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Precedence-Type Tests for the Comparison of Treatments with a Control

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Abstract: Precedence-type tests are proposed for comparing several treatments with a control. The null distributions of these test statistics are derived, and critical values for some combination of sample sizes are then presented. Next, the exact power function of these tests under the Lehmann alternative is derived and used to compare the power properties of the proposed test procedures. Finally, an example is presented to illustrate all the test procedures discussed here.

Keywords and phrases: Precedence test, Wilcoxon rank-sum test, life-testing, level of significance, power, Lehmann alternative

2.1 Introduction

In life-testing and reliability experiments, it is natural to compare several treatments with a standard treatment (control). For example, a manufacturer of electronic components may wish to compare $(k-1)$ new production processes with the standard process and then determine whether any of these new processes would produce more reliable components than the standard process. In many cases, the costs of production for the new processes are relatively high because they are under development, and so it would be desirable to have a statistical test procedure which allows the experimenter to make a decision early on in the life-test.

The precedence test, first proposed by Nelson (1963), is a distribution-free two-sample life-test (i.e., a special case when $k = 2$) based on the order of early failures. Assume that a random sample of n_1 units from distribution F_X and another independent sample of n_2 units from distribution F_Y are placed

J. Glaz et al. (eds.), *Scan Statistics: Methods and Applications*, Statistics for Industry and Technology, DOI 10.1007/978-0-8176-4749-0_2,

simultaneously on a life-testing experiment. Suppose the null hypothesis is that the two lifetime distributions are equal, and the alternative hypothesis of interest is that one distribution is stochastically larger than the other, say, F_X is stochastically larger than F_Y. This alternative corresponds to the situation wherein the Y-units are more reliable than the X-units. The experiment is terminated as soon as the r-th failure from the Y-sample is observed. Then, the precedence test statistic $P_{(r)}$ is defined simply as the number of failures from the X-sample that precede the r-th failure from the Y-sample. It is obvious that large values of $P_{(r)}$ lead to the rejection of the hypothesis that $F_X = F_Y$ and in favor of the above-mentioned alternative hypothesis. The precedence test will be useful (i) when a life-test involves expensive units as the units that had not failed could be used for some other testing purposes, and (ii) to make quick and reliable decisions early on in the life-testing experiment. Many authors have studied the power properties of the precedence test and have also proposed some alternative tests; see, for example, Eilbott and Nadler (1965), Shorack (1967), Nelson (1986, 1993), Lin and Sukhatme (1992), Balakrishnan and Frattina (2000), Balakrishnan and Ng (2001), Ng and Balakrishnan (2002, 2004), and van der Laan and Chakraborti (2001). A brief review of all these precedence-type tests is first presented in Section 2.2, while an elaborate discussion of precedence-type tests and their variants can be found in the review articles by Chakraborti and van der Laan (1996, 1997) and also in the recent book by Balakrishnan and Ng (2006).

In this work, different precedence-type test procedures are proposed for the k-sample problem. Specifically, suppose we have $(k-1)$ treatments that we wish to compare with a control, or $(k-1)$ new processes that we wish to compare with the standard process. With $F_1(x)$ denoting the lifetime distribution associated with the control (or the standard process) and $F_{i+1}(x)$ denoting the lifetime distribution associated with the i-th treatment (or the i-th new process) for $i = 1, 2, \ldots, k-1$, our null hypothesis is simply

$$H_0 : F_1(x) = F_2(x) = \cdots = F_k(x) \text{ for all } x. \tag{2.1}$$

We are specifically concerned with a stochastically ordered alternative of the form

$$H_1 : \{F_2(x) \le F_1(x)\} \cup \{F_3(x) \le F_1(x)\} \cup \cdots \cup \{F_k(x) \le F_1(x)\} \text{ for all } x, \\ \text{with at least one holding strictly for some } x. \tag{2.2}$$

Suppose k independent random samples of sizes $n_1, n_2, \ldots, n_k$ from $F_1(x)$, $F_2(x), \ldots, F_k(x)$, respectively, are placed simultaneously on a life-testing experiment. The experiment is terminated as soon as the r-th failure from $F_1(x)$ is observed. Then, the number of failures from $F_i(x)$, $i = 2, \ldots, k$, in between the failures from $F_1(x)$ are counted and their functions are used as test statistics for testing the hypothesis in (2.1).

The chapter is organized as follows. In Section 2.2, we review some results on the precedence-type tests which are considered in the subsequent sections. In Section 2.3, we propose the precedence-type tests, which include tests based on the precedence, weighted maximal precedence and minimum Wilcoxon rank-sum precedence test statistics, for testing the hypothesis in (2.1). The exact null distributions of the proposed test statistics are derived in Section 2.3, and critical values for some selected choices of sample sizes are also tabulated. Exact power properties of these tests under Lehmann alternatives are derived in Section 2.4. We then compare the power properties of the proposed precedence-type tests under Lehmann alternatives. Finally, an example is presented to illustrate all the tests discussed here.

2.2 Review of Precedence-Type Tests

The precedence-type test allows a simple and robust comparison of two distribution functions. Suppose there are two failure time distributions F_X and F_Y and that we are interested in testing

$$H_0^* : F_X = F_Y \quad \text{against} \quad H_1^* : F_X > F_Y. \tag{2.3}$$

Note that some specific alternatives such as the location-shift alternative and the Lehmann alternative are subclasses of the stochastically ordered alternative considered in (2.3).

Assume that a random sample of n_1 units from distribution F_X and another independent sample of n_2 units from distribution F_Y are placed simultaneously on a life-testing experiment. Let $X_1, \ldots, X_{n_1}$ denote the sample from F_X, and $Y_1, \ldots, Y_{n_2}$ denote the sample from F_Y. Let us denote the order statistics from the X- and Y-samples by $X_{1:n_1} \leq \cdots \leq X_{n_1:n_1}$ and $Y_{1:n_2} \leq \cdots \leq Y_{n_2:n_2}$, respectively. Further, let M_1 denote the number of X-failures before $Y_{1:n_2}$ and M_i the number of X-failures between $Y_{i-1:n_2}$ and $Y_{i:n_2}$, $i = 2, 3, \ldots, r$. Figure 2.1 gives a schematic representation of this precedence setup.

Note here that the idea of precedence-type test is closely related to that of a run, which is defined as an uninterrupted sequence. Wald and Wolfowitz (1940) used runs to establish a two-sample test for testing the hypothesis in (2.3). They suggested that one should combine the two samples, arrange the $n_1 + n_2$ observations in increasing order of magnitude, and replace the ordered values by 0 or 1 depending on whether it originated from the X-sample or the Y-sample, respectively. For example, in Figure 2.1, we have a binary sequence (1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1). Then, the total number of runs in that binary sequence is used as a test statistic to test the hypothesis in (2.3). Instead of

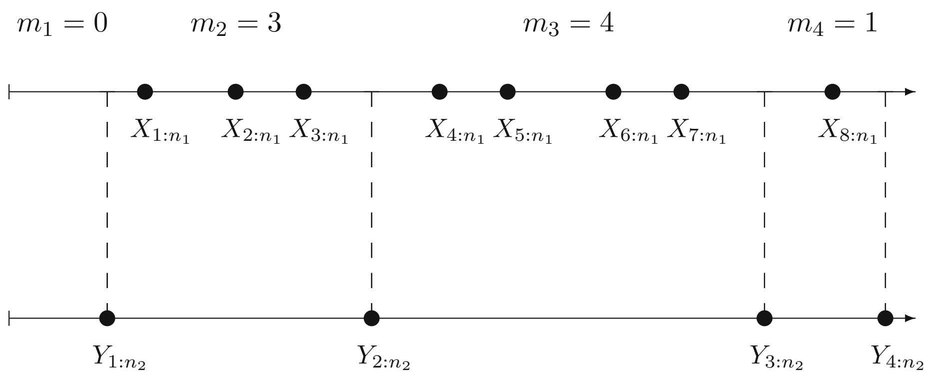

Figure 2.1. Schematic representation of a precedence life-test.

using the number of runs in the binary sequence, the precedence-type tests use the length of the runs of 0's (i.e., M_i, $i = 1, \ldots, n_2$) and their functions as test statistics for testing the hypotheses in (2.3). For extensive reviews on runs and applications, one may refer to Balakrishnan and Koutras (2002) and Fu and Lou (2003).

2.2.1 Precedence test

The precedence test statistic $P_{(r)}$ is defined simply as the number of failures from the X-sample that precede the r-th failure from the Y-sample, i.e.,

$$P_{(r)} = \sum_{j=1}^{r} M_j.$$

Large values of $P_{(r)}$ lead to the rejection of H_0^* and in favor of H_1^* in (2.3). In other words, H_0^* is rejected if $P_{(r)} \geq s$, where s is the critical value of the precedence test statistic for specific values of n_1, n_2, r and level of significance (α). For example, from Figure 2.1, with $r = 4$, the precedence test statistic takes on the value $P_{(4)} = \sum_{i=1}^{4} M_i = 0 + 3 + 4 + 1 = 8$. If we have $n_1 = n_2 = 10$ and we use the precedence test with $r = 4$, the near 5% critical value will be $s = 8$ with exact level of significance 0.035, in which case H_0^* would be rejected if there were at least 8 failures from the X-sample before the fourth failure from the Y-sample. Therefore, the null hypothesis that the two distributions are equal is rejected based on the precedence test in this example.

From Balakrishnan and Ng (2006, Theorem 4.1), we have the joint probability mass function of $(M_1, \ldots, M_r)$, under $H_0^* : F_X = F_Y$, to be

$$\Pr\left(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid H_0 : F_X = F_Y\right) = \frac{\binom{n_1 + n_2 - \sum_{j=1}^{r} m_j - r}{n_2 - r}}{\binom{n_1 + n_2}{n_2}}. \tag{2.4}$$

The null distribution and critical values of the precedence test statistic $P_{(r)}$ can be readily computed from (2.4). The critical values and their exact levels of significance (as close as possible to 5% and 10%) for different choices of r and the sample sizes n_1 and n_2 are presented, for example, in Balakrishnan and Ng (2006).

2.2.2 Weighted maximal precedence test

Balakrishnan and Frattina (2000) observed that a masking effect is present in the precedence test which has an adverse effect on its power properties. The maximal precedence test proposed by Balakrishnan and Frattina (2000) and Balakrishnan and Ng (2001) was specifically to avoid this masking problem. It is a test procedure based on the maximum number of failures occurring from the X-sample before the first, between the first and the second, ... , between the $(r-1)$-th and the r-th failures from the Y-sample. Then, Ng and Balakrishnan (2005) proposed the weighted maximal precedence test by giving a decreasing weight to m_j as j increases, which is given by

$$M_{(r)} = \max_{1 \leq j \leq r} (n_2 - j + 1) M_j. \tag{2.5}$$

It is also a test procedure suitable for testing the hypotheses in (2.3) with large values of $M_{(r)}$ leading to the rejection of H_0^* and in favor of H_1^* in (2.3). The null distribution of the weighted maximal precedence test statistic $M_{(r)}$ can also be obtained from (2.4). The critical values and their exact levels of significance (as close as possible to 5% and 10%) for different choices of r and the sample sizes n_1 and n_2 are presented, for example, in Balakrishnan and Ng (2006). For example, if we refer to Figure 2.1, with $r = 4$ and with $n_1 = n_2 = 10$, the critical value is 42 with exact level of significance 0.043 and the weighted maximal precedence test statistic is $M_{(4)} = \max(10 \times 0, 9 \times 3, 8 \times 4, 7 \times 1) = \max(0, 27, 32, 7) = 32$. Therefore, the null hypothesis that the two distributions are equal is not rejected based on the weighted maximal precedence test in this example.

2.2.3 Minimal Wilcoxon rank-sum precedence test

The Wilcoxon rank-sum test is a well-known nonparametric procedure for testing the hypotheses in (2.3) based on complete samples. For testing the hypotheses in (2.3), if complete samples of sizes n_1 and n_2 are available from F_X and F_Y, respectively, one can use the standard Wilcoxon's rank-sum statistic, proposed by Wilcoxon (1945), which is simply the sum of ranks of X-observations in the combined sample.

Ng and Balakrishnan (2002, 2004) proposed the Wilcoxon-type rank-sum precedence tests for testing the hypotheses in (2.3) in the context of precedence test described earlier, i.e., when the Y-sample is Type-II right censored. This test is a variation of the precedence test and a generalization of the Wilcoxon rank-sum test. In order to test the hypotheses in (2.3), instead of using the maximum of the frequencies of failures from the X-sample between the first r failures of the Y-sample, one could use the sum of the ranks of those failures. More specifically, suppose that $M_1, M_2, \ldots, M_r$ denote the number of X-failures that occurred before the first, between the first and the second, ..., between the $(r-1)$-th and the r-th Y-failures, respectively; see Figure 2.1. Let W be the rank-sum of the X-failures that occurred before the r-th Y-failure. The Wilcoxon's rank-sum test statistic will be smallest when all the remaining $\left(n_1 - \sum\limits_{j=1}^{r} M_j\right)$ X-failures occur between the r-th and $(r+1)$-th Y-failures. The test statistic in this case would be

$$\begin{aligned} W_{(r)} &= W + \left[\left(\sum_{j=1}^{r} M_j + r + 1\right) + \left(\sum_{j=1}^{r} M_j + r + 2\right) + \cdots + (n_1 + r)\right] \\ &= \frac{n_1(n_1 + 2r + 1)}{2} - \sum_{j=1}^{r}(r - j + 1)M_j. \end{aligned}$$

This is called the *minimal rank-sum statistic*. Note that in the special case of $r = n_2$ (that is, when we observe a complete sample), $W_{(n_2)}$ is equivalent to the classical Wilcoxon's rank-sum statistic. Small values of $W_{(r)}$ lead to the rejection of H_0^* and in favor of H_1^* in (2.3). The null distribution of the minimal Wilcoxon-type rank-sum precedence test statistic can once again be obtained from (2.4). The critical values and their exact levels of significance (as close as possible to 5% and 10%) for different choices of r and the sample sizes n_1 and n_2 are presented, for example, in Balakrishnan and Ng (2006).

For example, from Figure 2.1, when $n_1 = n_2 = 10$ and $r = 4$, we have

$$W_{(4)} = 2 + 3 + 4 + 6 + 7 + 8 + 9 + 11 + 13 + 14 = 77$$

and the critical value of the test is 81 with exact level of significance 0.050. Therefore, the null hypothesis that the two distributions are equal is not rejected based on the minimal Wilcoxon rank-sum precedence test in this example.

Ng and Balakrishnan (2002, 2004) observed that the large-sample normal approximation for the null distribution of these statistics is not satisfactory in the case of small or moderate sample sizes. For this reason, they developed an Edgeworth expansion to approximate the significance probabilities. They also derived the exact power function under the Lehmann alternative and examined the power properties of the minimal Wilcoxon-type rank-sum precedence test.

2.3 Test Statistics for Comparing $k - 1$ Treatments with Control

Suppose k independent random samples of sizes $n_1, n_2, \ldots, n_k$ from $F_1(x)$, $F_2(x), \ldots, F_k(x)$, respectively, are placed simultaneously on a life-testing experiment. When the sample sizes are all equal, we have a balanced case which usually provides a favorable setting for carrying out a precedence-type procedure for testing H_0 in (2.1) against the alternative in (2.2); however, the test can be carried out even in the unbalanced case, although the power of the test may be adversely affected in this case.

A precedence-type test procedure, for this specific testing problem, may be constructed as follows. After pre-fixing an r $(\leq n_1)$, the life-test continues until the r-th failure in the sample from the control group. We then observe $\boldsymbol{M}_2 = (M_{12}, M_{22}, \ldots, M_{r2}), \ldots, \boldsymbol{M}_k = (M_{1k}, M_{2k}, \ldots, M_{rk})$ from the $(k-1)$ treatments, where $M_{1i}, M_{2i}, \ldots, M_{ri}$ are the numbers of failures in the sample from the $(i-1)$-th treatment (for $i = 2, 3, \ldots, k$) before the first failure, between the first and second failures, ... , and between the $(r-1)$-th and r-th failures from the control group, respectively. The observed value of $\boldsymbol{M}_i$ is denoted by $\boldsymbol{m}_i$, $i = 2, \ldots, k$.

2.3.1 Tests based on precedence statistic

Let us consider

$$P_{(r)i} = \sum_{j=1}^{r} M_{ji} \quad \text{for} \quad i = 2, 3, \ldots, k \tag{2.6}$$

for the precedence statistic corresponding to the sample from the $(i-1)$-th treatment. For convenience of notation, let $M_{j\bullet} = \sum_{i=2}^{k} M_{ji}$ and denote its observed value by $m_{j\bullet}$, $j = 1, \ldots, r$. We may then propose the following precedence-type test statistics:

$$P_1 = \sum_{i=2}^{k} P_{(r)i} = \sum_{i=2}^{k} \sum_{j=1}^{r} M_{ji} = \sum_{j=1}^{r} M_{j\bullet} \tag{2.7}$$

and

$$P_2 = \min_{2\le i\le k} P_{(r)i} = \min_{2\le i\le k} \left\{ \sum_{j=1}^{r} M_{ji} \right\}. \tag{2.8}$$

The rationale for the use of the statistics in (2.7) and (2.8) is that, under the stochastically ordered alternative H_1 in (2.2), we would expect some of the precedence statistics $P_{(r)i}$ in (2.6) to be too small. Consequently, we will tend to reject H_0 in (2.1) in favor of H_1 in (2.2) for small values of P_1 and P_2 in which the critical values can be determined for specific values of k, r, $n_i, i = 1, 2, \ldots, k$, and pre-fixed level of significance α. Specifically, $\{0 \le P_1 \le c_{P_1}\}$ and $\{0 \le P_2 \le c_{P_2}\}$ will serve as critical regions, where c_{P_1} and c_{P_2} are determined such that

$$\Pr(P_1 \le c_{P_1} | H_0) = \alpha \quad \text{and} \quad \Pr(P_2 \le c_{P_2} | H_0) = \alpha. \tag{2.9}$$

The null distributions of the test statistics P_1 and P_2 can be expressed as

$$\Pr(P_1 = p_1 | H_0) = \sum_{p_{(r)2}=0}^{n_2} \cdots \sum_{p_{(r)k}=0}^{n_k} \Pr(P_{(r)i} = p_{(r)i}, i = 2, \ldots, k | H_0) I\left(\sum_{i=2}^{k} p_{(r)i} = p_1 \right) \tag{2.10}$$

for $p_1 = 0, 1, \ldots, \sum_{i=2}^{k} n_i$, and

$$\Pr(P_2 = p_2 | H_0) = \sum_{p_{(r)2}=0}^{n_2} \cdots \sum_{p_{(r)k}=0}^{n_k} \Pr(P_{(r)i} = p_{(r)i}, i = 2, \ldots, k | H_0) I\left(\min_{2\le i\le k} p_{(r)i} = p_2 \right) \tag{2.11}$$

for $p_2 = 0, 1, \ldots, \min_{2\le i\le k} n_i$, where $I(A)$ is the indicator function defined by

$$I(A) = \begin{cases} 1 \text{ if } A \text{ is true,} \\ 0 \text{ otherwise,} \end{cases}$$

and

$$\Pr(P_{(r)i} = p_{(r)i}, i = 2, \ldots, k | H_0) = \sum_{\boldsymbol{m}_2} \cdots \sum_{\boldsymbol{m}_k} \delta(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k) I\left(\sum_{j=1}^{r} m_{ji} = p_{(r)i}, i = 2, \ldots, k \right) \tag{2.12}$$

with

$$\sum_{\boldsymbol{m}_i} \stackrel{def.}{=} \sum_{m_{1i}=0}^{n_i} \sum_{m_{2i}=0}^{n_i - m_{1i}} \cdots \sum_{m_{ri}=0}^{n_i - \sum_{j=1}^{r-1} m_{ji}} \quad \text{for } i = 2, \ldots, k$$

and $\delta(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k)$ is the probability mass function of $(\boldsymbol{M}_2, \ldots, \boldsymbol{M}_k)$ under H_0 (see Appendix A)

$$\begin{aligned}\delta(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k) &= \Pr(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k | H_0 : F_1 = F_2 = \cdots = F_k) \\ &= \frac{1}{\binom{\sum_{i=1}^k n_i}{n_1, \ldots, n_k}} \left\{ \prod_{j=1}^r \binom{m_{j\bullet}}{m_{j2}, \ldots, m_{jk}} \right\} \\ &\quad \times \binom{\sum_{i=1}^k n_i - \sum_{j=1}^r m_{j\cdot} - r}{n_1 - r, n_2 - \sum_{j=1}^r m_{j2}, \ldots, n_k - \sum_{j=1}^r m_{jk}},\end{aligned}$$

where

$$\binom{a_1 + \cdots + a_l}{a_1, \ldots, a_l} = \frac{(a_1 + \ldots + a_l)!}{a_1! \ldots a_l!}.$$

From Equations (2.9)–(2.12), the critical values c_{P_1}, c_{P_2} and their exact levels of significance as close as possible to $\alpha = 5\%$ for $k = 3, 4$ with equal sample sizes $n_1 = \cdots = n_k = n$ and $r = 4(1)n$ were computed and are presented in Tables 2.1 and 2.2; similarly, for the unequal sample sizes $n_1 = 10, n_2 = \cdots = n_k = 15$; $n_1 = 15, n_2 = \cdots = n_k = 20$ and $r = 4(1)n_1$, the values are presented in Tables 2.3 and 2.4. Due to the heavy computational demand in going through all the possible outcomes, the critical values of the tests discussed in this section were obtained from the exact null distribution for $r \leq 8$ and through 20,000,000 Monte Carlo simulations for $r > 8$.

2.3.2 Tests based on weighted maximal precedence statistic

We can proceed similarly and propose weighted maximal precedence-type statistics for the testing problem discussed here. Once again, we terminate the life-test when the r-th failure occurs in the sample from the control group. Then, with $\boldsymbol{M}_i = (M_{1i}, M_{2i}, \ldots, M_{ri})$, for $i = 2, \ldots, k$, being observed from the $(k-1)$ treatments, where M_{ji} denotes the number of failures in the sample from the $(i-1)$-th treatment between the $(j-1)$-th and j-th failures from the control group, we may set

$$M_{(r)i} = \max_{1 \leq j \leq r} (n_1 - j + 1) M_{ji} \quad \text{for } \ i = 2, 3, \ldots, k$$

for the weighted maximal precedence statistic corresponding to the sample from the $(i-1)$-th treatment. We may then propose the weighted maximal precedence-type test statistics as

$$T_1 = \sum_{i=2}^k M_{(r)i} = \sum_{i=2}^k \max_{1 \leq j \leq r} (n_1 - j + 1) M_{ji} \tag{2.13}$$

Table 2.1. Near 5% critical values and exact levels of significance (l.o.s.) for P_1, P_2, T_1, T_2, W_1 and W_2 with $k = 3$, $n_1 = n_2 = n_3 = n = 10, 15$ and 20.

$n = 10$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	1	0.031	0	0.079	10	0.047	0	0.079	186	0.058	95	0.079
5	3	0.056	0	0.031	17	0.047	6	0.050	202	0.051	104	0.050
6	4	0.039	1	0.052	19	0.045	7	0.050	216	0.049	112	0.045
7	6	0.045	2	0.063	21	0.048	8	0.061	228	0.048	119	0.044
8	8	0.045	3	0.062	22	0.050	8	0.043	237	0.052	124	0.052
9	11	0.062	4	0.051	23	0.051	8	0.037	244	0.051	128	0.054
10	13	0.038	5	0.029	23	0.051	8	0.037	248	0.050	131	0.048
$n = 15$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	1	0.036	0	0.090	15	0.053	0	0.090	357	0.042	180	0.090
5	3	0.068	0	0.040	26	0.047	0	0.040	383	0.046	195	0.040
6	4	0.052	1	0.073	30	0.053	11	0.042	407	0.052	208	0.042
7	5	0.039	1	0.033	35	0.047	12	0.043	430	0.050	220	0.048
8	7	0.048	2	0.046	39	0.052	13	0.044	451	0.050	232	0.045
9	9	0.054	3	0.055	41	0.051	14	0.055	470	0.050	242	0.051
10	11	0.056	4	0.059	42	0.051	14	0.042	487	0.050	252	0.047
11	13	0.055	5	0.059	43	0.048	15	0.058	502	0.049	260	0.050
12	15	0.051	6	0.054	44	0.053	15	0.048	514	0.050	267	0.051
13	17	0.043	7	0.045	44	0.052	15	0.045	524	0.049	273	0.049
14	20	0.050	8	0.032	44	0.052	15	0.045	530	0.051	277	0.049
15	23	0.045	10	0.037	44	0.052	15	0.045	534	0.050	279	0.050
$n = 20$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	1	0.038	0	0.096	20	0.057	0	0.096	577	0.045	290	0.096
5	2	0.036	0	0.044	36	0.051	0	0.044	613	0.050	310	0.044
6	4	0.059	0	0.019	39	0.049	16	0.048	648	0.048	328	0.048
7	5	0.046	1	0.041	49	0.048	17	0.051	681	0.050	345	0.057
8	7	0.059	2	0.059	54	0.052	18	0.054	712	0.053	362	0.055
9	8	0.044	2	0.029	57	0.048	19	0.057	742	0.051	378	0.054
10	10	0.051	3	0.038	60	0.050	20	0.059	770	0.050	394	0.047
11	12	0.056	4	0.044	63	0.048	20	0.047	796	0.050	408	0.049
12	14	0.059	5	0.049	65	0.048	22	0.045	820	0.050	421	0.050
13	15	0.041	6	0.051	67	0.051	25	0.048	842	0.049	433	0.051
14	18	0.059	7	0.051	67	0.046	26	0.052	861	0.051	444	0.051
15	20	0.056	8	0.048	68	0.051	26	0.048	879	0.049	454	0.050
16	22	0.051	9	0.044	68	0.050	27	0.051	894	0.049	462	0.052
17	24	0.044	10	0.037	68	0.049	27	0.050	906	0.050	470	0.049
18	27	0.052	12	0.057	68	0.049	27	0.050	915	0.051	475	0.051
19	30	0.056	13	0.039	68	0.049	27	0.050	922	0.050	479	0.050
20	33	0.048	15	0.041	68	0.049	27	0.050	925	0.051	481	0.051

and

$$T_2 = \min_{2 \le i \le k} M_{(r)i} = \min_{2 \le i \le k} \left\{ \max_{1 \le j \le r} (n_1 - j + 1) M_{ji} \right\}. \tag{2.14}$$

Here again, the rationale for the use of the statistics in (2.13) and (2.14) is that, under the stochastically ordered alternative H_1 in (2.2), we would expect some of the weighted maximal precedence statistics $M_{(r)i}$ in (2.12) to be too

Table 2.2. Near 5% critical values and exact levels of significance (l.o.s.) for P_1, P_2, T_1, T_2, W_1 and W_2 with $k = 4$, $n_1 = n_2 = n_3 = n_4 = n = 10, 15$ and 20.

$n = 10$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	3	0.052	0	0.109	22	0.050	0	0.109	278	0.050	95	0.109
5	5	0.050	0	0.044	28	0.051	0	0.044	301	0.051	105	0.044
6	7	0.042	1	0.073	32	0.049	6	0.043	322	0.047	113	0.043
7	10	0.049	1	0.027	35	0.050	7	0.040	339	0.049	120	0.045
8	13	0.048	2	0.030	36	0.049	8	0.061	353	0.049	126	0.045
9	17	0.058	4	0.070	37	0.052	8	0.053	363	0.049	130	0.050
10	21	0.052	5	0.040	37	0.052	8	0.053	368	0.051	133	0.046
$n = 15$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	3	0.058	0	0.125	30	0.052	0	0.125	533	0.056	180	0.125
5	5	0.060	0	0.056	42	0.048	0	0.056	572	0.050	195	0.056
6	7	0.056	0	0.024	52	0.051	11	0.059	608	0.052	208	0.059
7	9	0.049	1	0.047	57	0.048	12	0.061	642	0.050	221	0.050
8	12	0.059	2	0.064	62	0.050	12	0.039	673	0.049	233	0.048
9	14	0.047	2	0.028	65	0.048	13	0.039	701	0.050	244	0.048
10	17	0.050	3	0.033	68	0.052	14	0.059	726	0.049	254	0.047
11	20	0.049	4	0.035	69	0.049	14	0.043	747	0.051	262	0.053
12	23	0.045	5	0.033	70	0.052	14	0.035	765	0.050	270	0.049
13	27	0.052	7	0.062	70	0.051	15	0.065	779	0.051	276	0.049
14	31	0.054	8	0.044	70	0.050	15	0.064	789	0.050	280	0.050
15	35	0.042	10	0.050	70	0.050	15	0.064	794	0.051	282	0.052
$n = 20$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	3	0.062	0	0.132	39	0.048	0	0.132	864	0.046	290	0.132
5	4	0.039	0	0.063	57	0.052	0	0.063	917	0.054	310	0.063
6	6	0.040	0	0.028	71	0.051	15	0.044	969	0.050	329	0.044
7	9	0.057	1	0.057	79	0.050	16	0.046	1018	0.051	347	0.043
8	11	0.051	1	0.027	87	0.050	17	0.049	1065	0.050	364	0.047
9	13	0.044	2	0.041	93	0.050	18	0.051	1109	0.050	380	0.052
10	16	0.051	3	0.053	97	0.048	19	0.053	1150	0.050	396	0.047
11	19	0.056	4	0.062	101	0.050	20	0.066	1188	0.051	410	0.052
12	21	0.045	5	0.067	104	0.051	20	0.052	1224	0.049	424	0.049
13	24	0.046	5	0.033	106	0.052	22	0.049	1256	0.050	436	0.052
14	27	0.046	6	0.034	107	0.049	22	0.042	1285	0.050	448	0.049
15	30	0.044	8	0.066	108	0.052	24	0.054	1310	0.050	458	0.050
16	34	0.052	9	0.060	108	0.050	24	0.049	1332	0.050	467	0.049
17	37	0.045	10	0.051	108	0.050	25	0.050	1350	0.050	474	0.051
18	41	0.048	11	0.039	108	0.050	25	0.050	1364	0.050	480	0.050
19	45	0.045	13	0.053	108	0.050	25	0.050	1374	0.050	484	0.051
20	50	0.046	15	0.055	108	0.050	25	0.050	1379	0.050	486	0.051

small. Therefore, we would reject H_0 in (2.1) in favor of H_1 in (2.2) for small values of T_1 and T_2 in which the critical values can be determined for specific values of $k, r, n_i, i = 1, 2, \ldots, k$, and pre-fixed level of significance α. Specifically, $\{0 \le T_1 \le c_{T_1}\}$ and $\{0 \le T_2 \le c_{T_2}\}$ will serve as critical regions, where c_{T_1} and c_{T_2} are determined such that

$$\Pr(T_1 \le c_{T_1} | H_0) = \alpha \quad \text{and} \quad \Pr(T_2 \le c_{T_2} | H_0) = \alpha. \tag{2.15}$$

Table 2.3. Near 5% critical values and exact levels of significance (l.o.s.) for P_1, P_2, T_1, T_2, W_1 and W_2 with $k = 3$, $n_1 = 10$, $n_2 = n_3 = 15$ and $n_1 = 15$, $n_2 = n_3 = 20$.

$n_1 = 10$, $n_2 = n_3 = 15$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	3	0.052	0	0.031	19	0.050	7	0.052	353	0.050	179	0.052
5	5	0.050	1	0.041	25	0.050	8	0.046	376	0.051	192	0.041
6	7	0.042	2	0.040	28	0.050	9	0.039	397	0.047	203	0.048
7	10	0.049	3	0.033	30	0.049	10	0.054	414	0.049	213	0.046
8	13	0.048	5	0.050	32	0.052	10	0.037	428	0.049	221	0.046
9	17	0.058	7	0.058	32	0.049	12	0.063	438	0.049	226	0.053
10	21	0.052	9	0.048	32	0.049	12	0.063	443	0.051	230	0.048
$n_1 = 15$, $n_2 = n_3 = 20$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	2	0.041	0	0.048	27	0.050	0	0.048	575	0.043	290	0.048
5	4	0.052	1	0.072	35	0.047	12	0.047	609	0.050	308	0.047
6	6	0.055	1	0.029	41	0.051	13	0.046	641	0.052	325	0.051
7	8	0.054	2	0.035	45	0.050	14	0.043	671	0.052	341	0.054
8	10	0.049	3	0.036	49	0.048	16	0.051	699	0.050	356	0.053
9	12	0.043	4	0.034	52	0.051	18	0.045	724	0.050	370	0.050
10	15	0.051	6	0.060	54	0.053	21	0.053	746	0.051	382	0.052
11	18	0.057	7	0.049	55	0.050	21	0.045	765	0.051	393	0.051
12	20	0.042	8	0.036	55	0.047	22	0.053	781	0.051	402	0.051
13	24	0.056	10	0.046	56	0.054	22	0.051	794	0.050	410	0.048
14	27	0.047	12	0.050	56	0.054	22	0.051	803	0.050	415	0.049
15	31	0.043	14	0.041	56	0.054	22	0.051	808	0.049	418	0.048

Table 2.4. Near 5% critical values and exact levels of significance (l.o.s.) for P_1, P_2, T_1, T_2, W_1 and W_2 with $k = 3$, $n_1 = 10$, $n_2 = n_3 = n_4 = 15$ and $n_1 = 15$, $n_2 = n_3 = n_4 = 20$.

$n_1 = 10$, $n_2 = n_3 = n_4 = 15$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	5	0.047	0	0.043	32	0.049	0	0.043	528	0.050	180	0.043
5	8	0.045	1	0.056	40	0.051	7	0.038	562	0.051	192	0.058
6	12	0.051	2	0.054	45	0.047	9	0.054	592	0.051	204	0.050
7	16	0.051	3	0.045	49	0.049	9	0.033	618	0.049	214	0.050
8	20	0.044	5	0.067	51	0.050	10	0.052	638	0.051	222	0.053
9	26	0.055	6	0.039	51	0.047	10	0.042	653	0.050	229	0.046
10	32	0.050	9	0.063	51	0.047	11	0.042	661	0.050	232	0.050
$n_2 = 15$, $n_2 = n_3 = n_4 = 20$												
	P_1		P_2		T_1		T_2		W_1		W_2	
r	c_{P_1}	l.o.s.	c_{P_2}	l.o.s.	c_{T_1}	l.o.s.	c_{T_2}	l.o.s.	c_{W_1}	l.o.s.	c_{W_2}	l.o.s.
4	4	0.046	0	0.067	43	0.052	0	0.067	860	0.052	290	0.067
5	7	0.055	0	0.025	56	0.050	11	0.041	911	0.052	309	0.041
6	10	0.057	1	0.041	66	0.050	12	0.040	959	0.051	326	0.052
7	13	0.055	2	0.049	73	0.049	14	0.061	1004	0.049	343	0.046
8	16	0.050	3	0.050	79	0.050	15	0.056	1045	0.049	358	0.050
9	20	0.056	4	0.047	83	0.050	16	0.050	1082	0.049	372	0.051
10	23	0.046	5	0.041	85	0.049	18	0.050	1114	0.051	385	0.049
11	27	0.046	7	0.066	87	0.050	20	0.055	1143	0.050	396	0.050
12	32	0.054	8	0.049	88	0.051	20	0.049	1166	0.051	406	0.048
13	36	0.046	10	0.062	88	0.050	20	0.045	1185	0.050	413	0.051
14	42	0.056	11	0.035	88	0.050	21	0.055	1198	0.050	419	0.048
15	48	0.054	14	0.054	88	0.050	21	0.055	1205	0.050	422	0.048

The null distributions of the test statistics T_1 and T_2 can be expressed as

$$\Pr(T_1 = t_1|H_0) \\ = \sum_{m_{(r)2}=0}^{n_2} \cdots \sum_{m_{(r)k}=0}^{n_k} \Pr(M_{(r)i} = m_{(r)i}, i = 2, \ldots, k|H_0) I\left(\sum_{i=2}^{k} m_{(r)i} = t_1\right) \tag{2.16}$$

for $t_1 = 0, 1, \ldots, \sum_{i=2}^{k} n_i$, and

$$\Pr(T_2 = t_2|H_0) \\ = \sum_{m_{(r)2}=0}^{n_2} \cdots \sum_{m_{(r)k}=0}^{n_k} \Pr(M_{(r)i} = m_{(r)i}, i = 2, \ldots, k|H_0) I\left(\min_{2\le i\le k} m_{(r)i} = t_2\right) \tag{2.17}$$

for $t_2 = 0, 1, \ldots, \min_{2\le i\le k} n_i$, where $\Pr(M_{(r)i} = m_{(r)i}|H_0)$ is

$$\Pr(M_{(r)i} = m_{(r)i}, i = 2, \ldots, k|H_0) \\ = \sum_{\boldsymbol{m}_2} \cdots \sum_{\boldsymbol{m}_k} \delta(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k) I\left(\max_{1\le j\le r} (n_1 - j + 1) m_{ji} = m_{(r)i}, i = 2, \ldots, k\right). \tag{2.18}$$

From Equations (2.15)–(2.18), the critical values c_{T_1}, c_{T_2} and their exact levels of significance as close as possible to $\alpha = 5\%$ for $k = 3, 4$ with equal sample sizes $n_1 = \cdots = n_k = n$ and $r = 4(1)n$ were computed and are presented in Tables 2.1 and 2.2; similarly, for the unequal sample sizes $n_1 = 10, n_2 = \cdots = n_k = 15$; $n_1 = 15, n_2 = \cdots = n_k = 20$ and $r = 4(1)n_1$, the values are presented in Tables 2.3 and 2.4.

2.3.3 Tests based on minimal Wilcoxon rank-sum precedence statistic

Similarly, we propose test procedures based on minimal Wilcoxon rank-sum precedence statistic for the testing problem discussed here. We set

$$W_{(r)i} = \frac{n_i(n_i + 2r + 1)}{2} - \sum_{j=1}^{r} (r - j + 1) M_{ji} \quad \text{for} \quad i = 2, 3, \ldots, k \tag{2.19}$$

for the minimal Wilcoxon rank-sum precedence statistic corresponding to the sample from the $(i-1)$-th treatment. We may then propose the minimal Wilcoxon rank-sum precedence statistics as

$$W_1 = \sum_{i=2}^{k} W_{(r)i}$$

and

$$W_2 = \max_{2 \le i \le k} W_{(r)i}.$$

Under the stochastically ordered alternative H_1 in (2.2), we would expect some of the minimal Wilcoxon rank-sum precedence statistics $W_{(r)i}$ in (2.19) to be large. Therefore, we would reject H_0 in (2.1) in favor of H_1 in (2.2) for large values of W_1 and W_2 in which the critical values can be determined for specific values of k, r, $n_i, i = 1, 2, \ldots, k$, and pre-fixed level of significance α. Specifically, $\{W_1 \ge c_{W_1}\}$ and $\{W_2 \ge c_{W_2}\}$ will serve as critical regions, where c_{W_1} and c_{W_2} are determined such that

$$\Pr(W_1 \ge c_{W_1} | H_0) = \alpha \quad \text{and} \quad \Pr(W_2 \ge c_{W_2} | H_0) = \alpha. \tag{2.20}$$

The null distributions of the test statistics W_1 and W_2 can be expressed as

$$\begin{aligned}
&\Pr(W_1 = w_1 | H_0) \\
&= \sum_{w_{(r)2}=l_2}^{u_2} \cdots \sum_{w_{(r)k}=l_k}^{u_k} \Pr(W_{(r)i} = w_{(r)i}, i = 2, \ldots, k | H_0) I\left(\sum_{i=2}^{k} w_{(r)i} = w_1\right)
\end{aligned} \tag{2.21}$$

for $w_1 = \min_{2 \le i \le k} l_i, \ldots, \max_{2 \le i \le k} u_i$, with $l_i = n_i(n_i+1)/2$, $u_i = (r+n_i)(r+n_i+1)/2 - r(r+1)/2$, and

$$\begin{aligned}
&\Pr(W_2 = w_2 | H_0) \\
&= \sum_{w_{(r)2}=l_2}^{u_2} \cdots \sum_{w_{(r)k}=l_k}^{u_k} \Pr(W_{(r)i} = w_{(r)i}, i = 2, \ldots, k | H_0) I\left(\max_{2 \le i \le k} w_{(r)i} = w_2\right)
\end{aligned} \tag{2.22}$$

for $w_2 = \min_{2 \le i \le k} l_i, \ldots, \min_{2 \le i \le k} u_i$, where $\Pr(W_{(r)i} = w_{(r)i} | H_0)$ is given by

$$\begin{aligned}
&\Pr(W_{(r)i} = w_{(r)i}, i = 2, \ldots, k | H_0) \\
&= \sum_{\boldsymbol{m}_2} \cdots \sum_{\boldsymbol{m}_k} \delta(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k) \\
&\qquad \times I\left(\frac{n_i(n_i+2r+1)}{2} - \sum_{j=1}^{r}(r-j+1)m_{ji} = w_{(r)i}, i = 2, \ldots, k\right).
\end{aligned} \tag{2.23}$$

From Equations (2.20)–(2.23), the critical values c_{W_1}, c_{W_2} and their exact levels of significance as close as possible to $\alpha = 5\%$ for $k = 3, 4$ with equal sample sizes $n_1 = \cdots = n_k = n$ and $r = 4(1)n$ were computed and are presented in

Tables 2.1 and 2.2; similarly, for the unequal sample sizes $n_1 = 10, n_2 = \cdots = n_k = 15$; $n_1 = 15, n_2 = \cdots = n_k = 20$ and $r = 4(1)n_1$, the values are presented in Tables 2.3 and 2.4.

2.4 Exact Power Under Lehmann Alternative

The Lehmann alternative $H_1 : [F_i(x)]^{\gamma_i} = F_1(x)$ for some γ_i, $i = 2, \ldots, k$, which was first proposed by Lehmann (1953), is a subclass of the alternative $H_1 : F_i(x) > F_1(x)$ when at least one $\gamma_i \in (0, 1)$ (see Gibbons and Chakraborti, 2003). In this section, we will derive an explicit expression for the power functions of the proposed test procedures under the Lehmann alternative.

When $\gamma_2 = \cdots = \gamma_k = \gamma$, for some $\gamma \in (0, 1)$, under the Lehmann alternative $H_1 : [F_i(x)]^\gamma = F_1(x)$, the probability mass function of $(\boldsymbol{M}_2, \ldots, \boldsymbol{M}_k)$ is (see Appendix B)

$$\begin{aligned}
&\delta^*(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k)\\
&= \Pr(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k | H_1 : [F_i]^\gamma = F_1, i = 2, \ldots, k)\\
&= \frac{\gamma^r n_1!}{(n_1 - r)!} \left\{ \prod_{i=2}^{k} \binom{n_i}{m_{1i}, m_{2i}, \ldots, m_{ri}, n_i - \sum_{j=1}^{r} m_{ji}} \right\}\\
&\quad \times \left\{ \prod_{j=1}^{r-1} B\left(m_{1\bullet} + \ldots + m_{j\bullet} + j\gamma, m_{j+1\bullet} + 1\right) \right\}\\
&\quad \times \left\{ \sum_{l=0}^{n_1-r} \binom{n_1 - r}{l} (-1)^l B\left(\sum_{j=1}^{r} m_{j\bullet} + (r + l)\gamma, \sum_{i=2}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet} + 1 \right) \right\},
\end{aligned} \tag{2.24}$$

where $B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}dx$ is the complete beta function. Note that the exact distribution of $(\boldsymbol{M}_2, \ldots, \boldsymbol{M}_k)$ under the general Lehmann alternative $H_1 : [F_k(x)]^{\gamma_k} = [F_{k-1}(x)]^{\gamma_{k-1}} = \cdots = [F_2(x)]^{\gamma_2} = [F_1(x)]$ can also be obtained. For the purpose of illustration, we present the result for $k = 3$ in Appendix B.

Under the Lehmann alternative, the probability mass functions of P_1, P_2, T_1, T_2, W_1 and W_2 can be computed from Equations (2.10), (2.11), (2.16), (2.17), (2.21) and (2.22), respectively, by replacing $\delta(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k)$ with $\delta^*(\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k)$ in Equations (2.12), (2.18) and (2.23). Here, we computed the power values of the proposed test procedures for $k = 3, 4$ with $n_1 = \cdots = n_k = 10$, $\gamma = 0.2(0.2)1.0$, $i = 2, \ldots, k$. Note that when $\gamma = 1.0$, the power values are precisely the exact levels of significance. These results are presented in Tables 2.5 and 2.6.

Table 2.5. Power values under Lehmann alternative for $k = 3$, $n_1 = n_2 = n_3 = 10$, $r = 4(1)10$ and $\gamma_2 = \gamma_3 = \gamma = 0.2(0.2)1.0$.

$\gamma = 1.0$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.031	0.079	0.047	0.079	0.058	0.079
	5	0.056	0.031	0.047	0.050	0.051	0.050
	6	0.039	0.052	0.045	0.050	0.049	0.045
	7	0.045	0.063	0.048	0.061	0.048	0.044
	8	0.045	0.062	0.050	0.043	0.052	0.052
	9	0.062	0.051	0.051	0.037	0.051	0.054
	10	0.038	0.029	0.051	0.037	0.050	0.048
$\gamma = 0.8$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.084	0.163	0.112	0.163	0.147	0.163
	5	0.133	0.073	0.115	0.114	0.134	0.114
	6	0.095	0.113	0.105	0.114	0.132	0.106
	7	0.102	0.126	0.105	0.131	0.129	0.106
	8	0.094	0.118	0.104	0.094	0.134	0.120
	9	0.112	0.092	0.106	0.082	0.130	0.121
	10	0.065	0.051	0.106	0.081	0.125	0.108
$\gamma = 0.6$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.221	0.334	0.263	0.334	0.346	0.334
	5	0.302	0.181	0.271	0.261	0.329	0.261
	6	0.231	0.246	0.240	0.259	0.326	0.252
	7	0.229	0.255	0.229	0.277	0.319	0.252
	8	0.202	0.229	0.219	0.208	0.323	0.275
	9	0.208	0.174	0.219	0.181	0.311	0.273
	10	0.118	0.094	0.218	0.181	0.299	0.246
$\gamma = 0.4$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.524	0.632	0.564	0.632	0.691	0.632
	5	0.612	0.438	0.567	0.557	0.678	0.557
	6	0.514	0.515	0.510	0.548	0.673	0.554
	7	0.488	0.505	0.480	0.544	0.661	0.553
	8	0.422	0.445	0.445	0.435	0.657	0.575
	9	0.390	0.340	0.433	0.385	0.636	0.563
	10	0.224	0.186	0.432	0.383	0.617	0.522
$\gamma = 0.2$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.918	0.945	0.926	0.945	0.973	0.945
	5	0.940	0.859	0.917	0.923	0.970	0.923
	6	0.893	0.886	0.879	0.907	0.967	0.924
	7	0.858	0.859	0.845	0.877	0.962	0.920
	8	0.784	0.792	0.793	0.779	0.957	0.922
	9	0.703	0.658	0.757	0.712	0.948	0.910
	10	0.455	0.407	0.756	0.705	0.939	0.886

2.5 Discussion

The results in Tables 2.5 and 2.6 show that the test procedures can detect the difference between two distributions effectively in most cases early in the life-testing experiment. Note that the desired level of significance may be impossible to attain for some test statistics when r is small, especially for the tests based on

Table 2.6. Power values under Lehmann alternative for $k = 4$, $n_1 = \cdots = n_4 = 10, r = 4(1)10$ and $\gamma_2 = \gamma_3 = \gamma_4 = \gamma = 0.2(0.2)1.0$.

$\gamma = 1.0$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.052	0.109	0.050	0.109	0.050	0.109
	5	0.050	0.044	0.051	0.044	0.051	0.044
	6	0.042	0.073	0.049	0.043	0.047	0.043
	7	0.049	0.027	0.050	0.040	0.049	0.045
	8	0.048	0.030	0.049	0.061	0.049	0.045
	9	0.058	0.070	0.052	0.053	0.049	0.050
	10	0.052	0.040	0.052	0.053	0.051	0.046
$\gamma = 0.8$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.135	0.216	0.127	0.216	0.141	0.216
	5	0.128	0.102	0.128	0.102	0.146	0.155
	6	0.108	0.151	0.119	0.102	0.136	0.102
	7	0.114	0.063	0.118	0.094	0.140	0.107
	8	0.105	0.066	0.113	0.129	0.137	0.107
	9	0.110	0.122	0.118	0.113	0.128	0.116
	10	0.088	0.067	0.118	0.113	0.136	0.106
$\gamma = 0.6$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.326	0.416	0.309	0.416	0.358	0.416
	5	0.307	0.238	0.304	0.238	0.368	0.332
	6	0.263	0.310	0.279	0.244	0.351	0.247
	7	0.257	0.156	0.267	0.223	0.354	0.258
	8	0.225	0.152	0.251	0.269	0.344	0.255
	9	0.210	0.219	0.255	0.238	0.322	0.265
	10	0.152	0.120	0.254	0.238	0.330	0.244
$\gamma = 0.4$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.665	0.719	0.645	0.719	0.723	0.719
	5	0.633	0.525	0.621	0.525	0.733	0.645
	6	0.566	0.596	0.577	0.541	0.714	0.549
	7	0.533	0.381	0.541	0.489	0.710	0.564
	8	0.461	0.349	0.507	0.519	0.692	0.553
	9	0.400	0.400	0.500	0.468	0.662	0.556
	10	0.273	0.226	0.499	0.467	0.664	0.520
$\gamma = 0.2$	r	P_1	P_2	T_1	T_2	W_1	W_2
	4	0.964	0.969	0.960	0.969	0.981	0.969
	5	0.950	0.906	0.943	0.906	0.981	0.953
	6	0.918	0.921	0.916	0.914	0.977	0.922
	7	0.884	0.794	0.879	0.857	0.974	0.924
	8	0.814	0.730	0.841	0.833	0.968	0.913
	9	0.717	0.710	0.816	0.776	0.958	0.906
	10	0.510	0.456	0.815	0.772	0.956	0.882

extrema (viz., P_2, T_2 and W_2). For instance, for $k = 4$, $n_1 = n_2 = n_3 = n_4 = 20$ and $r = 4$, the minimum level of significance attainable by the tests based on P_2, T_2 and W_2 are all equal to 0.132. It is, therefore, not possible to test the hypotheses in (2.1) at 5% level in this setting based on P_2, T_2 and W_2. For this reason, the tests based on the extrema of the precedence statistics from the treatments may not be applicable for small values of r in practice.

From Tables 2.5 and 2.6, we can observe that the power values of the tests increase with the number of treatments (i.e., $k-1$) as expected, but the power values do not increase with r under Lehmann alternatives. We can also see that the tests based on precedence statistics (P_1 and P_2) suffer from the masking effect. In other words, the power values of P_1 and P_2 decrease as r increases and the information given by a larger value of r is thus being masked. The tests based on weighted maximal precedence statistics (T_1 and T_2) and minimal Wilcoxon rank-sum precedence statistics (W_1 and W_2) reduce the masking effect that affects the performance of P_1 and P_2.

In comparing the power performance of tests based on the sum of the precedence statistics from the treatments (viz., P_1, T_1 and W_1) with those based on the extrema of the precedence statistics from the treatments (viz., P_2, T_2 and W_2), we observe that the former have better power performance than the latter. Furthermore, among all the tests discussed here, the test based on the sum of minimal Wilcoxon rank-sum precedence statistics among treatments (viz., W_1) seems to give overall the best power performance under the Lehmann alternative, and hence is the one that we recommend for the problem discussed here.

Further, the decrease in power values with increasing r also suggests that the test procedures based on the order of early failures can be more powerful than those based on a complete sample. In fact, r ($\leq n_1$) need not be large to provide reliable comparison between treatments and the control. This can save both time and experimental units in a life-testing experiment, which are clear advantages of precedence-type tests. One may be interested in maximizing the power with respect to r, i.e., to determine the best choice of r in designing the experiment. When prior information about the alternative is available, this task can be achieved by comparing the power values for different values of r. For example, for $k = 4$, $n_1 = n_2 = n_3 = n_4 = 10$, if prior information suggests $\gamma = 0.4$ for the Lehmann alternative, we would recommend the use of W_1 with $r = 6$ based on the power values presented in Table 2.6.

2.6 Illustrative Example

Let us consider X_2, X_3 and X_1 samples to be the data on appliance cord life in flex tests 1, 2 and 3, respectively, of Nelson (1982, p. 510). These three tests were done using two types of cord, viz., B6 and B7, where flex tests 1 and 2 were done with cord type B6 and test 3 was done with cord type B7. Suppose cord B7 was the standard production cord and B6 was proposed as a cost improvement. We will then be interested in testing the equality of the lifetime distributions of these cords. For these data, we have $k = 3$, $n_1 = n_2 = n_3 = 12$. Had we fixed $r = 8$, the experiment would have stopped as soon as the eighth failure from the X_1-sample (cord B7) had been observed, i.e., at 128.7 hours. The data are presented in Table 2.7. The observed values of $(m_{1i}, \ldots, m_{8i})$ and the values of the statistics $P_{(8)i}$, $M_{(8)i}$ and $W_{(8)i}$, $i = 2, 3$, are presented in Table 2.8.

Table 2.7. Appliance cord life data from Nelson (1982, p. 510) ($*$ denotes censored observations).

Test 1 (X_2) Cord B6	96.9	100.3	100.8	103.3	103.4	105.4	122.6	$*$	$*$	$*$	$*$	$*$
Test 2 (X_3) Cord B6	57.5	77.8	88.0	98.4	102.1	105.3	$*$	$*$	$*$	$*$	$*$	$*$
Test 3 (X_1) Cord B7	72.4	78.6	81.2	94.0	120.1	126.3	127.2	128.7	$*$	$*$	$*$	$*$

Table 2.8. Values of $(m_{1i}, \dots, m_{8i})$ and the statistics $P_{(8)i}$, $M_{(8)i}$ and $W_{(8)i}$ for $i = 2, 3$.

	m_{1i}	m_{2i}	m_{3i}	m_{4i}	m_{5i}	m_{6i}	m_{7i}	m_{8i}	$P_{(8)i}$	$M_{(8)i}$	$W_{(8)i}$
$i = 2$	0	0	0	0	6	1	0	0	7	48	147
$i = 3$	1	1	0	1	3	0	0	0	6	24	142

The near 5% critical values for $k = 3$, $n_1 = n_2 = n_3 = 12$ and $r = 8$ and their exact level of significance (in parentheses) for the test procedures discussed in the preceding sections are as follows:

P_1: 8 (0.061), P_2: 2 (0.033), T_1: 29 (0.048), T_2: 10 (0.044),
W_1: 317 (0.052), W_2: 164(0.056).

Then the test statistics and their p-values are

$$
\begin{aligned}
&P_1 = 13 \ (p\text{-value} = 0.363), && P_2 = 6 \ (p\text{-value} = 0.491),\\
&T_1 = 72 \ (p\text{-value} = 0.813), && T_2 = 24 \ (p\text{-value} = 0.697),\\
&W_1 = 289 \ (p\text{-value} = 0.398), && W_2 = 147 \ (p\text{-value} = 0.507),
\end{aligned}
$$

and so we will not reject the null hypothesis that the lifetime distributions of these cords are equal. This means that the cord B6 is not better than the cord B7. Incidentally, this finding agrees with that of Nelson (1982), who analyzed these data by assuming a normal model.

Appendix A: Probability Mass Function of $(M_2, \dots, M_k)$ Under the Null Hypothesis

Let the ordered failures from the control be $x_1 < x_2 < \cdots < x_r$. Consider the $(i-1)$-th treatment, conditional on the failures from the control. Then, the probability that there are m_{1i} failures from the treatment before x_1 and m_{ji} failures between x_{j-1} and x_j, $j = 2, \dots, r$, is given by the multinomial probability

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_i = \boldsymbol{m}_i \middle|\, x_1, \ldots, x_r\right) \\
&\quad = \Pr\left(M_{1i} = m_{1i}, \ldots, M_{ri} = m_{ri} \middle|\, x_1, \ldots, x_r\right) \\
&\quad = \binom{n_i}{m_{1i}, \ldots, m_{ri}, n_i - \sum_{j=1}^{r} m_{ji}} \\
&\qquad\qquad \times [F_i(x_1)]^{m_{1i}} \left\{ \prod_{j=2}^{r} [F_i(x_2) - F_i(x_1)]^{m_{ji}} \right\} [1 - F_i(x_r)]^{\left(n_i - \sum\limits_{j=1}^{r} m_{ji}\right)}.
\end{aligned}
$$

For fixed values of $x_1 < x_2 < \cdots < x_r$, due to the independence of the samples from the $(k-1)$ treatments, we readily have the conditional joint probability as

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle|\, x_1, \ldots, x_r\right) \\
&\quad = \left\{ \prod_{i=2}^{k} \binom{n_i}{m_{1i}, \ldots, m_{ri}, n_i - \sum_{j=1}^{r} m_{ji}} \right\} \\
&\qquad \times \left\{ \prod_{i=2}^{k} [F_i(x_1)]^{m_{1i}} \right\} \left\{ \prod_{i=2}^{k} \prod_{j=2}^{r} [F_i(x_j) - F_i(x_{j-1})]^{m_{ji}} \right\} \\
&\qquad \times \left\{ \prod_{i=2}^{k} [1 - F_i(x_r)]^{\left(n_i - \sum\limits_{j=1}^{r} m_{ji}\right)} \right\}.
\end{aligned}
$$

Now, we have the joint density of the first r order statistics from the control as

$$
f_{1,\ldots,r:n_1}(x_1, \ldots, x_r) = \frac{n_1!}{(n_1 - r)!} \left[\prod_{j=1}^{r} f_1(x_j) \right] [1 - F_1(x_r)]^{n_1 - r}, \quad x_1 < \ldots < x_r.
$$

As a result, we obtain the unconditional probability of $(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k)$ as

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k\right) \\
&\quad = C \int_{-\infty}^{\infty} \int_{-\infty}^{x_r} \cdots \int_{-\infty}^{x_2} \left\{ \prod_{i=2}^{k} [F_i(x_1)]^{m_{1i}} \right\} \left\{ \prod_{i=2}^{k} \prod_{j=2}^{r} [F_i(x_j) - F_i(x_{j-1})]^{m_{ji}} \right\} \\
&\qquad \times \left\{ \prod_{i=2}^{k} [1 - F_i(x_r)]^{\left(n_i - \sum\limits_{j=1}^{r} m_{ji}\right)} \right\} \\
&\qquad \times \left[\prod_{j=1}^{r} f_1(x_j) \right] [1 - F_1(x_r)]^{n_1 - r} dx_1 \cdots dx_r, \qquad (2.25)
\end{aligned}
$$

where

$$C = \frac{n_1!}{(n_1-r)!} \prod_{i=2}^{k} \binom{n_i}{m_{1i}, \ldots, m_{ri}, n_i - \sum_{j=1}^{r} m_{ji}}.$$

Under the null hypothesis, $H_0 : F_1(x) = F_2(x) = \cdots = F_k(x)$, by denoting $m_{j\bullet} = \sum_{i=2}^{k} m_{ji}$, the expression in (2.25) becomes

$$\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle|\, H_0\right) \\
&= C \int_{-\infty}^{\infty} \int_{-\infty}^{x_r} \cdots \int_{-\infty}^{x_2} \left\{ \prod_{i=2}^{k} [F_1(x_1)]^{m_{1i}} \right\} \left\{ \prod_{i=2}^{k} \prod_{j=2}^{r} [F_1(x_j) - F_1(x_{j-1})]^{m_{ji}} \right\} \\
&\quad \times \left\{ \prod_{i=2}^{k} [1 - F_1(x_r)]^{\left(n_i - \sum_{j=1}^{r} m_{ji}\right)} \right\} \\
&\quad \times \left[\prod_{j=1}^{r} f_1(x_j) \right] [1 - F_1(x_r)]^{n_1 - r} dx_1 \cdots dx_r \\
&= C \int_{-\infty}^{\infty} \int_{-\infty}^{x_r} \cdots \int_{-\infty}^{x_2} [F_1(x_1)]^{m_{1\bullet}} \left\{ \prod_{j=2}^{r} [F_1(x_j) - F_1(x_{j-1})]^{m_{j\bullet}} \right\} \\
&\quad \times [1 - F_1(x_r)]^{\left(\sum_{i=1}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet} - r\right)} \left[\prod_{j=1}^{r} f_1(x_j) \right] dx_1 \cdots dx_r.
\end{aligned}$$

Upon setting $u_i = F_1(x_i)$ for $i = 1, \ldots, r$, the above expression becomes

$$\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle|\, H_0\right) \\
&= C \int_0^1 \int_0^{u_r} \cdots \int_0^{u_2} u_1^{m_{1\bullet}} \left[\prod_{i=2}^{k} (u_j - u_{j-1})^{m_{j\bullet}} \right] \\
&\quad \times (1 - u_r)^{\left(\sum_{i=1}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet} - r\right)} du_1 \cdots du_r.
\end{aligned}$$

Using the transformation $w_1 = u_1/u_2$, we have

$$\begin{aligned}
\int_0^{u_2} u_1^{m_{1\bullet}} (u_2 - u_1)^{m_{2\bullet}} du_1 &= u_2^{m_{1\bullet} + m_{2\bullet}} \int_0^1 w_1^{m_{1\bullet}} (1 - w_1)^{m_{2\bullet}} dw_1 \\
&= u_2^{m_{1\bullet} + m_{2\bullet} + 1} B(m_{1\bullet} + 1, m_{2\bullet} + 1),
\end{aligned}$$

where, as before, $B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx$ is the complete beta function. Proceeding similarly and using the transformations $w_l = u_l/u_{l+1}$ for $l = 2, \ldots, r - 1$, we obtain

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle| H_0\right) \\
&= C \left\{ \prod_{j=1}^{r-1} B\left(m_{1\bullet} + \cdots + m_{j\bullet} + j, m_{j+1\bullet} + 1\right) \right\} \\
&\quad \times \int_0^1 u_r^{\left(\sum\limits_{j=1}^{r} m_{j\bullet} + r + 1\right)} (1 - u_r)^{\left(\sum\limits_{i=1}^{k} n_i - \sum\limits_{j=1}^{r} m_{j\bullet} - r\right)} du_r \\
&= C \left\{ \prod_{j=1}^{r-1} B\left(m_{1\bullet} + \cdots + m_{j\bullet} + j, m_{j+1\bullet} + 1\right) \right\} \\
&\quad \times B\left(\sum_{j=1}^{r} m_{j\bullet} + r, \sum_{i=1}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet} - r + 1 \right) \\
&= \frac{n_1!}{(n_1 - r)!} \left\{ \prod_{i=2}^{k} \binom{n_i}{m_{1i}, \ldots, m_{ri}, n_i - \sum_{j=1}^{r} m_{ji}} \right\} \\
&\quad \times \frac{\left(\sum\limits_{i=1}^{k} n_i - \sum\limits_{j=1}^{r} m_{j\bullet} - r \right)! m_{1\bullet}! \ldots m_{r\bullet}!}{\left(\sum\limits_{i=1}^{k} n_i \right)!} \\
&= \frac{1}{\binom{\sum_{i=1}^{k} n_i}{n_1, \ldots, n_k}} \left\{ \prod_{j=1}^{r} \binom{m_{j\bullet}}{m_{j2}, \ldots, m_{jk}} \right\} \\
&\quad \times \binom{\sum_{i=1}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet} - r}{n_1 - r, n_2 - \sum_{j=1}^{r} m_{j2}, \ldots, n_k - \sum_{j=1}^{r} m_{jk}}.
\end{aligned}
$$

Appendix B: Probability Mass Function of $(\boldsymbol{M}_2, \ldots, \boldsymbol{M}_k)$ Under the Lehmann Alternative

Under the Lehmann alternative H_1: $[F_k(x)]^{\gamma_k} = [F_{k-1}(x)]^{\gamma_{k-1}} = \cdots = [F_2(x)]^{\gamma_2} = F_1(x)$, for some $\gamma_i \in (0, 1)$, the expression in (2.25) can be expressed as follows:

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle| H_1 : F_k^{\gamma_k} = \cdots = F_2^{\gamma_2} = F_1\right) \\
&= C\gamma_k^r \int_{-\infty}^{\infty} \int_{-\infty}^{x_r} \cdots \int_{-\infty}^{x_2} \left\{ \prod_{i=2}^{k} [F_k(x_1)]^{m_{1i}\gamma_k/\gamma_i} \right\} \\
&\quad \times \left\{ \prod_{i=2}^{k} \prod_{j=2}^{r} [F_k^{\gamma_k/\gamma_i}(x_j) - F_k^{\gamma_k/\gamma_i}(x_{j-1})]^{m_{ji}} \right\} \\
&\quad \times \left\{ \prod_{i=2}^{k} [1 - F_i^{\gamma_k/\gamma_i}(x_r)]^{\left(n_i - \sum_{j=1}^{r} m_{ji}\right)} \right\} \left[\prod_{j=1}^{r} F_k^{\gamma_k - 1}(x_j) \right] \\
&\quad \times \left[\prod_{j=1}^{r} f_k(x_i) \right] [1 - F_k^{\gamma_k}(x_r)]^{n_1 - r} dx_1 \cdots dx_r. \qquad (2.26)
\end{aligned}
$$

In the special case when $\gamma_i = \gamma$ for $i = 2, \ldots, k$, the expression in (2.26) can be simplified as

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle| H_1 : F_k^{\gamma} = \cdots = F_2^{\gamma} = F_1\right) \\
&= C\gamma^r \int_{-\infty}^{\infty} \int_{-\infty}^{x_r} \cdots \int_{-\infty}^{x_2} [F_k(x_1)]^{m_{1\bullet} + \gamma - 1} \\
&\quad \times \left\{ \prod_{j=2}^{r} F_k^{\gamma - 1}(x_j) [F_k(x_j) - F_k(x_{j-1})]^{m_{j\bullet}} \right\} \\
&\quad \times [1 - F_k(x_r)]^{\left(\sum_{i=2}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet}\right)} \left[\prod_{j=1}^{r} f_k(x_i) \right] [1 - F_k^{\gamma}(x_r)]^{n_1 - r} dx_1 \cdots dx_r.
\end{aligned}
$$

Upon setting $u_i = F_k(x_i)$ for $i = 1, \ldots, r$, the above expression becomes

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k \middle| H_1 : F_k^{\gamma} = \cdots = F_2^{\gamma} = F_1\right) \\
&= C\gamma^r \int_0^1 \int_0^{u_r} \cdots \int_0^{u_2} u_1^{m_{1\bullet} + \gamma - 1} \left\{ \prod_{j=2}^{r} u_j^{\gamma - 1} (u_j - u_{j-1})^{m_{j\bullet}} \right\} \\
&\quad \times (1 - u_r)^{\left(\sum_{i=2}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet}\right)} (1 - u_r^{\gamma})^{n_1 - r} dx_1 \cdots dx_r.
\end{aligned}
$$

Adopting an approach similar to the one used in Appendix A, we obtain

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \ldots, \boldsymbol{M}_k = \boldsymbol{m}_k | H_1 : F_k^{\gamma} = \cdots = F_2^{\gamma} = F_1\right) \\
&= C\gamma^r \left\{ \prod_{j=1}^{r-1} B\left(m_{1\bullet} + \cdots + m_{j\bullet} + j\gamma, m_{j+1\bullet} + 1\right) \right\} \\
&\quad \times \int_0^1 u_r^{\left(\sum\limits_{j=1}^{r} m_{j\bullet} + r\gamma + 1\right)} (1-u_r)^{\left(\sum\limits_{i=2}^{k} n_i - \sum\limits_{j=1}^{r} m_{j\bullet}\right)} (1-u_r^{\gamma})^{n_1 - r} du_r \\
&= C\gamma^r \left\{ \prod_{j=1}^{r-1} B\left(m_{1\bullet} + \cdots + m_{j\bullet} + j\gamma, m_{j+1\bullet} + 1\right) \right\} \\
&\quad \times \left[\sum_{l=0}^{n_1 - r} \binom{n_1 - r}{l} (-1)^l \right. \\
&\qquad\qquad \left. \times \int_0^1 u_r^{\left(\sum\limits_{j=1}^{r} m_{j\bullet} + r\gamma + 1 + l\gamma\right)} (1-u_r)^{\left(\sum\limits_{i=2}^{k} n_i - \sum\limits_{j=1}^{r} m_{j\bullet}\right)} \right] du_r \\
&= C\gamma^r \left\{ \prod_{j=1}^{r-1} B\left(m_{1\bullet} + \cdots + m_{j\bullet} + j\gamma, m_{j+1\bullet} + 1\right) \right\} \\
&\quad \times \sum_{l=0}^{n_1 - r} \binom{n_1 - r}{l} (-1)^l B\left(\sum_{j=1}^{r} m_{j\bullet} + (r+l)\gamma, \sum_{i=2}^{k} n_i - \sum_{j=1}^{r} m_{j\bullet} + 1 \right).
\end{aligned}
$$

The exact distribution of $(\boldsymbol{M}_2, \ldots, \boldsymbol{M}_k)$, under the general Lehmann alternative $H_1 : [F_k(x)]^{\gamma_k} = [F_{k-1}(x)]^{\gamma_{k-1}} = \cdots = [F_2(x)]^{\gamma_2} = F_1(x)$, can be derived in a similar manner by expanding each term by the binomial formula, and the final expression would then involve multiple summation. For purposes of illustration, we present the result for $k = 3$. In this case, we have from Equation (2.26)

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \boldsymbol{M}_3 = \boldsymbol{m}_3 | H_1 : F_3^{\gamma_3} = F_2^{\gamma_2} = F_1\right) \\
&= C\gamma_3^r \int_{-\infty}^{\infty} \int_{-\infty}^{x_r} \cdots \int_{-\infty}^{x_2} [F_3(x_1)]^{m_{12}\gamma_3/\gamma_2} [F_3(x_1)]^{m_{13}} \\
&\quad \times \left\{ \prod_{j=2}^{r} [F_3^{\gamma_3/\gamma_2}(x_j) - F_3^{\gamma_3/\gamma_2}(x_{j-1})]^{m_{j2}} \right\} \left\{ \prod_{j=2}^{r} [F_3(x_j) - F_3(x_{j-1})]^{m_{j3}} \right\} \\
&\quad \times [1 - F_3^{\gamma_3/\gamma_2}(x_r)]^{\left(n_2 - \sum\limits_{j=1}^{r} m_{j2}\right)} [1 - F_3(x_r)]^{\left(n_3 - \sum\limits_{j=1}^{r} m_{j3}\right)} \\
&\quad \times \left\{ \prod_{j=1}^{r} [F_3(x_i)]^{\gamma_3 - 1} f_3(x_i) \right\} [1 - F_3^{\gamma_3}(x_r)]^{n_1 - r} dx_1 \cdots dx_r.
\end{aligned}
$$

Upon setting $u_i = F_3(x_i)$ for $i = 1, \ldots, r$, the preceding expression becomes

$$
\begin{aligned}
&\Pr\left(\boldsymbol{M}_2 = \boldsymbol{m}_2, \boldsymbol{M}_3 = \boldsymbol{m}_3 | H_1 : F_3^{\gamma_3} = F_2^{\gamma_2} = F_1\right)\\
&= C\gamma_3^r \int_0^1 \int_0^{u_r} \cdots \int_0^{u_2} u_1^{\left(\frac{m_{12}\gamma_3}{\gamma_2} + m_{13} + \gamma_3 - 1\right)}\\
&\quad \times \left\{ \prod_{j=2}^{r} u_j^{\gamma_3 - 1} \left(u_j^{\gamma_3/\gamma_2} - u_{j-1}^{\gamma_3/\gamma_2}\right)^{m_{j2}} (u_j - u_{j-1})^{m_{j3}} \right\}\\
&\quad \times \left(1 - u_r^{\gamma_3/\gamma_2}\right)^{\left(n_2 - \sum_{j=1}^{r} m_{j2}\right)} (1 - u_r)^{\left(n_3 - \sum_{j=1}^{r} m_{j3}\right)} du_1 \cdots du_r.
\end{aligned}
$$

The first integral with respect to u_1 can be expressed as

$$
\begin{aligned}
&\int_0^{u_2} u_1^{\left(m_{12}\frac{\gamma_3}{\gamma_2} + m_{13} + \gamma_3 - 1\right)} \left(u_2^{\gamma_3/\gamma_2} - u_1^{\gamma_3/\gamma_2}\right)^{m_{22}} (u_2 - u_1)^{m_{23}}\, du_1\\
&= \int_0^{u_2} u_1^{\left(\frac{m_{12}\gamma_3}{\gamma_2} + m_{13} + \gamma_3 - 1\right)}\\
&\qquad \times \left\{ \sum_{l_1=0}^{m_{22}} \binom{m_{22}}{l_2} (-1)^{l_1} u_2^{(m_{22} - l_1)\frac{\gamma_3}{\gamma_2}} u_1^{\left(\frac{l_1\gamma_3}{\gamma_2}\right)} \right\} (u_2 - u_1)^{m_{23}}\, du_1\\
&= u_2^{\left((m_{12} + m_{22})\frac{\gamma_3}{\gamma_2} + (m_{13} + m_{23}) + \gamma_3 - 1\right)}\\
&\quad \times \sum_{l_1=0}^{m_{22}} \binom{m_{22}}{l_2} (-1)^{l_1} B\left((m_{12} + l_1)\frac{\gamma_3}{\gamma_2} + m_{13} + \gamma_3, m_{23} + 1\right).
\end{aligned}
$$

Similarly, the j-th integral with respect to u_j $(j = 2, \ldots, r-1)$ becomes

$$
\begin{aligned}
&u_{j+1}^{\left((m_{12} + \ldots + m_{(j+1)2})\frac{\gamma_3}{\gamma_2} + (m_{13} + \ldots + m_{(j+1)3}) + \gamma_3 - 1\right)}\\
&\times \sum_{l_j=0}^{m_{(j+1)2}} \binom{m_{(j+1)2}}{l_j} (-1)^{l_j}\\
&\times B\left((m_{12} + \cdots + m_{j2} + l_j)\frac{\gamma_3}{\gamma_2} + (m_{13} + \cdots + m_{j3})\gamma_3, m_{(j+1)3} + 1\right),
\end{aligned}
$$

while the last integral with respect to u_r becomes

$$\int_0^{u_r} u_r^{\left(\left(\sum_{j=1}^r m_{j2}\right)\frac{\gamma_3}{\gamma_2}+\left(\sum_{j=1}^r m_{j3}\right)+\gamma_3-1\right)} (1-u_r^{\gamma_3/\gamma_2})^{\left(n_2-\sum_{j=1}^r m_{j2}\right)}$$

$$\times (1-u_r)^{\left(n_3-\sum_{j=1}^r m_{j3}\right)} (1-u_r^{\gamma_3})^{n_1-r} du_r$$

$$= \sum_{l_r=0}^{n_2-\sum_{j=1}^r m_{j2}} \sum_{l=0}^{n_1-r} \binom{n_2-\sum_{j=1}^r m_{j2}}{l_r}\binom{n_1-r}{l}(-1)^{l_r+l}$$

$$\times \int_0^1 u_r^{\left(\left(\sum_{j=1}^r m_{j2}\right)\frac{\gamma_3}{\gamma_2}+\left(\sum_{j=1}^r m_{j3}\right)+\gamma_3-1+l_r\frac{\gamma_3}{\gamma_2}+l\gamma_3\right)} (1-u_r)^{n_3-\sum_{j=1}^r m_{j3}} du_r$$

$$= \sum_{l_r=0}^{n_2-\sum_{j=1}^r m_{j2}} \sum_{l=0}^{n_1-r} \binom{n_2-\sum_{j=1}^r m_{j2}}{l_r}\binom{n_1-r}{l}(-1)^{l_r+l}$$

$$\times B\left(\left(\sum_{j=1}^r m_{j2}+l_r\right)\frac{\gamma_3}{\gamma_2}+\left(\sum_{j=1}^r m_{j3}\right)+(l+1)\gamma_3, n_3-\sum_{j=1}^r m_{j3}+1\right).$$

Combining all these expressions, we finally obtain

$$\Pr\left(\boldsymbol{M}_2=\boldsymbol{m}_2, \boldsymbol{M}_3=\boldsymbol{m}_3 | H_1 : F_3^{\gamma_3}=F_2^{\gamma_2}=F_1\right)$$

$$= C\gamma_3^r \sum_{l_1=0}^{m_{22}} \cdots \sum_{l_{r-1}=0}^{m_{r2}} \sum_{l_r=0}^{n_2-\sum_{j=1}^r m_{j2}} \sum_{l=0}^{n_1-r} \left\{\prod_{j=2}^r \binom{m_{j2}}{l_{j-1}}\right\}$$

$$\times \binom{n_2-\sum_{j=1}^r m_{j2}}{l_r}\binom{n_1-r}{l}(-1)^{\left(\sum_{j=1}^r l_j+l\right)}$$

$$\times \left\{\prod_{j=2}^r B\left(\left(\sum_{l^*=1}^j m_{l^*2}+l_j\right)\frac{\gamma_3}{\gamma_2}+\left(\sum_{l^*=1}^j m_{l^*3}\right)\gamma_3, m_{(j+1)3}+1\right)\right\}$$

$$\times B\left(\left(\sum_{j=1}^r m_{j2}+l_r\right)\frac{\gamma_3}{\gamma_2}+\left(\sum_{j=1}^r m_{j3}\right)+(l+1)\gamma_3, n_3-\sum_{j=1}^r m_{j3}+1\right).$$

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