alpha_0$, there exists $f \in F(x)$ such that

$$\|f\|_X = d^X(0, F(x)) \leq (l + \varepsilon)\|Ax\|_X \leq \|Ax\|_X$$

We take this time $K := \alpha B_{D(A)}$, the ball in $D(A)$ of radius $\alpha \geq \alpha_0$. Hence Lemma 13.2.6 states that whenever $\|x\|_{D(A)} = \alpha$ and $A^2x \in D(A^\star)$, then

$$T^H_{\alpha B_{D(A)}}(x) \; = \; \{ v \mid \; < A^2x, v > \leq 0\}$$

(and the whole space if not).

Inequalities

$$\begin{cases} < A^2x, f > = < Ax, Af > \leq \|Ax\|_X \; \|Af\|_{X^\star} \\ = \|Ax\|_X \; \|f\|_X \leq \|Ax\|_X^2 = < A^2x, Ax > \end{cases}$$

imply that for any $\alpha \geq \alpha_0$ and any $\|x\|_{D(A)} = \alpha$, there exists $f \in F(x)$ such that $Ax \in f - T_{\alpha B_{D(A)}(x)}$. □

More generally, we can obtain a viability theorem on closed subsets of H which are not necessarily bounded in X, thanks to the X-ellipticity which compensates this lack of boundedness:

Theorem 13.3.6 *Let us assume that*

$$\begin{cases} i) & \text{the injection from } X \text{ to } H \text{ is compact} \\ ii) & A \text{ is } X\text{-elliptic and symmetric} \\ iii) & F : H \leadsto H \text{ is } \textit{bounded} \text{ and upper} \\ & \text{hemicontinuous with closed convex images} \end{cases}$$

Let $\alpha_0 > 0$ be the number provided by Lemma 13.3.4. We assume that for some $\alpha \geq \alpha_0$,

$$\begin{cases} K \subset H \; \text{ is closed in } H \text{ and satisfies} \\ \forall \; \|x\|_X = \alpha, \;\; T^H_K(x) \cap T^H_{\alpha B_X}(x) \; = \; T^H_{K \cap \alpha B_X}(x) \end{cases}$$

Then $K \subset H$ is viable under $F - A$ whenever it is an H-viability domain in the sense that:

$$\forall \, x \in K \cap D(A), \;\; Ax \in F(x) - T^H_K(x)$$

Proof — The proof amounts to showing that for $\alpha > 0$ large enough, the intersection $K \cap \alpha B_X$ is still a viability domain which is bounded in X and closed in H. □

So, we have to provide sufficient conditions for the intersection of contingent cones to a closed subset and to a ball of X to be contained in the contingent cone to the intersection.

When K is convex, convex analysis provides the following:

Proposition 13.3.7 *Let K be a closed convex subset of H satisfying*

$$0 \in \mathrm{Int}_H(K + \alpha B_X) \tag{13.9}$$

Then property

$$\forall\, x \in D(A) \ \ \|x\|_X = \alpha, \ \ T_K^H(x) \cap T_{\alpha B_X}^H(x) \ = \ T_{K \cap \alpha B_X}^H(x)$$

holds true.

Indeed, assumption (13.9) implies that the intersection of the tangent cones is the tangent cone to the intersection. It follows in particular that $T_{K \cap \alpha B_X}^H(x) = H$ whenever $Ax \notin H$, i.e., whenever $x \notin D(A)$. Then it is enough to verify that

$$\forall\, x \in D(A) \cap K, \ \ Ax \in F(x) - T_K(x) \ \square$$

This happens in particular whenever a point of αB_X belongs to the interior (in H) of the closed convex subset $K \subset H$. In this case, we obtain the following consequence:

Theorem 13.3.8 *Let us assume that*

$$\begin{cases} i) & \text{the injection from } X \text{ to } H \text{ is compact} \\ ii) & A \text{ is } X\text{-elliptic and symmetric} \\ iii) & F : H \rightsquigarrow H \text{ is } \textit{bounded} \text{ and upper} \\ & \text{hemicontinuous with closed convex images} \\ iv) & K \subset H \text{ is closed, convex and} \\ & \exists\, x_0 \in X, \ \ \|x_0\|_X = \alpha, \ \text{ such that } x_0 \in \mathrm{Int}_H(K) \end{cases}$$

Then $K \subset H$ is viable under $F - A$ whenever it is an H-viability domain and there exists a viable equilibrium.

13.4 Distributed Control Systems

Let $\Omega \subset \mathbf{R}^n$ be a bounded open subset smooth enough for the Trace Theorem[10] to hold true. In order to define a "distributed" control system with feedbacks, we introduce a Hilbert space $\mathcal{U}$ of controls and

$$\begin{cases} i) & \text{a single-valued map } f : \mathbf{R} \times \mathcal{U} \to \mathbf{R} \\ ii) & \text{a set-valued map } U : \Omega \times \mathbf{R} \rightsquigarrow \mathcal{U} \\ iii) & \text{the Laplacian } \Delta := \sum_{i=1}^n \frac{\partial^2}{\partial \omega_i^2} \end{cases}$$

(for simplicity; we could have taken any elliptic differential operator).

The state of the distributed control system is governed by a nonlinear parabolic partial differential equation:

$$\begin{cases} i) & \frac{\partial}{\partial t} x(t,\omega) - \Delta x(t,\omega) = f(x(t,\omega), u(t,\omega)) \\ & \text{(the state equation)} \\ ii) & \text{for almost all } t, \omega, \;\; u(t,\omega) \in U(\omega, x(t,\omega)) \\ & \text{(state-dependent feedback controls)} \\ iii) & \forall\, t \in [0,T], \;\; x(t,\omega)|_{\partial\Omega} = 0 \\ & \text{(Dirichlet boundary conditions)} \\ iv) & \text{for almost all } \omega \in \Omega, \;\; x(0,\omega) = x_0(\omega) \\ & \text{(initial condition)} \end{cases} \tag{13.10}$$

We first make precise the spaces in which we are looking for solutions to this problem.

We recall that the Sobolev space $H^1(\Omega)$ is the space of functions $x(\cdot) \in L^2(\Omega)$ such that their partial derivatives (in the sense of distributions) $D_i x := \frac{\partial}{\partial \omega_i} x$ belong to $L^2(\Omega)$ for $i = 1, \ldots, n$. The (minimal) Sobolev space $H_0^1(\Omega)$ is the space of functions $x(\cdot) \in H^1(\Omega)$

[10] The Trace Theorem states that the trace operator $\gamma : x(\omega) \to x(\omega)|_{\partial\Omega}$ is a surjective continuous linear operator from the Sobolev space $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ and that its kernel $H_0^1(\Omega)$ is the closure of the space of infinitely differentiable functions with compact support in Ω. See for instance Chapters 7 and 9 of the text APPLIED FUNCTIONAL ANALYSIS by the author.

whose traces on the boundary vanish. Its dual is denoted by $H^{-1}(\Omega)$ and we adopt the identifications

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

From now on, we regard a function $(t,\omega) \to x(t,\omega)$ as a function $t \to x(t,\cdot)$ from $[0,T]$ to an adequate space of functions defined on Ω, as for instance Sobolev spaces.

We shall look for solutions in the space

$$\mathcal{W}^{\infty} := W^{\infty}(0,T; H_0^1(\Omega), H^{-1}(\Omega))$$

which is the Banach space of functions $x(\cdot) \in L^{\infty}(0,T; H_0^1(\Omega))$ whose derivatives belong to $L^{\infty}(T; H^{-1}(\Omega))$.

We identify the set-valued map U with the set-valued map (again denoted U) from $L^2(\Omega)$ to $L^2(\Omega;\mathcal{U})$ defined by

$$U(x(\cdot)) := \{u(\cdot) \in L^2(\Omega;\mathcal{U}) \mid \text{for almost all } \omega,\ u(\omega) \in U(\omega, x(\omega))\}$$

Corollary 13.4.1 *Let us assume that*

$$\begin{cases} i) & \Omega \text{ is bounded and enjoys the trace property} \\ ii) & f \text{ is continuous and affine with respect to } u \\ & \text{and satisfies } \sup_x |f(x,u)| \le c(\|u\|+1) \\ iii) & U \text{ is bounded and upper semicontinuous} \\ & \text{with closed convex images} \end{cases}$$

Then,

a/ — there exits a solution $(\overline{x}(\omega), \bar{u}(\omega))$ to the elliptic control problem

$$\begin{cases} i) & -\Delta\overline{x}(\omega) = f(\overline{x}(\omega), \bar{u}(\omega)) \\ & \text{(the elliptic state equation)} \\ ii) & \text{for almost all } \omega,\ \bar{u}(\omega) \in U(\omega, \overline{x}(\omega)) \\ & \text{(state-dependent feedback controls)} \\ iii) & \overline{x}(\omega)|_{\partial\Omega} = 0 \text{ (Dirichlet boundary conditions)} \end{cases}$$

b/ — for any initial state $x_0 \in H_0^1(\Omega)$, there exists a solution $x(\cdot) \in \mathcal{W}^{\infty}$ to the distributed control system (13.10).

Proof — We recall that $-\Delta$ is an H_0^1-elliptic operator from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ and that the injection from $H_0^1(\Omega)$ to $L^2(\Omega)$ is compact because Ω is bounded.

The set-valued map F defined on $L^2(\Omega)$ by

$$F(x) := \{f(\cdot, x(\cdot), u(\cdot))\}_{u(\cdot)\in U(x(\cdot))}$$

is a bounded upper hemicontinuous set-valued map with closed convex images. □

Let us consider now the viability problem for this control problem. For that purpose, we introduce the feedback R_K defined by

$$R_K(x) := \{\, u \in U(x) \quad \text{such that} \quad \Delta x + f(x,u) \in T_K(x) \,\}$$

Corollary 13.4.2 *We posit the assumptions of Corollary 13.4.1. Let $K \subset L^2(\Omega)$ be a closed subset such that some $\|x_0\|_{H_0^1(\Omega)} \leq \alpha$ belongs to its interior in $L^2(\Omega)$. Then K enjoys the viability property if and only if*

$$\forall\, x \in K, \; R_K(x) \neq \emptyset$$

and viable solutions are obtained through the regulation law

$$\text{for almost all } (t,\omega) \in [0,T] \times \Omega, \quad u(t,\omega) \in R_K(x(t,\omega))$$

13.5 Lyapunov Functions of Parabolic Inclusions

We now adapt to the case of parabolic differential inclusion the characterization of extended lower semicontinuous functions enjoying the Lyapunov property.

Theorem 13.5.1 *Let us assume that*

$$\left\{\begin{array}{ll} i) & \text{the injection from } X \text{ to } H \text{ is compact} \\ ii) & A \text{ is } X\text{-elliptic and symmetric} \\ iii) & F : H \rightsquigarrow H \text{ is \textit{bounded} and upper} \\ & \text{hemicontinuous with closed convex images} \end{array}\right. \tag{13.11}$$

and that the extended function $V : X \to \mathbf{R}_+ \cup \{+\infty\}$ *is convex, lower semicontinuous and continuous at* 0 *on* H. *If* V *is a Lyapunov function in the sense that for some* $a > 0$,

$$\forall\, x \in D(A) \cap \mathrm{Dom}(V), \quad \inf_{v \in F(x)} D_\uparrow V(x)(v - Ax) + aV(x) \leq 0 \quad (13.12)$$

then V *enjoys the Lyapunov property: for all* $x_0 \in \mathrm{Dom}(V)$, *there exists a solution to the parabolic operational differential inclusion (13.2) satisfying*

$$\forall\, t \geq 0, \ V(x(t)) \leq V(x_0)e^{-at}$$

Proof — We consider the system of operational differential inclusions

$$\begin{cases} i) & x'(t) + Ax(t) \in F(x(t)) \\ ii) & w'(t) + aw = 0 \end{cases} \quad (13.13)$$

where the viability subset is the epigraph $\mathcal{K} := \mathcal{E}p(V)$, which is a closed convex subset of $\mathcal{H} := H \times \mathbf{R}$.

We observe that the operator $\mathcal{A} := A \times a$ from $\mathcal{X} := X \times \mathbf{R}$ to $\mathcal{X}^\star = X^\star \times \mathbf{R}$ is elliptic, so that we can apply Theorem 13.3.8 with $\mathcal{F}(x) = F(x) \times \{0\}$, which is bounded whenever F is.

Since V is continuous at 0 on H, the element $(0, V(0)+\alpha)$ belongs to the intersection of the interior of the epigraph of V and the ball of radius $V(0) + \alpha$ in $\mathcal{X}$. Indeed, by taking $\delta < \alpha$, there exist $\gamma \in]0, \delta[$ such that, for all $z \in \gamma B_H$, $V(z) < V(0) + \alpha - \delta \leq V(0) + \alpha + \mu$ for any $|\mu| \leq \gamma$. This means that

$$(0, V(0) + \alpha) \ + \ \gamma B_{\mathcal{H}} \ \subset \ \mathcal{E}p(V)$$

Hence the assumptions of Theorem 13.3.8 are satisfied and we know that the epigraph of V is a viability domain of the system (13.13) if and only if

$$\forall\, (x, w) \in \mathcal{E}p(V) \cap (D(A) \times \mathbf{R}), \ \ (F(x) \times aw) \cap T_{\mathcal{E}p(V)}(x, w) \neq \emptyset$$

This is equivalent to inequality (13.12). □

Remark — We can relax the assumption that F is bounded by the condition:

$$\limsup_{\|x\|_X^2 + |w|^2 \to \infty} \ d^H(0, F(x)) / \sqrt{\|Ax\|_H^2 + a^2|w|^2} \ < \ 1$$

because we observe that

$$d^{\mathcal{H}}(0, \mathcal{F}(\vec{x}))/\|\mathcal{A}\vec{x}\|_{\mathcal{H}} = d^H(0, F(x))/\sqrt{\|Ax\|_H^2 + a^2|w|^2} \quad \square$$

Remark — We can also relax the assumption of continuity of V at 0 on H, by introducing the function V_α defined by

$$V_\alpha(z) := \inf_{\|x\|_X \leq \alpha} V(z-x) \leq V(z)$$

and by assuming that for any $\alpha > V(0)$,

$$V_\alpha \text{ is upper semicontinuous at } 0$$

Indeed, this means that for $\varepsilon := \alpha - V(0) > 0$, there exists $\delta \in]0, \varepsilon[$ such that for all $z \in \delta B_H$, $V_\alpha(z) \leq V_\alpha(0) + \varepsilon$. Hence, for any $(z, \mu) \in \delta B_{\mathcal{H}}$, there exists $x \in \alpha B_X$ such that

$$(z, \mu) = (z - x, V(0) + \varepsilon + \mu) + (x, -V(0) - \varepsilon) \in \mathcal{E}p(V) + \alpha B_{\mathcal{X}}$$

because

$$V(z-x) \leq V_\alpha(z) + 2\varepsilon \leq V_\alpha(0) + \varepsilon + \mu \leq V(0) + \varepsilon + \mu$$

and $|V(0) + \varepsilon| = \alpha$. $\square$

Chapter 14

Differential Games

Introduction

We consider two players, Xavier and Yvette, and a differential game whose dynamics are described by

$$\begin{cases} a) \begin{cases} i) & x'(t) &=& f(x(t),y(t),u(t)) \\ ii) & u(t) &\in& U(x(t),y(t)) \end{cases} \\ \\ b) \begin{cases} i) & y'(t) &=& g(x(t),y(t),v(t)) \\ ii) & v(t) &\in& V(x(t),y(t)) \end{cases} \end{cases}$$

where u, v, the controls, are regarded as *strategies* used by the players to govern the evolution of the states x, y of the game.

The rules of the game are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$, describing the constraints imposed by one player on the other. They replace the traditional intertemporal optimality and/or end-point criteria used in differential games.

The *playability domain* of the game $K \subset X \times Y$ is defined by:

$$K \ := \ \{(x,y) \in X \times Y \ \mid \ x \in P(y) \ \text{ and } \ y \in Q(x)\}$$

(We consider only the time-independent case for the sake of simplicity). We single out the following properties:

— The **playability property:** it states that for any initial state $(x_0, y_0) \in K$, there exists a solution to the differential game which is playable in the sense that

$$\forall\, t \geq 0, \quad x(t) \;\in\; P(y(t)) \;\;\&\;\; y(t) \;\in\; Q(x(t))$$

— **Xavier's discriminating property:** It states that for any initial state $(x_0, y_0) \in K$ and for any continuous closed loop strategy $\tilde{v}(\cdot,\cdot)$ played by Yvette, there exists a playable solution to the differential game.

— **Xavier's leading property:** It states that there exists a continuous closed loop strategy $\tilde{u}(\cdot,\cdot)$ played by Xavier such that for any initial state $(x_0, y_0) \in K$, there exists a playable solution to the differential game.

Our first task is to characterize the rules satisfying such properties as somewhat generalized solutions to Isaacs equations. Since the rules are set-valued maps and not functions, we may characterize them by the indicators Ψ_P and Ψ_Q of their graphs, defined by $\Psi_P(x,y) := 0$ when $x \in P(y)$ and $\Psi_P(x,y) := +\infty$ when $x \notin P(y)$. But these functions, which are only lower semicontinuous (when the graphs are closed) are not differentiable in the usual sense. Hence we must replace the concept of derivative by the one of contingent epiderivative in the Isaacs equations.

This being done, we shall interpret the solutions to contingent Isaacs equations in game theoretical terms and characterize the above properties of the rules P and Q by checking whether the function $\max(\Psi_P, \Psi_Q)$ is a solution to the corresponding contingent Isaacs equation.

We focus our attention in the second section to the playability property.

We shall characterize it by constructing *retroaction rules*

$$(x,y,v) \rightsquigarrow C(x,y;v) \;\;\&\;\; (x,y,u) \rightsquigarrow D(x,y;u)$$

which involve the contingent derivatives of the set-valued maps P and Q, with which we build the *regulation map* R mapping each $(x,y) \in K$ to the regulation set

$$R(x,y) \;\;=\;\; \{\, (u,v) \mid u \in C(x,y;v) \;\text{ and }\; v \in D(x,y;u) \,\}$$

The strategies belonging to $R(x,y)$ are called *playable.*

The Playability Theorem states that under technical assumptions, the playability property holds true if and only if

$$\forall\,(x,y)\in K,\ \ R(x,y)\ \neq\ \emptyset$$

and that playable solutions to the game are regulated by the *regulation law*:

$$\forall\, t\geq 0,\quad u(t)\ \in\ C(x(t),y(t);v(t))\ \ \&\ \ v(t)\ \in\ D(x(t),y(t);u(t))$$

We then deal in the third section with the construction of single-valued *playable feedbacks* $(\tilde{u},\tilde{v})$, such that the differential system

$$\begin{cases} x'(t) &= f(x(t),y(t),\tilde{u}(x(t),y(t))) \\ y'(t) &= g(x(t),y(t),\tilde{v}(x(t),y(t))) \end{cases}$$

has playable solutions for each initial state. By the Playability Theorem, they must be selections of the regulation map R in the sense that

$$\forall\,(x,y)\ \in\ K,\ \ (x,y)\mapsto(\tilde{u}(x,y),\tilde{v}(x,y))\in R(x,y)$$

We shall prove the existence of such continuous single-valued playable feedbacks, as well as more constructive, but discontinuous, playable feedbacks, such as the feedbacks associating the strategies of $R(x,y)$ with minimal norm (the playable slow feedbacks, as in Chapter 6). More generally, we shall show the existence of feedbacks (possibly set-valued) associating with any $(x,y)\in K$ the set of strategies $(u,v)\in R(x,y)$ which are solutions to a (static) optimization problem of the form:

$$(u,v)\in R(x,y)\ \ |\ \ \sigma(x,y;u,v)\leq \inf_{u',v'\in R(x,y)}\sigma(x,y;u',v')$$

or solutions to a noncooperative game of the form:

$$\forall(u',v')\in R(x,y),\ \ a(x,y;u,v')\leq a(x,y;u,v)\leq a(x,y;u',v)$$

In other words,

the players can implement playable solutions to the differential game by playing for each state $(x, y) \in K$ a static game on the strategies of the regulation subset $R(x, y)$.

We also consider in the fourth section the issue of finding *discriminating feedbacks* by providing for instance sufficient conditions implying that for all continuous feedback $\tilde{v}(x, y) \in V(x, y)$ played by Yvette, Xavier can find a feedback (continuous or of minimal norm) $\tilde{u}(x, y)$ such that the differential equation above has playable solutions for each initial state.

We address the question of whether Xavier has a leading role, i.e., the problem of constructing continuous *pure feedbacks* $\tilde{u}(x, y)$ which have the property of yielding playable solutions to the above differential game whatever the strategy played by Yvette.

The last section is devoted to closed loop decision rules, which *operate on the velocities of the strategies* (regarded as *decisions*) rather than on the controls. We need to provide first regulation maps which yield absolutely continuous strategies which are then almost everywhere differentiable. We distinguish among them the ones which guarantee or which allow victory or defeat of players adequately defined. The indicator functions of their graphs are characterized as solutions of contingent partial differential inequalities. We apply analogous selection procedures which yield closed loop decision rules allowing, say, a game to remain stable.

14.1 Contingent Isaacs Equations

Let us consider two players, Xavier and Yvette. Xavier acts on a state space X and Yvette on a state space Y. For doing so, they have access to some knowledge about the global state (x, y) of the system and are allowed to choose strategies u in a global state-dependent set $U(x, y)$ and v in a global state-dependent set $V(x, y)$ respectively.

But Xavier does not know Yvette's choice of controls v nor is Yvette assumed to know Xavier's controls.

Their actions on the state of the system are governed by the system of differential inclusions:

$$\left\{\begin{array}{ll} a) & \left\{\begin{array}{llcl} i) & x'(t) & = & f(x(t),y(t),u(t)) \\ ii) & u(t) & \in & U(x(t),y(t)) \end{array}\right. \\ \\ b) & \left\{\begin{array}{llcl} i) & y'(t) & = & g(x(t),y(t),v(t)) \\ ii) & v(t) & \in & V(x(t),y(t)) \end{array}\right. \end{array}\right. \qquad (14.1)$$

We now describe the influences (power relations) that Xavier exerts on Yvette and vice versa through *rules of the game.* They are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$ which are interpreted in the following way. When the state of Yvette is y, Xavier's choice is constrained to belong to $P(y)$. In a symmetric way, the set-valued map Q assigns to each state x the set $Q(x)$ of states y that Yvette can implement[1].

Hence, the *playability subset* of the game is the subset $K \subset X \times Y$ defined by:

$$K \; := \; \{\, (x,y) \in X \times Y \;\mid\; x \in P(y) \;\text{ and }\; y \in Q(x) \,\} \qquad (14.2)$$

Naturally, we must begin by providing sufficient conditions implying that the playability subset is nonempty. Since the playability subset is the subset of fixed-points (x, y) of the set-valued map $(x,y) \rightsquigarrow P(y) \times Q(x)$, we can use one of the many fixed point theorems to answer these types of questions[2].

From now on, *we shall assume that the playability subset associated with the rules P and Q is not empty.*

We can reformulate this differential game in a more compact form, by denoting

- by $z := (x,y) \in Z := X \times Y$ the global state,
- by $h(z,u,v) := (f(x,y,u,v), g(x,y,u,v))$ the values of the map $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ describing the dynamics of the game,

[1] We can extend the results below to the time-dependent case using the methods of Chapter 11.

[2] For instance, Kakutani's Fixed Point Theorem 3.7.7 furnishes such conditions: *Let $L \subset X$ and $M \subset Y$ be compact convex subsets and $P : M \rightsquigarrow L$ and $Q : L \rightsquigarrow M$ be closed maps with nonempty convex images. Then the playability subset is not empty.*

- by $L := \text{Graph}(P)$ Xavier's closed domain of definition,

- by $M := \text{Graph}(Q^{-1})$ Yvette's one and by $K := L \cap M$ the playability subset.

We shall also identify the set-valued maps U and V with their restrictions to L and M respectively by setting $U(z) := \emptyset$ whenever $z \notin L$ and $V(z) := \emptyset$ when $z \notin M$.

Hence the differential game can be written in the compact form

$$\left\{ \begin{array}{lrcl} i) & z'(t) & = & h(z(t), u(t), v(t)) \\ ii) & u(t) & \in & U(z(t)) \\ iii) & v(t) & \in & V(z(t)) \end{array} \right. \tag{14.3}$$

We denote by $\mathcal{S}(z_0)$ the subset of solutions $z(\cdot)$ to (14.3) starting at z_0.

Let us associate with this differential game the following four Hamilton-Jacobi-Isaacs partial differential equations:

$$\left\{ \begin{array}{lrcl} i) & \inf_{u \in U(z)} \inf_{v \in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \\ ii) & \sup_{u \in U(z)} \sup_{v \in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \\ iii) & \sup_{v \in V(z)} \inf_{u \in U(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \\ iv) & \inf_{u \in U(z)} \sup_{v \in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z, u, v) & = & 0 \end{array} \right.$$

We would like to study the properties of the solutions to these partial differential equations, and in particular, characterize the solutions which are indicators of closed subsets L. Hence we are led to weaken the concept of usual derivatives involved in these partial differential equations by replacing them by contingent epiderivatives[3].

[3]since any extended function $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ has contingent epiderivative, and in particular, indicators, for which we have the relation

$$D_{\uparrow}\Psi_L(z)(v) = \Psi_{T_L(z)}(v) := \left\{ \begin{array}{ll} 0 & \text{if } v \in T_L(z) \\ +\infty & \text{if } v \notin T_L(z) \end{array} \right.$$

Theorem 14.1.1 *Let us assume at least that $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ is continuous, has linear growth, and that the set-valued maps U and V are closed with linear growth.*

We assume that all extended functions Φ are nonnegative and contingently epidifferentiable[4] *and that their domains are contained in the intersection K of the domains of U and V.*

1 — If the values of the set-valued maps U and V are convex and if h is affine with respect to the controls, Φ is a solution to the contingent inequality

$$\inf_{u \in U(z)} \inf_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \leq 0 \tag{14.4}$$

if and only if

$$\forall\, z \in \mathrm{Dom}(\Phi),\ \exists\, z(\cdot) \in \mathcal{S}(z) \mid \forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$$

2 — Assume that h is uniformly Lipschitz with respect to x. Then Φ is a solution to the contingent inequality

$$\sup_{u \in U(z)} \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \leq 0 \tag{14.5}$$

if and only if

$$\forall\, z \in \mathrm{Dom}(\Phi),\ \forall\ \ z(\cdot) \in \mathcal{S}(z), \forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$$

3 — Assume that V is lower semicontinuous, that the values of U and V are convex and that h is affine with respect to u. Then Φ is a solution to the contingent inequality

$$\sup_{v \in V(z)} \inf_{u \in U(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \leq 0 \tag{14.6}$$

if and only if for any continuous closed-loop strategy $\tilde{v}(z) \in V(z)$ played by Yvette and any initial state $z \in \mathrm{Dom}(\Phi)$, there exists a solution $z(\cdot)$ to Xavier's control problem

$$\begin{cases} i) & z'(t) = h(z(t), u(t), \tilde{v}(z(t))) \\ ii) & u(t) \in U(z(t)) \end{cases}$$

[4]This means that for all $z \in \mathrm{Dom}(\Phi)$, $\forall\, v \in X$, $D_{\uparrow}\Phi(z)(v) > -\infty$ and that $D_{\uparrow}\Phi(z)(v) < \infty$ for at least one $v \in X$.

starting at z and satisfying $\forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$.

4 — *Assume that V is lower semicontinuous with convex values. Then* Φ *is a solution to the contingent inequality*

$$\inf_{u \in U(z)} \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \;\leq\; 0 \tag{14.7}$$

if and only if Xavier can play a closed-loop strategy $\tilde{u}(z) \in U(z)$ *such that, for any continuous closed-loop strategy* $\tilde{v}(z) \in V(z)$ *played by Yvette and for any initial state* $z \in \mathrm{Dom}(\Phi)$*, there exists a solution* $z(\cdot)$ *to*

$$z'(t) \;=\; h(z(t), \tilde{u}(z(t)), \tilde{v}(z(t))) \tag{14.8}$$

starting at z and satisfying for all $t \geq 0$, $\Phi(z(t)) \leq \Phi(z)$. *The converse is true if*

$$\begin{cases} B_{\Phi}(z) := \{\bar{u} \in U(z) \quad \textit{such that} \\ \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z,\bar{u},v)) \\ = \ \inf_{u \in U(z)} \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v))\} \end{cases}$$

is lower semicontinuous with closed convex values.

Proof

— The two first statements are translations of the theorems characterizing Lyapunov and global Lyapunov functions (see Chapter 9) applied to the differential inclusion $z'(t) \in H(z(t))$ where $H(z) := f(z, U(z), V(z))$.

— Let us prove the third one. Assume that Φ satisfies the stated property. Since V is lower semicontinuous with convex values, Michael's Theorem 6.5.7 implies that for all $z_0 \in \mathrm{Dom}(V)$ and $v_0 \in V(z_0)$, there exists a continuous selection $\tilde{v}(\cdot)$ of V such that $v(z_0) = v_0$. Then Φ enjoys the Lyapunov property for the set-valued map $H_{\tilde{v}}(z) := h(z, U(z), \tilde{v}(z))$, and thus, there exists $u_0 \in U(z_0)$ such that

$$D_{\uparrow}\Phi(z_0)(h(z_0, u_0, \tilde{v}(z_0))) \;\leq\; 0$$

Hence Φ is a solution to (14.6).

Conversely, assume that Φ is a solution to (14.6). Then for any closed-loop strategy $\tilde{v}$, the set-valued map $H_{\tilde{v}}$ satisfies the assumptions of the theorem characterizing Lyapunov functions, so that there exists a solution to the inclusion $z' \in H_{\tilde{v}}(z)$ for any initial state $z \in \mathrm{Dom}(\Phi)$ satisfying for all $t \geq 0$, $\Phi(z(t)) \leq \Phi(z)$.

— Consider finally the fourth statement. Assume that Xavier can find a continuous closed-loop strategy $\tilde{u}$ such that for any closed-loop strategy $\tilde{v}$, Φ enjoys the stated property. Since V is lower semicontinuous with convex values, Michael's Theorem implies that for all $z_0 \in \mathrm{Dom}(V)$ and $v_0 \in V(z_0)$, there exists a continuous selection $\tilde{v}(\cdot)$ of V such that $v(z_0) = v_0$. Since for any continuous closed-loop strategy $\tilde{v}(\cdot)$, Φ enjoys the Lyapunov property for the single-valued map $z \to h(z, \tilde{u}(z), \tilde{v}(z))$, we deduce that for all $z_0 \in \mathrm{Dom}(\Phi)$, there exists $u := \tilde{u}(z)$ such that for all $v \in V(z)$, $D_{\uparrow}\Phi(z)(h(x,u,v)) \leq 0$, so that Φ is a solution to (14.6).

Conversely, assume that the set-valued map B_Φ is lower semicontinuous with closed convex values. Hence Michael's Theorem implies that there exists a continuous selection $\tilde{u}$ of B_Φ. Then for any continuous closed-loop strategy $\tilde{v}(\cdot) \in V(\cdot)$, we deduce from (14.7) that Φ is a Lyapunov function for the single-valued map $z \to h(z, \tilde{u}(z), \tilde{v}(z))$, so that, for all $z \in \mathrm{Dom}(\Phi)$, there exists a solution $z(\cdot)$ to the system (14.8) satisfying for all $t \geq 0$, $\Phi(z(t)) \leq \Phi(z)$. □

Let L be a closed subset of the intersection K of the domains of U and V. The problem we investigate is of finding that one (or all) solution(s) $z(\cdot)$ of the game is (are) viable in L. There are several ways to achieve that purpose, according to the cooperative or noncooperative behavior of the players. Here, we shall investigate several of them.

Definition 14.1.2 *We shall say the a subset L enjoys:*

1 — the "playability property" if and only if

$$\forall\, z \in L,\ \exists\, z(\cdot) \in \mathcal{S}(z) \mid \forall\, t \geq 0,\ \ z(t) \in L$$

2 — the "winability property" if and only if

$$\forall\, z \in L,\ \forall\ \ z(\cdot) \in \mathcal{S}(z), \forall\, t \geq 0,\ \ z(t) \in L$$

3 — "Xavier's discriminating property" if and only if for any continuous closed-loop strategy $\tilde{v}(z) \in V(z)$ played by Yvette *and any initial state $z \in L$, there exists a solution $z(\cdot)$ to Xavier's control problem*

$$\begin{cases} i) & z'(t) = h(z(t), u(t), \tilde{v}(z(t))) \\ ii) & u(t) \in U(z(t)) \end{cases}$$

starting at z and viable in L.

4 — "Xavier's leading property" if and only if Xavier can play a closed-loop strategy $\tilde{u}(z) \in U(z)$ such that, for any continuous closed-loop strategy $\tilde{v}(z) \in V(z)$ played by Yvette and for any initial state $z \in L$, there exists a solution $z(\cdot)$ to (14.8) starting at z and viable in L.

We shall characterize these properties: for that purpose we associate with L the following set-valued maps:

— The regulation map R_L defined by

$$\forall\, z \in L,\ R_L(z) := \{\ (u, v) \in U(z) \times V(z) \mid h(z, u, v) \in T_L(z)\ \}$$

— Xavier's discriminating map A_L defined by

$$\forall\, z \in L,\ A_L(z, v) := \{\ u \in U(z) \mid (u, v) \in R_L(z)\ \}$$

— Xavier's leading map B_L defined by

$$\forall\, z \in L,\ B_L(z) := \bigcap_{v \in V(z)} A_L(z, v)$$

Definition 14.1.3 *We shall say that*

— *L is a playability domain if $\forall\, z \in L,\ R_L(z) \neq \emptyset$*
— *L is a winability domain if $\forall\, z \in L,\ R_L(z) := U(z) \times V(z)$*
— *L is a Xavier's discriminating domain if*

$$\forall\, z \in L, \forall\, v \in V(z),\ A_L(z, v) \neq \emptyset \tag{14.9}$$

— *L is a Xavier's leading domain if $\forall\, z \in L,\ B_L(z) \neq \emptyset$*

We begin by translating these properties in terms of contingent Isaacs' equations:

Proposition 14.1.4 *Let us assume that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps* U, V *are closed with linear growth.*

— L *is a playability domain if and only if* Ψ_L *is a solution to (14.4)*

— L *is a winability domain if and only if* Ψ_L *is a solution to (14.5)*

— L *is a discriminating domain for Xavier if and only if* Ψ_L *is a solution to (14.6)*

— L *is a leading domain for Xavier if and only if* Ψ_L *is a solution to (14.7)*

Therefore, Theorem 14.1.1 implies the following characterization of these domains:

Corollary 14.1.5 *Let us assume at least that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps are closed with linear growth.*

1 — If the values of the set-valued maps U *and* V *are convex and if* h *is affine with respect to the controls, then* L *enjoys the playability property if and only if it is a playability domain.*

2 — Assume that h *is uniformly Lipschitz with respect to* x*. Then* L *enjoys the winability property if and only if it is a winability domain.*

3 — Assume that V *is lower semicontinuous, that the values of* U *and* V *are convex and that* h *is affine with respect to* u*. Then* L *enjoys Xavier's discriminating property if and only if it is a discriminating domain for Xavier.*

4 — Assume that V *is lower semicontinuous with convex values. If* L *enjoys Xavier's leading property, then it is a leading domain for him. The converse is true if* B_L *is lower semicontinuous with closed convex values.*

The existence theorems of the viability and invariance kernels imply the following consequence:

Proposition 14.1.6 *Let us assume that* $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ *is continuous, has linear growth, and that the set-valued maps are closed with linear growth.*

1 — If the values of the set-valued maps U and V are convex and if h is affine with respect to the controls, then there exists a largest closed playability domain contained in L, whose indicator is the smallest lower semicontinuous solution to (14.4) larger than or equal to the indicator Ψ_L of L.

2 — Assume that h is uniformly Lipschitz with respect to x. Then there exists a largest closed winability domain contained in L, whose indicator is the smallest lower semicontinuous solution to (14.5) larger than or equal to the indicator Ψ_L of L.

14.2 Playable Differential Games

We now proceed with the case of the game described by (14.1), where the playability domain is defined from rules P and Q by

$$K \;:=\; \{\, (x,y) \in X \times Y \;\mid\; x \in P(y) \;\text{ and }\; y \in Q(x) \,\}$$

enjoys the *playability property*, which becomes in this case: for any initial state $(x_0, y_0) \in K$, there exists a solution to the differential game (14.1) which is *playable* in the sense that

$$\forall\, t \geq 0, \quad x(t) \;\in\; P(y(t)) \;\;\&\;\; y(t) \;\in\; Q(x(t))$$

We now need to define *playable rules*. For that purpose, we associate with the rules P and Q acting on the states *retroaction rules* C and D acting on the strategies defined in the following way:

Definition 14.2.1 *Xavier's* retroaction rule *is the set-valued map C defined by*

$$\begin{cases} C(x,y;v) \\ = \{\, u \in U(x,y) \mid f(x,y,u) \in DP(y,x)(g(x,y,v)) \,\} \end{cases}$$

and Yvette's retroaction rule *is the set-valued map D defined by*

$$\begin{cases} D(x,y;u) \\ = \{\, v \in V(x,y) \mid g(x,y,v) \in DQ(x,y)(f(x,y,u)) \,\} \end{cases}$$

We associate with them the regulation map *R defined by*

$$R(x,y) \;=\; \{\, (u,v) \mid u \in C(x,y;v) \;\;\&\;\; v \in D(x,y;u) \,\} \qquad (14.10)$$

The subset $R(x,y)$ is called the regulation set *and its elements are called* playable controls.

In other words, we have associated with each state (x,y) of the playability domain a static game on the strategies defined by the retroaction rules. This new game on strategies is playable if the subset $R(x,y)$ is nonempty. This property deserves a definition.

Definition 14.2.2 *We shall say that P and Q are* playable rules *if their graphs are closed, the playability domain K defined by (14.2) is nonempty and if for all pairs $(x,y) \in K$, the values $R(x,y)$ of the regulation map are nonempty.*

We still need a definition of transversality of the rules before stating an adequate characterization of playability.

Definition 14.2.3 *We shall say that the rules P and Q are* transversal *if for all $(x,y) \in K$ and for all perturbations $(e,f) \in X \times Y$, there exists (u,v) satisfying*

$$\begin{cases} i) & u \in DP(y,x)(v) + e \\ \\ ii) & v \in DQ(x,y)(u) + f \end{cases}$$

We shall say that they are strongly transversal *if*

$$\begin{cases} \forall (x,y) \in K,\ \exists c > 0, \delta > 0 \text{ such that } \forall (x',y') \in B_K((x,y),\delta), \\ \forall\, (e,f) \in X \times Y,\ \text{there exist solutions } (u,v) \text{ to the system} \\ \begin{cases} i) & u \in DP(y',x')(v) + e \\ ii) & v \in DQ(x',y')(u) + f \end{cases} \\ \text{satisfying} \\ \max(\|u\|, \|v\|) \ \leq \ \max(\|e\|, \|f\|) \end{cases}$$

We also assume that the rules are *sleek* (See Definition 5.1.4).

We shall now derive from Corollary 14.1.5 a characterization of the playability property.

Theorem 14.2.4 (Playability Theorem) *Let us assume that the functions f and g are continuous, affine with respect to the strategies and have a linear growth, that the feedback maps U and V are upper* *semicontinuous with compact convex images and have a linear growth and that the rules P and Q are sleek and transversal.*

Then the rules P and Q enjoy the playability property if and only if they are playable. *Furthermore, the strategies $u(\cdot)$ and $v(\cdot)$ which provide playable solutions obey the following* regulation law*: for every $t \geq 0$,*

$$u(t) \in C(x(t), y(t); v(t)) \ \& \ v(t) \in D(x(t), y(t); u(t)) \quad (14.11)$$

Proof — We apply Corollary 14.1.5 and prove that the playability subset of the differential game is a playability domain, i.e., that for any global state $(x, y) \in K$ of the system, there exist strategies u and v such that the pair $(f(x, y, u), g(x, y, v))$ belongs to the contingent cone $T_K(x, y)$.

Since K is the intersection of the graphs of Q and P^{-1}, we need to use a sufficient condition for the contingent cone to an intersection to be equal to the intersection of the contingent cones.

The graphs of Q and P^{-1} are sleek because the rules of the game are supposed to be so. Furthermore,

$$T_{\mathrm{Graph}(P^{-1})}(x, y) - T_{\mathrm{Graph}(Q)}(x, y) = X \times Y$$

because the maps P and Q are transversal: For any $(e, f) \in X \times Y$, there exists (u, v) such that $(u, v - f)$ belongs to the graph of Q and $(u + e, v)$ to the graph of P^{-1}, i.e., that $(e, f) = (u + e, v) - (u, v - f)$. We deduce that

$$\begin{cases} T_K(x, y) & = T_{\mathrm{Graph}(P^{-1})}(x, y) \cap T_{\mathrm{Graph}(Q)}(x, y) \\ & = \mathrm{Graph}(DP(y, x))^{-1} \cap \mathrm{Graph}(DQ(x, y)) \end{cases}$$

Therefore, K is a viability domain if and only if the regulation map R has nonempty values, i.e., if and only if the rules of the game are playable. □

The regulation law (14.11) describes how the players must behave to keep the state of the system playable. A first question arises: Do the domains of the set-valued maps

$$\begin{cases} i) & C(x, y) : v \rightsquigarrow C(x, y; v) \\ ii) & D(x, y) : u \rightsquigarrow D(x, y; u) \end{cases}$$

coincide with $U(x,y)$ and $V(x,y)$ respectively?

Proposition 14.2.5 *We posit the assumptions of Theorem 14.2.4. Let us assume that for all* $(x,y) \in K$,

$$\begin{cases} i) & \mathrm{Dom}(C(x,y)) = V(x,y) \\ ii) & \mathrm{Dom}(D(x,y)) = U(x,y) \end{cases} \tag{14.12}$$

Then the rules are playable.

Proof — We deduce it from Kakutani's Fixed Point Theorem, since the set $R(x,y)$ is the set of fixed points of the set-valued map

$$(u,v) \rightsquigarrow C(x,y;v) \times D(x,y;u)$$

defined on the convex compact subset $U(x,y) \times V(x,y)$ to itself. This set-valued map has non empty values by assumption, which are moreover convex since the rules P and Q being sleek, the graphs of the contingent derivatives $DP(x,y)$ and $DQ(x,y)$ are convex. They are also closed. This implies that the graph of $(u,v) \rightsquigarrow C(x,y;v) \times D(x,y;u)$ is closed. Hence we can apply Kakutani's Fixed Point Theorem[5]. □

14.3 Feedback Solutions

When we know the regulation law (14.11), *playing the game* amounts to choosing for each pair $(x,y) \in K$ playable strategies (u,v) in the regulation set $R(x,y)$ through *playable feedbacks*.

We begin by looking for single-valued playable feedbacks $(\tilde{u},\tilde{v})$, which are selections of the regulation map R in the sense that

$$\forall\, (x,y) \in K, \;\; (x,y) \mapsto (\tilde{u}(x,y), \tilde{v}(x,y)) \in R(x,y)$$

[5] We can also use Theorem 3.7.11 and replace condition (14.12) by a sufficient condition of the form:

$$\begin{cases} \forall\, (u,v) \in U(x,y) \times V(x,y), \\ 0 \in (f(x,y;u), g(x,y;v)) - T_K(x,y) - A(T_{U(x,y)}(u) \times T_{V(x,y)}(v)) \end{cases}$$

where A is a linear operator from $Z_X \times Z_Y$ to $X \times Y$. This provides many sufficient conditions for playability.

or, equivalently, solutions to the system

$$\forall\ (x,y) \in K, \quad \begin{cases} \tilde{u}(x,y) & \in & C(x,y;\tilde{v}(x,y)) \\ \text{and} & & \\ \tilde{v}(x,y) & \in & D(x,y;\tilde{u}(x,y)) \end{cases}$$

For instance, continuous selections of the set-valued map R provide continuous playable feedbacks $(\tilde{u}, \tilde{v})$ such that the system of differential equations

$$\begin{cases} x'(t) & = & f(x(t), y(t), \tilde{u}(x(t), y(t))) \\ y'(t) & = & g(x(t), y(t), \tilde{v}(x(t), y(t))) \end{cases} \tag{14.13}$$

does have solutions which are playable.

Michael's Continuous Selection Theorem, as well as other selection procedures we shall use, require the lower semicontinuity of the regulation map R.

Our next objective is then to provide criteria under which the regulation map is lower semicontinuous. For that purpose, we need to strengthen the concept of playable rules.

Definition 14.3.1 *We associate with any perturbation* (e, f) *the* retroaction rules $C_{(e,f)}$ *and* $D_{(e,f)}$ *defined by:*

$$\begin{cases} C_{(e,f)}(x,y;v) \\ = \{\, u \in U(x,y) \mid f(x,y;u) \in DP(y,x)(g(x,y,v) - f) + e \,\} \end{cases}$$

and

$$\begin{cases} D_{(e,f)}(x,y;u) \\ = \{\, v \in V(x,y) \mid g(x,y,v) \in DQ(x,y)(f(x,y;u) - e) + f \,\} \end{cases}$$

and regulation map $R_{(e,f)}$ *defined by*

$$R_{(e,f)}(x,y) \quad = \quad \{\, (u,v) \mid u \in C_{(e,f)}(x,y;v) \ \ \& \ \ v \in D_{(e,f)}(x,y;u) \,\}$$

We shall say that the rules P *and* Q *are* strongly playable *if*

$$\begin{cases} \forall (x,y) \in K,\ \exists \gamma > 0, \delta > 0 \ \text{ such that } \forall (x',y') \in B_K((x,y),\delta), \\ \forall\ (e,f) \in \gamma B,\ \ R_{(e,f)}(x',y') \ \neq \ \emptyset \end{cases}$$

Theorem 14.3.2 *Let us assume that the functions f and g are continuous, affine with respect to the strategies and have a linear growth, that the feedback maps U and V are upper semicontinuous with compact convex images and have a linear growth and that the rules P and Q are sleek, strongly transversal and strongly playable.*

Then the regulation map R is lower semicontinuous *with closed convex images.*

Consequently, there exist continuous playable feedbacks $(\tilde{u}, \tilde{v})$.

Proof — We use the Lower Semicontinuity Criterion of the intersection and the inverse image of lower semicontinuous set-valued maps (see Theorem 6.3.1).

First, we need to prove that the set-valued map

$$(x,y) \rightsquigarrow T_K(x,y) \;:=\; \text{Graph}(DP(y,x)^{-1}) \cap \text{Graph}(DQ(x,y))$$

is lower semicontinuous. But this follows from the strong transversality of the rules P and Q and the Lower Semicontinuity Criterion.

We observe that $U \times V$ being upper semicontinuous with compact values, it maps a neighborhood of each point to a compact set. Since we can write

$$R(x,y) = \{(u,v) \in (U \times V)(x,y) \mid (f(x,y;u), g(x,y;v)) \in T_K(x,y)\}$$

and since both $U \times V$ and T are lower semicontinuous with convex images, strong playability of the retroaction rules implies that the regulation map R is lower semicontinuous. □

Unfortunately, the proof of Michaels's Continuous Selection Theorem is not constructive. We would rather trade the continuity of the playable control with some explicit and computable property, such as $u^\circ(x,y)$ being the element of minimal norm in $R(x,y)$, or other properties. Hence we need to prove the existence of a solution to the differential equation (14.13) for such discontinuous feedbacks.

Theorem 6.6.6 on the regulation of control systems becomes

Theorem 14.3.3 *We posit the assumptions of Theorem 14.2.4 and we suppose that K is a playability domain.*

Let S_R be a selection procedure with convex images of the regulation map R. Then, for any initial state $(x_0, y_0) \in K$, there exists a playable solution starting at (x_0, y_0) to the differential inclusion

$$\begin{cases} i) & x'(t) = f(x(t), y(t); u(t)) \\ ii) & y'(t) = g(x(t), y(t); v(t)) \\ iii) & \text{for almost all } t, \ (u(t), v(t)) \in S(R(x(t), y(t))) \end{cases}$$

In particular, if for every (x, y) the intersection

$$S_R(x, y) \cap R(x, y) := (\tilde{u}(x, y)), \tilde{v}(x, y))$$

is single-valued, then the strategies $(x, y) \mapsto (\tilde{u}(x, y)), \tilde{v}(x, y))$ are single-valued playable feedback controls.

We can now multiply the possible corollaries, by supplying several instances of selection procedures of set-valued maps.

We begin by cooperative procedures, where the players agree on criteria $\sigma(x, y; \cdot, \cdot)$ for selecting strategies in the regulation sets $R(x, y)$.

Example— COOPERATIVE BEHAVIOR

Proposition 14.3.4 *We posit the assumptions of Theorem 14.3.2. Let σ be continuous on* Graph(R) *and convex with respect to the pair (u, v). Then, for any initial state $(x_0, y_0) \in K$, there exist a playable solution starting at (x_0, y_0) and playable strategies to the differential game (14.1) which are regulated by:*

$$\begin{cases} \text{for almost all } t \geq 0, \ (u(t), v(t)) \in R(x(t), y(t)) \text{ and} \\ \sigma(x(t), y(t); u(t), v(t)) = \inf_{u', v' \in R(x(t), y(t))} \sigma(x(t), y(t); u', v') \end{cases}$$

In particular, the game can be played by the slow feedbacks *of minimal norm:*

$$\begin{cases} (u^\circ(x, y), v^\circ(x, y)) \in R(x, y) \\ \|u^\circ(x, y)\|^2 + \|v^\circ(x, y)\|^2) = \min_{(u,v) \in R(x,y)}(\|u\|^2 + \|v\|^2) \end{cases}$$

Proof — We introduce the set-valued map S_R defined by:

$$S_R(x,y) := \{(u,v) \mid \sigma(x,y;u,v) \leq \inf_{(u',v') \in R(x,y)} \sigma(x,y;u',v')\}$$

which is a convex-valued *selection procedure* of R since R is lower semicontinuous (see Theorem 6.5.3). We then apply Theorem 14.3.3. We observe that when we take

$$\sigma(x,y;u,v) \ := \ \|u\|^2 + \|v\|^2$$

the selection procedure yields the elements of minimal norm. □

Example— NONCOOPERATIVE BEHAVIOR We can also choose strategies in the regulation sets $R(x,y)$ in a non cooperative way, as saddle points of a function $a(x,y;\cdot,\cdot)$.

Proposition 14.3.5 *We posit the assumptions of Theorem 14.3.2 and we suppose that K is a playability domain. Let us assume that $a : X \times Y \times U \times V \to \mathbf{R}$ satisfies*

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall (x,y,v) \in X \times V, \ \ u \mapsto a(x,y;u,v) \text{ is convex} \\ iii) & \forall (x,y;u) \in X \times U, \ \ v \mapsto a(x,y;u,v) \text{ is concave} \end{cases}$$

Then, for any initial state $(x_0, y_0) \in K$, there exist a playable solution starting at (x_0, y_0) and playable strategies to the differential game (14.1) which are regulated by: for almost all $t \geq 0$,

$$\begin{cases} i) \quad (u(t), v(t)) \ \in \ R(x(t), y(t)) \\ \\ ii) \quad \forall \ (u',v') \in R(x(t),y(t)), \\ \\ \quad\quad a(x(t),y(t);u(t),v') \leq a(x(t),y(t);u(t),v(t)) \\ \\ \quad\quad \leq a(x(t),y(t);u',v(t)) \end{cases}$$

Proof — The set-valued map S_R associating with any $(x,y) \in K$ the subset

$$\begin{array}{l} S_R(x,y) \ := \{(u,v) \ \text{such that} \\ \forall (u',v') \in R(x,y), \ \ a(x,u,v') \leq a(x,u',v)\} \end{array}$$

is a convex-valued selection procedure of R. The associated selection map $S(R(\cdot))$ associates with any $(x, y) \in X \times Y$ the subset

$$\begin{array}{l} S(R(x,y)) \;:=\; \{\, (u,v) \in R(x,y) \text{ such that} \\ \forall (u',v') \in R(x,y),\;\; a(x,y;u,v') \leq a(x,y;u',v) \,\} \end{array}$$

of saddle-points of $a(x, y; \cdot, \cdot)$ in $R(x, y)$. We then apply Theorem 14.3.3. □

14.4 Discriminating and Leading Feedbacks

We now address the question of finding criteria for the playability domain K to be Xavier's discriminating domain, and for finding Xavier's feedback strategies which are selections of the set-valued map $(x, y, v) \rightsquigarrow A(x, y, v) \subset U(x, y)$ defined by

$$A(x,y;v) \;:=\; \{\, u \in U(x,y) \mid (u,v) \in R(x,y) \,\}$$

Such feedbacks are called *discriminating feedbacks.* If we assume that Xavier has access to the strategies chosen by Yvette, he can keep the states of the system playable by "playing" a discriminating control whatever the choice of Yvette through a discriminating feedback. Then, we shall investigate whether we can find (possibly, single-valued) selections of such a set-valued map A, and for that, provide sufficient conditions for A to be lower semicontinuous.

We first observe that A can be written in the form

$$A(x,y;v) \;:=\; C(x,y;v) \cap (D(x,y))^{-1}(v)$$

The first assumption we must make for obtaining discriminating feedbacks for Xavier is that the domain of the set-valued maps $A(x, y; \cdot))$ are not empty. i.e., that

$$\left\{ \begin{array}{l} \forall\, v \in V(x,y),\;\; \exists\, u \in U(x,y) \;\; \text{such that} \\ f(x,y;u) \in DP(y,x)(g(x,y;v)) \cap DQ(x,y)^{-1}(g(x,y;v)) \end{array} \right.$$

We shall actually strengthen it a bit to get the lower semicontinuity of A, by assuming that

$$\left\{\begin{array}{l} \forall\,(x,y)\in K,\ \forall\, v\in V(x,y),\ \exists\,\delta>0,\ \exists\,\gamma>0\ \text{ such that} \\ \forall\,(x',y')\in B_K(x,y,\delta),\ \forall\, v'\in B(v,\delta)\cap V(x',y'), \forall\,\|e_i\|\leq\gamma \\ (i=1,2),\ \exists\, u\in U(x',y')\ \text{ such that }\ f(x',y';u)\ \text{belongs to} \\ (DP(y',x')(g(x',y';v'))-e_1)\cap(DQ(x',y')^{-1}(g(x',y';v'))-e_2) \end{array}\right. \tag{14.14}$$

Proposition 14.4.1 *We posit the assumptions of Theorem 14.3.2, where we replace strong playability by assumption (14.14), and we assume further that the norms of the closed convex processes $DP(y,x)$ and $DQ(x,y)^{-1}$ are bounded. Then the set-valued map A is lower semicontinuous.*

Proof — First, we have to prove that C is lower semicontinuous, and, for that purpose, that $(x,y,w)\rightsquigarrow DP(y,x)(w)$ is lower semicontinuous.

By Theorem 2.5.7, we know that it is sufficient to prove that

$$(x,y)\rightsquigarrow \text{Graph}(DP(y,x))\ \text{ is lower semicontinuous}$$

and that

$$\|DP(y,x)\| \ := \ \sup_{\|w\|\leq 1}\ \inf_{u\in DP(y,x)(w)}\|u\| \ < \ +\infty$$

This is the case because P is assumed to be sleek and because we have assumed that the norms of the derivatives are bounded.

Therefore, the set-valued map

$$(x,y,v)\rightsquigarrow DP(y,x)(g(x,y;v))$$

is also lower semicontinuous.

The Lower Semicontinuity Criterion and assumption (14.14) imply that $(x,y,v)\rightsquigarrow C(x,y;v)$ is lower semicontinuous.

The same proof shows that the map $(x,y,v)\rightsquigarrow DQ(x,y)^{-1}(v)$ is also lower semicontinuous. Since A is the intersection of these two set-valued maps, we apply again the Lower Semicontinuity Criterion to deduce that A is lower semicontinuous, which is possible thanks to assumption (14.14). □

Theorem 14.4.2 *We posit the assumptions of Theorem 14.2.4. For any continuous feedback control $(x,y) \mapsto \tilde{v}(x,y)$ played by Yvette, there exits a continuous single-valued feedback $\tilde{u}(x,y)$ played by Xavier such that the differential equation (14.13) has playable solutions for any initial state $(x_0,y_0) \in K$.*

More generally, let S_A be a convex-valued selection procedure of the set-valued map A. Then, for any continuous feedback control $(x,y) \mapsto \tilde{v}(x,y)$ played by Yvette, for any initial state $(x_0,y_0) \in K$, there exists a playable solution starting at (x_0,y_0) to the differential game

$$\begin{cases} i) & x'(t) = f(x(t),y(t);u(t)) \\ ii) & y'(t) = g(x(t),y(t);\tilde{v}(x(t),y(t))) \\ iii) & u(t) \in S(A(x(t),y(t);\tilde{v}(x(t),y(t)))) \end{cases}$$

where

$$S(A(x,y;\tilde{v}(x,y))) \;:=\; S_A(x,y;\tilde{v}(x,y)) \cap A(x,y;\tilde{v}(x,y))$$

In particular, if the selection procedure yields single-valued selections, then the control $\tilde{u}(x,y)$ defined by

$$\tilde{u}_{\tilde{v}}(x,y) \;:=\; S(A(x,y;\tilde{v}(x,y)))$$

is a single-valued feedback control.

This is the case, for instance, when we posit the assumptions of Proposition 14.4.1 and when Xavier plays the feedback control $u^\circ_{\tilde{v}}(x,y)$ of minimal norm in the set $A(x,y;\tilde{v}(x,y))$. In this case, there exists also a continuous control $\tilde{u}(x,y) \in A(x,y;\tilde{v}(x,y))$

Proof — Whenever Yvette plays a continuous feedback $\tilde{v}(x,y)$, K remains a playability domain for the system

$$\begin{cases} i) & x'(t) = f(x(t),y(t);u(t)) \\ ii) & y'(t) = g(x(t),y(t);\tilde{v}(x(t),y(t))) \\ iii) & u(t) \in S_A(x(t),y(t);\tilde{v}(x(t),y(t))) \end{cases}$$

So playable solutions to this system satisfy also the condition

$$u(t) \;\in\; A(x(t),y(t);\tilde{v}(x(t),y(t)))$$

so that actually,

$$u(t) \in S(A(x(t), y(t); \tilde{v}(x(t), y(t))))$$

When the set-valued map $(x, y) \rightsquigarrow A(x, y; \tilde{v}(x, y))$ is lower semicontinuous, it contains continuous selections $\tilde{u}(x, y)$ which therefore yield playable selections.

We can also use more constructive selection procedures of the set-valued map $(x, y) \rightsquigarrow A(x, y; \tilde{v}(x, y))$ with convex values and deduce that Xavier can implement playable solutions by playing strategies $u(t)$ in the selection $S(A(x(t), y(t); \tilde{v}(x(t), y(t))))$. □

A much better situation for Xavier occurs when he can find feedback strategies $\tilde{u}$ which are selections of the set-valued map B defined by

$$B(x, y) := \bigcap_{v \in V(x,y)} A(x, y; v)$$

In other words, such a feedback allows him to implement playable solutions whatever the control $v \in V(x, y)$ chosen by Yvette, since in this case the pair (u, v) belongs to the regulation set $R(x, y)$ for any v. Such feedbacks are called *pure feedbacks*.

In order to obtain continuous single-valued feedbacks, we need to prove the lower semicontinuity of the set-valued map B, which is an infinite intersection of lower semicontinuous set-valued maps.

Theorem 14.4.3 *We posit the assumptions of Proposition 14.4.1. We assume further that there exist positive constants δ and γ such that for all $(x', y') \in B_K((x, y), \delta)$, we have*

$$\begin{cases} \forall\, v \in V(x', y'),\ \forall\, e_v^i \in \gamma B,\ (i = 1, 2),\ \exists\, u \in U(x', y') \quad \text{such that} \\ \\ f(x', y'; u) \in DP(y', x'; v) + e_v^1 \\ \text{and} \\ g(x', y'; v) \in DQ(x', y'; u) + e_v^2 \end{cases} \tag{14.15}$$

Then the set-valued map B is lower semicontinuous and there exist continuous single-valued pure feedback strategies for Xavier.

Proof — We observe that V is upper semicontinuous with compact values, that A is lower semicontinuous and has its images in a fixed compact set, and that assumption (14.15) implies obviously that there exist positive constants δ and γ such that for all $(x',y') \in B_K((x,y),\delta)$, we have

$$cB \cap \bigcap_{y\in H(x'),z\in\gamma B} (F(x',y)-z) \neq \emptyset$$

This theorem follows then from Theorem 6.3.3 on the lower semicontinuity of an infinite intersection of lower semicontinuous set-valued maps. □

14.5 Closed Loop Decision Rules

Actually, although differential games can be played through retroaction rules, there are many games where players *act on the velocities of the strategies* regarded as *decisions* of players.

This leads us to introduce the following definition: We shall call *decisions* the derivatives of the strategies.

Then, in order to deal with decisions defined in such a sense, we must now assume that players use open-loop strategies $u(\cdot)$ and $v(\cdot)$ which are *absolutely continuous* and obey a growth condition of the type[6]

$$\begin{cases} i) & \|u'(t)\| \leq \rho(\|u(t)\|+1) \\ ii) & \|v'(t)\| \leq \sigma(\|v(t)\|+1) \end{cases} \quad (14.16)$$

We shall refer to them as "smooth open-loop controls", the non-negative parameters[7] ρ and σ being fixed once and for all. We denote by $\mathcal{K}$ the subset

$$\begin{cases} (z,u,v) \in \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \text{ such that} \\ u \in U(z) \;\&\; v \in V(z) \end{cases}$$

Instead of finding largest playability or winability domains in the state space, we shall look for analogous concepts in the state-strategy

[6] one can replace $\rho(\|u\|+1)$ by any continuous function $\varphi(u)$ with linear growth.
[7] or any other linear growth condition $\varphi(\cdot)$ or $\psi(\cdot)$.

space. We shall determine set-valued maps which allow players to win in the sense that either property

$$\forall\, t \geq 0, \;\; u(t) \;\in\; U(z(t)) \tag{14.17}$$

or property

$$\forall\, t \geq 0, \;\; v(t) \;\in\; V(z(t)) \tag{14.18}$$

or both hold. Roughly speaking, Xavier may win as long as his opponent allows him to choose at each instant $t \geq 0$ strategies $u(t)$ in the subset $U(z(t))$, and must lose if for any choice of open-loop controls, there exists a time $T > 0$ such that $u(T) \notin U(z(T))$.

Definition 14.5.1 *Let (u_0, v_0, z_0) be an initial situation such that initial strategies $u_0 \in U(z_0)$ and $v_0 \in V(z_0)$ of the two players are consistent with the initial state z_0.*

We shall say that

— *Xavier* must win *if and only if for all smooth open-loop strategies $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0, there exists a solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0 such that (14.17) is satisfied.*

— *Xavier* may win *if and only if there exist smooth open-loop strategies $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and a solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0 such that (14.17) is satisfied.*

— *Xavier* must lose *if and only if for any smooth open-loop strategy $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0, there exists a time $T > 0$ such that*

$$u(T) \;\notin\; U(z(T))$$

— *The initial situation is* stable *if and only if there exist open-loop strategies $u(\cdot)$ and $v(\cdot)$ starting at u_0 and v_0 and a solution $z(\cdot)$ to (14.3) and (14.16) starting at z_0 satisfying* both *relations (14.17) and (14.18).*

Naturally, if both Xavier and Yvette must win, then both relations (14.17) and (14.18) are satisfied. This is not necessarily the case when both Xavier and Yvette may win, and this is the reason why we need to introduce the concept of stability.

Table 14.1: The 10 areas of the domain of the differential game

$(z_0, u_0, v_0) \in$	Graph(S_U)	Graph(R_U)	$\mathcal{K}\backslash$Graph(R_U)
Graph(S_V)	Xavier must win Yvette must win	Xavier may win Yvette must win	Xavier must lose Yvette must win
Graph(R_V)	Xavier must win Yvette may win	? ? ? ? STABILITY ? ? ? ?	Xavier must lose Yvette may win
$\mathcal{K}\backslash$Graph(R_V)	Xavier must win Yvette must lose	Xavier may win Yvette must lose	Xavier must lose Yvette must lose

Theorem 14.5.2 *Let us assume that h is continuous with linear growth and that the graphs of U and V are closed. Let the growth rates ρ and σ be fixed.*

There exist five (possibly empty) closed set-valued feedback maps from $\mathbf{R}^n$ to $\mathbf{R}^p \times \mathbf{R}^q$ having the following properties:

— $R_U \subset U$ *is such that whenever* $(u_0, v_0) \in R_U(z_0)$, *Xavier* may win *and that whenever* $(u_0, v_0) \notin R_U(z_0)$, *Xavier* must lose

— *If h is Lipschitz,* $S_U \subset R_U$ *is the largest closed set-valued map such that whenever* $(u_0, v_0) \in S_U(z_0)$, *Xavier must win.*

— $S_V \subset R_V \subset V$, *which have analogous properties.*

— $R_{UV} \subset R_U \cap R_V$ *is the largest closed set-valued map such that any initial situation satisfying* $(u_0, v_0) \in R_{UV}(z_0)$ *is stable.*

Knowing these five set-valued feedback maps, we can split the domain $\mathcal{K}$ of initial situations in ten areas which describe the behavior of the differential game according to the position of the initial situation.

In particular, the complement of the graph of R_{UV} in the intersection of the graphs of R_U and R_V is the instability region, where either Xavier or Yvette may win, but not both together.

The problem is to characterize these five set-valued maps, the existence of which is now guaranteed, by solving the "contingent extension" of the partial differential equation[8]

[8]If Φ is a solution to this partial differential equation, one can check that

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)-\rho(\|u\|+1)\left\|\frac{\partial\Phi}{\partial u}\right\|-\sigma(\|v\|+1)\left\|\frac{\partial\Phi}{\partial v}\right\|\leq 0 \quad (14.19)$$

which can be written in the following way:

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)+\inf_{\|u'\|\leq\rho(\|u\|+1)}\frac{\partial\Phi}{\partial u}\cdot u'+\inf_{\|v'\|\leq\sigma(\|v\|+1)}\frac{\partial\Phi}{\partial v}\cdot v'\leq 0$$

We shall also introduce the partial differential equation[9]

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)+\rho(\|u\|+1)\left\|\frac{\partial\Phi}{\partial u}\right\|+\sigma(\|v\|+1)\left\|\frac{\partial\Phi}{\partial v}\right\|\leq 0 \quad (14.20)$$

which can be written in the following way:

$$\frac{\partial\Phi}{\partial z}\cdot h(z,u,v)+\sup_{\|u'\|\leq\rho(\|u\|+1)}\frac{\partial\Phi}{\partial u}\cdot u'+\sup_{\|v'\|\leq\sigma(\|v\|+1)}\frac{\partial\Phi}{\partial v}\cdot v'\leq 0$$

The link between the feedback maps and the solutions to the solutions to these partial differential equations is provided by the indicators of the graphs: we associate with the set-valued maps S_U, R_U and R_{UV} the functions Φ_U, Ψ_U and Ψ_{UV} from $\mathbf{R}^n\times\mathbf{R}^p\times\mathbf{R}^q$ to $\mathbf{R}_+\cup\{+\infty\}$ defined by

for any initial situation $(z_0,u_0,v_0)\in\mathrm{Dom}(\Phi)$, there exists a smooth solution $(z(\cdot),u(\cdot),v(\cdot))$ such that

$$t\rightarrow\Phi(z(t),u(t),v(t)) \text{ is nonincreasing}$$

This property remains true for the solutions to the contingent partial differential equation (14.22).

[9]We can check that if h is Lipschitz and Φ is a solution to this partial differential equation, for any initial situation $(z_0,u_0,v_0)\in\mathrm{Dom}(\Phi)$, any smooth solution $(z(\cdot),u(\cdot),v(\cdot))$ satisfies

$$t\rightarrow\Phi(z(t),u(t),v(t)) \text{ is non increasing}$$

This property remains true for the solutions to the contingent partial differential equation (14.23).

$$\begin{cases} i) & \Phi_U(z,u,v) & := \begin{cases} 0 & \text{if } (u,v) \in S_U(z) \\ +\infty & \text{if } (u,v) \notin S_U(z) \end{cases} \\ ii) & \Psi_U(z,u,v) & := \begin{cases} 0 & \text{if } (u,v) \in R_U(z) \\ +\infty & \text{if } (u,v) \notin R_U(z) \end{cases} \\ iii) & \Psi_{UV}(z,u,v) & := \begin{cases} 0 & \text{if } (u,v) \in R_{UV}(z) \\ +\infty & \text{if } (u,v) \notin R_{UV}(z) \end{cases} \end{cases} \tag{14.21}$$

and the functions Ψ_V and Φ_V associated to the set-valued map R_V and S_V in an analogous way.

These functions being only lower semicontinuous, but not differentiable, cannot be solutions to either partial differential equations (14.19) and (14.20). But we can use the *contingent epiderivatives* of any function $\Phi : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R} \cup \{+\infty\}$ and replace the partial differential equations (14.19) and (14.20) by the contingent partial differential equations

$$\inf_{\substack{\|u'\| \le \rho(\|u\|+1) \\ \|v'\| \le \sigma(\|v\|+1)}} D_\uparrow \Phi(z,u,v)(h(z,u,v),u',v') \le 0 \tag{14.22}$$

and

$$\sup_{\substack{\|u'\| \le \rho(\|u\|+1) \\ \|v'\| \le \sigma(\|v\|+1)}} D_\uparrow \Phi(z,u,v)(h(z,u,v),u',v') \le 0 \tag{14.23}$$

respectively.

Let Ω_U and Ω_V be the indicators of the graphs of the set-valued maps U and V defined by

$$\begin{cases} i) & \Omega_U(z,u,v) := \begin{cases} 0 & \text{if } u \in U(z) \\ +\infty & \text{if } u \notin U(z) \end{cases} \\ ii) & \Omega_V(z,u,v) := \begin{cases} 0 & \text{if } v \in V(z) \\ +\infty & \text{if } v \notin V(z) \end{cases} \end{cases}$$

Theorem 14.5.3 *We posit the assumptions of Theorem 14.5.2. Then*

— Ψ_U *is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to* Ω_U

— *Ψ_V is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to Ω_V*

— *Ψ_{UV} is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to* $\max(\Omega_U, \Omega_V)$

— *If h is Lipschitz, Φ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (14.23) larger than or equal to Ω_U*

— *If h is Lipschitz, Φ_V is the smallest lower semicontinuous solution to the contingent partial differential equation (14.23) larger than or equal to Ω_V*

If any of the above solutions is the constant $+\infty$, the corresponding feedback map is empty.

Proof of Theorem 14.5.2 — Let us denote by B the unit ball and introduce the set-valued map F defined by

$$H(z,u,v) \ := \ \{h(z,u,v)\} \times \rho(\|u\|+1)B \times \sigma(\|v\|+1)B$$

The evolution of the differential game described by equations (14.3) and (14.16) is governed by the differential inclusion

$$(z'(t), u'(t), v'(t)) \ \in \ H(z(t), u(t), v(t))$$

— Since the graph of U is closed, we take the graph of R_U to be the viability kernel of $\text{Graph}(U)\times\mathbf{R}^q$. Indeed, if $(u_0, v_0) \in R_U(z_0)$, there exists a solution to the differential inclusion remaining in the graph of U, i.e., Xavier may win. If not, all solutions starting at (z_0, u_0, v_0) must leave this domain in finite time.

The set-valued feedback map R_V is defined in an analogous way.

— For the same reasons, the graph of the set-valued feedback map R_{UV} is the viability kernel of the set $\mathcal{K}$ of initial situations.

— When h is Lipschitz, so is H. We define the graph of S_U as the invariance kernel of $\text{Graph}(U) \times \mathbf{R}^q$. □

Proof of Theorem 14.5.3 — We recall that thanks to the viability Theorem, a subset $L \subset \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ is a viability domain of F if and only if

$$\forall\, (z,u,v) \in L,\ T_L(z,u,v) \cap H(z,u,v) \ \neq \ \emptyset$$

Let Ψ_L denote the indicator of L. We know that the Viability Theorem can be reformulated in the following way:

The set L is a closed viability domain if and only if its indicator function Ψ_L is a solution to the contingent partial differential equation (14.22).

— Hence to say that the graph of R_U is the largest closed viability domain contained in the graph of U amounts to saying that its indicator Ψ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to the indicator Ω_U of Graph$(U) \times \mathbf{R}^q$. The same reasoning shows that indicator Ψ_V of R_V is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to Ω_V and that the indicator Ψ_{UV} of the graph of R_{UV} is the smallest lower semicontinuous solution to the contingent partial differential equation (14.22) larger than or equal to the indicator of $\mathcal{K}$, which is equal to $\max(\Omega_U, \Omega_V)$.

— We know that a closed subset $L \subset \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q$ is "invariant" by a Lipschitz set-valued map F if and only if

$$\forall\, (z,u,v) \in L,\ T_L(z,u,v) \ \subset\ H(z,u,v)$$

This condition can be reformulated in terms of contingent epiderivative of the indicator function Ψ_L of L by saying that

$$\forall\, (z,u,v) \in L, \quad \sup_{w \in H(z,u,v)} \ D_\uparrow \Psi_L(z,u,v)(w) = 0$$

Hence to say that the graph of S_U is the largest closed invariance domain contained in the graph of U amounts to saying that its indicator Φ_U is the smallest lower semicontinuous solution to the contingent partial differential equation (14.23) larger than or equal to the indicator Ω_U of Graph$(U) \times \mathbf{R}^q$. □

Let us denote by R one of the feedback maps R_U, R_V, R_{UV} and assume that the initial situation belongs to the graph of the set-valued feedback map R (when it is not empty). The theorem states only that there exists at least a solution $(z(\cdot), u(\cdot), v(\cdot))$ to the differential game such that

$$\forall\, t \geq 0,\ (u(t), v(t)) \ \in\ R(z(t))$$

To implement these strategies, players must *make decisions, i.e., to choose velocities of controls* in an adequate way:

We observe that stable solutions

Proposition 14.5.4 *The solutions to the game satisfying*

$$\forall\, t \geq 0,\ (u(t), v(t)) \in R(z(t))$$

are the solutions to the system of differential inclusions

$$\begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & (u'(t), v'(t)) \in G_R(z(t), u(t), v(t)) \end{cases} \tag{14.24}$$

where we have denoted by G_R *the* R-decision map *defined by*

$$G_R(z, u, v) \ := \ DR(z, u, v)(h(z, u, v)) \cap (\rho(\|u\| + 1)B \times \sigma(\|v\| + 1)B)$$

For simplicity, we shall set $G := G_R$ whenever there is no ambiguity.

Proof — Indeed, since the function $(z(\cdot), u(\cdot), v(\cdot))$ takes its values into Graph(R) and is absolutely continuous, then its derivative $(z'(\cdot), u'(\cdot), v'(\cdot))$ belongs almost everywhere to the contingent cone

$$T_{\text{Graph}(R)}(z(t), u(t), v(t)) \ := \ \text{Graph}(DR(z(t), u(t), v(t)))$$

We then replace $z'(t)$ by $h(z(t), u(t), v(t))$.

The converse holds true because equation (14.24) makes sense only if $(z(t), u(t), v(t))$ belongs to the graph of R. □

The question arises whether we can construct selection procedures of the decision components of this system of differential inclusions. It is convenient for this purpose to introduce the following definition.

Definition 14.5.5 (Closed Loop Decision Rules) *We say that a selection* $(\tilde{c}, \tilde{d})$ *of the contingent derivative of the smooth regulation map* R *in the direction* h *defined by: for all* $(z, u, v) \in$ Graph(R).

$$(\tilde{c}(z, u, v), \tilde{d}(z, u, v)) \in DR(z, u, v)(h(z, u, v) \tag{14.25}$$

is a closed loop decision rule.

The system of differential equations

$$\begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & u'(t) = c(z(t), u(t), v(t)) \\ iii) & v'(t) = d(z(t), u(t), v(t)) \end{cases} \tag{14.26}$$

is called the associated closed loop decision game.

Therefore, closed loop decision rules being given for each player, the closed loop decision system is just a system of ordinary differential equations.

It has solutions whenever the maps c and d are continuous (and if such is the case, they will be continuously differentiable).

But they also may exist when c or d or both are no longer continuous. This is the case when the decision map is lower semicontinuous thanks to Michael's Theorem:

Theorem 14.5.6 *Let us assume that the decision map $G := G_R$ is lower semicontinuous with non empty closed convex values on the graph of R. Then there exist continuous decision rules c and d, so that the decision system (14.26) has a solution whenever the initial situation $(u_0, v_0) \in R(z_0)$*

By using selection procedures (see Definition 6.5.2), we can obtain explicit decision rules which are not necessarily continuous, but for which the decision system (14.26) still has a solution.

Hence, we also obtain the following existence theorem for closed loop decision rules obtained through convex-valued selection procedures, which is analogous to Theorem 7.6.4.

Theorem 14.5.7 *Let S_G be a selection of the set-valued map G with convex values. Then, for any initial state $(z_0, u_0, v_0) \in \text{Graph}(R)$, there exists a solution starting at (z_0, u_0, v_0) to the associated system of differential inclusions*

$$\begin{cases} z'(t) = h(z(t), u(t), v(t)) \\ (u'(t), v'(t)) \in G(z(t), u(t), v(t)) \cap S_G(z(t), u(t), v(t)) \end{cases} \tag{14.27}$$

In particular, if for every (z,u,v) the intersection

$$S(G(z,u,v)) := (\tilde{c}(z,u,v), \tilde{d}(z,u,v))$$

is single-valued, then the strategies $(x,y) \mapsto (\tilde{c}(z,u,v), \tilde{d}(z,u,v))$ are single-valued closed-loop decision rules, for which decision system 14.26 has a solution for any initial state $(z_0,u_0,v_0) \in \text{Graph}(R)$.

We can now multiply the possible corollaries, since we have given several instances of selection procedures of set-valued maps.

Example— COOPERATIVE BEHAVIOR
Let $\sigma : \text{Graph}(G) \mapsto \mathbf{G}$ be continuous.

Corollary 14.5.8 *Let us assume that the set-valued map G is lower semicontinuous with nonempty closed convex images on* Graph(R). *Let σ be continuous on* Graph(G) *and convex with respect to the pair (u,v). Then, for all initial situations $(u_0,v_0) \in R(z_0)$, there exists a solution starting at (z_0,u_0,v_0) to the differential game (14.3)-(14.16) which are regulated by:*

$$\begin{cases} \text{for almost all } \geq 0, \quad (u'(t),v'(t)) \in G(z(t),u(t),v(t)) \text{ and} \\ \\ \sigma(z(t),u(t),v(t),u'(t),v'(t)) \\ = \inf_{u',v' \in G(z(t),u(t),v(t))} \sigma(z(t),u(t),v(t),u',v') \end{cases}$$

In particular, the game can be played by the heavy decision *of minimal norm:*

$$\begin{cases} (c^\circ(z,u,v), d^\circ(z,u,v)) \in G(z,u,v) \\ \\ \|c^\circ(z,u,v)\|^2 + \|d^\circ(z,u,v)\|^2) = \min_{(u',v') \in G(z,u,v)}(\|u'\|^2 + \|v'\|^2) \end{cases}$$

Example— NONCOOPERATIVE BEHAVIOR
We can also choose strategies in the regulation sets $G(z,u,v)$ in a noncooperative way, as saddle points of a function $a(z,u,v,\cdot,\cdot)$.

Corollary 14.5.9 *Let us assume that the set-valued map G is lower semicontinuous with nonempty closed convex images on* Graph(R)

and that $a : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}$ *satisfies*

$$\begin{cases} i) & a \text{ is continuous} \\ ii) & \forall(z,u,v,d), \quad c \mapsto a(z,u,v,c,d) \text{ is convex} \\ iii) & \forall(z,u,v,c), \quad d \mapsto a(z,u,v,c,d) \text{ is concave} \end{cases}$$

Then, for all initial situation $(u_0, v_0) \in R(z_0)$*, there exist solutions to the differential game (14.3)-(14.16) starting at* (z_0, u_0, v_0) *which are regulated by: for almost all* $t \geq 0$,

$$\begin{cases} i) & (u'(t), v'(t)) \in G(z(t), u(t), v(t)) \\ ii) & \forall\,(u', v') \in G(z(t), u(t), v(t)), \quad a(z(t), u(t), v(t), u'(t), v') \\ & \leq a(z(t), u(t), v(t), u'(t), v'(t)) \leq a(z(t), u(t), v(t), u', v'(t)) \end{cases}$$

Bibliographical Comments

Chapter 1

The Nagumo Theorem was proved in 1942 in [391] and has been rediscovered many times since (see [530, Yorke]). The distinction between viability and invariance properties appeared in [16].

The finite difference approximation is due to [461, Saint-Pierre].

The fourth section on replicator systems is taken from the wonderful book by [279, Hofbauer & Sigmund].

The fifth section on stochastic viability and invariance appeared in [20, Aubin & Da Prato], where one can find more details on the calculus of stochastic contingent sets.

Chapter 2

We refer to [28] for detailed historical notes on upper and lower limits (introduced by Painlevé in 1902, and then, popularized by the publication of the book by [310, Kuratowski].)

Upper and lower semicontinuous maps were introduced by Bouligand and Kuratowski in the early twenties.

Pseudo-Lipschitz set-valued maps have been introduced in [46, 47] in the framework of the Inverse Function Theorem in finite-dimensional spaces, and later, in [26] for Banach spaces, and have also been studied by Rockafellar in [452]. We refer to [255, Frankowska] and her forthcoming monograph [257] for an exhaustive exposition of inverse function theorems of first and/or higher order for set-valued maps.

The Convergence Theorem (or Mazur's Theorem) has been used by several authors studying differential inclusions with upper semicontinuous convex-valued right-hand sides with closed convex values.

Closed convex processes and their transposes have been introduced by Rockafellar [443] and further studied in [444,445]. Robinson-Ursescu's Theorem has been proved in [439,440,441,501] and the Uniform Boundedness and the Crossed Convergence Theorem in [38]. The Closed Image Theorem is taken from [22]. Linear processes and their invariance properties were studied in the framework of singular or implicit systems (including descriptor systems) in [254, Frankowska].

Chapter 3

One can find an extensive bibliography on differential inclusions and their applications to control and viability theory in the book [16] and the forthcoming monograph by [257, Frankowska] for instance. In a nutshell, the first independent investigations of differential inclusions began with the papers by Zaremba [539,540] and Marchaud [372,371], where the derivatives of the nondifferentiable solutions where taken in the contingent and paratingent sense respectively, following the definitions introduced at the beginning of the thirties by Bouligand. They were resurrected in the fifties and sixties by the Polish school around Ważewski [517,519,520,521,522] when he made the link between the Marchaud-Zaremba theory and optimal control problems. Among many references of the "Krakow school", mention only [336, Lasota-Olech], [337,338, Lasota-Opial], [393,394,396,397, Olech], [400, Opial]. In the same period, and independently, Filippov, in [229,227,228], and the Russian school around him, developed a series of fundamental theorems on differential inclusions. For more references, we refer to the recent English translation [231, Filippov] of [230, Filippov]. The importance of the Marchaud-Zaremba theory from another point of view has been pointed out in [66, Barbashin-Alimov]. The concepts of viability property and viability domains appeared in the framework of differential inclusions in [532, Yorke] and [83, Bebernes & Schuur] under the name of weak invariance and admissibility and also [261, Gautier]. The more general concepts of monotone solutions in [34,12, Aubin-Cellina-Nohel] (for convex viability domains and Marchaud maps) and in [17, Aubin-Clarke] (for any compact subset and continuous maps.) The invariance property by Lipschitz maps already appeared in [172, Clarke]. The proof of the Viability Theorem we use is due to Haddad [264,267,265] (which he actually devised in the case of functional differential inclusions.)

The concept of viability property is involved in the definition of stable bridge introduced in [293, Krasovski-Subbotin] (see its recent English augmented version [294]) and their school in Sverdlovsk in the framework of differential games. The concept of viability domain and the use of contingent derivatives of tubes appeared in [83, Bebernes & Schuur]. It was used in the context of differential games in [262], as well as the equivalence between the two first statements of Theorem 3.2.4. The proof given here is due to Frankowska (see [32].)

The viability property appeared independently in linear control theory under the name of *controlled invariance* or (A, B)-invariance and the invariance property under the name of *conditional invariance* in [77,78, Basile & Marro] (see also [528,389, Morse & Wonham], [466, Shumacher] and [524,525,526].) We refer to the book by [529, Wonham] for more details and bibliographical comments on these geometrical methods in linear control theory.

One can find generalizations of these results to lower semicontinuous maps by [127, Bressan]. See also [117,121] for related results.

Kurzhanski's Theorem 3.5.8 is taken from [328,311,312,313,314].

The detailed proof of the Convergence Theorem 3.6.5 can be found in [28, Chapter 7].

Derivatives of the solution map have been studied in [28, Chapter 10]. See this book for further bibliographical comments.

One can find in [7, Artstein] an interesting study on collective limit sets.

Theorem 3.7.4 seems to be new and Theorem 3.7.5 generalizes to differential inclusions results from [279, Hofbauer & Sigmund].

The proof of Equilibrium Theorem 3.7.6 can be found in [28, Chapter 3]. Ky Fan inequality appeared in [226] and the proofs of the minimax inequalities in [41]. For Fixed Point Theorems using properties of differential inclusions, see among other contributions Deimling's book [206] and also [207,204,209]. However, the literature on nonlinear analysis is too broad to give any fair account in this short review.

Theorem 3.8.1 on chaotic behavior generalizes a useful and powerful principle due to Saari. We refer to [458] for further details and many applications.

Chapter 4

The concept of viability kernel, although without being named this way, played a crucial role in [14,15, Aubin-Frankowska-Olech] in the framework of controllability and observability of closed convex processes. It appeared under this name for differential inclusions with

Marchaud right-hand side in [49], where its existence was proved and where it was used to tackle smoothness issues in differential inclusions. It played a crucial role ever since in the successive drafts of this book. Meanwhile, the concept of *zero dynamics* has been inspired by an interpretation of transmission zeros in terms of controlled invariant (viable) distributions published in [298, Krener-Isidori]. Its study has been taken up since in [132,133,135,136,137,138,139, Byrnes-Isidori], [380], [284], etc., and applied to problems such as stabilization, tracking and disturbance attenuation. The connections between the concepts of zero dynamics and viability kernels was established in [11]. See Isidori's monograph [285] and the forthcoming book [140, Byrnes-Isidori] for more details and further bibliographical comments.

The second part of Proposition 4.1.4 is due to Quincampoix, the concept of *permanence* has been introduced by Hofbauer and Sigmund and is extensively studied in their book [279] (see also [37]) and the concept of *fluctuation*, motivated by biological considerations, by Krivan in [303].

Some of the concepts of section 4.2 go back to Poincaré and [514, 515,516, Ważewski]. The results of section 4.3 are due to [430,432, Quincampoix] and Theorem 4.3.8 to Saint-Pierre [463]. The fact that solutions starting on the boundary of the exit tube remains on it is related to the properties of the contingent cones to reachable sets proved in [245,242, Frankowska].

The *zero dynamics algorithm*, called here the viability kernel algorithm, has been extensively studied by Byrnes and Isidori in [132,134,135,136,138,284,285,281]. It is a generalization of the structure algorithm introduced by Silverman in [478] and Basile & Marro in [77] for linear control systems. See also [382, Moore & Laub]. The "fast viability kernel algorithm" is due to [240,239, Frankowska-Quincampoix] where one can find further results (in the convex and polyhedral cases in particular) and examples. The concept of global contingent set and the modified viability domain algorithm appeared in [33]. Another viability domain algorithm has been proposed in [177, Colombo-Krivan]. The convergence of the viability kernels of finite-difference approximations to the viability kernel is taken from [35].

Chapter 5

It is impossible to give an exhaustive account of the many papers in which the various concepts of tangent cones appeared in the literature. The need to introduce *contingent* directions was felt by Bouligand in the thirties in order to differentiate nondifferentiable functions (see [114,113].) This was taken up by Zaremba [540] and Marchaud [371] to define solutions to differential inclusions. See also the paper [166] by Choquet.

The contingent cone was used under the names of *tangent cone* and *sequential tangent cone* in optimization and control theory in the fifties. Nagumo and after him, many authors used the contingent cone under its "lim inf formulation", often called "subtangent condition". See the book [374, Martin] for instance. We refer to [28, Chapter 4] and its bibliographical comments for further details on tangent cones to nonsmooth sets.

The fact that the contingent cone coincides with the Clarke tangent cone for sleek subsets was first proved in [17] in finite dimensional vector-spaces (among other characterizations). See [28] for further bibliographical details.

The calculus of tangent cones to intersections and inverse images appeared in [46] (in the finite dimensional case, generalizing formulas due to Rockafellar in [446] and under stronger transversality conditions) in [26] (in the case of Banach spaces.)

Theorem 5.1.11 is due to [368, Maderner].

The original proof of Filippov's Theorem appeared in [229]. It can be also found in [16] for set-valued maps continuous with respect to the time. A general proof (including the case when $T = \infty$) can be found in Frankowska's monograph [257]. The extension to operational differential inclusions in infinite dimensional spaces has been proved in [246,253, Frankowska].

Theorem 5.3.4 on local invariance under Lipschitz maps was proved in [172, Clarke]. The proof we give is taken from [16].

The viability property of the complement of the invariance kernel is proved in [430,432, Quincampoix], as well as the semi-permeability property of the boundary of the viability kernel and the viability property of the boundary of the invariance kernel. We refer to the forthcoming paper by [434, Quincampoix] for the study of invariance envelopes of closed subsets.

Section 5.6 on the victory and defeat domains of a target are due to [430,432, Quincampoix].

Most of the results of section 5.7, and in particular the characterization of the polar of a viability domain of a closed convex process, appeared in [14,15, Aubin-Frankowska-Olech] to study the duality relations between controllability and observability of closed convex processes.

This study has been pursued with analogous techniques in the framework of descriptor systems in [254, Frankowska].

The existence of eigenvectors of a closed convex process on cones with compact soles appeared in [14] for characterizing controllability of closed convex processes. Existence of positive eigenvectors and eigenvalues and an extension of the Perron-Frobenius Theorem can also be found in [22, Chapter 3].

Chapter 6

We refer to the thesis of [541, Zhao] for a first application of viability to the building of an actual robot arm called *Lola 80 Super.* built by the company *Ivo Lola Ribar*. Other applications of viability theory to the learning procedures of regulation laws by neural networks can be found in [469,470, Seube] and [467, Seube-Macias].

The lower semicontinuity criterion of a finite intersection of convex-valued lower semicontinuous maps can be found in [22,28] for instance and the lower semicontinuity criterion for an infinite intersection of lower semicontinuous set-valued maps was proved in the context of differential games [57].

The Continuous Selection Theorem is due to Michael [376,377, 378]. One can find in [28, Chapter 9] an exhaustive exposition of selection and parametrization theorems. See also other selection procedures in [118,125,260,260].

The concept of selection procedure is due to Frankowska (unpublished) and has been used in [24,25]. The existence of slow viable solutions is due to [225, Falcone & Saint-Pierre] and their approximation can be found in [461, Saint-Pierre].

Chapter 7

The concepts of *graphical derivatives* and their use in the extension of the Inverse Function Theorem have been initiated in [44] for the contingent derivative and in [46,47] for the circatangent derivatives. (There were however many definitions of *pointwise derivatives* of set-valued maps proposed in the literature such as [65, Banks-Jacobs] among many other papers on this topic. They are not well adapted to our purpose.)

The theorem on the regularity of solutions to differential inclusions and control systems appeared in [49] and has been taken up in [33]. The application to the case of inequality constraints appeared in [368, Maderner].

The first theorem on the existence of local viable solutions to second-order differential inclusions was published in [186, Cornet-Haddad]. We refer to [82, Bebernes & Keley] for boundary-value problems for differential inclusions.

The concept of heavy viable solutions and their studies have been investigated in [24,25,33] in collaboration with H. Frankowska.

Regularity of higher order, heavy solutions of high order, ramp controls and polynomial open-loop controls is the topic of [23].

Chapter 8

The tracking property and the general results related to it appeared in [29,32] in collaboration with H. Frankowska. The results dealing with the construction of trackers and observers, the decentralization property and hierarchical decomposition are taken from [59]. Theorems dealing with the existence of the largest set-valued contingent solution with linear growth to a decomposable system of partial differential inclusions are taken from the first part of [31,30, Aubin-Frankowska]. In the case of partial differential equations, the same result was obtained in [19,18, Aubin-Da Prato]. The tracking property has also been the topic of the papers [135,136,137,139] by Byrnes & Isidori.

For quasi-linear systems of partial differential equations, the existence and uniqueness, as well as the convergence of the viscosity method, has been proved in [19,18] in collaboration with Da Prato. More general results dealing with the fully nonlinear case has been announced by P.-L. Lions & Souganidis (private communication.) The existence of a solution in the case of partial differential inclusions appeared in [31,30], as well as further comparison results that were not included in this book.

The variational principle is taken from [29,32, Aubin-Frankowska].

It has been shown in [250,251,256] that "contingent solutions" are related by duality to the "viscosity solutions" introduced in the context of Hamilton-Jacobi equations by Crandall & Lions in [193] (see also [359] and the literature following these papers.)

The variational principle (Theorem 8.4.2) states that for systems of partial differential equations or inclusions, the contingent solutions are adaptations to the vector-valued case of viscosity solutions.

The convergence properties of the codifferential as well as a thorough study of codifferentials are exposed in [29,32, Aubin-Frankowska]. A discussion of results on upper limits of normal cones, which we deduce by polarity from the properties of upper and lower limits of tangent cones, can be found in [28, Chapter 5 & 7].

Other theorems on the lower limits of gradients of differentiable functions converging uniformly appeared in [28, Chapter 7]. They extend to the infinite-dimensional case a theorem of Crandall, Evans & P.-L. Lions [192] which allows the study of the stability of viscosity solutions to Hamilton-Jacobi equations.

Chapter 9

The use of contingent epiderivatives in some problems (related to the value function of optimal control problems, in particular) appeared in [44] and in [16, Chapter 6]. They were used later on for a further study of the dynamic programming approach, in [243,250,247,148,147]). It has been shown in [251] that "contingent solutions" are related by duality to the "viscosity solutions" introduced in the context of Hamilton-Jacobi equations by Crandall and Lions.

We refer to [28, Chapter 6] for a thorough presentation of epiderivatives of extended functions. The epidifferential calculus has been developed in [26]. It has been established by Clarke [174] in the case of locally Lipschitz functions and by Rockafellar [447,448,449, 451] under stronger transversality conditions than the ones used in [26].

Stability Theorem 9.2.5 uses epilimits of extended functions in a natural way. We refer to [28, Chapter 7] for a short introduction to epiconvergence and/or Γ-convergence and to the book [10, Attouch] and the forthcoming monograph [442, Rockafellar-Wets] for further references and an exhaustive presentation of this topic.

Section 9.3 on optimal Lyapunov functions is taken from [54].

Subsection 9.4.3 is due to [497, Tchou] and subsection 9.5.2 on comparison of solutions was inspired by [210, Deimling].

Chapter 10

The motivations for studying variational differential inequalities came from two distinct sources: mechanics, in the framework of partial differential equations, for which we refer to [84, Bensoussan-Lions] among many other references, and planing procedures in economics, initiated by papers by Malinvaud and other mathematical economists, among which we single out [275, Henry]. We present here results of [187,189,190, Cornet] with simpler proofs. Other projection methods motivated by biological problems have been proposed in [305, Krivan].

Section 10.2 on fuzzy differential inclusions is taken from [58]. Fuzzy sets have been introduced in 1965 by [538, Zadeh]. Soon after, Goguen proposed to generalize the idea of fuzzy sets by L-fuzzy sets, where the interval $[0, 1]$ in which the membership functions take their values is replaced by some abstract set L. Until now, only $[0, 1]$-fuzzy sets are considered in practice, despite the theoretical interest of general L-fuzzy sets. We had to use the scale $L = [0, \infty]$ in order to define and use convex fuzzy sets. In the first version of [58], $[0, \infty]$-fuzzy sets were called "hazy sets", and then, fuzzy sets, in order to avoid the increase of vocabulary. In their recent paper, [222, Dubois & Prade] proposed to call them "toll sets" and investigated how the basic concepts of $[0, 1]$-fuzzy sets are transferred to $[0, \infty]$-fuzzy sets (toll measures, toll logic, etc.). Other "fuzzification" of differential equations has been proposed in [221,287,111] for instance. For an introduction to fuzzy sets, we refer to [220, Dubois-Prade] among many other books.

The results of Section 10.3 dealing with numerical issues are taken from [461, Saint-Pierre]. See among other papers on numerical approximation of differential inclusions [508,509,510,511, Veliov]. The results of Section 10.3 on the Newton method can be found in [462, Saint-Pierre]. The idea of using differential equations and inclusions has been used very often by many authors for Newton methods and other continuation methods. It seems to go back at least to [513, Ważewski]. By the way, one can find in this paper an extension of the notion of contingent derivative for maps between metric spaces (allongements contingentiels et paratangentiels.) For a recent use of such ideas, we also refer to [392] by Olech, Parthasrathy and Ravindran.

Chapter 11

The viability theorem for maps depending measurably on time are due to [496, Tallos]. See also [210, Deimling] on this topic. For Lipschitz maps depending measurably on time, we refer to the forthcoming paper by [288, Kannai-Tallos].

Actually, the concept of viability tubes goes back to Ważewski under the name of *tuyau*, in the framework of a fundamental method to determine the viability kernel of a viability tube by a topological approach. See [514,515,516, Ważewski] [104, Bielecki & Kluczny], [292, Kluczny], for instance.

The concept of viability and invariance tubes are quite close to but distinct from the concepts of *reachable maps* or *funnels*, which extend to the set-valued case the concept of semi-groups of operators. The first study goes back to [69,70, Barbashin] and [453, Roxin]. The formula stating that $\overline{co}(F(t,x))$ is the infinitesimal generator of the semi-group of reachable maps (using contingent derivatives) is due to [244, Frankowska]. In the framework of differential games, the viability property of a tube is involved in the concept of *stable bridge* and the concept of reachable maps to *stable integral manifolds* of [294]. Contingent derivatives of tubes were used in this context in [262].

Analogous ideas using "pointwise derivatives" defined through Hausdorff distances, have been thoroughly investigated under the name of *funnel equations* in the soviet literature, by Krasovski and his collaborators ([296,293,295]), Kurzhanski ([326,329,330]), Tolstonogov [499], Panasyuk [408,410,411].

Approximation of viability tubes by *ellipsoids* have been initiated in [325, Kurzhanski] and extensively studied by Kurzhanski, Valyi [504], and their collaborators, with many numerical approximations made at IIASA. We refer for this approach to the series of papers [322,323] and the forthcoming monograph [323] by Kurzhanski and Valyi.

Chapter 12

The functional viability theorem is due to Haddad in the papers [264,265,267]. The characterization of the functional viability domains for delay and Volterra constraints is due to [223, Duluc-Vigneron].

Chapter 13

The results of this chapter are due to Shi Shuzhong. For more details on this topic, we refer to [472,473,475,476]. The extension of the Filippov Theorem to operational differential inclusions appeared in [253, Frankowska] for Lipschitz right-hand sides. One can find a study of elliptic and parabolic partial differential inclusions in [76, Bartuzel-Fryszkowski] for decomposable lower semicontinuous right-hand sides. Viability and uncertainty issues for parabolic equations are studied by Kurzhanski and Khapalov [315,316,317,318,319].

Chapter 14

The literature on differential games is too large to do it justice. The contents of this chapter is taken from [57,51,55] and has been extended in [431, Quincampoix] for the time dependent case. Further results on the of victory and defeat domains in the framework of differential games appear in the forthcoming paper [433, Quincampoix].

We refer to [91] for a result on barriers in the case of differential games and to [430] for any target of a control problem.

For a short survey of differential games, we refer to [96,97,98,99, 100, Bernhard] and his monograph [92]. See also [293,294, Krasovski-Subbotin] among many other references.

Set-valued solutions to funnel equations for differential games and their approximation by ellipsoids of set-valued solutions to funnel equations can be found in [322,323, Kurzhanski-Valyi].

For applications of differential games to robotics, we refer to [232, 479].

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