
Effective Batalin–Vilkovisky Theories, Equivariant Configuration Spaces and Cyclic Chains

Alberto S. Cattaneo^{1,*} and Giovanni Felder²

¹ Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstrasse 190,
8057 Zürich, Switzerland, alberto.cattaneo@math.uzh.ch

² Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland,
giovanni.felder@math.ethz.ch

Dedicated to Murray Gerstenhaber and Jim Stasheff

Abstract. Kontsevich’s formality theorem states that the differential graded Lie algebra of multidifferential operators on a manifold M is L_∞ -quasi-isomorphic to its cohomology. The construction of the L_∞ -map is given in terms of integrals of differential forms on configuration spaces of points in the upper half-plane. Here we consider configuration spaces of points in the disk and work equivariantly with respect to the rotation group. This leads to considering the differential graded Lie algebra of multivector fields endowed with a divergence operator. In the case of \mathbb{R}^d with standard volume form, we obtain an L_∞ -morphism of modules over this differential graded Lie algebra from cyclic chains of the algebra of functions to multivector fields. As a first application we give a construction of traces on algebras of functions with star-products associated with unimodular Poisson structures. The construction is based on the Batalin–Vilkovisky quantization of the Poisson sigma model on the disk and in particular on the treatment of its zero modes.

Key words: Poisson sigma model, BV quantization, Effective action, L_∞ algebras, Cyclic homology, Unimodular Poisson structures

Mathematics Subject Classification 2010: Primary 53D55; Secondary 81T40, 81T45, 81T70

1 Introduction

The Hochschild complex of any algebra with unit carries a differential graded Lie algebra structure introduced by Gerstenhaber [14]. In the case of the algebra of smooth functions on a manifold, one has a differential graded Lie subalgebra \mathfrak{g}_G of multidifferential operators, whose cohomology is the

graded Lie algebra \mathfrak{g}_S of multivector fields with Schouten–Nijenhuis bracket.¹ Kontsevich [17] showed that \mathfrak{g}_G and \mathfrak{g}_S are quasi-isomorphic as L_∞ -algebras, a notion introduced by Stasheff as the Lie version of A_∞ -algebras [28], see [24, 20]. A striking application of this result is the classification of formal associative deformations of the product of functions in terms of Poisson structures. Kontsevich’s L_∞ -quasi-isomorphism is given in terms of integrals over configuration spaces of points in the upper half-plane. As shown in [3], these are Feynman amplitudes of a topological quantum field theory known as the Poisson sigma model [16, 23].

In this paper we consider the case of a manifold M endowed with a volume form Ω . In this case \mathfrak{g}_S comes with a differential, the divergence operator $\operatorname{div}_\Omega$ of degree -1 . One considers then the differential graded Lie algebra $\mathfrak{g}_S^\Omega = \mathfrak{g}_S[v]$ where v is an indeterminate of degree 2, the bracket is extended by v -linearity and the differential is $v \operatorname{div}_\Omega$. The relevant topological quantum field theory is a BF theory (or Poisson sigma model with trivial Poisson structure) on a disk whose differential is the Cartan differential on S^1 -equivariant differential forms. This theory is described in Section 2. The new feature, compared to the original setting of Kontsevich’s formality theorem, is that zero modes are present. We use recent ideas of Losev, Costello and Mnev to treat them in the Batalin–Vilkovisky quantization scheme. This gives the physical setting from which the Feynman amplitudes are derived. In the remaining sections of this paper, which can be read independently of Section 2, we give a purely mathematical treatment of the same objects. The basic result is the construction for $M = \mathbb{R}^d$ of an L_∞ -morphism of \mathfrak{g}_S^Ω -modules from the module of negative cyclic chains $(C_{-\bullet}(A)[u], b + uB)$ to the trivial module $(\Gamma(\wedge^{-\bullet} TM), \operatorname{div}_\Omega)$. We also check that this L_∞ -morphism has properties needed to extend the result to general manifolds.

As in the case of Kontsevich’s theorem, the coefficients of the L_∞ -morphism are integrals of differential forms on configuration spaces. Whereas Kontsevich considers the spaces of n -tuples of points in the upper half-plane modulo the action of the group of dilations and horizontal translations, we consider the space of n -tuples of points in the unit disk and work equivariantly with respect to the action of the rotation group. The quadratic identities defining the L_∞ -relations are then proved by means of an equivariant version of the Stokes theorem.

As a first application we construct traces in deformation quantization associated with unimodular Poisson structures. Our construction can also be extended to the case of supermanifolds; the trace is then replaced by a non-degenerate cyclic cocycle (Calabi–Yau structure, see [18], Section 10.2, and [10]) for the A_∞ -algebra obtained by deformation quantization in [5]. Further applications will be studied in a separate publication [6]. In particular we will derive the existence of an L_∞ -quasi-isomorphism of \mathfrak{g}_S^Ω -modules from

¹ We use Tsygan’s notation [30]. Kontsevich’s notation [17] is $\mathcal{D}_{\text{poly}} = \mathfrak{g}_G$, $\mathcal{T}_{\text{poly}} = \mathfrak{g}_S$.

the complex \mathfrak{g}_S^Ω with the adjoint action to the complex of cyclic cochains with a suitable module structure. This is a module version of the Kontsevich–Shoikhet formality conjecture for cyclic cochains [26].

Notations and conventions

All vector spaces are over \mathbb{R} . We denote by S_n the group of permutations of n letters and by $\epsilon: S_n \rightarrow \{\pm 1\}$ the sign character. We write $|\alpha|$ for the degree of a homogeneous element α of a \mathbb{Z} -graded vector space. The sign rules for tensor products of graded vector spaces hold: if f and g are linear maps on graded vector spaces, $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$. The graded vector space $V[n]$ is V shifted by n : $V[n]^i = V^{n+i}$. There is a canonical map (the identity) $s^n: V[n] \rightarrow V$ of degree n . The graded symmetric algebra $S^\bullet V = \bigoplus_{n \geq 0} S^n V$ of a graded vector space V is the algebra generated by V with relations $a \cdot b = (-1)^{|a||b|} b \cdot a$, $a, b \in V$; the degree of a product of generators is the sum of the degrees. If $\sigma \in S_n$ is a permutation and $a_1, \dots, a_n \in V$, then $a_{\sigma(1)} \cdots a_{\sigma(n)} = \epsilon a_1 \cdots a_n$; we call $\epsilon = \epsilon(\sigma; a_1, \dots, a_n)$ the Koszul sign of σ and a_j . The exterior algebra $\bigwedge V$ is defined by the relations $a \wedge b = -(-1)^{|a||b|} b \wedge a$ on generators. We have a linear isomorphism $S^n(V[1]) \rightarrow (\bigwedge^n V)[n]$ given by $v_1 \cdots v_n \mapsto s^{-n}(-1)^{\sum(n-j)(|v_j|-1)} sv_1 \wedge \cdots \wedge sv_n$, $v_j \in V[1]$.

2 BV formalism and zero modes

This section provides the interested reader with some “physical” motivation for the constructions in this paper. It may be safely skipped by the reader who is only interested in the construction and not in its motivation.

The basic idea is to use the Batalin–Vilkovisky (BV) formalism in order to deal with theories with symmetries (like the Poisson sigma model). What is interesting for this paper is the case when “zero modes” are present.

It is well known in algebraic topology that structures may be induced on subcomplexes (in particular, on an embedding of the cohomology) like, e.g., induced differentials in spectral sequences or Massey products. It is also well known in physics that low-energy effective field theories may be induced by integrating out high-energy degrees of freedom. As observed by Losev [21] (and further developed by Mnev [22] and Costello [9]), the two things are actually related in terms of the BV approach to (topological) field theories. We are interested in the limiting case when the low-energy fields are just the zero modes, i.e., the critical points of the action functional modulo its symmetries.

Let \mathcal{M} be an SP-manifold, i.e., a graded manifold endowed with a symplectic form of degree -1 and a compatible Berezinian [25]. Let Δ be the corresponding BV-Laplace operator. The compatibility amounts to saying that Δ squares to zero and that it generates the BV bracket $(\ , \)$ (i.e., the Poisson bracket of degree 1 determined by the symplectic structure of degree -1): namely,

$$\Delta(AB) = (\Delta A)B + (-1)^{|A|} A\Delta B - (-1)^{|A|}(A, B). \tag{1}$$

Assume now that \mathcal{M} is actually a product of SP-manifolds \mathcal{M}_1 and \mathcal{M}_2 , with BV-Laplace operators Δ_1 and Δ_2 , $\Delta = \Delta_1 + \Delta_2$. The central observation is that for every Lagrangian submanifold \mathcal{L} of \mathcal{M}_2 and any function F on \mathcal{M} – for which the integral makes sense – one has

$$\Delta_1 \int_{\mathcal{L}} F = \int_{\mathcal{L}} \Delta F. \tag{2}$$

In infinite dimensions, where we would really like to work, this formula is very formal as both the integration and Δ are ill-defined. In finite dimensions, on the other hand, this is just a simple generalization of the fact that, for any differential form α on the Cartesian product of two manifolds M_1 and M_2 and any closed submanifold S of M_2 on which the integral of α converges, we have

$$d \int_S \alpha = \pm \int_S d\alpha,$$

where integration on S yields a differential form on M_1 . The correspondence with the BV language is obtained by taking $\mathcal{M}_{1,2} := T^*[-1]M_{1,2}$ and $\mathcal{L} := N^*[-1]S$ (where N^* denotes the conormal bundle). The Berezinian on \mathcal{M} is determined by a volume form $v = v_1 \wedge v_2$ on $M := M_1 \times M_2$, with v_i a volume form on M_i . Finally, Δ is $\phi_v^{-1} \circ d \circ \phi_v$, with $\phi_v: \Gamma(\wedge^\bullet TM) \rightarrow \Omega^{\dim M - \bullet}(M)$, $X \mapsto \phi_v(X) := \iota_X v$. The generalization consists in the fact that there are Lagrangian submanifolds of \mathcal{M}_2 not of the form of a conormal bundle; however, by a result of Schwarz [25], they can always be brought to this form by a symplectomorphism so that formula (2) holds in general.

In the application we have in mind, \mathcal{M}_2 (and so \mathcal{M}) is infinite-dimensional, but \mathcal{M}_1 is not. Thus, we have a well-defined BV-Laplace operator Δ_1 and try to make sense of Δ by imposing (2), following ideas of [21, 22] and, in particular, [9]. More precisely, we consider “BF-like” theories. Namely, let (\mathcal{V}, δ) and $(\tilde{\mathcal{V}}, \delta)$ be complexes with a nondegenerate pairing $\langle \ , \ \rangle$ of degree -1 which relates the two differentials:

$$\langle B, \delta A \rangle = \langle \delta B, A \rangle, \quad \forall A \in \mathcal{V}, B \in \tilde{\mathcal{V}}. \tag{3}$$

We set $\mathcal{M} = \mathcal{V} \oplus \tilde{\mathcal{V}}$ and define $S \in C^\infty(\mathcal{M})$ as

$$S(A, B) := \langle B, \delta A \rangle. \tag{4}$$

The pairing defines a symplectic structure of degree -1 on \mathcal{M} and the BV bracket with S is δ . In particular,

$$(S, S) = 0. \tag{5}$$

We denote by $\mathcal{H}(\tilde{\mathcal{H}})$ the δ -cohomology of $\mathcal{V}(\tilde{\mathcal{V}})$. Then we choose an embedding of $\mathcal{M}_1 := \mathcal{H} \oplus \tilde{\mathcal{H}}$ into \mathcal{M} and a complement \mathcal{M}_2 .

Example 1. Take $\mathcal{V} = \Omega(\Sigma)[1]$ and $\tilde{\mathcal{V}} = \Omega(\Sigma)[s-2]$, with Σ a closed, compact s -manifold, and $\delta = d$, the de Rham differential, on \mathcal{V} ; up to a sign, δ on $\tilde{\mathcal{V}}$ is also the de Rham differential if the pairing is defined by integration: $\langle \mathbf{B}, \mathbf{A} \rangle := \int_{\Sigma} \mathbf{B} \wedge \mathbf{A}$, $\mathbf{A} \in \mathcal{V}$, $\mathbf{B} \in \tilde{\mathcal{V}}$. In this case $\mathcal{M}_1 = H(\Sigma)[1] \oplus H(\Sigma)[s-2]$, with $H(\Sigma)$ the usual de Rham cohomology. A slightly more general situation occurs when Σ has a boundary; in this case, appropriate boundary conditions have to be chosen so that δ has an adjoint as in (3). Let $\partial\Sigma = \partial_1\Sigma \sqcup \partial_2\Sigma$ (each of the boundary components $\partial_{1,2}\Sigma$ may be empty). We then choose $\mathcal{V} = \Omega(\Sigma, \partial_1\Sigma)[1]$ and $\tilde{\mathcal{V}} = \Omega(\Sigma, \partial_2\Sigma)[s-2]$, where $\Omega(\Sigma, \partial_i\Sigma)$ denotes differential forms whose restrictions to $\partial_i\Sigma$ vanish. In this case, $\mathcal{M}_1 = H(\Sigma, \partial_1\Sigma)[1] \oplus H(\Sigma, \partial_2\Sigma)[s-2]$.

Example 2. Suppose that S^1 acts on Σ (and that the $\partial_i\Sigma$ s are invariant). Let \mathbf{v} denote the vector field on Σ generating the infinitesimal action. Let $\Omega_{S^1}(\Sigma, \partial\Sigma) := \Omega(\Sigma, \partial\Sigma)^{S^1}[u]$ denote the Cartan complex with differential $d_{S^1} = d - u\iota_{\mathbf{v}}$, where u is an indeterminate of degree 2. Then we may generalize Example 1 replacing $\Omega(\Sigma, \partial\Sigma)$ with $\Omega_{S^1}(\Sigma, \partial\Sigma)$.

Now suppose that \mathcal{H} (and so $\tilde{\mathcal{H}}$) is finite-dimensional, as in the examples above. In this case it is always possible to choose a BV-Laplacian Δ_1 on \mathcal{M}_1 . Once and for all we also choose a Lagrangian submanifold \mathcal{L} on which the infinite-dimensional integral makes sense in perturbation theory. Assuming $\Delta S = 0$, the first consequence of (2) and (5) is that the partition function

$$Z_0 = \int_{\mathcal{L}} e^{\frac{i}{\hbar}S}$$

is Δ_1 -closed. Actually, in the case at hand, Z_0 is constant on \mathcal{M}_1 .

For every functional \mathcal{O} on \mathcal{M} for which integration on \mathcal{L} makes sense, we define the expectation value

$$\langle \mathcal{O} \rangle_0 := \frac{\int_{\mathcal{L}} e^{\frac{i}{\hbar}S} \mathcal{O}}{Z_0}.$$

The second consequence of (2), and of the fact that Z_0 is constant on \mathcal{M}_1 , is the Ward identity

$$\Delta_1 \langle \mathcal{O} \rangle_0 = \left\langle \Delta \mathcal{O} - \frac{i}{\hbar} \delta \mathcal{O} \right\rangle_0, \tag{6}$$

where we have also used (1).

To interpret the Ward identity for $\mathcal{O} = \mathbf{B} \otimes \mathbf{A}$, we denote by $\{\theta^\mu\}$ a linear coordinate system on \mathcal{H} and by $\{\zeta_\mu\}$ a linear coordinate system on $\tilde{\mathcal{H}}$, such that their union is a Darboux system for the symplectic form on \mathcal{M}_1 with $\Delta_1 = \frac{\partial}{\partial \theta^\mu} \frac{\partial}{\partial \zeta_\mu}$. We next write $\mathbf{A} = \alpha_\mu \theta^\mu + \mathbf{a}$ and $\mathbf{B} = \beta^\mu \zeta_\mu + \mathbf{b}$ with $\mathbf{a} \oplus \mathbf{b} \in \mathcal{M}_2$. The left-hand side of the Ward identity is now simply $\Delta_1 \langle \mathbf{B} \otimes \mathbf{A} \rangle_0 = \sum_{\mu} (-1)^{|\beta^\mu|} \beta^\mu \otimes \alpha_\mu =: \phi$. On the assumption that the ill-defined BV-Laplacian Δ should be a second-order differential operator, the

first term $\langle \Delta(\mathbf{B} \otimes \mathbf{A}) \rangle_0$ on the right-hand side is ill-defined but constant on \mathcal{M}_1 ; we denote it by K . Since δ vanishes in cohomology and, as a differential operator, it can be extracted from the expectation value, (6) yields a constraint for the propagator

$$\omega := \frac{i}{\hbar} \langle \mathbf{b} \otimes \mathbf{a} \rangle_0; \tag{7}$$

namely,

$$\delta\omega = K - \phi.$$

From now on we assume that \mathcal{M} is defined in terms of differential forms as in Examples 1 and 2. In this case, ω is a distributional $(s - 1)$ -form on $\Sigma \times \Sigma$ while ϕ is a representative of the Poincaré dual of the diagonal D_Σ in $\Sigma \times \Sigma$. By the usual naive definition of Δ , K is equal to the delta distribution on D_Σ . Thus, the restriction of ω to the configuration space $C_2(\Sigma) := \Sigma \times \Sigma \setminus D_\Sigma$ is a smooth $(m - 1)$ -form satisfying $d\omega = \phi$. If Σ has a boundary, ω satisfies in addition the conditions $\iota_1^* \omega = \iota_2^* \omega = 0$ with ι_1 the inclusion of $\Sigma \times \partial_1 \Sigma$ into $\Sigma \times \Sigma$ and ι_2 the inclusion of $\partial_2 \Sigma \times \Sigma$ into $\Sigma \times \Sigma$. Denoting by $\pi_{1,2}$ the two projections $\Sigma \times \Sigma \rightarrow \Sigma$ and by $\pi_*^{1,2}$ the corresponding fiber-integrations, we may define $P: \Omega(\Sigma, \partial_1 \Sigma) \rightarrow \Omega(\Sigma, \partial_1 \Sigma)$ and $\tilde{P}: \Omega(\Sigma, \partial_2 \Sigma) \rightarrow \Omega(\Sigma, \partial_2 \Sigma)$ by $P(\sigma) = \pi_*^1(\omega \wedge \pi_1^* \sigma)$ and $\tilde{P}(\sigma) = \pi_*^2(\omega \wedge \pi_2^* \sigma)$. Then the equation for ω implies that P and \tilde{P} are parametrices for the complexes $\Omega(\Sigma, \partial_1 \Sigma)$ and $\Omega(\Sigma, \partial_2 \Sigma)$; namely, $dP + Pd = 1 - \varpi$ and $d\tilde{P} + \tilde{P}d = 1 - \tilde{\varpi}$, where ϖ and $\tilde{\varpi}$ denote the projections onto cohomology.

This characterization of the propagator of a “BF-like” theory also appears in [9]. Even though not justified in terms of the BV formalism, this choice of propagator was done before in [2] for Chern–Simons theory out of purely topological reasons, and later extended to BF theories in [7]. A propagator with these properties also appears in [13] for the Poisson sigma model on the interior of a polygon.

The quadratic action (4) is usually the starting point for a perturbative expansion. The first singularity that may occur comes from evaluating ω on D_Σ (“tadpole”). A mild form of renormalization consists in removing tadpoles or, in other words, in imposing that ω should vanish on D_Σ . By consistency, one has then to set K equal to the restriction of ϕ to D_Σ . In other words, one has to impose

$$\Delta(\mathbf{B}(x)\mathbf{A}(x)) = \psi(x) := \sum_{\mu} (-1)^{|\beta^\mu|} \beta^\mu(x) \alpha_\mu(x), \quad \forall x \in \Sigma. \tag{8}$$

Observe that ψ is a representative of the Euler class of Σ . By (1) and (8) one then obtains a well-defined version of Δ on the algebra $C^\infty(\mathcal{M})'$ generated by local functionals. This may be regarded as an asymptotic version (for the energy scale going to zero) of Costello’s regularized BV-Laplacian [9]. Actually,

Lemma 1. *$(C^\infty(\mathcal{M})', \Delta)$ is a BV algebra.*

We now restrict ourselves to the setting of the Poisson sigma model [16,23]. Namely, we assume Σ to be two-dimensional and take $\mathcal{V} = \Omega(\Sigma, \partial_1 \Sigma)[1] \otimes (\mathbb{R}^m)^*$ and $\tilde{\mathcal{V}} = \Omega(\Sigma, \partial_2 \Sigma) \otimes \mathbb{R}^m$. Here $(\mathbb{R}^m)^* \times \mathbb{R}^m$ is a local patch of the cotangent bundle of an m -dimensional target manifold M . (Whatever we say here and in the following may be globalized by taking \mathcal{M} to be the graded submanifold of $\text{Map}(T[1]\Sigma, T^*[1]M)$ defined by the given boundary conditions.) There is a Lie algebra morphism from the graded Lie algebra $\mathfrak{g}_S = \Gamma(\wedge^{\bullet+1} TM)$ of multivector fields on M to $C^\infty(\mathcal{M})'$ endowed with the BV bracket [4]: to $\gamma \in \Gamma(\wedge^k TM)$ it associates the local functional

$$S_\gamma = \int_\Sigma \gamma^{i_1, \dots, i_k}(\mathbf{B}) \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_k}.$$

Moreover, for $k > 0$, $(S, S_\gamma) = 0$. With the regularized version of the BV-Laplacian, we get

$$\Delta S_\gamma = \int_\Sigma \psi (\text{div}_\Omega \gamma)^{i_1, \dots, i_{k-1}}(\mathbf{B}) \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_{k-1}},$$

where div_Ω is the divergence with respect to the constant volume form Ω on \mathbb{R}^n . To account for this systematically, we introduce the differential graded Lie algebra $\mathfrak{g}_S^\Omega := \mathfrak{g}_S[v]$, where v is an indeterminate of degree two and the differential δ_Ω is defined as $v \text{div}_\Omega$ (and the Lie bracket is extended by v -linearity). To $\gamma \in \Gamma(\wedge^k TM)$ v^l we associate the local functional

$$S_\gamma = (-i\hbar)^l \int_\Sigma \psi^l \gamma^{i_1, \dots, i_k}(\mathbf{B}) \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_k}.$$

It is now not difficult to prove the following

Lemma 2. *The map $\gamma \mapsto S_\gamma$ is a morphism of differential graded Lie algebras from $(\mathfrak{g}_S^\Omega, [\ , \], \delta_\Omega)$ to $(C^\infty(\mathcal{M})', (\ , \), -i\hbar\Delta)$. Moreover, for every $\gamma \in \Gamma(\wedge^k TM)$ v^l with k or l strictly positive, we have $(S, S_\gamma) = 0$. If $\partial\Sigma = \emptyset$, the last statement holds also for $k = l = 0$.*

Observe that $\psi^2 = 0$ by dimensional reasons. However, in the generalization to the equivariant setting of Example 2, higher powers of ψ survive.

A first application of this formalism is the Poisson sigma model on Σ . If π is a Poisson bivector field (i.e., $\pi \in \Gamma(\wedge^2 TM)$, $[\pi, \pi] = 0$), then $S_\pi := S + S_\pi$ satisfies the master equation $(S_\pi, S_\pi) = 0$ but in general not the quantum master equation $\frac{1}{2}(S_\pi, S_\pi) + i\hbar\Delta S_\pi = 0$, which by (1) is equivalent to $\Delta e^{\frac{i}{\hbar} S_\pi} = 0$. Unless ψ is trivial² (which is, e.g., the case for Σ the upper half

² If the class of ψ is trivial, one may always choose bases in the embedded cohomologies so that $\psi = 0$. If one does not want to make this choice, one observes anyway that for $\psi = d\tau$ one has $-i\hbar\Delta S_\gamma = (S, S'_\gamma)$ with $S'_\gamma = (-i\hbar)^l \int_\Sigma \tau \psi^{l-1} \gamma^{i_1, \dots, i_k}(\mathbf{B}) \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_k}$, for $\gamma \in \Gamma(\wedge^k TM)$ v^l , $l > 0$. In the case at hand, one may then define a solution of the quantum master equation as $S + S_\pi + S'_\pi$.

plane, as in [3], or the torus), this actually happens only if π is divergence free. More generally, if π is unimodular [19], by definition we may find a function f such that $\text{div}_\Omega \pi = [\pi, f]$. So $\tilde{\pi} := \pi + \iota f$ is a Maurer–Cartan element in \mathfrak{g}_S^Ω (i.e., $\delta_\Omega \tilde{\pi} - \frac{1}{2}[\tilde{\pi}, \tilde{\pi}] = 0$). Hence $S_{\tilde{\pi}} := S + S_{\tilde{\pi}}$ satisfies the quantum master equation. It is not difficult to check that, for ψ nontrivial, the unimodularity of π is a necessary and sufficient condition for having a solution of the quantum master equation of the form $S + S_\pi + O(\hbar)$. For Σ the sphere this was already observed in [1] though using slightly different arguments.

We will now restrict ourselves to the case of interest for the rest of the paper: namely, Σ the disk and $\partial_2 \Sigma = \emptyset$. In this case $H(\Sigma)$ is one-dimensional and concentrated in degree 0 while $H(\Sigma, \partial \Sigma)$ is one-dimensional and concentrated in degree two. Thus, $\mathcal{H} = (\mathbb{R}^m)^*[-1]$ and $\mathcal{H} = \mathbb{R}^m$ which implies $\mathcal{M}_1 = T^*[-1]M$. Functions on \mathcal{M}_1 are then multivector fields on M but with reversed degree and the operator Δ_1 turns out to be the usual divergence operator div_Ω (which is now of degree +1) for the constant volume form. A first simple application is the expectation value

$$\text{tr } g := \frac{\int_{\mathcal{L}} e^{\frac{i}{\hbar} S_{\tilde{\pi}}} \mathcal{O}_g}{Z_0} = \left\langle e^{\frac{i}{\hbar} S_{\tilde{\pi}}} \mathcal{O}_g \right\rangle_0, \quad g \in C^\infty(M),$$

where $\tilde{\pi}$ is a Maurer–Cartan element corresponding to a unimodular Poisson structure and $\mathcal{O}_g(\mathbf{A}, \mathbf{B}) := g(\mathbf{B}(1))$, with 1 in $\partial \Sigma$ which we identify with the unit circle. Consider now $\text{tr}_2(g, h) := \left\langle e^{\frac{i}{\hbar} S_{\tilde{\pi}}} \mathcal{O}_{g,h} \right\rangle_0$, with $\mathcal{O}_{g,h} := g(\mathbf{B}(1)) \int_{\partial \Sigma \setminus \{1\}} h(\mathbf{B})$. By (1), we then have $\Delta_1 \text{tr}_2(g, h) = \left\langle e^{\frac{i}{\hbar} S_{\tilde{\pi}}} \delta \mathcal{O}_{g,h} \right\rangle_0$. Arguing as in [3], we see that the right-hand side corresponds to moving the two functions g and h close to each other (in the two possible ways) and by “bubbling” the disk around them; so we get

$$\Delta_1 \text{tr}_2(g, h) = \text{tr } g \star h - \text{tr } h \star g,$$

where \star is Kontsevich’s star product [17] which corresponds to the Poisson sigma model on the upper half plane [3]. Since Δ_1 is just the divergence operator with respect to the constant volume form Ω , for compactly supported functions we have the trace

$$\text{Tr } g := \int_M \text{tr } g \Omega.$$

More generally, we may work out the Ward identities relative to the quadratic action (4) (there is also an equivariant version for S^1 acting by rotations on Σ). Given a_0, a_1, \dots, a_p in $C^\infty(M)$ (or in $C^\infty(M)[u]$ for the equivariant version), we define

$$\mathcal{O}_{a_0, \dots, a_p} := a_0(\mathbf{B}(1)) \int_{t_1 < t_2 < \dots < t_p \in \partial \Sigma \setminus \{1\}} a_1(\mathbf{B}) \cdots a_p(\mathbf{B})$$

and

$$G_n(\gamma_1, \dots, \gamma_n; a_0, \dots, a_p) := \langle S_{\gamma_1} \dots S_{\gamma_n} \mathcal{O}_{a_0, \dots, a_p} \rangle_0,$$

$\gamma_i \in \mathfrak{g}_S^Q, i = 1, \dots, n$. By (6) we then have

$$-i\hbar \Delta_1 G_n(\gamma_1, \dots, \gamma_n; a_0, \dots, a_p) = -i\hbar \langle \Delta(S_{\gamma_1} \dots S_{\gamma_n} \mathcal{O}_{a_0, \dots, a_p}) \rangle_0 + \langle \delta(S_{\gamma_1} \dots S_{\gamma_n} \mathcal{O}_{a_0, \dots, a_p}) \rangle_0.$$

The left-hand side is just $(-i\hbar)$ times the divergence operator applied to the multivector field G_n . The first term on the right-hand side may then be computed as

$$\begin{aligned} & -i\hbar \langle \Delta(S_{\gamma_1} \dots S_{\gamma_n} \mathcal{O}_{a_0, \dots, a_p}) \rangle_0 = \\ & = \sum_{i=1}^n (-1)^{\sigma_i} G_n(\gamma_1, \dots, \delta_\Omega \gamma_i, \dots, \gamma_n; a_0, \dots, a_p) + \\ & -i\hbar \sum_{1 \leq i < j \leq n} (-1)^{\sigma_{ij}} G_{n-1}([\gamma_i, \gamma_j], \gamma_1, \dots, \hat{\gamma}_i, \dots, \hat{\gamma}_j, \dots, \gamma_n; a_0, \dots, a_p), \end{aligned}$$

where the caret denotes omission and

$$\begin{aligned} \sigma_i & := \sum_{c=1}^{i-1} |\gamma_c|, \\ \sigma_{ij} & := |\gamma_i| \sum_{c=1}^{i-1} |\gamma_c| + |\gamma_j| \sum_{c=1, c \neq i}^{j-1} |\gamma_c| + |\gamma_i| + 1, \end{aligned}$$

with $|\gamma| = k$ for $\gamma \in \Gamma(\wedge^k TM)[v]$. The second term on the right-hand side is a boundary contribution (in the equivariant sense if $\delta = d_{S^1} = d - u\iota_\nu$). By bubbling as in [3], some of the γ_i s collapse together with some of the consecutive a_k s and the result – which is Kontsevich’s formality map – is put back into G . The whole formula can then be interpreted as an L_∞ -morphism from the cyclic Hochschild complex to the complex of multivector fields regarded as L_∞ -modules over \mathfrak{g}_S^Q , as we are going to explain in the rest of the paper.

The only final remark is that $i\hbar$ occurs in this formula only as a book-keeping device. We define F_n by formally setting $i\hbar = 1$ in G_n .

3 Hochschild chains and cochains of algebras of smooth functions

Kontsevich’s theorem states that there is an L_∞ -quasi-isomorphism from the graded Lie algebra $\mathfrak{g}_S = \Gamma(\wedge^{\bullet+1} TM)$ of multivector fields on a smooth manifold M , with the Schouten–Nijenhuis bracket and trivial differential, to the differential graded Lie algebra \mathfrak{g}_G of multidifferential operators on M with Gerstenhaber bracket and Hochschild differential. Through Kontsevich’s morphism the Hochschild and cyclic chains become a module over \mathfrak{g}_S . In this section we review these notions as well as results and conjectures about them.

3.1 Multivector fields and multidifferential operators

Let \mathfrak{g}_S be the graded vector space $\mathfrak{g}_S = \bigoplus_{j \geq -1} \mathfrak{g}_S^j$ of multivector fields: $\mathfrak{g}_S^{-1} = C^\infty(M)$, $\mathfrak{g}_S^0 = \Gamma(TM)$, $\mathfrak{g}_S^1 = \Gamma(\wedge^2 TM)$, and so on. The Schouten–Nijenhuis bracket of multivector fields is defined to be the usual Lie bracket on vector fields and is extended to arbitrary multivector field by the Leibniz rule: $[\alpha \wedge \beta, \gamma] = \alpha \wedge [\beta, \gamma] + (-1)^{|\gamma| \cdot (|\beta|+1)} [\alpha, \gamma] \wedge \beta$, $\alpha, \beta, \gamma \in \mathfrak{g}_S$. The graded Lie algebra \mathfrak{g}_S is considered here as a differential graded Lie algebra with trivial differential.

The differential graded Lie algebra \mathfrak{g}_G of multidifferential operators is, as a complex, the subcomplex of the shifted Hochschild complex $\text{Hom}(A^{\otimes(\bullet+1)}, A)$ of the algebra $A = C^\infty(M)$ of smooth functions, consisting of multilinear maps that are differential operators in each argument. The Gerstenhaber bracket [14] on \mathfrak{g}_G is the graded Lie bracket $[\phi, \psi] = \phi \bullet_G \psi - (-1)^{|\phi| \cdot |\psi|} \psi \bullet_G \phi$ with Gerstenhaber product³

$$\phi \bullet_G \psi = \sum_{k=0}^n (-1)^{|\psi|(|\phi|-k)} \phi \circ (\text{id}^{\otimes k} \otimes \psi \otimes \text{id}^{\otimes |\phi|-k}). \quad (9)$$

The Hochschild differential can be written in terms of the bracket as $[\mu, \cdot]$, where $\mu \in \mathfrak{g}_G^1 = \text{Hom}(A \otimes A, A)$ is the multiplication in A .

The Hochschild–Kostant–Rosenberg map $\mathfrak{g}_S^\bullet \rightarrow \mathfrak{g}_G^\bullet$ induces an isomorphism of graded Lie algebras on cohomology. It is the identity on $\mathfrak{g}_S^{-1} = C^\infty(M) = \mathfrak{g}_G^{-1}$ and, for any vector fields ξ_1, \dots, ξ_n , it sends the multivector field $\xi_1 \wedge \dots \wedge \xi_n$ to the multidifferential operator

$$f_1 \otimes \dots \otimes f_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \xi_{\sigma(1)}(f_1) \dots \xi_{\sigma(n)}(f_n), \quad f_i \in A.$$

Although the HKR map is a chain map inducing a Lie algebra isomorphism on cohomology, it does not respect the Lie bracket at the level of complexes. The correct point of view on this problem was provided by Kontsevich in his formality conjecture, which he then proved in [17]. The differential graded Lie algebras \mathfrak{g}_S , \mathfrak{g}_G should be considered as L_∞ -algebras and the HKR map is the first component of an L_∞ -morphism. Let us recall the definitions.

3.2 L_∞ -algebras

For any graded vector space V let $S^+V = \bigoplus_{j=1}^\infty S^j V$ be the free coalgebra without counit cogenerated by V . The coproduct is $\Delta(a_1 \dots a_n) = \sum_{p=1}^{n-1} \sum_{\sigma} \pm a_{\sigma(1)} \dots a_{\sigma(p)} \otimes a_{\sigma(p+1)} \dots a_{\sigma(n)}$, with summation over shuffle permutations with Koszul signs. A coderivation of a coalgebra is an

³ The sign differs by a factor $(-1)^{|\phi| \cdot |\psi|}$ from the sign in [14]. We have chosen the convention making the induced bracket on cohomology equal to the standard Schouten–Nijenhuis bracket on multivector fields.

endomorphism D obeying $\Delta \circ D = (D \otimes \text{id} + \text{id} \otimes D) \circ \Delta$. Coderivations with the commutator bracket form a Lie algebra. What is special about the free coalgebra S^+V is that for any linear map $D: S^+V \rightarrow V$ there is a unique coderivation \tilde{D} such that $D = \pi \circ \tilde{D}$, where π is the projection onto $V = S^1V$. By definition an L_∞ -algebra is a graded vector space \mathfrak{g} together with a coderivation D of degree 1 of $S^+(\mathfrak{g}[1])$ obeying $[D, D] = 0$. A coderivation is thus given by a sequence of maps (the Taylor components) $D_n: S^n\mathfrak{g}[1] \rightarrow \mathfrak{g}[2]$ (or $\wedge^n\mathfrak{g} \rightarrow \mathfrak{g}[2-n]$), $n = 1, 2, \dots$, obeying quadratic relations. In particular D_1 is a differential and D_2 is a chain map obeying the Jacobi identity up to a homotopy D_3 . It follows that D_2 induces a Lie bracket on the D_1 -cohomology. Differential graded Lie algebras are L_∞ -algebras with $D_3 = D_4 = \dots = 0$. An L_∞ -morphism $(\mathfrak{g}, D) \rightsquigarrow (\mathfrak{g}', D')$ is a homomorphism $U: S^+\mathfrak{g}[1] \rightarrow S^+\mathfrak{g}'[1]$ of graded coalgebras such that $U \circ D = D' \circ U$. Homomorphisms of free coalgebras are uniquely defined by their composition with the projection $\pi': S^+\mathfrak{g}'[1] \rightarrow \mathfrak{g}'[1]$; thus U is uniquely determined by its Taylor components $U_n: S^n\mathfrak{g}[1] \rightarrow \mathfrak{g}'[1]$ (or $\wedge^n\mathfrak{g} \rightarrow \mathfrak{g}'[1-n]$): U_n is the restriction to $S^n\mathfrak{g}[1]$ of $\pi' \circ U$. Conversely, any such sequence U_n comes from a coalgebra homomorphism. The first relation between D, D' and U is that U_1 is a chain map.

Theorem 1. (*Kontsevich [17]*) *There is an L_∞ -morphism $\mathfrak{g}_S(M) \rightsquigarrow \mathfrak{g}_G(M)$ whose first Taylor component U_1 is the Hochschild–Kostant–Rosenberg map.*

If M is an open subset of \mathbb{R}^d the formula for the Taylor components U_n is explicitly given in [17] as a sum over Feynman graphs.

3.3 Multivector fields and differential forms

The algebra $\Omega^\bullet(M)$ of differential forms on a manifold M is a module over the differential graded Lie algebra $\mathfrak{g}_S(M)$ of multivector fields: a multivector field $\gamma \in \Gamma(\wedge^{p+1}TM)$ acts on forms as $\mathcal{L}_\gamma\omega = d\iota_\gamma + (-1)^p\iota_\gamma d$ generalizing Cartan’s formula for Lie derivatives of vector fields. Here d is the de Rham differential and the interior multiplication ι_γ is the usual multiplication if γ is a function and is the composition of interior multiplications of vector fields ξ_j if $\gamma = \xi_1 \wedge \dots \wedge \xi_k$. Moreover the action of $\mathfrak{g}_S(M)$ on $\Omega^\bullet(M)$ commutes with the de Rham differential and induces the trivial action on cohomology.

3.4 Hochschild cochains and cyclic chains

The algebras $\Omega^\bullet(M)$ and $H^\bullet(M)$ are cohomologies of the complexes of the Hochschild and of the periodic cyclic chains of $C^\infty(M)$. The normalized Hochschild chain complex of a unital algebra A is $C_\bullet(A) = A \otimes \bar{A}^{\otimes \bullet}$, where $\bar{A} = A/\mathbb{R}1$. If we denote by (a_0, a_1, \dots, a_p) the class of $a_0 \otimes \dots \otimes a_p$ in $C_p(A)$, the Hochschild differential is

$$b(a_0, \dots, a_p) = \sum_{i=0}^{p-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p (a_p a_0, a_1, \dots, a_{p-1}).$$

We set $C_p(A) = 0$ for $p < 0$. There is an HKR map $C_\bullet(A) \rightarrow \Omega^\bullet(M)$ given by

$$(a_0, \dots, a_p) \mapsto \frac{1}{p!} a_0 da_1 \cdots da_p. \tag{10}$$

It is a chain map if we consider differential forms as a complex with *trivial* differential. The HKR map induces an isomorphism on homology, provided we take a suitable completion of the tensor product $C^\infty(M)^{\otimes(p+1)}$, for example the jets at the diagonal of smooth maps $M^{p+1} \rightarrow \mathbb{R}$. On the Hochschild chain complex there is a second differential B of degree 1 and anticommuting with b , see [8]:

$$B(a_0, \dots, a_p) = \sum_{i=0}^p (-1)^{ip} (1, a_i, \dots, a_p, a_0, \dots, a_{i-1}).$$

The negative cyclic complex, in the formulation of [15], is $CC^-_\bullet(A) = C^-_\bullet(A)[u]$ with differential $b + uB$, where u is of degree 2. The extension of the HKR map by $\mathbb{R}[u]$ -linearity defines a quasi-isomorphism

$$(CC^-_\bullet(A), b + uB) \rightarrow (\Omega^{-\bullet}(M)[u], u d).$$

Now both $C(A)$ and $CC^-(A)$ are differential graded modules over the Lie algebra \mathfrak{g}_G of multidifferential operators. The action is the restriction of the action of cochains on chains $C^k(A) \otimes C_p(A) \rightarrow C_{p-k+1}(A)$, $\phi \otimes a \mapsto \phi \cdot a$, defined for any associative algebra with unit as

$$\begin{aligned} & (-1)^{(k-1)(p+1)} \phi \cdot (a_0, \dots, a_p) \\ &= \sum_{i=0}^{p-k+1} (-1)^{i(k-1)} (a_0, \dots, a_{i-1}, \phi(a_i, \dots, a_{i+k-1}), a_{i+k}, \dots, a_p) \\ &+ \sum_{i=p-k+2}^p (-1)^{ip} (\phi(a_i, \dots, a_p, a_0, \dots, a_{i+k-p-2}), a_{i+k-p-1}, \dots, a_{i-1}). \end{aligned}$$

This action extends by $\mathbb{R}[u]$ -linearity to an action on the negative cyclic complex.

3.5 L_∞ -modules

Let (\mathfrak{g}, D) be an L_∞ -algebra. The free $S^+ \mathfrak{g}[1]$ -comodule generated by a vector space V is $\hat{V} = S\mathfrak{g}[1] \otimes V$ with coaction $\Delta^V: \hat{V} \rightarrow S^+ \mathfrak{g}[1] \otimes \hat{V}$ defined as

$$\Delta^V(\gamma_1 \cdots \gamma_n \otimes v) = \sum_{p=1}^n \sum_{\sigma \in S_{p, n-p}} \pm \gamma_{\sigma(1)} \cdots \gamma_{\sigma(p)} \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(n)} \otimes v).$$

A coderivation of the L_∞ -module V is then an endomorphism D^V of \hat{V} obeying $\Delta^V \circ D^V = (D \otimes \text{id} + \text{id} \otimes D^V)\Delta^V$. An L_∞ -module is a coderivation D^V of degree 1 of \hat{V} obeying $D^V \circ D^V = 0$. A coderivation is uniquely determined by its composition with the projection $\hat{V} \rightarrow V$ onto the first direct summand and is thus given by its Taylor components $D_n^V : S^n \mathfrak{g}[1] \otimes V \rightarrow V[1]$. The lowest component D_0^V is then a differential on V and D_1^V a chain map inducing an honest action of the Lie algebra $H(\mathfrak{g}, D_1)$ on the cohomology $H(V, D_0^V)$. A morphism of L_∞ -modules $V_1 \rightarrow V_2$ over \mathfrak{g} is a degree 0 morphism of $S^+ \mathfrak{g}[1]$ -comodules $F : \hat{V}_1 \rightarrow \hat{V}_2$ intertwining the coderivations. The composition with the projection $\hat{V}_2 \rightarrow V_2$ gives rise to Taylor components

$$F_n : S^n \mathfrak{g}[1] \otimes V_1 \rightarrow V_2, \quad n = 0, 1, 2, \dots$$

that determine F completely. The lowest component F_0 is then a chain map inducing a morphism of $H(\mathfrak{g}, D_1)$ -modules on cohomology.

3.6 Tsygan and Kontsevich conjectures [30], [26]

Conjecture 1. There exists a quasi-isomorphism of L_∞ -modules

$$F : C_{-\bullet}(C^\infty(M)) \rightsquigarrow (\Omega^{-\bullet}(M), 0)$$

such that F_0 is the HKR map.

Conjecture 2. There exists a natural $\mathbb{C}[[u]]$ -linear quasi-isomorphism of L_∞ -modules

$$F : CC_{-\bullet}(C^\infty(M)) \rightsquigarrow (\Omega^{-\bullet}(M)[[u]], ud)$$

such that F_0 is the Connes quasi-isomorphism [8], given by the u -linear extension of the HKR map (10).

Conjecture 1 is now a theorem. Different proofs for $M = \mathbb{R}^d$ were given in [29] and [27]. Shoikhet’s proof [27] gives an explicit formula for the Taylor components of F in terms of integrals over configuration spaces on the disk and extends to general manifolds, as shown in [11].

Let us turn to Kontsevich’s formality conjecture for cyclic cochains, as quoted in [26]. Recall that a volume form $\Omega \in \Omega^d(M)$ on a d -dimensional manifold defines an isomorphism $\Gamma(\wedge^k TM) \rightarrow \Omega^{d-k}(M)$, $\gamma \mapsto \iota_\gamma \Omega$. The de Rham differential d on $\Omega^\bullet(M)$ translates to a differential div_Ω , the divergence operator of degree -1 . The divergence operator is a derivation of the bracket on $\mathfrak{g}_S = \Gamma(\wedge^{\bullet+1} TM)$ of degree -1 . Let us introduce the differential graded Lie algebra $\mathfrak{g}_S^\Omega = (\mathfrak{g}_S[v], \delta_\Omega)$, where v is a formal variable of degree 2. The bracket is the Schouten–Nijenhuis bracket and the differential is $\delta_\Omega = v \text{div}_\Omega$. The cyclic analogue of \mathfrak{g}_G is the differential graded Lie algebra

$$\mathfrak{g}_G^{\text{cycl}} = \left\{ \varphi \in \mathfrak{g}_G, \int_M a_0 \varphi(a_1, \dots, a_p) \Omega = (-1)^p \int_M a_p \varphi(a_0, \dots, a_{p-1}) \Omega \right\}.$$

Conjecture 3. For each volume form $\Omega \in \Omega^d(M)$ there exists an L_∞ -quasi-isomorphism of L_∞ -algebras $F: \mathfrak{g}_S^\Omega \rightsquigarrow \mathfrak{g}_G^{\text{cycl}}$.

Shoikhet [26] constructed a quasi-isomorphism of complexes $C_1: \mathfrak{g}_S^\Omega \rightarrow \mathfrak{g}_G^{\text{cycl}}$ and conjectural formulae for an L_∞ -morphism whose first component is C_1 in terms of integrals over configuration spaces. One consequence of the conjecture is the construction of cyclically-invariant star-products from divergenceless Poisson bivector fields. Such star-products were then constructed independently of the conjecture, see [12].

4 An L_∞ -morphism for cyclic chains

4.1 The main results

Let Ω be volume form on a manifold M and \mathfrak{g}_S^Ω be the differential graded Lie algebra $\mathfrak{g}_S[v]$ with Schouten bracket and differential $\delta_\Omega = v \operatorname{div}_\Omega$, see Section 3.6. The Kontsevich L_∞ -morphism composed with the canonical projection $\mathfrak{g}_S^\Omega \rightarrow \mathfrak{g}_S = \mathfrak{g}_S^\Omega/v\mathfrak{g}_S^\Omega$ is an L_∞ -morphism $\mathfrak{g}_S^\Omega \rightsquigarrow \mathfrak{g}_G$. Through this morphism the differential graded \mathfrak{g}_G -module $CC_\bullet^-(A)$ of negative cyclic chains of $A = C^\infty(M)$ becomes an L_∞ -module over \mathfrak{g}_S^Ω .

Theorem 2. *Let M be an open subset of \mathbb{R}^d with coordinates x_1, \dots, x_n and volume form $\Omega = dx_1 \cdots dx_d$. Let $A = C^\infty(M)$. Let $\Gamma(\wedge^{-\bullet} TM)$ be the differential graded module over \mathfrak{g}_S^Ω with differential $\operatorname{div}_\Omega$ and trivial \mathfrak{g}_S^Ω -action. Then there exists an $\mathbb{R}[u]$ -linear morphism of L_∞ -modules over \mathfrak{g}_S^Ω*

$$F: CC_{-\bullet}^-(A) \rightsquigarrow \Gamma(\wedge^{-\bullet} TM)[u],$$

such that

- (i) *The component F_0 of F vanishes on $CC_p(A)$, $p > 0$ and for $f \in A \subset CC_0^-(A)$, $F_0(f) = f$.*
- (ii) *For $\gamma \in \Gamma(\wedge^k TM)$, $\ell = 0, 1, 2, \dots$, $a = (a_0, \dots, a_p) \in CC_p^-(A)$,*

$$F_1(\gamma v^\ell; a) = \begin{cases} (-1)^p u^s \gamma \lrcorner H(a), & \text{if } k \geq p \text{ and } s = k + \ell - p - 1 \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\lrcorner: \Gamma(\wedge^k TM) \otimes \Omega^p(M) \rightarrow \Gamma(\wedge^{k-p} TM)$ is the contraction map and H is the HKR map (10).

- (iii) *The maps F_n are equivariant under linear coordinate transformations and $F_n(\gamma_1 \cdots \gamma_n; a) = \gamma_1 \wedge F_{n-1}(\gamma_2 \cdots \gamma_n; a)$ whenever $\gamma_1 = \sum (e_k^i x_k + d^i) \partial_i \in \mathfrak{g}_S \subset \mathfrak{g}_S^\Omega$ is an affine vector field and $\gamma_2, \dots, \gamma_n \in \mathfrak{g}_S^\Omega$.*

The proof of this Theorem is deferred to Section 6.3.

In explicit terms, F is given by a sequence of $\mathbb{R}[u]$ -linear maps $F_n: S^n \mathfrak{g}_S^{\Omega}[1] \otimes CC^-(A) \rightarrow \Gamma(\wedge^n TM)$, $\gamma \otimes a \mapsto F_n(\gamma; a)$, $n \geq 0$, obeying the following relations. For any $\gamma = \gamma_1 \cdots \gamma_n \in S^n \mathfrak{g}_S^{\Omega}[1]$, $a \in CC_p^-(A)$.

$$\begin{aligned}
 & F_n(\delta_{\Omega}\gamma; a) + (-1)^{|\gamma|+p} F_n(\gamma; (b + uB)a) \tag{11} \\
 & + \sum_{k=0}^{n-1} \sum_{\sigma \in S_{k, n-k}} (-1)^{|\gamma|-1} \epsilon(\sigma; \gamma) F_k(\gamma_{\sigma(1)} \cdots \gamma_{\sigma(k)}; U_{n-k}(\bar{\gamma}_{\sigma(k+1)} \cdots \bar{\gamma}_{\sigma(n)}) \cdot a) \\
 & + \sum_{i < j} \epsilon_{ij} F_{n-1}((-1)^{|\gamma_i|-1} [\gamma_i, \gamma_j] \cdot \gamma_1 \cdots \hat{\gamma}_i \cdots \hat{\gamma}_j \cdots \gamma_n; a) = \operatorname{div}_{\Omega} F_n(\gamma; a).
 \end{aligned}$$

Here $\bar{\gamma}_i$ denotes the projection of γ_i to $\mathfrak{g}_S[1] = \mathfrak{g}_S^{\Omega}[1]/v\mathfrak{g}_S^{\Omega}[1]$; $S_{p,q} \subset S_{p+q}$ is the set of (p, q) -shuffles and the signs $\epsilon(\sigma; \gamma), \epsilon_{ij}$ are the Koszul signs coming from the permutation of the $\gamma_i \in \mathfrak{g}_S[1]$; $|\gamma| = \sum_i |\gamma_i|$; the differential δ_{Ω} is extended to a degree 1 derivation of the algebra $S\mathfrak{g}_S^{\Omega}[1]$; the maps $U_k: S^k \mathfrak{g}_S[1] \rightarrow \mathfrak{g}_G[1]$ are the Taylor components of the Kontsevich L_{∞} -morphism of Theorem 1.

We give the explicit expressions of the maps F_n in Section 5. Before that we explore some consequences.

4.2 Maurer–Cartan elements

An element of degree 1 in \mathfrak{g}_S^{Ω} has the form $\tilde{\pi} = \pi + vh$ where π is a bivector field and h is a function. The Maurer–Cartan equation $\delta_{\Omega}\tilde{\pi} - \frac{1}{2}[\tilde{\pi}, \tilde{\pi}] = 0$ translates to

$$[\pi, \pi] = 0, \quad \operatorname{div}_{\Omega} \pi - [h, \pi] = 0.$$

Thus π is a Poisson bivector field whose divergence is a Hamiltonian vector field with Hamiltonian h . Such Poisson structures are called unimodular [19]. As explained in [17], Poisson bivector fields in $\epsilon\mathfrak{g}_S[[\epsilon]]$ are mapped to solution of the Maurer–Cartan equations in $\epsilon\mathfrak{g}_G[[\epsilon]]$, which are star-products, i.e., formal associative deformations of the pointwise product in $C^{\infty}(M)$:

$$f \star g = fg + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} U_n(\pi, \dots, \pi)(f \otimes g).$$

Here the function part of $\tilde{\pi}$ does not contribute as it is projected away in the L_{∞} -morphism $\mathfrak{g}_S^{\Omega} \rightsquigarrow \mathfrak{g}_G$.

If $\tilde{\pi} = \pi + vh$ is a Maurer–Cartan element in \mathfrak{g}_S^{Ω} then $\tilde{\pi}_{\epsilon} = \epsilon\pi + vh$ is a Maurer–Cartan element in $\mathfrak{g}_S^{\Omega}[[\epsilon]]$. The twist of F by $\tilde{\pi}$ then gives a chain map from the negative cyclic complex of the algebra $A_{\epsilon} = (C^{\infty}(M)[[\epsilon]], \star)$ to $\Gamma(\wedge TM)[u][[\epsilon]]$. In particular, we get a trace

$$f \mapsto \tau(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_M F_n(\tilde{\pi}_{\epsilon}, \dots, \tilde{\pi}_{\epsilon}; f)\Omega, \tag{12}$$

on the subalgebra of A_ϵ consisting of functions with compact support. Here there is a question of convergence since there are infinitely many terms contributing to each fixed power of ϵ . The point is that these infinitely many terms combine to exponential functions. More precisely we have the following result.

Proposition 1. *The trace (12) can be written as*

$$\tau(f) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \int_M H_n(\pi, h, f) e^h \Omega = \int_M f e^h \Omega + O(\epsilon)$$

where H_n is a differential polynomial in π, h, f .

The proof is based on the expression of F_n in terms of graphs. We postpone it to Section 5.5, after we introduce this formalism.

5 Feynman graph expansion of the L_∞ -morphism

In this section we construct the morphism of L_∞ -modules of Theorem 2. The Taylor components have the form

$$F_n(\gamma; a) = \sum_{\Gamma \in \mathcal{G}_{\mathbf{k}, m}} w_\Gamma F_\Gamma(\gamma; a).$$

Here $\gamma = \gamma_1 \cdots \gamma_n$, with $\gamma_i \in \Gamma(\wedge^{k_i} TM)[v]$, $\mathbf{k} = (k_1, \dots, k_n)$ and $a = (a_0, \dots, a_m) \in C_m(A)$. The sum is over a finite set $\mathcal{G}_{\mathbf{k}, m}$ of directed graphs with some additional structure. To each graph a weight $w_\Gamma \in \mathbb{R}[u]$, defined as an integral over a configuration space of points in the unit disk, is assigned.

We turn to the descriptions of the graphs and weights.

5.1 Graphs

Let $m, n \in \mathbb{Z}_{\geq 0}$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$. We consider directed graphs Γ with $n + m$ vertices with additional data obeying a set of rules. The data are a partition of the vertex set into three totally ordered subsets $V(\Gamma) = V_1(\Gamma) \sqcup V_2(\Gamma) \sqcup V_w(\Gamma)$, a total ordering of the edges originating at each vertex and the assignment of a nonnegative integer, the *degree*, to each vertex in $V_1(\Gamma)$. The rules are:

1. There are n vertices in $V_1(\Gamma)$. There are exactly k_i edges originating at the i th vertex of $V_1(\Gamma)$.
2. There are m vertices in $V_2(\Gamma)$. There are no edges originating at these vertices.
3. There is exactly one edge pointing at each vertex in $V_w(\Gamma)$ and no edge originating from it.

4. There are no edges starting and ending at the same vertex.
5. For each pair of vertices i, j there is at most one edge from i to j .

The last rule is superfluous, but since all graphs with multiple edges will have vanishing weight we may just as well exclude them from the start. This has the notational advantage that we may think of the edge set $E(\Gamma)$ as a subset of $V(\Gamma) \times V(\Gamma)$.

Two graphs are called equivalent if there is a graph isomorphism between them that respects the partition and the orderings. The set of equivalence classes is denoted $\mathcal{G}_{\mathbf{k},m}$.

The vertices in $V_1(\Gamma)$ are called vertices of the first type, those in $V_2(\Gamma)$ of the second type. The vertices in $V_b(\Gamma) = V_1(\Gamma) \cup V_2(\Gamma)$ are called black, those in $V_w(\Gamma)$ are called white. We denote by $E_b(\Gamma)$ the subset of $E(\Gamma)$ consisting of edges whose endpoints are black.

To each $\Gamma \in \mathcal{G}_{\mathbf{k},m}$ there corresponds a multivector field $F_\Gamma(\gamma; a)$ whose coefficients are differential polynomials in the components of γ_i, a_i . The rules are the same as in [17] except for the additional white vertices, representing uncontracted indices and the degrees d_i , that select the power d_i of v in γ_i . Let us consider for example the graph of Fig. 1 and suppose that the degrees of the two vertices of the first type are k and ℓ . The algebra of multivector fields on $M \subset \mathbb{R}^d$ is generated by $C^\infty(M)$ and anticommuting generators $\theta_\nu = \partial/\partial x_\nu$. Thus $\gamma \in \Gamma(\wedge^k TM)$ has the form

$$\gamma = \frac{1}{k!} \sum_{\nu_1, \dots, \nu_k} \gamma^{\nu_1 \dots \nu_k} \theta_{\nu_1} \dots \theta_{\nu_k}.$$

The components $\gamma^{\nu_1 \dots \nu_k} \in C^\infty(M)$ are skew-symmetric under permutation of the indices ν_i . The graph of Fig. 1, with the convention that the edges originating at each vertex are ordered counterclockwise, gives then

$$F_\Gamma(\gamma_1 v^k, \gamma_2 v^\ell; a_0, a_1, a_2) = \sum \gamma_1^{ij} \partial_j \gamma_2^{pqr} \partial_i a_0 \partial_p a_1 \partial_q a_2 \theta_r,$$

and is zero on other monomials in v .

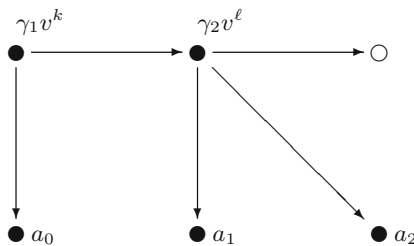


Fig. 1. A graph in $\mathcal{G}_{(2,3),3}$ with two vertices in V_1 of valencies $(2, 3)$, three in V_2 and one white vertex. The degrees of the vertices of the first type are k and ℓ

5.2 Equivariant differential forms on configuration spaces

Let Σ be a manifold with an action of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The infinitesimal action $\text{Lie}(S^1) = \mathbb{R} \frac{d}{dt} \rightarrow \Gamma(T\Sigma)$ is generated by a vector field $\mathbf{v} \in \Gamma(T\Sigma)$, the image of $\frac{d}{dt}$. The Cartan complex of S^1 -equivariant forms, computing the equivariant cohomology with real coefficients, is the differential graded algebra

$$\Omega_{S^1}^\bullet(\Sigma) = \Omega^\bullet(\Sigma)^{S^1}[u],$$

of polynomials in an undetermined u of degree 2 with coefficients in the S^1 -invariant smooth differential forms. The differential is $d_{S^1} = d - u\iota_{\mathbf{v}}$, where d is the de Rham differential and $\iota_{\mathbf{v}}$ denotes interior multiplication by \mathbf{v} , extended by $\mathbb{R}[u]$ -linearity. If Σ has an S^1 -invariant boundary $\partial\Sigma$ and $j: \partial\Sigma \rightarrow \Sigma$ denotes the inclusion map, then the relative equivariant complex is

$$\Omega_{S^1}^\bullet(\Sigma, \partial\Sigma) = \text{Ker}(j^*: \Omega_{S^1}^\bullet(\Sigma) \rightarrow \Omega_{S^1}^\bullet(\partial\Sigma)).$$

In the case of the unit disk we have:

Lemma 3. *Let $\bar{D} = \{z \in \mathbb{C}, |z| \leq 1\}$ be the closed unit disk.*

- (i) *The equivariant cohomology $H_{S^1}^\bullet(\bar{D})$ of \bar{D} is the free $\mathbb{R}[u]$ -module generated by the class of $1 \in \Omega^0(\bar{D})$.*
- (ii) *The relative equivariant cohomology $H_{S^1}^\bullet(\bar{D}, \partial\bar{D})$ of $(\bar{D}, \partial\bar{D})$ is the free $\mathbb{R}[u]$ -module generated by the class of*

$$\phi(z, u) = \frac{i}{2\pi} dz \wedge d\bar{z} + u(1 - |z|^2). \tag{13}$$

5.3 The propagator

The integrals over configuration spaces defining the L_∞ -morphism are constructed out of a propagator, a differential 1-form ω on $\bar{D} \times \bar{D} \setminus \Delta$, with a simple pole on the diagonal $\Delta = \{(z, z), z \in \bar{D}\}$ and defining the integral kernel of a homotopy contracting equivariant differential forms to a space of representatives of the cohomology. The explicit formula of the propagator associated to the choice of cocycles in Lemma 3 is given by

$$\omega(z, w) = \frac{1}{4\pi i} \left(d \ln \frac{(z-w)(1-z\bar{w})}{(\bar{z}-\bar{w})(1-\bar{z}w)} + z d\bar{z} - \bar{z} dz \right). \tag{14}$$

Lemma 4. *Let $p_i: \bar{D} \times \bar{D} \rightarrow \bar{D}$ be the projection to the i -th factor, $i = 1, 2$. The differential form $\omega \in \Omega_{S^1}^1(\bar{D} \times \bar{D} \setminus \Delta)$ has the following properties:*

- (i) *Let $j: \partial\bar{D} \times \bar{D} \rightarrow \bar{D} \times \bar{D}$ be the inclusion map. Then $j^*\omega = 0$.*
- (ii) *$d_{S^1}\omega = -p_1^*\phi$.*
- (iii) *As $z \rightarrow w$, $\omega(z, w) = (2\pi)^{-1} d \arg(z-w) + \text{smooth}$.*

(iv) As z and w approach a boundary point, $\omega(z, w)$ converges to the Kontsevich propagator $\omega_K(x, y) = (2\pi)^{-1}(\text{d arg}(x-y) - \text{d arg}(\bar{x}-y))$ on the upper half-plane H_+ from [17]. More precisely, for small $t > 0$ let $\varphi_t(x) = z_0 e^{itx}$ be the inclusion of a neighbourhood of $0 \in H_+$ into a neighbourhood of $z_0 \in \partial D$ in D . Then $\lim_{t \rightarrow 0} (\varphi_t \times \varphi_t)^* \omega = \omega_K$.

The proof is a simple computation left to the reader.

5.4 Weights

The weights are integrals of differential forms over configuration spaces $C_{n,m}^0(D)$ of n points in the unit disk $D = \{z \in \mathbb{C}, |z| < 1\}$ and $m + 1$ cyclically ordered points on its boundary $\partial \bar{D}$, the first of which is at 1:

$$C_{n,m}^0(D) = \{(z, t) \in D^n \times (\partial \bar{D})^m, z_i \neq z_j, (i \neq j), 0 < \arg(t_1) < \dots < \arg(t_m) < 2\pi\}.$$

The differential forms are obtained from the propagator ω , see (14), and the form ϕ , see (13). Let $\Gamma \in \mathcal{G}_{(k_1, \dots, k_n), m}$. The weight w_Γ of Γ is

$$w_\Gamma = \frac{1}{\prod_{i=1}^n k_i!} \int_{C_{n,m}^0(D)} \omega_\Gamma$$

where $\omega \in \Omega^\bullet(C_{n,m}^0(D))[u]$ is the differential form

$$\omega_\Gamma = \prod_{i \in V_1(\Gamma)} \prod_{(i,j) \in E_b(\Gamma)} \omega(z_i, z_j) \prod_{i \in V_1(\Gamma)} \phi(z_i, u)^{r_i}.$$

Here z_i is the coordinate of $z \in C_{n,m}^0(D)$ assigned to the vertex i of Γ : to the vertices of the first type we assign the points in the unit disk and to the vertices of the second type the points on the boundary. The assignment is uniquely specified by the ordering of the vertices in Γ . The number r_i is the degree of the vertex i plus the number of white vertices connected to it. The product over (i, j) is over the edges connecting black vertices to black vertices. For example, a point of $C_{2,3}^0(D)$ is given by coordinates $(z_1, z_2, 1, t_1, t_2)$ with $z_i \in D$ and $t_i \in S^1$. The differential form associated to the graph of Fig. 1, with degree assignments k, ℓ , is

$$\pm \omega_\Gamma = \omega(z_1, 1) \omega(z_1, z_2) \omega(z_2, t_1) \omega(z_2, t_2) \phi(z_1, u)^k \phi(z_2, u)^{\ell+1}.$$

The signs are tricky. A consistent set of signs may be obtained by the following procedure. View a multivector field $\gamma \in \mathfrak{g}_S[v]$ as a polynomial $\gamma(x, \theta, v)$ whose coefficients are functions on $T^*[1]M = M \times \mathbb{R}^d[1]$. Build a function in $C^\infty((T^*[1]M)^{n+m})[v_1, \dots, v_n]$:

$$\begin{aligned} &g(x^{(1)}, \theta^{(1)}, v_1, \dots, x^{(\bar{m})}) \\ &= \gamma_1(x^{(1)}, \theta^{(1)}, v_1) \cdots \gamma_n(x^{(n)}, \theta^{(n)}, v_n) a_0(x^{(\bar{0})}) \cdots a_m(x^{(\bar{m})}). \end{aligned}$$

Then

$$F_n(\gamma; a) = (-1)^{|\gamma| m} \int_{C_{n,m}^0(D)} i_{\Delta}^* \circ \exp(\Phi_n)(g)|_{v_1=\dots=v_n=0},$$

$$\Phi_n = \sum_{i \neq k} \omega(z_i, z_k) \sum_{\nu=1}^d \frac{\partial^2}{\partial \theta_{\nu}^{(i)} \partial x_{\nu}^{(k)}} + \sum_i \phi(z_i, u) \left(\sum_{\nu=1}^d \theta_{\nu} \frac{\partial}{\partial \theta_{\nu}^{(i)}} + \frac{\partial}{\partial v_i} \right).$$

The sums over i are from 1 to n and the sum over k is over the set $\{1, \dots, n, \bar{1}, \dots, \bar{m}\}$, with the understanding that $z_{\bar{j}} = t_j$. The map i_{Δ}^* is the restriction to the diagonal: its effect is to set all $x^{(i)}$ to be equal to x and all $\theta^{(i)}$ to be equal to θ . The integrand is then an element of the tensor product of graded commutative algebras $\Omega(C_{n,m}^0(D)) \otimes C^{\infty}(T^*[1]M)[u]$. The integral is defined as $\int(\alpha \otimes \gamma) = (\int \alpha)\gamma$ and the expansion of the exponential functions gives rise to a finite sum over graphs.

5.5 Proof of Proposition 1 on page 126

A vertex of a directed graph is called disconnected if there is no edge originating or ending at it.

Lemma 5. *Let \tilde{F}_n be defined as F_n except that the sum over graphs is restricted to the graphs without disconnected vertices of the first type. Then*

$$F_{k+n}((hv)^k \cdot \pi^n; f) = \sum_{s=0}^k \binom{k}{s} h^s \tilde{F}_{k-s+n}((hv)^{k-s} \cdot \pi^n; f).$$

Proof. For each fixed graph Γ_0 without disconnected vertices of the first type, we consider all graphs Γ contributing to F_{k+n} that reduce to Γ_0 after removing all disconnected vertices of the first type. The contribution to F_{k+n} of such a graph Γ is h^s times the contribution of Γ_0 , where s is the number of disconnected vertices of the first type. Indeed, each disconnected vertex in a graph Γ gives a factor h to F_{Γ} and a factor $\int_D \phi = 1$ to the weight w_{Γ} . The proof of the lemma is complete. \square

Let

$$H_n(\pi, h, f) = \sum_{r=0}^{\infty} \frac{1}{r!} \tilde{F}_{n+r}((hv)^r \cdot \pi^n; f).$$

In this sum there are finitely many terms since in the absence of disconnected vertices only derivatives of h can appear and the number of derivatives is bounded (by $2n$). Therefore $H_n(\pi, h, f)$ is a differential polynomial in π, h, f . We conclude that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} F_n(\hat{\pi}^n; f) &= \sum_{n,k=0}^{\infty} \frac{\epsilon^n}{k!n!} F_{k+n}((hv)^k \pi^n; f) \\ &= \sum_{n,r,s=0}^{\infty} \frac{\epsilon^n}{r!s!n!} h^s \tilde{F}_{n+r}((hv)^r \pi^n; f) = e^h \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n(\pi, h, f). \end{aligned}$$

This concludes the proof of Proposition 1.

6 Equivariant differential forms on configuration spaces and Stokes theorem

6.1 Configuration spaces and their compactifications

We consider three types of configuration spaces of points, the first two appearing in [17].

(i) *Configuration spaces of points in the plane.* Let $\text{Conf}_n(\mathbb{C}) = \{z \in \mathbb{C}^n, z_i \neq z_j, (i \neq j)\}$, $n \geq 2$. The three-dimensional real Lie group G_3 of affine transformations $w \mapsto aw + b, a > 0, b \in \mathbb{C}$ acts freely on the manifold $\text{Conf}_n(\mathbb{C})$. We set $C_n(\mathbb{C}) = \text{Conf}_n(\mathbb{C})/G_3$ ($n \geq 2$). It is a smooth manifold of dimension $2n - 3$. We fix the orientation defined by the volume form $d\varphi_2 \wedge \bigwedge_{j \geq 3} d\text{Re}(z_j) \wedge d\text{Im}(z_j)$, with the choice of representatives with $z_1 = 0, z_2 = e^{i\varphi_2}$.

(ii) *Configuration spaces of points in the upper half-plane.* Let $H_+ = \{z \in \mathbb{C}, \text{Im}(z) > 0\}$ be the upper half-plane. Let $\text{Conf}_{n,m}(H_+) = \{(z, x) \in H_+^n \times \mathbb{R}^m, z_i \neq z_j, (i \neq j), t_1 < \dots < t_m\}$, $2n + m \geq 2$. The two-dimensional real Lie group G_2 of affine transformations $w \mapsto aw + b, a > 0, b \in \mathbb{R}$ acts freely on the manifold $\text{Conf}_{n,m}(H_+)$. We set $C_{n,m}(H_+) = \text{Conf}_{n,m}(H_+)/G_2$ ($2n + m \geq 2$). It is a smooth manifold of dimension $2n + m - 2$. If $n \geq 1$, we fix the orientation by choosing representatives with $z_1 = i$ and taking the volume form $dt_1 \wedge \dots \wedge dt_m \wedge \bigwedge_{j \geq 2} d\text{Re}(z_j) \wedge d\text{Im}(z_j)$. If $m \geq 2$, we fix the orientation defined by the volume form $(-1)^m dt_2 \wedge \dots \wedge dt_{m-1} \wedge \bigwedge_{j \geq 1} d\text{Re}(z_j) \wedge d\text{Im}(z_j)$, with the choice of representatives with $t_1 = 0, t_m = 1$. If $m \geq 2$ and $n \geq 1$, it is easy to check that the two orientations coincide.

(iii) *Configuration spaces of points in the disk.* Let $D = \{z \in \mathbb{C}, |z| < 1\}$ be the unit disk, $S^1 = \partial D$ the unit circle. Let $C_{n,m+1}(D) = \{(z, x) \in D^n \times (S^1)^{m+1}, z_i \neq z_j, (i \neq j), \arg(t_0) < \dots < \arg(t_m) < \arg(t_0) + 2\pi\}$, $m \geq 0$. The circle group acts freely on $C_{n,m+1}(D)$ by rotations. We do not take a quotient here, since the differential forms we will introduce are not basic, and work equivariantly instead. Instead of the quotient we consider the section $C_{n,m}^0(D) = \{(z, x) \in C_{n,m+1}(D), t_0 = 1\}$, ($m \geq 1$). It is a smooth manifold of dimension $2n + m$. The orientation of $C_{n,m+1}(D)$ is defined by $d\arg(t_0) \wedge \dots \wedge d\arg(t_m) \wedge \bigwedge_{j=1}^n d\text{Re}(z_j) \wedge d\text{Im}(z_j)$. The orientation of $C_{n,m}^0(D)$ is defined by $d\arg(t_1) \wedge \dots \wedge d\arg(t_m) \wedge \bigwedge_{j=1}^n d\text{Re}(z_j) \wedge d\text{Im}(z_j)$.

As in [17], compactifications $\bar{C}_n(\mathbb{C}), \bar{C}_{n,m}(H_+), \bar{C}_{n,m+1}(D), \bar{C}_{n,m}^0(D)$ of these spaces as manifolds with corners are important. Their construction is the same as in [17]. Roughly speaking, one adds strata of codimension 1 corresponding to limiting configurations in which a group of points collapses to a point, possibly on the boundary, in such a way that within the group the relative position after rescaling remains fixed. Higher codimension strata correspond to collapses of several groups of points possibly within each other. The main point is that the Stokes theorem applies for smooth top differential forms on manifold with corners, and for this only codimension 1 strata are important.

Let us describe the codimension 1 strata of $C_{n,m}^0(D)$.

Strata of type I. These are strata where a subset A of $n' \geq 2$ out of n points z_i in the interior of the disk collapse at a point in the interior of the disk, the relative position of the collapsing points is described by a configuration on the plane and the remaining points and the point of collapse are given by a configuration on the disk. This stratum is thus

$$\partial_A \bar{C}_{n,m}^0(D) \simeq \bar{C}_{n'}(\mathbb{C}) \times \bar{C}_{n-n'+1,m}^0(D). \tag{15}$$

Strata of type II. These are strata where a subset A of n' out of n points z_i and a subset B of the m points t_i collapse at a point on the boundary of the disk ($2n' + m' \geq 2$). The relative position of the collapsing points is described by a configuration on the upper half-plane and the remaining points and the point of collapse are given by a configuration on the disk. This stratum is thus

$$\partial_{A,B} \bar{C}_{n,m}^0(D) \simeq \bar{C}_{n',m'}(H_+) \times \bar{C}_{n-n',m-m'+1}^0(D). \tag{16}$$

6.2 Forgetting the base point and cyclic shifts

Let $j_0: C_{n,m}^0(D) \rightarrow C_{n,m}(D)$ be the map $(z, 1, t_1, \dots, t_m) \mapsto (z, t_1, \dots, t_m)$ forgetting the base point $t_0 = 1$. It is an orientation preserving open embedding.

The cyclic shift $\lambda: C_{n,m}^0(D) \rightarrow C_{n,m}^0(D)$ is the map

$$\lambda: (z_1, \dots, z_n, 1, t_1, \dots, t_m) \mapsto (z_1, \dots, z_n, 1, t_m, t_1, \dots, t_{m-1}).$$

It is a diffeomorphism preserving the orientation if m is odd and reversing the orientation if m is even. The following fact is then easily checked.

Lemma 6. *The collection of maps $j_k = j_0 \circ \lambda^{ok}$, $k = 0, \dots, m - 1$, defines an embedding $j: C_{n,m}^0(D) \sqcup \dots \sqcup C_{n,m}^0(D) \rightarrow C_{n,m}(D)$ with dense image. The restriction of j to the k th copy of $C_{n,m}^0(D)$ multiplies the orientation by $(-1)^{(m-1)k}$.*

6.3 Proof of Theorem 2 on page 124

The proof uses the Stokes theorem as in [17]. The new features are: (i) the differential forms in the integrand are not closed and (ii) an equivariant version of the Stokes theorem is used.

We first compute the differential of the differential form associated to a graph Γ .

Lemma 7. *Let $\partial_e \Gamma$ be the graph obtained from Γ by adding a new white vertex $*$ and replacing the black-to-black edge $e \in E_b(\Gamma)$ by an edge originating at the same vertex as e but ending at $*$. Then*

$$d_{S^1} \omega_\Gamma = \sum_{e \in E_b(\Gamma)} (-1)^{\#e} \omega_{\partial_e \Gamma},$$

where $\#e = k$ if $e = e_k$ and e_1, \dots, e_N are the edges of Γ in the ordering specified by the ordering of the vertices and of the edges at each vertex.

Proof. This follows from the fact that d_{S^1} is a derivation of degree 1 of the algebra of equivariant forms and Lemma 4, (ii). \square

The next lemma is an equivariant version of the Stokes theorem.

Lemma 8. *Let $\omega \in \Omega_{S^1}^\bullet(\bar{C}_{n,m+1}(D))$. Denote also by ω its restriction to $\bar{C}_{n,m}^0(D) \subset \bar{C}_{n,m+1}(D)$ embedded as the subspace where $t_0 = 1$ and to the codimension 1 strata $\partial_i C_{n,m}^0(D)$ of $C_{n,m}^0(D)$. Then*

$$\int_{\bar{C}_{n,m}^0(D)} d_{S^1} \omega = \sum_i \int_{\partial_i \bar{C}_{n,m}^0(D)} \omega - u \int_{\bar{C}_{n,m+1}(D)} \omega.$$

Proof. Write $d_{S^1} = d - u \iota_\nu$. For $u = 0$ the claim is just the Stokes theorem for manifolds with corners. Let us compare the coefficients of u . The action map restricts to a diffeomorphism $f: S^1 \times \bar{C}_{n,m}^0(D) \rightarrow \bar{C}_{n,m+1}(D)$. Since ω is S^1 -invariant, $\iota_\nu \omega$ is also invariant and we have $f^* \omega = 1 \otimes \omega + dt \otimes \iota_\nu \omega \in \Omega(S^1) \otimes \Omega(\bar{C}_{n,m}^0(D)) \subset \Omega(S^1 \times \bar{C}_{n,m}^0(D))$, where t is the coordinate on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Thus

$$\int_{\bar{C}_{n,m+1}(D)} \omega = \int_{S^1 \times \bar{C}_{n,m}^0(D)} dt \otimes \iota_\nu \omega = \int_{\bar{C}_{n,m}^0(D)} \iota_\nu \omega. \quad \square$$

Finally we use Lemma 6 to reduce the integral over $\bar{C}_{n,m+1}(D)$ to integrals over $\bar{C}_{n,m+1}^0(D)$. We obtain:

Lemma 9. *Let $\omega \in \Omega_{S^1}^\bullet(\bar{C}_{n,m+1}(D))$ and let j_k be the maps defined in Lemma 6. Then*

$$\int_{\bar{C}_{n,m+1}(D)} \omega = \sum_{k=0}^m (-1)^{mk} \int_{\bar{C}_{n,m+1}^0(D)} j_k^* \omega.$$

We can now complete the proof of Theorem 2. We first prove the identity (11), starting from the right-hand side. Suppose that $a = (a_0, \dots, a_m) \in C_{-m}(A)$, $\gamma = \gamma_1 \cdots \gamma_n$, with $\gamma_i \in \Gamma(\wedge^{k_i} TM)$. It is convenient to identify $\Gamma(\wedge TM)$ with $C^\infty(M)[\theta_1, \dots, \theta_n]$ where θ_i are anticommuting variables, so that $\text{div}_\Omega = \sum \partial^2 / \partial t_i \partial \theta_i$. It follows that for any $F \in \mathcal{G}_{\mathbf{k},m}$, $\text{div}_\Omega F_\Gamma(\gamma; a)$ can be written as a sum (with signs) of terms $F_{\Gamma'}(\gamma; a)$, where Γ' is obtained from

Γ by identifying a white vertex with a black vertex and coloring it black. Some of these graphs Γ' have an edge connecting a vertex to itself and contribute to $F_n(\delta_\Omega \gamma; a)$. The remaining ones yield, in the notation of Lemma 7:

$$\operatorname{div}_\Omega F_n(\gamma; a) - F_n(\delta_\Omega \gamma; a) = \sum_{(\Gamma, e)} (-1)^{\sharp_e} w_{\partial_e \Gamma} F_\Gamma(\gamma; a).$$

The summation is over pairs (Γ, e) where $\Gamma \in \mathcal{G}_{\mathbf{k}, m}$ and $e \in E_b(\Gamma)$ is a black-to-black edge. By Lemmas 8 and 9,

$$\sum_{e \in E_b(\Gamma)} (-1)^{\sharp_e} w_{\partial_e \Gamma} = \sum_i \int_{\partial_i C_{n, m}^0(D)} \omega_\Gamma - u \sum_{k=0}^m (-1)^{km} \int_{\bar{C}_{n, m+1}^0} j_k^* \omega_\Gamma.$$

The second term on the right-hand side, containing the sum over cyclic permutations, gives rise to $F_{n+1}(\gamma; Ba)$. The first term is treated as in [17]: the strata of type I (see Section 6.1) give zero by Kontsevich’s lemma (see [17], Theorem 6.5) unless the number n' of collapsing interior points is 2. The sum over graphs contributes then to the term with the Schouten bracket $[\gamma_i, \gamma_j]$ in (11). The strata of type II such that $n - k > 0$ interior points approach the boundary give rise to the term containing the components of the Kontsevich L_∞ -morphism U_{n-k} . Finally the strata of type II in which only boundary points collapse give the term with Hochschild differential $F_{n-1}(\gamma; ba)$. This proves (11).

Property (i) is clear: F_0 is a sum over graphs with vertices of the second type only. These graphs have no edges. Thus the only case for which the weight does not vanish is when the configuration space is 0-dimensional, namely, when there is only one vertex. Property (ii) is checked by an explicit calculation of the weight. The only graphs with a nontrivial weight have edges connecting the vertex of the first type with white vertices or to vertices of the second type. There must be at least p edges otherwise the weight vanishes for dimensional reasons. In this case, i.e., if $k \geq p$, the integral computing the weight is

$$w_\Gamma = \int \phi(z, u)^{\ell+k-p} \omega(z, t_1) \cdots \omega(z, t_p), \tag{17}$$

with integration over $z \in D$, $t_i \in S^1$, $0 < \arg(t_1) < \cdots < \arg(t_p) < 2\pi$. The integral of the product of the 1-forms ω is a function of z that is independent of z , as is easily checked by differentiating with respect to z , using the Stokes theorem and the boundary conditions of ω . Thus it can be computed for $z = 0$. Since $\omega(0, t_i) = \frac{i}{2\pi} d \arg(t_i)$ the integral is $1/p!$. The remaining integral over z can then be performed. Set $\ell + k - p = s + 1$. This power must be positive otherwise the integral vanishes for dimensional reasons.

$$\int_D \phi(z, u)^{s+1} = \frac{i}{2\pi} (s+1) u^s \int_D (1 - |z|^2)^s dz \wedge d\bar{z} = u^s, \quad s \geq 0,$$

and we obtain $w_\Gamma = u^s/p!$. We turn to Property (iii). The equivariance under linear coordinate transformations is implicit in the construction. The graphs

contributing to $F_n(\gamma_1 \cdots ; a)$ for linear γ_1 are of two types: either the vertex associated with γ_1 has exactly one ingoing and one outgoing edge or it has an outgoing edge pointing to a white vertex and there are no incoming edges. The graphs of the second type contribute to $\gamma_1 \wedge F_{n-1}(\cdots ; a)$, since their weight factorize as $1 = \int_D \phi$ times the weight of the graphs obtained by omitting the vertex associated to γ_1 and the white vertex connected to it. The claim then follows from the following vanishing lemma.

Lemma 10. (i) For all $z, z' \in \bar{D}$, $\int_{w \in D} \omega(z, w)\omega(w, z') = 0$.
(ii) For all $z \in \bar{D}$, $\int_{w \in D} \omega(z, w)\phi(w, u) = 0$.

Proof. (i) We reduce the first claim to the second: consider the integral

$$I(z, z') = \int_{w_1, w_2 \in D} d(\omega(z, w_1)\omega(w_1, w_2)\omega(w_2, z')).$$

On the one hand, $I(z, z')$ can be evaluated by using Stokes's theorem, giving three terms all equal up to sign to the integral appearing in (i). On the other hand, the differential can be evaluated explicitly giving

$$I(z, z') = - \int_{w_1, w_2 \in D} \omega(z, w_1)\omega(w_1, w_2)\phi(w_2, 0).$$

The integral over w_2 vanishes if (ii) holds. The proof of (ii) is an elementary computation that uses the explicit expression of ω and ϕ . Alternatively, one shows that $\int_{w \in D} \omega(z, w)\phi(w, u)$ is a closed 1-form on the disk that vanishes on the boundary, is invariant under rotations and odd under diameter reflections. Therefore it vanishes. We leave the details to the reader. \square

Acknowledgments

We are grateful to Francesco Bonechi, Damien Calaque, Kevin Costello, Florian Schätz, Carlo Rossi, Jim Stasheff, Thomas Willwacher and Marco Zambon for useful comments. This work been partially supported by SNF Grants 20-113439 and 200020-105450, by the European Union through the FP6 Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652), and by the European Science Foundation through the MISGAM program. The first author is grateful to the Erwin Schrödinger Institute, where part of this work was done, for hospitality.

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