11

Gabor Bases and Frames

In this chapter we will consider the construction and properties of the class of *Gabor frames* for the Hilbert space $L^2(\mathbf{R})$. The analysis and application of Gabor systems is one part of the field of *time-frequency analysis*, which is more broadly explored in Gröchenig's text [Grö01].

In Chapter 10 we focused on systems of weighted exponentials $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ and systems of translates $\{g(x-k)\}_{k\in\mathbf{Z}}$. Each of these systems is generated by applying a single type of operation (modulation or translation) to a single generating function (φ or g). The resulting sequences have many applications, but their closed spans can only be proper subspaces of $L^2(\mathbf{R})$. In contrast, Gabor systems incorporate both modulations and translations, and can be frames for all of $L^2(\mathbf{R})$.

Gabor systems were briefly introduced in Example 8.10 and are defined precisely as follows.

Definition 11.1. A *lattice Gabor system*, or simply a *Gabor system* for short, is a sequence in $L^2(\mathbf{R})$ of the form

$$\mathcal{G}(g,a,b) \;=\; \{e^{2\pi ibnx}g(x-ak)\}_{k,n\in\mathbf{Z}},$$

where $g \in L^2(\mathbf{R})$ and $a,\, b > 0$ are fixed. We call g the *generator* or the *atom* of the system, and refer to a, b as the *lattice parameters*. $\diamondsuit$

More generally, an "irregular" Gabor system is a sequence of the form $\mathcal{G}(g,\Lambda) = \big\{e^{2\pi ibx}g(x-a)\big\}_{(a,b)\in\Lambda}$, where Λ is an arbitrary countable set of points in $\mathbf{R}^2$. Lattice Gabor systems have many attractive features and applications, and are much easier to analyze than irregular Gabor systems, so we focus on lattice systems for most of this chapter. For more details on irregular Gabor systems, we refer to [Grö01] or the survey paper [Hei07].

We are especially interested in Gabor systems that form frames or Riesz bases for $L^2(\mathbf{R})$. Naturally, if $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$, then we call it a *Gabor frame*, and if it is a Riesz basis, then we call it a *Gabor Riesz basis* or an *exact Gabor frame*.

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Gabor systems are named after Dennis Gabor (1900–1979), who was awarded the Nobel prize for his invention of holography. In his paper [Gab46], Gabor proposed using the Gabor system $\mathcal{G}(\phi, 1, 1)$ generated by the Gaussian function $\phi(x) = e^{-\pi x^2}$. Von Neumann [vN32, p. 406] had earlier claimed (without proof) that $\mathcal{G}(\phi, 1, 1)$ is complete in $L^2(\mathbf{R})$, i.e., its finite linear span is dense. Gabor conjectured (incorrectly, as we will see) that every function in $L^2(\mathbf{R})$ could be represented in the form

$$f = \sum_{k,n \in \mathbf{Z}} c_{kn}(f)\, M_n T_k \phi \tag{11.1}$$

for some scalars $c_{kn}(f)$; see [Gab46, Eq. 1.29]. This is one reason why general families $\mathcal{G}(g, a, b)$ are named in his honor (see [Jan01] for additional historical remarks and references).

Von Neumann's claim of completeness was proved in [BBGK71], [Per71], and [BGZ75]. However, completeness is a weak property and does not imply the existence of expansions of the form given in equation (11.1). Reading a bit extra into what von Neumann and Gabor actually wrote, possibly they expected that $\mathcal{G}(\phi, 1, 1)$ would be a Schauder basis or a Riesz basis for $L^2(\mathbf{R})$. In fact, $\mathcal{G}(\phi, 1, 1)$ is neither, as it is overcomplete in the sense that any single element may be removed and still leave a complete system. In fact, the excess is precisely 1, because this system becomes incomplete as soon as two elements are removed. However, even with one element removed, the resulting exact system forms neither a Schauder basis nor a Riesz basis; cf. [Fol89, p. 168]. In fact, Janssen proved in [Jan81] that Gabor's conjecture that each $f \in L^2(\mathbf{R})$ has an expansion of the form in equation (11.1) is true, but he also showed that the series converges only in the sense of tempered distributions—not in the norm of L^2—and the coefficients c_{kn} grow with k and n (see also [LS99]).

Today we realize that there are no "good" Gabor Riesz bases $\mathcal{G}(g, a, b)$ for $L^2(\mathbf{R})$. Indeed, the *Balian–Low Theorem*, which we mentioned in Chapter 8 and will consider in detail in Section 11.8, implies that only "badly behaved" atoms g can generate Gabor Riesz bases. On the other hand, redundant Gabor frames with nice generators do exist, and they provide us with useful tools for many applications. We will study the construction and special properties of Gabor frames in this chapter.

11.1 Time-Frequency Shifts

We recall the following operations on functions $f \colon \mathbf{R} \to \mathbf{C}$.

$$\begin{aligned}
\text{Translation:} \quad (T_a f)(x) &= f(x - a), & a \in \mathbf{R}.\\
\text{Modulation:} \quad (M_b f)(x) &= e^{2\pi i b x} f(x), & b \in \mathbf{R}.\\
\text{Dilation:} \quad (D_r f)(x) &= r^{1/2} f(rx), & r > 0.
\end{aligned}$$

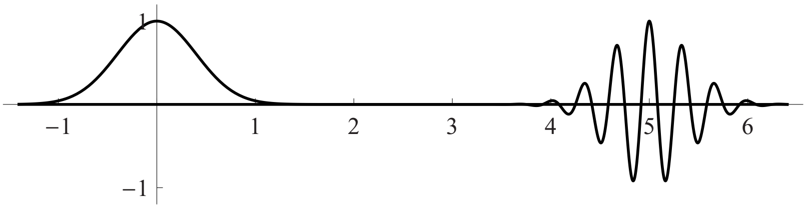

Fig. 11.1. The Gaussian window $\phi(x) = e^{-\pi x^2}$ and the real part of the time-frequency shift $M_3T_5\phi$.

We often think of the independent variable $x \in \mathbf{R}$ as representing time, and hence refer to translation as a *time shift*. We call modulation a *frequency shift*, and say that a composition of translation and modulation is a *time-frequency shift* (see the illustration in Figure 11.1). Thus, a Gabor system $\mathcal{G}(g,a,b)$ is a set of time-frequency shifts of the atom g:

$$\mathcal{G}(g,a,b) \;=\; \bigl\{M_{bn}T_{ak}g\bigr\}_{k,n\in\mathbf{Z}}.$$

Unfortunately, the translation and modulation operators do not commute in general. Being careful with the ordering of composition and evaluation, we compute that

$$\begin{aligned} T_aM_bf(x) \;&=\; (T_a(M_bf))(x) \\ &=\; (M_bf)(x-a) \\ &=\; e^{2\pi ib(x-a)}f(x-a) \\ &=\; e^{-2\pi iab}\,e^{2\pi ibx}f(x-a) \\ &=\; e^{-2\pi iab}\,M_bT_af. \end{aligned}$$

The pesky *phase factor* $e^{-2\pi iab}$ has modulus 1, but we only have $e^{-2\pi iab} = 1$ when $ab \in \mathbf{Z}$. Hence M_b and T_a only commute when the product ab is integer. Even so, by Exercise 11.3, $\{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ is a frame if and only if $\{T_{ak}M_{bn}g\}_{k,n\in\mathbf{Z}}$ is a frame, so in this sense the ordering of T_{ak} and M_{bn} is not important in many circumstances. However, we must still be careful to respect these phase factors in our calculations, as they do create significant difficulties at times (as in Section 11.9).

The product ab of the lattice generators appears in many calculations involving Gabor systems. It is usually the product ab that is important, rather than the individual values of a and b, because by dilating g we can change the value of a at the expense of a complementary change to b. This is made precise in the next lemma.

Lemma 11.2. *Fix $g \in L^2(\mathbf{R})$ and $a,\, b \in \mathbf{R}$. Then given $r > 0$, $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ if and only if $\mathcal{G}(D_rg, a/r, br)$ is a frame for $L^2(\mathbf{R})$.*

Proof. Using the dilation $D_r g(x) = r^{1/2} g(rx)$, we have

$$\begin{aligned} D_r(M_{bn}T_{ak}g)(x) &= r^{1/2}(M_{bn}T_{ak}g)(rx) \\ &= r^{1/2}e^{2\pi ibnrx}g(rx-ak) \\ &= r^{1/2}e^{2\pi ibnrx}g(r(x-ak/r)) \\ &= M_{bnr}T_{ak/r}(D_r g)(x). \end{aligned}$$

Thus $\mathcal{G}(D_r g, a/r, br)$ is the image of $\mathcal{G}(g,a,b)$ under the dilation D_r. The result then follows from the fact that D_r is a unitary mapping of $L^2(\mathbf{R})$ onto itself. □

If $\mathcal{G}(g,a,b)$ is a Gabor frame, then its frame operator is

$$Sf = \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \langle f, M_{bn}T_{ak}g\rangle \, M_{bn}T_{ak}g.$$

The frame operator commutes with M_{bn} and T_{ak} for k, $n \in \mathbf{Z}$ (Exercise 11.3). A consequence of this is that S^{-1} also commutes with M_{bn} and T_{ak}, so we have $S^{-1}(M_{bn}T_{ak}g) = M_{bn}T_{ak}(S^{-1}g)$. Therefore the canonical dual of $\mathcal{G}(g,a,b)$ is another Gabor frame.

Lemma 11.3. *If $\mathcal{G}(g,a,b)$ is a Gabor frame for $L^2(\mathbf{R})$, then its canonical dual frame is $\mathcal{G}(\widetilde{g},a,b)$ where $\widetilde{g} = S^{-1}g$.* ♢

To each Gabor system $\mathcal{G}(g,a,b)$ we will associate the a-periodic function G_0 defined by

$$G_0(x) = \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 = \sum_{k\in\mathbf{Z}} |T_{ak}g(x)|^2, \qquad x \in \mathbf{R}.$$

Implicitly, G_0 depends on g and a. Note that G_0 is the a-periodization of $|g|^2$ in the sense of Exercise 10.13, and by that exercise we have $G_0 \in L^1[0,a]$ and

$$\int_0^a G_0(x)\,dx = \int_{-\infty}^{\infty} |g(x)|^2\,dx = \|g\|_{L^2}^2. \tag{11.2}$$

Exercises

11.1. Given $g \in L^2(\mathbf{R})$, show that $\{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{R})$ if and only if $\{T_{ak}M_{bn}g\}_{k,n\in\mathbf{Z}}$ is a frame, and in this case their frame operators coincide.

11.2. (a) Use the fact that $T_aM_b = e^{-2\pi iab}M_bT_a$ to show that the set $\{T_aM_b\}_{a,b\in\mathbf{R}}$ of time-frequency shift operators is not closed under compositions, and hence does not form a group.

(b) Define

$$\mathbb{H}_1 = \{e^{2\pi i t} T_a M_b\}_{a,b,t\in\mathbf{R}},$$

and show that $\mathbb{H}_1$ is a nonabelian group under composition of operators.

(c) Define

$$\mathbb{H}_2 = \mathbf{R}^3 = \{(a,b,t)\}_{a,b,t\in\mathbf{R}}.$$

Show that $\mathbb{H}_2$ is a nonabelian group with respect to the operation

$$(a,b,t) * (c,d,u) = (a+c, b+d, t+u+bc).$$

Show further that $\mathbb{H}_2$ is isomorphic to $\mathbb{H}_1$.

(d) Define

$$\mathbb{H}_3 = \left\{ \begin{bmatrix} 1 & b & t \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \right\}_{a,b,t\in\mathbf{R}}.$$

Show that $\mathbb{H}_3$ is a nonabelian group with respect to multiplication of matrices, and $\mathbb{H}_3$ is isomorphic to $\mathbb{H}_1$.

(e) Show that $a\mathbf{Z} \times b\mathbf{Z} \times \{0\} = \{(ak, bn, 0)\}_{k,n\in\mathbf{Z}}$ is not a subgroup of $\mathbb{H}_2$, but the countable subset $a\mathbf{Z}\times b\mathbf{Z}\times ab\mathbf{Z} = \{(ak, bn, abj)\}_{k,n,j\in\mathbf{Z}}$ is a subgroup.

(f) As a set, $\mathbb{H}_2 = \mathbf{R}^3$, and hence has a natural topology. In fact, $\mathbb{H}_2$ is an example of a *locally compact group* (LCG). Every LCG has associated left and right *Haar measures* (and these are unique up to scalar multiples). Show that the left Haar measure for $\mathbb{H}_2$ is $da\,db\,dt$, which means that for every $(c,d,u) \in \mathbb{H}_2$ we have

$$\iiint F\big((c,d,u) * (a,b,t)\big)\, da\, db\, dt = \iiint F(a,b,t)\, da\, db\, dt$$

for every integrable function F on $\mathbb{H}_2 = \mathbf{R}^3$. Show that the right Haar measure is also $da\,db\,dt$. Thus, even though $\mathbb{H}_2$ is nonabelian, its left and Haar right measures coincide (such an LCG is said to be *unimodular*).

Remark: The (isomorphic) groups $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{H}_3$ are called the *Heisenberg group*. The properties of the Heisenberg group should be contrasted with those of the affine group discussed in Exercise 12.2.

11.3. Let $\mathcal{G}(g,a,b)$ be a Gabor frame for $L^2(\mathbf{R})$.

(a) Show that the frame operator S commutes with M_{bn} and T_{ak} for all k, $n \in \mathbf{Z}$, and use this to show that S^{-1} also commutes with M_{bn} and T_{ak}.

(b) Show that the canonical dual frame of $\mathcal{G}(g,a,b)$ is the Gabor frame $\mathcal{G}(\widetilde{g},a,b)$ where $\widetilde{g} = S^{-1}g$.

(c) Suppose that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$. Show that $\mathcal{G}(g,a,b)$ is a Riesz basis if and only if $\langle g, \widetilde{g}\rangle = 1$.

(d) Show that the canonical Parseval frame of a lattice Gabor frame is another lattice Gabor frame. Specifically, if $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ and we set $g^\sharp = S^{-1/2}g$, where S is the frame operator, then $\mathcal{G}(g^\sharp,a,b)$ is a Parseval frame for $L^2(\mathbf{R})$.

11.4. Fix $g \in L^2(\mathbf{R})$ and $a, b > 0$. Recall from equations (9.2) and (9.3) that the Fourier transform interchanges translation with modulation. Use this to show that

$$\mathcal{G}(g,a,b) \text{ is a frame} \quad \Longleftrightarrow \quad \mathcal{G}(\widehat{g},b,a) \text{ is a frame.}$$

11.2 Painless Nonorthogonal Expansions

The simplest example of a Gabor frame is

$$\mathcal{G}(\chi_{[0,1]},1,1) \;=\; \bigl\{e^{2\pi inx}\,\chi_{[k,k+1]}(x)\bigr\}_{k,n\in\mathbf{Z}}.$$

If we fix a particular k, then by Example 1.52 we know that the sequence $\bigl\{e^{2\pi inx}\,\chi_{[k,k+1]}(x)\bigr\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[k,k+1]$. Hence the Gabor system $\mathcal{G}(\chi_{[0,1]},1,1)$ is simply the union of orthonormal bases for $L^2[k,k+1]$ over all $k \in \mathbf{Z}$, and consequently $\mathcal{G}(\chi_{[0,1]},1,1)$ is an orthonormal basis for $L^2(\mathbf{R})$.

Unfortunately, this Gabor system is not very useful in practice. The generator $\chi_{[0,1]}$ is very well localized in the time domain in the sense that it is zero outside of a finite interval. However, it is discontinuous, and this means that the expansion of a smooth function in the orthonormal basis $\mathcal{G}(\chi_{[0,1]},1,1)$ will not converge any faster than the expansion of a discontinuous function. From another viewpoint, the problem with the function $g = \chi_{[0,1]}$ is that its Fourier transform is a modulated sinc function:

$$\widehat{g}(\xi) \;=\; e^{-\pi i\xi}\,\frac{\sin \pi\xi}{\pi\xi}.$$

Thus $\widehat{g}$ decays only on the order of $1/|\xi|$ and is not even integrable. We want to find Gabor frames generated by functions that are both smooth and well localized.

We can try to create "better" Gabor systems by using a different atom g or different lattice parameters a, b. If we stick to functions g that are compactly supported in an interval of length $1/b$, then it is quite easy to create Gabor frames $\mathcal{G}(g,a,b)$ for $L^2(\mathbf{R})$, and we can even do so with smooth, compactly supported generators if we choose a and b appropriately. This was first done by Daubechies, Grossmann, and Meyer [DGM86], who referred to these as *Painless Nonorthogonal Expansions.*

Theorem 11.4 (Painless Nonorthogonal Expansions). *Fix a, $b > 0$ and $g \in L^2(\mathbf{R})$.*

(a) *If $0 < ab \le 1$ and $\operatorname{supp}(g) \subseteq [0, b^{-1}]$, then $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ if and only if there exist constants A, $B > 0$ such that*

$$Ab \;\le\; \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 \;\le\; Bb \text{ a.e.} \tag{11.3}$$

In this case, A, B are frame bounds for $\mathcal{G}(g,a,b)$.

(b) *If* $0 < ab < 1$, *then there exist* g *supported in* $[0, b^{-1}]$ *that satisfy equation* (11.3) *and are as smooth as we like (even infinitely differentiable).*

(c) *If* $ab = 1$, *then any* g *that is supported in* $[0, b^{-1}]$ *and satisfies equation* (11.3) *must be discontinuous.*

(d) *If* $ab > 1$ *and* g *is supported in* $[0, b^{-1}]$, *then equation* (11.3) *is not satisfied and* $\mathcal{G}(g, a, b)$ *is incomplete in* $L^2(\mathbf{R})$.

Proof. (a) Suppose that $\mathrm{supp}(g) \subseteq [0, b^{-1}]$ and equation (11.3) holds. Exercise 8.4 tells us that in order to show that $\mathcal{G}(g, a, b)$ is a frame, we need only establish that the frame bounds hold on a dense subset of $L^2(\mathbf{R})$. So, let us consider functions f in the dense subspace $C_c(\mathbf{R})$ (actually, continuity is not needed here, we could just as well restrict our attention to functions that are bounded and compactly supported). Since $g \in L^2(\mathbf{R})$ is supported within $[0, b^{-1}]$, the translated function $T_{ak}g$ belongs to $L^2(I_k)$, where $I_k = [ak, ak + b^{-1}]$. Since f is bounded, the product $f \cdot T_{ak}\overline{g}$ also belongs to $L^2(I_k)$. Now, $\{e^{2\pi i n x}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2[0, 1]$, so by making a change of variables it follows that

$$\{b^{1/2} e_{bn}\}_{n \in \mathbf{Z}} \;=\; \{b^{1/2} e^{2\pi i b n x}\}_{n \in \mathbf{Z}}$$

is an orthonormal basis for $L^2(I_k)$. Applying the Plancherel Equality (and keeping in mind that $T_{ak}g$ is supported in I_k), we therefore have

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)\, g(x - ak)|^2\, dx \;&=\; \int_{ak}^{ak+b^{-1}} \bigl|f(x)\, T_{ak}\overline{g(x)}\bigr|^2\, dx \\
&=\; \bigl\|f \cdot T_{ak}\overline{g}\bigr\|_{L^2(I_k)} \\
&=\; \sum_{n \in \mathbf{Z}} \bigl|\bigl\langle f \cdot T_{ak}\overline{g},\, b^{1/2} e_{bn}\bigr\rangle_{L^2(I_k)}\bigr|^2 \\
&=\; b \sum_{n \in \mathbf{Z}} \left| \int_{ak}^{ak+b^{-1}} f(x)\, \overline{g(x - ak)}\, e^{-2\pi i b n x}\, dx \right|^2 \\
&=\; b \sum_{n \in \mathbf{Z}} \left| \int_{-\infty}^{\infty} f(x)\, \overline{e^{2\pi i b n x}\, g(x - ak)}\, dx \right|^2 \\
&=\; b \sum_{n \in \mathbf{Z}} \bigl|\bigl\langle f,\, M_{bn} T_{ak} g \bigr\rangle\bigr|^2.
\end{aligned} \tag{11.4}$$

Hence, using Tonelli's Theorem to interchange the sum and integral,

$$\begin{aligned}\sum_{k,n\in\mathbf{Z}} \left|\langle f,\, M_{bn}T_{ak}g\rangle\right|^2 &= b^{-1}\sum_{k\in\mathbf{Z}}\int_{-\infty}^{\infty} |f(x)\,g(x-ak)|^2\,dx \\ &= b^{-1}\int_{-\infty}^{\infty} |f(x)|^2 \sum_{k\in\mathbf{Z}} |g(x-ak)|^2\,dx \qquad (11.5) \\ &\ge \int_{-\infty}^{\infty} |f(x)|^2\,A\,dx \;=\; A\,\|f\|_{L^2}^2.\end{aligned}$$

A similar computation shows that the upper frame bound estimate also holds for f. Since $C_c(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, we conclude that $\mathcal{G}(g,a,b)$ is a frame with frame bounds A, B.

We will improve on the converse implication in Theorem 11.6, so we omit the proof here.

(b) Suppose that $0 < ab < 1$, and let g be any continuous function such that $g(x) = 0$ outside of $[0, b^{-1}]$ and $g(x) > 0$ on $(0, b^{-1})$. For example, we could let g be the hat function supported on $[0, b^{-1}]$. Because $a < b^{-1}$, it follows that the a-periodic function $G_0(x) = \sum |g(x-ak)|^2$ is continuous and strictly positive at every point. Consequently, $0 < \inf G_0 \le \sup G_0 < \infty$, so $\mathcal{G}(g,a,b)$ is a frame by part (a).

There are many functions g that satisfy these requirements and are more smooth, even infinitely differentiable. For concrete examples, see Exercise 11.9.

(c) If $ab = 1$ then $a = b^{-1}$. If $\mathrm{supp}(g) \subseteq [0, b^{-1}] = [0, a]$ then $T_{ak}g$ is supported in $[ka, (k+1)a]$. If g is continuous then $g(0) = g(a) = 0$. Since the intervals $[ka, (k+1)a]$ overlap at at most one point, it follows that G_0 is continuous and $G_0(ka) = 0$ for every $k \in \mathbf{Z}$. Part (a) therefore implies that $\mathcal{G}(g,a,b)$ cannot be a frame.

(d) If $ab > 1$ then $a > b^{-1}$. Hence $G_0(x) = \sum |g(x-ak)|^2$ is zero on $[b^{-1}, a]$, so $\mathcal{G}(g,a,b)$ cannot be a frame. In fact, the function $\chi_{[b^{-1},a]}$ is orthogonal to every element of $\mathcal{G}(g,a,b)$, so this Gabor system is incomplete. □

Note that it is the product ab that is important in Theorem 11.4 because, by Lemma 11.2, we can change the value of a at the expense of a complementary change to b. Also, by translating g we can replace $[0, b^{-1}]$ by any interval of length b^{-1}.

Here is a more constructive approach to the proof of Theorem 11.4(b).

Example 11.5. For simplicity, assume that $\frac{1}{2} < ab < 1$. Then for any given x, the series $G_0(x) = \sum |g(x-ak)|^2$ contains at most two nonzero terms. Define a continuous function g supported on $[0, b^{-1}]$ by setting

$$g(x)^2 = \begin{cases} 0, & x < 0, \\ \text{linear}, & x \in [0, b^{-1} - a], \\ 1, & x \in [b^{-1} - a, a], \\ \text{linear}, & x \in [a, b^{-1}], \\ 0, & x > b^{-1}. \end{cases}$$

For this g we have $G_0(x) = 1$ for every $x \in \mathbf{R}$ (see Figure 11.2). Hence $\mathcal{G}(g, a, b)$ is a b^{-1}-tight frame, and by rescaling we can make it a Parseval frame if we wish. By using a smoother g, we can similarly create Parseval Gabor frames with generators that are as smooth as we like (Exercise 11.10). The construction becomes more complicated if $ab < \frac{1}{2}$ because there are more overlaps to consider, but the idea can be extended to any values of a, b with $0 < ab < 1$. ◇

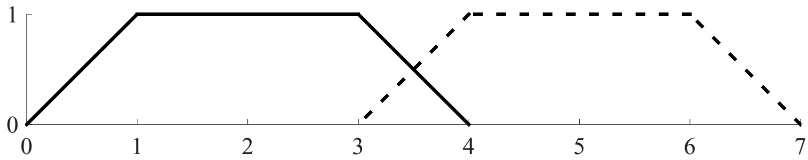

Fig. 11.2. Graphs of $g(x)^2$ and $g(x-a)^2$ from Example 11.5 using $a = 3$ and $b = 1/4$.

We summarize some of the important points in the Painless Nonorthogonal Expansions construction.

- If $0 < ab < 1$ then we can construct nice atoms g (smooth and compactly supported) such that $\mathcal{G}(g, a, b)$ is a frame or even a Parseval frame for $L^2(\mathbf{R})$.
- If $ab = 1$ then there exist Gabor frames $\mathcal{G}(g, a, b)$ for $L^2(\mathbf{R})$, but all of the frames constructed using the methods of this section have generators g that are discontinuous.
- If $ab > 1$ then no Gabor system with $\operatorname{supp}(g) \subseteq [0, b^{-1}]$ can be a frame for $L^2(\mathbf{R})$, and in fact $\mathcal{G}(g, a, b)$ must be incomplete in this case.

Exercise 11.6 refines these observations further, yielding the following additional facts.

- If $0 < ab < 1$ then the frames constructed in this section are redundant (not exact).
- If $ab = 1$ then the frames constructed in this section are exact and hence are Riesz bases for $L^2(\mathbf{R})$.

In the following sections, we will see that the properties listed above apply not only to the "Painless" constructions, but to all Gabor systems $\mathcal{G}(g, a, b)$. The analysis will not be quite as painless and will require new insights, but we will see that there are no "nice" Gabor Riesz bases $\mathcal{G}(g, a, b)$ at all, whereas there are many "well-behaved" redundant Gabor frames. Although it lies outside the scope of this volume, we remark that the utility of redundant Gabor frames extends far beyond the Hilbert space setting. Specifically, if $\mathcal{G}(g, a, b)$ is a Gabor frame that is generated by a function g that has sufficient simultaneous concentration in both time and frequency, then $\mathcal{G}(g, a, b)$ will be a frame not only for $L^2(\mathbf{R})$ but also for an entire range of associated function spaces $M^{p,q}_s(\mathbf{R})$ ($1 \le p,\ q \le \infty,\ s \in \mathbf{R}$) known as *modulation spaces.* These spaces quantify time-frequency concentration of functions (and distributions), and arise naturally in problems that involve both time and frequency. We refer to the text by Gröchenig [Grö01] for a beautiful development of this rich subject.

Exercises

11.5. Show that $\mathcal{G}(\chi_{[0,1]}, 1, 1)$ is an orthonormal basis for $L^2(\mathbf{R})$. Also show that $\mathcal{G}(\chi_{[0,1]}, a, 1)$ is a frame for $L^2(\mathbf{R})$ if and only if $0 < a \le 1$.

Remark: Amazingly, there is no known explicit characterization of the set of points (a, b) such that $\mathcal{G}(\chi_{[0,1]}, a, b)$ is a frame for $L^2(\mathbf{R})$, see [Jan03].

11.6. Assume that the hypotheses of part (a) of Theorem 11.4 are satisfied, i.e., $0 < ab \le 1$, $\operatorname{supp}(g) \subseteq [0, b^{-1}]$, and equation (11.3) holds. Prove the following statements about the frame $\mathcal{G}(g, a, b)$.

(a) The frame operator is pointwise multiplication by $b^{-1}G_0$, i.e., $Sf = b^{-1}G_0 f$ for $f \in L^2(\mathbf{R})$.

(b) The canonical dual frame is $\mathcal{G}(\widetilde{g}, a, b)$ where $\widetilde{g} = bg/G_0$.

(c) If $ab = 1$ then $\mathcal{G}(g, a, b)$ is a Riesz basis for $L^2(\mathbf{R})$.

(d) If $0 < ab < 1$ then $\mathcal{G}(g, a, b)$ is a redundant frame for $L^2(\mathbf{R})$.

11.7. Show that if $g \in C_c(\mathbf{R})$ is not the zero function, then there exist some $a,\ b > 0$ such that $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbf{R})$.

11.8. Let $g \in C_c(\mathbf{R})$ satisfy $\operatorname{supp}(g) = [0, b_0^{-1}]$ and $g(x) > 0$ for $x \in (0, b_0^{-1})$. Show that $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbf{R})$ for $0 < a < b_0^{-1}$ and $0 < b < b_0$.

11.9. This exercise will construct a compactly supported, infinitely differentiable function on the real line. Define $f(x) = e^{-1/x^2}\, \chi_{(0,\infty)}(x)$.

(a) Show that for every $n \in \mathbf{N}$, there exists a polynomial p_n of degree $3n$ such that

$$f^{(n)}(x) \;=\; p_n(x^{-1})\, e^{-x^{-2}}\, \chi_{(0,\infty)}(x).$$

Conclude that f is infinitely differentiable, every derivative of f is bounded, and $f^{(n)}(x) = 0$ for every $x \le 0$ and $n \ge 0$.

(b) Show that if $a < b$, then $g(x) = f(x-a)\, f(b-x)$ is infinitely differentiable, is zero outside of (a,b), and is strictly positive on (a,b).

11.10. Let $0 < ab < 1$ be fixed. By Exercise 11.9, there exists a function $f \in C_c^\infty(\mathbf{R})$ supported in $[0, b^{-1}]$ such that $f > 0$ on $(0, b^{-1})$.

(a) Set $F_0(x) = \sum_{k\in\mathbf{Z}} |f(x-ak)|^2$ and show that $g = f/F_0^{1/2}$ is infinitely differentiable, compactly supported, and satisfies $\sum_{k\in\mathbf{Z}} |g(x-ak)|^2 = 1$ everywhere.

(b) Show that there exists a function $g \in C_c^\infty(\mathbf{R})$ such that $\mathcal{G}(g,a,b)$ is a Parseval frame for $L^2(\mathbf{R})$.

11.3 The Nyquist Density and Necessary Conditions for Frame Bounds

Theorem 11.4, the Painless Nonorthogonal Expansions construction, gives necessary and sufficient conditions for the existence of Gabor frames $\mathcal{G}(g,a,b)$ when the atom g is supported in an interval of length $1/b$. This equivalence does not extend to general functions in $L^2(\mathbf{R})$. Still, the necessary part of the theorem does extend, as follows.

Theorem 11.6. *If $g \in L^2(\mathbf{R})$ and $a,\, b > 0$ are such that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ with frame bounds $A,\, B > 0$, then we must have $Ab \le G_0 \le Bb$ a.e., i.e.,*

$$Ab \;\le\; \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 \;\le\; Bb \quad \text{a.e.} \tag{11.6}$$

In particular, g must be bounded.

Proof. The proof is similar to the proof of part (a) of Theorem 11.4. However, now we do not know the support of g, so instead we restrict our attention to functions f that are bounded and supported in an interval I of length $1/b$. In this case the product $f \cdot T_{ak}\overline{g}$ belongs to $L^2(I)$. Since $\{b^{1/2}e_{bn}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$, it follows, just as in equation (11.4), that

$$b\sum_{n\in\mathbf{Z}} \bigl|\langle f,\, M_{bn}T_{ak}g\rangle\bigr|^2 \;=\; \int_{-\infty}^{\infty} |f(x)\, g(x-ak)|^2\, dx.$$

Applying the lower frame bound for $\mathcal{G}(g,a,b)$, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2\, G_0(x)\, dx \;&=\; \sum_{k\in\mathbf{Z}} \int_{-\infty}^{\infty} |f(x)\, g(x-ak)|^2\, dx \\
&=\; b \sum_{k,n\in\mathbf{Z}} \bigl|\langle f,\, M_{bn}T_{ak}g\rangle\bigr|^2
\end{aligned}$$

$$\geq\ bA\,\|f\|_{L^2}^2$$
$$=\ bA\int_{-\infty}^{\infty}|f(x)|^2\,dx.$$

Thus, for every bounded $f \in L^2(I)$ we have

$$\int_{-\infty}^{\infty}|f(x)|^2\,\big(G_0(x)-bA\big)\,dx\ \geq\ 0. \tag{11.7}$$

Now, if $G_0(x) < bA$ on some subset E of I that has positive measure, then we could take $f = \chi_E$ and obtain a contradiction to equation (11.7). Therefore we must have $G_0 \geq bA$ a.e. on I, and a similar calculation using the upper frame bound gives $G_0 \leq bB$ a.e. on I. Since I is an arbitrary interval of length $1/b$ and since the real line can be covered by countably many translates of I, we conclude that $bA \leq G_0 \leq bB$ a.e. on $\mathbf{R}$. □

Combining Theorem 11.6 with Exercise 11.3 gives several interesting corollaries for Gabor frames. Note that the statements in the next corollary apply to all Gabor frames $\mathcal{G}(g,a,b)$, not just those with compactly supported atoms g.

Corollary 11.7 (Density and Frame Bounds). *Fix $g \in L^2(\mathbf{R})$ and a, $b > 0$. If $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ with frame bounds A, B, then the following statements hold.*

(a) $Aab \leq \|g\|_{L^2}^2 \leq Bab$.

(b) *If $\mathcal{G}(g,a,b)$ is a Parseval frame, then $\|g\|_{L^2}^2 = ab$.*

(c) $0 < ab \leq 1$.

(d) $\langle g, \widetilde{g}\,\rangle = ab$, *where $\widetilde{g} = S^{-1}g$ is the generator of the canonical dual frame.*

(e) *$\mathcal{G}(g,a,b)$ is a Riesz basis if and only if $ab = \langle g, \widetilde{g}\,\rangle = 1$.*

Proof. (a), (b) Integrating equation (11.6) over the interval $[0,a]$, we have

$$Aab\ =\ \int_0^a Ab\,dx\ \leq\ \int_0^a\sum_{k\in\mathbf{Z}}|g(x-ak)|^2\,dx\ \leq\ \int_{-\infty}^{\infty}|g(x)|^2\,dx\ =\ \|g\|_{L^2}^2.$$

A similar calculation shows that $\|g\|_{L^2}^2\ \leq\ Bab$. If the frame is Parseval then $A = B = 1$.

(c) By Exercise 11.3, if we set $g^\sharp = S^{-1/2}g$ then $\mathcal{G}(g^\sharp,a,b)$ is a Parseval frame. Part (b) therefore implies that $\|g^\sharp\|_{L^2}^2 = ab$. On the other hand, the elements of a Parseval frame can have at most unit norm (see Exercise 7.5), so we must have $\|g^\sharp\|_{L^2}^2 \leq 1$. Hence $ab \leq 1$.

(d) Combining $\|g^\sharp\|_{L^2}^2 = ab$ with the fact that $S^{-1/2}$ is self-adjoint,

$$\langle g,\widetilde{g}\,\rangle\ =\ \langle g, S^{-1/2}S^{-1/2}g\rangle\ =\ \langle S^{-1/2}g, S^{-1/2}g\rangle\ =\ \|g^\sharp\|_{L^2}^2\ =\ ab.$$

(e) This follows by combining part (d) with Corollary 8.23 (see also Exercise 11.3). □

Parts (a) and (b) of Corollary 11.7 were proved by Daubechies in her seminal paper [Dau90]. The first proof of part (c) was given by Ramanathan and Steger [RS95] as a special case of their results on irregular Gabor systems. The simple proof of part (c) given here appears to have been first presented by Balan [Bal98], but has been independently discovered several times.

Looking at parts (c) and (e) of Corollary 11.7 a little more closely, we see that the value of ab separates Gabor frames into three categories:

- If $ab > 1$ then $\mathcal{G}(g, a, b)$ is not a frame.
- If $\mathcal{G}(g, a, b)$ is a frame and $ab = 1$ then it is a Riesz basis.
- If $\mathcal{G}(g, a, b)$ is a frame and $0 < ab < 1$ then it is a redundant frame.

We saw in Section 11.2 that this trichotomy held for the Painless Nonorthogonal Expansions, and now we see that it holds for all Gabor systems. The value $1/(ab)$ is called the *density* of the Gabor system $\mathcal{G}(g, a, b)$, because the number of points of $a\mathbf{Z} \times b\mathbf{Z}$ that lie in a given ball in $\mathbf{R}^2$ is asymptotically $1/(ab)$ times the volume of the ball as the radius increases to infinity. We refer to the density $1/(ab) = 1$ as the *critical density* or the *Nyquist density.*

In fact, the trichotomy for the Painless Nonorthogonal Expansions was even more pronounced. We proved in Theorem 11.4(d) that if $ab > 1$ and $g \in L^2(\mathbf{R})$ is supported in $[0, b^{-1}]$ then $\mathcal{G}(g, a, b)$ is *incomplete.* In contrast, Corollary 11.7 only tells us that $\mathcal{G}(g, a, b)$ cannot be a frame, which is a weaker statement. Although it is more difficult to prove, it is true that if g is *any* function in $L^2(\mathbf{R})$ and $ab > 1$ then $\mathcal{G}(g, a, b)$ must be incomplete in $L^2(\mathbf{R})$. The first explicit proof of this fact was given by Baggett [Bag90], using the representation theory of the discrete Heisenberg group. It was also proved by Daubechies for the case that ab is rational [Dau90], and she also pointed out that a proof for general $ab > 1$ can be inferred from results of Rieffel [Rie81] on the coupling constants of C^*-algebras.

There is still a surprise left for us in the case $ab > 1$. Comparing Theorem 11.6 to Theorem 10.19 we see some suspiciously similar equations. Theorem 11.6 tells us that if $\mathcal{G}(g, a, b) = \{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ is a Gabor frame for $L^2(\mathbf{R})$ with frame bounds A, B, then

$$Ab \;\le\; \sum_{k\in\mathbf{Z}} |g(x - ak)|^2 \;\le\; Bb \;\; \text{a.e.}$$

After making the appropriate changes of variable (see Exercise 11.12), Theorem 10.19 says that $\mathcal{T}(g) = \{T_{ak}g\}_{k\in\mathbf{Z}}$ is a Riesz basis for its closed span with frame bounds A, B if and only if

$$Aa \;\le\; \sum_{k\in\mathbf{Z}} |\widehat{g}(\xi - \tfrac{k}{a})|^2 \;\le\; Ba \;\; \text{a.e.,} \tag{11.8}$$

where $\widehat{g}$ is the Fourier transform of g. Coordinating properly between g and $\widehat{g}$, a and $\frac{1}{a}$, and b and $\frac{1}{b}$, we find that there are Riesz sequences of translates of g and $\widehat{g}$ associated with every Gabor frame, even redundant frames!

Theorem 11.8. *Assume $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbf{R})$ with frame bounds A, B, and let $\widehat{g}$ be the Fourier transform of g. Then the following statements hold.*

(a) $Aa \le \sum_{n\in\mathbf{Z}} |\widehat{g}(\xi - bn)|^2 \le Ba$ *a.e.*

(b) $\{T_{n/b}g\}_{n\in\mathbf{Z}}$ *is a Riesz sequence in $L^2(\mathbf{R})$ with frame bounds Aab, Bab (as a frame for its closed span).*

(c) $\{T_{k/a}\widehat{g}\}_{k\in\mathbf{Z}}$ *is a Riesz sequence in $L^2(\mathbf{R})$ with frame bounds Aab, Bab (as a frame for its closed span).*

Proof. (a) Suppose that $\mathcal{G}(g, a, b)$ is a frame. Exercise 11.4 shows that the image of $\mathcal{G}(g, a, b)$ under the Fourier transform is $\mathcal{G}(\widehat{g}, b, a)$. Since the Fourier transform is unitary, $\mathcal{G}(\widehat{g}, b, a)$ must be a frame with the same frame bounds as $\mathcal{G}(g, a, b)$. Statement (a) then follows by applying Theorem 11.6 to $\mathcal{G}(\widehat{g}, b, a)$.

(b) Write part (a) as

$$\frac{Aab}{b} \le \sum_{n\in\mathbf{Z}} |\widehat{g}(\xi - bn)|^2 \le \frac{Bab}{b} \quad \text{a.e.}$$

Comparing this to equation (11.8), we see that $\{T_{n/b}g\}_{n\in\mathbf{Z}}$ is a Riesz basis for its closed span, and the frame bounds are Aab, Bab.

(c) This follows by applying part (b) to the frame $\mathcal{G}(\widehat{g}, b, a)$. □

Thus, even if $\mathcal{G}(g, a, b)$ is a *redundant* frame (which cannot have a biorthogonal sequence), $\{T_{n/b}g\}_{n\in\mathbf{Z}}$ is a *Riesz sequence* and therefore has a biorthogonal sequence! Although we will not prove it, Theorem 11.8 is actually only a part of a result that seems *very* surprising (at least when first encountered).

Theorem 11.9 (Duality Principle). *Given $g \in L^2(\mathbf{R})$ and a, $b > 0$, the following statements are equivalent.*

(a) $\mathcal{G}(g, a, b) = \{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ *is a frame for $L^2(\mathbf{R})$, with frame bounds A, B.*

(b) $\mathcal{G}(g, 1/b, 1/a) = \{M_{k/a}T_{n/b}g\}_{k,n\in\mathbf{Z}}$ *is a Riesz sequence in $L^2(\mathbf{R})$, with frame bounds Aab, Bab (as a frame for its closed span).* ◊

Thus, the property of being a frame with respect to the lattice $a\mathbf{Z} \times b\mathbf{Z}$ is dual to the property of being a Riesz sequence with respect to the lattice $\frac{1}{b}\mathbf{Z} \times \frac{1}{a}\mathbf{Z}$ (which is called the *adjoint lattice* to $a\mathbf{Z} \times b\mathbf{Z}$). In spirit, this is similar to the fact that if the rows of a rectangular $m \times n$ matrix span $\mathbf{R}^n$, then its columns are linearly independent vectors in $\mathbf{R}^m$, and conversely.

Independent and essentially simultaneous proofs of Theorem 11.9 were published by Daubechies, H. Landau, and Z. Landau [DLL95], Janssen [Jan95], and Ron and Shen [RS97], each with a completely different technique.

Theorem 11.9 gives us the following addition to the "trichotomy facts" discussed previously.

Corollary 11.10. *If $\mathcal{G}(g,a,b)$ is a Riesz sequence in $L^2(\mathbf{R})$, then $ab \ge 1$.*

Proof. If $\mathcal{G}(g,a,b)$ is a Riesz sequence, then $\mathcal{G}(g,1/b,1/a)$ is a frame by Theorem 11.9. Corollary 11.7 therefore implies that $\frac{1}{b}\frac{1}{a} \le 1$, so $ab \ge 1$. □

For additional discussion and extensive references related to the material of this section we refer to the survey paper [Hei07].

Exercises

11.11. Fix $g \in L^2(\mathbf{R})$ and $a,\, b > 0$.

(a) Show that $\mathcal{G}(g,a,b)$ is a Riesz basis for $L^2(\mathbf{R})$ if and only if it is a frame and $ab = 1$.

(b) Show that $\mathcal{G}(g,a,b)$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if it is a tight frame, $ab = 1$, and $\|g\|_{L^2} = 1$.

11.12. Given $g \in L^2(\mathbf{R})$, show that $\mathcal{T}(g) = \{T_k g\}_{k\in\mathbf{Z}}$ is a Riesz basis for its closed span with frame bounds A, B if and only if equation (11.8) holds.

11.13. Suppose that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$. Without appealing to Theorem 11.9, show that $\{M_{k/a}\widehat{g}\}_{n\in\mathbf{Z}}$ and $\{M_{n/b}\widehat{g}\}_{n\in\mathbf{Z}}$ are Riesz sequences in $L^2(\mathbf{R})$.

11.14. Assuming Theorem 11.9, show that $\mathcal{G}(g,a,b)$ is a tight frame for $L^2(\mathbf{R})$ if and only if $\mathcal{G}(g,1/b,1/a)$ is an orthogonal sequence in $L^2(\mathbf{R})$.

11.4 Wiener Amalgam Spaces

Now we introduce a family of Banach spaces that will play an important role in our further analysis of Gabor frames. While the L^p spaces are ubiquitous in analysis, one of their limitations is that the L^p-norm is defined by a "global" criterion alone. As the following example shows, we can rearrange functions in many ways that do not change their L^p-norms but do change other properties.

Example 11.11. Recall that the box function $\chi_{[0,1)}$ generates a Gabor system $\mathcal{G}(\chi_{[0,1)},1,1)$ that is an orthonormal basis for $L^2(\mathbf{R})$. Although the box function has the disadvantage of being discontinuous, it at least has the advantage of being well localized in time.

Now let us create a new function by dividing the interval $[0,1)$ into the infinitely many pieces $[0,\frac{1}{2})$, $[\frac{1}{2},\frac{3}{4})$, $[\frac{3}{4},\frac{7}{8})$, ... and then "sending those pieces off to infinity." That is, we define

$$g = \chi_{[0,\frac{1}{2})} + T_1\chi_{[\frac{1}{2},\frac{3}{4})} + T_2\chi_{[\frac{3}{4},\frac{7}{8})} + \cdots \tag{11.9}$$

$$= \chi_{[0,\frac{1}{2})} + \chi_{[1+\frac{1}{2},1+\frac{3}{4})} + \chi_{[2+\frac{3}{4},2+\frac{7}{8})} + \cdots . \tag{11.10}$$

Not only is this function discontinuous, but it does not decay at infinity. Even so, it has exactly the same L^p-norm as $\chi_{[0,1)}$, and because we translated the "pieces" by integers it follows that $\mathcal{G}(g,1,1)$ is also an orthonormal basis for $\mathcal{G}(g,1,1)$ (Exercise 11.16). However, we cannot distinguish between the well localized function $\chi_{[0,1)}$ and the poorly localized function g by considering their L^p-norms $\|\chi_{[0,1)}\|_{L^p}$ and $\|g\|_{L^p}$. ♢

The amalgam spaces are determined by a norm which amalgamates, or mixes, a local criterion for membership with a global criterion. Or, it may be more precise to interpret the norm as giving a global criterion for a local property of the function. Special cases were first introduced by Wiener [Wie26], [Wie33]. A more general class of amalgams, named *Wiener amalgam spaces*, was introduced and extensively studied by Feichtinger, with some of the main papers being [FG85], [Fei87], [Fei90]. We refer to [Hei03] for an introductory survey of amalgam spaces with references to the original papers. We will need the following simple amalgams, which mix a local L^p criterion with a global ℓ^q criterion.

Definition 11.12 (Wiener Amalgam Spaces). Given $1 \le p \le \infty$ and $1 \le q < \infty$, the *Wiener amalgam space* $W(L^p, \ell^q)$ consists of those functions $f \in L^p(\mathbf{R})$ for which the norm

$$\|f\|_{W(L^p,\ell^q)} \;=\; \biggl(\sum_{k\in\mathbf{Z}} \|f\cdot\chi_{[k,k+1]}\|_{L^p}^q\biggr)^{1/q}$$

is finite. For $q = \infty$ we substitute the ℓ^∞-norm for the ℓ^q-norm above, i.e.,

$$\|f\|_{W(L^p,\ell^\infty)} \;=\; \sup_{k\in\mathbf{Z}} \|f\cdot\chi_{[k,k+1]}\|_{L^p}.$$

We also define

$$W(C,\ell^q) \;=\; \bigl\{f \in W(L^\infty,\ell^q) : f \text{ is continuous}\bigr\},$$

and we impose the norm $\|\cdot\|_{W(L^\infty,\ell^q)}$ on $W(C,\ell^q)$. ♢

Thus a function in $W(L^p,\ell^q)$ is locally an L^p function, and globally the values $\|f\cdot\chi_{[k,k+1]}\|_{L^p}$ decay in an ℓ^q manner. The space $W(L^\infty,\ell^2)$ made an appearance earlier in this volume; see Lemma 10.24.

Note that $W(L^p,\ell^p) = L^p(\mathbf{R})$. By Exercise 11.15, $W(L^p,\ell^q)$ and $W(C,\ell^q)$ are Banach spaces.

The space $W(L^\infty,\ell^1)$ will be especially important to us in the coming pages. A function g in this space is "locally bounded" and has an "ℓ^1-type decay" at infinity.

Here are some of the properties of $W(L^\infty,\ell^1)$. In particular, part (d) of this result says that the intervals $[k,k+1]$ in the definition of the amalgam norm can be replaced by intervals $[ak,a(k+1)]$ in the sense of giving an equivalent norm on the space. The constants in this norm equivalence will be expressed in terms of the numbers

$$C_a = \max\{1+a,\, 2\}.$$

Theorem 11.13. (a) *$W(L^\infty,\ell^1)$ is contained in $L^p(\mathbf{R})$ for $1 \le p \le \infty$, and is dense in $L^p(\mathbf{R})$ for $1 \le p < \infty$.*

(b) *$W(L^\infty,\ell^1)$ is closed under translations, and for each $b \in \mathbf{R}$ we have*

$$\|T_b f\|_{W(L^\infty,\ell^1)} \le 2\,\|f\|_{W(L^\infty,\ell^1)}. \tag{11.11}$$

(c) *$W(L^\infty,\ell^1)$ is an ideal in $L^\infty(\mathbf{R})$ with respect to pointwise products, i.e.,*

$$f \in L^\infty(\mathbf{R}),\ g \in W(L^\infty,\ell^1) \implies fg \in W(L^\infty,\ell^1),$$

and

$$\|fg\|_{W(L^\infty,\ell^1)} \le \|f\|_{L^\infty}\,\|g\|_{W(L^\infty,\ell^1)}. \tag{11.12}$$

(d) *Given $a > 0$,*

$$|||f|||_a = \sum_{k\in\mathbf{Z}} \|f\cdot\chi_{[ak,a(k+1)]}\|_{L^\infty}$$

is an equivalent norm for $W(L^\infty,\ell^1)$, with

$$\frac{1}{C_{1/a}}\,|||f|||_a \le \|f\|_{W(L^\infty,\ell^1)} \le C_a\,|||f|||_a. \tag{11.13}$$

Proof. We will prove the upper inequality in equation (11.13), and assign the remainder of the proof as Exercise 11.17.

Fix $a > 0$, and define

$$I_k = \{n \in \mathbf{Z} : [k,k+1] \cap [an,a(n+1)] \ne \emptyset\},$$
$$J_n = \{k \in \mathbf{Z} : [k,k+1] \cap [an,a(n+1)] \ne \emptyset\}.$$

If $a \ge 1$ then $|J_n| \le 1+a$, while if $0 < a \le 1$ then $|J_n| \le 2$. Hence $|J_n| \le C_a$, independently of n. Therefore

$$\begin{aligned}
\|f\|_{W(L^\infty,\ell^1)} &= \sum_{k\in\mathbf{Z}} \|f\cdot\chi_{[k,k+1]}\|_{L^\infty}\\
&\le \sum_{k\in\mathbf{Z}}\sum_{n\in I_k} \|f\cdot\chi_{[an,a(n+1)]}\|_{L^\infty}\\
&= \sum_{n\in\mathbf{Z}}\sum_{k\in J_n} \|f\cdot\chi_{[an,a(n+1)]}\|_{L^\infty}\\
&\le C_a \sum_{n\in\mathbf{Z}} \|f\cdot\chi_{[an,a(n+1)]}\|_{L^\infty}. \quad \square
\end{aligned}$$

Rewording part of Theorem 11.13(d) gives us the following inequality.

Corollary 11.14. *If* $f \in W(L^\infty, \ell^1)$ *and* $a > 0$, *then*

$$\sum_{k \in \mathbf{Z}} \|T_{ak}f \cdot \chi_{[0,a]}\|_{L^\infty} \;\le\; C_{1/a}\, \|f\|_{W(L^\infty,\ell^1)}.$$

Proof. We simply have to note that

$$\sum_{k \in \mathbf{Z}} \|T_{ak}f \cdot \chi_{[0,a]}\|_{L^\infty} \;=\; \sum_{k \in \mathbf{Z}} \|f \cdot \chi_{[ak,a(k+1)]}\|_{L^\infty}$$

and apply the lower inequality in equation (11.13). □

While the periodization of a generic function in $L^1(\mathbf{R})$ is integrable over a period (Exercise 10.13), the periodization of a function $g \in W(L^\infty, \ell^1)$ is bounded.

Lemma 11.15. *Fix* $a > 1$. *If* $g \in W(L^\infty, \ell^1)$ *then its* a*-periodization*

$$\varphi(x) \;=\; \sum_{n \in \mathbf{Z}} g(x + an) \;=\; \sum_{n \in \mathbf{Z}} T_{an}g(x)$$

is a*-periodic, bounded, and satisfies*

$$\|\varphi\|_{L^\infty} \;=\; \Big\|\sum_{n \in \mathbf{Z}} T_{an}g\Big\|_{L^\infty} \;\le\; C_{1/a}\, \|g\|_{W(L^\infty,\ell^1)}. \tag{11.14}$$

Proof. The function φ is a-periodic and integrable by Exercise 10.13. Using the periodicity, we therefore have

$$\|\varphi\|_{L^\infty} \;=\; \|\varphi \cdot \chi_{[0,a]}\|_{L^\infty} \;=\; \Big\|\sum_{n \in \mathbf{Z}} T_{an}g \cdot \chi_{[0,a]}\Big\|_{L^\infty} \;\le\; C_{1/a}\, \|g\|_{W(L^\infty,\ell^1)},$$

where the final inequality comes from Corollary 11.14. □

Exercises

11.15. Prove that $W(L^p, \ell^q)$ is a Banach space for each p, q, and $W(C, \ell^q)$ is a closed subspace of $W(L^\infty, \ell^q)$.

11.16. Let g be the function defined in Example 11.11. Show that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbf{R})$, but $g \notin W(L^\infty, \ell^1)$.

11.17. Complete the proof of Theorem 11.13.

11.5 The Walnut Representation

The Painless Nonorthogonal Expansions give us many examples of Gabor frames, but they are limited by the requirement that the atom g be supported in an interval of length $1/b$. This support assumption produces some "miraculous cancellations" that allow us to write the frame condition in very simple terms. Indeed, equation (11.5) tells us that if g is supported in $[0, b^{-1}]$ then

$$\sum_{k,n\in\mathbf{Z}} \bigl|\langle f, M_{bn}T_{ak}g\rangle\bigr|^2 \;=\; b^{-1}\int_{-\infty}^{\infty} |f(x)|^2\, G_0(x)\, dx.$$

While the left-hand side of this equation is quite complicated, involving both time shifts of g and multiplications by complex exponentials $e^{2\pi ibnx}$, the right-hand side is extremely simple, involving a single multiplication. Even the function G_0 is quite simple, being built purely out of translates of g:

$$G_0(x) \;=\; \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 \;=\; \sum_{k\in\mathbf{Z}} |T_{ak}g(x)|^2.$$

Upon closer examination, what lies behind the miraculous cancellations in the Painless Nonorthogonal Expansions is the Plancherel Equality: g is supported in $[0, b^{-1}]$ and $\{b^{1/2}e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0, b^{-1}]$. If the support of g is not contained in a single interval of length b^{-1}, then the analysis of the frame condition becomes much more involved. The Walnut Representation [Wal92] is a result of this analysis, and it provides a fundamental characterization of the frame operator for Gabor systems with a much broader class of atoms g. The idea is simply that we break an arbitrary function g into pieces of length b^{-1}, analyze each piece, and paste the pieces back together. In the end we obtain a representation of the frame that is expressed purely in terms of translation operators—no modulations! This representation plays a fundamental role in time-frequency analysis, especially in the extension of the frame properties of Gabor systems from $L^2(\mathbf{R})$ to other function spaces.

A forerunner of the Walnut Representation was used by Daubechies in her paper [Dau90]. Walnut's work appears in [Wal89], [Wal92], [Wal93], and some of it is also summarized in the survey paper [HW89]. We will develop the Walnut Representation in $L^2(\mathbf{R})$, and refer to the text [Grö01] for extensions beyond the Hilbert space setting.

The delicate part of the proof of the Walnut Representation lies in pasting the pieces back together. Here, it becomes necessary to place a mild restriction on g. Specifically, we need g to lie in the Wiener amalgam space $W(L^\infty, \ell^1)$. This excludes functions that have extremely poor decay at infinity, like the one given in equation (11.9), but still leaves us with a very large class of atoms to choose from. Given this restriction, we can define a family of correlation functions associated with g, of which G_0 is only the first member.

Definition 11.16. Given $g \in W(L^\infty, \ell^1)$ and $a,\, b > 0$, we define associated *correlation functions* G_n by

$$G_n(x) \;=\; \sum_{k \in \mathbf{Z}} g(x-ak)\, \overline{g(x-ak-\tfrac{n}{b})}, \qquad n \in \mathbf{Z}. \qquad \diamondsuit$$

In particular, $G_0(x) = \sum_{k\in\mathbf{Z}} |g(x-ak)|^2$.

Note how both the usual lattice $a\mathbf{Z} \times b\mathbf{Z}$ and the adjoint lattice $\frac{1}{b}\mathbf{Z} \times \frac{1}{a}\mathbf{Z}$ from the Duality Principle play a role in the definition of the correlation functions!

It is often useful to write G_n in the forms

$$G_n \;=\; \sum_{k\in\mathbf{Z}} T_{ak}g \cdot T_{ak+\frac{n}{b}}\bar{g} \;=\; \sum_{k\in\mathbf{Z}} T_{ak}\big(g \cdot T_{\frac{n}{b}}\bar{g}\big). \tag{11.15}$$

Thus G_n is the a-periodization of $g \cdot T_{\frac{n}{b}}\bar{g}$. Since g belongs to $W(L^\infty, L^1)$, it is bounded, and therefore the product $g \cdot T_{\frac{n}{b}}\bar{g}$ belongs to $W(L^\infty, \ell^1)$ by Theorem 11.13(c). Applying Lemma 11.15 to this function, we see that G_n is well defined, a-periodic, and bounded. The next lemma shows that the L^∞-norms of the G_n are actually very well controlled.

Lemma 11.17. *If $g \in W(L^\infty, \ell^1)$ then $G_n \in L^\infty(\mathbf{R})$ and*

$$\sum_{n\in\mathbf{Z}} \|G_n\|_{L^\infty} \;\le\; 2\, C_{1/a}\, C_b\, \|g\|^2_{W(L^\infty,\ell^1)}.$$

Proof. By Lemma 11.15, using the form of G_n given in equation (11.15) we see that

$$\|G_n\|_{L^\infty} \;=\; \bigg\|\sum_{k\in\mathbf{Z}} T_{ak}\big(g\cdot T_{\frac{n}{b}}\bar{g}\big)\bigg\|_{L^\infty} \;\le\; C_{1/a}\, \big\|g \cdot T_{\frac{n}{b}}\bar{g}\big\|_{W(L^\infty,\ell^1)}.$$

Since $|\bar{g}| = |g|$, we therefore have

$$\begin{aligned}
\sum_{n\in\mathbf{Z}} \|G_n\|_{L^\infty} \;&\le\; C_{1/a} \sum_{n\in\mathbf{Z}} \big\|g\cdot T_{\frac{n}{b}}g\big\|_{W(L^\infty,\ell^1)} \\
&=\; C_{1/a} \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \big\|g\cdot \chi_{[k,k+1]} \cdot T_{\frac{n}{b}}g \cdot \chi_{[k,k+1]}\big\|_{L^\infty} \\
&\le\; C_{1/a} \sum_{k\in\mathbf{Z}} \|g\cdot\chi_{[k,k+1]}\|_{L^\infty} \bigg(\sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\|_{L^\infty}\bigg).
\end{aligned}$$

The series in parentheses on the last line resembles the $W(L^\infty, \ell^1)$ norm of $T_{\frac{n}{b}}g$, but it is not since the summation is over n instead of k. Instead, after some work similar to that used in the proof of Theorem 11.13(d), we see that

$$\begin{aligned}\sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\|_{L^\infty} &= \sum_{n\in\mathbf{Z}} \|g\cdot\chi_{[-\frac{n}{b}+k,-\frac{n}{b}+k+1]}\|_{L^\infty}\\ &\le 2C_b \sum_{m\in\mathbf{Z}} \|g\cdot\chi_{[m,m+1]}\|_{L^\infty}\\ &= 2C_b\,\|g\|_{W(L^\infty,\ell^1)}.\end{aligned}$$

The main issue in the computation above is that an interval of the form $[m, m+1]$ intersects at most $2C_b$ intervals of the form $[-\frac{n}{b}+k, -\frac{n}{b}+k+1]$ with $n \in \mathbf{Z}$. Hence

$$\begin{aligned}\sum_{n\in\mathbf{Z}} \|G_n\|_{L^\infty} &\le 2\,C_{1/a}\,C_b \sum_{k\in\mathbf{Z}} \|g\cdot\chi_{[k,k+1]}\|_{L^\infty}\,\|g\|_{W(L^\infty,\ell^1)}\\ &= 2\,C_{1/a}\,C_b\,\|g\|^2_{W(L^\infty,\ell^1)}. \quad \square\end{aligned}$$

Now we can derive the Walnut Representation. While not every function in $W(L^\infty, \ell^1)$ will generate a Gabor frame, the next theorem tells us that $\mathcal{G}(g, a, b)$ will always be a Bessel sequence, no matter what values of a, $b > 0$ that we choose. Therefore $\mathcal{G}(g, a, b)$ has a well-defined frame operator that maps $L^2(\mathbf{R})$ into itself, and the Walnut Representation realizes this frame operator solely in terms of translations. A simple trick that we will employ several times in the proof is to write

$$\sum_{n\in\mathbf{Z}} \int_0^{b^{-1}} h\big(x-\tfrac{n}{b}\big)\,dx \;=\; \int_{-\infty}^{\infty} h(x)\,dx \;=\; \int_0^{b^{-1}} \sum_{n\in\mathbf{Z}} h\big(x-\tfrac{n}{b}\big)\,dx.$$

This is valid for any function $h \in L^1(\mathbf{R})$.

Theorem 11.18 (Walnut Representation). *Let $g \in W(L^\infty, \ell^1)$ and a, $b > 0$ be given. Then $\mathcal{G}(g, a, b)$ is a Bessel sequence, and its frame operator is given by*

$$Sf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} f\cdot G_n, \qquad f \in L^2(\mathbf{R}). \tag{11.16}$$

Proof. Lemma 11.17 implies that the series

$$Lf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} \cdot G_n$$

converges absolutely in $L^2(\mathbf{R})$ for each $f \in L^2(\mathbf{R})$. Moreover,

$$\|Lf\| \;\le\; b^{-1} \sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}} f\|_{L^2}\,\|G_n\|_{L^\infty} \;\le\; B\,\|f\|_{L^2}$$

where

$$B \;=\; \frac{2}{b}\,C_{1/a}\,C_b\,\|g\|^2_{W(L^\infty,\ell^1)}.$$

Hence L is a bounded operator on $L^2(\mathbf{R})$.

By Theorem 7.4, to show that $\mathcal{G}(g,a,b)$ is a Bessel sequence we only need to establish that the Bessel bound holds on a dense subspace of $L^2(\mathbf{R})$. We will show that B is a Bessel bound on the dense subspace $C_c(\mathbf{R})$.

Fix $f \in C_c(\mathbf{R})$ and $k \in \mathbf{Z}$. Then $f \cdot T_{ak}\bar{g}$ is bounded and compactly supported, so its b^{-1}-periodization

$$F_k(x) \;=\; \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}$$

belongs to $L^2[0,b^{-1}]$ (and in fact is bounded). Since F_k is b^{-1}-periodic, we have $F_k\bigl(x-\frac{j}{b}\bigr) = F_k(x)$ for $j \in \mathbf{Z}$.

Using the fact that $\{b^{1/2}e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,b^{-1}]$, we compute that

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \\
&\qquad= \sum_{n\in\mathbf{Z}} \left|\int_{-\infty}^{\infty} f(x)\,e^{-2\pi ibnx}\,\overline{g(x-ak)}\,dx\right|^2 \\
&\qquad= \sum_{n\in\mathbf{Z}} \left|\int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,e^{-2\pi ibn\left(x-\frac{j}{b}\right)}\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}\,dx\right|^2 \\
&\qquad= \sum_{n\in\mathbf{Z}} \left|\int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}\,e^{-2\pi ibnx}\,dx\right|^2 \\
&\qquad= \sum_{n\in\mathbf{Z}} \bigl|\bigl\langle F_k,\, e_{bn}\bigr\rangle_{L^2[0,b^{-1}]}\bigr|^2 \\
&\qquad= \bigl\|F_k\bigr\|^2_{L^2[0,b^{-1}]} \\
&\qquad= b^{-1}\int_0^{b^{-1}} |F_k(x)|^2\,dx.
\end{aligned}$$

Assuming that we can interchange the integral and sum as indicated, and using the fact that F_n is b^{-1}-periodic, we therefore have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \\
&\qquad= b^{-1}\sum_{k\in\mathbf{Z}} \int_0^{b^{-1}} F_k(x)\,\overline{F_k(x)}\,dx \\
&\qquad= b^{-1}\sum_{k\in\mathbf{Z}} \int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)\,F_k\bigl(x-\tfrac{j}{b}\bigr)}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= b^{-1} \sum_{k\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{g(x-ak)\, F_k(x)}\, dx \\
&= b^{-1} \sum_{k\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{g(x-ak)} \sum_{j\in\mathbf{Z}} \overline{f\big(x-\tfrac{j}{b}\big)}\, g\big(x-ak-\tfrac{j}{b}\big)\, dx \\
&= b^{-1} \sum_{j\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{f\big(x-\tfrac{j}{b}\big)} \sum_{k\in\mathbf{Z}} \overline{g(x-ak)}\, g\big(x-ak-\tfrac{j}{b}\big)\, dx \\
&= b^{-1} \sum_{j\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{f\big(x-\tfrac{j}{b}\big)\, G_j(x)}\, dx \\
&= \Big\langle f,\, b^{-1} \sum_{j\in\mathbf{Z}} T_{\frac{j}{b}} f \cdot G_j(x) \Big\rangle_{L^2(\mathbf{R})} \\
&= \langle f,\, Lf \rangle.
\end{aligned}
$$

The interchanges in order can be justified by using Fubini's Theorem (Exercise 11.18). Since T is bounded, we conclude that $\mathcal{G}(g,a,b)$ is a Bessel sequence, and

$$
\begin{aligned}
\langle f, Sf\rangle &= \sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \\
&= \langle f,\, Lf\rangle \\
&\le \|f\|_{L^2}\, \|Lf\|_{L^2} \\
&\le B\, \|f\|_{L^2}^2.
\end{aligned}
$$

Hence the Bessel bound holds on $C_c(\mathbf{R})$.

This also shows us that $\langle f, Sf\rangle = \langle f, Lf\rangle$ for all $f \in C_c(\mathbf{R})$. Since $C_c(\mathbf{R})$ is dense and both S and L are bounded, we conclude that $\langle f, Sf\rangle = \langle f, Lf\rangle$ for all $f \in L^2(\mathbf{R})$. Since S is self-adjoint, Corollary 2.16 therefore implies that $S = L$. □

We emphasize the contrast between the appearance of the Gabor frame operator in its original form and in the Walnut Representation:

$$
\sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}g \;=\; Sf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} f \cdot G_n.
$$

Aside from the fact that the Walnut Representation contains a single summation, it also contains no complex exponentials. If f and g are real valued then every term on the right-hand side of the line above is real valued, while the terms on the left-hand side need not be.

One of the consequences of Theorem 11.18 is that if $g \in W(L^\infty, \ell^1)$ then $\mathcal{G}(g,a,b)$ will be a frame for all small enough values of a and b [HW89, Thm. 4.1.8].

We end this section by mentioning another fundamental representation of the Gabor frame operator. This is the Janssen Representation (also known as the *Dual Lattice Representation*), which expresses the frame operator as a superposition of time-frequency shift operators [Jan95], [DLL95]. The hypotheses required for the Janssen Representation are slightly different than those of the Walnut Representation. Note the explicit role played by the adjoint lattice in this representation.

Theorem 11.19 (Janssen Representation). *Let $g \in L^2(\mathbf{R})$ and $a,\, b > 0$ be given. If*

$$\sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \bigl|\bigl\langle g,\, M_{\frac{k}{a}} T_{\frac{n}{b}} g\bigr\rangle\bigr| \;<\; \infty, \tag{11.17}$$

then

$$S \;=\; \frac{1}{ab} \sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \bigl\langle g,\, M_{\frac{k}{a}} T_{\frac{n}{b}} g\bigr\rangle\, M_{\frac{k}{a}} T_{\frac{n}{b}},$$

where the series converges absolutely in operator norm. ◊

Equation (11.17) is referred to as *Condition A*. It is close but not identical to the requirement that g belong to $W(L^\infty, \ell^1)$. The *Feichtinger algebra* S_0, which equals the modulation space M^1, is a smaller subspace on which both conditions are satisfied simultaneously. The Feichtinger algebra has many other useful properties, e.g., it is closed under both convolution and pointwise products, and in most cases it is the class from which we should choose generators g for Gabor frames [Grö01].

Exercises

11.18. Justify the use of Fubini's Theorem in the proof of Theorem 11.18.

11.19. This exercise gives a perturbation theorem for Gabor frames.

(a) Let $g \in L^2(\mathbf{R})$ and $a,\, b > 0$ be such that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$. Show that there exists a $\delta > 0$ such that if $h \in L^2(\mathbf{R})$ and $\|g-h\|_{W(L^\infty,\ell^1)} < \delta$, then $\mathcal{G}(h,a,b)$ is a frame for $L^2(\mathbf{R})$.

(b) Does part (a) remain valid if we replace the amalgam norm $\|\cdot\|_{W(L^\infty,\ell^1)}$ by the L^2-norm $\|\cdot\|_{L^2}$?

11.6 The Zak Transform

The *Zak transform* is a fundamental tool for analyzing Gabor frames, especially at the critical density ($ab = 1$). The Zak transform was first introduced by Gel'fand [Gel50]. As with many useful notions, it has been rediscovered many times and goes by a variety of names. Weil [Wei64] defined a Zak transform for locally compact abelian groups, and this transform is often called the *Weil–Brezin map* in representation theory and abstract harmonic analysis, e.g., [Sch84], [AT85]. Zak rediscovered this transform, which he called the *k-q transform*, in his work on quantum mechanics, e.g., [Zak67], [BGZ75]. The terminology "Zak transform" has become customary in applied mathematics and signal processing. For more information, we refer to Janssen's influential article [Jan82] and survey [Jan88], or Gröchenig's text [Grö01, Chap. 8].

In this section we define the Zak transform and examine some of its most interesting properties. In the following sections we will see how the Zak transform can be used to analyze Gabor systems, and how the unusual properties of the Zak transform are related to the Balian–Low Theorem. We will be concentrating in this section on the critical density, $ab = 1$. By dilating g, we can reduce this further to $a = b = 1$, so we simply fix $a = b = 1$ now.

The Gabor system $\mathcal{G}(\chi_{[0,1]}, 1, 1) = \{M_n T_k \chi_{[0,1]}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$. Let

$$Q \;=\; [0,1]^2$$

denote the unit square in $\mathbf{R}^2$, and consider the sequence

$$\{E_{nk}\}_{k,n\in\mathbf{Z}}, \qquad \text{where } E_{nk}(x,\xi) = e^{2\pi i n x}\, e^{-2\pi i k \xi}. \tag{11.18}$$

This sequence is contained in the Hilbert space $L^2(Q)$, whose norm and inner product are given by

$$\|F\|_{L^2(Q)}^2 \;=\; \int_0^1\!\int_0^1 |F(x,\xi)|^2\, dx\, d\xi$$

and

$$\langle F, G\rangle \;=\; \int_0^1\!\int_0^1 F(x,\xi)\,\overline{G(x,\xi)}\, dx\, d\xi.$$

By Theorem B.10 or by direct calculation, $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$. We can define a unitary map by sending the elements of one orthonormal basis to another orthonormal basis, and this is precisely what we do to define the Zak transform (see Exercise 11.20).

Definition 11.20 (Zak Transform). The *Zak transform* is the unique unitary map $Z\colon L^2(\mathbf{R}) \to L^2(Q)$ that satisfies

$$Z(M_n T_k \chi_{[0,1]}) \;=\; E_{nk}, \qquad k,n \in \mathbf{Z}. \qquad \diamondsuit \tag{11.19}$$

Now we give an equivalent formulation of the Zak transform that will help us to extend its domain to spaces other than $L^2(\mathbf{R})$.

Theorem 11.21. *Given $f \in L^2(\mathbf{R})$, we have*

$$Zf(x,\xi) \;=\; \sum_{j\in\mathbf{Z}} f(x-j)\, e^{2\pi i j \xi}, \qquad (x,\xi) \in Q, \tag{11.20}$$

where this series converges unconditionally in the norm of $L^2(Q)$.

Proof. A direct calculation shows that if $f \in L^2(\mathbf{R})$ and $j \neq \ell$ then the functions $f(x+j)\, e^{2\pi i j\xi}$ and $f(x+\ell)\, e^{2\pi i \ell\xi}$ are orthogonal elements of $L^2(Q)$. Therefore, if F is any finite subset of $\mathbf{Z}$ then

$$\begin{aligned} \Big\| \sum_{j\in F} f(x-j)\, e^{2\pi i j\xi} \Big\|_{L^2(Q)}^2 &= \sum_{j\in F} \|f(x-j)\, e^{2\pi i j\xi}\|_{L^2(Q)}^2 \\ &= \sum_{j\in F} \int_0^1 \int_0^1 |f(x-j)\, e^{2\pi i j\xi}|^2\, dx\, d\xi \\ &= \sum_{j\in F} \int_0^1 |f(x-j)|^2\, dx. \end{aligned} \tag{11.21}$$

Since $f \in L^2(\mathbf{R})$, the series $\sum_{j\in\mathbf{Z}} \int_0^1 |f(x-j)|^2\, dx$ converges unconditionally and equals $\|f\|_{L^2}^2$. Consequently, the series appearing on the right-hand side of equation (11.20) converges unconditionally in $L^2(Q)$, and if we set $Uf(x,\xi) = \sum_{j\in\mathbf{Z}} f(x-j)\, e^{2\pi i j\xi}$ then it follows from equation (11.21) that $\|Uf\|_{L^2(Q)} = \|f\|_{L^2}$. This operator U is an isometry, so to show that $U = Z$ we simply have to show that $U(M_n T_k \chi_{[0,1]}) = E_{nk}$ for all $k,\, n \in \mathbf{Z}$. To see this, note that if $(x,\xi) \in Q$ then $\chi_{[0,1]}(x-j) = 0$ for all $j \neq 0$, so

$$\begin{aligned} U(M_n T_k \chi_{[0,1]})(x,\xi) &= \sum_{j\in\mathbf{Z}} M_n T_k \chi_{[0,1]}(x-j)\, e^{2\pi i j\xi} \\ &= \sum_{j\in\mathbf{Z}} e^{2\pi i n(x-j)} \chi_{[0,1]}(x-j-k)\, e^{2\pi i j\xi} \\ &= e^{2\pi i n(x+k)}\, e^{-2\pi i k\xi} \;=\; E_{nk}(x,\xi), \end{aligned}$$

where we have used the fact that $e^{2\pi i n k} = 1$. □

It will be important for us to consider the Zak transform on domains other than $L^2(\mathbf{R})$, and the correct spaces are precisely the Wiener amalgam spaces $W(L^p,\ell^1)$ introduced in Section 11.4. The next theorem shows that the Zak transform maps $W(L^p,\ell^1)$ into $L^p(Q)$, and maps $W(C,\ell^1)$ into $C(Q)$, the space of continuous functions on $Q = [0,1]^2$.

Theorem 11.22. (a) *If $1 \le p \le \infty$ then for each $f \in W(L^p,\ell^1)$ the series*

$$Zf(x,\xi) \;=\; \sum_{j\in\mathbf{Z}} f(x-j)\, e^{2\pi i j\xi}, \qquad (x,\xi) \in Q, \tag{11.22}$$

converges absolutely in $L^p(Q)$, and Z is a bounded mapping of $W(L^p,\ell^1)$ into $L^p(Q)$.

(b) *For each $f \in W(C,\ell^1)$ the series in equation* (11.22) *converges absolutely in $C(Q)$ with respect to the uniform norm, and Z is a bounded mapping of $W(C,\ell^1)$ into $C(Q)$.*

Proof. (a) If $f \in W(L^p, \ell^1)$ with p finite then

$$\begin{aligned}\sum_{j\in\mathbf{Z}} \|f(x-j)\, e^{2\pi i j\xi}\|_{L^p(Q)} &= \sum_{j\in\mathbf{Z}} \biggl(\int_0^1\int_0^1 |f(x-j)\, e^{2\pi i j\xi}|^p\, dx\, d\xi\biggr)^{1/p} \\ &= \sum_{j\in\mathbf{Z}} \|f\cdot\chi_{[j,j+1]}\|_{L^p} \;<\; \infty,\end{aligned}$$

so the series defining Zf converges absolutely in $L^p(Q)$. A similar calculation holds if $p=\infty$, and these calculations also show that $\|Zf\|_{L^p(Q)} \le \|f\|_{W(L^p,\ell^1)}$.

(b) If $f \in W(C,\ell^1) \subseteq W(L^\infty,\ell^1)$ then we have by part (a) that $Zf \in L^\infty(Q)$, and the series defining Zf converges absolutely in the uniform norm. As each term $f(x-j)\, e^{2\pi i j\xi}$ is continuous on Q and the uniform limit of continuous functions is continuous, Zf is continuous on Q. □

Remark 11.23. In particular, the Zak transform maps $L^1(\mathbf{R}) = W(L^1,\ell^1)$ continuously into $L^1(Q)$, and it is injective by Exercise 11.22. However, that exercise also shows that the range of $Z\colon L^1(\mathbf{R}) \to L^1(Q)$ is a dense but proper subspace of $L^1(Q)$. A consequence of this is that $Z^{-1}\colon \mathrm{range}(Z) \to L^1(\mathbf{R})$ is unbounded, in contrast to the fact that Z is a unitary mapping of $L^2(\mathbf{R})$ onto $L^2(Q)$. Readers familiar with interpolation will recognize that since Z maps $L^1(Q)$ boundedly into itself and $L^2(Q)$ boundedly into itself, it extends to a bounded map of $L^p(Q)$ into itself for each $1 \le p \le 2$. However, if $1 \le p < 2$ then Z is not surjective and Z^{-1} is unbounded. ◊

Given $f \in L^2(\mathbf{R})$, we can extend the domain of Zf from $Q = [0,1]^2$ to all of $\mathbf{R}^2$ in a natural way. In all of the preceding arguments, nothing is changed if we replace the unit square Q with a translated square $Q+z$, where $z \in \mathbf{R}^2$. Moreover, if Q and $Q+z$ overlap then the two definitions of Zf will coincide almost everywhere on $Q\cap(Q+z)$ (and everywhere if Zf is continuous). Hence Zf has a unique extension from Q to the entire plane $\mathbf{R}^2$. This is similar to how a function on $[0,1)$ is extended to a 1-periodic function on $\mathbf{R}$, as in Notation 4.23. However, there is an interesting twist here, because Zf on $\mathbf{R}^2$ is *not* obtained by extending Zf periodically from Q. Instead, Zf satisfies the following rather peculiar *quasiperiodicity relations* (Exercise 11.21).

Theorem 11.24. *If $f \in L^2(\mathbf{R})$ or $f \in W(L^p,\ell^1)$, then for $m, n \in \mathbf{Z}$ we have*

$$Zf(x+m, \xi+n) \;=\; e^{2\pi i m\xi}\, Zf(x,\xi),$$

where the equality holds pointwise everywhere on $\mathbf{R}^2$ if Zf is continuous, and almost everywhere otherwise. ◊

Definition 11.25 (Quasiperiodicity). We say that a function F on $\mathbf{R}^2$ that satisfies

$$F(x+m, \xi+n) \;=\; e^{2\pi i m \xi}\, F(x, \xi) \text{ a.e.}, \qquad m, n \in \mathbf{Z}, \tag{11.23}$$

is *quasiperiodic*. We refer to equation (11.23) as the *quasiperiodicity relations* for F. ♢

Quasiperiodicity has a rather unexpected implication: No continuous quasiperiodic function can be nonzero everywhere. Since a complete justification of this statement requires some facts from complex analysis or algebraic geometry, we will be content to appeal to authority for the justification of certain steps in the proof that we present.

Theorem 11.26. *A continuous quasiperiodic function F must vanish at some point of Q.*

Proof. First we give a standard direct argument that proves the theorem but does little to illuminate the mystery of why a zero must exist. Suppose that F was quasiperiodic and continuous on $\mathbf{R}^2$ but everywhere nonzero. Because $\mathbf{R}^2$ is simply connected, there exists a continuous function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$F(x, \xi) \;=\; |F(x, \xi)|\, e^{2\pi i \varphi(x, \xi)}, \qquad (x, \xi) \in \mathbf{R}^2.$$

Students of complex analysis may recognize that this continuous logarithm φ can be constructed directly, and its existence also follows from general topological lifting principles [Grö01]. Applying this logarithm to the quasiperiodicity relations, we see that for each m, $n \in \mathbf{Z}$ there exists an integer $\kappa(m, n)$ such that

$$\varphi(x+m, \xi+n) \;=\; \varphi(x, \xi) + m\xi + \kappa(m, n), \qquad (x, \xi) \in \mathbf{R}^2.$$

Hence

$$\begin{aligned} 0 \;&=\; \big(\varphi(0,0) - \varphi(1,0)\big) + \big(\varphi(1,0) - \varphi(1,1)\big) \\ &\qquad\qquad + \big(\varphi(1,1) - \varphi(0,1)\big) + \big(\varphi(0,1) - \varphi(0,0)\big) \\ &=\; \big(-0 - \kappa(1,0)\big) + \big(-0 - \kappa(0,1)\big) + \big(1 + \kappa(1,0)\big) + \big(0 + \kappa(0,1)\big) \\ &=\; 1, \end{aligned}$$

which is a contradiction.

Now we give another argument, due to Janssen [Jan05], that is perhaps more revealing. Suppose that F is continuous, quasiperiodic, and everywhere nonzero on $\mathbf{R}^2$. Then for each fixed $x \in \mathbf{R}$, the function $F_x(\xi) = F(x, \xi)$ is continuous, 1-periodic, and nonzero on $\mathbf{R}$. As ξ varies from 0 to 1, the values $F_x(\xi)$ trace out a closed curve J_x in the complex plane that never intersects the origin. Such a curve has a well-defined *winding number* N_x that is an integer representing the total number of times the curve J_x travels counterclockwise around the origin. Now, since F is continuous, the curves J_x deform continuously as we vary x. Further, since

$$F_1(\xi) = F(1,\xi) = e^{2\pi i\xi}F(0,\xi) = e^{2\pi i\xi}F_0(\xi),$$

the curve J_1 winds one more time around the origin than does J_0. However, there is no way to continuously deform a curve that winds N_0 times around the origin into one that winds $N_1 = N_0 + 1$ times around the origin without having the curve pass through the origin at some time. Hence there must be at least one value of x such that the curve J_x passes through the origin, which says that $F(x,\xi) = 0$ for some ξ. □

Figure 11.3 illustrates the idea of the second proof of Theorem 11.26. We can think of the curve J_x as being a rubber band wound N_x times around the origin. The rubber band is stretched and moved as x varies, but always lies in the complex plane. It can cross itself, but it cannot be cut. The left side of Figure 11.3 shows the curve J_0 for the specific example $F_0(\xi) = 1 + i + e^{2\pi i\xi}$. This curve is a circle that does not contain the origin, and so has winding number $N_0 = 0$. The curve J_1 traced out by the function $F_1(\xi) = e^{2\pi i\xi}F_0(\xi)$ is shown on the right side of Figure 11.3. The point $F_1(\xi)$ is located at the same distance from the origin as $F_0(\xi)$, but has been rotated counterclockwise by an angle of $2\pi\xi$ radians. As a consequence, J_1 makes one extra trip around the origin, so has winding number $N_1 = 1$. There is no way to deform the left-hand rubber band into the right-hand one without passing through the origin in the process.

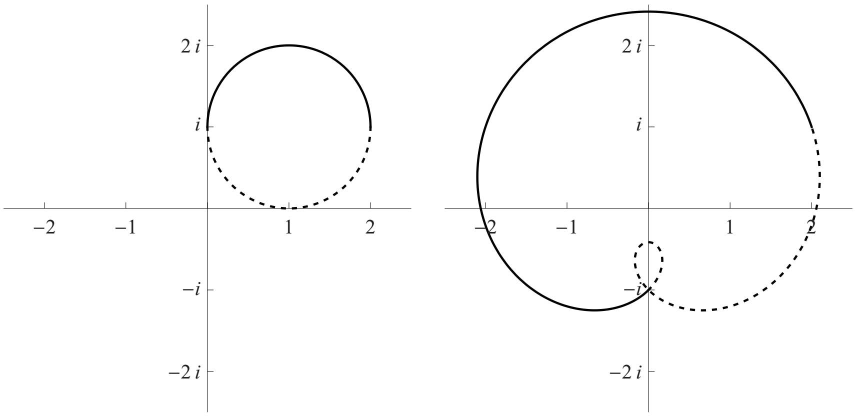

Fig. 11.3. Plots of the complex-valued functions $F_0(\xi) = 1 + i + e^{2\pi i\xi}$ and $F_1(\xi) = e^{2\pi i\xi}\,F_0(\xi)$ for $0 \le \xi \le 1$. The graph is shown as a solid line for $0 \le \xi \le 1/2$, and as a dashed line for $1/2 \le \xi \le 1$. The winding number of the left-hand graph is zero, while it is one for the right-hand graph.

Remark 11.27. Note that the domain of the function F in Theorem 11.26 is the plane $\mathbf{R}^2$, and F is required to be continuous on the entire plane. Applying the quasiperiodicity relations, this is the same as requiring that F

be continuous on the closed square $Q = [0,1]^2$. It is not enough to assume that F is quasiperiodic and continuous on $[0,1)^2$. For example, if we set $F = 1$ on $[0,1)^2$ then we can extend it to a quasiperiodic function on $\mathbf{R}^2$ by defining

$$F(x+m, \xi+n) = e^{2\pi i m\xi}, \qquad x, \xi \in [0,1),\ m, n \in \mathbf{Z}.$$

This function F is quasiperiodic, but it is not continuous on $\mathbf{R}^2$, and it has no zeros on $\mathbf{R}^2$. ♢

Example 11.28. The third *Jacobi theta function* is

$$\theta_3(z,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} \cos(4\pi k z) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{4\pi i k z},$$

where $0 \le q < 1$ and $z \in \mathbf{C}$ [Rai60]. With q fixed, $\theta_3(\cdot, q)$ is analytic on the entire complex plane.

Fix $r > 0$ and let φ_r be the Gaussian function $\varphi_r(x) = e^{-rx^2}$. Since $\varphi_r \in W(C, \ell^1)$, we know that $Z\varphi_r$ is continuous and therefore has a zero. The Zak transform of φ_r is

$$\begin{aligned} Z\varphi_r(x,\xi) &= \sum_{j\in\mathbf{Z}} \varphi_r(x-j)\, e^{2\pi i j\xi} \\ &= \sum_{j\in\mathbf{Z}} e^{-rx^2}\, e^{2rxj}\, e^{-rj^2}\, e^{2\pi i j \xi} \\ &= e^{-rx^2} \sum_{j\in\mathbf{Z}} (e^{-r})^{j^2}\, e^{4\pi i j (\frac{\xi}{2} - \frac{irx}{2\pi})} \\ &= e^{-rx^2}\, \theta_3\big(\tfrac{\xi}{2} - \tfrac{irx}{2\pi},\, e^{-r}\big). \end{aligned}$$

In particular, $Z\varphi_r$ is infinitely differentiable on $\mathbf{R}^2$.

Now, the zeros of $\theta_3(\cdot, q)$ occur precisely at the points

$$z_{mn} = \frac{1}{4} + \frac{\tau}{4} + \frac{m}{2} + \frac{n\tau}{2},$$

where $q = e^{\pi i \tau}$, $\mathrm{Im}(\tau) > 0$. Since $e^{-r} = e^{\pi i (ir)}$, it follows that $Z\varphi_r(x,\xi) = 0$ if and only if

$$\frac{\xi}{2} - \frac{irx}{2\pi} = \frac{1}{4} + \frac{ir}{4\pi} + \frac{m}{2} + \frac{irn}{2\pi},$$

i.e., $(x,\xi) = (-n-1/2, m+1/2)$. Thus $Z\varphi_r$ has a single zero in the unit square $Q = [0,1]^2$, at the point $(1/2, 1/2)$. ♢

Exercises

11.20. Prove that there is a unique unitary operator that satisfies equation (11.19).

11.21. Prove Theorem 11.24.

11.22. If $f \in L^1(\mathbf{R})$ then $Zf \in W(L^1, \ell^1) = L^1(Q)$ by Theorem 11.22. Prove the following statements.

(a) $f(x) = \int_0^1 Zf(x,\xi)\,d\xi$ for almost every $x \in \mathbf{R}$.

(b) If Zf is continuous, then f is continuous.

(c) Z is an injective mapping of $L^1(\mathbf{R})$ into $L^1(Q)$, and the range of $Z\colon L^1(\mathbf{R}) \to L^1(Q)$ is a proper, dense subspace of $L^1(Q)$.

(d) $Z^{-1}\colon L^1(Q) \to L^1(\mathbf{R})$ is unbounded.

11.23. Suppose that $f \in L^2(\mathbf{R})$ is such that Zf is continuous.

(a) Show that if f is even then $Zf(1/2, 1/2) = 0$.

(b) Show that if f is odd then $Zf(0,0) = Zf(0,1/2) = Zf(1/2,0) = 0$.

(c) Show that if f is real valued then $Zf(x, 1/2) = 0$ for some $x \in [0,1]$.

11.7 Gabor Systems at the Critical Density

Now we will use the Zak transform to analyze Gabor systems at the critical density. As before, it suffices to consider $a = b = 1$. In this section we will characterize those Gabor systems $\mathcal{G}(g,1,1)$ that are exact, Riesz bases, or orthonormal bases in terms of the Zak transform of g, and in the next section we will use this characterization to prove some versions of the Balian–Low Theorem.

The utility of the Zak transform is that it converts a Gabor system $\mathcal{G}(g,1,1) = \{M_nT_kg\}_{k,n\in\mathbf{Z}}$ into a system of weighted exponentials on $\mathbf{R}^2$. Recall from equation (11.18) that E_{nk} denotes the two-dimensional complex exponential function $E_{nk}(x,\xi) = e^{2\pi inx}\,e^{-2\pi ik\xi}$.

Theorem 11.29. *If $g \in L^2(\mathbf{R})$, then*

$$Z(M_nT_kg) \;=\; E_{nk}\,Zg \;\; a.e., \qquad k,n \in \mathbf{Z}.$$

Proof. Using the fact that $e^{-2\pi in(j-k)} = 1$ for integer j, k, n, we compute that

$$\begin{aligned} Z(M_nT_kg)(x,\xi) &= \sum_{j\in\mathbf{Z}} (M_nT_kg)(x-j)\,e^{2\pi ij\xi} \\ &= \sum_{j\in\mathbf{Z}} e^{2\pi in(x-j)}\,g(x-k-j)\,e^{2\pi ij\xi} \\ &= \sum_{j\in\mathbf{Z}} e^{2\pi in(x-j+k)}\,g(x-j)\,e^{2\pi i(j-k)\xi} \end{aligned}$$

$$= e^{2\pi inx}\, e^{-2\pi ik\xi} \sum_{j\in\mathbf{Z}} g(x-j)\, e^{2\pi ij\xi}$$

$$= E_{nk}(x,\xi)\, Zg(x,\xi).$$

The series above converge in $L^2(Q)$, not pointwise, but this does not affect the calculation. □

As a consequence, we obtain another proof that a sequence of regular translations $\{g(x-ak)\}_{k\in\mathbf{Z}}$ cannot be complete in $L^2(\mathbf{R})$ (compare Exercise 10.18).

Corollary 11.30. *If $g \in L^2(\mathbf{R})$ and $a > 0$, then $\{g(x-ak)\}_{k\in\mathbf{Z}}$ is incomplete in $L^2(\mathbf{R})$.*

Proof. By dilating g, it suffices to take $a = 1$, so our sequence is $\mathcal{T}(g) = \{T_k g\}_{k\in\mathbf{Z}}$. Taking $n = 0$ in Theorem 11.29, the image of this sequence under the Zak transform is

$$Z\mathcal{T}(g) = \{E_{0k}\, Zg\}_{k\in\mathbf{Z}} = \bigl\{e^{-2\pi ik\xi}\, Zg(x,\xi)\bigr\}_{k\in\mathbf{Z}}.$$

Taking finite linear combinations and L^2 limits, it follows that every element of $\overline{\mathrm{span}}(Z\mathcal{T}(g))$ has the form $p(\xi)Zg(x,\xi)$ for some function p. However, not every element of $L^2(Q)$ has this form (why?), so $Z\mathcal{T}(g)$ is incomplete in $L^2(Q)$. Since Z is unitary, $\mathcal{T}(g)$ is therefore incomplete in $L^2(\mathbf{R})$. □

Since the Zak transform is unitary, it preserves basis and frame properties. Consequently, $\mathcal{G}(g,1,1)$ is exact, a frame, a Riesz basis, or an orthonormal basis if and only if the same is true of $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$. Now, the system of unweighted exponentials $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$, and a Riesz basis is the image of an orthonormal basis under a topological isomorphism, so if $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$ is to be a Riesz basis then the mapping that sends E_{nk} to $E_{nk}\, Zg$ must extend to be a topological isomorphism of $L^2(Q)$ onto itself. The only way that the multiplication operation $U(F) = F \cdot Zg$ on $L^2(Q)$ can be a topological isomorphism is if $0 < \inf|Zg| \le \sup|Zg| < \infty$. Extending this idea gives us the following characterization of Gabor systems at the critical density. Note that this result is very much a two-dimensional version of Theorem 10.10!

Theorem 11.31. *Let $g \in L^2(\mathbf{R})$ be fixed.*

(a) *$\mathcal{G}(g,1,1)$ is complete in $L^2(\mathbf{R})$ if and only if $Zg \ne 0$ a.e.*

(b) *$\mathcal{G}(g,1,1)$ is minimal in $L^2(\mathbf{R})$ if and only if $1/Zg \in L^2(Q)$. In this case, $\mathcal{G}(g,1,1)$ is exact and its biorthogonal system is $\mathcal{G}(\widetilde{g},1,1)$ where $\widetilde{g} \in L^2(\mathbf{R})$ satisfies $Z\widetilde{g} = 1/\overline{Zg}$.*

(c) *$\mathcal{G}(g,1,1)$ is a Bessel sequence in $L^2(\mathbf{R})$ if and only if $Zg \in L^\infty(\mathbf{R})$, and in this case $B = \|Zg\|^2_{L^\infty}$ is a Bessel bound.*

(d) $\mathcal{G}(g,1,1)$ *is a frame for* $L^2(\mathbf{R})$ *if and only if there exist* A, $B > 0$ *such that* $A \le |Zg(x,\xi)|^2 \le B$ *a.e. In this case* $\mathcal{G}(g,1,1)$ *is a Riesz basis and* A, B *are frame bounds.*

(e) $\mathcal{G}(g,1,1)$ *is an orthonormal basis for* $L^2(\mathbf{R})$ *if and only if* $|Zg(x,\xi)| = 1$ *a.e.*

Proof. Much of the proof is similar to the proof of Theorem 10.10. Therefore we will prove some statements and assign the remainder as Exercise 11.24.

(a) Suppose that $Zg \ne 0$ a.e. If we can show that $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$ is complete in $L^2(Q)$, then it follows from the unitarity of Z that $\mathcal{G}(g,1,1)$ is complete in $L^2(\mathbf{R})$.

So, suppose that $F \in L^2(Q)$ is such that $\langle F, E_{nk}\, Zg\rangle_{L^2(Q)} = 0$ for each $k, n \in \mathbf{Z}$. Let $G = F \cdot \overline{Zg}$. Then $G \in L^1(Q)$, and its Fourier coefficients with respect to the orthonormal basis $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ are

$$\begin{aligned}\widehat{G}(n,k) \;=\; \langle G, E_{nk}\rangle_{L^2(Q)} &= \int_0^1\int_0^1 F(x,\xi)\,\overline{Zg(x,\xi)}\,\overline{E_{nk}(x,\xi)}\,dx\,d\xi \\ &= \langle F, E_{nk}\, Zg\rangle_{L^2(Q)} \\ &= 0.\end{aligned}$$

Although $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is not a basis for $L^1(Q)$, a two-dimensional analogue of Theorem 4.25 implies that functions in $L^1(Q)$ are uniquely determined by their Fourier coefficients. Since the Fourier coefficients of G agree with those of the zero function, we conclude that $G = 0$ a.e. As $G = F\cdot\overline{Zg}$ and $Zg \ne 0$ a.e., it follows that $F = 0$ a.e. Hence $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$ is complete.

(b) Suppose that $1/Zg \in L^2(Q)$. Then we must have $Zg \ne 0$ a.e., so $\mathcal{G}(g,1,1)$ is complete by statement (a). Also, since Z is surjective, there exists some function $\widetilde{g} \in L^2(Q)$ such that $Z\widetilde{g} = 1/\overline{Zg}$. We compute that

$$\begin{aligned}\langle M_nT_kg,\, M_{n'}T_{k'}\widetilde{g}\rangle &= \langle E_{nk}\, Zg,\, E_{n'k'}\, Z\widetilde{g}\rangle_{L^2(Q)} \\ &= \langle E_{nk}\, Zg,\, E_{n'k'}/\overline{Zg}\rangle_{L^2(Q)} \\ &= \langle E_{nk},\, E_{n'k'}\rangle_{L^2(Q)} \\ &= \delta_{nn'}\,\delta_{kk'}.\end{aligned}$$

Hence $\mathcal{G}(\widetilde{g},1,1)$ is biorthogonal to $\mathcal{G}(g,1,1)$. Thus $\mathcal{G}(g,1,1)$ is both minimal and complete, so it is exact. □

As is the case for the systems of weighted exponentials considered in Theorem 10.10, the characterization of Gabor systems $\mathcal{G}(g,1,1)$ that are Schauder bases for $L^2(\mathbf{R})$ is a more subtle problem. In [HP06] it was shown that if

$\mathcal{G}(g, a, b)$ is a Schauder basis then we must have $ab = 1$ (and therefore can reduce to the case $a = b = 1$), and $\mathcal{G}(g, 1, 1)$ is a Schauder basis for $L^2(\mathbf{R})$ if and only if $|Zg|^2$ is a *product* $\mathcal{A}_2$ *weight* for $L^2(Q)$.

From Theorem 11.31 we obtain the following corollary, whose implications will be explored in the next section. We let $C^1(\mathbf{R}^2)$ denote the set of all differentiable functions $F\colon \mathbf{R}^2 \to \mathbf{C}$ whose partial derivatives $\partial F/\partial x$ and $\partial F/\partial \xi$ are both continuous.

Corollary 11.32. *Fix* $g \in L^2(\mathbf{R})$.

(a) *If* Zg *is continuous on* Q *(and hence on* $\mathbf{R}^2$*), then* $\mathcal{G}(g, 1, 1)$ *not a frame or a Riesz basis for* $L^2(\mathbf{R})$.

(b) *If* $Zg \in C^1(\mathbf{R}^2)$ *then* $\mathcal{G}(g, 1, 1)$ *not exact in* $L^2(\mathbf{R})$.

Proof. (a) This follows immediately from Theorem 11.31 and the fact that any continuous quasiperiodic function must have a zero.

(b) If Zg is differentiable on $\mathbf{R}^2$ then it is continuous and therefore has at least one zero in Q by Theorem 11.26. For simplicity, assume that this zero is located at the origin. The C^1 hypothesis implies that Zg is Lipschitz on a neighborhood of the origin, i.e., there exist $C > 0$ and $\delta > 0$ such that

$$x^2 + \xi^2 < \delta \implies |Zg(x, \xi) - Zg(0, 0)| \le C\,|(x, \xi) - (0, 0)|,$$

where $|\cdot|$ is the Euclidean norm on $\mathbf{R}^2$. Since $Zg(0, 0) = 0$, by switching to polar coordinates we find that the integral of $1/|Zg|^2$ over the open ball $B_\delta(0)$ is

$$\begin{aligned}\iint_{B_\delta(0)} \frac{1}{|Zg(x, \xi)|^2}\, dx\, d\xi &\ge \frac{1}{C^2} \iint_{B_\delta(0)} \frac{1}{x^2 + \xi^2}\, dx\, d\xi \\ &= \frac{1}{C^2} \int_0^{2\pi} \int_0^{\delta} \frac{1}{r^2}\, r\, dr\, d\theta = \infty.\end{aligned}$$

Hence $1/Zg \notin L^2(Q)$, so $\mathcal{G}(g, 1, 1)$ is not exact. □

Exercises

11.24. Prove the remaining statements in Theorem 11.31.

11.25. (a) Let $p(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$ be a trigonometric polynomial. Show that if $|p| = 1$ a.e., then $p(x) = c_n e^{2\pi i n x}$ for some n between $-N$ and N.

(b) Suppose that $g \in L^2(\mathbf{R})$ is compactly supported. Show that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|g| = \chi_E$ for some bounded set $E \subseteq \mathbf{R}$ that satisfies $\sum_{k \in \mathbf{Z}} \chi_E(x - k) = 1$ a.e.

11.8 The Balian–Low Theorem

In this section we will prove the two versions of the Balian–Low Theorem given in Theorem 8.12, which state that all Gabor frames at the critical density are "bad" in some sense. We begin with the following simple result, which was proved in [Hei90] and first appeared in journal form in [BHW95]. Recall from Theorem 11.31 that when $a = b = 1$, a Gabor system $\mathcal{G}(g, 1, 1)$ is a frame for $L^2(\mathbf{R})$ if and only if it is a Riesz basis.

Theorem 11.33 (Amalgam BLT). *If $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbf{R})$ then $g \notin W(C, \ell^1)$. Specifically, either*

$$g \text{ is not continuous} \qquad \text{or} \qquad \sum_{k \in \mathbf{Z}} \|g \cdot \chi_{[k,k+1]}\|_{L^\infty} \;=\; \infty.$$

Moreover, we also have $\widehat{g} \notin W(C, \ell^1)$, where $\widehat{g}$ is the Fourier transform of g.

Proof. We have already done the work earlier in the chapter. If $g \in W(C, \ell^1)$ then Theorem 11.22 implies that $Zg \in C(Q)$. Corollary 11.32 therefore implies that $\mathcal{G}(g, 1, 1)$ cannot be a frame for $L^2(\mathbf{R})$, simply because Zg must have a zero. The same reasoning transfers to $\widehat{g}$ by applying Exercise 11.4. □

Thus, if g is to generate a Riesz basis at the critical density, then either g must be discontinuous or it must have poor decay at infinity, and similarly $\widehat{g}$ is either discontinuous or has poor decay.

Example 11.34. We saw in Example 11.28 that the Zak transform of the Gaussian function $\phi(x) = e^{-\pi x^2}$ is continuous, so $\mathcal{G}(\phi, 1, 1)$ cannot be a Riesz basis for $L^2(\mathbf{R})$. Since $Z\phi$ is bounded, $\mathcal{G}(\phi, 1, 1)$ is a Bessel sequence, but it does not have a positive lower frame bound. On the other hand, $\mathcal{G}(\phi, 1, 1)$ is complete since $Z\phi$ has only a single zero in Q and therefore $Z\phi$ is nonzero almost everywhere. Because $Z\phi$ is infinitely differentiable, Corollary 11.32 implies that $\mathcal{G}(\phi, 1, 1)$ is not exact, and therefore it has a positive excess. An argument similar to the one presented in Example 5.9(c) can be used to show that $\mathcal{G}(g, 1, 1)$ is overcomplete by precisely one element. That is, if we remove any single element from $\mathcal{G}(\phi, 1, 1)$ then it will still be complete, but if we remove two elements then it becomes incomplete. In particular,

$$\mathcal{G}(\phi, 1, 1) \setminus \{\phi\} \;=\; \{M_n T_k \phi\}_{(k,n) \neq (0,0)}$$

is exact, but it is not a Schauder basis or a frame for $L^2(\mathbf{R})$ (see [Fol89, p. 168]). ♢

The theorem originally stated by Balian [Bal81] and independently by Low [Low85] quantifies the "unpleasantness" of a Gabor orthonormal basis generator in a different manner than Theorem 11.33. In contrast to the Amalgam BLT, the hypotheses of their theorem do not imply that Zg is continuous, which is why the proof is more difficult. A gap in the original proofs was filled by Coifman, Daubechies, and Semmes in [Dau90]. At the same time, they also extended the proof from orthonormal bases to Riesz bases, yielding the following result that we call the "Classical" Balian–Low Theorem.

Theorem 11.35 (Classical BLT). *If $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbf{R})$ then*

$$\biggl(\int_{-\infty}^{\infty} |xg(x)|^2\,dx\biggr)\biggl(\int_{-\infty}^{\infty} |\xi\widehat{g}(\xi)|^2\,d\xi\biggr) \;=\; \infty. \qquad \diamondsuit \tag{11.24}$$

Before discussing the proof of Theorem 11.35, we make some remarks on what it says qualitatively and how its conclusions compare to those of the Amalgam BLT.

The Fourier transform is a unitary mapping of $L^2(\mathbf{R})$ onto itself, so if g belongs to $L^2(\mathbf{R})$ then so does $\widehat{g}$. The celebrated *Classical Uncertainty Principle* of quantum mechanics takes the following mathematical form: We must always have

$$\biggl(\int_{-\infty}^{\infty} |xg(x)|^2\,dx\biggr)\biggl(\int_{-\infty}^{\infty} |\xi\widehat{g}(\xi)|^2\,d\xi\biggr) \;\ge\; \frac{1}{4\pi}\int_{-\infty}^{\infty} |g(x)|^2\,dx. \tag{11.25}$$

A proof of this inequality is sketched in Exercise 11.32. The left-hand side of equation (11.25) may be finite or infinite, but it can never be smaller than the right-hand side. The Gaussian function $\phi(x) = e^{-\pi x^2}$ achieves equality in equation (11.25), and the only functions that do so are translated and modulated Gaussians of the form

$$c\,e^{2\pi i\xi_0 x}\,e^{-r(x-x_0)^2}, \qquad c\in\mathbf{C},\;\; r>0.$$

The Classical BLT states that if $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbf{R})$ then not only do we have the bound given in equation (11.25), but the left-hand side of that equation must actually be infinite. Thus the generator of a Gabor Riesz basis must "maximize uncertainty."

One important feature of the Fourier transform is that it interchanges the roles of smoothness and decay. Roughly speaking, the smoother that g is, the faster that $\widehat{g}$ must decay at infinity, and the faster that g decays, the smoother that $\widehat{g}$ must be. If g decays well at infinity then we should have $\int |xg(x)|^2\,dx < \infty$. For example, if g is bounded and for x large enough we have $|g(x)| \le C|x|^{-p}$ where $p > 3/2$, then $\int |xg(x)|^2\,dx$ will be finite. Thus, to say that $\int |xg(x)|^2\,dx = \infty$ is to say that g does not decay rapidly, at least in some integrated average sense, and therefore $\widehat{g}$ is not very smooth. Similarly, if $\int |\xi\widehat{g}(\xi)|^2\,d\xi = \infty$ then $\widehat{g}$ does not decay well and hence g is not very smooth. The Classical BLT implies that if $\mathcal{G}(g,1,1)$ is a Riesz basis then at least one of these things must happen, and so g is a "bad function" (at least in terms of Gabor theory).

Qualitatively, the Classical and Amalgam BLTs have similar conclusions: The generator of a Gabor Riesz basis at the critical density is either not smooth or it has poor decay. The two theorems *quantify* this statement in somewhat different ways. While there is a good deal of overlap, neither conclusion implies the other, so the two BLTs are distinct theorems [BHW95].

We will give an elegant proof of Theorem 11.35 due to Battle [Bat88] for the case that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis. This proof relies on the operator theory that underlies the proof of the Classical Uncertainty Principle, and with some work the proof can be extended to Gabor systems that are Riesz bases, see [DJ93]. For some variations on the proof and more extensive discussion we refer to the survey paper [BHW95].

Proof (of Theorem 11.35 for orthonormal bases). The quantum mechanics operators of position and momentum are, in mathematical terms,

$$Pf(x) \;=\; xf(x) \qquad \text{and} \qquad Mf(x) \;=\; \frac{1}{2\pi i}\, f'(x). \tag{11.26}$$

These operators obviously do not map $L^2(\mathbf{R})$ into itself. We can make them well defined by restricting their domains to appropriate dense subsets of $L^2(\mathbf{R})$, but even if we do this, these operators are unbounded with respect to L^2-norm (Exercise 11.28). Still, these are key operators in harmonic analysis and quantum mechanics.

Suppose that $g \in L^2(\mathbf{R})$ is such that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbf{R})$. In particular, this implies

$$\langle g, M_n T_k g\rangle \;=\; \delta_{0k}\, \delta_{0n}, \qquad k, n \in \mathbf{Z}.$$

If either $\int |xg(x)|^2\, dx \;=\; \infty$ or $\int |\xi\widehat{g}(\xi)|^2\, d\xi \;=\; \infty$ then equation (11.24) holds trivially, so suppose that both of these quantities are finite. In terms of the position operator, this means $Pg \in L^2(\mathbf{R})$ and $P\widehat{g} \in L^2(\mathbf{R})$.

Since g and Pg both belong to $L^2(\mathbf{R})$, for k, $n \in \mathbf{Z}$ we compute that

$$\begin{aligned}
&\bigl\langle Pg,\, M_n T_k g\bigr\rangle \qquad\qquad (11.27)\\
&= \int_{-\infty}^{\infty} xg(x)\, e^{-2\pi i n x}\, \overline{g(x-k)}\, dx\\
&= \int_{-\infty}^{\infty} g(x)\, \overline{e^{2\pi i n x}\, (x-k)\, g(x-k)}\, dx + k \int_{-\infty}^{\infty} g(x)\, \overline{e^{2\pi i n x}\, g(x-k)}\, dx\\
&= \bigl\langle g,\, M_n T_k P g\bigr\rangle \;+\; k \bigl\langle g,\, M_n T_k g\bigr\rangle\\
&= \bigl\langle g,\, M_n T_k P g\bigr\rangle \;+\; k\, \delta_{0k}\, \delta_{0n}\\
&= \bigl\langle g,\, M_n T_k P g\bigr\rangle \;+\; 0. \qquad\qquad (11.28)
\end{aligned}$$

The adjoint of M_n is M_{-n}, and likewise the adjoint of T_k is T_{-k}. Further, M_n and T_k commute because we are at the critical density. Therefore

$$\bigl\langle g,\, M_n T_k P g\bigr\rangle \;=\; \bigl\langle T_{-k} M_{-n} g,\, Pg\bigr\rangle \;=\; \bigl\langle M_{-n} T_{-k} g,\, Pg\bigr\rangle. \tag{11.29}$$

Combining equations (11.28) and (11.29), we see that

$$\langle Pg, M_nT_kg\rangle = \langle M_{-n}T_{-k}g, Pg\rangle. \tag{11.30}$$

Our next goal is to perform a similar calculation using Mg instead of Pg.

Because the Fourier transform interchanges smoothness with decay, the hypotheses $g, P\widehat{g} \in L^2(\mathbf{R})$ imply that g has a certain amount of smoothness. Specifically, Theorem 9.27(b) states that g is *absolutely continuous* on any finite interval, $g'(x)$ exists a.e., $g' \in L^2(\mathbf{R})$, and

$$\widehat{g'}(\xi) = 2\pi i\xi\,\widehat{g}(\xi) = 2\pi i\,P\widehat{g}(\xi) \text{ a.e.}$$

In particular, $Mg \in L^2(\mathbf{R})$ and

$$(Mg)^\wedge = \Big(\frac{1}{2\pi i}g'\Big)^\wedge = P\widehat{g}.$$

Since the Fourier transform is unitary on $L^2(\mathbf{R})$, we "switch to the Fourier side" and apply equation (11.30) to compute that

$$\begin{aligned}
\langle Mg, M_nT_kg\rangle &= \langle (Mg)^\wedge, (M_nT_kg)^\wedge\rangle \\
&= \langle P\widehat{g}, T_nM_{-k}\widehat{g}\rangle \\
&= \langle P\widehat{g}, M_{-k}T_n\widehat{g}\rangle \\
&= \langle M_kT_{-n}\widehat{g}, P\widehat{g}\rangle \\
&= \langle (T_{-k}M_{-n}g)^\wedge, (Mg)^\wedge\rangle \\
&= \langle T_{-k}M_{-n}g, Mg\rangle \\
&= \langle M_{-n}T_{-k}g, Mg\rangle.
\end{aligned} \tag{11.31}$$

By expanding Pg and Mg in the orthonormal basis $\{M_nT_kg\}_{k,n\in\mathbf{Z}}$ and applying equations (11.30) and (11.31) we obtain

$$\begin{aligned}
\langle Mg, Pg\rangle &= \Big\langle \sum_{k,n\in\mathbf{Z}} \langle Mg, M_nT_kg\rangle\, M_nT_kg,\ Pg\Big\rangle \\
&= \sum_{k,n\in\mathbf{Z}} \langle Mg, M_nT_kg\rangle\,\langle M_nT_kg, Pg\rangle \\
&= \sum_{k,n\in\mathbf{Z}} \langle M_{-n}T_{-k}g, Mg\rangle\,\langle Pg, M_{-n}T_{-k}g\rangle \\
&= \sum_{k,n\in\mathbf{Z}} \langle Pg, M_nT_kg\rangle\langle M_nT_kg, Mg\rangle \\
&= \langle Pg, Mg\rangle.
\end{aligned}$$

However, we will show that we also have

$$\bigl\langle Mg,\, Pg \bigr\rangle \;=\; \langle Pg, Mg\rangle - \frac{1}{2\pi i}, \tag{11.32}$$

which is a contradiction. To see that equation (11.32) holds, first write

$$\bigl\langle Mg,\, Pg \bigr\rangle \;=\; \frac{1}{2\pi i} \int_{-\infty}^{\infty} g'(x)\, \overline{xg(x)}\, dx.$$

Integration by parts is valid for absolutely continuous functions (Theorem 9.28). Setting $u(x) = \overline{g(x)}$ and $v(x) = xg(x)$, we therefore compute that

$$\begin{aligned}
&\int_a^b g'(x)\, \overline{xg(x)}\, dx \\
&= \int_a^b \bigl(xg'(x) + g(x)\bigr)\, \overline{g(x)}\, dx \;-\; \int_a^b g(x)\, \overline{g(x)}\, dx \\
&= \biggl(b\, |g(b)|^2 - a\, |g(a)|^2 \;-\; \int_a^b xg(x)\, \overline{g'(x)}\, dx\biggr) \;-\; \int_a^b |g(x)|^2\, dx.
\end{aligned}$$

If we fix a, then each of the integrals appearing above converges to a finite value as $b \to \infty$. Consequently, $b\,|g(b)|^2$ must converge as $b \to \infty$. However, since g is square integrable, this limit must be zero (Exercise 11.27). A similar argument applies as $a \to -\infty$, so we have

$$\begin{aligned}
\bigl\langle Mg,\, Pg \bigr\rangle \;&=\; \frac{1}{2\pi i} \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b g'(x)\, \overline{xg(x)}\, dx \\
&=\; \frac{1}{2\pi i} \lim_{\substack{a\to-\infty\\ b\to\infty}} \biggl(-\int_a^b xg(x)\, \overline{g'(x)}\, dx \;-\; \int_a^b |g(x)|^2\, dx\biggr) \\
&=\; \int_{-\infty}^{\infty} Pg(x)\, \overline{Mg(x)}\, dx \;-\; \frac{1}{2\pi i}\, \|g\|_{L^2}^2 \\
&=\; \langle Pg, Mg\rangle \;-\; \frac{1}{2\pi i}.
\end{aligned}$$

This gives our contradiction. □

However, this is not the end of the story on bases related to time-frequency shifts. A remarkable construction known as *Wilson bases* yields orthonormal bases for $L^2(\mathbf{R})$ (as well as unconditional bases for the modulation spaces $M^{p,q}_s(\mathbf{R})$) generated by appropriate linear combinations of time-frequency shifts of "nice" functions. For details on this topic we refer to Gröchenig's text [Grö01].

Exercises

11.26. Let $\phi(x) = e^{-\pi x^2}$ be the Gaussian function. Show that $\mathcal{G}(\phi, 1, 1)$ is ℓ^2-independent, i.e., if $c = (c_{kn})_{k,n\in\mathbf{Z}} \in \ell^2(\mathbf{Z}^2)$ and $\sum c_{kn} M_n T_k \phi = 0$, then $c_{kn} = 0$ for every k and n. Note that since $\mathcal{G}(\phi, 1, 1)$ is a Bessel sequence, if we let R denote the synthesis operator for $\mathcal{G}(\phi, 1, 1)$ then ℓ^2-independence is equivalent to the statement that $R\colon \ell^2(\mathbf{Z}^2) \to L^2(\mathbf{R})$ is injective.

11.27. Show that if $g \in L^2(\mathbf{R})$ and $\lim_{x\to\infty} x\,|g(x)|^2$ exists, then this limit must be zero.

11.28. Let P, M be the position and momentum operators introduced in equation (11.26). These operators are not defined on all of $L^2(\mathbf{R})$. Instead, define domains

$$\begin{aligned} D_P &= \{f \in L^2(\mathbf{R}) : xf(x) \in L^2(\mathbf{R})\}, \\ D_M &= \{f \in L^2(\mathbf{R}) : f \text{ is differentiable and } f' \in L^2(\mathbf{R})\}, \end{aligned}$$

which are dense subspaces of $L^2(\mathbf{R})$. Restricted to these domains, P maps D_P into $L^2(\mathbf{R})$ and M maps D_M into $L^2(\mathbf{R})$. Show that P and M are unbounded even when restricted to these domains, i.e.,

$$\sup_{\substack{f\in D_P,\\ \|f\|_{L^2}=1}} \|Pf\|_{L^2} = \infty = \sup_{\substack{f\in D_M,\\ \|f\|_{L^2}=1}} \|Mf\|_{L^2}.$$

11.29. Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space introduced in Definition 9.18. Show that the position and momentum operators map $\mathcal{S}(\mathbf{R})$ into itself, and are self-adjoint when restricted to this domain, i.e.,

$$\langle Pf, g\rangle = \langle f, Pg\rangle \qquad \text{and} \qquad \langle Mf, g\rangle = \langle f, Mg\rangle$$

for all $f \in \mathcal{S}(\mathbf{R})$. (The Schwartz space is a convenient dense subspace of $L^2(\mathbf{R})$, but can be replaced in this problem by some larger subspaces of $L^2(\mathbf{R})$ if desired.)

11.30. The *commutator* of position and momentum is the operator $[P, M] = PM - MP$. Show that $[P, M] = -\frac{1}{2\pi i} I$ in the sense that $[P, M]f = -\frac{1}{2\pi i} f$ for all differentiable functions f. How does this relate to equation (11.32)?

11.31. This exercise will give an abstract operator-theoretic version of the Uncertainty Principle.

Let S be a subspace of a Hilbert space H, and let A, $B\colon S \to H$ be linear but possibly unbounded operators. By replacing S with the smaller space $\mathrm{domain}(AB) \cap \mathrm{domain}(BA)$ if necessary, we may assume that A, B, AB, and BA are all defined on S.

(a) Show that if A, B are self-adjoint in the sense that

$$\forall\, f, g \in S, \qquad \langle Af, g\rangle \;=\; \langle f, Ag\rangle \quad \text{and} \quad \langle Bf, g\rangle \;=\; \langle f, Bg\rangle,$$

then

$$\forall\, f \in S, \quad \|Af\|\,\|Bf\| \;\geq\; \frac{1}{2}\,\bigl|\bigl\langle [A,B]f,\, f\bigr\rangle\bigr|,$$

where $[A,B] = AB - BA$ is the commutator of A and B.

(b) Show that equality holds in part (a) if and only if $Af = icBf$ for some $c \in \mathbf{R}$.

11.32. Apply Exercises 11.29–11.31 to the position and momentum operators P and M to derive the Classical Uncertainty Principle,

$$\|xg(x)\|_{L^2}\,\|\xi\,\widehat{g}(\xi)\|_{L^2} \;\geq\; \frac{1}{4\pi}\,\|g\|_{L^2}^2, \tag{11.33}$$

for $g \in \mathcal{S}(\mathbf{R})$.

Remark: An extension by density argument can be used to prove that equation (11.33) extends to all $g \in L^2(\mathbf{R})$, or integration by parts for absolutely continuous functions can be used to prove directly that equation (11.33) holds whenever $\|xg(x)\|_{L^2}\,\|\xi\,\widehat{g}(\xi)\|_{L^2}$ is finite, see [Heil].

11.33. Modify Battle's argument to prove the *Weak BLT*: If $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbf{R})$ then

$$\|xg(x)\|_{L^2}\,\|\xi\widehat{g}(\xi)\|_{L^2}\,\|x\widetilde{g}(x)\|_{L^2}\,\|\xi\widehat{\widetilde{g}}(\xi)\|_{L^2} \;=\; \infty,$$

where $\widetilde{g}$ is the dual system generator from Theorem 11.31(b).

Remark: It requires some work, but it can be shown that the Weak BLT implies Theorem 11.35; see [DJ93] or the survey paper [BHW95].

11.9 The HRT Conjecture

In the final section of this chapter we will present an open problem related to Gabor systems that is so very simple to state yet is still unsolved, at least as of the time of writing. This conjecture first appeared in print in 1996 [HRT96]. As this topic is more personal to me than some of the others that appear in this volume, I will often speak more directly to the reader in this section than usual.

In the previous sections we saw many results dealing with Gabor systems $\mathcal{G}(g,a,b)$ that are complete, a frame, exact, a Riesz basis, an orthonormal basis, and so forth. Yet we have not yet asked what may be the most basic questions of all: Are Gabor systems finitely independent? Given any collection of vectors in a vector space, surely one of the very first properties that we would like to determine is whether these vectors are independent or dependent. For *lattice* Gabor systems $\mathcal{G}(g,a,b)$, the answer is known (though the proof is nontrivial!). The next theorem is due to Linnell [Lin99], and partially answers a question first posed in [HRT96].

Theorem 11.36. *If $g \in L^2(\mathbf{R})\backslash\{0\}$ and $a,\ b > 0$, then $\mathcal{G}(g,a,b)$ is finitely linearly independent.* $\diamondsuit$

We will discuss Theorem 11.36 and its proof a little later. Assuming the validity of Theorem 11.36, Exercise 11.36 shows how to extend it a little further, as follows.

Corollary 11.37. *Let A be an invertible 2×2 matrix, choose $z \in \mathbf{R}^2$, and set $\Lambda = A(\mathbf{Z}^2) + z$. Then for any nonzero $g \in L^2(\mathbf{R})$,*

$$\mathcal{G}(g,\Lambda) = \{M_b T_a g\}_{(a,b)\in\Lambda}$$

is finitely linearly independent. $\diamondsuit$

When A is an invertible matrix, we call $A(\mathbf{Z}^2)$ a *full-rank lattice* in $\mathbf{R}^2$. Thus $A(\mathbf{Z}^2) + z$ is a rigid translate of a full-rank lattice. In particular, if we choose any three noncollinear points in $\mathbf{R}^2$, then we can always find A and z so that $\Lambda = A(\mathbf{Z}^2) + z$ contains these three points (Exercise 11.34). Therefore any set of three noncollinear time-frequency shifts of a nonzero $g \in L^2(\mathbf{R})$ is linearly independent, and the collinear case can be addressed by other arguments (Exercise 11.37). Since one point is trivial and two points are always collinear, we obtain the following corollary.

Corollary 11.38. *Let $N = 1,\ 2$, or 3. If $g \in L^2(\mathbf{R})\backslash\{0\}$ and (p_k, q_k) for $i = 1, \dots, N$ are distinct points in $\mathbf{R}^2$, then*

$$\{e^{2\pi i q_k x} g(x - p_k) : k = 1, \dots, N\}$$

is linearly independent. $\diamondsuit$

Thus, any collection of up to three distinct time-frequency shifts of a function $g \in L^2(\mathbf{R})$ is linearly independent. Surely four points cannot be much more difficult—how hard can it be to show that a set of four vectors in a vector space are linearly independent? It is not that hard if we have four specific vectors in hand, but we are asking a somewhat more general question. If we let (p_k, q_k) for $i = 1,2,3,4$ be any set of four distinct points in $\mathbf{R}^2$, we want to know if

$$\{M_{q_k} T_{p_k} g : k = 1,2,3,4\} = \{e^{2\pi i q_k x} g(x - p_k) : k = 1,2,3,4\}$$

is linearly independent for every nonzero function $g \in L^2(\mathbf{R})$. *The answer to this question is not known!*

One difficulty is that four noncollinear distinct points in $\mathbf{R}^2$ need not lie on a translate of a full-rank lattice. For example, because the distances between the following points are not rationally related, there is no matrix A and point z so that the four points

$$\{(0,0),\ (1,0),\ (0,1),\ (\sqrt{2},\sqrt{2})\}$$

are contained in $A(\mathbf{Z}^2)+z$. Forgetting about generic sets of four points, what about just *this* particular set of points? If $g \in L^2(\mathbf{R})$ is not the zero function, must the set of time-frequency translates of g determined by those four points be independent, i.e., must

$$\bigl\{g(x),\, g(x-1),\, e^{2\pi i x}g(x),\, e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\bigr\}$$

be linearly independent? I don't know, and neither does anyone else.

Conjecture 11.39 (HRT Subconjecture). *If $g \in L^2(\mathbf{R})\backslash\{0\}$ then*

$$\bigl\{g(x),\; g(x-1),\; e^{2\pi i x}g(x),\; e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\bigr\} \qquad (11.34)$$

is linearly independent.

There's nothing special about $\sqrt{2}$ in this choice of four points; the answer is still unknown if we replace the two instances of $\sqrt{2}$ in equation (11.34) by some other irrational numbers (on the other hand, Ziemowit Rzeszotnik has shown me his unpublished proof that $\bigl\{g(x),\; g(x-1),\; e^{2\pi i x}g(x), e^{2\pi i\sqrt{2}x}g(x)\bigr\}$ is independent for each nonzero $g \in L^2(\mathbf{R})$, and the recent paper [Dem10] addresses the case of any four points that lie on two parallel lines).

The answer to Conjecture 11.39 is known for some special classes of functions $g \in L^2(\mathbf{R})$, and for those functions for which the answer is known the answer is always yes, linear independence holds.

Example 11.40. Suppose that $g \in L^2(\mathbf{R})$ is supported within the halfline $[0,\infty)$, i.e., $g(x) = 0$ for almost every $x < 0$, and suppose also that g is not the zero function on $[0,1]$. If the collection of time-frequency translates in equation (11.34) is dependent then there exist scalars a, b, c, d, not all zero, such that

$$ag(x) + bg(x-1) + ce^{2\pi i x}g(x) + de^{2\pi i\sqrt{2}x}g(x-\sqrt{2}) \;=\; 0 \text{ a.e.} \qquad (11.35)$$

Note that the functions $g(x)$ and $e^{2\pi i x}g(x)$ are supported within $[0,\infty)$, while $g(x-1)$ is supported in $[1,\infty)$ and $e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})$ is supported in $[\sqrt{2},\infty)$. Therefore, if we only consider points x between 0 and 1 then equation (11.35) reduces to

$$(a + ce^{2\pi i x})\, g(x) \;=\; 0 \quad \text{for a.e. } x \in [0,1].$$

However, if either a or c is nonzero then $a + ce^{2\pi i x} \neq 0$ for almost every x, so $g(x) = 0$ a.e. on $[0,1]$, which contradicts our assumptions on g. Therefore we must have $a = c = 0$. But then

$$bg(x-1) + de^{2\pi i\sqrt{2}x}g(x-\sqrt{2}) \;=\; 0 \text{ a.e.,}$$

which contradicts the fact that any set of two time-frequency translates of g must be independent. ◇

Conjecture 11.39 is a special case of the following conjecture, first made in [HRT96].

Conjecture 11.41 (HRT Conjecture). *If $g \in L^2(\mathbf{R})$ is not the zero function and $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbf{R}^2$, then*

$$\mathcal{G}(g, \Lambda) \;=\; \bigl\{M_{q_k} T_{p_k} g\bigr\}_{k=1}^{N}$$

is a linearly independent set of functions in $L^2(\mathbf{R})$. $\diamondsuit$

Conjecture 11.41 is also known as the Linear Independence Conjecture for time-frequency shifts. Despite having been worked on by a large number of groups, there is a scarcity of hard results. The main papers specifically dealing with the HRT Conjecture appear to be [HRT96], [Lin99], [Kut02], [Bal08], [BS09], [Dem10], [DG10], [DZ10], and there is also a survey paper on the topic [Hei06].

Some partial results on the HRT Conjecture are known. For example, the idea of Example 11.40 extends to any finite number of points, so independence in the HRT Conjecture is known to hold if we add the extra assumption that g is compactly supported or is only nonzero within a halfline $(-\infty, a]$ or $[a, \infty)$; see Exercise 11.38. On the other hand, it is quite surprising that there are very few partial results based on smoothness or decay conditions on g. In particular, the HRT Conjecture is open even if we impose the extra hypothesis that g lie in the Schwartz class $\mathcal{S}(\mathbf{R})$, i.e., g is infinitely differentiable and $x^m g^{(n)}(x) \to 0$ as $x \to \pm\infty$ for every m, $n \in \mathbf{N}$. While the HRT Conjecture is known to be true for some Schwartz class functions, such as those that are compactly supported, it is not known whether or not it holds for *every* nonzero Schwartz class function.

Let us return to lattice Gabor systems and Theorem 11.36 in particular, and try to illustrate why the proof of that theorem is nontrivial. Consider the case of three specific points in $\mathbf{R}^2$, say

$$\Lambda \;=\; \bigl\{(0,0),\ (a,0),\ (0,1)\bigr\}. \tag{11.36}$$

We will address the "difficult case" where a is irrational.

Example 11.42. Suppose that $g \in L^2(\mathbf{R})\backslash\{0\}$ is such that

$$\mathcal{G}(g, \Lambda) \;=\; \bigl\{g(x),\ g(x-a),\ e^{2\pi i x} g(x)\bigr\}$$

is linearly dependent, where $a > 0$ is irrational. Then there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1 g(x) + c_2 g(x-a) + c_3 e^{2\pi i x} g(x) \;=\; 0 \text{ a.e.}$$

If any one of c_1, c_2, c_3 is zero then we reduce to only two time-frequency shifts, so we assume that c_1, c_2, c_3 are all nonzero. Dividing through by c_2, we can further assume that $c_2 = 1$. Rearranging,

$$g(x-a) \;=\; \bigl(-c_1 - c_3\, e^{2\pi i x}\bigr)\, g(x) \;=\; m(x)\, g(x) \text{ a.e.,} \tag{11.37}$$

where $m(x) = -c_1 - c_3\, e^{2\pi i x}$. Note that m is a 1-periodic trigonometric polynomial. Iterating equation (11.37), for integer $n > 0$ we obtain

$$\begin{aligned} |g(x-na)| \;&=\; |m(x-(n-1)a) \cdots m(x-a)\, m(x)\, g(x)| \\ &=\; |g(x)| \prod_{j=0}^{n-1} |m(x-ja)| \\ &=\; |g(x)|\, e^{n \cdot \frac{1}{n} \sum_{j=0}^{n-1} p(x-ja)} \quad \text{a.e.,} \end{aligned} \tag{11.38}$$

where $p(x) = \ln|m(x)|$. Since g is square integrable, if $g(x-na)$ grows with n then we might hope to obtain a contradiction, although we must be careful since g is only defined almost everywhere.

Now, p is 1-periodic, so $p(x-ja) = p(x-ja \bmod 1)$, where $t \bmod 1$ denotes the fractional part of t. A consequence of the fact that a is irrational is that the points $\{x - ja \bmod 1\}_{j=0}^{\infty}$ form a dense subset of $[0,1)$. In fact, they are "well distributed" in a technical sense due to the fact that $x \mapsto x + a \bmod 1$ is an *ergodic mapping* of $[0,1)$ onto itself (i.e., only subsets of measure 0 or measure 1 can be invariant under this map). Hence the quantity $\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja)$ is like a Riemann sum approximation to $\int_0^1 p(x)\,dx$, except that the rectangles with height $p(x-ja)$ and width $\frac{1}{n}$ are distributed "randomly" around $[0,1)$ instead of uniformly, possibly even with overlaps or gaps (see Figure 11.4). Still, the ergodicity ensures that the Riemann sum analogy is a good one in the limit. Specifically, the *Birkhoff Ergodic Theorem* [Wal82] implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} p(x-ja) \;=\; \int_0^1 p(x)\,dx \;=\; C \text{ a.e.} \tag{11.39}$$

The fact that $C = \int_0^1 p(x)\,dx$ exists and is finite follows from the fact that any singularities of p correspond to zeros of the well-behaved function m (Exercise 11.42). So, if we fix $\varepsilon > 0$, then $\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja) \geq (C-\varepsilon)$ for n large enough. Let us ignore the fact that "large enough" depends on x (or, by applying Egoroff's Theorem, restrict to a subset where the convergence in equation (11.39) is uniform). Substituting into equation (11.38) then yields

$$|g(x-na)| \;\geq\; e^{(C-\varepsilon)n}\, |g(x)|, \qquad n \text{ large.}$$

Considering x in a set of positive measure where g is nonzero and using the fact that $g \in L^2(\mathbf{R})$, we conclude that $C - \varepsilon < 0$. This is true for every $\varepsilon > 0$, so $C \leq 0$. A converse argument based on the relation $g(x) = m(x+a)\, g(x+a)$ similarly yields the inequality $C \geq 0$. This still allows the possibility that $C = 0$, but a slightly more subtle argument presented in [HRT96] also based on ergodicity yields the full result. The case a is rational is more straightforward, since then the points $x - ja \bmod 1$ repeat themselves. $\diamondsuit$

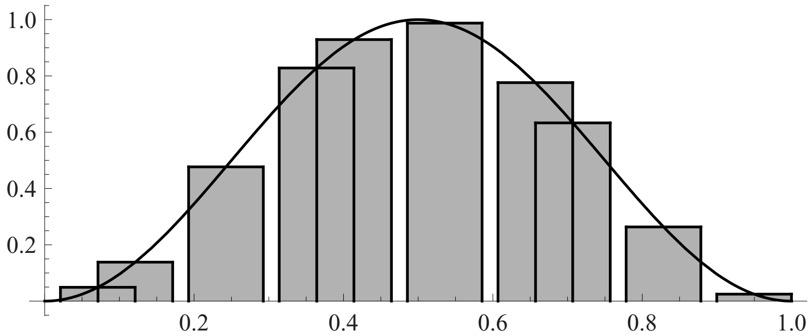

Fig. 11.4. The area of the boxes (counting overlaps) is $\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja)$.

By applying the techniques used in Exercises 11.35 and 11.36, the HRT Conjecture for three noncollinear points can always be reduced to the HRT Conjecture for the three points given in equation (11.36) for some $a \neq 0$. However, the argument given in Example 11.42 is limited to only three points so we have not proved that the HRT Conjecture is valid for all lattice Gabor systems. Still, our argument does suggest why the proof of Theorem 11.36 is nontrivial. There is no obvious way to extend the technique of Example 11.42 to apply to four points in general position. It particular, the argument depends critically on the recurrence relation that appears in equation (11.37), and this recurrence relation is a consequence of the fact that there are only two distinct translations appearing in the collection $\{g(x),\ g(x-a),\ e^{2\pi ix}g(x)\}$. Specifically, $g(x)$ and $e^{2\pi ix}g(x)$ are translated by zero, while $g(x-a)$ is translated by a. As soon as we have three or more distinct translates, the recurrence relation becomes much more complicated (too complicated to use?). Indeed, Linnell's proof takes a quite different approach, relying on the fact that the operators M_{bn}, T_{ak} with $k,\, n \in \mathbf{Z}$ generate a von Neumann algebra (see [Lin99]).

So, we attack the HRT Conjecture from a different angle. Fix any set $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ of finitely many distinct points in $\mathbf{R}^2$, and define

$$S_\Lambda \;=\; \{g \in L^2(\mathbf{R}) \;:\; \mathcal{G}(g,\Lambda) \text{ is independent}\}. \tag{11.40}$$

The HRT Conjecture is that $S_\Lambda = L^2(\mathbf{R})\backslash\{0\}$. While we don't know that this is the case, we do know that S_Λ is dense in $L^2(\mathbf{R})$. For example, S_Λ contains all compactly supported functions in $L^2(\mathbf{R})$ (Exercise 11.38) and all finite linear combinations of Hermite functions (Exercise 11.40), each of which is a dense subset of $L^2(\mathbf{R})$. Perhaps we can apply some kind of perturbation argument to show that S_Λ actually contains all nonzero functions in $L^2(\mathbf{R})$. The next theorem is an attempt in this direction.

Theorem 11.43. *Assume that* $g \in L^2(\mathbf{R})$ *and* $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ *are such that* $\mathcal{G}(g,\Lambda)$ *is linearly independent. Then there exists an* $\varepsilon > 0$ *such that* $\mathcal{G}(h,\Lambda)$ *is independent for any* $h \in L^2(\mathbf{R})$ *with* $\|g-h\|_{L^2} < \varepsilon$.

Proof. Define the linear mapping $T\colon \mathbf{C}^N \to L^2(\mathbf{R})$ by

$$T(c_1,\dots,c_N) \;=\; \sum_{k=1}^N c_k\, M_{q_k} T_{p_k} g.$$

Note that T is injective since $\mathcal{G}(g,\Lambda)$ is independent. Therefore T is a linear bijection of $\mathbf{C}^N$ onto range(T), which is an N-dimensional subspace of $L^2(\mathbf{R})$. Since linear operators on finite-dimensional spaces are continuous, both $T\colon \mathbf{C}^N \to \mathrm{range}(T)$ and $T^{-1}\colon \mathrm{range}(T) \to \mathbf{C}^N$ are bounded. As all norms on $\mathbf{C}^N$ are equivalent, it follows that there exist constants A, $B > 0$ such that

$$A \sum_{k=1}^N |c_k| \;\le\; \biggl\| \sum_{k=1}^N c_k\, M_{q_k} T_{p_k} g \biggr\|_{L^2} \;\le\; B \sum_{k=1}^N |c_k|, \qquad (c_1,\dots,c_N) \in \mathbf{C}^N.$$

Therefore, if $\|g-h\|_{L^2} < A$, then for any $(c_1,\dots,c_N) \in \mathbf{C}^N$ we have

$$\begin{aligned}
\biggl\| \sum_{k=1}^N c_k\, M_{q_k} T_{p_k} h \biggr\|_{L^2} \;&\ge\; \biggl\| \sum_{k=1}^N c_k\, M_{q_k} T_{p_k} g \biggr\|_{L^2} \;-\; \biggl\| \sum_{k=1}^N c_k\, M_{q_k} T_{p_k} (h-g) \biggr\|_{L^2} \\
&\ge\; A \sum_{k=1}^N |c_k| \;-\; \sum_{k=1}^N |c_k|\, \|M_{q_k} T_{p_k}(h-g)\|_{L^2} \\
&=\; (A - \|h-g\|_{L^2}) \sum_{k=1}^N |c_k|.
\end{aligned}$$

Consequently, if $\sum_{k=1}^N c_k\, M_{q_k} T_{p_k} h = 0$ a.e. then $c_k = 0$ for every k. □

Thus, the set S_Λ defined in equation (11.40) is actually an open subset of $L^2(\mathbf{R})$. Plus, we know that it is dense—so isn't it all of $L^2(\mathbf{R})$? No, we can't conclude that. For example, $\mathbf{R}\backslash\{\pi\}$ is an open and dense but proper subset of the real line. Therefore, we still don't know whether the HRT Conjecture is valid for all nonzero $g \in L^2(\mathbf{R})$. On the other hand, this does tell us that any counterexamples are "rare" in some sense.

I've worked hard on the HRT Conjecture but haven't solved it. If you solve it, please let me know! One word of warning—the problem seems to be much harder than it looks. I've produced dozens of incorrect proofs myself, and seen many more. Many of the errors in these proofs are related to the fact that the translation and modulation operators T_a, M_b do not commute for most values of a and b. I hope you enjoy this charming little problem, but beware of the pesky phase factor in the relation $T_a M_b = e^{-2\pi i a b}\, M_b T_a$.

Exercises

11.34. Show that if (p_i, q_i) for $i = 1, 2, 3$ are three noncollinear points in $\mathbf{R}^2$, then there exist an invertible 2×2 matrix A and a point $z \in \mathbf{R}^2$ such that $\Lambda = A(\mathbf{Z}^2) + z$ contains those three points.

11.35. Fix $g \in L^2(\mathbf{R})\backslash\{0\}$, and let $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ be any set of finitely many distinct points in $\mathbf{R}^2$. Define $\mathcal{G}(g, \Lambda) = \{M_{q_k} T_{p_k} g\}_{k=1}^N$.

(a) Fix $z \in \mathbf{R}^2$. Show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(g, \Lambda + z)$ is linearly independent.

(b) Given $r \in \mathbf{R}$, let $S_r = \left[\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right]$, so multiplication by the matrix S_r is a shear operation on $\mathbf{R}^2$. Define $h(x) = e^{\pi i r x^2} g(x)$, and show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(h, S_r(\Lambda))$ is linearly independent.

(c) Let $R = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, so multiplication by R is a counterclockwise rotation of $\mathbf{R}^2$ by 90 degrees. Show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(\check{g}, R(\Lambda))$ is linearly independent, where $\check{g}$ is the inverse Fourier transform of g.

(d) Given $a \neq 0$, let $D_a = \left[\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right]$, so multiplication by D_a is a dilation by a on the x_1-axis and a corresponding dilation by $1/a$ on the x_2-axis. Define $h(x) = g(x/a)$, and show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(h, D_a(\Lambda))$ is linearly independent.

(e) Show that if A is a 2×2 matrix with $\det(A) = 1$, then A can be written as a product of matrices of the form S_r, R, and D_a.

Remark: This factorization is related to the fact that every 2×2 matrix with determinant 1 is a *symplectic matrix*. In contrast, not every $2d \times 2d$ matrix with determinant 1 is symplectic when $d > 1$. As a consequence, the HRT Conjecture becomes even more intractable in higher dimensions.

11.36. Assuming Theorem 11.36, prove Corollary 11.37.

11.37. Fix $g \in L^2(\mathbf{R})\backslash\{0\}$.

(a) Show that if $\{q_k\}_{k=1}^N$ is any set of finitely many distinct real numbers, then $\{M_{q_k} g\}_{k=1}^N$ is linearly independent.

(b) Show that if $\{p_k\}_{k=1}^N$ is any set of finitely many distinct real numbers, then $\{T_{p_k} g\}_{k=1}^N$ is linearly independent.

(c) Show that if $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct but collinear points in $\mathbf{R}^2$, then $\mathcal{G}(g, \Lambda) = \{M_{q_k} T_{p_k} g\}_{k=1}^N$ is linearly independent.

11.38. Suppose that $g \in L^2(\mathbf{R})\backslash\{0\}$ is supported within some halfline, either $(-\infty, a]$ or $[a, \infty)$ where $a \in \mathbf{R}$. Show that if $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbf{R}^2$, then $\mathcal{G}(g, \Lambda) = \{M_{q_k} T_{p_k} g\}_{k=1}^N$ is linearly independent.

11.39. The nth *Hermite function* H_n is

$$H_n(x) \;=\; e^{\pi x^2}\, D^n e^{-2\pi x^2}, \qquad n \geq 0,$$

where D^n denotes the nth derivative operator.

(a) Prove that

$$H_{n+1}(x) \;=\; H_n'(x) - 2\pi x\, H_n(x), \qquad n \geq 0. \tag{11.41}$$

(b) Use equation (11.41) to show that $H_n(x) = p_n(x)\, e^{-\pi x^2}$, where p_n is a polynomial of degree n whose leading coefficient is $(-4\pi)^n$. Consequently, each H_n is infinitely differentiable and has exponential decay at infinity, and $\mathrm{span}\{H_n\}_{n\geq 0} = \big\{p(x)\, e^{-\pi x^2} : p \text{ is a polynomial}\big\}$.

Remark: It can be shown that $\{H_n\}_{n\geq 0}$ is an orthogonal (but not orthonormal) basis for $L^2(\mathbf{R})$. Hence $\mathrm{span}\{H_n\}_{n\geq 0}$ is a dense subspace of $L^2(\mathbf{R})$.

11.40. Let $\phi(x) = e^{-\pi x^2}$ be the Gaussian function, and let $h(x) = p(x)\, e^{-\pi x^2}$ where p is any nontrivial polynomial. Show that if $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbf{R}^2$, then $\mathcal{G}(h, \Lambda) = \{M_{q_k} T_{p_k} h\}_{k=1}^N$ is linearly independent.

11.41. Assume that $g \in L^2(\mathbf{R})$ and $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ are such that $\mathcal{G}(g, \Lambda)$ is linearly independent. Show that there exists $\varepsilon > 0$ such that $\mathcal{G}(g, \Lambda')$ is independent for any set $\Lambda' = \{(\alpha_k', \beta_k')\}_{k=1}^N$ with $|\alpha_k - \alpha_k'|$, $|\beta_k - \beta_k'| < \varepsilon$ for $k = 1, \dots, N$.

11.42. Suppose that m is differentiable and $m(0) = 0$. Set $p(x) = \ln|m(x)|$, and show that $\int_{-\delta}^{\delta} p(x)\, dx$ exists and is finite if $\delta > 0$ is small enough.