Chapter 2
Conventional Sliding Modes

This chapter considers the development of conventional sliding mode methods. The chapter describes the early work to define the notion of the solution of differential equations with discontinuous right-hand sides and the concept of "equivalent control" as a means to describe the reduced-order dynamics while a sliding motion is taking place. The main focus of the chapter is on the development of sliding mode design techniques for uncertain linear systems—specifically systems which can be thought of as predominantly linear in a characteristic, or nonlinear systems which can be modeled well (at least locally) by a linear system. For such systems, sliding surfaces formed from linear combinations of the states are considered (i.e., hyperplanes in the state space). In this chapter we consider different explicit design methods which can be used to synthesize hyperplanes which give appropriate closed-loop dynamics when a sliding motion is induced. Different classes of control law are then developed to guarantee the existence of a sliding motion in finite time and to ensure the sliding motion can be maintained in the face of uncertainty. The majority of the chapter is based on the assumption that state information is available for use in the control law. This is convenient and indeed mirrors the development of the ideas since their inception. However, the assumption that all the state, are available is somewhat impractical from an engineering perspective, and in the later sections we consider the case when only output information is available. The impact of this is studied both in terms of the constraints this imposes on the choice of sliding surfaces and the associated control laws.

2.1 Introduction

This chapter will discuss "conventional" sliding modes—or to be more precise first-order sliding modes when viewed in the context of higher-order sliding. Consider a general state-space system

$$\dot{x} = f(x, u, d) \tag{2.1}$$

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where $x \in \mathbb{R}^n$ is a vector which represents the state and $u \in \mathbb{R}^m$ is the control input. It is assumed that $f(\cdot)$ is differentiable with respect to x and absolutely continuous with respect to time. The quantity $d \in \mathbb{R}^q$ represents external bounded disturbances/uncertainties within the system. Consider a surface in the state space given by

$$\mathcal{S} = \{x : \sigma(x) = 0\} \tag{2.2}$$

A formal definition of an ideal sliding mode will now be given generalizing Definition 1.3 from Chap. 1.

Definition 2.1. An ideal sliding mode is said to take place on Eq. (2.2) if the states $x(t)$ evolve with time such that $\sigma(x(t_r)) = 0$ for some finite $t_r \in \mathbb{R}^+$ and $\sigma(x(t)) = 0$ for all $t > t_r$.

During a sliding mode, $\sigma(t) = 0$ for all $t > t_r$. Intuitively this dynamical collapse implies the motion of the system when confined to $\mathcal{S}$ will be of reduced dynamical order. From a control systems perspective the capacity to analyze the dynamics of the reduced-order motion is important.

If the control action in $u = u(x)$ Eq. (2.1) is discontinuous, the differential equation describing the resulting closed-loop system written as

$$\dot{x}(t) = f^c(x) \tag{2.3}$$

is such that the function $f^c : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is discontinuous with respect to the state vector. The classical theory of differential equations is now not applicable since Lipschitz assumptions are usually employed to guarantee the existence of a unique solution. The solution concept proposed by Filippov for differential equations with discontinuous right-hand sides constructs a solution as the "average" of the solutions obtained from approaching the point of discontinuity from different directions.

2.1.1 *Filippov Solution*

Consider initially the case when the system has a single input and $\sigma : \mathbb{R}^n \mapsto \mathbb{R}$. Suppose x_0 is a point of discontinuity on $\mathcal{S}$ and define $f_-^c(x_0)$ and $f_+^c(x_0)$ as the limits of $f^c(x)$ as the point x_0 is approached from opposite sides of the tangent to $\mathcal{S}$ at x_0. The solution proposed by Filippov is given by

$$\dot{x}(t) = (1-\alpha) f_-^c(x) + \alpha f_+^c(x) \tag{2.4}$$

where the scalar $0 < \alpha < 1$.

The scalar α is chosen so that

$$f_a^c := (1-\alpha) f_-^c + \alpha f_+^c$$

is tangential to $\mathcal{S}$ (see Fig. 2.1).

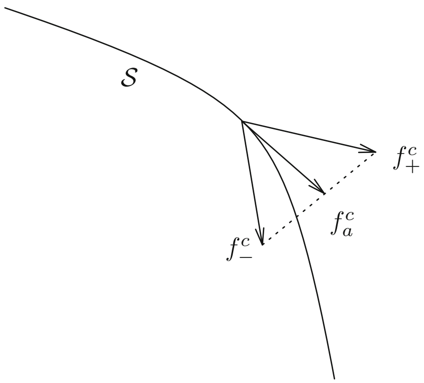

Fig. 2.1 A schematic of the Filippov construction

Remark 2.1. From the discussion above, it is clear the Filippov solution is an average solution of the two "velocity" vectors at the point x_0.

Equation (2.4) can be thought of as a differential equation whose right-hand side is defined as the convex set

$$F(x) = \left\{(1-\alpha) f_-^c + \alpha f_+^c \ : \ \text{for all } \alpha \in \begin{bmatrix} 0 & 1 \end{bmatrix}\right\}$$

and thus

$$\dot{x}(t) \in F(x)$$

The values of α which ensure $\dot{\sigma}(t) = 0$ can be computed explicitly from Eq. (2.4). For simplicity further suppose $\sigma = Sx$ where $S^T \in \mathbb{R}^n$. Then explicitly

$$\dot{\sigma} = S\dot{x} = (1-\alpha)Sf_-^c + \alpha Sf_+^c$$

In order to maintain $\sigma = 0$, the scalar α must satisfy

$$(1-\alpha)Sf_-^c + \alpha Sf_+^c = 0$$

and consequently (and uniquely)

$$\alpha = \frac{Sf_-^c}{Sf_-^c - Sf_+^c}$$

so that

$$\dot{x}(t) = \frac{Sf_-^c f_+^c - Sf_+^c f_-^c}{Sf_-^c - Sf_+^c}$$

A formal definition for the generic case of multi-input systems is:

Definition 2.2. A differential inclusion $\dot{x} \in F(x)$, $x \in \mathbb{R}^n$, is called a *Filippov differential inclusion* if the vector set $F(x)$ is nonempty, closed, convex, locally bounded, and upper-semi-continuous. The latter condition means that the maximal distance of the points of $F(x)$ from the set $F(y)$ vanishes when $x \to y$. Solutions are defined as absolutely continuous functions of time satisfying the inclusion almost everywhere.

Recall in this context that a function is absolutely continuous if and only if it can be represented as a Lebesgue integral of some integrable function.[1] Thus, such a function is almost everywhere differentiable. Solutions of Filippov differential inclusions always exist and have most of the well-known standard properties except the uniqueness.

Definition 2.3. It is said that a differential equation $\dot{x} = f(x)$ with a locally bounded Lebesgue-measurable right-hand side is understood in the Filippov sense, if it is replaced by a special Filippov differential inclusion $\dot{x} \in F(x)$, where

$$F(x) = \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co}\, f(O_\delta(x) \setminus N) \tag{2.5}$$

Here μ is the Lebesgue measure, $O_\delta(x)$ is the δ-vicinity of x, and $\overline{co} M$ denotes the convex closure of M.

Note that any surface or curve has zero Lebesgue measure. Thus, values on any such set do not affect the Filippov solutions. In the most usual case, when f is continuous almost everywhere, the procedure is to take $F(x)$ being the convex closure of the set of all possible limit values of f at a given point x, obtained when its continuity point y tends to x. In the general case approximate-continuity points y are taken (one of the equivalent definitions by Filippov).[2] A solution of $\dot{x} = f(x)$ is defined as a solution of $\dot{x} \in F(x)$. Obviously, values of f on any set of measure 0 do not influence the Filippov solutions. Note that with continuous f the standard definition is obtained. The nonautonomous case is reduced to the considered one introducing the fictitious equation $\dot{t} = 1$.

In order to better understand the definition, consider the case when the number of limit values $f_1, \ldots, f_n$ at the point x is finite. Then any possible Filippov velocity has the form $\dot{x} = \lambda_1 f_1 + \cdots + \lambda_n f_n$, $\lambda_1 + \cdots + \lambda_n = 1$, $\lambda_i \geq 0$, and can be considered as a mean value of the velocity taking on the values f_i during the fraction of time $\lambda_i \Delta t$ of the current infinitesimal time interval Δt.

[1] For details see [158].

[2] For details see [158].

2.1.2 Concept of Equivalent Control

One way to undertake this analysis is by the so-called *equivalent control* method attributed to Utkin. This defines the equivalent control as the control action necessary to maintain an ideal sliding motion on $\mathcal{S}$. The idea is to exploit the fact that in conventional sliding modes both $\sigma = \dot{\sigma} = 0$. The constraint on the derivative of σ can be written as

$$\dot{\sigma} = \frac{\partial \sigma}{\partial x}\frac{dx}{dt} = \frac{\partial \sigma}{\partial x} f(x, u, d) = 0$$

This represents an algebraic equation in x, u, and d, and by definition, the equivalent control signal $u_{eq}(t)$, which is the continuous control function required to maintain sliding, is the solution to

$$\frac{\partial \sigma}{\partial x} f(x, u_{eq}, d) = 0 \tag{2.6}$$

For example, consider the affine system

$$\dot{x} = f(x) + g(x)u + d \tag{2.7}$$

The specific structure which has been imposed here ensures that for a given x the control input appears linearly. Consequently Eq. (2.6) simplifies to

$$\frac{\partial \sigma}{\partial x} f(x) + \frac{\partial \sigma}{\partial x} g(x) u_{eq} + \frac{\partial \sigma}{\partial x} d = 0 \tag{2.8}$$

and so, provided $\frac{\partial \sigma}{\partial x} g(x)$ is nonsingular, from Eq. (2.6)

$$u_{eq} = -\left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x} f(x) - \left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x} d \tag{2.9}$$

The closed-loop response is given by substituting the expression in Eq. (2.9) into Eq. (2.7) to yield

$$\dot{x} = \left(I - g(x)\left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x}\right) f(x) + \left(I - \left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x}\right) d \tag{2.10}$$

Example 2.1. Consider a system where the state vector is given by $x = [x_1, x_2]^T$ with the structure of (2.7) so that

$$f(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ \phi(x_1, x_2, t) \end{bmatrix} \tag{2.11}$$

The scalar disturbance term $\phi(x_1, x_2, t)$ comprises dry and viscous friction as well as any other unknown resistance forces, and it is assumed to be bounded, i.e., $|\phi(x_1, x_2, t)| \leq L$. Now considering the surface $\sigma = x_2 + cx_1$, it implies

$$\frac{\partial \sigma}{\partial x} = \left[\frac{\partial \sigma}{\partial x_1}, \frac{\partial \sigma}{\partial x_2} \right] = [c, 1]$$

and consequently the equivalent control is given by Eq. (2.9), i.e.,

$$u_{eq} = -cx_2 - \phi(x_1, x_2, t)$$

Example 2.2. Consider the following multi-input multi-output (MIMO) linear system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_3 + u_2 \\ \dot{x}_2 &= x_2 + 3x_3 + u_1 - u_2 \\ \dot{x}_3 &= x_1 + x_3 - u_1 \end{aligned}$$

and the corresponding outputs surfaces

$$\begin{aligned} \sigma_1 &= -x_1 + 10x_3 \\ \sigma_2 &= x_3 + x_2 \end{aligned}$$

Applying the concept of equivalent control we need to find the dynamic of sliding modes in the intersection of the output surfaces σ_1 and σ_2, i.e.,

$$\begin{aligned} \dot{\sigma}_1 &= 9x_1 - x_2 + 9x_3 - 10u_1 + u_2 \\ \dot{\sigma}_2 &= x_1 + x_2 + 4x_3 - u_2 \end{aligned}$$

From the invariance conditions $\dot{\sigma}_1 = 0, \sigma_1 = 0$, and $\dot{\sigma}_2 = 0, \sigma_2 = 0$ we obtain:

$$\begin{aligned} u_{1eq} &= -x_1 - 2x_2 - 4x_3 \\ u_{2eq} &= x_1 + 1.3x_3 \end{aligned}$$

and the reduced dynamics of the original system is given by

$$\begin{aligned} x_1 &= 10x_3 \\ x_2 &= -x_3 \\ \dot{x}_3 &= -0.3x_3 \end{aligned}$$

Remark 2.2. There are several points to note from the analysis given above:

- The expression for the equivalent control u_{eq} in Eq. (2.9) comes from formally solving Eq. (2.6), considered as an algebraic equation. It is therefore quite independent of the control signal which is actually applied. The control signal which is physically applied to the plant can be discontinuous in nature. However, the solution to Eq. (2.9) will always be smooth.

- The equivalent control u_{eq} in Eq. (2.9) depends on the disturbance $d(t)$ which will generally be unknown. Consequently Eq. (2.9) will not be physically implementable.
- The control signal in Eq. (2.9) is best thought of as an abstract concept to facilitate the creation of an expression for the reduced-order system in Eq. (2.10), hence establishing a differential equation from which the stability of the closed-loop system can be studied.

Another crucial property of sliding mode control systems can now be demonstrated, namely its robustness—or more precisely, its invariance to a certain class of uncertainty. Suppose the disturbance d acts in the channels of the inputs so that

$$d(t) = g(x)\xi(t) \tag{2.12}$$

for some (unknown) signal $\xi(t)$. Then it is easy to see that in Eq. (2.10)

$$\begin{aligned}\left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)d &= \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)g(x)\xi \\ &= g(x)\xi - g(x)\xi \\ &= 0\end{aligned}$$

and so Eq. (2.10) collapses to

$$\dot{x} = \left(I - g(x)\left(\frac{\partial\sigma}{\partial x}g(x)\right)^{-1}\frac{\partial\sigma}{\partial x}\right)f(x) \tag{2.13}$$

The closed-loop (reduced-order) system given in Eq. (2.13) is completely independent of ξ. This *invariance property* has motivated research in sliding mode control.

Clearly from Eq. (2.10) the choice of the surface affects the dynamics of the reduced-order motion. In terms of control system design, the selection of the surface is one of the key design choices. Later in this chapter an alternative viewpoint and design framework will be given which is more amenable from the perspective of synthesizing choices for $\mathcal{S}$.

Example 2.3. Consider a second-order system representing a DC motor:

$$\dot{\theta}(t) = w(t) \tag{2.14}$$

$$\dot{w}(t) = \frac{F(t)}{J} + \frac{K_t}{J}u(t) \tag{2.15}$$

where θ represents the shaft position and w is the angular rotation speed. The scalar J represents the inertia of the shaft, $F(t)$ represents the effects of dynamic friction and K_t represents the motor constant. Assume all the coefficients are unknown but bounded so that $0 < \underline{J} \leq J \leq \bar{J}$, $|F(t)| \leq \bar{F}$ and the motor constant $\underline{K}_t \leq K_t \leq \bar{K}_t$.

Suppose a switching function σ is defined as

$$\sigma = w + m\theta \tag{2.16}$$

where m is a positive design scalar. During the sliding motion if $\sigma \equiv 0$, then combining Eqs. (2.14) and (2.16) gives

$$\dot{\theta}(t) = w(t) = -m\theta(t)$$

and a first-order system is obtained which is *independent of the uncertainty* associated with $F(t)$, J, and K_t. The closed-loop solution is given by

$$\theta(t) = \theta_0 e^{-m(t-t_s)} \tag{2.17}$$

where θ_0 represents the value of $\theta(\cdot)$ at the time instant t_s at which sliding is achieved. Clearly in Eq. (2.17) the effect of the uncertainty has been totally rejected and robust closed-loop performance has been achieved.

Figures 2.2–2.4 are associated with simulations where $\frac{K_t}{J} = 0.45$ and $m = 1$. The controller regulates the shaft position back to zero from an initial displacement of 1 rad.

The next section focuses on linear (in fact uncertain linear) system representations which have been more well studied in the literature and yield systematic tractable methods for the design of $\mathcal{S}$.

2.2 State-Feedback Sliding Surface Design

Consider the nth-order linear time-invariant system with m inputs given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.18}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $1 \leq m \leq n$. Without loss of generality it can be assumed that the input distribution matrix B has full rank. Define a switching function $\sigma : \mathbb{R} \to \mathbb{R}^m$ to be

$$\sigma(x) = Sx(t) \tag{2.19}$$

where $S \in \mathbb{R}^{m\times n}$ is of full rank and let $\mathcal{S}$ be the hyperplane defined by

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ Sx = 0\} \tag{2.20}$$

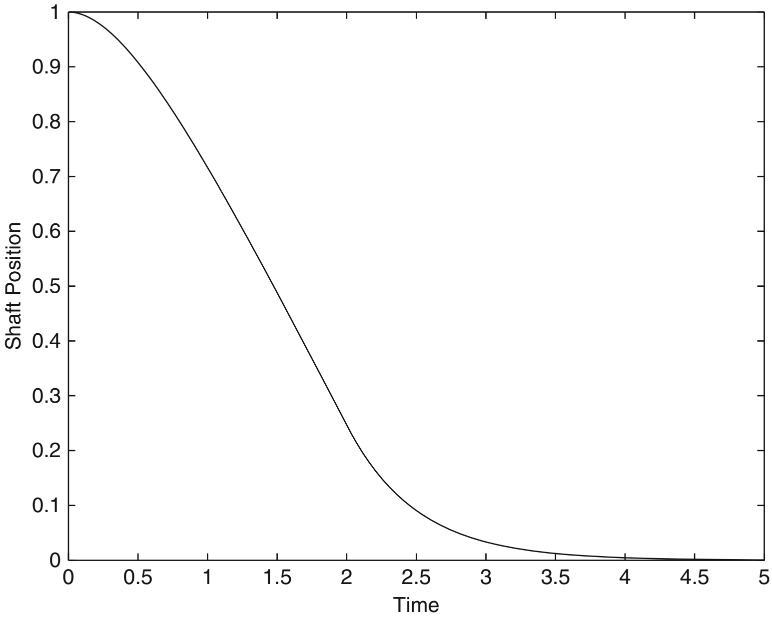

Fig. 2.2 Evolution of shaft position

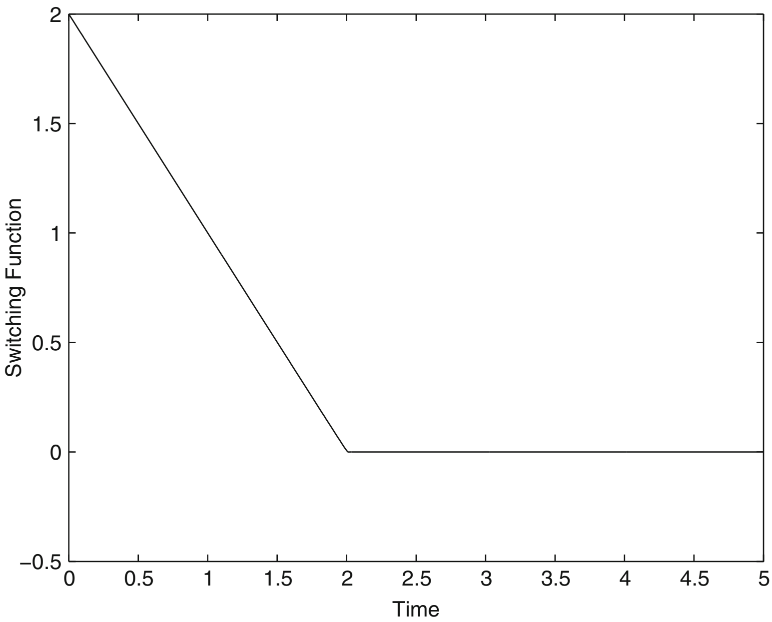

Fig. 2.3 Evolution of switching function

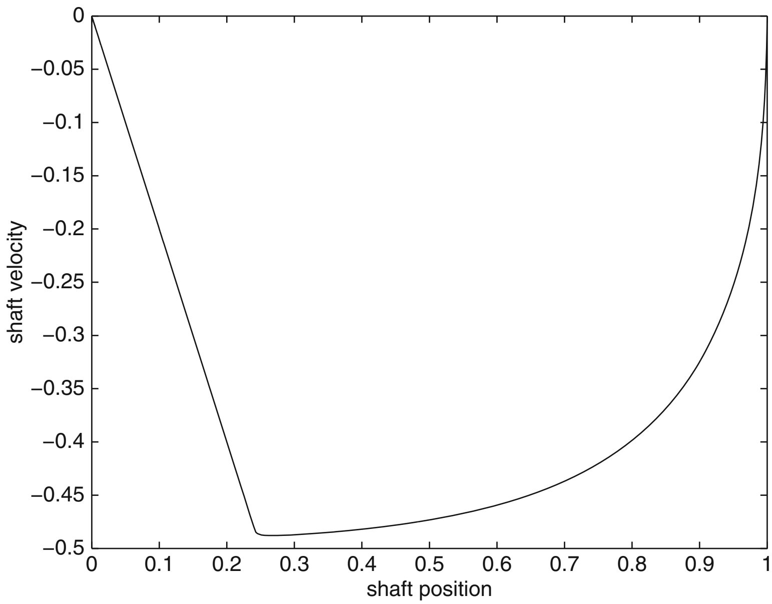

Fig. 2.4 Phase portrait

This implies the switching function $\sigma(x)$ is a linear combination of the states. Also from Eq. (2.20) it follows that the sliding motion is associated with the null space of the matrix S. Also note that the number of rows of the matrix S corresponds to the number of columns of the input distribution matrix B and consequently the matrix SB is square.

Suppose u represents a sliding mode control law where the changes in control strategy depend on the value of the switching function $\sigma(x)$. It is natural to explore the possibility of choosing the control action and selecting the switching strategy so that an *ideal sliding motion* takes place on the hyperplane, i.e., there exists a time t_r such that

$$\sigma(x) = Sx(t) = 0 \qquad \text{for all } t > t_r \tag{2.21}$$

Suppose at time $t = t_r$ the system states lie on the surface $\mathcal{S}$ and an ideal sliding motion takes place. This can be expressed mathematically as $Sx(t) = 0$ and $\dot{\sigma}(t) = S\dot{x}(t) = 0$ for all $t \geq t_r$. Substituting for $\dot{x}(t)$ from Eq. (2.18) gives

$$S\dot{x}(t) = SAx(t) + SBu(t) = 0 \qquad \text{for all } t \geq t_r \tag{2.22}$$

Suppose the matrix S is designed so that the square matrix SB is nonsingular (in practice this is easily accomplished since B is full rank and S is a free parameter).

The equivalent control, written as u_{eq}, as argued above, is the unique solution to the algebraic equation (2.22), namely

$$u_{eq}(t) = -(SB)^{-1}SAx(t) \tag{2.23}$$

This represents the control action which is required to maintain the states on the switching surface. Substituting the expression for the equivalent control into Eq. (2.18) results in a free motion

$$\dot{x}(t) = \left(I_n - B(SB)^{-1}S\right)Ax(t) \qquad \text{for all } t \geq t_r \text{ and } Sx(t_r) = 0 \tag{2.24}$$

It can be seen from Eq. (2.24) that the sliding motion is a control independent free motion which depends on the choice of sliding surface.

Now suppose the system (2.18) is uncertain:

$$\dot{x}(t) = Ax(t) + Bu(t) + B\xi(t, x, u) \tag{2.25}$$

where $\xi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^m$ is unknown but bounded and encapsulates any nonlinearities or uncertainties in the system. Uncertainty which acts in the channel of the inputs is often referred to as *matched uncertainty*. Suppose a control law can be found for the system in Eq. (2.25) which induces a sliding motion on Eq. (2.20). Arguing as before, the equivalent control is in this case given by

$$u_{eq}(t) = -(SB)^{-1}SAx(t) - \xi(t, x, u) \tag{2.26}$$

The closed-loop sliding motion is given by substituting Eq. (2.26) in Eq. (2.25) and yields

$$\dot{x}(t) = \left(I_n - B(SB)^{-1}S\right)Ax(t) \tag{2.27}$$

This motion is completely independent of the uncertainty. Although the sliding motion is clearly dependent on the matrix S, how to select S to achieve a specific design goal is not transparent. One way to see the effect is to first transform the system into a suitable *regular form*.

2.2.1 Regular Form

In this section a coordinate transformation is introduced to create a special structure in the input distribution matrix. Since by assumption $\text{rank}(B) = m$ there exists an orthogonal matrix $T_r \in \mathbb{R}^{n \times n}$ such that

$$T_r B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{2.28}$$

where $B_2 \in \mathbb{R}^{m \times m}$ and is nonsingular. The matrix T_r can be obtained via so-called QR factorization.[3] This means a design algorithm can be created to deliver a change of coordinates in which a specific structure is imposed on the input distribution matrix.

Remark 2.3. This is not the only way to achieve the partition in Eq. (2.28). In principle, any nonsingular matrix which partitions the input distribution matrix can be employed—indeed later in the chapter a different approach based on orthogonal complements will be used. One advantage of the method of QR factorization is that the method generates an orthogonal matrix T_r. Consequently the associated coordinate transformation is orthogonal which means it has good numerical conditioning and also Euclidean distance is preserved.

Let $z = T_r x$ and partition the new coordinates so that

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{2.29}$$

where $z_1 \in \mathbb{R}^{n-m}$ and $z_2 \in \mathbb{R}^m$. The system matrices (A, B) in the original coordinates become $A \leftrightarrow T_r A T_r^T$ and $B \leftrightarrow T_r B$ in the "z" coordinates. Now the linear system (2.18) can be written as

$$\dot{z}_1(t) = A_{11} z_1(t) + A_{12} z_2(t) \tag{2.30}$$

$$\dot{z}_2(t) = A_{21} z_1(t) + A_{22} z_2(t) + B_2 u(t) \tag{2.31}$$

in which

$$T_r A T_r^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

The representation in Eqs. (2.30) and (2.31) is referred to as *regular form*. Suppose the matrix defining the switching function in the new coordinate system is compatibly partitioned as

$$S T_r^T = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{2.32}$$

where $S_1 \in \mathbb{R}^{m \times (n-m)}$ and $S_2 \in \mathbb{R}^{m \times m}$. Since $SB = S_2 B_2$ it follows that a necessary and sufficient condition for the matrix SB to be nonsingular is that $\det(S_2) \neq 0$ since $\det(SB) = \det(S_2 B_2) = \det(S_2)\det(B_2)$ and therefore

$$\det(SB) \neq 0 \;\Leftrightarrow\; \det(S_2) \neq 0$$

since $\det(B_2) \neq 0$ by construction. By design assume this to be the case. During an ideal sliding motion

$$S_1 z_1(t) + S_2 z_2(t) = 0 \qquad \text{for all } t > t_s \tag{2.33}$$

[3]For details of QR factorization methods see [177].

and therefore exploiting the non-singularity of S_2, the relationship in Eq. (2.33), can be rewritten as

$$z_2(t) = -M z_1(t) \tag{2.34}$$

where $M = S_2^{-1} S_1$. Substituting in Eq. (2.30) gives

$$\dot{z}_1(t) = (A_{11} - A_{12} M) z_1(t) \tag{2.35}$$

This equation is a straightforward expression describing the reduced-order dynamics in terms of the design freedom associated with the sliding surface.

If Eq. (2.30) is considered in isolation with z_1 thought of as the state vector and z_2 as a "virtual" control input, then Eq. (2.34) can be thought of as a state-feedback control law for Eq. (2.30). Consequently the dynamics describing the sliding motion in Eq. (2.35) can be thought of as the closed-loop system applying the feedback control law (2.34)–(2.30). It can be seen from Eq. (2.35) that S_2 has no direct effect on the dynamics of the sliding motion and acts only as a scaling factor for the switching function. A common choice for S_2, however, which stems from the so-called *hierarchical* design procedure, is to let $S_2 = \Lambda B_2^{-1}$ for some diagonal design matrix $\Lambda \in \mathbb{R}^{m \times m}$ which implies $SB = \Lambda$. By selecting M and S_2 the switching function in Eq. (2.32) is completely determined.

Remark 2.4. The matrix S of the switching function $\sigma(x) = Sx(x)$ has the form:

$$S = S_2 \begin{bmatrix} M & I_{m \times m} \end{bmatrix} T_r. \tag{2.36}$$

There exist two major techniques for the design of the matrix M; these are

- Eigenvalue placement method
- Linear-quadratic minimization

2.2.2 Eigenvalue Placement

Single-input systems represented by the pair (A, B) where $B \in \mathbb{R}^n$ can be written in the so-called controllability canonical form[4]

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -a_1 & -a_2 & \dots & \dots & -a_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.37}$$

[4]See for example [47].

where the scalars a_i are the coefficients of the characteristic equation of the A matrix:

$$\lambda^n + a_n\lambda^{n-1} + \cdots + a_2\lambda + a_1 = 0$$

For this general system an appropriate switching function is

$$\sigma(x) = s_1x_1 + s_2x_2 + \ldots + s_{n-1}x_{n-1} + x_n \qquad (2.38)$$

where the scalars s_i are to be chosen. Partition the state space associated with Eq. (2.37) into the first $n-1$, and the last equation, so that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} x_n$$

During the sliding motion $\sigma(x) = 0$ and so from Eq. (2.38) the last coordinate can be expressed as

$$x_n = -s_1x_1 - s_2x_2 - \ldots - s_{n-1}x_{n-1}$$

Now substituting for x_n in Eq. (2.2.2) yields a description of the sliding motion as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ -s_1 & \ldots & \ldots & \ldots & -s_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Therefore the characteristic equation of the sliding motion is

$$\lambda^{n-1} + s_{n-1}\lambda^{n-2} + \ldots + s_2\lambda + s_1 = 0$$

The scalars $s_1, \ldots s_n$ should therefore be chosen to make the polynomial above Hurwitz. More generally for multi-input systems, the regular form from Eqs. (2.30) and (2.31) must be relied upon. For a hyperplane parameterized as in Eq. (2.36) the sliding motion is governed by the system matrix $(A_{11} - A_{12}M)$ in Eq. (2.35) where the matrices A_{11} and A_{12} are associated with the regular form in Eqs. (2.30) and (2.31). In the context of designing a regulatory system, the matrix $(A_{11} - A_{12}M)$ must have stable eigenvalues. The switching surface design problem can therefore be considered to be one of choosing a state-feedback matrix M to stabilize the

reduced-order system associated with the pair (A_{11}, A_{12}). Because of the special structure of the regular form, it can be shown that the pair (A_{11}, A_{12}) is controllable if and only if (A, B) is controllable.[5] Consequently if the original pair (A, B) is controllable then the problem of synthesizing the matrix S associated with the switching function can always be solved to ensure the associated hyperplane $\mathcal{S}$ yields a stable sliding motion. This is important; otherwise, the method would lack credibility. Any eigenvalue pole placement methods can be employed.
The eigenvalue placement algorithm is demonstrated on the following example.

Example 2.4. Consider the linear system:

$$\dot{x} = Ax + Bu = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Apply the transformation $z = T_r x$, where T_r is obtained by QR factorization, such that

$$\bar{B} = T_r B = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sqrt{6} \\ \sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

to create the equivalent system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \bar{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \bar{B}u = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{0} \\ B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $\bar{A} = T_r A T_r^T$ and $T_r T_r^T = I_{n\times n}$. After some computations,

$$\bar{A}_{11} = [3], \;\; \bar{A}_{12} = \begin{bmatrix} -\frac{1}{6}\sqrt{18} & -\frac{1}{2}\sqrt{6} \end{bmatrix}, \;\; \bar{A}_{21} = \begin{bmatrix} 0 \\ \frac{1}{3}\sqrt{6} \end{bmatrix}, \;\; \bar{A}_{22} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4}\sqrt{12} \\ -\frac{5}{12}\sqrt{12} & -\frac{1}{2} \end{bmatrix}$$

"Now setting $z_2 = -M z_1$ where $M = [M_{11} \;\; M_{12}]^T$, the dynamics for z_1 are given by:

$$\dot{z}_1 = (\bar{A}_{11} - A_{12}M)z_1 = (3 + \frac{1}{6}\sqrt{18}M_{11} + \frac{1}{2}\sqrt{6}M_{12})z_1$$

Provided the eigenvalue $\alpha = 3 + \frac{1}{6}\sqrt{18}M_{11} + \frac{1}{2}\sqrt{6}M_{12} < 0$, stability of z_1 is ensured. For example setting $\alpha = -1$, implies $M = [-\frac{18}{\sqrt{18}} \;\; -\frac{2}{\sqrt{6}}]^T$. Finally with $S_2 = I_{m\times m} = 1$, the matrix S is given by:

$$S = S_2[M \;\; I_{m\times m}]T_r = \begin{bmatrix} -3.1162 & -1.409 & -2.8233 \end{bmatrix}"$$

[5]See, for example, Sect. 3.4 in [67].

2.2.3 Quadratic Minimization

Consider the problem of minimizing the quadratic performance index

$$J = \frac{1}{2}\int_{t_s}^{\infty} x(t)^T Q x(t)\, dt \tag{2.39}$$

where Q is both symmetric and positive definite, and t_s is the time at which the sliding motion commences. The objective is to minimize Eq. (2.39) subject to the system equation (2.18) under the assumption that sliding takes place. Notice this is quite different from the "classical" LQR problem formulation which includes a penalty weighting on the control effort. Here no penalty cost on the control is imposed, and this represents a so-called cost-free control problem.

It is assumed that the state of the system at time t_s, given by $x(t_s)$, is a known initial condition, and the closed-loop system is stable such that $x(t) \to 0$ as $t \to \infty$. To solve this problem, the matrix Q from Eq. (2.39) is transformed and partitioned compatibly with the z coordinates from Eq. (2.29) so that

$$T_r Q T_r^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{2.40}$$

In the "z" coordinates, the cost J in Eq. (2.39) can be written as

$$J = \frac{1}{2}\int_{t_s}^{\infty} z_1(t)^T Q_{11} z_1(t) + 2z_1(t)^T Q_{12} z_2(t) + z_2(t)^T Q_{22} z_2(t)\, dt \tag{2.41}$$

If the component z_1 is considered to be the state vector and z_2 the "virtual control" input then this expression represents a "traditional" mixed cost LQR problem associated with the state-space representation in Eq. (2.30) since the term $z_1^T Q_{12} z_2$ involves a mix of the state vector and the virtual control. To avoid this complication, a trick can be employed to "eliminate" the cross term. Define a new "virtual control" input as

$$v := z_2 + Q_{22}^{-1} Q_{12}^T z_1 \tag{2.42}$$

After algebraic manipulation, Eq. (2.41) may then be written as

$$J = \frac{1}{2}\int_{t_s}^{\infty} z_1^T \hat{Q} z_1 + v^T Q_{22} v\, dt \tag{2.43}$$

where

$$\hat{Q} := Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T \tag{2.44}$$

Note that $\hat{Q}$ represents part of the Schur complement of Eq. (2.40). Recall the constraint equation (the null-space dynamics associated with the regular form) may be written as

$$\dot{z}_1(t) = A_{11} z_1(t) + A_{12} z_2(t) \tag{2.45}$$

Eliminating the z_2 contribution from Eq. (2.45) and using Eq. (2.42), after writing the differential equations in terms of the virtual control, the modified constraint equation becomes

$$\dot{z}_1(t) = \hat{A} z_1(t) + A_{12} v(t) \tag{2.46}$$

where

$$\hat{A} = A_{11} - A_{12} Q_{22}^{-1} Q_{12}^T \tag{2.47}$$

The positive definiteness of Q ensures from Schur complement arguments that $Q_{22} > 0$, so that Q_{22}^{-1} exists, and also that $\hat{Q} > 0$. Furthermore, the controllability of the original (A, B) pair ensures the pair $(\hat{A}, A_{12})$ is controllable. Consequently, the problem becomes one of minimizing the functional (2.43) subject to the system (2.46). This can be interpreted as a standard LQR optimal state-regulator problem. A necessary condition to ensure a solution to the LQR problem is that the pair $(\hat{A}, \hat{Q}^{1/2})$ is detectable[6] and then after solving the Riccati equation

$$\hat{P}\hat{A}^T + \hat{A}\hat{P} + \hat{Q} - \hat{P} A_{12} Q_{22}{}^{-1} A_{12}^T \hat{P} = 0 \tag{2.48}$$

the matrix parameterizing the hyperplane is

$$M = Q_{22}^{-1} Q_{12}^T + Q_{22}{}^{-1} A_{12}^T \hat{P} \tag{2.49}$$

Robustness of the LQR Sliding Surface Design

An advantage of this approach compared to pole placement is that the LQR optimization method inherits robustness. Suppose in fact *unmatched uncertainty* is present in the system so that Eq. (2.30) becomes

$$\dot{z}_1(t) = A_{11}(I + \Delta_1) z_1(t) + A_{12}(I + \Delta_1) z_2(t) \tag{2.50}$$

where Δ_1 and Δ_2 represent (unknown) multiplicative perturbations. Suppose Q is chosen so that $Q_{12} = 0$ and $Q_{22} = qI_m$ where q is a positive scalar. If a sliding mode is enforced, the reduced-order sliding motion will be given by

$$\dot{z}_1(t) = \left(A_{11}(I + \Delta_1) - \frac{1}{q} A_{12}(I + \Delta_2) A_{12}^T P_1 \right) z_1(t) \tag{2.51}$$

where P_1 is the symmetric positive definite solution to

$$P_1 A_{11}^T + A_{11} P_1 + Q_{11} - \frac{1}{q} P_1 A_{12} A_{12}^T P_1 = 0 \tag{2.52}$$

---

[6]A more detailed description of this approach is given in Sect. 4.2.2 in [67] and details about LQR methods appear in Appendix C.

Under some conditions on the perturbations it will be shown that Eq. (2.50) is stable. Assume Δ_1 is sufficiently small that

$$P_1\Delta_1 + \Delta_1^T P_1 < Q_{11} \tag{2.53}$$

and consider $V_1 = z_1^T P_1 z_1$ as a candidate Lyapunov function. Then

$$\begin{aligned}\dot{V} &\le z_1^T \frac{1}{q}\left(P_1 A_{12} A_{12}^T P_1 - P_1 A_{12}(I + \Delta_2) A_{12}^T P_1 \; - P_1 A_{12}(I + \Delta_2^T) A_{12}^T P_1\right) z_1 \\ &= -z_1^T \frac{1}{q} P_1 A_{12}\left(\Delta_2 + \Delta_2^T + I\right) A_{12}^T P_1 z_1 \end{aligned} \tag{2.54}$$

after substituting from the Riccati equation from Eq. (2.52) and the bound on the Δ_1 inequality from Eq. (2.53). Consequently if the uncertainty Δ_2 satisfies

$$\Delta_2 + \Delta_2^T + I > 0 \tag{2.55}$$

then $\dot{V} < 0$ for $z_1 \neq 0$ and the sliding motion remains stable. Some special cases can be considered:

- Unstructured Perturbations: Consider the expression

  $$\Theta := 2(\Delta_2 + \frac{1}{2}I)^T(\Delta_2 + \frac{1}{2}I)$$

  Clearly $\Theta \ge 0$ for all Δ_2. Expanding the right hand side it follows

  $$2\Delta_2^T\Delta_2 + \Delta_2 + \Delta_2^T + \frac{1}{2}I \ge 0$$

  which implies

  $$\Delta_2 + \Delta_2^T + I \ge -2\Delta_2^T\Delta_2 + \frac{1}{2}I = 2(\frac{1}{4} - \Delta_2^T\Delta_2) \tag{2.56}$$

  for all Δ_2. However if $\|\Delta_2\| < \frac{1}{2}$, the right hand side of (2.56) is positive and $\Delta_2 + \Delta_2^T + I > 0$.
- Structured Perturbations: Suppose the uncertainty is structured and has the special form $\Delta = \text{diag}(\delta_1, \ldots \delta_m)$ where the δ_i are scalars. Then Eq. (2.55) is equivalent to

  $$2\delta_i + 1 > 0 \quad \text{for } i = 1 \ldots m$$

  and stability is ensured if $\delta_i > -\frac{1}{2}$ for $i = 1 \ldots m$.

  The analysis confirms the robustness of the LQR sliding surface design.

Example 2.5. Consider Example 2.4, now we will design the sliding surface with LQ minimization. Consider the matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

After transformation we have

$$T_r Q T_r^T = \begin{bmatrix} 2 & -\frac{1}{6}\sqrt{18} & -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{18} & \frac{3}{2} & \frac{1}{12}\sqrt{12} \\ -\frac{1}{6}\sqrt{6} & \frac{1}{12}\sqrt{12} & \frac{5}{2} \end{bmatrix}$$

the elements of the new matrix Q are given by

$$Q_{11} = [2], Q_{12} = \begin{bmatrix} -\frac{1}{6}\sqrt{18} & -\frac{1}{6}\sqrt{6} \end{bmatrix},$$
$$Q_{21} = \begin{bmatrix} -\frac{1}{6}\sqrt{18} \\ -\frac{1}{6}\sqrt{6} \end{bmatrix}, Q_{22} = \begin{bmatrix} \frac{3}{2} & \frac{1}{12}\sqrt{12} \\ \frac{1}{12}\sqrt{12} & \frac{5}{2} \end{bmatrix}$$

and after direct computations we have $\hat{A} = 2$ and $\hat{Q} = 1.6364$. Now we can use the *lqr* command of MATLAB in order to obtain the matrix M:

$$M = \begin{bmatrix} -2.032 - 2.3464 \end{bmatrix}$$

Finally we obtain the matrix S:

$$S = \begin{bmatrix} -2.032 & -2.3464 & 1 \end{bmatrix} T_r = \begin{bmatrix} -3.089 & 0.492 & -0.922 \end{bmatrix}$$

2.3 State-Feedback Relay Control Law Design

In this section, although previously multi-input systems were considered at the outset, here the development of control laws for single-input systems will be considered first before multi-input generalizations are considered.

2.3.1 Single-Input Nominal Systems

Using the nomenclature developed in the previous section, suppose $m = 1$, i.e., the system is single input in nature. Assume that the switching function $\sigma(x) = Sx$ has already been defined. In this situation the matrix S is a row vector of the same order as the states. The objective is to force $\sigma \to 0$ in *finite time* and to ensure $\sigma \equiv 0$ for all subsequent time. From the nominal representation in Eq. (2.18) it follows

$$\dot{\sigma}(t) = S\dot{x}(t) = SAx(t) + SBu(t) \tag{2.57}$$

The objective is, through feedback control, to turn the equation above into the differential equation

$$\dot{\sigma}(t) = -\eta \text{sign}(\sigma(t)) \tag{2.58}$$

or equivalently

$$\sigma\dot{\sigma}(t) = -\eta|\sigma(t)| \tag{2.59}$$

For $\sigma(0) \neq 0$ the solution to the equation above becomes zero in finite time. This can easily be seen by the change of variable $V = \frac{1}{2}\sigma^2$. Clearly $\dot{V} = \sigma\dot{\sigma}$ and $|\sigma| = \sqrt{2V}$. Consequently the equation above becomes

$$\dot{V} = -\sqrt{2}\eta V^{1/2}$$

This implies

$$V^{1/2}(t) = V^{1/2}(0) - \sqrt{2}\eta t$$

and therefore at time $t_r = V^{1/2}(0)/(\sqrt{2}\eta)$, it follows $V^{1/2}(t_r) = 0$. Notice that $V = \frac{1}{2}\sigma^2$ can be viewed as a Lyapunov function for the system (2.59) because

$$\dot{V} = \sigma\dot{\sigma} = -\eta\sigma\text{sign}(\sigma) = -\eta|\sigma| < 0$$

when $\sigma \neq 0$. Comparing Eqs. (2.57) and (2.59) it is clear that choosing

$$u(t) = \underbrace{-(SB)^{-1}SAx(t)}_{u_{eq}(t)} - \eta(SB)^{-1}\text{sign}(\sigma) \tag{2.60}$$

as the control law in Eq. (2.57) creates in closed loop the system in (2.59). The simple control law in Eq. (2.60) thus ensures σ is driven to zero in finite time—in fact in $|\sigma(0)|/\eta$ units of time.

2.3.2 *Single-Input Perturbed Systems*

This can be easily extended to the case of systems with bounded matched uncertainty. Now consider the uncertain linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + B\xi(t, x) \tag{2.61}$$

where the (unknown) function $\xi : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^m$ represents matched uncertainty. With the same choice of switching function

$$\dot{\sigma}(t) = S\dot{x}(t) = SAx(t) + SB\xi(t, x) + SBu(t) \tag{2.62}$$

In this situation consider the control law

$$u(t) = -(SB)^{-1}SAx(t) - (\eta + \rho(t, x)|SB|)(SB)^{-1}\text{sign}(\sigma(t)) \tag{2.63}$$

where the scalar function $\rho(t, x)$ represents *a known* upper bound on the (unknown) uncertainty $\xi(t, x)$. Substituting for Eq. (2.60) in Eq. (2.62) gives

$$\begin{aligned}\dot{\sigma}(t) &= SAx(t) + SB\xi(t,x) + SBu(t) \\ &= SAx(t) + SB\xi(t,x) - SAx(t) - (\eta + \rho(t,x)|SB|)\text{sign}(\sigma) \\ &= SB\xi(t,x) - \eta\text{sign}(\sigma) - \rho(t,x)|SB|\text{sign}(\sigma)\end{aligned} \tag{2.64}$$

However since

$$SB\xi(t,x) \leq |SB\xi(t,x)| = |SB||\xi(t,x)| \leq |SB|\rho(t,x)$$

it follows from Eq. (2.64) that

$$\begin{aligned}\sigma\dot{\sigma}(t) &= \sigma SB\xi(t,x) - \eta\sigma\text{sign}(\sigma) - \rho(t,x)|SB|\sigma\text{sign}(\sigma) \\ &\leq |\sigma||SB|\rho(t,x) - \eta|\sigma| - \rho(t,x)|SB||\sigma| \\ &= -\eta|\sigma|\end{aligned} \tag{2.65}$$

and once again σ is driven to zero in less than $|\sigma(0)|/\eta$ units of time.

Consider next a system with what might be described as multiplicative uncertainty:

$$\dot{x}(t) = Ax(t) + B(1+\delta)u(t) + B\xi(t,x) \tag{2.66}$$

where $\delta \in (-\delta_0, \delta_1)$ with known scalars $0 < \delta_0 < 1$ and $\delta_1 > 0$. Compared to Eq. (2.61) it is clear there is now uncertainty associated with the control signal. Note the limitation that $\delta_0 < 1$ prevents a "change of polarity" with respect to the control. Proceeding as before

$$\dot{\sigma}(t) = S\dot{x}(t) = SAx(t) + SB\xi(t,x) + SB\delta u(t) + SBu(t) \tag{2.67}$$

Consider the control law

$$u(t) = u_l(t) + u_n(t), \tag{2.68}$$

where as before the linear term $u_l(t) = -(SB)^{-1}SAx(t)$ and the nonlinear term (with a new modulation function) is $u_n(t) = -(SB)^{-1}\bar{\rho}(t,x)\text{sign}(\sigma(t))$ where

$$\bar{\rho}(t) = (\eta + |SB|(\rho(t,x) + \delta_1|u_l(t)|))(1-\delta_0)^{-1} \tag{2.69}$$

As before the scalar function $\rho(t, x)$ represents a known upper bound on the (unknown) uncertainty $\xi(t, x)$. Note that the modulation function in the nonlinear term also depends on δ_0. Substituting for Eq. (2.68) in Eq. (2.62) gives

$$\begin{aligned}\dot{\sigma}(t) &= SAx(t) + SB\xi(t,x) + SBu(t) \\ &= SB\xi(t,x) + SB(1+\delta)u_n(t) + SB\delta u_l(t)\end{aligned} \tag{2.70}$$

Consequently

$$\begin{aligned}\sigma\dot{\sigma}(t) &= \sigma SB\xi(t,x) + \sigma SB(1+\delta)u_n(t) + \sigma SB\delta u_l(t)\\ &\leq |\sigma||SB||\xi(t,x)| - (1+\delta)\bar{\rho}(t,x)\text{sign}(\sigma) + |\sigma||SB||\delta||u_l(t)|\\ &\leq |\sigma||SB||\xi(t,x)| - (1+\delta)\bar{\rho}(t,x)|\sigma| + |\sigma||SB|\delta_1|u_l(t)|\\ &\leq |\sigma||SB||\xi(t,x)| - (1-\delta_0)\bar{\rho}(t,x)|\sigma| + |\sigma||SB|\delta_1|u_l(t)| \end{aligned} \tag{2.71}$$

Substituting for $\bar{\rho}(t,x)$ from Eq. (2.69) yields

$$\sigma\dot{\sigma}(t) \leq -\eta|\sigma| \tag{2.72}$$

and a sliding mode is guaranteed to be achieved in finite time, and subsequently maintained.

Example 2.6. Consider the state-space model

$$\dot{x}(t) = Ax(t) + bu(t) \tag{2.73}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

These represent the equations of motion of a hot-air balloon where the control input is the fuel flow into the burner and the first component represents the altitude of the balloon.[7] The open-loop poles are $\{0, -1, -2\}$.

The aim is to select a switching function defined by

$$S = \begin{bmatrix} s_1 & s_2 & 1 \end{bmatrix}$$

or equivalently

$$\sigma(x) = s_1x_1 + s_2x_2 + x_3$$

to ensure the reduced-order sliding motion confined to $\mathcal{S}$ is stable, and meets any design specifications. While sliding, when $\sigma = 0$,

$$x_3 = -\begin{bmatrix} s_1 & s_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.74}$$

Because of the special form of the state space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3 \tag{2.75}$$

$$x_3 = -\begin{bmatrix} s_1 & s_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.76}$$

[7] Taken from [86].

Simplifying Eqs. (2.74) and (2.75)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -s_1 & -2-s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.77}$$

Equations (2.75) and (2.74) represent a second-order system in which x_3 has the role of the control variable and $[\ s_1 \quad s_2\ ]$ is a full state-feedback matrix. The characteristic equation of Eq. (2.77) is

$$\det \begin{bmatrix} \lambda & -1 \\ s_1 & \lambda + 2 + s_2 \end{bmatrix} = 0$$

in other words

$$\lambda^2 + (2 + s_2)\lambda + s_1 = 0 \tag{2.78}$$

Choosing the required sliding mode poles to be $\{-1 \pm j\}$ gives a desired characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

Comparing coefficients with Eq. (2.78) gives $s_1 = 2$ and $s_2 = 0$ and the resulting switching function

$$\sigma(x) = 2x_1 + x_3 \tag{2.79}$$

A control law must be developed such that the reachability condition (2.59) is satisfied. It follows (in this case) that

$$\dot{\sigma} = 2\dot{x}_1 + \dot{x}_3$$

Now substituting from the original equations

$$\dot{\sigma} = 2 \underbrace{x_2}_{\dot{x}_1} + \underbrace{-x_3 + 10u}_{\dot{x}_3}$$

Now choose

$$u = -\frac{1}{5}x_2 + \frac{1}{10}x_3 - \frac{\eta}{10}\,\mathrm{sign}(\sigma) \tag{2.80}$$

where η is a positive scalar. It follows that

$$\dot{\sigma} = -\eta\,\mathrm{sign}(\sigma) \quad \Rightarrow \quad \sigma\dot{\sigma} = -\eta|\sigma|$$

Hence Eq. (2.80) is an appropriate variable structure controller which induces a sliding motion. The plot of the switching function is shown in Fig. 2.5.

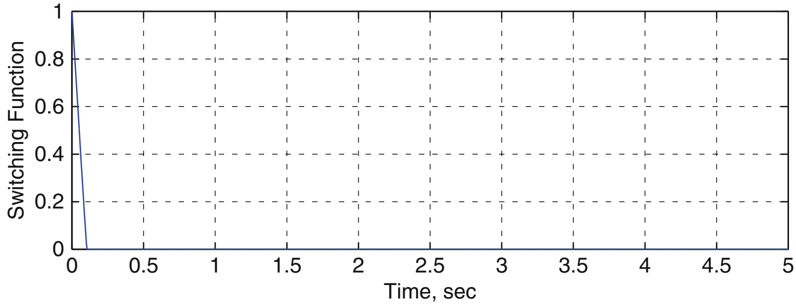

Fig. 2.5 Switching function

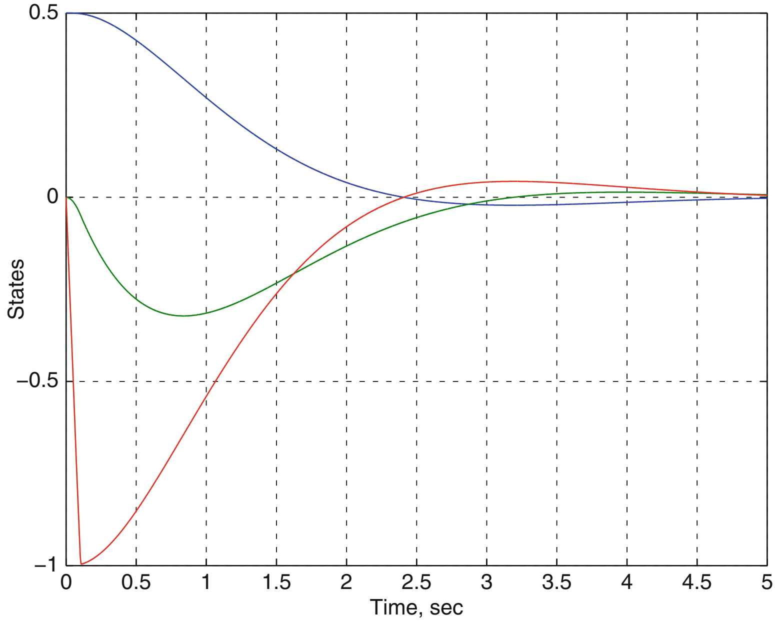

Fig. 2.6 States time history

Note that in finite time—approximately 0.1 s—the switching function has become zero. Furthermore once zero (at which point the hyperplane has been reached) the states are forced to remain on the surface. The state's evolution is presented in Fig. 2.6. It is clear from Fig. 2.6 that x_1 and x_2 approach zero as time increases in the sliding mode.

2.3.3 Relay Control for Multi-input Systems

In this subsection these ideas are extended to multi-input systems. Many different multivariable control structures exist and fundamentally the key thing that must be achieved is a multivariable version of the sign function.

In the multi-input case, once again

$$\dot{\sigma}(t) = S\dot{x}(t) = SAx(t) + SBu(t) \tag{2.81}$$

The "complication" now is the fact that SB is a square *matrix* and not a scalar. A simple way to circumvent this is to enforce a structure on SB through choice of S while at the same time ensuring appropriate properties for the reduced-order sliding motion. It will be argued that in fact S can always be chosen so that $SB = I_m$ while simultaneously ensuring appropriate dynamics for the sliding mode.

Recall from Eq. (2.32) that the sliding surface hyperplane matrix can be parameterized as

$$S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} T_r \tag{2.82}$$

where the matrices S_2 and S_1 were linked via the parameter $M = S_2^{-1}S_1$ and T_r was the orthogonal matrix used as the basis of the coordinate transformation to achieve regular form. Equivalently Eq. (2.82) above can be written as

$$S = S_2 \begin{bmatrix} M & I_m \end{bmatrix} T_r \tag{2.83}$$

In the above, for a given pair (A, B), the matrix T_r is established from QR reduction based in B. Furthermore the dynamics of the reduced-order motion depend solely on the choice of matrix M which may be viewed as a state-feedback gain for the null-space system in (2.30). Clearly the choice of S_2 does not affect the dynamics of the sliding motion and it is this design freedom which is exploited to ensure that $SB = I_m$. For a given (A, B) and design parameter M, it follows

$$\begin{aligned} SB &= \begin{bmatrix} S_1 & S_2 \end{bmatrix} T_r B \\ &= \begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad \text{from Eq. (2.28)} \\ &= S_2 B_2 \end{aligned} \tag{2.84}$$

Consequently choosing $S_2 = B_2^{-1}$ ensures the sliding motion dynamics are specified as $(A_{11} - A_{12}M)$ and simultaneously $SB = I_m$.

Now the multi-input structure in Eq. (2.81) can be decomposed into m independent equations. Specifically exploiting the fact that $SB = I_m$, Eq. (2.81) can be written componentwise as

$$\dot{\sigma}_i(t) = (SA)_i x(t) + u_i(t) \tag{2.85}$$

where $(SA)_i$ is the ith row of the $m \times n$ matrix SA and u_i and σ_i are the ith components of the vectors u and σ, respectively. Now as discussed in the previous section, m independent single-input controllers can be designed for each of the components u_i.

2.4 State-Feedback Unit-Vector Control

Of the many different multivariable sliding mode control structures which exist, the one that will be considered here is the *unit-vector* approach. This has the advantage of being an inherently multi-input approach. Consider an uncertain system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t, x, u) \tag{2.86}$$

where the function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, which represents the uncertainties or nonlinearities, satisfies the so-called *matching condition*, i.e.,

$$f(t, x, u) = B\xi(t, x, u) \tag{2.87}$$

where $\xi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and is unknown but satisfies

$$\|\xi(t, x, u)\| \leq k_1\|u\| + \alpha(t, x), \tag{2.88}$$

where $1 > k_1 \geq 0$ is a known constant and $\alpha(\cdot)$ is a known function.

2.4.1 Design in the Presence of Matched Uncertainty

The proposed control law comprises two components: a linear component to stabilize the nominal linear system and a discontinuous component. Specifically

$$u(t) = u_l(t) + u_n(t) \tag{2.89}$$

where the linear component is given by

$$u_l(t) = -\Lambda^{-1}(SA - \Phi S)\,x(t) \tag{2.90}$$

where $\Phi \in \mathbb{R}^{m\times m}$ is any stable design matrix and $\Lambda = SB$. The nonlinear component is defined to be

$$u_n(t) = -\rho(t, x)\Lambda^{-1}\frac{P_2\sigma(t)}{\|P_2\sigma(t)\|}, \tag{2.91}$$

where $P_2 \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix satisfying the Lyapunov equation

$$P_2 \Phi + \Phi^T P_2 = -I \tag{2.92}$$

and the scalar function $\rho(t, x)$, which depends only on the magnitude of the uncertainty, is any function satisfying

$$\rho(t, x) \geq (\|\Lambda\|(k_1 \|u_l\| + \alpha(t, x)) + \gamma) / (1 - k_1 \|\Lambda\| \|\Lambda^{-1}\|) \tag{2.93}$$

where $\gamma > 0$ is a design parameter. In this equation it is assumed that the scaling parameter S_2 has been chosen so that $\Lambda = SB$ has the property that

$$k_1 \|\Lambda\| \|\Lambda^{-1}\| < 1 \tag{2.94}$$

A necessary condition for Eq. (2.94) to hold is that $k_1 < 1$ because $\|\Lambda\| \|\Lambda^{-1}\| > 1$ for all choices of S.

Before demonstrating that the above controller induces a sliding motion, it will first be established that any scalar modulation function satisfying Eq. (2.93) bounds the uncertain term $\xi(t, x, u)$.

Rearranging Eq. (2.93) gives

$$\begin{aligned} \rho(t, x) &\geq \|\Lambda\|(k_1 \|u_l\| + \alpha(t, x)) + \gamma + k_1 \|\Lambda\| \|\Lambda^{-1}\| \rho(t, x) \\ &\geq \|\Lambda\|(k_1 \|\Lambda^{-1}\| \rho(t, x) + k_1 \|u_l\| + \alpha(t, x)) + \gamma \\ &\geq \|\Lambda\|(k_1 \|u\| + \alpha(t, x)) + \gamma \\ &\geq \|\Lambda\| \|\xi(t, x, u)\| + \gamma \end{aligned} \tag{2.95}$$

In obtaining the third inequality the fact that

$$u = u_l - \rho(t, x)\Lambda^{-1} \frac{P_2 \sigma}{\|P_2 \sigma\|} \quad \Rightarrow \quad \|u\| < \|u_l\| + \rho(t, x) \|\Lambda^{-1}\|$$

is used. Inequality (2.95) demonstrates $\rho(t, x)$ is greater in magnitude than the matched uncertainty occurring in Eq. (2.87). Substituting for the control law in Eq. (2.86) yields

$$\begin{aligned} \dot{\sigma} &= SAx(t) + \Lambda u + \Lambda \xi(t, x, u) \\ &= \Phi \sigma - \rho(t, x) \frac{P_2 \sigma}{\|P_2 \sigma\|} + \Lambda \xi(t, x, u) \end{aligned} \tag{2.96}$$

It will now be shown that $V(\sigma) = \sigma^T P_2 \sigma$ guarantees quadratic stability for the switching states σ, and in particular

$$\begin{aligned} \dot{V} &= \sigma^T (P_2 \Phi + \Phi^T P_2) \sigma - 2\rho \|P_2 \sigma\| + 2\sigma^T P_2 \Lambda \xi \\ &\leq \sigma^T (P_2 \Phi + \Phi^T P_2) \sigma - 2\rho \|P_2 \sigma\| + 2\|P_2 \sigma\| \|\Lambda\| \|\xi\| \\ &= -\sigma^T \sigma - 2\|P_2 \sigma\|(\rho(t, x) - \|\Lambda\| \|\xi\|) \\ &\leq -\sigma^T \sigma - 2\gamma \|P_2 \sigma\| \end{aligned} \tag{2.97}$$

Assuming that the closed-loop system has no finite-escape time during the reaching phase, then this control law guarantees that the switching surface is reached in finite time despite the disturbance or uncertainty. Once the sliding motion is attained, it is completely independent of the uncertainty.

Example 2.7. Consider the satellite dynamics given by

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + u_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + u_3 \end{aligned} \tag{2.98}$$

where I_1, I_2, and I_3 represent the moments of inertia around the principal axes of the body. The variables ω_1, ω_2, and ω_3 are the angular velocities, which are measurable. The variables u_1, u_2, and u_3 are the control input torques. Defining $x_1 = \omega_1$, $x_2 = \omega_2$, $x_3 = \omega_3$, and $x = [x_1, x_2, x_3]^T$, the system in (2.98) has the representation:

$$\dot{x} = Bu + B\xi(x) = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix} \begin{bmatrix} (I_2 - I_3)x_2x_3 \\ (I_3 - I_1)x_3x_1 \\ (I_1 - I_2)x_1x_2 \end{bmatrix}$$

$$\sigma = Sx$$

where σ is the sliding output and $\xi(x)$ represents the nonlinearities which satisfy the *matching condition*. Furthermore $\xi(x)$ satisfies $||\xi(x)|| \leq \alpha(x) \leq c$ in a domain $x \in \Omega \subset \mathbb{R}^3$ that includes the origin, where c is a known constant. Then the problem is to stabilize the equilibrium point $x = 0$ of the satellite in finite time. Based on Eq. (2.89), the proposed control law is:

$$u = (SB)^{-1}(\Phi S)x - \rho(x)(SB)^{-1}\frac{P_2\sigma}{||P_2\sigma||}$$

Choosing $S = \text{diag}(l_1, l_2, l_3)$ it follows $SB = I_{3\times3}$. If $\Phi = -\frac{1}{2}I_{3\times3}$, $P_2 = I_{3\times3}$ is the solution of the Lyapunov equation:

$$P_2\Phi + \Phi^T P_2 = -I \tag{2.99}$$

Now choosing the Lyapunov function $V = \sigma^T P_2\sigma$ we have

$$\dot{V} \leq -||\sigma||^2 - 2\gamma||P_2\sigma|| < -2\gamma||\sigma|| < 0$$

and finally

$$\dot{V} < -2\gamma V^{\frac{1}{2}}$$

where $\gamma = \rho - c > 0$. For simulation purposes we consider $\rho = 3$. Therefore the proposed control law guarantees that the sliding surface is reached in finite time,

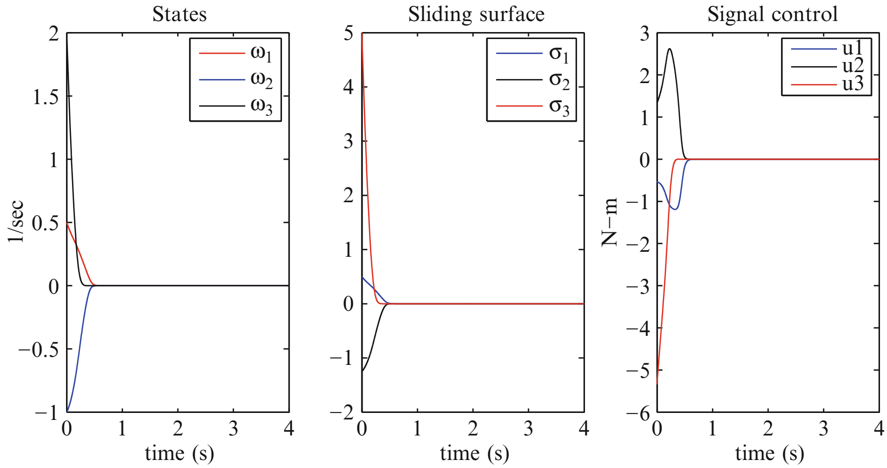

Fig. 2.7 Stabilization of the equilibrium point of the satellite

which in this case means that x equals zero in finite time. The system in Eq. (2.98) has been simulated with the control law (2.89) using the initial conditions $x(0) = [0.5, -1, 2]^T$ and the parameters $I_1 = 1\,\text{kgm}^2$, $I_2 = 0.8\,\text{kgm}^2$, and $I_3 = 0.4\,\text{kgm}^2$. The results obtained with the proposed control law are presented in Fig. 2.7. In order to attenuate chattering the discontinuous portion of the control signal is approximated by $\frac{P_2\sigma}{||P_2\sigma||} \approx \frac{P_2\sigma}{||P_2\sigma||+\epsilon}$, with $\epsilon \ll 1$.

2.4.2 Design in the Presence of Unmatched Uncertainty

If the uncertainty does not meet the matching requirements, after transformation into regular form, in the "z" coordinates, a system of the form

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + f_u(t, z_1, z_2) \tag{2.100}$$

$$\dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + B_2u(t) + f_m(t, z_1, z_2) \tag{2.101}$$

is obtained where $f_m(t, z_1, z_2)$ and $f_u(t, z_1, z_2)$ represent the matched and unmatched components of the uncertainty, respectively. As argued in the earlier sections, the effects of the matched uncertainty $f_m(t, z_1, z_2)$ can be canceled. This section considers the null-space dynamics in Eq. (2.100). If a sliding motion can be induced on $\mathcal{S}$, then $z_2 = -Mz_1$ and the reduced-order motion is governed by

$$\dot{z}_1(t) = (A_{11} - A_{12}M)z_1(t) + f_u(t, z_1, -Mz_1) \tag{2.102}$$

Because of the presence of the term $f_u(t, z_1, -Mz_1)$, stability of the system in Eq. (2.102) is not guaranteed. However, if the linear component is dominant, then bounds on $f_u(t, z_1, -Mz_1)$ can be obtained to guarantee that stability is maintained. Many different approaches and assumptions can be made: here a Lyapunov approach will be adopted. Specifically it will be assumed that

$$\|f_u(t, z_1, z_2)\| \leq \mu\|z\| \tag{2.103}$$

where μ is a positive scalar.

Since, by choice of the sliding surface, the matrix $(A_{11} - A_{12}M)$ is stable, there exists a symmetric positive definite matrix P_1 such that

$$P_1(A_{11} - A_{12}M) + (A_{11} - A_{12}M)^T P_1 = -I_{n-m}$$

It can be shown that if

$$\mu\sqrt{(1 + \|M\|^2} < 1/(2\|P_1\|) \tag{2.104}$$

then Eq. (2.102) is stable while sliding. To establish this, first the constraint in Eq. (2.103) will be written in terms of $\|z_1\|$. Since during the sliding motion $z_2 = -Mz_1$ it follows

$$\|z\|^2 = \left\|\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\|^2 \leq \|z_1\|^2 + \|z_2\|^2 \leq \|z_1\|^2 + \|Mz_1\|^2 \leq (\|M\|^2 + 1)z_1^2$$

and consequently

$$\|z\| \leq \sqrt{(\|M\|^2 + 1)}\|z_1\|$$

Create from the symmetric positive definite matrix P_1 a candidate Lyapunov function $V(z_1) = z_1^T P_1 z_1$. It follows

$$\begin{aligned}
\dot{V} &= z_1^T P_1 \dot{z}_1 + \dot{z}_1^T P_1 z_1 \\
&= z_1^T \left(P_1(A_{11} - A_{12}M) + (A_{11} - A_{12}M)^T P_1\right) z_1 + 2z_1^T P_1 f_u(t, z_1, -Mz_1) \\
&\leq -z_1^T z_1 + 2\|z_1\|\|P_1\|\|f_u(t, z_1, -Mz_1)\| \\
&\leq -z_1^T z_1 + 2\|z_1\|\|P_1\|\mu\|z\| \\
&\leq -z_1^T z_1 + 2\|z_1\|\|P_1\|\mu\sqrt{(\|M\|^2 + 1)}\|z_1\| \\
&\leq \|z_1\|^2(2\mu\|P_1\|\sqrt{(\|M\|^2 + 1)} - 1)
\end{aligned} \tag{2.105}$$

If the inequality in Eq. (2.104) holds then

$$\dot{V} \leq \|z_1\|^2(2\|P_1\|\mu\sqrt{(\|M\|^2 + 1)} - 1) < 0 \tag{2.106}$$

and so the reduced-order motion is stable.

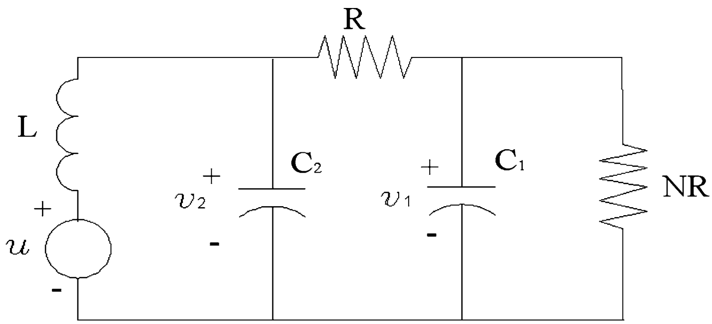

Fig. 2.8 Chua's circuit

Some modifications need to be made to the control law gain (2.93) to ensure a sliding motion can be achieved and maintained. Now

$$\dot{\sigma}(t) = SA(t) + SBu(t) + Sf(t,x) \tag{2.107}$$

It can be shown using similar arguments to those deployed earlier that

$$\rho(t,x) \geq \frac{\|S_2\| \left(\|M\|\mu\|x(t))\|\|u_l(t)\| + \alpha(t,x)\right) + \gamma}{(1 - k_3\|\Lambda\|\|\Lambda^{-1}\|\|B_2^{-1}\|)} \tag{2.108}$$

guarantees the existence of a sliding motion.

Example 2.8. Chua's circuit consists of one inductor, two capacitors, and one piecewise-linear nonlinear resistor; see Fig. 2.8. The normalized dynamic equations of the circuit are:

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1 - f(x_1)) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2 + u \end{aligned} \tag{2.109}$$

where $x_1 = v_1$, $x_2 = v_2$, and x_3 is the current through the inductor; α and β are known parameters and $f(x_1)$ is a function that depends on the nonlinear resistor; this function represents the *unmatched uncertainty*. We consider the nonlinear function $f(x_1) = -\frac{1}{5}x_1(1 - x_1^2)$. Here it is assumed that $|x_1| < 1$, which implies $|f(x_1)| \leq \frac{1}{5}|x_1| + \frac{1}{5}|x_1^2| \leq \frac{2}{5}|x_1|$; then inequality (2.103) is satisfied with $\mu = \frac{2}{5}$. The goal of this example is to stabilize the equilibrium point $x = [0,0,0]^T$ of Chua's circuit. First define $z_1 = [x_1, x_2]^T$ and $z_2 = x_3$; then Chua's circuit has the following representation:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{0} \\ 1 \end{bmatrix} u - \begin{bmatrix} \bar{f}(z_1) \\ \bar{0} \end{bmatrix}$$

with the following definitions:

$$A_{11} = \begin{bmatrix} -\alpha & \alpha \\ 1 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & -\beta \end{bmatrix}, \quad A_{22} = 0, \quad \bar{f}(z_1) = \begin{bmatrix} f(x_1) \\ 0 \end{bmatrix}$$

Setting the sliding output $\sigma_1 = z_2 + Mz_1$ the reduced-order dynamic is given by:

$$\dot{z}_1 = (A_{11} - A_{12}M)z_1 - \bar{f}(z_1).$$

Because of the term $\bar{f}(z_1)$, stability is not guaranteed. However since by choice of the sliding surface, the matrix $(A_{11} - A_{12}M)$ is stable, there exists a symmetric positive matrix P_1 such that

$$P_1(A_{11} - A_{12}M) + (A_{11} - A_{12}M)^T P_1 = -I_{2\times 2}$$

Now the candidate Lyapunov function $V = z_1^T P_1 z_1$ is proposed. It follows:

$$\dot{V} \leq ||z_1||^2(2||P_1||\mu\sqrt{||M||^2 + 1} - 1)$$

Then if the inequality $(2||P_1||\mu\sqrt{||M||^2 + 1} - 1) < 0$ is satisfied the reduced-order dynamic is stable. Typical system parameters are chosen to be $\alpha = 9.1/7$ and $\beta = -8/7$. For $M = [2, 2]$ we obtain

$$P_1 = \begin{bmatrix} 0.3553 & -0.0293 \\ -0.0293 & 0.1764 \end{bmatrix}, \quad (2||P_1||\mu\sqrt{||M||^2 + 1} - 1) = -0.1854$$

and inequality (2.104) is satisfied. Now with the control law

$$u = -(A_{21} - A_{22}M)z_1 - M(A_{11} - A_{12}M)z_1 - \rho(z)\frac{\sigma_1}{||\sigma_1||}$$

and the candidate Lyapunov function $V = \frac{1}{2}\sigma_1^2$, we have

$$\dot{V} \leq -(\rho(z) - \mu||M||||z||)||\sigma_1||$$

If the inequality $\rho(z) > \mu||M||||z||$ is satisfied, finite time convergence is ensured, provided during the reaching phase $|x_1(t)| < 1$. For simulation purposes $\rho = 4 > \mu||M||||z|| = 1.14$. Using the initial conditions $x_1(0) = 0.4$, $x_2(0) = 0.2$, and $x_3(0) = 0.5$, simulations were performed with the proposed control law and the results are presented in Fig. 2.8. In order to attenuate chattering the discontinuous portion of the control signal is approximated by $\rho(z)\frac{\sigma_1}{||\sigma_1||} \approx \rho(z)\frac{\sigma_1}{||\sigma_1||+\epsilon}$, with $\epsilon \ll 1$ (Fig. 2.9).

In engineering situations, tracking problems are often encountered whereby (usually) the output of the system is required to follow a predefined reference signal. In the following section an integral action based method is considered for output tracking.

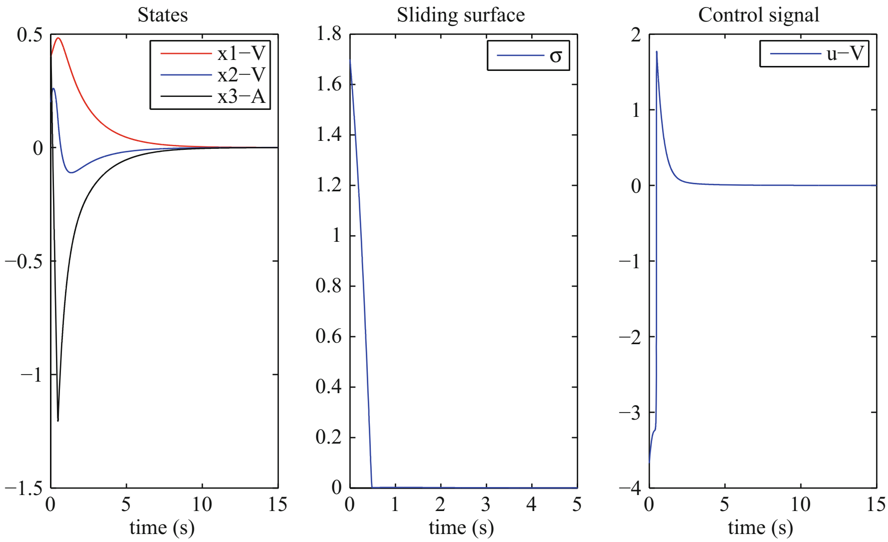

Fig. 2.9 Stabilization of the equilibrium point of Chua's circuit

2.5 Output Tracking with Integral Action

Consider the development of a tracking control law for the nominal linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.110}$$

$$y(t) = Cx(t) \tag{2.111}$$

which is assumed to be square, i.e., it has the same number of inputs and outputs. In addition, for convenience, assume the matrix pair (A, B) is in regular form. The control law described here utilizes an *integral action* methodology. Consider the introduction of additional states $x_r \in \mathbb{R}^m$ satisfying

$$\dot{x}_r(t) = r(t) - y(t) \tag{2.112}$$

where the differentiable signal $r(t)$ satisfies

$$\dot{r}(t) = \Gamma\,(r(t) - R) \tag{2.113}$$

with $\Gamma \in \mathbb{R}^{m\times m}$ a stable design matrix and R a constant demand vector. Augment the states with the integral action states and define

$$\tilde{x} = \begin{bmatrix} x_r \\ x \end{bmatrix} \tag{2.114}$$

The associated system and input distribution matrices for the *augmented system* are

$$\tilde{A} = \begin{bmatrix} 0 & -C \\ 0 & A \end{bmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{2.115}$$

assuming the pair (A, B) is in regular form, the pair $(\tilde{A}, \tilde{B})$ is in regular form. The proposed controller seeks to induce a sliding motion on the surface

$$\mathcal{S} = \{\tilde{x} \in \mathbb{R}^{n+m} \ : \ S\tilde{x} = S_r r\} \tag{2.116}$$

where $S \in \mathbb{R}^{m\times(n+p)}$ and $S_r \in \mathbb{R}^{m\times m}$ are design parameters which govern the reduced-order motion. Partition the hyperplane system matrix as

$$S = \begin{array}{c} \quad \overset{n}{\leftrightarrow} \quad \overset{m}{\leftrightarrow} \\ \begin{bmatrix} S_1 & S_2 \end{bmatrix} \end{array} \tag{2.117}$$

and the system matrix as

$$\tilde{A} = \begin{array}{c} \quad \overset{n}{\leftrightarrow} \quad \overset{m}{\leftrightarrow} \\ \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \end{array} \begin{array}{l} \updownarrow n \\ \updownarrow m \end{array} \tag{2.118}$$

and assume $\Lambda = S\tilde{B}$ is nonsingular. If a controller exists which induces an ideal sliding motion on $\mathcal{S}$ and the augmented states are partitioned as

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.119}$$

where $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$, then the ideal sliding motion is given by

$$\dot{x}_1(t) = (\tilde{A}_{11} - \tilde{A}_{12}M)x_1(t) + \left(\tilde{A}_{12}S_2^{-1}S_r + B_r\right) r(t) \tag{2.120}$$

where $M = S_2^{-1}S_1$ and $B_r = \begin{bmatrix} I_m & 0_{n\times m} \end{bmatrix}^T$. In order for the design methods described earlier to be valid, it is necessary for the matrix pair $(\tilde{A}_{11}, \tilde{A}_{12})$ to be completely controllable.

Remark 2.5. Necessary conditions on the original system are that (A, B, C) is completely controllable and has no invariant zeros at the origin.[8]

The development that follows mirrors the approach in Sect. 2.3.3 where Φ is any stable design matrix. The overall control law is then given by

[8]For details see Sect. 4.4.2 in [67] and Appendix C.

$$u = u_l(\tilde{x}, r) + u_n(\tilde{x}, r) \tag{2.121}$$

where the continuous control

$$u_l(\tilde{x}, r) = -\Lambda^{-1}(SA - \Phi S)\,\tilde{x}(t) \tag{2.122}$$

and the discontinuous control vector

$$u_n(\sigma, r) = -\rho_c(u_l, y)\Lambda^{-1}\frac{P_2(S\tilde{x} - S_r r)}{\|P_2(S\tilde{x} - S_r r)\|} \tag{2.123}$$

where P_2 is a symmetric positive definite matrix satisfying

$$P_2\Phi + \Phi^T P_2 = -I \tag{2.124}$$

The positive scalar function which multiplies the unit-vector component can be obtained from arguments similar to those in Sect. 2.3. It follows that, in terms of the original coordinates,

$$u_n(\tilde{x}, r) = L\tilde{x} + L_r r + L_{\dot{r}}\dot{r} \tag{2.125}$$

with the gains defined as

$$L = -\Lambda^{-1}(S\tilde{A} - \Phi S) \tag{2.126}$$

$$L_r = -\Lambda^{-1}(\Phi S_r + S_1 B_r) \tag{2.127}$$

$$L_{\dot{r}} = \Lambda^{-1} S_r \tag{2.128}$$

The parameter S_r can take any value and does not affect the stability of the closed-loop system. One common choice is to let $S_r = 0$ for simplicity. Another option, which has been found to give good results with practical applications, is to choose S_r so that at steady state the integral action states are zero in the absence of any uncertainty.

Up to this point it has been assumed that all the states are available for use in the control law. This hypothesis will be dropped in the remaining sections of Chap 2.

2.6 Output-Based Hyperplane Design

Consider the linear system in Eq. (2.18) and suppose that only the measured outputs

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.129}$$

$$y(t) = Cx(t) \tag{2.130}$$

where $C \in \mathbb{R}^{p\times n}$ are available. The methods described earlier in this chapter are now no longer directly applicable since the state vector is not directly available. One approach is to use an observer (this will be discussed in later chapters) to estimate the states and then to use the estimate in place of the real states. This is conceptually straightforward but has potential pitfalls—particularly if linear observers are used. It is well understood that in linear systems (with linear control laws and linear observers), the robustness associated with the feedback law can be destroyed by the introduction of an observer as part of the feedback loop. Also the introduction of an observer will add significant computational costs in terms of implementation. Instead let us consider the possibility of introducing controllers in the spirit of Sect. 2.5 subject to the constraint that only output information is available.

2.6.1 Static Output-Feedback Hyperplane Design

The state-feedback control strategies described earlier are not immediately employable here. Firstly, it is intuitively likely that since the hyperplane design problem resolved itself into a state-feedback control paradigm for the fictitious triple (A_{11}, A_{12}), in the situation when only output information is available this will become some form of restricted state-feedback design problem: in fact a static output-feedback design problem for a certain triple (A_{11}, A_{12}, C_1). Secondly the control laws described in the previous section involved a linear state-feedback control component. This is unlikely to be achievable in an output-feedback context.

Here, the situation when there are more outputs than inputs is considered, since in the square case, no design freedom exists in terms of selecting the dynamics of the sliding motion. Two assumptions will be made:

(A1) The parameter CB is full rank.
(A2) Any invariant zeros of (A, B, C) have negative real parts.

Remark 2.6. The dependence on invariant zeros is not perhaps surprising and the presence of zeros plays an important role in linear systems theory. It is also worth noting that for appropriate initial conditions associated with a particular zero direction $y(t)$ can be made zero for all time with an appropriate control linear control input although the state itself is not zero. This has clear links with the concept of a sliding motion (without the robustness properties).

These assumptions will be central to the output-feedback-based sliding mode control used here. The following lemma provides a canonical form for the system triple (A, B, C) which will be used in the subsequent analysis:

Lemma 2.1. *Let (A, B, C) be a linear system with $p > m$ and* rank$(CB) = m$*. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:*

(a) The system matrix can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{2.131}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ *and the (fictitious) pair* (A_{11}, C_1) *is detectable where*

$$C_1 = \begin{bmatrix} 0 & I_{p-m} \end{bmatrix} \tag{2.132}$$

(b) The input distribution matrix has the form

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad \text{where } B_2 \in \mathbb{R}^{m\times m} \text{ and is nonsingular} \tag{2.133}$$

(c) The output distribution matrix has the form

$$C = \begin{bmatrix} 0 & T \end{bmatrix} \quad \text{where } T \in \mathbb{R}^{p\times p} \text{ and is orthogonal} \tag{2.134}$$

Remark 2.7. It can be shown that the unobservable modes of (A_{11}, C_1) are in fact the invariant zeros of the triple (A, B, C).

The idea is to first achieve a regular form structure for the input distribution matrix and then to exploit the fact that CB is full rank to create another transformation which preserves the structure of B while enforcing the partition nature of the output distribution matrix C, which is a feature of the canonical form.

Remark 2.8. Note, this can be viewed as a special case of the traditional regular form discussed earlier which was used as the basis for switching function design in the state-feedback case.

Remark 2.9. Clearly the existence of unstable zeros renders difficulties. In fact unstable invariant zeros means the techniques described in this section are not applicable. While this is not ideal, it must be remembered in linear systems the presence of right half plane zeros limits the performance that can be imposed on the closed-loop system.[9]

Under the premise that only output information is available, the switching function must be of the form

$$\sigma(x) = FCx(t) \tag{2.135}$$

where $F \in \mathbb{R}^{m\times p}$. Suppose a controller exists which induces a stable sliding motion on

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ FCx = 0\} \tag{2.136}$$

[9] This situation is discussed further in "Notes and References" in Chap. 8.

For a first-order sliding motion to exist on $\mathcal{S}$, the equivalent control will be given by solving

$$\dot{\sigma} = FCAx(t) + FCBu_{eq}(t) = 0$$

For an unique equivalent control to exist, the matrix $FCB \in \mathbb{R}^{m\times m}$ must have full rank: this implies that rank$(CB) = m$ since rank(FCB) $\leq$rank(CB). In all the analysis which follows it is assumed without loss of generality that the system is already in the canonical form of Lemma 2.1. Define matrices F_1 and F_2 such that

$$\begin{array}{cc} \overset{p-m}{\leftrightarrow} & \overset{m}{\leftrightarrow} \\ \left[\; F_1 \right. & \left. F_2 \;\right] \end{array} = FT \tag{2.137}$$

where $F_2 \in \mathbb{R}^{m\times m}$ is assumed to be nonsingular. Notice this structure has a relationship to the hyperplane matrix parametrization given in Eq. (2.83) since Eq. (2.137) can be re-written as

$$F = F_2 \left[\; K \quad I_m \;\right] T^T \tag{2.138}$$

where $K = F_2^{-1}F_1$. The structure in Eq. (2.138) may not be particularly intuitive, but it nicely isolates the design freedom present in the problem. The matrix F_2 is essentially a scaling matrix which is square and invertible and it plays no role in determining the dynamics of the sliding motion. Furthermore it is analogous to the role the matrix S_2 plays in the switching function expansion in Eq. (2.83). The other design parameter is K. This is analogous to the matrix M from Eq. (2.83). It is clear from the dimension of the matrices F and M that the former has only $(p-m)\times(n-m)$ elements while the latter has $m \times (n-m)$. This reduction in parametrization results in less design flexibility—which is the price to pay for only having output rather than state information.

Based on Eq. (2.138) the matrix which defines the switching function can then be written as

$$FC = \left[\, F_1C_1 \;\; F_2 \,\right]$$

where

$$C_1 = \left[\, 0_{(p-m)\times(n-p)} \;\; I_{(p-m)} \,\right] \tag{2.139}$$

In this way $FCB = F_2B_2$ and $\det(F_2) \neq 0 \Leftrightarrow \det(FCB) \neq 0$. It follows during sliding $\sigma = FCx(t) = 0$ and

$$F_1C_1z_1 + F_2z_2 = 0$$

which substituting in the null-space equations yields

$$\begin{aligned} \dot{z}_1(t) &= (A_{11} - A_{12}F_2^{-1}F_1C_1)z_1(t) \\ &= (A_{11} - A_{12}KC_1)z_1(t) \end{aligned} \tag{2.140}$$

since $K = F_2^{-1}F_1$. Consequently the problem of designing a suitable hyperplane is equivalent to an output-feedback problem for the system (A_{11}, A_{12}, C_1).

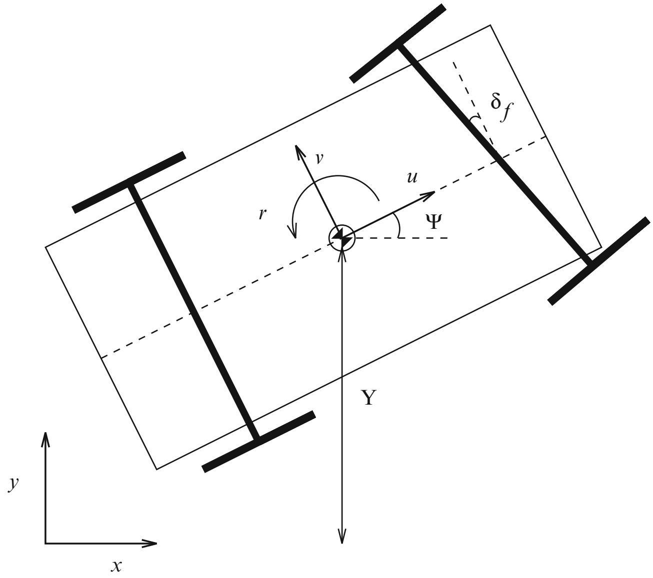

Fig. 2.10 Schematic of the vehicle

Remark 2.10. If the pair (A_{11}, C_1) is observable and the triple (A_{11}, A_{12}, C_1) satisfies the Kimura–Davison conditions[10]

$$m + p \geq n + 1 \tag{2.141}$$

output-feedback pole placement methods can be used to place the poles appropriately.

Example 2.9. Consider the fourth-order system

$$A = \begin{bmatrix} -3.9354 & 0 & 0 & -14.7110 \\ 0 & 0 & 0 & 1.0000 \\ 1.0000 & 14.9206 & 0 & 1.6695 \\ 0.7287 & 0 & 0 & -2.1963 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.8116 \end{bmatrix} \tag{2.142}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.143}$$

This represents a linearization of the rigid body dynamics of a passenger vehicle (Fig. 2.10). The first state is an average of the lateral velocity v and yaw rate r; the second state represents Ψ, the vehicle orientation; the third state, Y, is the lateral

[10]See "Notes and References" at the end of the chapter.

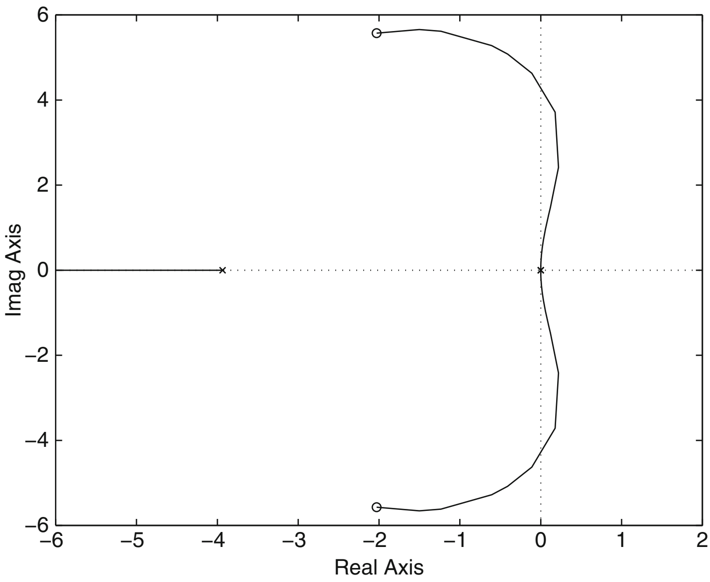

Fig. 2.11 Sliding mode poles as a root locus

deviation from the intended lane position, and the fourth state, r, is the yaw rate. The input to the system, δ_f, is the angular position of the front wheels relative to the chassis.

Notice this is already in the canonical form of Eqs. (2.131)–(2.134), and so

$$A_{11} = \begin{bmatrix} -3.9354 & 0 & 0 \\ 0 & 0 & 0 \\ 1.0000 & 14.9206 & 0 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} -14.7110 \\ 1.0000 \\ 1.6695 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{2.144}$$

The so-called Kimura–Davison conditions are not met for this example since $m + p = 3 < 4 = n$. In this case $p - m = 1$ and $m = 1$ and so the hyperplane matrix can be parameterized as

$$F = \begin{bmatrix} k & 1 \end{bmatrix}$$

where k is a scalar design parameter. The sliding motion is determined by a classical root locus of the "plant" $G_p = C_1(sI - A_{11})^{-1}A_{12}$ in series with a gain "k" in a unity feedback configuration . The root locus plot is given in Fig. 2.11.

For all values of $k > 2.58$, all the sliding mode poles lie in the LHP, and so, this constitutes an appropriate solution to the existence problem.

2.6.2 Static Output-Feedback Control Law Development

Having designed the surface, it is necessary to develop a controller to induce and sustain a sliding motion. Perhaps a natural choice would be a control structure of the form

$$u(t) = -Gy(t) - \rho(t, y)\frac{FCx(t)}{\|FCx(t)\|} \tag{2.145}$$

where the quantity $\rho(t, y)$ must upper bound the uncertainty. A common design methodology is based on synthesizing a static output-feedback gain G numerically to ensure the so-called reachability condition is satisfied.

To facilitate the analysis, an additional, switching function dependent coordinate transformation will be made. Let $z \mapsto T_K z = \hat{z}$ where

$$T_K := \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix} \tag{2.146}$$

with C_1 defined in (2.139). In this new coordinate system, the system triple $(\hat{A}, \hat{B}, F\hat{C})$ has the property that

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \qquad F\hat{C} = \begin{bmatrix} 0 & I_m \end{bmatrix} \tag{2.147}$$

where B_2 is defined in (2.133). The matrix $\hat{A}_{11} = A_{11} - A_{12}KC_1$ which is assumed to be stable by choice of K.
Furthermore

$$\hat{C} = \begin{bmatrix} 0_{p\times(n-p)} & \hat{T} \end{bmatrix} \tag{2.148}$$

where

$$\hat{T} := \begin{bmatrix} (T_1 - T_2K) & T_2 \end{bmatrix} \tag{2.149}$$

and T_1 and T_2 represent the first $p - m$ and last m columns of the matrix T from Eq. (2.134). Notice that $\hat{T}$ is nonsingular. Define a partition of $\hat{A}_{21}$ from Eq. (2.147) as

$$\hat{A}_{21} = \begin{bmatrix} \overset{\overset{n-p}{\leftrightarrow}}{\hat{A}_{211}} & \overset{\overset{p-m}{\leftrightarrow}}{\hat{A}_{212}} \end{bmatrix} \tag{2.150}$$

Ideally the degrees of freedom in selecting the controller should be determined numerically so that the reachability condition

$$\dot{\sigma}^T\sigma < 0 \tag{2.151}$$

is satisfied where $\sigma(x) = FCx(t)$ is the switching function. If Eq. (2.151) can be satisfied, then the sliding surface $\mathcal{S}$ is *globally attractive*.

Assuming an appropriate switching surface has been designed to solve the existence problem, the linear part of the control law can be chosen as

$$G = -\gamma F, \qquad \gamma > 0 \tag{2.152}$$

For a large enough scalar γ it can be shown that a sliding motion is obtained in finite time from any initial condition. However the reachability condition $\dot{\sigma}^T \sigma < 0$, where $\sigma(t) = Fy(t)$, only holds in a compact domain around the origin. Outside this domain the controller behaves as a variable structure controller with the property that it forces the state trajectories into the invariant domain (sometimes referred to as the "sliding patch") in finite time. Inside this domain the reachability condition $\dot{\sigma}^T \sigma < 0$ holds and so sliding occurs in finite time. *Provided the existence problem can be solved, no additional structural or system conditions need to be imposed.*

In practical situations the shortcoming of this controller is that γ must be large and hence the controller takes on a "high gain" characteristic.

Without loss of generality, write the linear feedback gain as

$$G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \hat{T}^{-1} \tag{2.153}$$

where $\hat{T}$ is from Eq. (2.149) and $G_1 \in \mathbb{R}^{m \times (p-m)}$ and $G_2 \in \mathbb{R}^{m \times m}$. Define a symmetric positive definite block diagonal matrix

$$P := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0 \tag{2.154}$$

where $P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $P_2 \in \mathbb{R}^{m \times m}$. Then it is possible to find a matrix P as in Eq. (2.154), and a gain matrix G so that

$$PA_c + A_c^T P < 0 \tag{2.155}$$

where $A_c = \hat{A} - \hat{B}G\hat{C}$; then the control law will induce a sliding motion on the surface $\mathcal{S}$ inside the domain (the sliding patch)

$$\Omega = \{(\hat{z}_1, \hat{z}_2) \ : \ \|\hat{z}_1\| < \eta \gamma_0^{-1}\}$$

where $\gamma_0 = \|P_2(\hat{A}_{21} - G_1 C_1)\|$ and $\hat{z}_1 \in \mathbb{R}^{(n-m)}$, $\hat{z}_2 \in \mathbb{R}^m$ represent a partition of the state $\hat{z}$.

From the point of view of control law design, a requirement is to make

$$\|P_2(\hat{A}_{21} - G_1 C_1)\| \tag{2.156}$$

small to make the sliding patch Ω large.

The block diagonal structure in Eq. (2.154), together with the canonical form in Eq. (2.147), effectively guarantees a solution to the structural constraint

$P\hat{B} = (F\hat{C})^T$, which in turn ensures the transfer function matrix $F\hat{C}(sI - \hat{A})^{-1}\hat{B}$ is strictly positive real.[11]

Example 2.10. Consider the example from Sect. 2.9. From the root locus in Fig. 2.11 it can be seen that high gain is needed to improve the damping of the dominant complex conjugate pair. With a value of $\gamma = 100$ the sliding motion poles are governed by

$$\{-1.9241 \pm 5.6081i, \; -167.0320\}$$

and

$$\hat{A}_{21} = \begin{bmatrix} 100.7287 & 1492.0600 & -16474.8680 \end{bmatrix}$$

It can be shown for

$$G = \begin{bmatrix} 1095.7134 & 208.2982 \end{bmatrix}$$

that

$$\text{eig}(A - BGC) = \{-162.0054, \; -12.4248, \; -0.3736 \pm 5.0794i\}$$

and an associated Lyapunov matrix can be found which also satisfies the structural constraint.

Remark 2.11. Although from a control theory point of view this example demonstrates the theory is valid and that a static output-feedback controller does exist, the resulting scheme may not be practical. Here the gain G is large and the sliding motion will be governed by two quite poorly damped dominant complex eigenvalues. This motivates the consideration of compensator-based output-feedback sliding mode controller design.

2.6.3 Dynamic Output-Feedback Hyperplane Design

So far, only the static output feedback case has been considered. In certain circumstances, the subsystem triple (A_{11}, A_{12}, C_1) is known to be infeasible with respect to static output-feedback stabilization. In such situations, consider the introduction of a dynamic compensator. Specifically, let

$$\dot{z}_c(t) = Hz_c(t) + Dy(t) \tag{2.157}$$

where the matrices $H \in \mathbb{R}^{q \times q}$ and $D \in \mathbb{R}^{q \times p}$ are to be determined. Define a new hyperplane in the augmented state space, formed from the plant and compensator state spaces, as

$$\mathcal{S}_c = \{(z, z_c) \in \mathbb{R}^{n+q} \; : \; F_c z_c + FCz = 0\} \tag{2.158}$$

[11] For details and definitions see [175]. This is also discussed in Appendix B.

where $F_c \in \mathbb{R}^{m\times q}$ and $F \in \mathbb{R}^{m\times p}$. Define $D_1 \in \mathbb{R}^{q\times(p-m)}$ and $D_2 \in \mathbb{R}^{q\times m}$ as

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} = DT \tag{2.159}$$

then the compensator can be written as

$$\dot{z}_c(t) = Hz_c(t) + D_1C_1z_1(t) + D_2z_2(t) \tag{2.160}$$

where C_1 is defined in Eq. (2.139). Assume that a control action exists which forces and maintains motion on the hyperplane $\mathcal{S}_c$ given in Eq. (2.158). As before, in order for a unique equivalent control to exist, the square matrix F_2 must be invertible. By writing $K = F_2^{-1}F_1$ and defining $K_c = F_2^{-1}F_c$ then the system matrix governing the reduced-order sliding motion, obtained by eliminating the coordinates z_2, can be written as

$$\dot{z}_1(t) = (A_{11} - A_{12}KC_1)z_1(t) - A_{12}K_cz_c(t) \tag{2.161}$$

$$\dot{z}_c(t) = (D_1 - D_2K)C_1z_1(t) + (H - D_2K_c)z_c(t) \tag{2.162}$$

It follows that stability of the sliding motion depends only on the matrix

$$\begin{bmatrix} A_{11} - A_{12}KC_1 & -A_{12}K_c \\ (D_1 - D_2K)C_1 & H - D_2K_c \end{bmatrix} \tag{2.163}$$

As in the uncompensated case, it is necessary for the pair (A_{11}, C_1) to be detectable. To simplify the design problem, at the expense of removing some of the design flexibility, one can specifically choose $D_2 = 0$ in Eq. (2.163).

The resulting matrix in Eq. (2.163) can be viewed as the negative feedback interconnection of the "plant" $G_p(s) = C_1(sI - A_{11})^{-1}A_{12}$ and the "compensator"

$$K(s) = K + K_c(sI - H)^{-1}D_1 \tag{2.164}$$

Note that this still has a very generalized structure and any linear design paradigm that creates an internally stabilizing closed loop can be employed to synthesize the sliding mode compensator matrices D_1 and H, and a hyperplane, represented by the matrices K and K_c,

In certain situations it is advantageous to consider the feedback configuration in Fig. 2.12 and to design a compensator $K(s)$ using any suitable paradigm to yield appropriate closed-loop performance. From the state-space realization of $K(s)$ in Eq. (2.164), the parameters K, K_c, D_1, and H can be identified. If the quantity $p - m$ is small, using "classical control" ideas, very simple compensators may be found which give good closed-loop performance to the fictitious feedback system in Fig. 2.12 which governs the sliding mode performance of the real system.

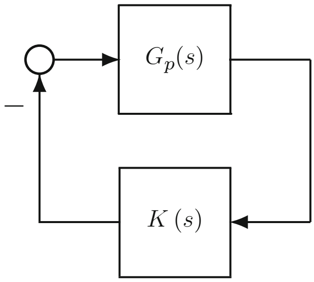

Fig. 2.12 A general linear feedback configuration

2.6.4 Dynamic Output-Feedback Control Law Development

Example 2.11. The third-order "plant" from Eq. (2.144) has a double pole at the origin. In terms of classical control, this suggests the use of a lead compensator. Choosing

$$K(s) = \frac{(s + 0.5)}{(s + 10)} \tag{2.165}$$

in unity feedback with $G_p(s)$ gives closed-loop poles at

$$\{-12.1413, \ -0.9656, \ -1.2490 \pm 0.9718\}$$

This has improved the damping ratio of the dominant complex pair. A realization of the compensator in Eq. (2.165) is $H = -10$, $D_1 = 1$, $K_c = -9.5$, and $K = 1$. Using these values (and $D_2 = 0$) in the formulae in Eqs. (2.138) and (2.159) gives

$$F_a := \left[\begin{array}{cc} F_c & F \end{array}\right] = \left[\begin{array}{c|cc} -11.7059 & 1.2322 & 1.2322 \end{array}\right] \tag{2.166}$$

and

$$D = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$$

Using these matrices, an appropriate controller to induce a sliding motion may be obtained by using the method described earlier for the augmented system

$$A_a = \left[\begin{array}{cc} H & DC \\ 0 & A \end{array}\right] \qquad B_a = \left[\begin{array}{c} 0 \\ B \end{array}\right] \qquad C_a = \left[\begin{array}{cc} 1 & 0 \\ 0 & C \end{array}\right] \tag{2.167}$$

where the switching function matrix F_a is defined in Eq. (2.166). Choosing

$$G = \left[\begin{array}{ccc} -45.9050 & 4.7749 & 0.3392 \end{array}\right]$$

gives

$$\text{eig}(A_a - B_a G C_a) = \{-8.6605, \ -3.0494 \pm 1.9424i, \ -0.8238 \pm 0.3124\}$$

2.6.5 Case Study: Vehicle Stability in a Split-Mu Maneuver

This section will show how these output-feedback sliding mode ideas can be applied to a realistic control problem. One of the areas of active research in the automotive industry is vehicle chassis control. The principal aims are to increase vehicle safety, maneuverability, and passenger comfort while reducing the work load on the driver. Most modern vehicles have antilock brake systems (ABS) which prevent the wheels locking under heavy braking. However, if a so-called split-mu surface is encountered, in which the traction between the road surface and the tire on one wheel becomes significantly reduced compared to the others, for instance, if a patch of ice is encountered, then any braking maneuver will introduce a sudden unexpected yawing moment. At high speed the driver will not have sufficient time to react, and a potentially extremely dangerous spin may occur.

This case study will consider a steer-by-wire system which will aim to maintain heading and vehicle stability in such a situation. An eighth-order nonlinear model of the vehicle, wheels, and road/tire interaction has been developed.[12] This model has been tuned to be representative of a typical family saloon. A linearization has been performed about an operating condition which represents a straight line trajectory at $15\,\mathrm{ms}^{-1}$ longitudinal velocity, corresponding wheel velocities, and zero lateral velocity, yaw rate, and yaw angle. The linearization assumes a uniform friction coefficient on each wheel. Considering yaw rate and lateral deviation as measured variables, a reduced-order linear model of the rigid body states is given in Eqs. (2.142) and (2.143).

The scenario that will be considered is an emergency stop on a split-mu surface. A simple ABS system has been incorporated into the nonlinear vehicle model. This is designed to bring the vehicle to a standstill in as short a time as possible. The steer-by-wire system will alter the front wheel position (the control input) in an effort to brake in a straight line. The key requirement is, therefore, to keep the third state, Y, in Eq. (2.143) as near to zero as possible.

From the canonical form in Eq. (2.144) the output of the fictitious system $G_p(s) = C_1(sI - A_{11})^{-1}A_{12}$ (as far as the switching function design is concerned) is, in the real system, the output of interest, Y. Because $G_p(s)$ has a double integrator characteristic no integral action is needed to achieve a steady-state error of zero.

The closed-loop simulation obtained from a fully nonlinear model is shown in Fig. 2.13. The controller manages to stop the vehicle from developing an excessive yaw angle. Also, the lateral deviation, Y, is halted with a peak of 17 cm and is regulated to zero. The controller develops and maintains sufficient yaw angle as is necessary to counteract the yawing moment induced by the asymmetric ABS braking. The input signal which the controller utilizes to perform this is shown in Fig. 2.14. It is also demonstrated that a sliding motion is maintained throughout the maneuver.

[12] For details of the model see [72, 108].

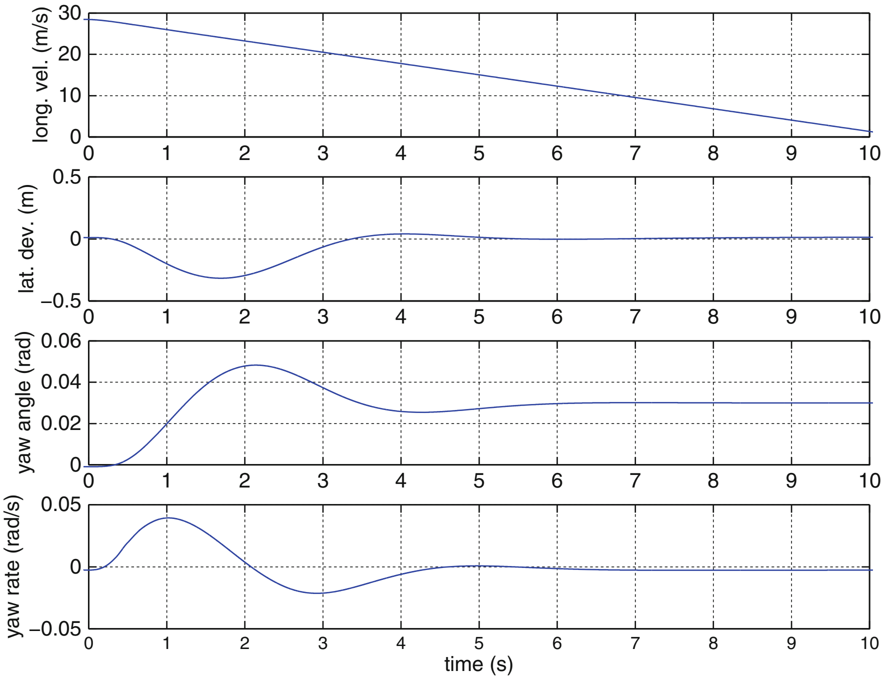

Fig. 2.13 Vehicle states and braking torques for closed-loop simulation

2.7 Integral Sliding Mode Control

As discussed earlier in this chapter, sliding mode control techniques are very useful for the controller design in systems with matched disturbances/parametric uncertainties. The system's compensated dynamics become insensitive to these matched disturbances and uncertainties under sliding mode control. This property of insensitivity is only achieved when the sliding surface is reached and the sliding motion is established. In this section we explore a method to compensate for the disturbance without the presence of a reaching phase.

For known linear systems, traditional controllers, including proportional-plus-derivative (PD), proportional-plus-integral-plus-derivative (PID), and optimal linear-quadratic regulator (LQR), can be successfully designed to achieve ideal closed-loop dynamics. Also known, nonlinear systems (with certain structures) can be controlled using, for instance, feedback linearization, backstepping, or any other Lyapunov-based nonlinear technique. Once a system is subjected to external bounded disturbances, it is natural to try to compensate such effects by means of an auxiliary (sliding mode) control while the original controller compensates for the unperturbed system.

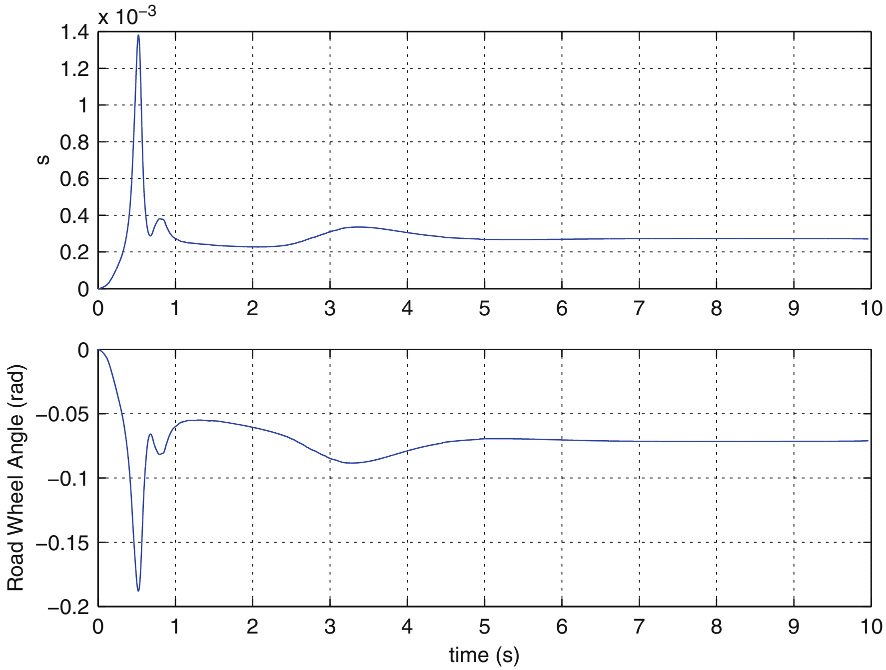

Fig. 2.14 Switching function and control signal

In this section we will design a so-called integral sliding mode (ISM) auxiliary controller compensating for the disturbance from $t \geq 0$, while retaining the order of the uncompensated system. This can be achieved assuming that the initial conditions are known.

2.7.1 Problem Formulation

Let us consider the following controlled uncertain system represented by the state-space equation

$$\dot{x} = f(x) + B(x)u + \phi(x,t), \tag{2.168}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, and $x(0) = x_0$. The function $\phi(x,t)$ represents the uncertainties affecting the systems, due to parameter variations, unmodeled dynamics, and/or exogenous disturbances.

Let $u = u_0$ be a nominal control designed for Eq. (2.168) assuming $\phi = 0$, where u is designed to achieve a desired task, whether stabilization, tracking, or an optimal control problem. Thus, the trajectories of the ideal system ($\phi = 0$) are given by the solutions of

$$\dot{x}_0 = f(x_0) + B(x_0)u_0. \tag{2.169}$$

When $\phi \neq 0$ the trajectories of Eqs. (2.168) and (2.169) are different. The trajectories of Eq. (2.169) satisfy some specified requirements, whereas the trajectories of Eq. (2.168) might have a quite different and possibly even undesirable performance (depending on ϕ).

In order to design the controller assume that:

(A1) Rank $(B(x)) = m$ for all $x \in \mathbb{R}^n$.
(A2) The disturbance $\phi(x,t)$ is matched and there exists a vector $\xi(x,t) \in \mathbb{R}^m$ such that $\phi(x,t) = B(x)\,\xi(x,t)$.
(A3) A known upper bound for $\xi(x,t)$ can be found, i.e.,

$$\|\xi(x,t)\| \leq \xi^{+}(x,t). \tag{2.170}$$

Obviously, the second restriction is needed to exactly compensate ϕ. If ϕ were known exactly, it would be enough to chose $u = -\xi$. However, since ξ is uncertain, some other restrictions are needed in order to eliminate the influence of ϕ. Here the sliding mode approach is used to replace exact knowledge of ϕ.

2.7.2 *Control Design Objective*

Now the problem is *to design a control law* such that $x(0) = x_0(0)$, and *guarantees the identity* $x(t) = x_0(t)$ for all $t \geq 0$. Comparing Eqs. (2.168) and (2.169), it can be seen that the objective is achieved only if the equivalent control is equal to minus the uncertainty (i.e., $u_{1_{eq}} = -\xi$). Thus, the control objective can be reformulated in the following terms: *design the control* $u = u(t)$ as

$$u(t) = u_0(t) + u_1(t), \tag{2.171}$$

where $u_0(t)$ is the nominal control part designed for Eq. (2.169) and $u_1(t)$ is the ISM control part compensating for the unmeasured matched uncertainty $\phi(x,t)$, starting from $(t = 0)$.

2.7.3 *Linear Case*

Let us consider the linear time-invariant case:

$$\dot{x}(t) = Ax(t) + B\left(u_0(t) + u_1(t)\right) + \phi(t,x), \qquad \phi(t,x) = B\xi(t,x) \tag{2.172}$$

In this case the vector function σ can be defined as

$$\sigma(x) = G\left(x(t) - x(0)\right) - G\int_0^t \left(Ax(\tau) + Bu_0(\tau)\right) d\tau, \tag{2.173}$$

where $G \in \mathbb{R}^{m \times n}$ is a projection matrix satisfying the condition

$$\det(GB) \neq 0$$

The time derivative of σ has the form

$$\dot{\sigma}(x) = GB(u_1 + \xi)$$

The control u_1 is taken as the unit vector

$$u_1 = -\rho(t,x)\frac{(GB)^T\sigma}{\|(GB)^T\sigma\|} \tag{2.174}$$

where $\rho(t,x) \geq \|\xi^+(t,x)\|$. Taking $V = \frac{1}{2}\sigma^T\sigma$ and in view of Eq. (2.170) the derivative of V on time is

$$\begin{aligned}\dot{V} &= \sigma^T GB(u_1 + \gamma)\\ &\leq -\|(GB)^T\sigma\|(M - \xi^+) < 0\end{aligned}$$

Hence, the ISM is guaranteed.

Example 2.12. Let us consider the following system:

$$\dot{x} = Ax + B\,(u_0 + u_1) + \phi \tag{2.175}$$

representing a linearized model of an inverted pendulum on a cart, where x_1 and x_2 are the cart and pendulum positions, respectively, and x_3 and x_4 are the respective velocities. The matrices A and B are taken with the following values:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1.25 & 0 & 0\\ 0 & 7.55 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0\\ 0\\ 0.19\\ 0.14 \end{bmatrix},$$

and the control $u_0 = u_0^*$ is designed for the nominal system, where u_0^* solves the following optimal problem subject to an LQ performance index:

$$J\,(u_0) = \int_0^\infty x_0^T\,(t)\,Qx_0\,(t) + u_0^T\,(t)\,Ru_0\,(t)\,dt$$

where $u_0^* = \arg\min J\,(u_0)$. It is known (see Appendix C) that the solution of this problem is the following:

$$u_0^*\,(x) = -R^{-1}B^T Px = -Kx,$$

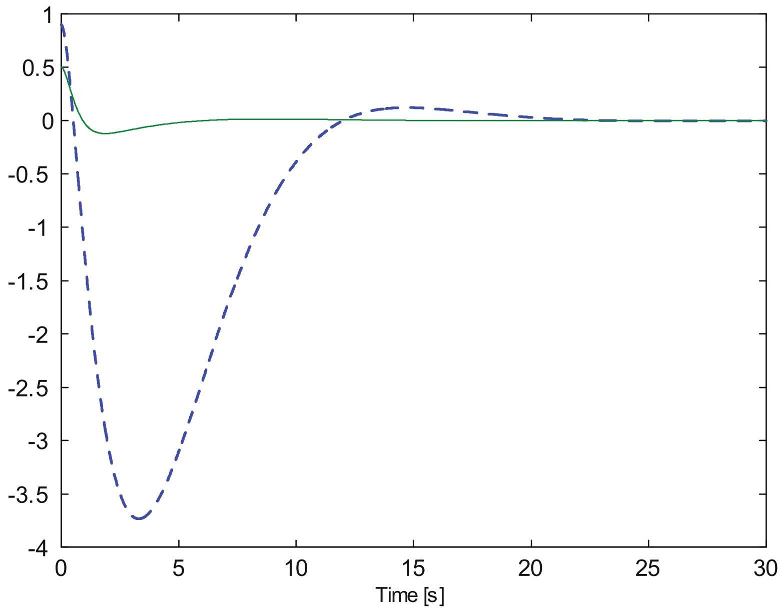

Fig. 2.15 States x_1 (*dashed*) and x_2 (*solid*) using ISM in the presence of matched uncertainties

where P is a symmetric positive definite matrix that is the solution of the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P = -Q.$$

For the considered matrices A and B and taking $Q = I$ and $R = 1$, we have

$$K = \begin{bmatrix} -1 & 131.36 & -4.337 & 48.47 \end{bmatrix}.$$

In the simulation $\phi = B\xi$ with $\xi = 2\sin(0.5t) + 0.1\cos(10t)$ and the ISM control is

$$u_1 = -5\,\mathrm{sign}(\sigma),$$

where σ is designed according to Eq. (2.173). Now, the only restriction on the matrix G is $\det GB \neq 0$. One simple choice is $G = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$; thus we obtain $GB = 0.19$, which obviously is different from zero.

Figure 2.15 shows the position of the cart and the pendulum. We can see that there is no influence of the disturbance ξ due to the compensation effect caused by the ISM control part u_1.

2.7.4 ISM Compensation of Unmatched Disturbances

To solve the problems of the reaching phase and of robustness against unmatched uncertainties/disturbances simultaneously, the main idea, as in the conventional sliding mode case, has been the combination of ISM and other robust techniques. However, in practice we also need to ensure that the compensation (a) does not amplify unmatched uncertainties/disturbances and ideally (b) minimizes the effect of the unmatched uncertainties/disturbances.

Consider the linear system (2.172), together with assumption (A1) and the assumption that there exists an upper bound for the perturbation $\phi(x,t)$. It is not assumed that this perturbation is matched. For that reason, before we obtain the sliding motion equations and try to understand how the uncertainty affects it, it is convenient to project the perturbation $\phi(x,t)$ into matched and unmatched spaces.

Let $B^{\perp} \in \mathbb{R}^{n\times(n-m)}$ be a full rank matrix whose image is orthogonal to the image of B, i.e., $B^T B^{\perp} = 0$. As a consequence the matrix $\begin{bmatrix} B & B^{\perp} \end{bmatrix}$ is nonsingular. Furthermore $\operatorname{rank}(I-BB^+) = n-m$, where $B^+ = \left(B^T B\right)^{-1} B^T$, and $(I - BB^+)B = 0$. Therefore, the columns of $B^{\perp}$ can be formed from the linearly independent columns of $(I - BB^+)^T$. Thus, let $\xi\,(x,t) \in \mathbb{R}^m$ and $\mu\,(x,t) \in \mathbb{R}^{n-m}$ be the vectors defined by

$$\begin{bmatrix} \xi\,(x,t) \\ \mu\,(x,t) \end{bmatrix} = \begin{bmatrix} B & B^{\perp} \end{bmatrix}^{-1} \phi\,(x,t) \tag{2.176}$$

Thus, Eq. (2.172) takes the following form:

$$\dot{x} = Ax + B\,(u_1 + u_0) + B\xi + B^{\perp}\mu \tag{2.177}$$

Then selecting σ as in Eq. (2.173), we have

$$\dot{\sigma} = GB\,(u_1 + \xi) + GB^{\perp}\mu$$

The control component u_1 should be designed as in Eq. (2.174) under the assumption GB is nonsingular. Let the modulation function $\rho \geq \xi^+ + (GB)^+ GB^{\perp}\mu$. The equivalent control obtained from solving $\dot{\sigma} = 0$ is given by the equation

$$u_{1_{\text{eq}}} = -\xi - (GB)^{-1}\, GB^{\perp}\mu$$

Substitution of $u_{1_{\text{eq}}}$ in Eq. (2.177) yields the sliding motion equation:

$$\dot{x} = Ax + Bu_0 + (I - B\,(GB)^{-1}\, G)B^{\perp}\mu$$

Define $\bar{d} := (I - B\,(GB)^{-1}\,G)B^{\perp}\mu$. Taking $G = B^T$ or $G = B^+$, we get $\bar{d} = B^{\perp}\mu$, i.e., the application of the sliding mode controller has not affected the unmatched disturbance part.

Now the question is, is it possible to select G to ensure the norm of $\bar{d}$ is less than the norm of $B^{\perp}\mu$? This is addressed in the following proposition.

Proposition 2.1 *The set of matrices* $\left\{G = QB^T : Q \in \mathbb{R}^{m\times m} \text{ and } \det(Q) \neq 0\right\}$ *is the solution of the optimization problem*

$$G^* = \arg\min_{G\in\bar{G}} \|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu, \mu \neq 0\|$$

where $\bar{G} = \{G \in \mathbb{R}^{m\times n} : \det(GB) \neq 0\}$.

Proof. Since $B^{\perp}\mu$ and $B\,(GB)^{-1}\,GB^{\perp}\mu$ are orthogonal vectors, the norm of the vector $\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\|$ is always greater than $\|B^{\perp}\mu\|$. Indeed,

$$\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\|^2 = \|B^{\perp}\mu\|^2 + \|B\,(GB)^{-1}\,GB^{\perp}\mu\|^2$$

That is,

$$\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\| \geq \|B^{\perp}\mu\| \tag{2.178}$$

If identity (2.178) is achieved, then the norm of $\|(I - B\,(GB)^{-1}\,G)B^{\perp}\mu\|$ is minimized with respect to G. The identity is obtained, if and only if $B\,(GB)^{-1}\,GB^{\perp}\mu{=}0$. Or equivalently, since $\text{rank}(B) = m$, $GB^{\perp}\mu = 0$, i.e., $G = QB^T$, where Q is nonsingular. This completes the proof. □

Remark 2.12. Notice that the control law itself is not modified in order to optimize the effect of the unmatched uncertainties, and moreover, an optimal solution for G^* is simple: the simplest choice is $G^* = B^T$, but $B^+ = \left(B^T B\right)^{-1} B^T$ is another possibility.

Proposition 2.2 *For an optimal matrix G^*, the Euclidean norm of the disturbance is not amplified, that is,*

$$\|\phi\,(t)\,\| \geq \|(I - B\left(G^* B\right)^{-1} G^*)B^{\perp}\mu\,(t)\,\| \tag{2.179}$$

Proof. From Proposition 2.1 it follows that

$$\|(I - B\left(G^* B\right)^{-1} G^*)B^{\perp}\mu\,(t)\,\| = \|(I - BB^+)B^{\perp}\mu\,(t)\,\| = \|B^{\perp}\mu\,(t)\,\| \tag{2.180}$$

Now, since $\phi\,(t) = B\xi + B^{\perp}\mu$, and $B^T B^{\perp} = 0$, we obtain the equation

$$\|\phi\,(t)\,\|^2 = \|B\xi\,(t) + B^{\perp}\mu\,(t)\,\|^2 = \|B\xi\,(t)\,\|^2 + \|B\mu\,(t)\,\|^2 \geq \|B\mu\,(t)\,\|^2 \tag{2.181}$$

Hence, comparing Eqs. (2.180) and (2.181), we can obtain Eq. (2.2). □

Example 2.13. Consider system Eq. (2.175) with the uncertainty $\phi(x,t)$ shown below:

$$\phi(x,t) = \begin{bmatrix} 0 & 0 & 2\sin(0.5t) + 0.1\cos(10t) & 0.1\sin(1.4t) \end{bmatrix}^T$$

The first step is to project the perturbation $\phi(x,t)$ into the matched and unmatched spaces using the expression in Eq. (2.176). Note the selection of the matrix $B^{\perp}$ is not unique: one simple choice is given by

$$B^{\perp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.14 \\ 0 & 0 & 0.19 \end{bmatrix}$$

In this way ξ and μ in system (2.177) become

$$\xi = 3.41(2\sin(0.5t) + 0.1\cos(10t)) + 2.51(0.1\sin(1.4t)),$$

$$\mu = \begin{bmatrix} 0 \\ 0 \\ 3.41\,[0.1\sin(1.4t)] - 2.51\,[2\sin(0.5t) + 0.1\cos(10t)] \end{bmatrix}$$

The control law is the same as in Example 2.12, except for the choice of matrix G, which according to Proposition 2.1 is optimal if $G = B^T = \begin{bmatrix} 0 & 0 & 0.19 & 0.14 \end{bmatrix}$ or $G = B^{+}$. Here we consider three cases: the case when we use $G = B^T$, the case when the ISM control is not applied, and the case when G is not selected in an optimal manner. For this last case, we use $G = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ as in Example 2.12. For the first case, the states x_1 and x_2 (the positions) are depicted in Fig. 2.16; there we can see that the uncertainties do not significantly affect the trajectories of the system. To compare the effect of the ISM in presence of unmatched disturbances, Fig. 2.17 shows the trajectories of x_1 and x_2 when the ISM control part is omitted ($u = u_0$). It is clear that in this case, the disturbances affect the system considerably compared with the trajectories of Fig. 2.17 where a well-designed ISM control (with an optimal G) reduces significantly the effect of the disturbances. Figure 2.18 shows the effect of the matrix G in the design of the sliding surface. In Fig. 2.18 we compare the behavior of the variable x_2 when G is badly chosen.

2.8 Notes and References

A very readable concise treatment of some of the material in this chapter appears in *The Control Handbook* [58].

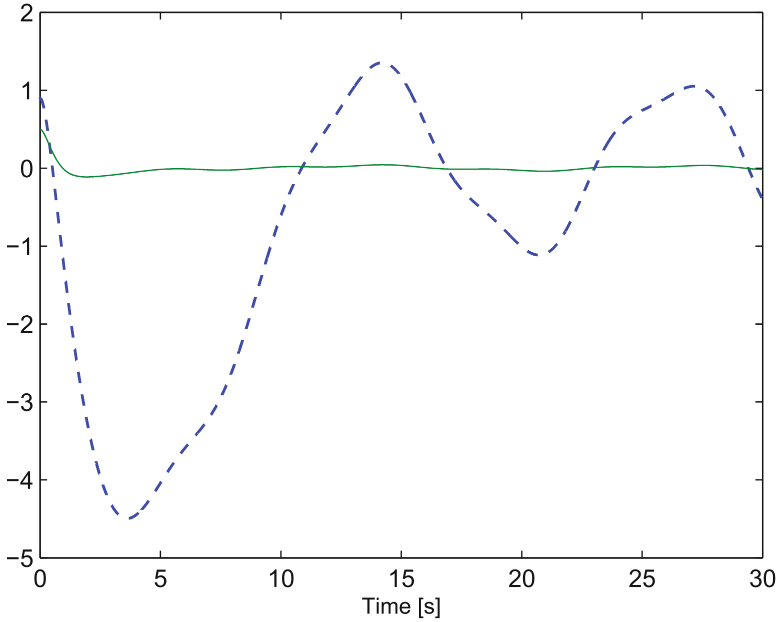

Fig. 2.16 States x_1 (*dashed*) and x_2(*solid*) without ISM compensator

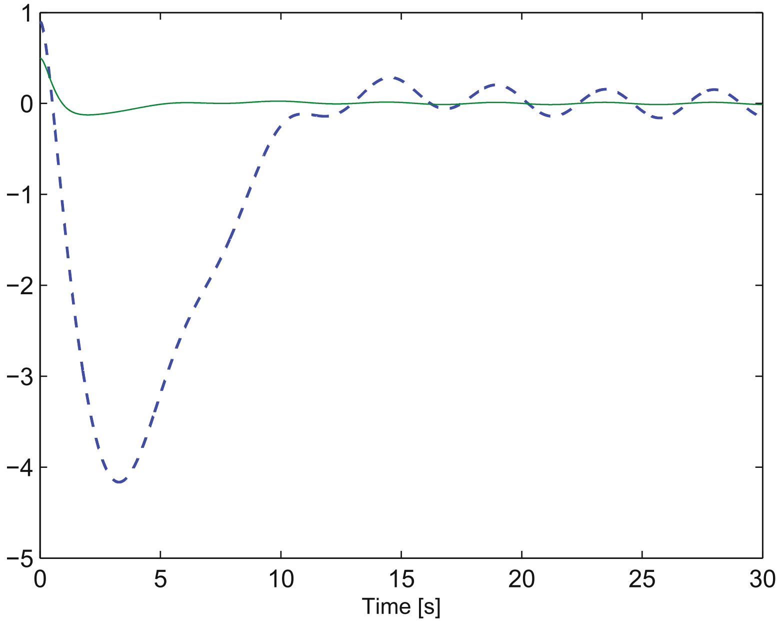

Fig. 2.17 States x_1 (*dashed*) and x_2 (*solid*) for $G = B^T$

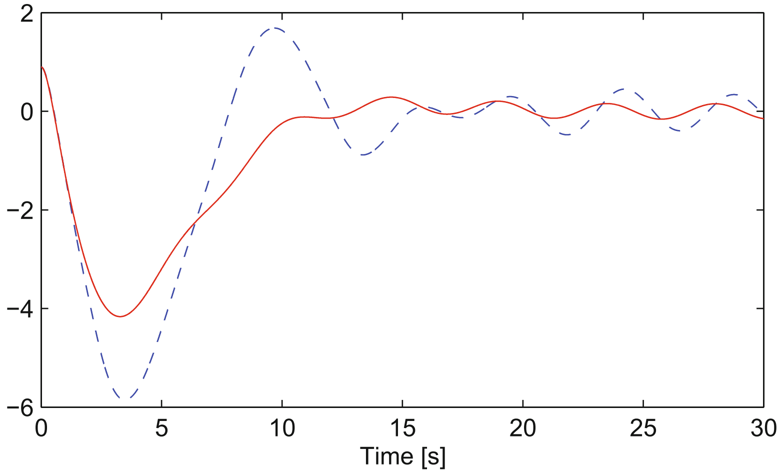

Fig. 2.18 Trajectories of the position x_2 for $G = B^T$ (*solid*) and $G = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ (*dashed*)

Regular form-based methods were first introduced in [136, 137]. Linear hyperplane design methods for a class of single-input systems are given in [2, 171]. Another approach to the hyperplane design, which can be used for treating the unmatched uncertainties, is based on the linear methods of [149]. For details of different design methods for the hyperplane matrix M see Chap. 4 in [63, 67].

The control structure considered in Sect. 2.3.2 is essentially that of Ryan and Corless from [157] and is described as the *unit-vector* approach. The unit-vector structure appears first in [103]. The precise relationship between the control law in Eqs. (2.90) and (2.91) and the original description of Ryan and Corless is described in Sect. 3.6.3 in [67].

A variety of approximations of the discontinuous control functions are described and analyzed in the literature [40]. Power law approximations are given in Ryan and Corless [157], and state based approximations are given in Tomizuka [49]. DeJager compares different approximation methods in [60]. A specific treatment on chattering reduction is also included in [171].

The so-called equivalent control method is attributed to Utkin [182]. The solution concept proposed by Filippov [81] for differential equations with discontinuous right-hand sides constructs a solution as the "average" of the solutions obtained from approaching the point of discontinuity from different directions. The definition for the solution to the differential inclusion given in Definition 2.2 is from [81].

Details of the proof of Lemma 2.1 and the canonical form that can be achieved are discussed in [67]. A common design methodology for output-feedback sliding mode controller design [13, 73, 109, 110] is based on synthesizing a static output-feedback gain G numerically to ensure the so-called reachability condition is satisfied. In [66], assuming an appropriate switching surface has been designed to solve the existence problem, the linear part of the control law was chosen as a scaling of the switching function, thought of as a feedback gain. The static output-feedback control law development in Sect. 2.6.2 is based on [71]. Example 2.9 is taken from the sliding mode output-feedback paper [72]. In the case study relating to vehicle control, the aims of increasing vehicle safety, maneuverability, and passenger comfort, while reducing the work load on the driver are discussed in detail in [1]. The limitations of some static output-feedback sliding mode controllers are discussed in [68].

The ISM concept was proposed independently by Matthews and DeCarlo [140] and Utkin and Shi [185]. In this chapter we followed the approach of Matthews and DeCarlo [140]. The ISM approach described in Sect. 2.7 can be easily extended to the class of affine nonlinear systems given by

$$\dot{x} = f(x) + B(x)(u_0 + u_1 + \xi)$$

For a detailed analysis of this case see [43]. Additional material for the advance study of the ISM approach can be found in many publications including [156, 192]. In particular, various combinations of ISM with $\mathcal{H}_\infty$ are studied in [43–45, 191]. The use of ISM schemes for "robustification" of solutions of LQR problems can be found in [93, 154] (a multi-model optimization problem); see also the design of robust output LQR control in [21, 26] and multiplant LQR control in [27]. The ISM controllers are widely used in robotic applications [50, 151] when it is necessary to track the reference trajectories. The corresponding references can be found in the books [186, 187] and the paper [59] and references therein.

An interesting application avenue exploiting the robustness properties of sliding modes with respect to matched uncertainties is the area of fault tolerant control. In such scenarios, actuator faults appear naturally within the control channels of the plant and can be accommodated "automatically" by sliding mode controllers [6, 7, 105, 162].

Although not discussed in this book, the main ideas and techniques of discrete-time sliding mode control can be found [3, 14, 97, 141, 187].

2.9 Exercises

Exercise 2.1. Consider the linear system

$$\dot{x} = Ax + Bu$$
$$\sigma = Gx$$

with σ as the sliding variable. Find the equivalent control u_{eq} and the sliding mode equations when

(a) $A = \begin{bmatrix} 2 & 19 \\ 3 & 29 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ and $G = \begin{bmatrix} 9 & 12 \end{bmatrix}$

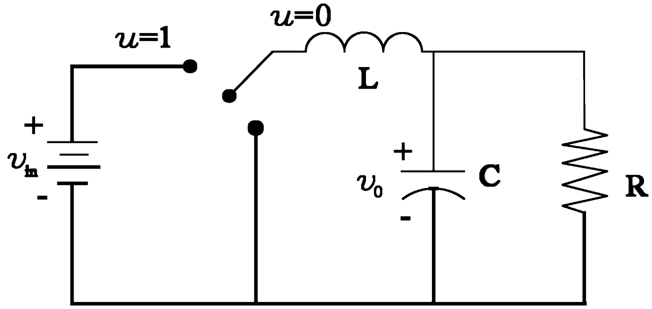

Fig. 2.19 DC–DC buck converter

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 9 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 1 & 29 & 0 \\ 1 & 12 & 0 \end{bmatrix}$

Exercise 2.2.
Consider the system given by

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_3 + u_1 + 10u_2 \\ \dot{x}_2 &= x_2 + 3x_3 + u_1 - 2u_2 \\ \dot{x}_3 &= x_1 + x_3 - u_1 \\ \sigma_1 &= x_1 + 10x_2,\ \sigma_2 = x_1 + 5x_2 \end{aligned}$$

Find the system dynamics in the sliding mode $\sigma_1 = \sigma_2 = 0$.

Exercise 2.3. Consider the DC–DC buck converter in Fig. 2.19 which belongs to the class of attenuation circuits; the corresponding dynamic equations are given by:

$$\begin{aligned} L\frac{di}{dt} &= -v + uV_{in} \\ C\frac{dv}{dt} &= i - \frac{v}{R} \end{aligned}$$

where i is the current through the inductor L, v is the voltage across the capacitor C, V_{in} is the input voltage, and $u \in \{0, 1\}$ is the switching control signal. The goal is to stabilize the output voltage v at the desired level v_d. This goal is to be achieved via stabilization of the inductor current i at the desired level $i_d = \frac{v_d}{R}$ using sliding mode control, for this purpose:

(a) Setting $\sigma = i - i_d$ and using the control input $u = \frac{1}{2}(1 - \text{sign}(\sigma))$, find the equivalent control and the sliding mode dynamics.
(b) Considering $L = 20\,\text{mH}$, $C = 20\,\mu\text{F}$, $R = 30\,\Omega$, $V_{in} = 15\,\text{V}$, $v_d = 10\,\text{V}$, and the initial conditions $i(0) = 0.1\,\text{A}$ and $v(0) = 5\,\text{V}$, confirm the efficacy of the controller by simulations.

Exercise 2.4. Consider the linear system

$$\dot{x} = Ax + B(u + f)$$

with:

$$A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

that is stabilized by the sliding mode controller $u = -(6||x|| + 3)\text{sign}(\sigma)$ in the presence of the bounded disturbance $|f| \leq 2$. Design the sliding variable σ assuming $f = 0$, considering the two cases:

(a) LQR minimization, with $Q = I$ and $R = 1$
(b) Eigenvalue assignment, considering the eigenvalue $\lambda = -2$

Confirm your design by simulations, using $f = 2\sin(t)$ and the initial conditions $x_1(0) = 2$ and $x_2(0) = 1$.

Exercise 2.5. Using the sliding variable $\sigma = x_1$ and the control $u = -\text{sign}(\sigma)$, find the sliding mode equation for the systems given below, by using the equivalent control method and Filippov method for the cases:

(a)

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= (2u^2 - 1)x_2 \end{aligned}$$

(b)

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= (u - 2u^3)x_2 \end{aligned}$$

(c)

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= (u - 2u^2)x_2 \end{aligned}$$

Exercise 2.6. Consider the linear system given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \frac{1}{3} & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 8/3 & 1 \\ 4 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Design an output-feedback sliding mode control to stabilize the system at the origin using $u = -Ksign(\sigma)$; use the eigenvalue assignment algorithm in order to design the sliding variable σ, and consider the eigenvalues, $\lambda_{1,2} = -2 \pm j5$. Considering

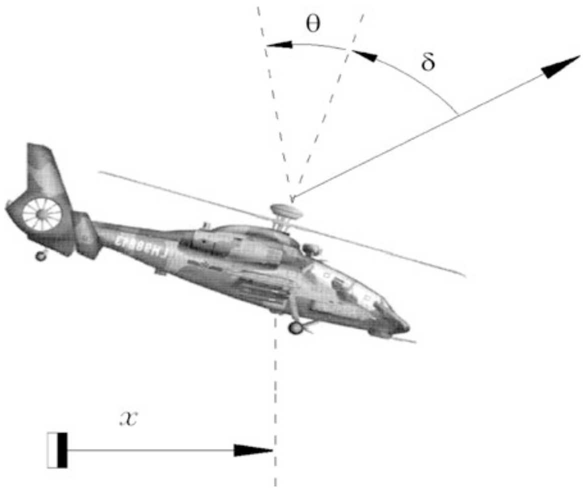

Fig. 2.20 Helicopter pitch angle, θ, control

the initial condition $x(0) = [1, 2, 3]^T$, obtain the appropriate value of K to ensure a reaching time $t_r = 2.5$ s and confirm your design by simulation.

Exercise 2.7. Consider the Chua's circuit of Example 2.7 with the output $y{=}x_1$. Propose a control law based on sliding modes in order to achieved the output tracking of $y_r = 5\sin(2t)$. Confirm your design by simulation.

Exercise 2.8. The equations of motion of the high-performance helicopter in Fig. 2.20 are given by:

$$\begin{aligned}\ddot{\theta} &= -a_1\dot{\theta} - b_1\dot{x} + n\delta \\ \ddot{x} &= g\theta - a_2\dot{\theta} - b_2\dot{x} + g\delta\end{aligned}$$

where x is the translation in the horizontal direction. Design a sliding mode controller in terms of the rotor thrust angle δ that forces the pitch angle θ to asymptotically follow the reference profile $\theta_d = \frac{\pi}{6} + \frac{\pi}{12}\sin(t)$. Considering the initial conditions $\theta(0) = 0.2$ rad, $\dot{\theta} = 0$ rad/s, $x(0) = 0$ m, and $\dot{x}(0) = 50$ m/s, confirm your design by simulations of system with the parameters $a_1 = 0.415$, $a_2 = 0.0198$, $b_1 = 0.0111$, $b_2 = 1.43$, $n = 6.27$, and $g = 9.81$.

Exercise 2.9. Consider the satellite in Example 2.6 and design an ISM control $u = u_0 + u_1$. Preserve the linear control $u_0 = u_l$ as in Example 2.6 in order to stabilize the equilibrium point $(0, 0, 0)^T$ for the ideal system (without nonlinear disturbance terms). The control law u_0 should be designed using the LQR minimization algorithm. Design the ISM control component u_1 as in Eq. (2.174) to compensate the matched disturbances. Consider Q and R as identity matrices

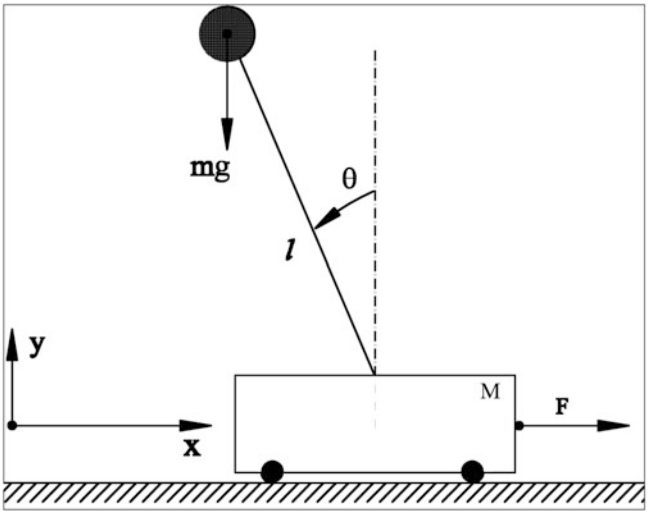

Fig. 2.21 Cart–pendulum

of appropriate dimensions. Simulate the system where for simulation purposes consider the initial condition $x(0) = [0.5, -1, 2]^T$ and the parameters $I_1 = 1\,\text{kgm}^2$, $I_2 = 0.8\,\text{kgm}^2$, and $I_3 = 0.4\,\text{kgm}^2$.

Exercise 2.10. Consider the following linear system subject to external disturbances:

$$\dot{x} = Ax + Bu + \phi$$

where $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\phi = \begin{bmatrix} 3\sin(t) \\ 4\cos(t) \\ 4\cos(t) \end{bmatrix}$

(a) Identify if the disturbance vector ϕ is matched. If it is not, then project the disturbance into matched and unmatched terms.

(b) Design the control $u = u_0 + u_1$. The linear control u_0 is to be designed for nominal system (without disturbances) in order to stabilize the equilibrium point $(0, 0, 0)$ using the LQR algorithm. Consider Q and R identity matrices with appropriate dimensions. Design an ISM control component u_1 as in Eq. (2.174) to compensate the matched disturbances, and select the matrix $G = B^+ = (B^T B)^{-1} B^T$ that helps to accommodate the unmatched disturbance term. Confirm the effectiveness of the controller design via simulations. For simulation purposes consider the initial conditions $x(0) = (1, -2, 3)^T$.

Exercise 2.11. Consider the cart–pendulum system in Fig. 2.21 that consists of a cart of mass M that moves along the axis x, with a ball of mass m at the end of a rigid massless pendulum of length l. Shown as inputs are a horizontal force $F = u$

acting on the cart and a force d acting on the ball perpendicular to the pendulum. The output signals are the angles θ and the position of the ball $y = x + l\sin(\theta)$. The two linearized equations about the equilibrium point $(x, \theta) = (0, \pi)$ are

$$\begin{array}{rcl} (M+m)\ddot{x} + ml\ddot{\theta} & = & u + d \\ \ddot{x} + l\ddot{\theta} - g\theta & = & \frac{1}{m}d \end{array}$$

Design an ISM control based on a LQR algorithm (see the explanation in Exercise 2.10) in order to stabilize the unstable equilibrium point. For simulation purposes use $d = 2\sin(t)$ and initial condition $(x(0), \theta(0)) = (-0.7, \frac{\pi}{3})$.