7

Bessel Sequences and Bases in Hilbert Spaces

In this chapter and the next we focus on bases and basis-like systems in Hilbert spaces. Our goal in this chapter is to understand bounded unconditional bases in Hilbert spaces, but in order to do this, we first need to study sequences that need not be bases but which do have a property that is reminiscent of Bessel's Inequality for orthonormal bases. These *Bessel sequences* will also be very useful to us in Chapter 8 when we consider *frames* in Hilbert spaces.

7.1 Bessel Sequences in Hilbert Spaces

Bessel sequences are defined as follows.

Definition 7.1 (Bessel Sequence). A sequence $\{x_n\}$ in a Hilbert space H is a *Bessel sequence* if

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n \rangle|^2 \;<\; \infty. \qquad \diamondsuit$$

Thus, if $\{x_n\}$ is a Bessel sequence, then the *analysis operator* C that takes an element x to the sequence of coefficients $Cx = \big(\langle x, x_n\rangle\big)$ maps H into ℓ^2. By applying either the Uniform Boundedness Principle or the Closed Graph Theorem, this mapping must be bounded. The next theorem, whose proof is Exercise 7.2, states several additional properties possessed by Bessel sequences (parts (a)–(c) of this exercise can also be derived by applying Exercise 3.8 with $X = H$ and $p = 2$).

Theorem 7.2. *Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H. If we define $Cx = \big(\langle x, x_n\rangle\big)$ for $x \in H$, then the following statements hold.*

(a) *C is a bounded mapping of H into ℓ^2, and therefore there exists a constant $B > 0$ such that*

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n \rangle|^2 \;\le\; B\,\|x\|^2. \tag{7.1}$$

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(b) *If* $(c_n) \in \ell^2$, *then the series* $\sum c_n x_n$ *converges unconditionally in* H, *and* $Rc = \sum c_n x_n$ *defines a bounded map of* ℓ^2 *into* H.

(c) $R = C^*$ *and* $\|R\| = \|C\| \le B^{1/2}$. *Consequently,*

$$\forall\, (c_n) \in \ell^2, \quad \left\|\sum_n c_n x_n\right\|^2 \;\le\; B \sum_n |c_n|^2.$$

(d) *If* $\{x_n\}$ *is complete, then* C *is injective and* range(R) *is dense in* H. ♢

Comparing equation (7.1) to Bessel's Inequality for orthonormal bases (Theorem 1.49), we see the motivation for the name "Bessel sequence." However, a Bessel sequence need not be orthonormal and need not be a basis (Exercise 7.1).

Definition 7.3. Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H.

(a) A constant B such that equation (7.1) holds is called a *Bessel bound* or an *upper frame bound* for $\{x_n\}$ (compare Definition 8.2). The smallest such constant B is called the *optimal Bessel bound.*

(b) The operator $C\colon H \to \ell^2$ defined in Theorem 7.2 is called the *analysis operator* or the *coefficient mapping* for $\{x_n\}$, and its adjoint $R\colon \ell^2 \to H$ is the *synthesis operator* or the *reconstruction operator* for $\{x_n\}$.

(c) The *frame operator* for $\{x_n\}$ is $S = RC\colon H \to H$.

(d) The *Gram operator* or *Gram matrix* for $\{x_n\}$ is $G = CR\colon \ell^2 \to \ell^2$. ♢

Note that the optimal Bessel bound is precisely $\|C\|^2$.

We will study *frames* in detail in Chapter 8. These are Bessel sequences which also possess a "lower frame bound" in the sense that there is a constant $A > 0$ such that $A\,\|x\|^2 \le \sum |\langle x, x_n\rangle|^2$ for $x \in H$. The synthesis operator for a frame is sometimes called the *pre-frame operator* (and this terminology is sometimes applied to generic Bessel sequences as well).

Since the analysis and synthesis operators associated to a Bessel sequence $\{x_n\}$ are bounded, the frame and Gram operators are bounded as well. Moreover, $S = C^*C = RR^*$ and $G = CC^* = R^*R$ are self-adjoint and positive in the sense of Definition 2.14. By definition,

$$Sx \;=\; RCx \;=\; \sum_n \langle x, x_n\rangle\, x_n, \qquad x \in H,$$

and therefore

$$\langle Sx, x\rangle \;=\; \sum_n |\langle x, x_n\rangle|^2. \tag{7.2}$$

In particular, an orthonormal basis is a Bessel sequence, and the frame operator for an orthonormal basis is $S = I$. However, there exist Bessel sequences whose frame operator is $S = I$ but which are neither orthonormal nor bases (see Exercise 7.1).

We have the following equivalent characterizations of Bessel sequences (see Exercise 7.3).

Theorem 7.4. *Let $\{x_n\}$ be a sequence in a Hilbert space H, and let $\{\delta_n\}$ be the standard basis for ℓ^2. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a Bessel sequence in H.*

(b) *There exists a constant $B > 0$ and a dense set $E \subseteq H$ such that*

$$\forall\, x \in E, \quad \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2.$$

(c) *There exists a constant $B > 0$ such that*

$$\forall\, N \in \mathbf{N}, \quad \forall\, c_1, \dots, c_N \in \mathbf{F}, \quad \Bigl\| \sum_{n=1}^{N} c_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2. \tag{7.3}$$

(d) *The series $\sum c_n x_n$ converges for each sequence $(c_n) \in \ell^2$.*

(e) *There exists a bounded operator $R\colon \ell^2 \to H$ such that $R\delta_n = x_n$ for each $n \in \mathbf{N}$.*

(f) *There exists an orthonormal sequence $\{e_n\}$ in H and a bounded operator $T \in \mathcal{B}(H)$ such that $Te_n = x_n$ for each $n \in \mathbf{N}$.*

Further, when these hold, the operator R appearing in part (e) *is the synthesis operator for $\{x_n\}$, and $\overline{\operatorname{span}}\{x_n\} = \overline{\operatorname{range}(R)}$.* $\diamondsuit$

Now we consider the Gram operator G associated with a Bessel sequence. Since G is a bounded mapping of ℓ^2 into itself, it can be represented as multiplication by an infinite matrix. We identify the Gram operator and the matrix that represents it. The form of this matrix is given in the next result, whose proof is Exercise 7.4.

Theorem 7.5. *Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H. Then the matrix for the Gram operator G is*

$$G \;=\; \bigl[\,\langle x_n, x_m\rangle\,\bigr]_{m,n\in\mathbf{N}}. \qquad \diamondsuit$$

That is, if we think of $c = (c_n) \in \ell^2$ as a column vector, then Gc is the product of the infinite matrix $\bigl[\,\langle x_n, x_m\rangle\,\bigr]_{m,n\in\mathbf{N}}$ with the vector $c = (c_n)$. The mth entry of Gc is $(Gc)_m = \sum_n c_n \langle x_n, x_m\rangle$.

We can extend the notion of a Gram matrix to sequences that are not Bessel. Given any sequence $\{x_n\}$ in a Hilbert space H, we call $G = \bigl[\,\langle x_n, x_m\rangle\,\bigr]_{m,n\in\mathbf{N}}$ the *Gram matrix* or the *Gramian* for $\{x_n\}$. However, it is important to note that this matrix need not define a bounded mapping on ℓ^2. In fact, the following converse to Theorem 7.5 shows that this happens exactly for Bessel sequences.

Theorem 7.6. *Let $\{x_n\}$ be a sequence in a Hilbert space H, and let G be its Gram matrix. If either:*

(a) *G is a bounded map of $\big(c_{00}, \|\cdot\|_{\ell^2}\big)$ into ℓ^2, i.e., there exists a constant $B > 0$ such that $\|Gc\|_{\ell^2} \le B\,\|c\|_{\ell^2}$ for all finite sequences c, or*

(b) *multiplication by G is a well-defined mapping of ℓ^2 into itself, i.e., for each $c = (c_n) \in \ell^2$ the series $(Gc)_m = \sum_n c_n \langle x_n, x_m\rangle$ converges for each $m \in \mathbf{N}$ and the sequence $Gc = \big((Gc)_m\big)_{m\in\mathbf{N}}$ belongs to ℓ^2,*

then $\{x_n\}$ is a Bessel sequence.

Proof. (a) Choose any finite sequence $c = (c_1, \dots, c_N, 0, 0, \dots) \in c_{00}$. Then

$$\begin{aligned}
\langle Gc, c\rangle &= \sum_{m=1}^{N} (Gc)_m\,\overline{c_m} \\
&= \sum_{m=1}^{N} \left(\sum_{n=1}^{N} \langle x_n, x_m\rangle\, c_n\right) \overline{c_m} \\
&= \sum_{m=1}^{N} \sum_{n=1}^{N} c_n\,\langle x_n, x_m\rangle\,\overline{c_m} \\
&= \left\langle \sum_{n=1}^{N} c_n x_n,\ \sum_{m=1}^{N} c_m x_m \right\rangle \\
&= \left\| \sum_{n=1}^{N} c_n x_n \right\|^2.
\end{aligned}$$

On the other hand,

$$\langle Gc, c\rangle \;\le\; \|Gc\|_{\ell^2}\,\|c\|_{\ell^2} \;\le\; B\,\|c\|_{\ell^2}^2 \;=\; B \sum_{n=1}^{N} |c_n|^2.$$

Combining these two estimates, we see that equation (7.3) holds, and therefore Theorem 7.4 implies that $\{x_n\}$ is a Bessel sequence.

(b) The well-defined hypothesis of this part precisely fulfills the hypotheses of Exercise 2.34. That exercise therefore implies that $c \mapsto Gc$ is a bounded mapping on ℓ^2, so we conclude from part (a) that $\{x_n\}$ is a Bessel sequence. □

If $\{x_n\}$ is a Bessel sequence, then it follows from the proof of Theorem 7.6, or directly from the fact that $G = R^*R$, that we have the useful equality

$$\forall\, c = (c_n) \in \ell^2, \quad \langle Gc, c\rangle \;=\; \|Rc\|^2 \;=\; \left\| \sum_n c_n x_n \right\|^2.$$

Example 7.7. Consider the sequence of monomials $\{x^k\}_{k\ge 0}$. By Example 1.29 or Theorem 5.6, the monomials are complete but are not a basis for $C[0,1]$, and by Exercise 5.2, the same is true in the space $L^2[0,1]$. The Gram matrix for the monomials is

$$G = \big[\langle x^n, x^m\rangle\big]_{m,n\ge 0} = \big[\tfrac{1}{m+n+1}\big]_{m,n\ge 0} = \mathcal{H},$$

which is the famous *Hilbert matrix*. It is not obvious, but the Hilbert matrix determines a bounded mapping on $\ell^2(\mathbf{N}\cup\{0\})$. Exercise 7.12 shows that $\|\mathcal{H}\| \le 4$, and in fact it is known that the operator norm of the Hilbert matrix is precisely $\|\mathcal{H}\| = \pi$ [Cho83]. Theorem 7.6 therefore implies that $\{x^k\}_{k\ge 0}$ is a Bessel sequence in $L^2[0,1]$. $\diamondsuit$

All Bessel sequences must be bounded above in norm (Exercise 7.5), but not all norm-bounded sequences are Bessel sequences (see Exercise 7.1). On the other hand, we end this section by making use of Orlicz's Theorem to prove that all unconditional bases that are norm-bounded above are examples of Bessel sequences. Various examples of other systems that are or are not Bessel sequences are considered in the Exercises.

Theorem 7.8. *Let H be a Hilbert space. Every unconditional basis for H that is norm-bounded above is a Bessel sequence in H.*

Proof. Let $\{x_n\}$ be an unconditional basis for H such that $\sup\|x_n\| < \infty$, and let $\{y_n\}$ be its biorthogonal system in H. By Theorem 4.13 we have for each n that $1 \le \|x_n\|\,\|y_n\| \le 2\mathcal{C}$ where $\mathcal{C}$ is the basis constant. Hence $\inf\|y_n\| > 0$.

By Exercise 6.4, $\{y_n\}$ is an unconditional basis for H and $\{x_n\}$ is its biorthogonal sequence. Therefore, given $x \in H$, the series $x = \sum \langle x, x_n\rangle\, y_n$ converges unconditionally. By Orlicz's Theorem (Theorem 3.16), it follows that

$$\sum_n |\langle x, x_n\rangle|^2\,\|y_n\|^2 = \sum_n \big\|\langle x, x_n\rangle\, y_n\big\|^2 < \infty.$$

Consequently, since $\{y_n\}$ is norm-bounded below, $\sum |\langle x, x_n\rangle|^2 < \infty$ for each $x \in H$. Therefore $\{x_n\}$ is a Bessel sequence. $\square$

Exercises

7.1. Let H be a separable Hilbert space. For each of the following, construct a sequence $\{x_n\}$ that has the specified property.

(a) A bounded sequence that is not a Bessel sequence.

(b) A Bessel sequence that is a nonorthogonal basis for H.

(c) A Bessel sequence that is not a basis for H but has frame operator $S = I$.

(d) A Bessel sequence such that $\{x_n\}_{n\in\mathbf{N}\setminus F}$ is complete for every finite $F \subseteq \mathbf{N}$.

(e) An unconditional basis that is not a Bessel sequence.

(f) A normalized conditional basis that is a Bessel sequence.

(g) A normalized conditional basis that is not a Bessel sequence.

(h) A basis that is Bessel but whose biorthogonal sequence is not Bessel.

7.2. Give a direct proof of Theorem 7.2.

7.3. Prove Theorem 7.4.

7.4. Prove Theorem 7.5.

7.5. Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H and let B be a Bessel bound.

(a) Show that $\|x_n\|^2 \le B$ for every $n \in \mathbf{N}$. Thus Bessel sequences are bounded above in norm.

(b) Show that if $\|x_m\|^2 = B$ for any particular m, then $x_n \perp x_m$ for all $n \ne m$.

7.6. Let H, K be Hilbert spaces. Show that if $\{x_n\}$ is a Bessel sequence in H and $L \in \mathcal{B}(H, K)$, then $\{Lx_n\}$ is a Bessel sequence in K.

7.7. Suppose that H is a Hilbert space contained in another Hilbert space K. Given a sequence $\{x_n\}$ in H, show that $\{x_n\}$ is a Bessel sequence in H if and only if it is a Bessel sequence in K.

7.8. Let $\{x_n\}$ be a sequence in a Hilbert space H.

(a) If $\sum |\langle x, x_n\rangle|^2 < \infty$ for all x in a dense set $E \subseteq H$, must $\{x_n\}$ be a Bessel sequence?

(b) If there exists a constant $B > 0$ such that $\sum |\langle x, x_n\rangle|^2 \le B\,\|x\|^2$ for all x in a complete set $E \subseteq H$, must $\{x_n\}$ be a Bessel sequence?

7.9. Show that a sequence $\{x_n\}$ in a Hilbert space H is a Bessel sequence if either of the following two conditions holds:

(a) $\sum_m \sum_n |\langle x_m, x_n\rangle|^2 < \infty$, or

(b) $\sup_m \sum_n |\langle x_m, x_n\rangle| < \infty$.

Observe that hypothesis (a) is quite restrictive, e.g., it is not satisfied by any infinite orthonormal sequence.

7.10. Suppose that $\{x_n\}$ is a Bessel sequence that is a basis for a Hilbert space H. Let $\{y_n\}$ be the biorthogonal sequence, and let B be a Bessel bound.

(a) Show that

$$\forall\, x \in H, \quad \frac{1}{B}\,\|x\|^2 \;\le\; \sum_n |\langle x, y_n\rangle|^2.$$

We say that $\{y_n\}$ has a *lower frame bound* of B^{-1}; compare Definition 8.2. Note that $\{y_n\}$ need not be a Bessel sequence; see Exercise 7.1(h).

(b) Show that for all $N \in \mathbf{N}$ and $c_1, \dots, c_N \in \mathbf{F}$ we have

$$\frac{1}{B} \sum_{n=1}^{N} |c_n|^2 \leq \left\| \sum_{n=1}^{N} c_n y_n \right\|^2.$$

7.11. Let $\{x_n\}$, $\{y_n\}$ be Bessel sequences in separable Hilbert spaces H, K, respectively. Show that the tensor product sequence $\{x_m \otimes y_n\}_{m,n \in \mathbf{N}}$ is a Bessel sequence in $H \otimes K = \mathcal{B}_2(H, K)$ (see Appendix B for definitions).

7.12. The Hilbert matrix is

$$\mathcal{H} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \\ 1/3 & 1/4 & 1/5 & 1/6 & \\ 1/4 & 1/5 & 1/6 & 1/7 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Define

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & 0 & \\ 1/4 & 1/4 & 1/4 & 1/4 & \\ \vdots & & & & \ddots \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/2 & 1/3 & 1/4 & \\ 1/3 & 1/3 & 1/3 & 1/4 & \\ 1/4 & 1/4 & 1/4 & 1/4 & \\ \vdots & & & & \ddots \end{bmatrix},$$

and prove the following statements.

(a) $L = CC^*$, so $L \geq 0$ (i.e., L is a positive operator).

(b) $I - (I - C)(I - C)^* = \text{diag}(1, 1/2, 1/3, 1/4, \dots)$, the diagonal matrix with entries 1, 1/2, ... on the diagonal.

(c) $\|(I - C)\|^2 = \|(I - C)(I - C)^*\| \leq 1$.

(d) $\|C\| \leq 2$ and $\|L\| \leq 4$.

Remark: It is a fact (though not so easy to prove) that if A, B are symmetric matrices and $a_{ij} \leq b_{ij}$ for all $i, j \in \mathbf{N}$, then $\|A\| \leq \|B\|$. Consequently, $\|\mathcal{H}\| \leq \|L\| \leq 4$.

7.2 Unconditional Bases and Riesz Bases in Hilbert Spaces

Let H be a separable Hilbert space. We saw in Example 4.21 that all orthonormal bases in H are equivalent. We will show in this section that the class of bases that are equivalent to orthonormal bases coincides with the class of bounded unconditional bases for H, and we will discuss some of the properties of such bases.

Definition 7.9 (Riesz Basis). Let $\{x_n\}$ be a sequence in a Hilbert space H.

(a) $\{x_n\}$ is a *Riesz basis* if it is equivalent to some (and therefore every) orthonormal basis for H.

(b) $\{x_n\}$ is a *Riesz sequence* if it is a Riesz basis for its closed span in H. ◊

Note that all Riesz bases are equivalent since all orthonormal bases are equivalent. Also, since all orthonormal bases are Bessel sequences, any Riesz basis $\{x_n\}$ must be a Bessel sequence (see Exercise 7.6). Hence we have at hand the tools discussed in Section 7.1. In particular, if $\{x_n\}$ is a Riesz basis, then we know that the analysis operator $Cx = \big(\langle x, x_n\rangle\big)$ is a bounded mapping of H into ℓ^2, and its adjoint is the synthesis operator $Rc = \sum c_n x_n$ for $c = (c_n) \in \ell^2$, where this series converges unconditionally in H.

As with bases or unconditional bases, the image of a Riesz basis under a topological isomorphism is a Riesz basis.

Lemma 7.10. *Riesz bases are preserved by topological isomorphisms. Specifically, if $\{x_n\}$ is a Riesz basis for a Hilbert space H and $T\colon H \to K$ is a topological isomorphism, then $\{Tx_n\}$ is a Riesz basis for K.*

Proof. Since H possesses a basis, it is separable. Therefore K, being topologically isomorphic to H, is separable as well. By Exercise 1.71, all separable Hilbert spaces are isometrically isomorphic, so there exists an isometry Z that maps H onto K. Further, by the definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $U\colon H \to H$ such that $Ue_n = x_n$. Since Z is an isometric isomorphism, the sequence $\{Ze_n\}$ is an orthonormal basis for K. Hence, TUZ^{-1} is a topological isomorphism of K onto itself which has the property that $TUZ^{-1}(Ze_n) = TUe_n = Tx_n$. Hence $\{Tx_n\}$ is equivalent to an orthonormal basis for K, so we conclude that $\{Tx_n\}$ is a Riesz basis for K. □

This yields one half of our characterization of Riesz bases.

Theorem 7.11. *Every Riesz basis for a Hilbert space H is a bounded unconditional basis for H.*

Proof. Let $\{x_n\}$ be a Riesz basis for a Hilbert space H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$ for every n. However, $\{e_n\}$ is a bounded unconditional basis, and bounded unconditional bases are preserved by topological isomorphisms by Lemma 6.2(b), so $\{x_n\}$ must be a bounded unconditional basis for H. □

Before presenting the converse to this result, we prove that Riesz bases are interchangeable with their dual systems in the following sense.

Lemma 7.12. *Let $\{x_n\}$ be a basis for a Hilbert space H, with biorthogonal system $\{y_n\}$. Then the following statements are equivalent.*

(a) $\{x_n\}$ *is a Riesz basis for* H.

(b) $\{y_n\}$ *is a Riesz basis for* H.

(c) $\{x_n\} \sim \{y_n\}$.

Proof. (a) $\Rightarrow$ (b), (c). If $\{x_n\}$ is a Riesz basis for H, then $\{x_n\} \sim \{e_n\}$ for some orthonormal basis $\{e_n\}$ of H. By Corollary 5.23, $\{x_n\}$ and $\{e_n\}$ have equivalent biorthogonal systems. However, $\{e_n\}$ is biorthogonal to itself, so this implies $\{y_n\} \sim \{e_n\} \sim \{x_n\}$. Hence $\{y_n\}$ is equivalent to $\{x_n\}$, and $\{y_n\}$ is a Riesz basis for H.

(b) $\Rightarrow$ (a), (c). By Corollary 5.22, $\{y_n\}$ is a basis for H with biorthogonal system $\{x_n\}$. Therefore, this argument follows symmetrically.

(c) $\Rightarrow$ (a), (b). Assume that $\{x_n\} \sim \{y_n\}$. Then there exists a topological isomorphism $T\colon H \to H$ such that $Tx_n = y_n$ for every n. Given $x \in H$, we therefore have

$$x \;=\; \sum_n \langle x, y_n\rangle\, x_n \;=\; \sum_n \langle x, Tx_n\rangle\, x_n,$$

so

$$\langle Tx, x\rangle \;=\; \Big\langle \sum_n \langle x, Tx_n\rangle\, Tx_n,\; x \Big\rangle \;=\; \sum_n |\langle x, Tx_n\rangle|^2 \;\geq\; 0.$$

Thus T is a continuous and positive linear operator on H, and therefore has a continuous and positive square root $T^{1/2}$ by Theorem 2.18. Similarly, T^{-1} is positive and has a positive square root. Consequently, $T^{1/2}$ is a topological isomorphism. Further, $T^{1/2}$ is self-adjoint, so

$$\langle T^{1/2}x_m, T^{1/2}x_n\rangle \;=\; \langle x_m, T^{1/2}T^{1/2}x_n\rangle \;=\; \langle x_m, Tx_n\rangle \;=\; \langle x_m, y_n\rangle \;=\; \delta_{mn}.$$

Hence $\{T^{1/2}x_n\}$ is an orthonormal sequence in H, and it is complete since $\{x_n\}$ is complete and $T^{1/2}$ is a topological isomorphism. Therefore $\{x_n\}$ is the image of the orthonormal basis $\{T^{1/2}x_n\}$ under the topological isomorphism $T^{-1/2}$, so $\{x_n\}$ is a Riesz basis. By symmetry, $\{y_n\}$ is a Riesz basis as well. □

Now we can prove that Riesz bases and bounded unconditional bases are equivalent, and we also give several other equivalent formulations of Riesz bases. We include the proofs of more implications than are strictly necessary. Additional characterizations of Riesz bases will be given in Theorem 8.32.

Theorem 7.13. *Let $\{x_n\}$ be a sequence in a Hilbert space H. Then the following statements are equivalent.*

(a) $\{x_n\}$ *is a Riesz basis for* H.

(b) $\{x_n\}$ *is a bounded unconditional basis for* H.

(c) $\{x_n\}$ *is a basis for* H, *and*

$$\sum_n c_n x_n \text{ converges} \iff \sum_n |c_n|^2 < \infty.$$

(d) $\{x_n\}$ *is complete in* H *and there exist constants* A, $B > 0$ *such that*

$$\forall\, c_1, \dots, c_N, \quad A \sum_{n=1}^N |c_n|^2 \le \left\| \sum_{n=1}^N c_n x_n \right\|^2 \le B \sum_{n=1}^N |c_n|^2. \tag{7.4}$$

(e) *There is an equivalent inner product* $(\cdot,\cdot)$ *for* H *such that* $\{x_n\}$ *is an orthonormal basis for* H *with respect to* $(\cdot,\cdot)$.

(f) $\{x_n\}$ *is a complete Bessel sequence and possesses a biorthogonal system* $\{y_n\}$ *that is also a complete Bessel sequence.*

(g) $\{x_n\}$ *is complete, and multiplication of vectors in* ℓ^2 *by the Gram matrix* $G = \big[\langle x_n, x_m\rangle\big]_{m,n\in\mathbf{N}}$ *defines a topological isomorphism of* ℓ^2 *onto itself.*

Proof. (a) ⇒ (b). This is Theorem 7.11.

(a) ⇒ (e). If $\{x_n\}$ is a Riesz basis for H, then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Tx_n = e_n$ for every n. Define

$$(x, y) = \langle Tx, Ty\rangle \qquad \text{and} \qquad |||x|||^2 = (x, x) = \langle Tx, Tx\rangle = \|Tx\|^2.$$

It is easy to see that $(\cdot,\cdot)$ is an inner product for H, and by applying Exercise 2.37 we obtain $\|T^{-1}\|^{-1}\,\|x\| \le |||x||| \le \|T\|\,\|x\|$. Hence $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms for H, and so $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are equivalent inner products. Since

$$(x_m, x_n) = \langle Tx_m, Tx_n\rangle = \langle e_m, e_n\rangle = \delta_{mn},$$

the sequence $\{x_n\}$ is orthonormal with respect to $(\cdot,\cdot)$. Suppose $x \in H$ satisfies $(x, x_n) = 0$ for every n. Then $0 = (x, x_n) = \langle Tx, Tx_n\rangle = \langle Tx, e_n\rangle$ for every n, so $Tx = 0$ since $\{e_n\}$ is complete with respect to $\langle\cdot,\cdot\rangle$. Since T is a topological isomorphism, we therefore have $x = 0$, so $\{x_n\}$ is complete with respect to $(\cdot,\cdot)$. A complete orthonormal sequence is an orthonormal basis, so statement (e) holds.

(a) ⇒ (g). Suppose that $\{x_n\}$ is a Riesz basis for H. Since all Riesz bases and orthonormal bases are equivalent, there exists a topological isomorphism $T\colon \ell^2 \to H$ such that $T\delta_n = x_n$, where $\{\delta_n\}$ is the standard basis for ℓ^2. Note that since $\{x_n\}$ is a Bessel sequence, T is precisely the synthesis operator R for $\{x_n\}$. Hence $G = R^*R = T^*T$ is also a topological isomorphism.

(b) ⇒ (f). Suppose that $\{x_n\}$ is a bounded unconditional basis for H, and let $\{y_n\}$ be its biorthogonal system. Since H is reflexive, Exercise 6.4 implies that $\{y_n\}$ is also an unconditional basis for H. Also, by Theorem 4.13,

$1 \le \|x_n\|\,\|y_n\| \le 2\mathcal{C}$ where $\mathcal{C}$ is the basis constant for $\{x_n\}$. Hence $\{y_n\}$ is a bounded unconditional basis for H. All bounded unconditional bases are Bessel sequences by Theorem 7.8, so statement (f) follows.

(c) $\Rightarrow$ (a). Let $\{e_n\}$ be an orthonormal basis for H. Then, by Theorem 4.20, statement (c) implies that $\{x_n\} \sim \{e_n\}$, so $\{x_n\}$ is a Riesz basis for H.

(d) $\Rightarrow$ (c). Suppose that statement (d) holds. Taking $c_m = 1$ and $c_n = 0$ for $n \ne m$, we see from equation (7.4) that $\|x_m\|^2 \ge B^{-1}$. Hence each x_m is nonzero. Choose any $M < N$, and scalars $c_1, \dots, c_N$. Then, by equation (7.4),

$$\left\|\sum_{n=1}^{M} c_n x_n\right\|^2 \;\le\; B \sum_{n=1}^{M} |c_n|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2 \;\le\; \frac{B}{A}\left\|\sum_{n=1}^{N} c_n x_n\right\|^2.$$

Since $\{x_n\}$ is complete and every x_n is nonzero, Theorem 5.17 implies that $\{x_n\}$ is a basis for H.

It remains to show that $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$. Given a sequence of scalars (c_n) and $M < N$, we have by equation (7.4) that

$$A \sum_{n=M+1}^{N} |c_n|^2 \;\le\; \left\|\sum_{n=M+1}^{N} c_n x_n\right\|^2 \;\le\; B \sum_{n=M+1}^{N} |c_n|^2.$$

Therefore, $\sum c_n x_n$ is a Cauchy series in H if and only if $\sum |c_n|^2$ is a Cauchy series of real numbers. Hence one series converges if and only if the other series converges.

(e) $\Rightarrow$ (d). Suppose that $(\cdot,\cdot)$ is an equivalent inner product for H such that $\{x_n\}$ is an orthonormal basis with respect to $(\cdot,\cdot)$. Let $|||\cdot|||$ denote the norm induced by $(\cdot,\cdot)$. Then there exist constants A, $B > 0$ such that

$$\forall\, x \in H, \quad A\,|||x|||^2 \;\le\; \|x\|^2 \;\le\; B\,|||x|||^2. \tag{7.5}$$

Given $x \in H$, we have the orthonormal basis expansion $x = \sum (x, x_n)\, x_n$, where the series converges with respect to $|||\cdot|||$. Since $\|\cdot\|$ is equivalent to $|||\cdot|||$, this series also converges with respect to $\|\cdot\|$. Hence $\mathrm{span}\{x_n\}$ is dense and therefore $\{x_n\}$ is complete, with respect to both norms.

Now choose any scalars $c_1, \dots, c_N$. Then by the Plancherel Equality (Theorem 1.50), $|||\sum_{n=1}^{N} c_n x_n|||^2 = \sum_{n=1}^{N} |c_n|^2$. Combined with equation (7.5), this implies that

$$A \sum_{n=1}^{N} |c_n|^2 \;\le\; \left\|\sum_{n=1}^{N} c_n x_n\right\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2,$$

so statement (d) holds.

(f) $\Rightarrow$ (b). Suppose that $\{x_n\}$ and $\{y_n\}$ are biorthogonal Bessel systems that are each complete in H. Given $x \in H$, we have $\big(\langle x, y_n\rangle\big) \in \ell^2$ since $\{y_n\}$

is Bessel. Hence $z = \sum \langle x, y_n \rangle x_n$ converges unconditionally by Theorem 7.2. By biorthogonality, $\langle z, y_n \rangle = \langle x, y_n \rangle$ for every n, and so $z = x$ since $\{y_n\}$ is complete. Thus $x = \sum \langle x, y_n \rangle x_n$ with unconditional convergence. Biorthogonality ensures that this representation is unique, so $\{x_n\}$ is an unconditional basis for H. Both $\{x_n\}$ and $\{y_n\}$ are bounded above in norm since they are Bessel sequences. Also, $1 \le \|x_n\| \, \|y_n\| \le 2\mathcal{C}$, where $\mathcal{C}$ is the basis constant for $\{x_n\}$, so $\{x_n\}$ and $\{y_n\}$ are bounded below in norm. Therefore $\{x_n\}$ is a bounded unconditional basis for H.

(f) $\Rightarrow$ (g). Suppose that $\{x_n\}$, $\{y_n\}$ are biorthogonal sequences that are each complete Bessel sequences. Let C, R be the analysis and synthesis operators for $\{x_n\}$, and let D, V be the analysis and synthesis operators for $\{y_n\}$. These are all bounded since $\{x_n\}$ and $\{y_n\}$ are Bessel. By biorthogonality, if $c \in \ell^2$, then

$$CVc = \left(\left\langle \sum_n c_n y_n, \, x_m \right\rangle \right)_{m \in \mathbf{N}} = (c_m)_{m \in \mathbf{N}} = c.$$

Further, if $x \in H$, then $RDx = \sum \langle x, y_n \rangle x_n$, and biorthogonality implies that $\langle RDx, y_n \rangle = \langle x, y_n \rangle$ for each n. Since $\{y_n\}$ is complete, this implies that $RDx = x$. Symmetric arguments show that VC and DR are identity operators as well. Finally, $G = CR$, so $L = DV$ is a bounded operator that satisfies

$$GL = CRDV = CV = I \qquad \text{and} \qquad LG = DVCR = DR = I.$$

Hence G has a bounded two-sided inverse, and therefore is a topological isomorphism.

(g) $\Rightarrow$ (d). Assume that $\{x_n\}$ is complete and the Gram matrix G defines a topological isomorphism of ℓ^2 onto itself. Then G is bounded, so we have by Theorem 7.6 that $\{x_n\}$ is a Bessel sequence, and therefore $\langle Gc, c \rangle = \left\| \sum c_n x_n \right\|^2 \ge 0$ for all $c = (c_n) \in \ell^2$. Hence G is a positive operator on ℓ^2, and in fact it is positive definite since it is a topological isomorphism. Exercise 2.45 therefore implies that $|||c||| = \langle Gc, c \rangle$ is an equivalent norm on ℓ^2. Hence there exist constants A, $B > 0$ such that $A\,|||c|||^2 \le \|c\|_{\ell^2}^2 \le B\,|||c|||^2$ for all $c \in \ell^2$, and this implies that statement (d) holds. □

Exercises

7.13. Given a Riesz basis $\{x_n\}$ in a Hilbert space H, prove that the following statements are equivalent.

(a) $\sum c_n x_n$ converges.

(b) $\sum c_n x_n$ converges unconditionally.

(c) $\sum |c_n|^2 < \infty$.

7.14. Show that every basis for a finite-dimensional vector space V is a Riesz basis for V (with respect to any inner product on V).

7.15. Exhibit an unconditional basis for a Hilbert space H that is not a Riesz basis for H.

7.16. Show that if $\{x_n\}$ is a complete sequence in a Hilbert space H that satisfies $\|\sum_{n=1}^N c_n x_n\|^2 = \sum_{n=1}^N |c_n|^2$ for any $N \in \mathbf{N}$ and $c_1, \dots, c_N \in \mathbf{F}$, then $\{x_n\}$ is an orthonormal basis for H.

7.17. Let $\{x_n\}$, $\{y_n\}$ be Riesz bases for Hilbert spaces H, K, respectively. Show that the tensor product sequence $\{x_m \otimes y_n\}_{m,n\in\mathbf{N}}$ is a Riesz basis for $H \otimes K = \mathcal{B}_2(H, K)$ (see Appendix B for definitions).

7.18. Let $\{x_n\}$ be an orthonormal basis for a Hilbert space H. Suppose $\{y_n\}$ is a sequence in H and there exists $0 < \lambda < 1$ such that

$$\left\|\sum_{n=1}^N c_n (x_n - y_n)\right\|^2 \le \lambda \sum_{n=1}^N |c_n|^2, \qquad N \in \mathbf{N},\ c_1, \dots, c_N \in \mathbf{F}.$$

Show that $\{y_n\}$ is a Riesz basis for H.

7.19. Let $\{x_n\}$ be an orthonormal basis for a Hilbert space H. Let $T_k \in \mathcal{B}(H)$ and $a_{nk} \in \mathbf{F}$ be such that

$$\lambda = \sum_{k=1}^\infty \|T_k\| \left(\sup_n |a_{nk}|\right) < 1.$$

Assume that the series

$$y_n = x_n + \sum_{k=1}^\infty a_{nk}\, T_k e_n$$

converges for each $n \in \mathbf{N}$. Show that $\{y_n\}$ is a Riesz basis for H.

7.20. In this exercise we will use the abbreviation $e_b(x) = e^{2\pi i b x}$, where $b \in \mathbf{R}$. Also, we identify the Hilbert space $L^2(\mathbf{T})$ with $L^2[-\frac{1}{2}, \frac{1}{2}]$.

Fix $\lambda_n \in \mathbf{C}$ and assume that

$$\delta = \sup_{n\in\mathbf{Z}} |n - \lambda_n| < \infty.$$

(a) Define bounded linear operators T_k on $L^2[-\frac{1}{2}, \frac{1}{2}]$ by

$$T_k f(x) = x^k f(x).$$

Show that the operator norm of T_k is $\|T_k\| = 2^{-k}$.

(b) Define

$$a_{nk} \;=\; -\frac{\bigl(2\pi i(\lambda_n - n)\bigr)^k}{k!}.$$

Show that

$$e_n - e_{\lambda_n} \;=\; \sum_{k=1}^{\infty} a_{nk}\, T_k e_n, \qquad n \in \mathbf{Z},$$

where the series converge absolutely in $L^2[-\frac{1}{2}, \frac{1}{2}]$.

(c) Show that if

$$\delta \;<\; (\ln 2)/\pi \;\approx\; 0.22\ldots,$$

then $\{e^{2\pi i \lambda_n x}\}_{n \in \mathbf{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$.

Remark: This result is due to Duffin and Eachus [DE42], but it is not quite the best possible. *Kadec's* $\frac{1}{4}$*-Theorem* [Kad64] states that if $\delta < \frac{1}{4}$ then $\{e^{2\pi i \lambda_n x}\}_{n \in \mathbf{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$, and it is known that $\frac{1}{4}$ is the optimal value. For a more detailed discussion, we refer to [You01].