Chapter 7
Observation and Identification via HOSM Observers

Control systems normally perform under uncertainties/disturbances and with measurement signals corrupted by noise. For systems with reliable models and noisy measurements, a filtration approach (Kalman filters, for example) is efficient. However, as shown in Chap. 3, sliding mode observers based on first-order sliding modes are effective in the presence of uncertainties/disturbances. Nevertheless, as discussed in that chapter, they are only applicable when the relative degree of the outputs with respect to the uncertainties/disturbances is one, and differentiation of noisy outputs signals is not needed.

Unfortunately, even for observation of mechanical systems with measured positions, the estimation of velocities—i.e., the derivatives of position—is necessary. The uncertainties/disturbances in mechanical systems are in the equations for accelerations and have relative degree two with respect to the measured positions. This means that differentiators which can provide the best possible accuracy in the presence of sampling steps and noise are needed for the general case of observation of control systems working under uncertainties/disturbances. HOSM differentiators are one class of such differentiators. In this chapter we will show how to design these HOSM observers for different types of systems.

In this chapter sliding mode based observers are presented as an alternative to the problem of observation of perturbed systems. In particular, high-order sliding mode (HOSM) based observers can be considered as a successful technique for the state observation of perturbed systems, due to their high precision and robust behavior with respect to parametric uncertainties. The existence of a direct relationship between differentiation and the observability problem makes sliding mode based differentiators a technique that can be applied directly for state reconstruction. Even when the differentiators appear as a natural solution to the observation problem, the use of the system knowledge for the design of an observation strategy results in a reduction in the magnitude of the gains for the sliding mode compensation terms. Moreover, complete or partial knowledge of the system model facilitates the application of the techniques to parametric reconstruction or disturbance reconstruction.

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7.1 Observation/Identification of Mechanical Systems

This section will begin by focusing on observation and identification of mechanical systems, which have been the focus of many studies throughout the years. Recent research on these systems has produced many important applications such as telesurgery with the aid of robotic manipulators, missile guidance and defense, and space shuttle control. The general model of second-order mechanical systems is derived from the Euler–Lagrange equations which are obtained from an energy analysis of such systems. They are commonly expressed in matrix form as

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + P(\dot{\mathbf{q}}) + G(\mathbf{q}) + \Delta(t, \mathbf{q}, \dot{\mathbf{q}}) = \tau \tag{7.1}$$

where $\mathbf{q} \in \mathbb{R}^n$ is a vector of generalized coordinates, $M(\mathbf{q})$ is the inertia matrix, $C(\mathbf{q}, \dot{\mathbf{q}})$ is the matrix of Coriolis and centrifugal forces, $P(\dot{\mathbf{q}})$ is Coulomb friction, which possibly contains relay terms depending on $\dot{\mathbf{q}}$, $G(\mathbf{q})$ is the term associated with the gravitational forces, $\Delta(t, \mathbf{q}, \dot{\mathbf{q}})$ is an uncertainty term; and τ is the generalized torque/force produced by the actuators. The control input τ is assumed to be given by some known feedback function. Note that $M(\mathbf{q})$ is invertible since $M(\mathbf{q}) = M^T(\mathbf{q})$ is strictly positive definite. All the other terms are supposed to be uncertain, but the corresponding nominal functions $M_n(\mathbf{q})$**,** $C_n(\mathbf{q}, \dot{\mathbf{q}})$**,** and $P_n(\dot{\mathbf{q}})$, $G_n(\mathbf{q})$ are assumed known.

Introducing new variables $x_1 = \mathbf{q}$, $x_2 = \dot{\mathbf{q}}$, $u = \tau$, the model (7.1) can be rewritten in the state-space form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(t, x_1, x_2, u) + \xi(t, x_1, x_2, u), \qquad u = U(t, x_1, x_2), \\ y &= x_1, \end{aligned} \tag{7.2}$$

where the nominal part of the system dynamics is represented by the function

$$f(t, x_1, x_2, u) = -M_n^{-1}(x_1)[C_n(x_1, x_2)x_2 + P_n(x_2) + G_n(x_1) - u]$$

containing the known nominal functions M_n, C_n, G_n, P_n, while the uncertainties are lumped in the term $\xi(t, x_1, x_2, u)$. The solutions to system (7.3) are understood in a Filippov sense. It is assumed that the function $f(t, x_1, x_2, U(t, x_1, x_2))$ and the uncertainty $\xi(t, x_1, x_2, U(t, x_1, x_2))$ are Lebesgue measurable function of t and uniformly bounded in any compact region of the state space x_1, x_2 .

In order to apply a state-feedback controller or to simply perform system monitoring, knowledge of the coordinate x_2 is required. Moreover, in the general case, for the design of the controller, it is necessary to know the parameters of the system. The tasks considered in this section are to design a finite-time convergent observer of the velocity $\dot{\mathbf{q}}$ for the original system (7.1), when only the position $\mathbf{q}$ and the nominal model are available, as well as the development of an identification algorithm to obtain the system parameters through knowledge of only the state x_1 (i.e., $\mathbf{q}$) and the input $u(t)$. Only the scalar case $x_1, x_2 \in R$ is considered for the sake of simplicity.

7.1.1 Super-Twisting Observer

One of the popular second-order sliding mode algorithms which offer a finite reaching time and which can be used for sliding mode based observation is the super-twisting algorithm considered in Chap. 4. The proposed super-twisting observer has the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + z_1 \\ \dot{\hat{x}}_2 &= f(t, x_1, \hat{x}_2, u) + z_2 \end{aligned} \tag{7.3}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the state estimates while the correction variables z_1 and z_2 are output error injections of the form

$$\begin{aligned} z_1 &= \lambda |x_1 - \hat{x}_1|^{1/2} \operatorname{sign}(x_1 - \hat{x}_1) \\ z_2 &= \alpha \operatorname{sign}(x_1 - \hat{x}_1) \end{aligned} \tag{7.4}$$

Taking $\tilde{x}_1 = x_1 - \hat{x}_1$ and $\tilde{x}_2 = x_2 - \hat{x}_2$ we obtain the error equations

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 - \lambda |\tilde{x}_1|^{1/2} \operatorname{sign}(\tilde{x}_1) \\ \dot{\tilde{x}}_2 &= F(t, x_1, x_2, \hat{x}_2) - \alpha \operatorname{sign}(\tilde{x}_1) \end{aligned} \tag{7.5}$$

where

$$F(t, x_1, x_2, \hat{x}_2) = f(t, x_1, x_2, u) - f(t, x_1, \hat{x}_2, u) \tag{7.6}$$

$$+\xi(t, x_1, x_2, y) \tag{7.7}$$

Suppose that the system states are bounded, then the existence of a constant f^+ is ensured, such that the inequality

$$|F(t, x_1, x_2, \hat{x}_2)| < f^+ \tag{7.8}$$

holds for any possible t, x_1, x_2 and $|\hat{x}_2| \leq 2 \sup |x_2|$.

According to Sect. 4.3.2 the parameters of observer α and λ could be selected as $\alpha = a_1 f^+$ and $\lambda = a_2 (f^+)^{1/2}$, where $a_1 = 1.1$, $a_2 = 1.5$. Convergence of the observer states $(\hat{x}_1, \hat{x}_2)$ from Eqs. (7.3) and (7.4) to the system state variables (x_1, x_2) in Eq. (7.3) occurs in finite time, from the theorem in Sect. 4.3.2. All other theorems from Sect. 4.3.2 are also true for the observer Eqs. (7.3) and (7.4).

The standard 2-sliding-mode-based differentiator from Sect. 4.3.2 could be also implemented here to estimate the velocity. In this case, if the accelerations in the mechanical system are bounded, the constant f^+ can be found as the double maximal possible acceleration of the system. For the proposed observer, the design of the gains α and λ is based on an estimate of $F(t, x_1, x_2, \hat{x}_2, u)$. This means that the observer design in Eqs. (7.3) and (7.4) takes into account (partial) knowledge of systems dynamics and is more accurate.

A pendulum, a classical example of a mechanical system, is now used to illustrate the effectiveness of the proposed observer Eqs. (7.3) and (7.4).

Example 7.1. Consider a pendulum system with Coulomb friction and external perturbation given by the equation

$$\ddot{\theta} = \frac{1}{J}\tau - \frac{g}{L}\sin(\theta) - \frac{V_s}{J}\dot{\theta} - \frac{P_s}{J}\,\mathrm{sign}(\dot{\theta}) + v \tag{7.9}$$

where the values $m = 1.1$, $g = 9.815$, $L = 0.9$, $J = mL^2 = 0.891$, $V_S = 0.18$ and $P_s = 0.45$ are the system parameters for simulation purposes, and v is an uncertain external perturbation satisfying $|v| \leq 1$. The function $v = 0.5\sin(2t) + 0.5\cos(5t)$ was used in simulation. Now let Eq. (7.9) be driven by the twisting controller

$$\tau = -30\,\mathrm{sign}(\theta - \theta_d) - 15\,\mathrm{sign}(\dot{\theta} - \dot{\theta}_d) \tag{7.10}$$

where $\theta_d = \sin(t)$ and $\dot{\theta}_d = \cos(t)$ are the reference signals. The system can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \tfrac{1}{J}\tau - \tfrac{g}{L}\sin(x_1) - \tfrac{V_s}{J}x_2 - \tfrac{P_s}{J}\,\mathrm{sign}(x_2) + v \end{aligned}$$

Thus, the proposed velocity observer has the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + 1.5(f^+)^{1/2}|\tilde{x}_1|^{1/2}\,\mathrm{sign}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \tfrac{1}{J_n}\tau - \tfrac{g}{L_n}\sin(x_1) - \tfrac{V_{sn}}{J_n}\hat{x}_2 + 1.1f^+\,\mathrm{sign}(x_1 - \hat{x}_1) \end{aligned}$$

where $m_n = 1$, $L_n = 1$, $J_n = m_n L_n^2 = 1$, $V_{sn} = 0.2$, and $P_{sn} = 0.5$ are the "known" nominal values of the parameters and f^+ is to be assigned. Assume also that it is known that the real parameters differ from the assumed known values by not more than 10 %. The initial values $\theta = x_1 = \hat{x}_1 = 0$ and $\dot{\theta} = x_2 = 1$, $\hat{x}_2 = 0$ were taken at $t = 0$. Noting that $0 \leq \theta \leq 2\pi$, θ belongs to a compact set (a ring), and obviously the dynamic system in (7.9) is BIBS stable.

Easy calculations show that the given controller yields $|\tau| \leq 45$ and the inequality $|\dot{\theta}| \leq 70$ is guaranteed when the nominal values of the parameters and their maximal possible deviations are taken into account. Taking $|x_2| \leq 70$, $|\hat{x}_2| \leq 140$ it follows that $|F| = |\frac{1}{J}\tau - \frac{g}{L}\sin(x_1) - \frac{V_s}{J}x_2 - \frac{P_s}{J}\,\mathrm{sign}(x_2) + v - \frac{1}{J_n}\tau + \frac{g}{L_n}\sin(x_1) + \frac{V_{sn}}{J_n}\hat{x}_2| < 60 = f^+$. Therefore, the observer parameters $\alpha = 66$ and $\lambda = 11.7$ were chosen. Simulations show that $f^+ = 6$ and the respective values $\alpha = 6.6$ and $\lambda = 4$ are sufficient. Note that the terms $\frac{mgL}{J}\sin(x_1)$ and $\frac{1}{J}\tau$ would have to be taken into account when selecting the differentiator parameters, when using the techniques from Chap. 4, causing much larger coefficients to be

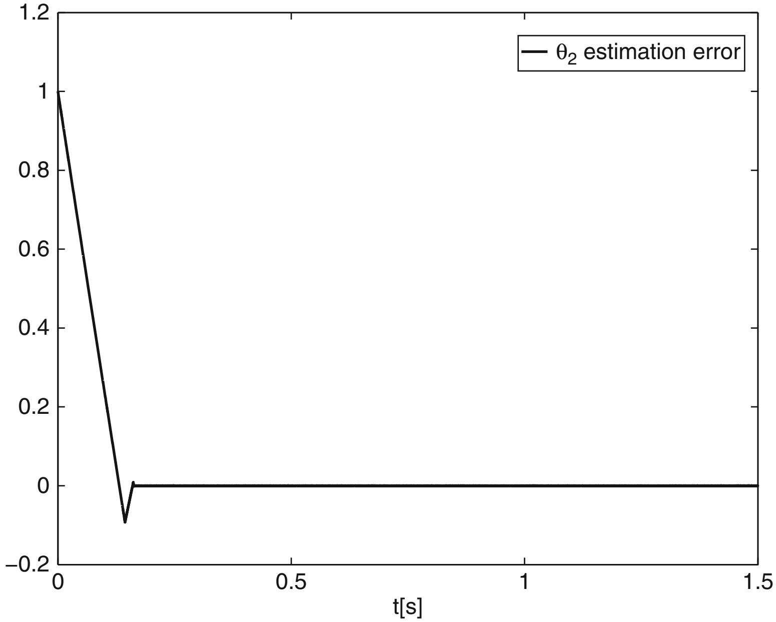

Fig. 7.1 Estimation error for x_2

used. The finite-time convergence of the velocity observation error to the origin is demonstrated in Fig. 7.1. Figure 7.2 illustrates the same convergence by comparing the estimated velocity to the real one. Finally, Fig. 7.3 shows the convergence of the observer dynamics in the $\tilde{x}_1$ versus $\tilde{x}_2$ plane.

7.1.2 Equivalent Output Injection Analysis

It is a well-known fact (see Chaps. 1–3) that the equivalent injection term contains information about the disturbances/uncertainties and unknown inputs in a system and can therefore be used for their reconstruction. This important concept is used in the analysis below to reconstruct unknown inputs in mechanical systems. Moreover, the problem of parameter estimation is also addressed in the latter part of this subsection.

Equivalent Output Injection

The finite-time convergence to the second-order sliding mode set ensures that there exists a time $t_0 > 0$ such that for all $t \geq t_0$ the following identity holds:

$$0 \equiv \dot{\tilde{x}}_2 \equiv \Delta F(t, x_1, x_2, \hat{x}_2, u) + \xi(t, x_1, x_2, u) - (\alpha_1 \text{sign}(\tilde{x}_1))_{eq}$$

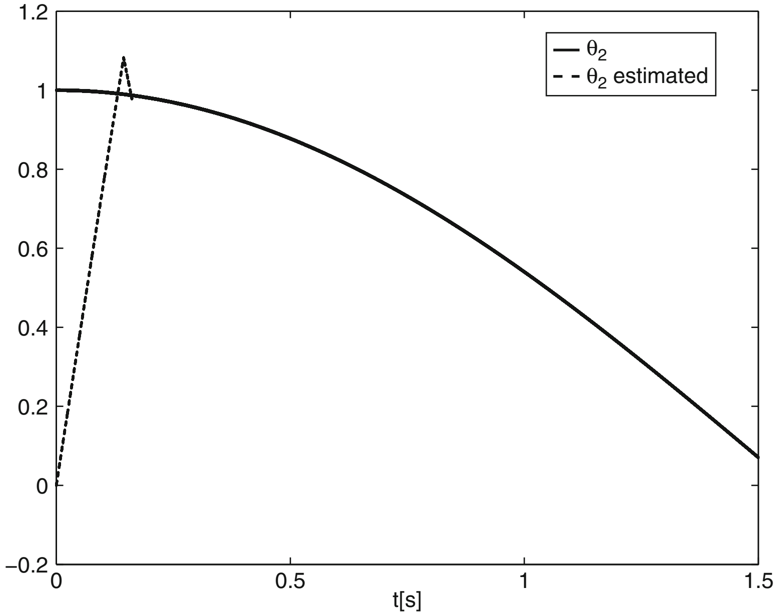

Fig. 7.2 Real and estimated velocity

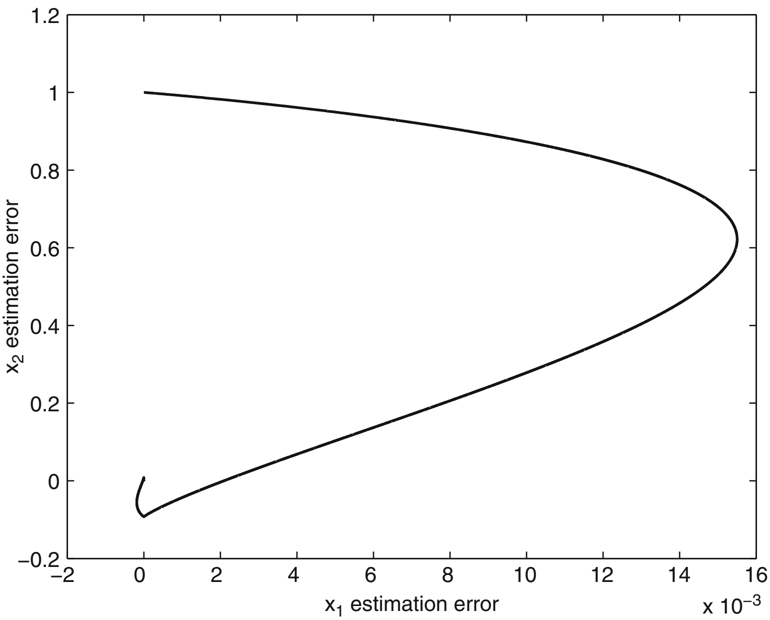

Fig. 7.3 Convergence to 0 of both $\tilde{x}_1$ and $\tilde{x}_2$

Notice that $\Delta F(t, x_1, x_2, \hat{x}_2, u) = f(t, x_1, x_2, u) - f(t, x_1, \hat{x}_2, u) = 0$ because $\hat{x}_2 = x_2$. Then the equivalent output injection z_{eq} is given by

$$z_{eq}(t) \equiv (\alpha_1 \text{sign}(\tilde{x}_1))_{eq} \equiv \xi(t, x_1, x_2, u) \tag{7.11}$$

Recall that the term $\xi(t, x_1, x_2, u)$ is composed of uncertainties and perturbations. This term may be written as

$$\xi(t, x_1, x_2, u) = \zeta(t) + \Delta F(t, x_1, x_2, u) \tag{7.12}$$

where $\zeta(t)$ is an external perturbation term and $\Delta F(t, x_1, x_2, u)$ embodies the parameter uncertainties.

Theoretically, the equivalent output injection is the result of an infinite frequency switching of the discontinuous term $\alpha_1 \text{sign}(\tilde{x}_1)$. Nevertheless, the realization of the observer produces high (finite) switching frequency making the application of a filter necessary. To eliminate the high-frequency component we will use the filter of the form:

$$\bar{\tau}\dot{\bar{z}}_{eq}(t) = -\bar{z}_{eq}(t) + \alpha_1 \text{sign}(\tilde{x}_1) \tag{7.13}$$

where $\bar{\tau} \in \mathbb{R}$ and $h \ll \tau \ll 1$ with h being the sampling step. It is possible to rewrite z_{eq} as the result of a filtering process in the following form

$$z_{eq}(t) = \bar{z}_{eq}(t) + \varepsilon(t) \tag{7.14}$$

where $\varepsilon(t) \in \mathbb{R}^n$ is the difference caused by the filtration process and $\bar{z}_{eq}(t)$ is the filtered version of $z_{eq}(t)$. It can be shown the $\lim_{\bar{\tau}\to 0, h/\bar{\tau}\to 0} \bar{z}_{eq}(\tau, h) = z_{eq}(t)$. In other words, the equivalent injection can be obtained by appropriate low-pass filtering of the discontinuous injection signals.

Perturbation Identification

Consider the case where the nominal model is totally known, i.e., for all $t > t_0$ the uncertain part $\Delta F(t, x_1, x_2, u) = 0$. The equivalent output injection takes the form

$$\bar{z}_{eq}(t) = (\alpha_1 \text{sign}(\tilde{x}_1))_{eq} = \zeta(t) \tag{7.15}$$

The result of the filtering process satisfies $\lim_{\bar{\tau}\to 0, h/\bar{\tau}\to 0} \bar{z}_{eq}(\tau) = \zeta(t)$. Then, any bounded perturbation can be identified, even in the case of discontinuous perturbations, by directly using the output of the filter. This is illustrated in the next example where a smooth continuous signal and a discontinuous perturbation are identified.

Example 7.2. Consider the mathematical model of the pendulum in Example 7.1 given by

$$\ddot{\theta} = \frac{1}{J}u - \frac{MgL}{2J}\sin(\theta) - \frac{V_s}{J}\dot{\theta} + v(t)$$

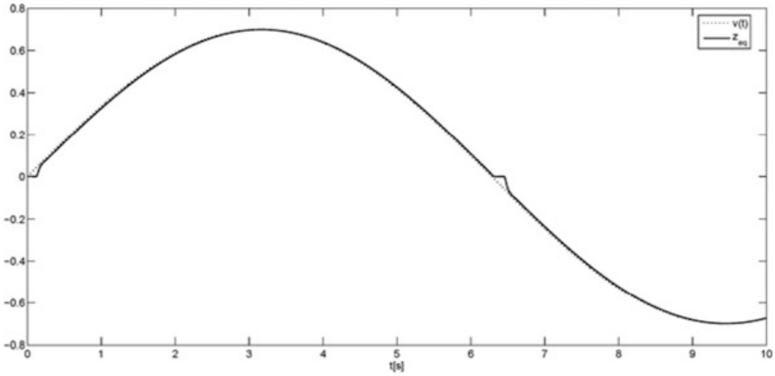

Fig. 7.4 Sinusoidal external perturbation identification

where $m = 1.1$ is the pendulum mass, $g = 9.815$ is the gravitational force, $L = 0.9$ is the pendulum length, $J = mL^2 = 0.891$ is the arm inertia, $V_S = 0.18$ is the pendulum viscous friction coefficient, and $v(t)$ is a bounded disturbance term. Assume that the angle θ is available for measurement. Introducing the variables $x_1 = \theta$, $x_2 = \dot{\theta}$ and the measured output $y = \theta$, the pendulum equation can be written in the state-space form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{J}u - \frac{mgL}{2J}\sin(x_1) - \frac{V_s}{J}x_2 + v(t) \\ y &= x_1 \end{aligned}$$

Suppose that all the system parameters ($m = 1.1$, $g = 9.815$, $L = 0.9$, $J = mL^2 = 0.891$, $V_S = 0.18$) are well known. The super-twisting observer for this system has the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \alpha_2|\tilde{x}_1|^{1/2}\,\text{sign}(\tilde{x}_1) \\ \dot{\hat{x}}_2 &= \frac{1}{J}u - \frac{mgL}{2J}\sin(x_1) - \frac{V_s}{J}\hat{x}_2 + \alpha_1\,\text{sign}(\tilde{x}_1) \\ \tilde{x}_1 &= y - \hat{x}_1 \end{aligned}$$

and the equivalent output injection in this case is given by

$$z_{eq} = (\alpha_1 \text{sign}(\tilde{x}_1))_{eq} = v(t)$$

using a low-pass filter with $\bar{\tau} = 0.02[s]$. For a sinusoidal external perturbation, the identification is shown in Fig. 7.4. Using a filter $\bar{\tau} = 0.002[s]$ the perturbation identification for a discontinuous signal is shown in Fig. 7.5.

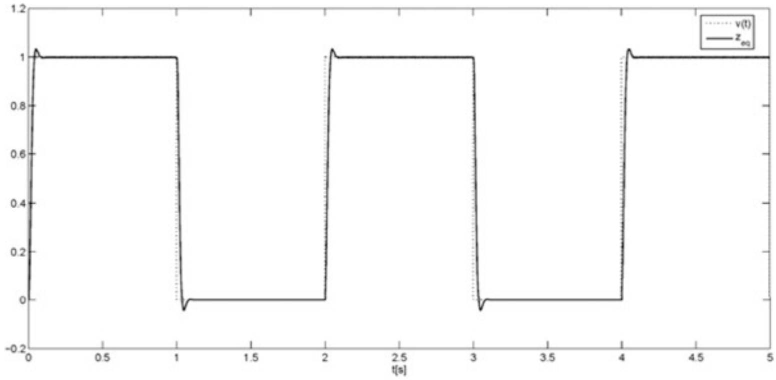

Fig. 7.5 Discontinuous perturbation identification

7.1.3 Parameter Identification

A problem that often arises in many control tasks is the uncertainty associated with the values of certain parameters, or even, in some cases, a complete lack of knowledge. In this situation, schemes to provide estimates of the unknown parameters at each instant are required. Although many algorithms have been developed to generate these estimates, the first step is usually to obtain a parametric model in which the desired parameters are concentrated in what is called the *unknown parameter vector* (denoted in most literature by θ). The interaction of these parameters within the system can then be expressed in *regressor form* as a linear combination of θ and a *regressor* which is a vector of known linear or nonlinear functions.

Regressor Form

Consider the nominal case when the system is not affected by disturbances and the only perturbations present are in the form of parametric uncertainties, i.e., $\zeta(t) = 0$ and $\xi(t, x_1, x_2, u) = \Delta F(t, x_1, x_2, u)$. The system acceleration (i.e., $\dot{x}_2$) can be represented as the sum of a well-known part and an uncertain part:

$$\dot{x}_2 = f(t, x_1, x_2, u) + \Delta F(t, x_1, x_2, u)$$

where $f(t, x_1, x_2, u) \in \mathbb{R}^n$ is the known part of the system and $\Delta F(t, x_1, x_2, u)$ is the uncertain part. Using the regressor notation[1] we can write the uncertain part as

$$\Delta F(t, x_1, x_2, u) = \theta(t)\varphi(t, x_1, x_2, u)$$

[1]For details see [173].

where $\theta(t) \in \mathbb{R}^{n \times l}$ is composed of the values of the uncertain parameters m, C, G, P and $\varphi(t, x_1, x_2, u) \in \mathbb{R}^l$ is the corresponding regressor. The system in Eq. (7.3) takes the form

$$\begin{aligned} \dot{x}_1 &= 7x_2 \\ \dot{x}_2 &= f(t, x_1, x_2, u) + \theta(t)\varphi(t, x_1, x_2, u), \quad u = U(t, x_1, \hat{x}_2) \\ y &= x_1 \end{aligned} \tag{7.16}$$

and the observer can be rewritten as

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \alpha_2 \lambda |\tilde{x}_1|^{1/2} \text{sign}(\tilde{x}_1) \\ \dot{\hat{x}}_2 &= f(t, x_1, \hat{x}_2, u) + \bar{\theta}(t)\varphi(t, x_1, \hat{x}_2, u) + \alpha_1 \text{sign}(\tilde{x}_1) \end{aligned} \tag{7.17}$$

where $\bar{\theta} \in \mathbb{R}^{n \times l}$ is a matrix of nominal values of the matrix $\theta(t)$. The error dynamics becomes

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 - \alpha_2 \lambda(\tilde{x}_1) \text{sign}(\tilde{x}_1) \\ \dot{\tilde{x}}_2 &= \theta(t) - \varphi(t, x_1, x_2, u) - \bar{\theta}(t))\varphi(t, x_1, \hat{x}_2, u) - \alpha_1 \text{sign}(\tilde{x}_1) \end{aligned} \tag{7.18}$$

The task is to design an algorithm which provides parameter identification for the original system (7.1), when only the position x_1 is measurable and the nominal model $\bar{\theta}(t)\varphi(t, x_1, x_2, u)$ is known.

Time-Invariant Parameter Identification

Consider the case when the system parameters are time invariant, i.e., $\theta(t) = \theta$. During the sliding motion, the equivalent output injection can be represented in the form

$$z_{eq}(t) = (\alpha_1 \text{sign}(\tilde{x}_1))_{eq} = (\theta - \bar{\theta})\varphi(t, x_1, x_2, u) \tag{7.19}$$

Notice that $\alpha_1 \text{sign}(\tilde{x}_1)$ is a known term and finite-time convergence of the observer guarantees $\varphi(t, x_1, \hat{x}_2, u) = \varphi(t, x_1, x_2, u)$ for all $t \geq t_0$. Equation (7.19) represents a linear regression model where the vector parameters to be estimated are $(\theta - \bar{\theta})$. To obtain the real system parameters θ a linear regression algorithm could be proposed from Eq. (7.19).

The recursive least-squares algorithm applied for parameter identification of dynamical systems is usually designed using discretization of the regressor and derivative of the states in order to obtain the regressor form. Then the algorithm is applied in discrete form.

In mechanical system observation and identification, we deal with a data set of a continuous-time nature. That is why the implementation of any standard discretization scheme is related to unavoidable losses of existing information. This produces a systematic error—basically caused by the estimation of the derivatives of the process. As shown above, the proposed second-order sliding mode technique provides an estimation of the derivatives, converging in finite time, that avoids any additional errors arising from any standard discretization scheme implementation. Below we present a continuous-time version of the least-squares algorithm based on the proposed second-order sliding mode observation scheme. Notice that the proposed algorithm can be implemented in analog devices directly. Defining $\Delta_\theta := \theta - \bar{\theta}$, post-multiplying Eq. (7.19) by $\varphi^T(t, x_1, x_2, u)$ (written for notational simplicity as $\varphi^T(t)$). Now, using the auxiliary variable σ for integration in time, the average values of Eq. (7.19) take the form

$$\frac{1}{t}\int_0^t \bar{z}_{eq}(\sigma)\varphi^T(\sigma)d\sigma = \Delta_\theta \frac{1}{t}\int_0^t \varphi(\sigma)\varphi(\sigma)^T d\sigma \tag{7.20}$$

where $\bar{z}_{eq}$ is obtained from Eq. (7.13). Therefore, the system parameters can be estimated from Eq. (7.20) by

$$\hat{\Delta}_\theta = \left[\int_0^t \bar{z}_{eq}(\sigma)\varphi^T(\sigma)d\sigma\right]\left[\int_0^t \varphi(\sigma)\varphi^T(\sigma)d\sigma\right]^{-1} \tag{7.21}$$

where $\hat{\Delta}_\theta$ is the estimate of Δ_θ. For any square matrix the following equalities hold

$$\begin{gathered}\Gamma^{-1}(t)\Gamma(t) = I,\\ \Gamma^{-1}(t)\dot{\Gamma}(t) + \dot{\Gamma}^{-1}(t)\Gamma(t) = 0\end{gathered} \tag{7.22}$$

Let us define $\Gamma(t) = \left[\int_0^t \varphi(\sigma)\varphi^T(\sigma)d\sigma\right]^{-1}$; then using Eq. (7.22) we can rewrite Eq. (7.21) in the form

$$\dot{\hat{\Delta}}_\theta = \left[\int_0^t \bar{z}_{eq}(\sigma)\varphi^T(\sigma)d\sigma\right]\dot{\Gamma}(t) + \bar{z}_{eq}(t)\varphi^T(t)\Gamma(t)$$

Now, using Eq. (7.20) we can write

$$\dot{\hat{\Delta}}_\theta = \hat{\Delta}_\theta\Gamma^{-1}(t)\dot{\Gamma}(t) + \bar{z}_{eq}(t)\varphi^T(t)\Gamma(t)$$

The equalities in Eq. (7.22) allow us to write a dynamic expression for estimating Δ_θ as

$$\dot{\hat{\Delta}}_\theta = \left[-\hat{\Delta}_\theta\varphi(t) + \bar{z}_{eq}(t)\right]\varphi^T(t)\Gamma(t) \tag{7.23}$$

In the same way, a dynamic form to find $\Gamma(t)$ is given by

$$\dot{\Gamma}(t) = -\Gamma(t)\varphi(t)\varphi^T(t)\Gamma(t) \tag{7.24}$$

The average values of the real $z_{eq}(t)$, without filtering, satisfy the equality

$$\int_0^t z_{eq}(\sigma)\varphi^T(\sigma)d\sigma = \Delta_\theta\int_0^t \varphi(\sigma)\varphi^T(\sigma)d\sigma$$

then

$$\Delta_\theta = \left[\int_0^t z_{eq}(\sigma)\varphi^T(\sigma)d\sigma \right] \Gamma(t)$$

Substituting from Eq. (7.14), the real values of the parameter vector Δ_θ satisfies

$$\Delta_\theta = \left[\int_0^t \bar{z}_{eq}(\sigma)\varphi^T(\sigma)d\sigma + \int_0^t \varepsilon(\sigma)\varphi^T(\sigma)d\sigma \right] \Gamma(t) \tag{7.25}$$

Let us assume $\bar{z}_{eq}(t) = \hat{\Delta}_\theta \varphi(t)$. In this case Eq. (7.25) becomes

$$\Delta_\theta = \left[\hat{\Delta}_\theta \int_0^t \varphi(\sigma)\varphi^T(\sigma)d\sigma + \int_0^t \varepsilon(\sigma)\varphi^T(\sigma)d\sigma \right] \Gamma(t)$$

which can be written as

$$\Delta_\theta = \hat{\Delta}_\theta + \left[\int_0^t \varepsilon(\sigma)\varphi^T(\sigma)d\sigma \right] \Gamma(t) \tag{7.26}$$

From Eqs. (7.21) and (7.26) it is possible to define the convergence conditions:

$$\sup ||t\Gamma(t)|| < \infty, \tag{7.27}$$

$$||\frac{1}{t} \int_0^t \varepsilon(\sigma)\varphi^T(\sigma)d\sigma|| \to 0 \quad \text{as} \quad t \to \infty \tag{7.28}$$

Condition (7.27), known as the persistent excitation condition,[2] requires the non-singularity of the matrix $\Gamma^{-1}(t) = \int_0^t \varphi(\sigma)\varphi^T(\sigma)d\sigma$. To avoid this restriction, introduce the term ρI where $0 < \rho \ll 1$ and I is the unit matrix, and redefine $\Gamma^{-1}(t)$ as

$$\Gamma^{-1}(t) = \int_0^t \left(\varphi(\sigma)\varphi^T(\sigma)d\sigma\right) + \rho I$$

In this case the value of $\Gamma^{-1}(t)$ is always nonsingular.

Notice that the introduction of the term ρI is equivalent to setting the initial conditions of Eq. (7.24), as

$$\Gamma(0) = \rho^{-1} I, \quad 0 < \rho\text{-small enough}$$

The introduction of the term ρ ensures the condition $\sup||t\Gamma(t)|| < \infty$ but does not guarantee the convergence of the estimated parameters to their real values. The convergence of the estimated values to the real ones is ensured by the *persistent excitation condition*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \left(\varphi(\sigma)\varphi(\sigma)^T d\sigma\right) > 0$$

[2]See, for example, [173].

The condition in Eq. (7.28) relates to the filtering process, and it gives the convergence quality of the identification. How fast the term $\frac{1}{t}\int_0^t \varepsilon(\sigma)\varphi(\sigma)^T d\sigma$ converges to zero dictates how fast the parameters will be estimated. The above can be summarized in Theorem 7.1.

Theorem 7.1. *The algorithm in Eqs. (7.23), (7.24) ensures the convergence of* $\hat{\Delta}_\theta \to \Delta_\theta$ *if conditions (7.27), (7.28) are satisfied*

Remark 7.1. The effect of noise sensitivity in the suggested procedure can be easily seen from (7.28):

$$\frac{1}{t}\int_0^t \varepsilon(\sigma)\,\varphi^T(\sigma)\,d\sigma \to 0 \text{ when } t \to \infty$$

The term $\varepsilon(t)$ in (7.14) includes all error effects caused by observation noise (if there is any) and errors in the realization of the equivalent output injection. One can see that if $\varepsilon(t)$ and $\varphi(t)$ are uncorrelated and are "on average" equal to zero, i.e.,

$$\frac{1}{t}\int_0^t \varepsilon(\sigma)\,d\sigma \to 0,\ \frac{1}{t}\int_0^t \varphi(\sigma)\,d\sigma \to 0$$

then the effect of noise vanishes.

The pendulum system is once again used to illustrate the previous algorithm.

Example 7.3. Consider the model of a pendulum from Example 7.1 with Coulomb friction given by the equation

$$\ddot{\theta} = \frac{1}{J}u - \frac{MgL}{2J}\sin(\theta) - \frac{V_s}{J}\dot{\theta} - \frac{P_s}{J}\operatorname{sign}(\dot{\theta})$$

where $m = 1.1$ is the pendulum mass, $g = 9.815$ is the gravitational force, $L = 0.9$ is the arm length, $J = mL^2 = 0.891$ is the arm inertia, $V_S = 0.18$ is the viscous friction coefficient, and $P_s = 0.45$ models the Coulomb friction coefficient. Suppose that the angle θ is available for measurement. Introducing the variables $x_1 = \theta$, $x_2 = \dot{\theta}$, the state-space representation for the system becomes

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \frac{1}{J}u - \frac{mgL}{2J}\sin(x_1) - \frac{V_s}{J}x_2 - \frac{P_s}{J}\operatorname{sign}(x_2),\\
y &= x_1
\end{aligned}$$

where $a_1 = \frac{mgL}{2J} = 5.4528$, $a_2 = \frac{V_s}{J} = 0.2020$, and $a_3 = \frac{P_s}{J} = 0.5051$ are the unknown parameters. Design the super-twisting-based observer as

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \alpha_2|\tilde{x}_1|^{1/2}\operatorname{sign}(\tilde{x}_1),\\
\dot{\hat{x}}_2 &= \frac{1}{J}u - \bar{a}_1\sin(x_1) - \bar{a}_2\hat{x}_2 - \bar{a}_3\operatorname{sign}(x_2) + \alpha_1\operatorname{sign}(\tilde{x}_1),
\end{aligned}$$

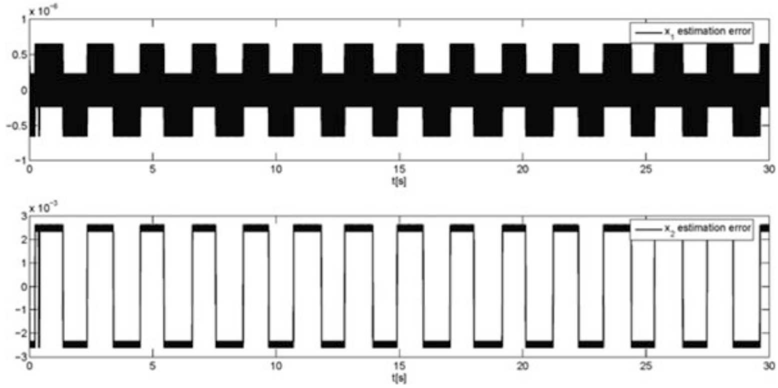

Fig. 7.6 x_1, x_2 estimation error LTI case

$$\tilde{x}_1 = y - \hat{x}_1$$

where $\bar{a}_1 = 2, \bar{a}_2 = \bar{a}_3 = 0.1$ are the nominal values of the unknown parameters. Let the control signal be generated by the twisting controller

$$u = -30\,\text{sign}(\theta - \theta_d) - 15\,\text{sign}(\dot{\theta} - \dot{\theta}_d), \tag{7.29}$$

where the reference signal is $\theta_d = 0.3\sin(3t + \pi/4) + 0.3\sin(1/2t + \pi)$. For a sampling time of $\Delta = 0.0001$ the state estimation error is shown in Fig. 7.6. In this case the identification variables are given by

$$\begin{aligned}
z_{eq} &= (\alpha_1 \text{sign}(\tilde{x}_1))_{eq} \\
\Delta_\theta &= [-a_1 + \bar{a}_1 \quad - a_2 + \bar{a}_2 \quad - a_3 + \bar{a}_3] \\
\Delta_\theta &= [-3.4528 \quad -0.1020 \quad -0.4051] \\
\varphi &= \begin{bmatrix} \sin(x_1) \\ x_2 \\ \text{sign}(x_2) \end{bmatrix}
\end{aligned}$$

It is now possible to use φ, the nonlinear regressor, to generate the dynamic adaptation gain $\Gamma(t)$ using Eq. (7.24). From Eq. (7.19), the value of $\bar{z}_{eq}$ is given by

$$\bar{z}_{eq} = z_{eq} = (\alpha_1 \text{sign}(\tilde{x}_1))_{eq}$$

The dynamic estimate of the parameter error vector Δ_θ, which contains all the necessary information to retrieve the desired parameter vector θ, can be generated by implementing the algorithm in Eq. (7.23). Figure 7.7 shows the convergence of the estimated parameters to the real parameter values.

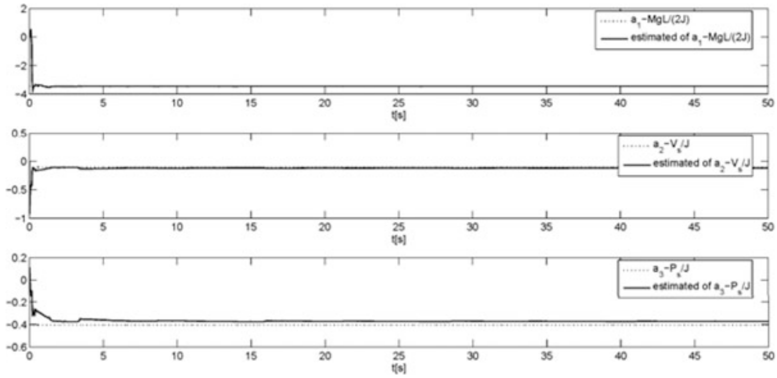

Fig. 7.7 Parameter identification for LTI case

7.2 Observation in Single-Output Linear Systems

The observer design problem for the general case of linear time-invariant systems will now be addressed in this section. The non-perturbed case will be revisited and then both unknown input and (exact) state estimation in the more complex perturbed case will be studied.

7.2.1 *Non-perturbed Case*

Consider a linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}\tag{7.30}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ are the system state and the output, $u \in \mathbb{R}^m$ is the known control, and the known matrices A, B, C have suitable dimensions. It is assumed that the pair (A, C) is observable. A standard Luenberger observer for this system is given by

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y})\\ \hat{y} &= C\hat{x}\end{aligned}\tag{7.31}$$

where $L \in \mathbb{R}^{n\times 1}$ is a gain matrix chosen such that $(A - LC)$ is a Hurwitz matrix. Such a gain matrix L always exists because of the assumed observability of the system, and it ensures asymptotic convergence to zero of the estimation error $e = x - \hat{x}$.

It is important to remark that without any disturbance, the standard Luenberger observer is sufficient to reconstruct asymptotically the states.

7.2.2 Perturbed Case

Now assume that the linear time-invariant system in Eq. (7.30) is perturbed by an external disturbance $\zeta(t)$. The perturbed linear time-invariant system is given by

$$\begin{aligned} \dot{x} &= Ax + Bu + D\zeta(t), \quad D \neq 0 \\ y &= Cx \end{aligned} \tag{7.32}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}^m$, and $\zeta \in \mathbb{R}$ is an unknown input (disturbance). The corresponding matrices A, B, C, D have suitable dimensions. The unknown input $\zeta(t)$ is assumed to be a bounded Lebesgue-measurable function, $|\zeta(t)| \leq \zeta^+$, $\zeta^+ > 0$.

The equations are understood in the Filippov sense in order to provide for possibility to use discontinuous signals in observers. It is assumed also that all the inputs which are considered allow the existence and extension of the solution to the whole semi-axis $t \geq 0$.

Suppose that the Luenberger observer Eq. (7.31) is used to estimate the states. The dynamics of the estimation error $e = x - \hat{x}$ are given by

$$\begin{aligned} \dot{e} &= (A - LC)e + D\zeta \\ y_e &= Ce \end{aligned}$$

In order to analyze the convergence properties of this observer, consider the Lyapunov-like function $V = e^T P_e e$, where $P_e = P_e^T > 0$ has suitable dimensions. Computing the first derivative of V we obtain

$$\dot{V} = e^T P_e \left((A - LC)e + D\zeta\right] + \left[(A - LC)e + D\zeta\right)^T P_e e$$

Suppose the matrix P_e is the solution of the Lyapunov equation

$$P_e(A - LC) + (A - LC)^T P_e = -H \tag{7.33}$$

for some $H = H^T > 0$, then the first derivative of V becomes

$$\dot{V} = -e^T He + 2(D\zeta)^T P_e e$$

The condition $\dot{V} \leq 0$ is satisfied for all the estimation error satisfying the inequality

$$||e|| > \frac{2\zeta^+ ||D^T P_e||}{||H||}$$

This last inequality implies that the proposed Luenberger observer only can ensure the convergence of the estimation error to a bounded region around the origin. As a consequence, standard Luenberger observer cannot be applied for state reconstruction on perturbed systems.

The task in this section is to build an observer guaranteeing asymptotic (and preferably exact finite-time convergent) estimation of the states and the unknown input. Obviously, it can be assumed without loss of generality that the known input u is equal to zero (i.e., $u(t) = 0$).

It is very important to establish conditions when the unknown input can be reconstructed along with estate estimation. For this reason, several definitions will be introduced to study the state observation problem for perturbed linear systems. It is assumed in the following definitions that $u = 0$.

Definition 7.1 ([107]). System (7.32) is called strongly observable if for any initial state $x(0)$ and any input $\zeta(t)$, $y(t) \equiv 0$ with $\forall t \geq 0$ implies that also $x \equiv 0$.

Definition 7.2 ([107]). The system is strongly detectable , if for any $\zeta(t)$ and $x(0)$ it follows that $y(t) \equiv 0$ with $\forall t \geq 0$ implies $x \to 0$ with $t \to \infty$.

It is important to remark that these two definitions are not directly related to the structure of the system. However, important consequences on the system structure can be established.

Theorem 7.2. *The system (7.32) is strongly observable if and only if the output y has relative degree n with respect to the unknown input $\zeta(t)$, i.e., it has no invariant zeros.*

Proof. Let matrix P be defined by

$$P = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Strong observability of the system requires observability, and therefore $rank\, P = n$. The observability implies the existence of a relative degree r associated with the output y with respect to the unknown input ζ. Indeed, otherwise $PD = 0$ and therefore $D = 0$. Then the coordinate transformation $x_O = Px$ turns system (7.32) into

$$\begin{aligned} \dot{x}_O &= A_O x_O + B_O u(t) + D_O \zeta(t) \\ y(t) &= C_O x_O \end{aligned} \tag{7.34}$$

where

$$A_O = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \tag{7.35}$$

$$D_O = [CD, \dots, CA^{n-2}D, CA^{n-1}D]^T \tag{7.36}$$

$$C_O = [1, 0, \dots, 0] \tag{7.37}$$

and the a_j, $j = 1, \dots, n$ are some constants. The vector B_O does not have any specific form. Recall that u is assumed to be zero. When $r = n$ only the last component of D_O is not zero. It is obvious that in that case the identity $y \equiv 0$ implies $x_O \equiv 0$.

Assume now that $r < n$. That means that some nontrivial zero dynamics exists, which corresponds to nontrivial solutions satisfying $y \equiv 0$ and contradicts the conditions for strong observability. This ends the proof of the theorem. □

7.2.3 Design of the Observer for Strongly Observable Systems

The importance of the property of strong observability for the type of linear systems described by Eq. (7.32) lies in the fact that it ensures the existence of the sliding mode state observer. This part of the chapter will explore the design of such an observer.

Assumption 7.1. *System (7.32) has the relative degree n with respect to the unknown input $\zeta(t)$ (i.e., the system is strongly observable).*

The observer is built in the form

$$\dot{z} = Az + Bu + L(y - Cz) \tag{7.38}$$

$$\hat{x} = z + Kv \tag{7.39}$$

$$\dot{v} = W(y - Cz, v) \tag{7.40}$$

where $z, \hat{x} \in \mathbb{R}^n$, $\hat{x}$ is the estimation of x and the matrix $L = [l_1, l_2, \dots, l_n]^T \in \mathbb{R}^n$ is a correction factor chosen so that the eigenvalues of the matrix $A - LC$ have negative real parts. (Such an L exists due to Assumption 7.1 and Theorem 7.2.)

The proposed observer is actually composed of two parts. Equation (7.38) is a traditional Luenberger observer providing the boundedness of the difference $z - x$ in the presence of the unknown bounded input ζ. System (7.40) is based on high-order sliding modes and ensures the finite-time convergence of the resulting estimation error to zero.

Suppose that only the states are to be estimated and that Assumption 7.1 holds. Note that in the simplest case when $n = 1$ the only observable coordinate coincides with the measured output and, therefore, only the input estimation problem makes sense, requiring Assumption 7.1. Thus assume that $n > 1$.

Since the pair (C, A) is observable, arbitrary stable values can be assigned to the eigenvalues of the matrix $(A - LC)$, choosing an appropriate gain matrix L. Obviously the pair $(C, A - LC)$ is also observable and therefore its observability matrix

$$\tilde{P} = \begin{bmatrix} C \\ C(A-LC) \\ C(A-LC)^2 \\ \vdots \\ C(A-LC)^{n-1} \end{bmatrix} \tag{7.41}$$

is nonsingular. Set the gain matrix $K = \tilde{P}^{-1}$ and assign

$$\hat{x} = z + \tilde{P}^{-1} v \tag{7.42}$$

The nonlinear part of the observer Eq. (7.40) is chosen as

$$\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_n M^{1/n} |v_1 - y + Cz|^{(n-1)/n} \operatorname{sign}(v_1 - y + Cz) + v_2 \\
\dot{v}_2 &= w_2 = -\alpha_{n-1} M^{1/(n-1)} |v_2 - w_1|^{(n-2)/(n-1)} \operatorname{sign}(v_2 - w_1) + v_3 \\
&\vdots \\
\dot{v}_{n-1} &= w_{n-1} = -\alpha_2 M^{1/2} |v_{n-1} - w_{n-2}|^{1/2} \operatorname{sign}(v_{n-1} - w_{n-2}) + v_n \\
\dot{v}_n &= -\alpha_1 M \operatorname{sign}(v_n - w_{n-1})
\end{aligned} \tag{7.43}$$

where v_i, z_i, and w_i are the components of the vectors $v, z \in \mathbb{R}^n$, and $w \in \mathbb{R}^{n-1}$, respectively. The parameter M must be chosen sufficiently large, and in particular $M > |d|\zeta^+$ must be satisfied, where $d = CA^{n-1}D$. The constants α_i are chosen recursively and must be sufficiently large (see Chap. 6 for a more detailed discussion). In particular, one of the possible choices is $\alpha_1 = 1.1, \alpha_2 = 1.5, \alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, in a situation when $n \leq 6$. Note that Eq. (7.43) has a recursive form, useful for the parameter adjustment.

Recall that $x_O = Px$ is the vector of canonical observation coordinates and $e_O = P(\hat{x} - x)$ is the canonical observation error. With this in mind, the following theorem summarizes the exact finite-time convergence properties of the designed observer.

Theorem 7.3. *Let Assumption 7.1 be satisfied and the output be measured subject to noise, which is a Lebesgue-measurable function of time with maximal magnitude ε. Then with properly chosen α_j's, and a sufficiently large M, the state x of the system is estimated in finite time by the observer Eqs. (7.38), (7.41), (7.42)* *and (7.43). With sufficiently small ε the observation errors $e_{Oi} = \hat{x}_{Oi} - \hat{x}_{Oi} = CA^{i-1}(\hat{x}_i - x_i)$ are of the order of $\varepsilon^{(n-i+1)/n}$, i.e., they satisfy the inequalities $|e_{Oi}| \leq \Gamma_i \varepsilon^{(n-i+1)/n}$ for some constants $\Gamma_i > 0$ depending only on the observer, the system parameters, and the input upper bound. A level of accuracy of the order of $\varepsilon^{1/n}$ is obtained in noncanonical coordinates due to the mix of coordinates. In particular, the state x is estimated* ***exactly*** *and in* ***finite time*** *in the absence of noises.*

Remark 7.2. It is worth noting that using a Kalman filter instead of a Luenberger observers in this algorithm may be beneficial in the presence of measurement noise.

The finite-time convergence of the observation error, which is guaranteed using a Luenberger observer, allows us to address the problem of reconstructing ζ, the unknown input to the system.

Identification of the Unknown Input

Now let $v \in \mathbb{R}^{\bar{n}+1}$, where $\bar{n} = n + k$, satisfy the nonlinear differential equation (7.40) in the form

$$\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_{\bar{n}+1} M^{1/(\bar{n}+1)} |v_1 - y + Cz|^{(\bar{n})/(\bar{n}+1)} \operatorname{sign}(v_1 - y + Cz) + v_2 \\
\dot{v}_2 &= w_2 = -\alpha_{\bar{n}} M^{1/(\bar{n})} |v_2 - w_1|^{(\bar{n}-1)/(\bar{n})} \operatorname{sign}(v_2 - w_1) + v_3 \\
&\vdots \\
\dot{v}_n &= -\alpha_{k+2} M^{1/(k+2)} |v_n - w_{n-1}|^{(k+1)/(k+2)} \operatorname{sign}(v_n - w_{n-1}) + v_{n+1} \qquad (7.44) \\
&\vdots \\
\dot{v}_{\bar{n}} &= w_{\bar{n}} = -\alpha_2 M^{1/2} |v_{\bar{n}-1} - w_{\bar{n}-2}|^{1/2} \operatorname{sign}(v_{\bar{n}+1} - w_{\bar{n}-2}) + v_{\bar{n}} \\
\dot{v}_{\bar{n}+1} &= -\alpha_1 M \operatorname{sign}(v_{\bar{n}+1} - w_{\bar{n}}) \qquad (7.45)
\end{aligned}$$

where M is a sufficiently large constant. As described previously, the nonlinear differentiator has a recursive form, and the parameters α_i are chosen in the same way. In particular, one of the possible choices is $\alpha_1 = 1.1, \alpha_2 = 1.5, \alpha_3 = 2, \alpha_4 = 3, \alpha_5 = 5, \alpha_6 = 8$, in the situation when $n+k \leq 5$. In any computer realization one has to calculate the internal auxiliary variables w_j, $j = 1, \ldots, n+k$, using only the simultaneously sampled current values of y, z_1, and v_j. The equality $\bar{e} = \omega$ is established in finite time, where ω is the truncated vector

$$\omega = (v_1, \ldots, v_n)^T$$

Thus, in the case of unknown input reconstruction, the corresponding observer Eq. (7.39) is now modified and takes the form

$$\hat{x} = z + \tilde{P}^{-1}\omega \qquad (7.46)$$

where $\tilde{P}$ is the observability matrix previously defined. The estimation of the input ζ is given as

$$\hat{\zeta} = \frac{1}{d}\left(v_{n+1} - (a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)\right) \tag{7.47}$$

where $s^n - a_n s^{n-1} - \cdots - a_1 = (-1)^n \det(A - LC - sI)$ defines the characteristic polynomial of the matrix $A - LC$.

An example is now presented to illustrate the effectiveness of the proposed observer Eq. (7.46) for both state and unknown input reconstruction in linear systems.

Example 7.4. Consider system (7.32) with matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix}$$

$$B = D = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T, C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

and initial conditions $x(0) = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$. Note that A is not stable since its eigenvalues are $-3, -2, -1, 1$. The relative degree r with respect to the unknown input equals 4. As a consequence, the system is strongly observable. The unknown input

$$\zeta = \cos(0.5t) + 0.5\sin(t) + 0.5$$

is used for pedagogical purposes. It is obviously a bounded smooth function with bounded derivatives. It is also assumed that $u = 0$. Furthermore, let the output of the system be affected by a deterministic noise of the form

$$\bar{w} = 0.1\sin(1037|\cos(687t)|)$$

of amplitude $\epsilon = 0.1$. The correction factor $L = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$ places the eigenvalues of $A - LC$ at $\{-1, -2, -3, -4\}$. The gain matrix $\tilde{P}^{-1}$ is thus given by

$$\tilde{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 5 & 5 & 5 & 1 \end{bmatrix} \tag{7.48}$$

The parameters $\alpha_1 = 1.1, \alpha_2 = 1.5, \alpha_3, \alpha_4 = 3, M = 2$ are chosen, and the nonlinear part of the observer takes the form

$$\dot{v}_1 = w_1 = -3 \cdot 2^{1/4} |v_1 - y + Cz|^{(3)/4} \operatorname{sign}(v_1 - y + Cz) + v_2$$
$$\dot{v}_2 = w_2 = -2 \cdot 2^{1/3} |v_2 - w_1|^{2/3} \operatorname{sign}(v_2 - w_1) + v_3$$

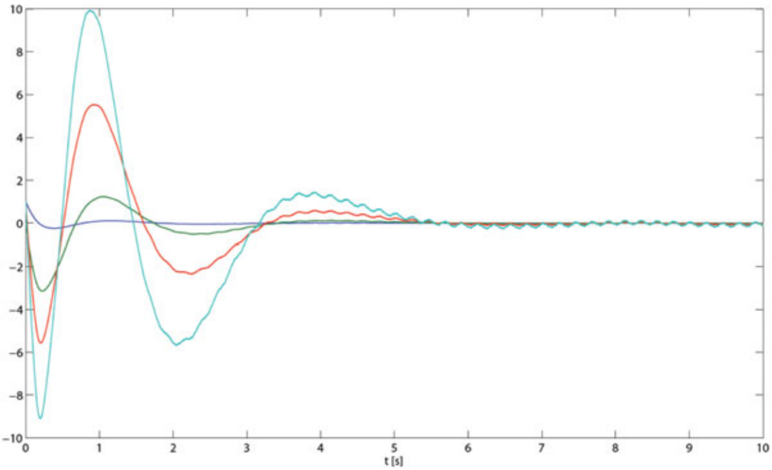

Fig. 7.8 State estimation errors in the presence of a deterministic noise of amplitude 10^{-1}

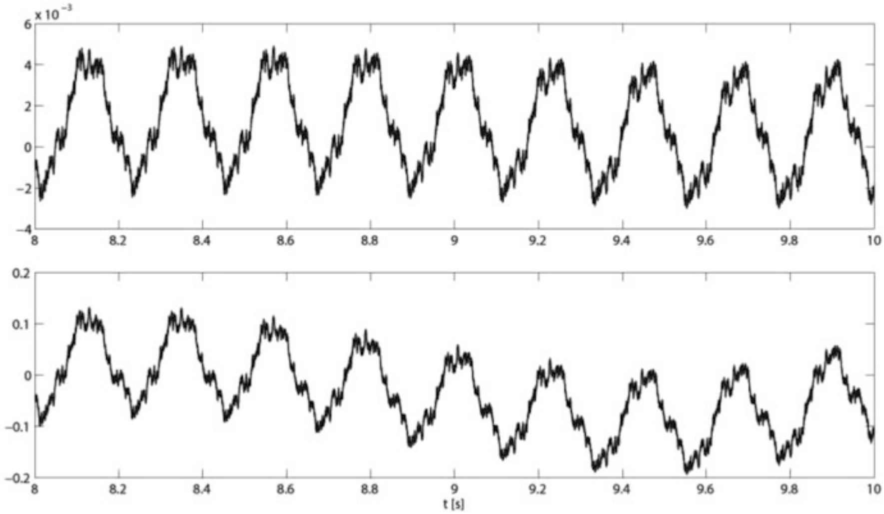

Fig. 7.9 Detail of observer error graphs. Estimation error of x_2 (*above*). Estimation error of x_4 (*below*)

$$\dot{v}_3 = w_{n-1} = -1.5 \cdot 2^{1/2}|v_3 - w_2|^{1/2}\,\text{sign}(v_3 - w_2) + v_4$$
$$\dot{v}_4 = -1.1 \cdot 2\,\text{sign}(v_4 - w_3)$$

The observer performance and finite-time convergence for the sampling time interval $\tau = 0.001$ are depicted in Fig. 7.8. Figure 7.9 shows the details of the state convergence. Note that the estimation error associated with x_2 converges to a bounded region of order $5 \cdot 10^{-3}$, while the estimation error in x_4 converges to a

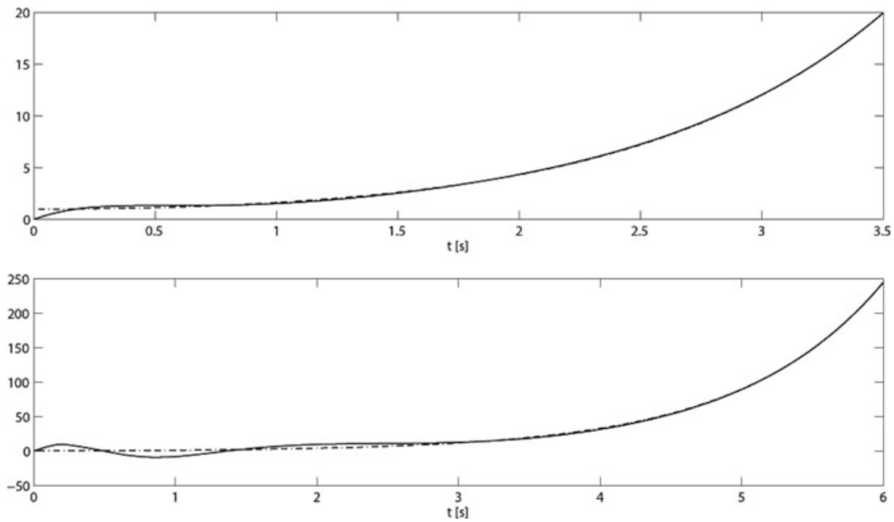

Fig. 7.10 Convergence of $\hat{x}_1, \hat{x}_4$ to x_1 and x_4

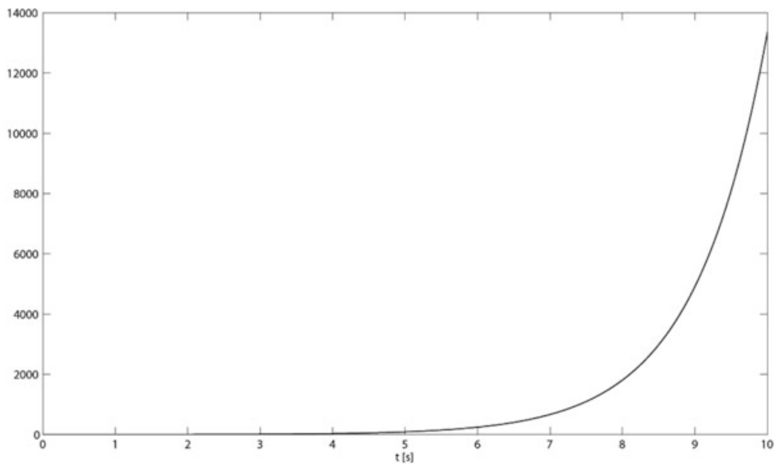

Fig. 7.11 System coordinates

bounded region of order $2 \cdot 10^{-1}$. The transient process is shown in Fig. 7.10 for the states x_1 and x_4. It is seen from Fig. 7.11 that the system trajectories and their derivatives of any order tend to infinity. Thus, the differentiator could not perform the observation alone. Figure 7.12 shows the effect of discretization in observation. The sampling time intervals $\tau = 0.0001$ and $\tau = 0.01$ were taken in the absence of noises.

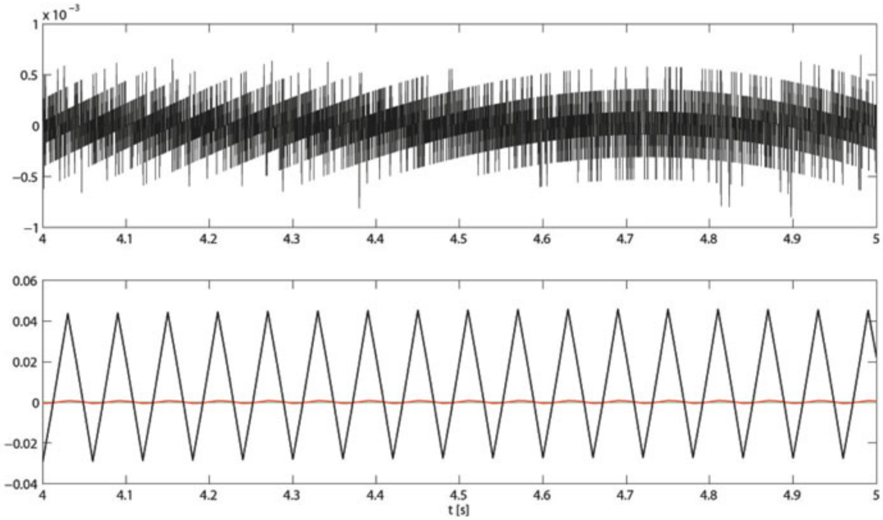

Fig. 7.12 Observer errors (detail) with sampling intervals $\tau = 0.0001$ (*above*) and $\tau = 0.01$ (*below*)

Consider now the input ζ as a bounded function with a Lipschitz derivative, $k = 1$. Both the state x and the disturbance ζ are now estimated.

The linear part of the observer Eq. (7.40) is designed in the same way as before with $L = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$ and $\tilde{P}$ given by Eq. (7.48). Finally, the parameters for Eq. (7.45) are chosen as $\alpha_1 = 1.1, \alpha_2 = 1.5, \alpha_3 = 2, \alpha_4 = 3, \alpha_5 = 5$, and $\alpha_6 = 8$ along with $M = 1$. Finite-time convergence of estimated states to the real states is shown in Fig. 7.13 with the sampling interval $\tau = 0.001$. The unknown input estimation is obtained using the relation (7.47) and it is demonstrated in Fig. 7.14. The effects of discretization are shown in Fig. 7.15 based on the sampling intervals $\tau = 0.0001$ and $\tau = 0.01$.

7.3 Observers for Single-Output Nonlinear Systems

The sliding mode algorithms given in the last section can be extended to the unknown input reconstruction problem for a more general single-output nonlinear case. A differentiator-based scheme is used. This requires a system transformation into a canonical basis. The unknown inputs are expressed as a function of the transformed states and thus a diffeomorphism must be established to recover the unknown input in the original states.

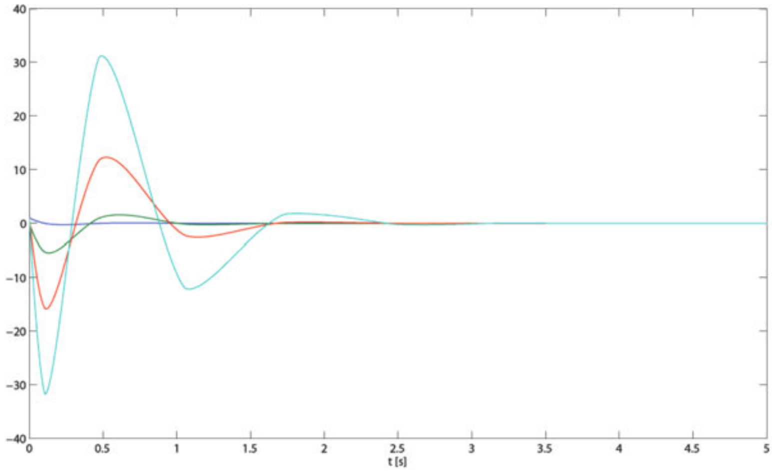

Fig. 7.13 Observer errors for the unknown input estimation case

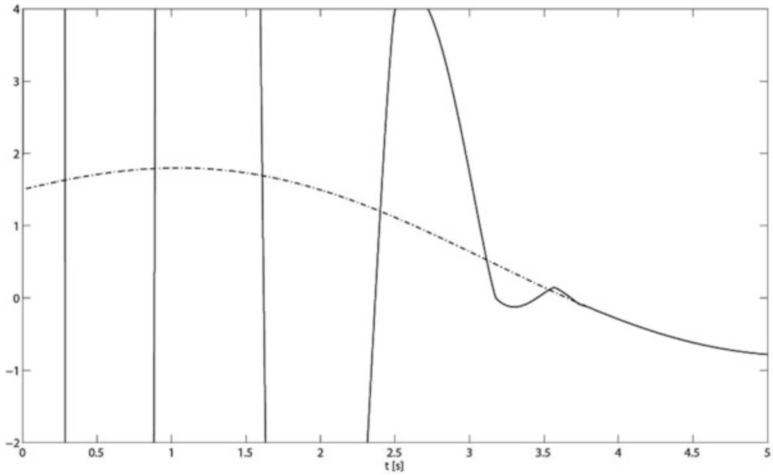

Fig. 7.14 Unknown input estimation

7.3.1 Differentiator-Based Observer

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\varphi(t) \\ y &= h(x) \end{aligned} \tag{7.49}$$

where $f(x) : \Omega \to \mathbb{R}^n$, $g(x) \in \mathbb{R}^n$, $h(x) : \mathbb{R}^n \to \mathbb{R}$ are smooth scalar and vector functions defined on an open set $\Omega \subset \mathbb{R}^n$. The states, outputs, and unknown inputs are given by $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, and $\varphi(t) \in \mathbb{R}$.

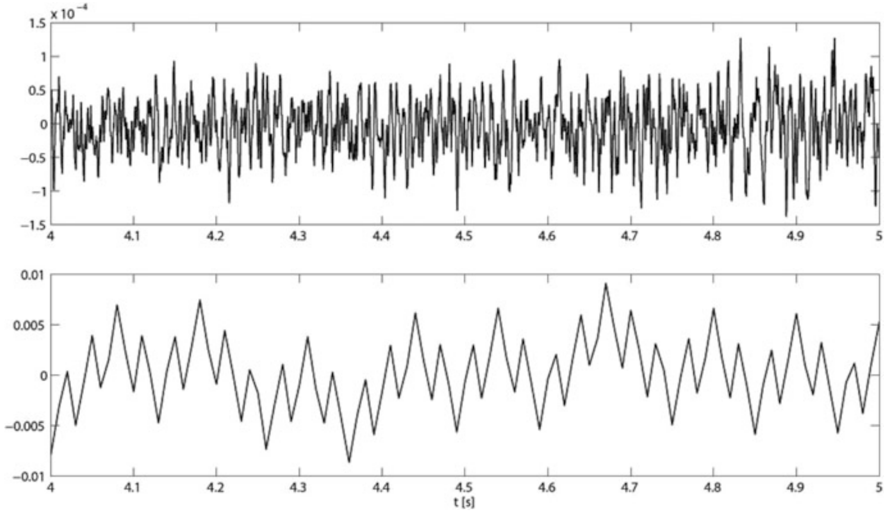

Fig. 7.15 Unknown input estimation error with $\tau = 0.0001$ (*above*) and 0.01 (*below*)

Assumption 7.2. *For any point $x \in \Omega$ it is satisfied that the output y has relative degree n with respect to the disturbance $\varphi(t)$, i.e.,*

$$\begin{aligned} &L_g L_f^k h(x) = 0, \quad k < n-1 \\ &L_g L_f^{n-1} h(x) \neq 0 \end{aligned} \tag{7.50}$$

This assumption means that system (7.49) does not have internal dynamics. The problem then is to design a finite-time convergent observer that generates the estimates $\hat{x}$, $\hat{\varphi}(t)$ for the state x and the disturbance $\varphi(t)$ given only the measurements $y = h(x)$. In order to accomplish this, system (7.49) must first undergo a transformation into a canonical form which we shall now proceed to describe.

System Transformation

The system in Eq. (7.49) with relative degree n can be represented in a new basis that is introduced as follows:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix} \in \mathbb{R}^n \tag{7.51}$$

It is well known[3] that if Assumption 7.2 is satisfied, then the mapping

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \tag{7.52}$$

defines a local diffeomorphism in a neighborhood of any point $x \in \bar{\Omega} \subset \Omega$, which means

$$x = \Phi^{-1}(\xi)$$

This is an important property since the signals obtained for the transformed system cannot be interpreted for the original system otherwise. Taking into account Eqs. (7.51) and (7.52), the system (7.49) with relative degree n can be written in the form

$$\dot{\xi} = \Lambda\xi + \psi(\xi) + \lambda(\xi, \varphi(t)) \tag{7.53}$$

where

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{7.54}$$

and

$$\psi(\xi) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ L_f^n h(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ L_f^n h(\Phi^{-1}(\xi)) \end{pmatrix} \tag{7.55}$$

and

$$\lambda(\xi, \varphi(t)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ L_g L_f^{n-1} h(x)\varphi(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ L_g L_f^{n-1} h(\Phi^{-1}(\xi))\varphi(t) \end{pmatrix} \tag{7.56}$$

In order to obtain the necessary derivatives of the output, the following higher-order sliding mode observation/differentiation algorithm (from Chaps. 4 and 6) is used.

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[3] See, for example, [112].

Higher-Order Sliding Mode Observer/Differentiator

The derivatives ξ_i, $i = 1, \ldots, n$ of the measured outputs $y = h(x)$ can be estimated in finite time by the higher-order sliding mode differentiator. This can be written in the form

$$\begin{aligned}
\dot{z}_0 &= v_0 = z_1 - \kappa_0 |z_0 - y|^{\frac{n}{n+1}} \operatorname{sign}(z_0 - y) \\
\dot{z}_1 &= v_1 = z_2 - \kappa_1 |z_1 - v_0|^{\frac{n-1}{n}} \operatorname{sign}(z_1 - v_0) \\
&\ldots \\
\dot{z}_i &= v_i = z_i - \kappa_i |z_i - v_{i-1}|^{\frac{n-i}{n-i+1}} \operatorname{sign}(z_i - v_{i-1}) \\
&\ldots \\
\dot{z}_n &= -\kappa_n \operatorname{sign}(z_n - v_{n-1})
\end{aligned} \tag{7.57}$$

The choice of κ_i, $i = 0, \ldots, n$ is discussed in Chap. 6 (see, for instance, Eq. (6.29)). Therefore, the following estimates are available in finite time:

$$\hat{\xi} = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \vdots \\ \hat{\xi}_n \end{pmatrix} = \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \\ \vdots \\ \hat{\phi}_n(x) \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} = \Phi(x) \tag{7.58}$$

The finite-time estimation $\hat{x}$ of the state x can be easily obtained from Eqs. (7.52) and (7.58) as

$$\hat{x} = \Phi^{-1}(\hat{\xi}) \tag{7.59}$$

With a proper estimation of the system states achieved we can now concentrate on the other observation objective and proceed to identify the disturbance.

7.3.2 *Disturbance Identification*

Combining Eq. (7.59) and the last coordinate of the transformed system in Eq. (7.53), we obtain

$$\dot{\xi}_n = L_f^n h(\Phi^{-1}(\xi)) + L_g L_f^{n-1} h(\Phi^{-1}(\xi))\varphi(t) \tag{7.60}$$

Since the exact finite-time estimate $\hat{\dot{\xi}}_n$ of $\dot{\xi}_n$ is available via the high-order sliding mode differentiator Eq. (7.57), and using the estimate $\hat{\xi}$ of ξ in Eq. (7.60), the finite-time estimate $\hat{\varphi}(t)$ of the disturbance can be obtained from

$$\hat{\varphi}(t) = (L_g L_f^{n-1} h(\Phi^{-1}(\hat{\xi})))^{-1} \left[\hat{\dot{\xi}}_n - L_f^n h(\Phi^{-1}(\hat{\xi})) \right] \tag{7.61}$$

Example 7.5. Consider a satellite system which is modeled as

$$\dot{\rho} = v$$
$$\dot{v} = \rho\omega^2 - \frac{k_g M}{\rho^2} + d$$
$$\dot{\omega} = -\frac{2v\omega}{\rho} - \frac{\theta\omega}{m}$$

In the equations above ρ is the distance between the satellite and the Earth's center, v is the radial speed of the satellite with respect to the Earth, Ω is the angular velocity of the satellite around the Earth, m and M are the mass of the satellite and the Earth, respectively, k_g represents the universal gravity coefficient, and θ is the damping coefficient. The quantity d which affects the radial velocity equation is assumed to be a disturbance which is to be reconstructed/estimated. Let $x := \text{col}(x_1, x_2, x_3) := (\rho, v, \omega)$. The satellite system can then be rewritten as follows:

$$\dot{x} = \begin{pmatrix} x_2 \\ x_1 x_3^2 - \frac{k_1}{x_1^2} \\ -\frac{2x_2x_3}{x_1} - k_2 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} d(t) \tag{7.62}$$

$$y = x_1 \tag{7.63}$$

where y is the system output, $k_1 = k_g M$ and $k_2 = \theta/m$. By direct computation, it follows that

$$L_g h(x) = 0, \quad L_g L_f h(x) = 1$$

and thus the system in Eqs. (7.62) and (7.63) has global relative degree 2. Choose the coordinate transformation as $T : \xi_1 = x_1, \xi_2 = x_2, \eta = x_1^2 x_3$. Note that for $x_1 \neq 0$, this transformation is invertible and an analytic expression for the inverse can be obtained as $x_1 = \xi_1, x_2 = \xi_2, x_3 = \frac{\eta}{\xi_1^2}$. Since $x_1 = \rho$ is the distance of the satellite from the center of the Earth $x_1 \neq 0$. It follows that in the new coordinates $\text{col}(\xi_1, \xi_2, \eta)$ the system from Eqs. (7.62) and (7.63) can be described by

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \frac{\eta^2}{\xi_1^3} - \frac{k_1}{\xi_1^2} + d$$
$$\dot{\eta} = -k_2 \eta$$

In this system the internal dynamics (given by the last equation) are linear and asymptotically stable, since $k_2 > 0$; and therefore the system itself is locally detectable. The higher-order sliding mode differentiator is described by

$$\begin{aligned}
\dot{z}_0^1 &= v_0^1 \\
v_0^1 &= -\lambda_0^1 |z_0^1 - y|^{2/3} \text{sign}(z_0^1 - y) + z_1^1 \\
\dot{z}_1^1 &= v_1^1 \\
v_1^1 &= -\lambda_1^1 |z_1^1 - v_0^1|^{1/2} \text{sign}(z_1^1 - v_0^1) + z_2^1 \\
\dot{z}_2^1 &= -\lambda_2^1 \text{sign}(z_2^1 - v_1^1)
\end{aligned}$$

where $\text{col}(z_0^1, z_1^2)$ or $\text{col}(z_0^1, v_0^1)$ give an estimate of ξ and the estimate for η can be obtained from the equation $\dot{\hat{\eta}} = -k_2\hat{\eta}$. Therefore, the estimate of the disturbance $d(t)$ is available online, and can be obtained from the expression

$$\hat{d} = \dot{\hat{\xi}}_2 - \frac{\hat{\eta}^2}{\hat{\xi}_1^3} + \frac{k_1}{\hat{\xi}_1^2}$$

In the simulations, the parameters have been chosen as follows: $m = 10, M = 5.98 \times 10^{24}, k_g = 6.67 \times 10^{-11}$, and $\theta = 2.5 \times 10^{-5}$. For simulation purposes, the disturbance $d(t) = \exp^{-0.002t} \sin(0.02t)$ has been introduced. The differentiator gains $\lambda_i^j j$ have been chosen as $\lambda_0^1 = 2$ and $\lambda_1^1 = \lambda_2^1 = 1$. In the following simulation, the initial values $x_0 = (10^7, 0, 6.3156 \times 10^{-4})$ are used for the plant states (in the original coordinates) while for the observer $z_0 = (1.001 \times 10^7, 0, 1)$ and $\hat{\eta}_0 = 6.3156 \times 10^{-4}$ (in the transformed coordinate system). Figures 7.16 and 7.17 show that the states and the disturbance signal $d(t)$ can be reconstructed faithfully.

7.4 Regulation and Tracking Controllers Driven by SM Observers

7.4.1 Motivation

The higher-order sliding mode observers presented in this chapter provide both theoretically exact observation and unknown input identification. This means that using such observers we cannot only observe the system states but also compensate matched uncertainties/disturbances (theoretically) exactly. Therefore, higher-order sliding mode observers create a situation in which two quite distinct control strategies can be used for the compensation of the matched uncertainties/disturbances:

- Sliding mode control based on observed system states
- The direct/continuous compensation of the uncertainties/disturbances based on its reconstructed/identified values

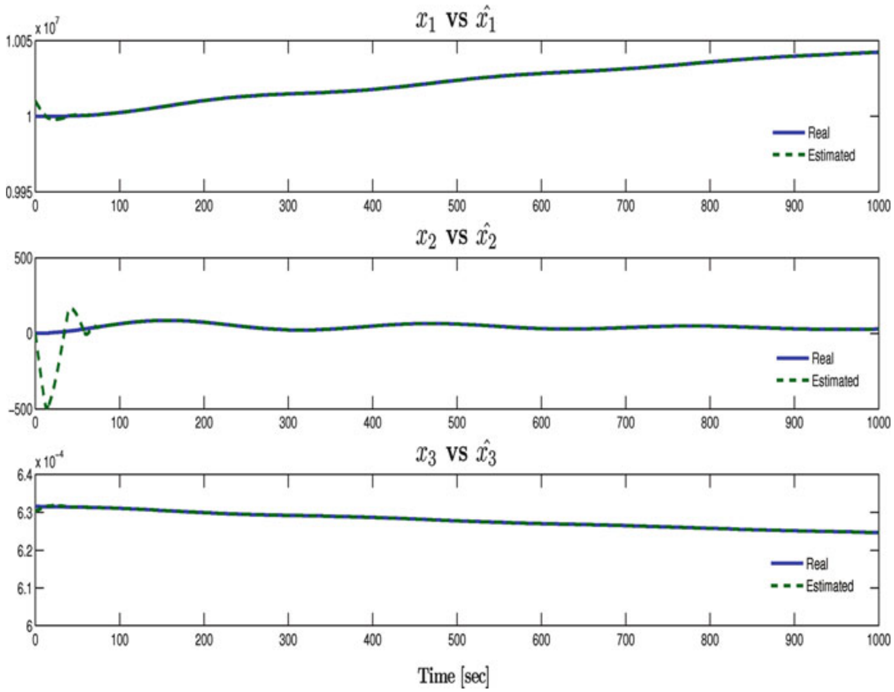

Fig. 7.16 The response of system states and their estimates

Both types of compensation are theoretically exact. In this section the control algorithms based on two proposed strategies are derived and compared. Finally, recommendations for the proper use of proposed algorithms are made which depend on the agility of the actuators, the parameters of discrete time implementation, and the measurement noise.

7.4.2 Problem Statement

Consider the linear time-invariant system with unknown inputs (LTISUI) of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t) + w(t)) \\ y(t) &= Cx(t) \end{aligned} \tag{7.64}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ $(1 \le p < n)$ are the state vector, the control, and the output of the system, respectively. The unknown inputs are represented by the function vector $w(t) \in \mathbb{R}^q$. Furthermore, $\text{rank}(C) = p$ and $\text{rank}(B) = m$. The following conditions are assumed to be fulfilled henceforth:

A1. The pair (A, B) is controllable.

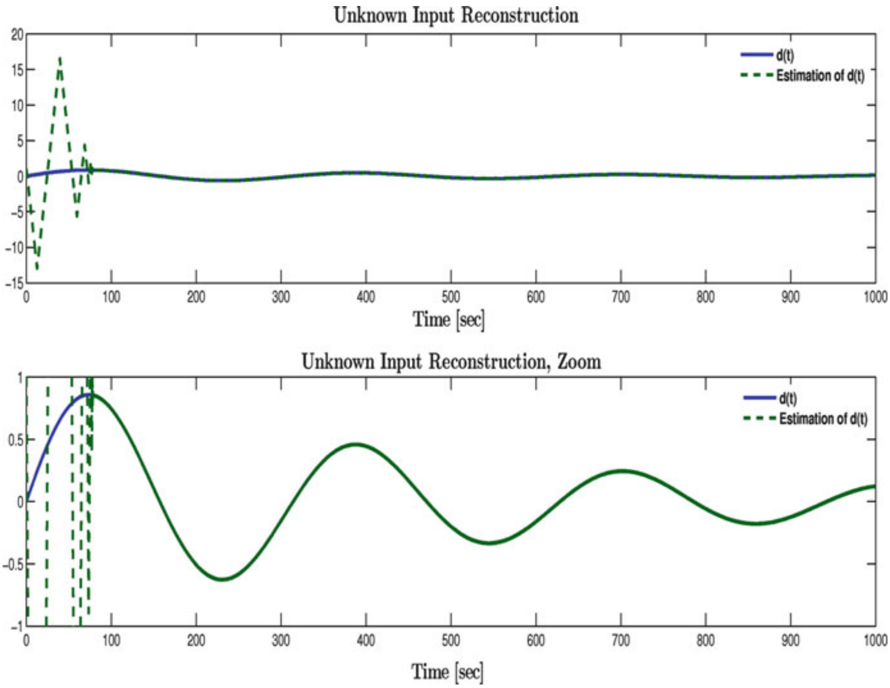

Fig. 7.17 The disturbance $d(t)$ and its reconstruction signal $\hat{d}(t)$

A2. For $u = 0$, the system is strongly observable (or the triple (A, C, B) has no invariant zeros).

A3. $w(t)$ is absolutely continuous, and there exists a constant w^+ such that $\|w(t)\| \leq w^+$.

In this section we will firstly derive an observer-based robust output control for system (7.64) of the form

$$u(t) = u_0(t) + u_1(t) \tag{7.65}$$

where $u_0(t)$ is a nominal control designed for the nominal system (i.e., $w(t) = 0$) and $u_1(t)$ is a compensator of the unknown input vector $w(t)$.

7.4.3 Theoretically Exact Output-Feedback Stabilization (EOFS)

Here, a compensation control law is designed based on the estimated states and the unknown input identification. Consider the nominal system

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t) \tag{7.66}$$

The control signal $u_0(t)$ is a stabilizing state-feedback control for the nominal system,

$$u_0(t) = -Kx(t)$$

where the gain K can be designed using any control strategy.

Let us design the second part of the control input Eq. (7.65) as

$$u_1(t) = -\hat{w}(t)$$

where $\hat{w}(t)$ is the identified unknown input.

Theoretically, assuming exact observation and identification, the equalities $\hat{x}(t) \equiv x(t)$ and $\hat{w}(t) \equiv w(t)$ hold after a finite time T. When the EOFS control law is given by

$$u(t) = -Kx(t) - \hat{w}(t) \tag{7.67}$$

applied to system (7.64) it yields the following closed-loop dynamic equation: $\dot{x}(t) = (A - BK)x(t)$. Theoretically the continuous control $u_1(t)$ exactly compensates the matched perturbations and the solutions for systems (7.64) and (7.66) coincide.

7.4.4 *Output Integral Sliding Mode Control*

In this subsection, we propose applying the ISM method using the estimated states obtained by the HOSM observer. We will call it output integral sliding mode control (OISMC). Consider a control input of the form (7.65), where $u_0(t)$ is the nominal control for the system without uncertainties Eq. (7.66). Let the nominal control be $u_0(t) = -Kx(t)$. The compensator $u_1(t)$ should be designed to reject the disturbance $w(t)$ in the sliding mode on the manifold $\{x: s(x,t) = 0\}$, so that the equivalent control $u_{1eq} = -w(t)$. The switching function $s(x,t)$ is defined as $s(x,t) = s_0(x,t) + \zeta(x,t)$, where $s, s_0, \zeta \in \mathbb{R}^m$ and $s_0(x,t) = B^+x(t)$ $(B^+ = (B^TB)^{-1}B^T)$ and the integral part ζ is selected such that $x(t) = x_0(t)$ for all $t \in [T, \infty)$. In other words, from $t_0 = T$ the system state belongs to the sliding surface, where the equivalent sliding mode control $u_{\text{eq}}(t)$ should compensate for the unknown input, that is, $u_{\text{eq}}(t) = -w(t)$. To achieve this purpose, ζ is determined from the equation $\dot{\zeta}(t) = -B^+\left(Ax(t) + Bu_0(t)\right)$, with $\zeta(t_0) = -B^+s(x(t_0))$. The switching surface takes the form

$$s(x(t), t) = B^+\left[x(t) - x(t_0) - \int_{t_0}^{t} [Ax(\tau) + Bu_0(x,\tau)]\, d\tau\right]$$

where $t_0 \geq T$. The compensator $u_1(t)$ is designed as a discontinuous unit-vector control $u_1(t) = -\rho\frac{s(x(t),t)}{\|s(x(t),t)\|}$. Thus, the sliding mode manifold $s(x,t)$ is attractive from t_0 if $\rho > w^+ \geq \|w(t)\|$. Finally, the control law Eq. (7.65) is designed as follows:

$$u(x,t) = -Kx(t) - \rho \frac{s(x,t)}{\|s(x,t)\|} \tag{7.68}$$

Again, in the ideal case, system (7.64) with $u(x,t)$ given by Eq. (7.68) takes the form of Eq. (7.66).

7.4.5 Precision of the Observation and Identification Processes

Suppose that we would like to realize the observation with a sampling step Δ while considering that a deterministic noise signal $n(t)$ (a Lebesgue-measurable function of time with a maximal magnitude η) is present in the system output. Let

$$f(t) = f_0(t) + n(t), \ \left\| f_0^{(i+1)}(t) \right\| < L, \ \|n(t)\| \leq \varepsilon \tag{7.69}$$

From Theorem 6.2, the error caused by the sampling time τ in the absence of noise for an ith-order HOSM differentiator given by Equation (6.28) is

$$\left\| f_0^{(j)}(t) - z_j(t) \right\| \leq O(\tau^{i-j+1}) \text{ for } j = 0, \ldots, i \tag{7.70}$$

and the differentiator error caused by a deterministic upper bounded noise will be

$$\left\| f_0^{(j)}(t) - z_j(t) \right\| \leq O(\varepsilon^{\frac{i-j+1}{i+1}}) \text{ for } j = 0, \ldots, i \tag{7.71}$$

Here, we are dealing with an $(\alpha + k - 1)$th-order HOSM differentiator. To recover the estimated state, $(k-1)$ differentiations are needed. From expressions (7.70) and (7.71) it follows that the observation error caused by the sampling time τ is $O(\tau^{\alpha+1})$, while the observation error caused by a deterministic upper bounded noise is $O(\varepsilon^{\frac{\alpha+1}{\alpha+k}})$.

It is clear that k differentiations are needed in order to recover the estimated unknown input. Therefore, from (7.70) the sampling step identification error will be $O(\tau^{\alpha})$, and the deterministic noise identification error Eq. (7.71) will be $O\left(\varepsilon^{\frac{\alpha}{\alpha+k}}\right)$.

The next proposition analyzes the total effect of both sampling step and deterministic noise errors:

Proposition 7.1. *Let us assume $\delta \leq k_{\Delta}\Delta$, and $\eta \leq k_{\eta}\Gamma\Delta^{i+1}$ with k_{Δ}, k_{η}, Δ some positive constants. Then after a finite-time, the HOSM observation and identification error will be $O\left(\Delta^{\alpha+1}\right)$ and $O\left(\Delta^{\alpha}\right)$, respectively.*

Remark 7.3. Table 7.1 summarizes the observation and identification errors when the unknown input $w(t)$ satisfies (7.69). Then, an $(\alpha + k - 1)$th-order differentiator is used to improve precision.

Table 7.1 Precision due to sampling step and bounded noise

Error	Sampling step δ	Bounded noise η	Total effect δ
Observation	$O\left(\delta^{\alpha+1}\right)$	$O\left(\eta^{\frac{\alpha+1}{\alpha+k}}\right)$	$O\left(\Delta^{\alpha+1}\right)$
Identification	$O\left(\delta^{\alpha}\right)$	$O\left(\eta^{\frac{\alpha}{\alpha+k}}\right)$	$O\left(\Delta^{\alpha}\right)$
HOSM differentiator	$O\left(\delta\right)$	$O\left(\eta^{\frac{1}{\alpha+k}}\right)$	$O\left(\Delta\right)$

EOFS Realization Error. Theoretically, the perturbations are exactly compensated in finite time. Nevertheless, in the previous section we discussed how the discretization and deterministic output noise present in the observation and identification processes affect the compensation accuracy. Furthermore, an additional error, due to the actuator time constant μ, will cause an error of order $O(\mu)$. Now, the EOFS controller stabilization error may be estimated by

$$\epsilon = O\left(\mu\right) + O\left(\Delta^{\alpha+1}\right) + O\left(\Delta^{\alpha}\right) \tag{7.72}$$

OISMC Realization Error. As we have seen, when the observation, identification, and control processes are free from nonidealities, both controllers, EOFS and OISMC, give identical results. However, in the practical case, the errors appearing in the complete control process should be taken into account. In the case of the OISMC the stabilization error is the sum of the observation error plus the control error, i.e.,

$$\epsilon = O(\mu) + O(\Delta^{a+1}) \tag{7.73}$$

Now, we analyze the accuracy of the HOSM observer and the identification procedure, combined with both control methodologies. Recall that we are using a $(\alpha + k - 1)$th-order HOSM differentiator and that we need the $(k - 1)$th and kth derivatives for the state observation and unknown input identification, respectively. Consider the following cases:

(a) $O\left(\Delta^{\alpha}\right) \ll O\left(\mu\right)$, i.e., the controller execution error is greater than the identification process error. In such a case, it would be suitable to use the EOFS strategy to avoid chattering.
(b) $O\left(\Delta^{\alpha+1}\right) \ll O\left(\mu\right) \ll O\left(\Delta^{\alpha}\right)$, i.e., the error related to the actuator time constant is less than the identification process error. Thus, the error in the EOFS control strategy is mainly determined by the identification error. In this case, OISMC strategy could be a better solution for systems tolerant to chattering with oscillation frequencies of order $O(\frac{1}{\mu})$.
(c) $O\left(\mu\right) \ll O\left(\Delta^{\alpha+1}\right)$, i.e., the error caused by the actuator time constant is less than the observation error. Once again, the precision of the EOFS controller is determined by the precision of the identification process $O(\Delta^{\alpha})$, and the precision of the OISMC controller is determined by the accuracy of the observation process $O(\Delta^{\alpha+1})$. However, it should be noted that in this case the use of the OISMC controller could amplify the observer noise.

7.5 Notes and References

The super-twisting-based observer for mechanical systems was first presented in [25, 52].

The design of HOSM observers for strongly observable and detectable linear systems with unknown inputs is suggested in [96] and in [24]. For a proof of Theorem 7.4, see [94]. A step-by-step differentiator approach for linear systems with stable invariant zero is presented in [84].

The design of observers for nonlinear systems with unknown inputs for the case when the relative degree of unknown inputs with respect to measured outputs is well defined is given in [95]. The method in [95] does not require the system to be strongly observable, but the internal dynamics must be asymptotically stable. The first approach to state observation presented in Sect. 7.2 (but for a class of nonlinear systems) was given in [42]. The first paper in which the approach in Sect. 7.3.1 was presented for state observation is [41]. In [41] step-by-step differentiation was used. The work in [83] also uses step-by-step differentiation. Subsequently in [53, 54] HOSM differentiators were applied for the design of HOSM observers, which requires only the transformation of the observability Jacobian, but does not require the inversion of observability map. Parameter identification methods are also studied in [48].

The satellite system example is taken from [95], although the original model is from Marino and Tomei [180].

For details of the recursive least-squares algorithm see, for example, [173]. The proof of Theorem 7.1 is given in [173].

The comparison of effectiveness of HOSM-based uncertainty identification and compensation versus sliding mode based uncertainty compensation, is presented in [79]. HOSM observer based control for the compensation of unmatched uncertainties was developed in [80].

State estimation and input reconstruction in nonminimum phase causal nonlinear systems using higher-order sliding mode observers is studied in [166].

Automotive applications of sliding mode disturbance observer- based control can be found in the book [111].

7.6 Exercises

Exercise 7.1. The pendulum-cart system (see Fig. 7.18), when restricted to a two-dimensional motion, can be described by the following set of equations:

$$(M + m)\dot{x} + ml\ddot{\theta} = u(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl \tag{7.74}$$

where M and m are the mass of the cart and the pendulum, respectively, l is the pendulum length, $\theta(t)$ is its deviation from the vertical, and $x(t)$ represents the horizontal displacement of the cart. The system parameters are given as $M = 2$[kg],

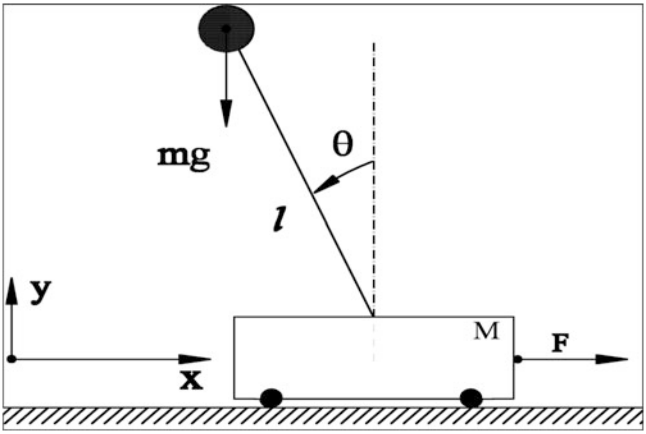

Fig. 7.18 The pendulum-cart system

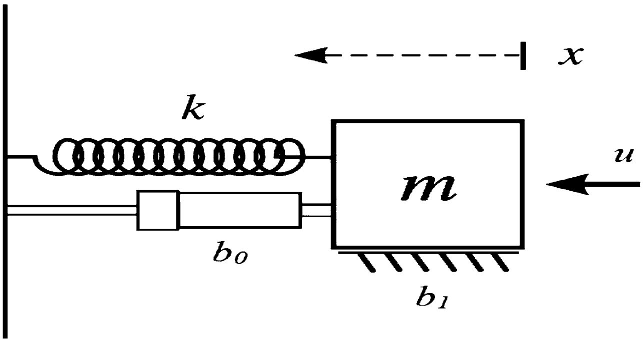

Fig. 7.19 Mass-spring-damper system

$m = 0.1$[kg], and $l = 0.5$[m]. Given the measured outputs θ and x, design a super-twisting observer for $\dot{\theta}$ and $\dot{x}$ for the uncontrolled case, $u = 0$. Assume the system's initial conditions are $\theta(0) = 0.3$[rad], $\dot{\theta}(0) = 0.03$[rad/s], $x(0) = 0$[m], and $\dot{x}(0) = 0.1$ [m/s]. Confirm the efficacy of the estimation algorithm via simulation if $\hat{\theta}(0) = 0$ [rad], $\hat{\dot{\theta}}(0) = 0$[rad/s], $\hat{x}(0) = 0$[m], and $\hat{\dot{x}} = 0$[m/s].

Exercise 7.2. Given the following mass-spring-damper system with friction (see Fig. 7.19)

$$m\ddot{x} + b_0\dot{x} + b_1 \text{sign}(\dot{x}) + kx = u \tag{7.75}$$

where $m = 1$ [kg] , $b_0 = 0.1$ [kg/s] , $b_1 = 0.05$ $\left[\text{kg/s}^2\right]$, and $k = 0.5$ [kg/s], design a feedback twisting control which achieves the tracking objective

$$x \to x_d = 0.7 \sin(2.3t) + 1.8 \sin(6.4t) \tag{7.76}$$

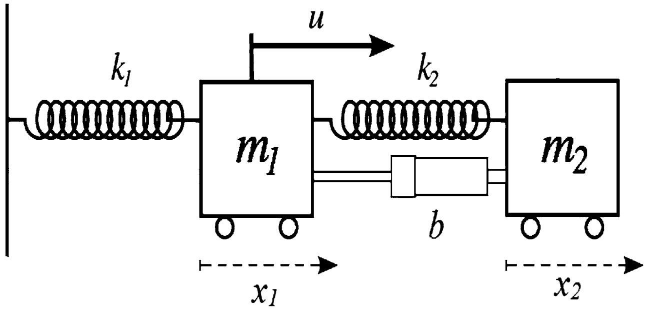

Fig. 7.20 Mass–spring–damper fourth-order system

Consider the only output of the system to be $y = x$ and design a super-twisting observer with $\hat{x} = 0$ and $\dot{\hat{x}} = 0$ to estimate $\dot{x}$ for the system starting at rest. Verify the convergence of the state estimates via simulation.

Exercise 7.3. Consider system (7.75) with unknown coefficients m, b_0, and k that is controlled by the tracking twisting controller to follow the command profile Eq. (7.76). Use the super-twisting observer with $\hat{x} = 0$ and $\dot{\hat{x}} = 0$ to estimate the mass velocity $\dot{x}$ as well as the unknown parameters for the system starting at rest. Compare the simulation results with those obtained in Exercise 7.2.

Exercise 7.4. Assuming that x_1 is measured, design the twisting tracking controller as in Exercise 7.2 that drives x_1 to follow the desired trajectory in Eq. (7.76), for the system shown in Fig. 7.20 and described by

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -500x_1 + 150x_3 + 1.2u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= 200x_1 - 600x_3 - 20x_4 \\
y &= x_1
\end{aligned}
$$

Estimate x_2 via the super-twisting state observer and reconstruct the position of the second mass, x_3, treated as a disturbance in the first two equations. Use $x_0 = [0.4, -3.0, 0.28, 0]^T$ as initial conditions for the system and $\hat{x}_0 = [0, 0, 0, 0]^T$ for the observer in your simulations.

Exercise 7.5. Consider a flux-controlled DC motor whose dynamics are given by the following equations:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.1x_2 + 0.1x_3u + \zeta \\
\dot{x}_3 &= -2x_3 - 0.2x_2u + 200
\end{aligned}
\tag{7.77}
$$

where x_1 is the rotor position, x_2 is the angular velocity, x_3 is the current in the motor armature, and ξ is a disturbance. Assume that the system is affected by a perturbation signal $\zeta \neq 0$ and that both the position x_1 and the internal dynamics x_3 are measured. A zero-average random noise of amplitude $\varepsilon = 0.001$ is assumed to affect the measurement of x_1. Apply the concept of equivalent control to identify $\zeta = \cos(0.5t) + 0.5\sin(t) + 0.5$ and the discontinuous signal shown in Fig. 7.5, given the initial conditions $x_0 = [1000, 4\pi, 100]^T$ and $\hat{x}_0 = [0, 0, 0]^T$. Perform simulations to confirm the proper reconstruction of ζ.

Exercise 7.6. If a rotary spring is attached to the axis of the DC motor described in Exercise 7.5 then its dynamics may be represented by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a_1x_1 - a_2x_2 + 0.1x_3u\\ \dot{x}_3 &= -2x_3 - 0.2x_2u + 200\end{aligned} \tag{7.78}$$

Design an appropriate controller u such that it satisfies the persistent excitation condition and estimate the unknown parameters a_1 and a_2 using the second-order sliding mode parameter estimator. Use $x_0 = [10, 4\pi, 100]^T$ and $\hat{x}_0 = [0, 0, 0]^T$ for simulation purposes.

Exercise 7.7. Consider the following LTI system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 12 & -4 & -15 & 5 & 3 \end{bmatrix} x + \begin{bmatrix} 0\\0\\0\\0\\1 \end{bmatrix} u + \begin{bmatrix} 0\\0\\0\\0\\1 \end{bmatrix} \zeta(t),$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} x + \mu$$

with initial conditions $x(0) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T$. Assume that the output is affected by a deterministic noise

$$\mu = 0.02\cos\left(1024 \cdot |\sin(606t)|\right)$$

Identify the constant unknown input $\zeta = 12\pi$ using the sliding mode observer for strongly observable systems assuming the control input is $u = 0$. Furthermore, apply feedback control to place the system poles at $-2, -4, -6, -1, -3.5$, and identify the sawtooth wave $\zeta(t) = 2\left(\frac{t}{0.5} - \text{floor}\left(\frac{t}{0.5} + \frac{1}{2}\right)\right)$. In both cases, use a Luenberger observer as the linear part of the observer and compare the results through simulations using a Kalman filter. The initial conditions for the observer should be $\hat{x}_0 = [0, 0, 0, 0, 0]^T$ in all cases.

Exercise 7.8. Consider the DC motor in Exercise 7.78 described by Eq. (7.77). Assume that a zero-average random noise of amplitude $\varepsilon = 0.001$ affects the measurement of x_1. Verify the strong observability of the system and apply the sliding mode observer to identify $\zeta = \cos(0.5t) + 0.5\sin(t) + 0.5$ and the discontinuous signal shown in Fig. 7.5 given the initial conditions $x_0 = [0, 4\pi, 100]^T$ and $\hat{x}_0 = [0, 0, 0]^T$. Compare the results with those obtained using the observer in Exercise 7.5 by means of simulation.

Exercise 7.9. Consider the following third-order nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= -2x_1 - x_2 + x_3 \\
\dot{x}_2 &= x_1 \\
\dot{x}_3 &= -{x_3}^2 - 2x_3\left(\frac{2x_1 + \sin(x_2)}{2 + \cos(x_3)}\right) + d(t) \\
y &= x_2
\end{aligned} \tag{7.79}$$

where $d(t) = \sin(3.18t) + 2\sin(7.32t) + 0.5\cos(0.79t)$ represents an unknown input. Design a differentiator-based observer that generates estimates of the state x and the unknown input $d(t)$, given the output y. Show the efficacy of the observer by means of simulations, considering the initial conditions $x(0) = [1,\ 2,\ -1]^T$ and $\hat{x}(0) = [0,\ 0,\ 0]^T$.

Exercise 7.10. The chaotic Chua's circuit can be described by the following state equations:

$$\begin{aligned}
\dot{x}_1 &= -acx_1 + ax_2 - a{x_1}^3 + d(t) \\
\dot{x}_2 &= x_1 - x_2 + x_3 \\
\dot{x}_3 &= -bx_2 \\
y &= x_3
\end{aligned} \tag{7.80}$$

The parameters of the system are chosen as $a = 10, b = 16, c = -0.143$. Consider the unknown input term as $d(t) = 0.5\sin(t)$. Generate an estimate for the unknown input using the differentiator-based observer. Realize the corresponding simulations given the initial conditions $x(0) = [0.1\ 0.1\ 0.1]^T$ and $\hat{x}(0) = [0,\ 0,\ 0]^T$.

Exercise 7.11. Develop a differentiator-based observer when Assumption 7.2 is not satisfied, i.e., in the case when the output y has relative degree $r < n$ with respect to the unknown input $\varphi(t)$.

Exercise 7.12. Necessary and sufficient conditions for the strong observability of linear and nonlinear systems single-output systems were given in this chapter. Formulate the equivalent conditions for strong observability in multiple-input, multiple-output systems.