6

Unconditional Bases in Banach Spaces

A Schauder basis provides unique series representations $x = \sum \langle x, a_n \rangle x_n$ of each vector in a Banach space. However, conditionally convergent series are delicate in many respects. For example, if $x = \sum \langle x, a_n \rangle x_n$ converges conditionally and (λ_n) is a bounded sequence of scalars, then the series $\sum \lambda_n \langle x, a_n \rangle x_n$ may not converge. Unconditionality is an important property, and in many applications we greatly prefer a basis that is unconditional over one that is conditional. Therefore we study unconditional bases in more detail in this chapter.

6.1 Basic Properties and the Unconditional Basis Constant

We can reformulate unconditionality of a basis as follows (see Exercise 6.1).

Lemma 6.1. *Given a sequence $\{x_n\}$ in a Banach space X, the following two statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for X.*

(b) *$\{x_{\sigma(n)}\}$ is a basis for X for every permutation σ of $\mathbf{N}$.*

In this case, if $\{a_n\}$ is the sequence of coefficient functionals for $\{x_n\}$, then $\{a_{\sigma(n)}\}$ is the sequence of coefficient functionals for $\{x_{\sigma(n)}\}$. $\diamondsuit$

By Lemma 4.18, topological isomorphisms preserve the property of being a basis. The same is true of unconditional bases (see Exercise 6.2).

Lemma 6.2. (a) *Unconditional bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is an unconditional basis for a Banach space X and $T\colon X \to Y$ is a topological isomorphism, then $\{Tx_n\}$ is an unconditional basis for Y.*

(b) *Bounded unconditional bases are likewise preserved by topological isomorphisms.* $\diamondsuit$

C. Heil, *A Basis Theory Primer: Expanded Edition*, Applied and Numerical Harmonic Analysis, 177
DOI 10.1007/978-0-8176-4687-5_6,

Recall from Definition 4.19 that two bases $\{x_n\}$ and $\{y_n\}$ are *equivalent* if there exists a topological isomorphism T such that $Tx_n = y_n$ for every n. We will see in Section 7.2 that all bounded unconditional bases for a Hilbert space are equivalent, and in fact they are equivalent to orthonormal bases. Up to isomorphisms, the only other infinite-dimensional Banach spaces that have a basis and in which all bounded unconditional bases are equivalent are the sequence spaces c_0 and ℓ^1 [LP68], [LZ69].

Notation 6.3. We will associate three types of partial sum operators with a given unconditional basis $\{x_n\}$ for a Banach space X. Let $\{a_n\}$ be the biorthogonal system to $\{x_n\}$. First, to each finite set $F \subseteq \mathbf{N}$ we associate the partial sum operator $S_F \colon X \to X$ defined by

$$S_F(x) \;=\; \sum_{n\in F} \langle x, a_n\rangle\, x_n, \qquad x \in X.$$

Second, to each finite set $F \subseteq \mathbf{N}$ and each set of scalars $\mathcal{E} = \{\varepsilon_n\}_{n\in F}$ satisfying $\varepsilon_n = \pm 1$ for all n, we associate the operator $S_{F,\mathcal{E}} \colon X \to X$ defined by

$$S_{F,\mathcal{E}}(x) \;=\; \sum_{n\in F} \varepsilon_n\, \langle x, a_n\rangle\, x_n, \qquad x \in X.$$

Finally, to each finite set $F \subseteq \mathbf{N}$ and each collection of scalars $\Lambda = \{\lambda_n\}_{n\in F}$ satisfying $|\lambda| \le 1$ for all n, we associate the operator $S_{F,\Lambda} \colon X \to X$ defined by

$$S_{F,\Lambda}(x) \;=\; \sum_{n\in F} \lambda_n\, \langle x, a_n\rangle\, x_n, \qquad x \in X.$$

Note that while the operators S_F are projections in the sense that $S_F^2 = S_F$, the operators $S_{F,\mathcal{E}}$ and $S_{F,\Lambda}$ need not be projections in this sense. ◇

Applying Theorem 3.10, we obtain the following facts about unconditional bases, where the suprema are implicitly taken over all F, $\mathcal{E}$, Λ described in Notation 6.3. The proof of this result is Exercise 6.3.

Theorem 6.4. *If $\{x_n\}$ is an unconditional basis for a Banach space X, then the following statements hold.*

(a) *The following three quantities are finite for each $x \in X$:*

$$|||x||| \;=\; \sup_F \|S_F(x)\|,$$

$$|||x|||_{\mathcal{E}} \;=\; \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}(x)\|,$$

$$|||x|||_{\Lambda} \;=\; \sup_{F,\Lambda} \|S_{F,\Lambda}(x)\|.$$

(b) *The following three numbers are finite:*

$$\mathcal{K} \;=\; \sup_F \|S_F\|, \qquad \mathcal{K}_{\mathcal{E}} \;=\; \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}\|, \qquad \mathcal{K}_{\Lambda} \;=\; \sup_{F,\Lambda} \|S_{F,\Lambda}\|.$$

(c) $|||\cdot||| \le |||\cdot|||_{\mathcal{E}} \le 2\,|||\cdot|||$ *and* $\mathcal{K} \le \mathcal{K}_{\mathcal{E}} \le 2\mathcal{K}$.

(d) *If* $\mathbf{F} = \mathbf{R}$ *then* $|||\cdot|||_{\mathcal{E}} = |||\cdot|||_{\Lambda}$ *and* $\mathcal{K}_{\mathcal{E}} = \mathcal{K}_{\Lambda}$.

(e) *If* $\mathbf{F} = \mathbf{C}$ *then* $|||\cdot|||_{\mathcal{E}} \le |||\cdot|||_{\Lambda} \le 2\,|||\cdot|||_{\mathcal{E}}$ *and* $\mathcal{K}_{\mathcal{E}} \le \mathcal{K}_{\Lambda} \le 2\mathcal{K}_{\mathcal{E}}$.

(f) $|||\cdot|||$, $|||\cdot|||_{\mathcal{E}}$, *and* $|||\cdot|||_{\Lambda}$ *form norms on* X, *each equivalent to the initial norm* $\|\cdot\|$, *with*

$$\|\cdot\| \;\le\; |||\cdot||| \;\le\; \mathcal{K}\,\|\cdot\|,$$

$$\|\cdot\| \;\le\; |||\cdot|||_{\mathcal{E}} \;\le\; \mathcal{K}_{\mathcal{E}}\,\|\cdot\|,$$

$$\|\cdot\| \;\le\; |||\cdot|||_{\Lambda} \;\le\; \mathcal{K}_{\Lambda}\,\|\cdot\|. \qquad \diamondsuit$$

Notation 6.5. Given an unconditional basis $\{x_n\}$ for a Banach space X, we will let the constants $\mathcal{K}$, $\mathcal{K}_{\mathcal{E}}$, and $\mathcal{K}_{\Lambda}$ and the norms $|||\cdot|||$, $|||\cdot|||_{\mathcal{E}}$, and $|||\cdot|||_{\Lambda}$ be as described in Theorem 6.4. $\diamondsuit$

Definition 6.6 (Unconditional Basis Constant). If $\{x_n\}$ is an unconditional basis for a Banach space X, then the number $\mathcal{K}_{\mathcal{E}}$ is called the *unconditional basis constant* for $\{x_n\}$. $\diamondsuit$

Comparing the number $\mathcal{K}$ to the basis constant $\mathcal{C}$ from Definition 4.14, we see that $\mathcal{C} \le \mathcal{K}$. In fact, if we let $\mathcal{C}_\sigma$ be the basis constant for the permuted basis $\{x_{\sigma(n)}\}$, then $\mathcal{K} = \sup \mathcal{C}_\sigma$, where we take the supremum over all permutations σ of $\mathbf{N}$.

The unconditional basis constant $\mathcal{K}_{\mathcal{E}}$ implicitly depends on the norm for X, and changing the norm to some other equivalent norm may change the value of the basis constant. For example, the unconditional basis constant for $\{x_n\}$ with respect to the equivalent norm $|||\cdot|||_{\mathcal{E}}$ is precisely 1 (compare Theorem 4.15 for the analogous statement for the basis constant).

Exercises

6.1. Prove Lemma 6.1.

6.2. Prove Lemma 6.2.

6.3. Prove Theorem 6.4.

6.4. Let $\{x_n\}$ be an unconditional basis for a Banach space X, with associated coefficient functionals $\{a_n\}$.

(a) Prove that $\{a_n\}$ is an unconditional basic sequence in X^*.

(b) Show that if X is reflexive, then $\{a_n\}$ is an unconditional basis for X^*.

6.5. Use Orlicz's Theorem to prove that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ cannot be an unconditional basis for $L^p(\mathbf{T})$ when $1 \le p < 2$. Argue by duality to show that it also cannot be an unconditional basis when $2 < p < \infty$. (See Chapter 14 for proof that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a basis for $L^p(\mathbf{T})$ when $1 < p < \infty$, but is not a basis for $L^1(\mathbf{T})$ or $C(\mathbf{T})$.)

6.2 Characterizations of Unconditional Bases

The next result gives several equivalent formulations of unconditional bases. We include the proofs of more implications than are strictly needed, in order to illustrate some different approaches to the proof.

Theorem 6.7. *Let $\{x_n\}$ be a complete sequence in a Banach space X such that $x_n \neq 0$ for every n. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for X.*

(b) $\exists\, C_1 \geq 1, \quad \forall\, c_1, \dots, c_N, \quad \forall\, \varepsilon_1, \dots, \varepsilon_N = \pm 1,$

$$\left\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \right\| \;\leq\; C_1 \left\| \sum_{n=1}^{N} c_n x_n \right\|. \tag{6.1}$$

(c) $\exists\, C_2 \geq 1, \quad \forall\, b_1, \dots, b_N, \quad \forall\, c_1, \dots, c_N,$

$$|b_1| \leq |c_1|, \dots, |b_N| \leq |c_N| \quad \Longrightarrow \quad \left\| \sum_{n=1}^{N} b_n x_n \right\| \;\leq\; C_2 \left\| \sum_{n=1}^{N} c_n x_n \right\|.$$

(d) $\exists\, 0 < C_3 \leq 1 \leq C_4 < \infty, \quad \forall\, c_1, \dots, c_N,$

$$C_3 \left\| \sum_{n=1}^{N} |c_n|\, x_n \right\| \;\leq\; \left\| \sum_{n=1}^{N} c_n x_n \right\| \;\leq\; C_4 \left\| \sum_{n=1}^{N} |c_n|\, x_n \right\|.$$

(e) *$\{x_n\}$ is a basis, and for each bounded sequence of scalars $\Lambda = (\lambda_n)$ there exists a continuous linear operator $T_\Lambda \colon X \to X$ such that $T_\Lambda(x_n) = \lambda_n x_n$ for all $n \in \mathbf{N}$.*

Further, in case these hold, the best constant C_1 in equation (6.1) *is the unconditional basis constant $C_1 = \mathcal{K}_\mathcal{E} = \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}\|$.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\{x_n\}$ is an unconditional basis for X, with coefficient functionals $\{a_n\}$. Choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$, and set $x = \sum_{n=1}^{N} c_n x_n$. Then $\langle x, a_n \rangle = c_n$ if $n \leq N$, while $\langle x, a_n \rangle = 0$ if $n > N$. Therefore

$$\sum_{n=1}^{N} \varepsilon_n c_n x_n \;=\; \sum_{n \in F} \varepsilon_n \langle x, a_n \rangle\, x_n \;=\; S_{F,\mathcal{E}}(x),$$

where $F = \{1, \dots, N\}$ and $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_N\}$. By definition of $|\!|\!|\cdot|\!|\!|_\mathcal{E}$ and by Theorem 6.4(f), we therefore have

$$\left\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \right\| \;=\; \|S_{F,\mathcal{E}}(x)\| \;\leq\; |\!|\!|x|\!|\!|_\mathcal{E} \;\leq\; \mathcal{K}_\mathcal{E}\, \|x\| \;=\; \mathcal{K}_\mathcal{E} \left\| \sum_{n=1}^{N} c_n x_n \right\|.$$

Thus statement (b) holds with $C_1 = \mathcal{K}_{\mathcal{E}}$.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds, and let σ be any permutation of $\mathbf{N}$. We must show that $\{x_{\sigma(n)}\}$ is a basis for X. By hypothesis, $\{x_{\sigma(n)}\}$ is complete with every element nonzero. Therefore, by Theorem 5.17 it suffices to show that there is a constant C_σ such that

$$\forall\, N \ge M, \quad \forall\, c_{\sigma(1)}, \dots, c_{\sigma(N)}, \quad \biggl\|\sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)}\biggr\| \;\le\; C_\sigma \,\biggl\|\sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)}\biggr\|.$$

To this end, fix any $N \ge M$ and choose any scalars $c_{\sigma(1)}, \dots, c_{\sigma(N)}$. Define $c_n = 0$ for $n \notin \{\sigma(1), \dots, \sigma(N)\}$. Let $L = \max\{\sigma(1), \dots, \sigma(N)\}$, and define

$$\varepsilon_n = 1 \qquad \text{and} \qquad \gamma_n \;=\; \begin{cases} 1, & \text{if } n \in \{\sigma(1), \dots, \sigma(M)\}, \\ -1, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \biggl\|\sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)}\biggr\| &= \biggl\|\sum_{n=1}^{L} \Bigl(\frac{\varepsilon_n + \gamma_n}{2}\Bigr) c_n x_n\biggr\| \\ &\le \frac{1}{2}\biggl\|\sum_{n=1}^{L} \varepsilon_n c_n x_n\biggr\| + \frac{1}{2}\biggl\|\sum_{n=1}^{L} \gamma_n c_n x_n\biggr\| \\ &\le \frac{C_1}{2}\biggl\|\sum_{n=1}^{L} c_n x_n\biggr\| + \frac{C_1}{2}\biggl\|\sum_{n=1}^{L} c_n x_n\biggr\| \\ &= C_1 \biggl\|\sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)}\biggr\|. \end{aligned}$$

This is the desired result, with $C_\sigma = C_1$.

(a) $\Rightarrow$ (c). Suppose that $\{x_n\}$ is an unconditional basis for X, with coefficient functionals $\{a_n\}$. Choose any scalars $c_1, \dots, c_N$ and $b_1, \dots, b_N$ such that $|b_n| \le |c_n|$ for every n. Define $x = \sum_{n=1}^{N} c_n x_n$, and note that $c_n = \langle x, a_n\rangle$. Let λ_n be such that $b_n = \lambda_n c_n$. Since $|b_n| \le |c_n|$ we can take $|\lambda_n| \le 1$ for every n. Therefore, if we define $F = \{1, \dots, N\}$ and $\Lambda = \{\lambda_1, \dots, \lambda_N\}$, then

$$\sum_{n=1}^{N} b_n x_n \;=\; \sum_{n \in F} \lambda_n c_n x_n \;=\; \sum_{n \in F} \lambda_n \,\langle x, a_n\rangle\, x_n \;=\; S_{F,\Lambda}(x).$$

Hence

$$\biggl\|\sum_{n=1}^{N} b_n x_n\biggr\| \;=\; \|S_{F,\Lambda}(x)\| \;=\; |||x|||_{\Lambda} \;\le\; \mathcal{K}_{\Lambda}\,\|x\| \;=\; \mathcal{K}_{\Lambda}\biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|.$$

Thus statement (c) holds with $C_2 = \mathcal{K}_\Lambda$.

(b) $\Rightarrow$ (c). Suppose that statement (b) holds. Choose any $N > 0$, and any scalars b_n, c_n such that $|b_n| \le |c_n|$ for each $n = 1, \dots, N$. Let $|\lambda_n| \le 1$ be such that $b_n = \lambda_n c_n$. Let $\alpha_n = \mathrm{Re}(\lambda_n)$ and $\beta_n = \mathrm{Im}(\lambda_n)$. Since the α_n are real and satisfy $|\alpha_n| \le 1$, Carathéodory's Theorem (Theorem 3.13) implies that we can find scalars $t_m \ge 0$ and signs $\varepsilon_m^n = \pm 1$, for $m = 1, \dots, N+1$ and $n = 1, \dots, N$, such that

$$\sum_{m=1}^{N+1} t_m \;=\; 1 \qquad \text{and} \qquad \sum_{m=1}^{N+1} \varepsilon_m^n\, t_m \;=\; \alpha_n \quad \text{for } n = 1, \dots, N.$$

Hence,

$$\begin{aligned}
\Bigl\|\sum_{n=1}^N \alpha_n c_n x_n\Bigr\| &= \Bigl\|\sum_{n=1}^N \sum_{m=1}^{N+1} \varepsilon_m^n t_m c_n x_n\Bigr\| \\
&= \Bigl\|\sum_{m=1}^{N+1} t_m \sum_{n=1}^N \varepsilon_m^n c_n x_n\Bigr\| \\
&\le \sum_{m=1}^{N+1} t_m \Bigl\|\sum_{n=1}^N \varepsilon_m^n c_n x_n\Bigr\| \\
&\le \sum_{m=1}^{N+1} t_m\, C_1 \Bigl\|\sum_{n=1}^N c_n x_n\Bigr\| \\
&= C_1 \Bigl\|\sum_{n=1}^N c_n x_n\Bigr\|.
\end{aligned}$$

A similar formula holds for the imaginary parts β_n (which are zero if $\mathbf{F} = \mathbf{R}$), so

$$\begin{aligned}
\Bigl\|\sum_{n=1}^N b_n x_n\Bigr\| &= \Bigl\|\sum_{n=1}^N \lambda_n c_n x_n\Bigr\| \\
&\le \Bigl\|\sum_{n=1}^N \alpha_n c_n x_n\Bigr\| + \Bigl\|\sum_{n=1}^N \beta_n c_n x_n\Bigr\| \\
&\le 2C_1 \Bigl\|\sum_{n=1}^N c_n x_n\Bigr\|.
\end{aligned}$$

Therefore statement (c) holds with $C_2 = 2C_1$.

(c) $\Rightarrow$ (a). Suppose that statement (c) holds, and let σ be any permutation of $\mathbf{N}$. We must show that $\{x_{\sigma(n)}\}$ is a basis for X. By hypothesis, $\{x_{\sigma(n)}\}$ is complete in X and every element $x_{\sigma(n)}$ is nonzero. Therefore, by Theorem 5.17 it suffices to show that there is a constant C_σ such that

$$\forall\, N \ge M, \quad \forall\, c_{\sigma(1)}, \dots, c_{\sigma(N)}, \quad \biggl\| \sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)} \biggr\| \;\le\; C_\sigma \,\biggl\| \sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)} \biggr\|.$$

To this end, fix any $N \ge M$ and choose any scalars $c_{\sigma(1)}, \dots, c_{\sigma(N)}$. Define $c_n = 0$ for $n \notin \{\sigma(1), \dots, \sigma(N)\}$. Let $L = \max\{\sigma(1), \dots, \sigma(N)\}$ and define

$$\lambda_n \;=\; \begin{cases} 1, & \text{if } n \in \{\sigma(1), \dots, \sigma(M)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \biggl\| \sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)} \biggr\| \;&=\; \biggl\| \sum_{n=1}^{L} \lambda_n c_n x_n \biggr\| \\ &\le\; C_2 \,\biggl\| \sum_{n=1}^{L} c_n x_n \biggr\| \\ &=\; C_2 \,\biggl\| \sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)} \biggr\|. \end{aligned}$$

This is the desired result, with $C_\sigma = C_2$.

(c) $\Rightarrow$ (d). Assume that statement (c) holds, and choose any scalars $c_1, \dots, c_N$. Let $b_n = |c_n|$. Then we have both $|b_n| \le |c_n|$ and $|c_n| \le |b_n|$, so statement (c) implies

$$\biggl\| \sum_{n=1}^{N} b_n x_n \biggr\| \;\le\; C_2 \,\biggl\| \sum_{n=1}^{N} c_n x_n \biggr\| \qquad \text{and} \qquad \biggl\| \sum_{n=1}^{N} c_n x_n \biggr\| \;\le\; C_2 \,\biggl\| \sum_{n=1}^{N} b_n x_n \biggr\|.$$

Therefore statement (d) holds with $C_3 = 1/C_2$ and $C_4 = C_2$.

(d) $\Rightarrow$ (c). Assume that statement (d) holds. Choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$. Then, by statement (d),

$$\biggl\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \biggr\| \;\le\; C_4 \,\biggl\| \sum_{n=1}^{N} |\varepsilon_n c_n|\, x_n \biggr\| \;=\; C_4 \,\biggl\| \sum_{n=1}^{N} |c_n|\, x_n \biggr\| \;\le\; \frac{C_4}{C_3} \,\biggl\| \sum_{n=1}^{N} c_n x_n \biggr\|.$$

Hence statement (c) holds with $C_2 = C_4/C_3$.

(a) $\Rightarrow$ (e). Let $\{x_n\}$ be an unconditional basis for X, with coefficient functionals $\{a_n\}$. Let (λ_n) be any bounded sequence of scalars, and let $M = \sup |\lambda_n|$. Fix any $x \in X$. Then the series $x = \sum \langle x, a_n \rangle\, x_n$ converges unconditionally. Hence, by Theorem 3.10(f), the series $T_\Lambda(x) = \sum \lambda_n \langle x, a_n \rangle\, x_n$ converges. Clearly $T_\Lambda \colon X \to X$ defined in this way is linear, and we have

$$\|T_\Lambda(x)\| \;=\; M\left\|\sum_n \frac{\lambda_n}{M}\langle x,a_n\rangle\, x_n\right\| \;\le\; M\mathcal{K}_\Lambda \left\|\sum_n \langle x,a_n\rangle\, x_n\right\| \;=\; M\mathcal{K}_\Lambda\,\|x\|.$$

Therefore T_Λ is continuous. Finally, the biorthogonality of $\{x_n\}$ and $\{a_n\}$ ensures that $T_\Lambda(x_n) = \lambda_n x_n$ for every n.

(e) $\Rightarrow$ (a). Suppose that statement (e) holds. Since $\{x_n\}$ is a basis, there exists a biorthogonal sequence $\{a_n\} \subseteq X^*$ such that the series $x = \sum \langle x,a_n\rangle\, x_n$ converges and is the unique expansion of x in terms of the vectors x_n. We must show that this series converges unconditionally. Let $\Lambda = (\lambda_n)$ be any sequence of scalars such that $|\lambda_n| \le 1$ for every n. Then, by hypothesis, there exists a continuous mapping $T_\Lambda\colon X \to X$ such that $T_\Lambda(x_n) = \lambda_n x_n$ for every n. The continuity of T_Λ implies that

$$T_\Lambda(x) \;=\; T_\Lambda\!\left(\sum_n \langle x,a_n\rangle\, x_n\right) \;=\; \sum_n \langle x,a_n\rangle\, T_\Lambda(x_n) \;=\; \sum_n \lambda_n \langle x,a_n\rangle\, x_n.$$

That is, the rightmost series on the line above converges for every choice of bounded scalars, so Theorem 3.10(f) tells us that the series $x = \sum \langle x,a_n\rangle\, x_n$ converges unconditionally. □

Exercises

6.6. Let X be a real Banach space, and suppose that $\{x_n\}$ is an unconditional basis for X with unconditional basis constant $\mathcal{K}_\mathcal{E} = 1$. Given $x = \sum a_n x_n$ and $y = \sum b_n y_n$ in X, declare that $x \le y$ if $a_n \le b_n$ for every n. Show that $\le$ is a partial order on X, and X is a Banach lattice in the sense of Definition 3.35. Using the notation of that definition, show that $x \vee y = \sum \max\{a_n, b_n\}\, x_n$, $x \wedge y = \sum \min\{a_n, b_n\}\, x_n$, and $|x| = \sum |a_n|\, x_n$.

6.7. Set $F = \mathbf{R}$. The Haar system is an orthonormal basis for $L^2[0,1]$, so by Exercise 6.6 there is a partial ordering $\le$ on $L^2[0,1]$ induced by this unconditional basis. There is also the ordinary partial ordering $\le$ on $L^2[0,1]$ defined by $f \le g$ if $f(t) \le g(t)$ for a.e. t. Do these two orderings coincide?

6.3 Conditionality of the Schauder System in $C[0,1]$

We saw in Section 4.5 that the Schauder system is a basis for $C[0,1]$. Now we will show that this basis is conditional. We do this indirectly—we will not explicitly construct an element of $C[0,1]$ whose basis representation converges conditionally, but rather will use Theorem 6.7 to demonstrate that the unconditional basis constant for the Schauder system must be infinite.

Using the notation of Section 4.3, the elements of the Schauder system are the box function $\chi = \chi_{[0,1]}$, the function $\ell(t) = t$, and the dilated and

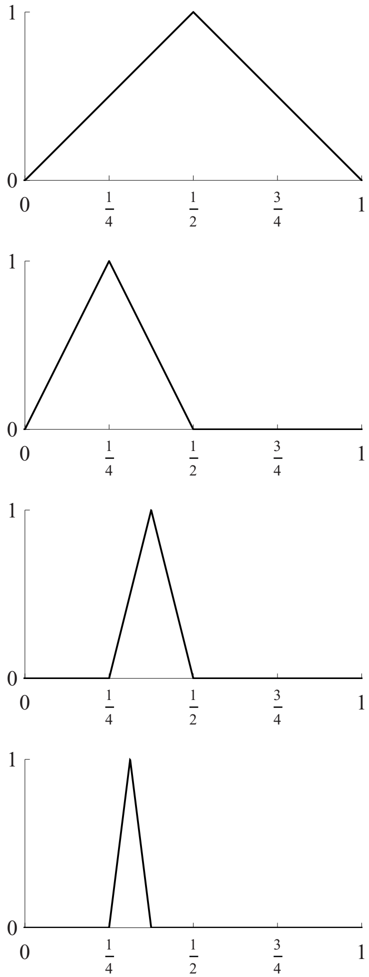

Fig. 6.1. From top to bottom: The functions t_1, t_2, t_3, and t_4.

translated hat functions $s_{n,k}(t) = W(2^n t - k)$, where W is the hat function of height 1 supported on $[0,1]$. We select a subsequence of the Schauder system by defining:

$$
\begin{aligned}
t_1 &= s_{0,0} \qquad &&\bigl(\text{hat function on } I_1 = \bigl[0,1\bigr]\bigr),\\
t_2 &= s_{1,0} \qquad &&\bigl(\text{hat function on } I_2 = \bigl[0,\tfrac{1}{2}\bigr]\bigr),\\
t_3 &= s_{2,1} \qquad &&\bigl(\text{hat function on } I_3 = \bigl[\tfrac{1}{4},\tfrac{1}{2}\bigr]\bigr),\\
t_4 &= s_{3,2} \qquad &&\bigl(\text{hat function on } I_4 = \bigl[\tfrac{1}{4},\tfrac{3}{8}\bigr]\bigr),
\end{aligned}
$$

$$t_5 \;=\; s_{4,5} \qquad \bigl(\text{hat function on } I_5 = \bigl[\tfrac{5}{16}, \tfrac{3}{8}\bigr]\bigr),$$

$$t_6 \;=\; s_{5,10} \qquad \bigl(\text{hat function on } I_6 = \bigl[\tfrac{5}{16}, \tfrac{11}{32}\bigr]\bigr),$$

etc., where we alternate choosing the left or right half of I_{N-1} as the interval I_N on which the hat function t_N is supported (see Figure 6.1).

Now consider the function $g_N = \sum_{n=1}^N t_n$. Our goal is not to show that g_N converges uniformly (in fact, it does not), but rather to compute its norm and to compare this to the norm of $h_N = \sum_{n=1}^N (-1)^{n+1} t_n$ (see the illustration in Figure 6.2).

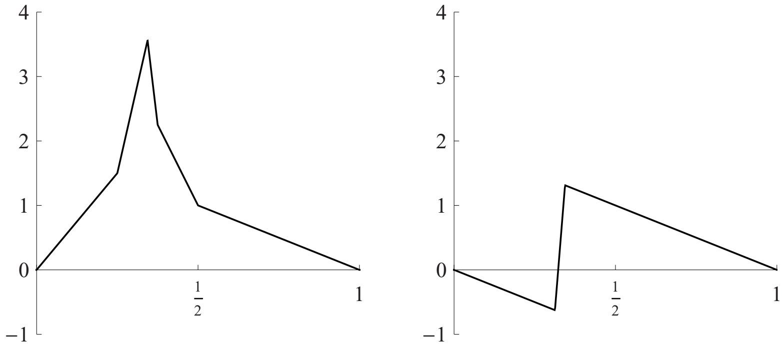

Fig. 6.2. The functions g_5 (left) and h_5 (right).

The functions g_{N-1} and g_N agree everywhere except on the interval I_N. Let μ_N be the midpoint of I_N. The function g_{N-1} is linear on the interval I_N, and g_N achieves its global maximum at the midpoint μ_N. By construction, for $N \ge 3$ one endpoint of I_N is μ_{N-2} and the other is μ_{N-1}. Letting $a_N = g_N(\mu_N)$ be the global maximum of g_N, we have

$$a_N \;=\; 1 + \frac{a_{N-1} + a_{N-2}}{2}.$$

By Exercise 6.8, a_N increases without bound as $N \to \infty$.

On the other hand, a similar analysis of $h_N = \sum_{n=1}^N (-1)^{n+1} t_n$ shows that we always have $|h_N(t)| \le 2$ (Exercise 6.8), so $b_N = \|h_N\|_\infty \le 2$. Consequently there can be no finite constant C such that

$$\biggl\|\sum_{n=1}^N t_n\biggr\|_\infty \;=\; a_N \;\le\; C b_N \;=\; C\,\biggl\|\sum_{n=1}^N (-1)^{n+1} t_n\biggr\|_\infty, \qquad N \in \mathbf{N}.$$

Considering hypothesis (c) of Theorem 6.7, we conclude that the Schauder system cannot be unconditional.

Exercises

6.8. Show that $a_N \to \infty$ and $0 \le b_N \le 2$ for each N.

6.4 Conditionality of the Haar System in $L^1[0,1]$

By Theorem 5.18, the Haar system is a basis for $L^p[0,1]$ for each $1 \le p < \infty$, at least with respect to the ordering given in equation (5.9). We will show that this basis is conditional when $p = 1$ by taking an indirect approach similar to the one we used to prove that the Schauder system is conditional.

Set $\chi = \chi_{[0,1]}$, and let $\psi_{n,k}$ be as defined in Example 1.54. For this proof, we only need to deal with the elements of the Haar system that are nonzero at the origin. Normalizing so that each function has unit L^1-norm, these are the functions χ and

$$k_n \;=\; 2^{n/2}\psi_{n,0} \;=\; 2^n\left(\chi_{[0,2^{-n-1})} - \chi_{[2^{-n-1},2^{-n})}\right), \qquad n \ge 0.$$

Fix $N > 0$ and define

$$f_N \;=\; \chi \;+\; \sum_{n=0}^{2N} k_n.$$

Examining the graphs of the functions k_n, we see that there is a great deal of cancellation in this sum, leaving us with

$$f_N \;=\; 2^{2N+1}\,\chi_{[0,2^{-2N-1})}.$$

In particular, f_N is a unit vector in $L^1[0,1]$.

Now we form a "subseries" of the series defining f_N. Specifically, we take

$$g_N \;=\; \sum_{\substack{n=0\\ n \text{ even}}}^{2N} k_n.$$

Looking at the graphs in Figure 6.3, we see that $g_0 = -1$ on $[\frac{1}{2}, 1)$, $g_1 = 4 - 1 = -3$ on $[\frac{1}{8}, \frac{1}{4})$, and $g_2 = 1 + 4 - 16 = -11$ on $[\frac{1}{32}, \frac{1}{16})$. In general, since k_n is -1 only on an interval where each of $k_0, \dots, k_{n-1}$ are identically 1, we see that

$$g_N(x) \;=\; \left(\sum_{n=0}^{N-1} 4^n\right) - 4^N \;=\; -\frac{2}{3}\,4^N - \frac{1}{3}, \qquad \tfrac{1}{2}\,4^{-N} \le x < 4^{-N}.$$

Therefore the L^1-norm of g_N on this particular interval is

$$\int_{\frac{1}{2}4^{-N}}^{4^{-N}} |g_N(t)|\,dt \;=\; \left(\frac{2}{3}\,4^N + \frac{1}{3}\right)\frac{1}{2}\,4^{-N} \;\ge\; \frac{1}{3}.$$

However, $g_N = g_{N-1}$ on the interval $[4^{-N}, 1]$, so the total L^1-norm of g_N is at least

$$\|g_N\|_{L^1} \ \geq \ \sum_{n=0}^{N} \int_{\frac{1}{2}\,4^{-n}}^{4^{-n}} |g_n(t)|\,dt \ \geq \ \frac{N+1}{3}.$$

Since $\|f_N\|_{L^1} = 1$ for every N, criterion (c) of Theorem 6.7 implies that the Haar system cannot be an unconditional basis for $L^1[0,1]$.

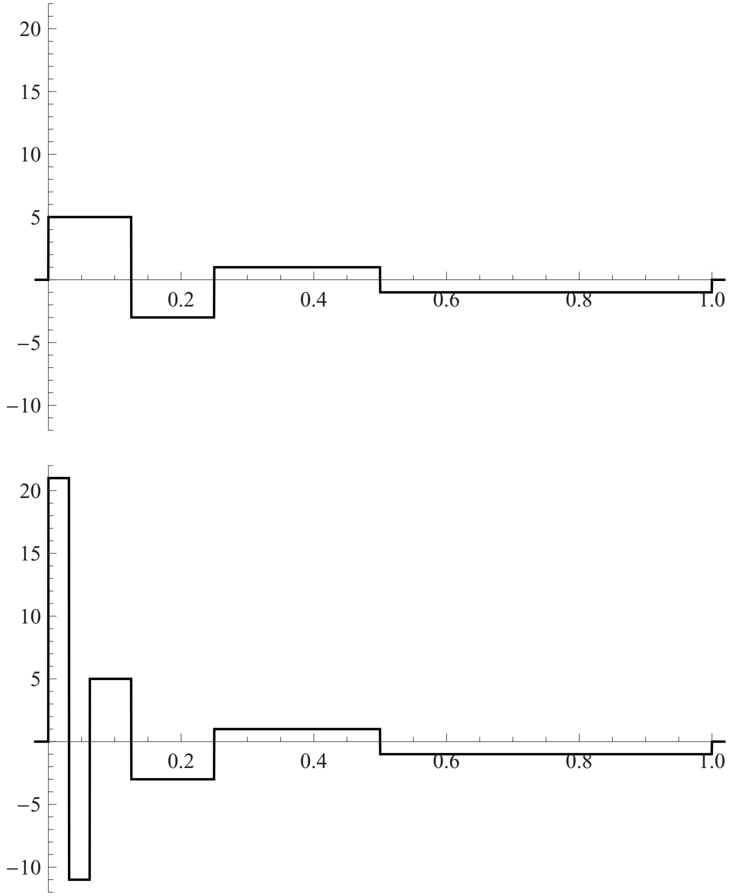

Fig. 6.3. The functions g_1 (top) and g_2 (bottom).

The facts that the Schauder system is conditional in $C[0,1]$ and the Haar system is conditional in $L^1[0,1]$ are special cases of the deeper fact that these two spaces contain no unconditional bases whatsoever! For proof, we refer to [LT77], [Sin70].