Applied and Numerical Harmonic Analysis

Christopher Heil

A Basis Theory Primer

Expanded Edition

Birkhäuser

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In memory of my mother, Agnes Jean Heil,
and my sister, Cynthia Ann Moon

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

John J. Benedetto
Series Editor
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Part II Bases and Frames

Preface

A basis for a Banach space is a set of elementary building blocks that can be put together in a unique way to obtain any given element of the space. Bases are central to the study of the geometry of Banach spaces, which is a rich and beautiful classical subject in analysis. Bases and their relatives are also of key importance in classical and applied harmonic analysis, where they are used for decomposing and manipulating functions, operators, signals, images, and other objects.

In 1986 I was a young graduate student at the University of Maryland, College Park, just beginning to learn about harmonic analysis from my advisor, John Benedetto. I was at that awkward point of realizing that he wanted me to call him by his first name, yet feeling uncomfortable using any address less formal than "Dr. Benedetto" (for a long time I avoided this issue by never speaking to John unless he was already looking at me). One day John returned from a conference with news about an exciting new development, called "wavelets." Together with John's other graduate students of the time (David Walnut, William Heller, Rodney Kerby, and Jean-Pierre Gabardo), we began learning and thinking about wavelets. Through John we obtained early preprints of many papers that ultimately had a deep impact on the field. I recall spending many hours studying a xerox copy of a long preprint by Ingrid Daubechies, some 50 pages of meticulously handwritten single-spaced mathematics and exposition that eventually became published as the paper [Dau90]. I was especially fascinated by the use of *frames* in this paper, both wavelet frames and time-frequency (Gabor) frames. Frames are basis-like systems, but they allow the possibility of nonunique representations, and hence can incorporate redundancy. While uniqueness seems at first glance to be a property that we cannot live without, redundancy can actually be a useful and even essential property in many settings.

While reading these early papers, it became clear to our group that it would be important to understand the precise relationships between bases and frames, and John assigned me the task of becoming our resident expert on basis theory. Thus began a journey into the classical field of the geometry of Banach spaces. My instructors were the beautiful but comprehensive (even encyclopedic) volumes on bases in Banach spaces by Singer [Sin70], Lindenstrauss and Tzafriri [LT77], [LT79], and Marti [Mar69] (and if I had been aware of it at the time, the text by Diestel [Die84] would also be on this list). Additionally, while not specifically a basis theory volume, the elegant text on nonharmonic Fourier analysis by Young (now published as a "Revised First Edition" [You01]) provided a deep yet gentle introduction to both bases and frames that perfectly complemented the basis theory books mentioned above. A 1987 handwritten survey of what I learned about bases and frames circulated for some years among John's students and colleagues. This survey was finally typed in 1997 and was the original incarnation of "A Basis Theory Primer." Over the years, John and many others have asked me to turn that survey into a proper book, and the present volume is the result. The core material of the first Basis Primer has been greatly expanded and polished (hopefully benefiting from some 20 years of reflection on these subjects). Many new topics have been added, including chapters on Gabor bases and frames, wavelet bases and frames, and Fourier series (which are bases of complex exponentials). Introductory chapters on Banach spaces and functional analysis have also been included, which make the text almost entirely self-contained. A solutions manual for this volume is also available for instructors upon request at the Birkhäuser website.

Outline and Goals

A *primer* is a old-fashioned word for a school book,[1] and this is a text for learning the theory of bases and frames and some of their appearances in classical and applied harmonic analysis. Extensive exercises complement the text and provide opportunities for learning by doing (hints for selected exercises appear at the end of the volume).

The text is divided into four parts. Part I reviews the functional analysis that underlies most of the concepts presented in the later parts of the text. Part II presents the abstract theory of bases and frames in Banach and Hilbert spaces. It begins with the classical topics of convergence, Schauder bases, biorthogonal systems, and unconditional bases, and concludes with more "modern" topics, such as Riesz bases and frames in Hilbert spaces. Part III is devoted to a study of concrete systems that form frames or bases for various Hilbert spaces. These include systems of weighted exponentials, systems of translations, Gabor systems, and wavelets. Each of these play important roles both in mathematics and in applications such as digital signal processing. It has become common to refer to these ideas as being part of the field of "applied harmonic analysis." Part IV is concerned with the theory of Fourier series, which is usually considered to be part of "classical harmonic analysis," although it is also widely applicable. Our presentation emphasizes

[1] The correct pronunciation is *prim-er*, not *prime-er*.

the role played by bases, which is a different viewpoint than is taken in most discussions of Fourier series.

In summary, Parts I and II deal with the abstract development of bases and frames, while Parts III and IV apply these concepts to particular situations in applied and classical harmonic analysis. The mathematical tools needed to understand the abstract theory are the basic ideas of functional analysis and operator theory, which are presented in Part I. Measure theory makes only limited appearances in Parts I and II, and is used in those parts mostly to give examples. However, the theory of Lebesgue measure and Lebesgue integration is needed throughout Parts III and IV, and therefore a short review of basic ideas from measure theory is given in Appendix A. A small number of proofs and exercises in the text use the concepts of compact operators or tensor products of Hilbert spaces, and these topics are briefly reviewed in Appendix B.

We discuss each part of the text in more detail below.

Part 1: A Primer on Functional Analysis

This part presents the background material that is needed to develop the theory of bases, frames, and applications in the remainder of the text. Most of this background is drawn from functional analysis and operator theory. Chapter 1 introduces Banach and Hilbert spaces and basic operator theory, while Chapter 2 presents more involved material from functional analysis, such as the Hahn–Banach Theorem and the Uniform Boundedness Principle. Proofs of most results are included, as well as extensive exercises.

Part II: Bases and Frames

Part II develops the abstract theory of bases and frames in Banach and Hilbert spaces. Chapter 3 begins with a detailed account of convergence of infinite series in Banach spaces, focusing especially on the meaning of unconditional convergence. Chapter 4 defines bases and derives their basic properties, one of the most important of which is the fact that the coefficient functionals associated with a basis are automatically continuous. The Schauder, Haar, and trigonometric systems are studied as concrete examples of bases in particular Banach spaces. Generalizations of bases that allow weak or weak* convergence instead of norm convergence are also discussed.

Chapter 5 continues the development of basis theory by examining the subtle distinctions between true bases, which provide unique infinite series representations of vectors in Banach spaces, and exact systems, which possess the minimality and completeness properties enjoyed by bases but which do not yield the infinite series representations that bases provide. This understanding allows us to extend the basis properties of the Haar system to $L^p[0,1]$, to derive results on the stability of bases under perturbations, and to characterize the basis properties of the sequence of coefficient functionals associated with a basis.

In Chapter 6 we consider bases that have the important extra property that the basis expansions converge unconditionally, i.e., regardless of ordering. We see that the Schauder system, which was proved by Schauder to be a basis for the space $C[0,1]$ of continuous functions on $[0,1]$, fails to be an unconditional basis for that space, as does the Haar system in $L^1[0,1]$.

Chapter 7 turns to the study of bases in the setting of Hilbert spaces. Unlike generic Banach spaces, Hilbert spaces have a notion of orthogonality, and we can take advantage of this additional structure to derive much more concrete results. We consider orthonormal bases, Riesz bases, unconditional bases, and the more general concept of Bessel sequences in Hilbert spaces. A key role is played by the *analysis operator*, which breaks a vector into a sequence of scalars that (we hope) captures all the information about a vector in a discrete fashion, much as a musical score is a discrete representation of a symphony as a "sequence of notes." Conversely, the *synthesis operator* forms (infinite) linear combinations of special vectors (perhaps basis vectors), much as the musicians in an orchestra create the symphony by superimposing their musical notes. When these special vectors form a basis, we have unique representations. Analysis and synthesis are injective in this case, and their composition is the identity operator.

Chapter 8 is inspired by a simple question: Why do we need *unique* representations? In many circumstances it is enough to know that we have a set of vectors that we can use as building blocks in the sense that any vector in the space can be represented as *some* suitable superposition of these building blocks. This leads us to define and study frames in Hilbert spaces, which provide such nonunique representations. Moreover, frames also have important "stability" properties such as unconditional convergence and the existence of an equivalent discrete-type norm for the space via the analysis operator. Frames provide a mathematically elegant means for dealing with nonunique or redundant representations, and have found many practical applications both in mathematics and engineering.

Part III: Bases and Frames in Applied Harmonic Analysis

In Part III of the text, we focus on concrete systems that form bases or frames in particular Hilbert spaces. The material in this part, and also in Part IV, does require more fluency with the tools of measure theory than was needed in Parts I or II. Appendix A contains a brief review, without proofs, of the main results from measure theory that are used in the text, such as the Lebesgue Dominated Convergence Theorem and Fubini's Theorem.

Much of what we do in Part III is related to what is today called "applied harmonic analysis." The Fourier transform on the real line is a fundamental tool for the analysis of many of these systems, and therefore Chapter 9 presents a short review of this topic. For the purposes of this volume, the most essential facts about the Fourier transform are that:

(i) the Fourier transform is a unitary mapping of $L^2(\mathbf{R})$ onto itself, and

(ii) the operations of translation and modulation are interchanged when the Fourier transform is applied.

Chapter 10 explores several important frames that are related to the trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$. The trigonometric system is an orthonormal basis for the Hilbert space $L^2[0,1]$, but by embedding it into $L^2(\mathbf{R})$ and applying the Fourier transform we magically obtain a fundamental result in signal processing known as the Classical Sampling Theorem: A bandlimited function $f \in L^2(\mathbf{R})$ with $\mathrm{supp}(\widehat{f}\,) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ can be recovered from its sample values $\{f(n)\}_{n\in\mathbf{Z}}$. Moreover, the fact that $\{e^{2\pi inbx}\}_{n\in\mathbf{Z}}$ is a frame for $L^2[0,1]$ when $0 < b < 1$ translates into a statement about stable reconstruction by oversampling. Also discussed in Chapter 10 are two closely related types of systems, namely systems of weighted exponentials $\{e^{2\pi inx}\varphi(x)\}_{n\in\mathbf{Z}}$ in $L^2[0,1]$ and systems of translates $\{g(x-k)\}_{k\in\mathbf{Z}}$ in $L^2(\mathbf{R})$.

In Chapter 11 we analyze *Gabor systems* in $L^2(\mathbf{R})$. These are generated by simple *time-frequency shifts* of a single function, and have the form $\{e^{2\pi ibnx}g(x-ak)\}_{k,n\in Z}$ where $g \in L^2(\mathbf{R})$ and a, $b > 0$ are fixed. Thus a Gabor system incorporates features from both systems of weighted exponentials and systems of translates. The elements $e^{2\pi ibnx}g(x-ak)$ of a Gabor system are much like notes of different frequencies played at different times that are superimposed to create a symphony. The theory of Gabor frames and bases, which is named in honor of the Nobel prize winner Dennis Gabor, is not only mathematically beautiful but has a great utility in mathematics, physics, and engineering. We will see that the mathematical formulation of the quantum mechanical uncertainty principle forces us to rely on Gabor frames that are *not* bases—redundancy is essential to these frames. There do exist Gabor systems that are orthonormal or Riesz bases for $L^2(\mathbf{R})$, but the generator g of such a system cannot be a very "nice" function. Such a generator cannot simultaneously be continuous and have good decay at infinity; more precisely, the Heisenberg product $\|xg(x)\|_{L^2}\,\|\xi\widehat{g}(\xi)\|_{L^2}$ that appears in the uncertainty principle must be infinite.

Wavelet systems, which are the topic of Chapter 12, are also simply generated from a single function, but through *time-scale* shifts instead of time-frequency shifts. A wavelet system has the form $\{a^{n/2}\psi(a^nx - mbk)\}_{k,n\in Z}$, where $\psi \in L^2(\mathbf{R})$, $a > 1$, and $b > 0$ are fixed. The Haar system is an example of a wavelet orthonormal basis, and has been known since 1910. Unfortunately, the Haar system, which is the wavelet system with $a = 2$, $b = 1$, and $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, is generated by a discontinuous function. The "wavelet revolution" of the 1980s began with the discovery of orthonormal wavelet bases generated by very nice functions (in striking contrast to the nonexistence of "nice" Gabor bases). In particular, we will encounter generators ψ that are either m-times differentiable and compactly supported, or are infinitely differentiable and have compactly supported Fourier transforms.

Part IV: Fourier Series

In Part IV of this volume, which consists of Chapters 13 and 14, we see how basis theory relates to that part of classical harmonic analysis that deals with Fourier series. The main goal of these chapters is to prove that the trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is a basis for $L^p[0,1]$ for each $1 < p < \infty$, although we will see that this basis is conditional when $p \neq 2$.

The basis properties of the trigonometric system are a beautiful application of the machinery developed earlier in Part II. However, to properly work with Fourier series we need an additional set of tools that were not required in the preceding chapters. This is one reason that Fourier series have been placed into a separate portion of the text. The tools we need, including convolution, approximate identities, and Cesàro summation, are quite elegant in themselves, not to mention very important in digital signal processing as well as mathematics. Chapter 13 is devoted to developing these tools, which then provide the foundation for our analysis of the basis properties of the trigonometric system in Chapter 14.

Course Outlines

This text is a learning tool, suitable for independent study or as the basis for an advanced course. There are several options for building a course around this text, two of which are listed below.

Course 1: Functional Analysis, Bases, and Frames. A course on functional analysis, bases, and frames could focus on Parts I and II. This would be ideal for students who have not already had an in-depth course on functional analysis. Additionally, the material in Parts I and II does not require deep familiarity with Lebesgue measure or integration. Part I develops the most important tools of functional analysis and operator theory, and then Part II applies these tools to develop the theory of bases and frames.

Course 2: Bases, Frames, Applied Harmonic Analysis, and Fourier Series. A course for students already familiar with functional analysis can begin with Part II, and treat Chapters 1 and 2 as a quick reference guide on functional analysis and operator theory. This course would emphasize bases and frames and their roles in applied and classical harmonic analysis. The abstract theory of bases and frames contained in Part II of the text would not require expertise in measure theory, while the applications in Parts III and IV will require fluency with Lebesgue measure and integral on the part of the reader.

A solutions manual for instructors is available upon request; instructions for obtaining a copy are given on the Birkhäuser website.

Further Reading

As the title emphasizes, this volume is a primer rather than an exhaustive treatment of the subject. There are many possible directions for the reader who wishes to learn more, including those listed below.

- *Functional Analysis.* Chapters 1 and 2 provide an introduction to operator theory and functional analysis. More detailed and extensive development of these topics is available in texts such as Conway [Con90], Folland [Fol99], Gohberg and Goldberg [GG01], and Rudin [Rud91], to name only a few.
- *Classical Basis Theory.* For the student interested in bases and the geometry of Banach spaces there are a number of classic texts available, including the volumes mentioned above by Singer [Sin70], Lindenstrauss and Tzafriri [LT77], [LT79], Marti [Mar69], and Diestel [Die84]. These books contain an enormous amount of material on basis theory. Most of the proofs on bases that we give in Chapters 3–6 are either adapted directly from or are inspired by the proofs given in these texts.
- *Frame Theory.* The recent text [Chr03] by Christensen provides a thorough and accessible introduction to both frames and Riesz bases. My own "Basis and Frame Primer" was the text by Young [You01], and his volume is still a gem that I highly recommend. Moreover, Young's text is a standard reference on sampling theory and nonharmonic Fourier series, and it provides a wealth of fascinating and useful historical notes.
- *Gabor Systems and Time-Frequency Analysis.* Gabor systems form one part of the modern theory of time-frequency analysis, which is itself a part of applied harmonic analysis. For a much more extensive account of time-frequency analysis than appears in this volume I highly recommend the essential text [Grö01] by Gröchenig.
- *Wavelet Theory.* There are now many texts on wavelet theory, but the volume by Daubechies [Dau92] is a classic. We also recommend Hernández and Weiss [HW96], especially for wavelet theory in function spaces other than $L^2(\mathbf{R})$. The text [Wal02] by my mathematical sibling Walnut provides an introduction to wavelet theory and many of its applications. Moreover, Walnut's text avoids measure theory and so is suitable for a course aimed at upper-level undergraduate students. For wavelet theory from the engineering point of view, we mention the texts by Mallat [Mal09], Strang and Nguyen [SN96], and Vetterli and Kovačević [VK95].
- *Fourier Series.* Chapters 13 and 14 delve into classical harmonic analysis, proceeding just far enough to develop the tools needed to understand the basis properties of the trigonometric system on $[0, 1]$, the one-dimensional torus. These same tools are the foundation of much of harmonic analysis, both on the torus, the real line, and abstractly. For further reading on harmonic analysis we suggest the volumes by Benedetto [Ben97], Katznelson [Kat04], Grafakos [Gra04], or the author's forthcoming text [Heil].

Acknowledgments

A text such as this is indebted to and has as its foundation many classic and recent volumes in Banach space theory, applied harmonic analysis, and classical harmonic analysis. Many of those volumes have been mentioned already in this preface, and others that have influenced the writing are listed in the references. A few results due to the author have been included in Chapters 8–12, but aside from these, this volume is an exposition and introduction to results due to many others. As my goal was to make a text to learn from, I have not attempted to provide detailed historical accounts or attributions of results.

Lost in the depths of time are the names of the many people who provided feedback on my original basis primer surveys, both the handwritten 1987 notes and the typed 1997 version. I recall helpful criticisms from Jae Kun Lim and Georg Zimmermann, and thank everyone who commented on those manuscripts.

It is a pleasure to thank some of the many people who helped me in the process of writing this new version of the basis primer. Amit Einav, Ole Christensen, and David Walnut provided extensive feedback and identified many typographical errors in various parts of the manuscript. Kathy Merrill rescued me from my limited fluency with Mathematica by providing the wedding cake wavelet set pictured in Figure 12.4. Discussions with Radu Balan and Alex Powell during a snowy walk in Banff, Canada influenced the writing of Chapter 4.

Two people deserve special thanks. Lili Hu carefully read the entire manuscript and provided detailed criticisms of my arguments. The fact that she was not afraid to tell me that what I had written was stupid saves me from much embarrassment. Shahaf Nitzan provided invaluable "big picture" feedback as well as many local comments. The overall organization of the text has greatly benefited from her extensive remarks and feedback.

Finally, my thanks go to John Benedetto, both for instigating the first basis primer and for encouraging me to do something with it.

Atlanta, Georgia *Christopher Heil*
January 8, 2010

General Notation

We use the symbol $\square$ to denote the end of a proof, and the symbol $\diamondsuit$ to denote the end of a definition, remark, or example, or the end of the statement of a theorem whose proof will be omitted.

The set of natural numbers will be denoted by $\mathbf{N} = \{1, 2, 3, \dots\}$. Also, $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$ is the set of integers, $\mathbf{Q}$ is the set of rational numbers, $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of complex numbers. On occasion, we formally use the *extended real numbers* $\mathbf{R} \cup \{-\infty, \infty\}$. For example, the infimum and supremum of a set of real numbers $\{a_n\}_{n\in\mathbf{N}}$ always exist as extended real numbers, i.e., we always have $-\infty \le \inf a_n \le \sup a_n \le \infty$.

Most of the abstract results of Parts I and II of this volume apply simultaneously to real vector spaces and complex vector spaces. We therefore let $\mathbf{F}$ denote a generic choice of scalar field, i.e., $\mathbf{F}$ can be either $\mathbf{R}$ or $\mathbf{C}$, according to context. When we turn from abstract theory to the study of concrete systems in Parts III and IV we will fix the scalar field as $\mathbf{F} = \mathbf{C}$.

The *real part* of a complex number $z = a + ib$ (a, $b \in \mathbf{R}$) is $\mathrm{Re}(z) = a$, and the *imaginary part* is $\mathrm{Im}(z) = b$. We say that z is *rational* if both its real and imaginary parts are rational numbers. The *complex conjugate* of z is $\bar{z} = a - ib$. The *polar form* of z is $z = re^{i\theta}$ where $r > 0$ and $\theta \in [0, 2\pi)$. The *modulus*, or *absolute value*, of z is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = r$, and its *argument* is $\arg(z) = \theta$.

If S is a subset of a set X, then its complement is $X \backslash S = \{x \in X : x \notin S\}$. The cardinality of a set A is denoted by $|A|$.

Given a set X and points $x_i \in X$ for $i \in I$, we let $\{x_i\}_{i\in I}$ denote the sequence indexed by I. We often write $\{x_i\}_{i\in I} \subseteq X$, although it should be noted that $\{x_i\}_{i\in I}$ is a sequence and not just a set. Technically, a sequence $\{x_i\}_{i\in I}$ is shorthand for the mapping $i \mapsto x_i$, and therefore the vectors in a sequence need not be distinct. In particular, a sequence $\{x_i\}_{i\in I}$ where $x_i = x$ for every $i \in I$ is called a *constant sequence.*

Sequences or series with unspecified limits are assumed to be over the natural numbers. That is, we use the shorthand notation

$$(c_n) = (c_n)_{n\in\mathbf{N}}, \qquad \{x_n\} = \{x_n\}_{n\in\mathbf{N}}, \qquad \sum_n x_n = \sum_{n=1}^{\infty} x_n.$$

We generally use the notation (c_n) to denote a sequence of scalars and $\{x_n\}$ to denote a sequence of vectors.

A series $\sum c_n$ of real or complex scalars *converges* if $\lim_{N\to\infty} \sum_{n=1}^{N} c_n$ exists as a real or complex number. If (c_n) is a sequence of *nonnegative real scalars*, we use the notation $\sum c_n < \infty$ to mean that the series $\sum c_n$ converges. A bi-infinite series $\sum_{n=-\infty}^{\infty} c_n$ converges if $\sum_{n=0}^{\infty} c_n$ and $\sum_{n=1}^{\infty} c_{-n}$ both converge (convergence of series is examined in detail in Chapter 3).

Let X and Y be sets. We write $f\colon X \to Y$ to denote a function with domain X and codomain Y. The *image* or *range* of f is $\operatorname{range}(f) = f(X) = \{f(t) : t \in X\}$. A function $f\colon X \to Y$ is *injective*, or *1-1*, if $f(a) = f(b)$ implies $a = b$. It is *surjective*, or *onto*, if $f(X) = Y$. It is *bijective* if it is both injective and surjective. If $S \subseteq X$, then the restriction of f to the domain S is denoted by $f|_S$. Given $A \subseteq X$, the *direct image* of A under f is $f(A) = \{f(t) : t \in A\}$, the set of all images of elements of A. If $B \subseteq Y$, then the *inverse image* of B under f is $f^{-1}(B) = \{t \in X : f(t) \in B\}$, the set of all elements whose image lies in B. Given $b \in \mathbf{R}$, a function $f\colon \mathbf{R} \to Y$ is *b-periodic* if $f(t + b) = f(t)$ for all $t \in \mathbf{R}$.

The *support* of a continuous function $f\colon \mathbf{R} \to \mathbf{F}$ is the closure in $\mathbf{R}$ of the set $\{t \in \mathbf{R} : f(t) \neq 0\}$. Hence a continuous function has compact support if it is zero outside of some finite interval.

We let $C(\mathbf{R})$ denote the space of all continuous functions $f\colon \mathbf{R} \to \mathbf{F}$, and $C_c(\mathbf{R})$ is the subspace consisting of the continuous, compactly supported functions. Likewise $C^m(\mathbf{R})$ is the space of all m-times differentiable functions such that $f, f', \dots, f^{(m)}$ are all continuous, and $C_c^m(\mathbf{R})$ contains those functions $f \in C^m(\mathbf{R})$ that have compact support. $C^\infty(\mathbf{R})$ is the space of infinitely differentiable functions on $\mathbf{R}$, and $C_c^\infty(\mathbf{R})$ is the subspace of infinitely differentiable, compactly supported functions.

Given A, $B \subseteq \mathbf{R}$ and c, $x \in \mathbf{R}$, we define $A + x = \{a + x : a \in A\}$, $A + B = \{a + b : a \in A,\, b \in B\}$, and $cA = \{ca : a \in A\}$.

Given a set X, the *characteristic function* of a subset $A \subseteq X$ is the function $\chi_A\colon X \to \mathbf{R}$ defined by

$$\chi_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

The *Kronecker delta* is

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

We let δ_n denote the sequence $\delta_n = (\delta_{nk})_{k\in\mathbf{N}}$. That is, the nth component of the sequence δ_n is 1, while all other components are zero. We call δ_n the *nth standard basis vector*.

Whenever we speak of measure or measurability, it is with respect to Lebesgue measure on subsets of the real line. The Lebesgue measure of a measurable set $E \subseteq \mathbf{R}$ is denoted by $|E|$. Some standard terminology related to Lebesgue measure is as follows. Let E be a measurable subset of $\mathbf{R}$ and let $f\colon E \to \mathbf{F}$ be a scalar-valued function defined on E. We say that f is *bounded* if there exists a real number M such that $|f(t)| \le M$ for every $t \in E$. We say f is *essentially bounded* if f is measurable and there exists a real number M such that $|f(t)| \le M$ almost everywhere, i.e., if the set $Z = \{t \in E : |f(t)| > M\}$ has Lebesgue measure zero.

A property is said to hold *almost everywhere* (abbreviated *a.e.*) if the Lebesgue measure of the set on which the property fails is zero. For example, if f, $g\colon E \to \mathbf{F}$ and $\{t \in E : f(t) \neq g(t)\}$ has measure zero, then we write $f = g$ a.e. Also, although we do not define the support of functions that are not continuous, we will say that a measurable function $f\colon \mathbf{R} \to \mathbf{F}$ has *compact support* if $f(t) = 0$ for almost all t outside of some finite interval. We write $\operatorname{supp}(f) \subseteq [a, b]$ to mean that $f(t) = 0$ for a.e. $t \notin [a, b]$.

All integrals in this volume are Lebesgue integrals. The Lebesgue integral of a bounded piecewise continuous function on a finite interval coincides with its Riemann integral (and, more generally, the two integrals coincide for any bounded function f on $[a, b]$ that is continuous at almost every point).

Part I

A Primer on Functional Analysis

1

Banach Spaces and Operator Theory

The tools that we need to develop the abstract theory of bases and frames are largely drawn from the field of functional analysis. Therefore, the first two chapters of this volume survey the basic definitions and theorems from functional analysis that will be used throughout this volume. This chapter reviews foundational material that is likely to be at least partially familiar to most readers, while Chapter 2 delves into more advanced topics in functional analysis.

Proofs of most theorems are either included or assigned as exercises (with hints appearing at the end of the volume). Some of the deeper results are stated without proof. There are many texts that the reader can turn to for proofs of those theorems and for more detailed information, including such classics as [Con90], [Fol99], [GG01], [RS80], [Rud91], and many others.

Many of the concrete examples of bases and frames that we will see in this volume occur in the setting of sequence spaces such as ℓ^p or function spaces such as $L^p(\mathbf{R})$. Some familiarity with basic real analysis, especially Lebesgue measure and integration, is needed to fully appreciate the L^p examples. References for background and details on real analysis and measure theory include [Fol99], [Roy88], [Rud87], [WZ77]. A brief review of Lebesgue measure and integration is presented in Appendix A.

1.1 Definition and Examples of Banach Spaces

We assume that the reader is familiar with vector spaces (which are also called *linear spaces*). The scalar field associated with the vector spaces in this volume will always be either the real line $\mathbf{R}$ or the complex plane $\mathbf{C}$. As noted in the opening section on General Notation, we will use the symbol $\mathbf{F}$ to denote a generic choice of one of these two fields.

A norm on a vector space quantifies the idea of the "size" of a vector.

Definition 1.1. A vector space X is called a *normed linear space* if for each $x \in X$ there is a (finite) real number $\|x\|$, called the *norm* of x, such that:

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(a) $\|x\| \geq 0$ for all $x \in X$,

(b) $\|x\| = 0$ if and only if $x = 0$,

(c) $\|cx\| = |c|\,\|x\|$ for all $x \in X$ and scalars c, and

(d) the *Triangle Inequality*: $\|x + y\| \leq \|x\| + \|y\|$ holds for all x, $y \in X$.

Given a norm $\|\cdot\|$, we refer to the number $\|x-y\|$ as the *distance* between the vectors x and y. We call

$$B_r(x) \;=\; \bigl\{y \in X : \|x - y\| < r\bigr\}$$

the *open ball in X centered at x with radius r.* ◇

On occasion we will also deal with *seminorms* which, by definition, must satisfy properties (a), (c), and (d) of Definition 1.1, but need not satisfy property (b). For example, if we define $\|x\| = |x_1|$ for $x = (x_1, x_2) \in \mathbf{F}^2$, then $\|\cdot\|$ is a seminorm on $\mathbf{F}^2$, but it is not a norm on $\mathbf{F}^2$.

Given a normed space X, it is usually clear from context what norm we mean to use on X. Therefore, we usually just write $\|\cdot\|$ to denote the norm on X. However, when there is a possibility of confusion we may write $\|\cdot\|_X$ to specify that this norm is the norm on X, or we may write "the space $(X, \|\cdot\|)$" to emphasize that $\|\cdot\|$ represents the norm on X.

In addition to $\|\cdot\|$, we sometimes use symbols such as $|\cdot|$, $|\!|\!|\cdot|\!|\!|$, or $\rho(\cdot)$ to denote a norm or seminorm.

Definition 1.2. Let X be a normed linear space.

(a) A sequence of vectors $\{x_n\}$ in X *converges* to $x \in X$ if we have $\lim_{n\to\infty} \|x - x_n\| = 0$, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, n \geq N, \quad \|x - x_n\| < \varepsilon.$$

In this case, we write either $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

(b) A sequence of vectors $\{x_n\}$ in X is a *Cauchy sequence* in X if we have $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. More precisely, this means that

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, m, n \geq N, \quad \|x_m - x_n\| < \varepsilon. \qquad \diamond$$

Every convergent sequence in a normed space is a Cauchy sequence (see Exercise 1.2). However, the converse is not true in general (consider Exercise 1.18).

Definition 1.3. We say that a normed space X is *complete* if it is the case that every Cauchy sequence in X is a convergent sequence. A complete normed linear space is called a *Banach space*. ◇

Sometimes we need to be explicit about which scalar field is associated with a Banach space X. We say that X is a *real Banach space* if it is a Banach space over the real field (i.e., $\mathbf{F} = \mathbf{R}$), and similarly it is a *complex Banach space* if $\mathbf{F} = \mathbf{C}$.

All Cauchy and convergent sequences in a normed space are bounded above in the following sense (see Exercise 1.2).

Definition 1.4. A sequence $\{x_n\}$ in a Banach space X is:

(a) *bounded below* if $\inf \|x_n\| > 0$,

(b) *bounded above* if $\sup \|x_n\| < \infty$,

(c) *normalized* if $\|x_n\| = 1$ for all n. ♢

To emphasize that the boundedness discussed in Definition 1.4 refers to the norm of the elements of the sequence, we will sometimes say that $\{x_n\}$ is *norm*-bounded below, etc. Also, we sometimes use the term "bounded" without the qualification "above" or "below." In most cases, we only mean that the sequence is bounded above. However, in certain contexts we may require that the sequence be bounded both above and below. For example, this is what we mean when we define a "bounded basis" in Definition 4.5. This more restricted meaning for "bounded" is always stated explicitly in a definition.

The simplest example of a Banach space is the scalar field $\mathbf{F}$, where the norm on $\mathbf{F}$ is the absolute value. We will take as given the fact that $\mathbf{F}$ is complete with respect to absolute value. There are infinitely many norms on $\mathbf{F}$, but they are all positive scalar multiples of absolute value (Exercise 1.1), so we always assume that the norm on $\mathbf{F}$ is the absolute value.

Example 1.5. The next simplest example of a Banach space is $\mathbf{F}^d$, the set of all d-tuples of scalars, where d is a positive integer. There are many choices of norms for $\mathbf{F}^d$. Writing a generic vector $v \in \mathbf{F}^d$ as $v = (v_1, \dots, v_d)$, each of the following defines a norm on $\mathbf{F}^d$, and $\mathbf{F}^d$ is complete with respect to each of these norms:

$$|v|_p \;=\; \begin{cases} \bigl(|v_1|^p + \cdots + |v_d|^p\bigr)^{1/p}, & 1 \le p < \infty, \\ \max\{|v_1|, \dots, |v_d|\}, & p = \infty. \end{cases} \tag{1.1}$$

The *Euclidean norm* $|v|$ of a vector $v \in \mathbf{F}^d$ is the norm corresponding to the choice $p = 2$, i.e.,

$$|v| \;=\; |v|_2 \;=\; \sqrt{|v_1|^2 + \cdots + |v_d|^2}.$$

This particular norm has some extra algebraic properties that we will discuss further in Section 1.5. The fact that $|\cdot|_p$ is a norm on $\mathbf{F}^d$ is immediate for the cases $p = 1$ and $p = \infty$. For $1 < p < \infty$ the only norm property that is not obvious is the Triangle Inequality, and this can be shown by using exactly the same argument that we use later to prove Theorem 1.13. The proof that $\mathbf{F}^d$ is complete with respect to these norms is assigned as Exercise 1.3. ♢

The following example shows that we can construct norms on any finite-dimensional vector space.

Example 1.6. Let V be a finite-dimensional vector space. Then there exists a finite set of vectors $\mathcal{B} = \{x_1, \dots, x_d\}$ that is a basis for V, i.e., $\mathcal{B}$ spans V and $\mathcal{B}$ is linearly independent. Each $x \in V$ can be written as $x = \sum_{k=1}^{d} c_k(x)\, x_k$ for a unique choice of scalars $c_k(x)$. Given $1 \le p \le \infty$, if we set

$$\|x\|_p \;=\; \begin{cases} \big(|c_1(x)|^p + \cdots + |c_d(x)|\big)^{1/p}, & 1 \le p < \infty, \\[2mm] \max\big\{|c_1(x)|, \dots, |c_d(x)|\big\}, & p = \infty, \end{cases}$$

then $\|\cdot\|_p$ is a norm on V and V is complete with respect to this norm (see Exercise 1.4 and Theorem 1.14). ♢

The preceding examples illustrate the fact that there can be many norms on any given space.

Definition 1.7. Suppose that X is a normed linear space with respect to a norm $\|\cdot\|$ and also with respect to another norm $|||\cdot|||$. These norms are *equivalent* if there exist constants C_1, $C_2 > 0$ such that

$$\forall\, x \in X, \quad C_1\, \|x\| \;\le\; |||x||| \;\le\; C_2\, \|x\|. \qquad ♢$$

Note that if $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on X, then they define the same convergence criterion in the sense that

$$\lim_{n\to\infty} \|x - x_n\| = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} |||x - x_n||| = 0.$$

Any two of the norms $|\cdot|_p$ on $\mathbf{F}^d$ are equivalent (see Exercise 1.3). This is a special case of the following theorem, whose proof is more subtle and will be omitted (see [Con90, Thm. 3.1]).

Theorem 1.8. *If V is a finite-dimensional vector space, then any two norms on V are equivalent.* ♢

Now we give some examples of infinite-dimensional Banach spaces. We have not yet presented the tools that are needed to prove that these are Banach spaces, but will do so in Section 1.2 (see Theorem 1.13).

Example 1.9. In this example we consider some vector spaces whose elements are infinite sequences of scalars $x = (x_k) = (x_k)_{k\in\mathbf{N}}$ indexed by the natural numbers.

(a) Given $1 \le p < \infty$, we define ℓ^p to be the space of all infinite sequences of scalars that are p-summable, i.e.,

$$\ell^p \;=\; \ell^p(\mathbf{N}) \;=\; \Big\{x = (x_k) : \sum_k |x_k|^p < \infty\Big\}. \tag{1.2}$$

We will see in Theorem 1.14 that this is a Banach space with respect to the norm

$$\|x\|_{\ell^p} \;=\; \|(x_k)\|_{\ell^p} \;=\; \biggl(\sum_k |x_k|^p\biggr)^{1/p}. \tag{1.3}$$

(b) We define ℓ^∞ to be the space of all bounded infinite sequences of scalars:

$$\ell^\infty \;=\; \ell^\infty(\mathbf{N}) \;=\; \bigl\{x = (x_k) : (x_k) \text{ is a bounded sequence}\bigr\}.$$

This is a Banach space with respect to the *sup-norm*

$$\|x\|_{\ell^\infty} \;=\; \|(x_k)\|_{\ell^\infty} \;=\; \sup_k |x_k|. \qquad \diamondsuit$$

We can obtain analogous Banach spaces $\ell^p(I)$ by replacing the index set $\mathbf{N}$ by another countable index set I. For example, $\ell^p\bigl(\{1,\dots,d\}\bigr)$ is simply the vector space $\mathbf{F}^d$ with the norm defined in equation (1.3). Another example is $\ell^p(\mathbf{Z})$, which is used extensively in Chapters 9–14. The elements of $\ell^p(\mathbf{Z})$ are bi-infinite sequences of scalars that are p-summable (or bounded if $p = \infty$). Exercise 1.15 shows how to define $\ell^p(I)$ for uncountable index sets I.

Example 1.10. Now we consider some vector spaces whose elements are scalar-valued functions on a measurable domain $E \subseteq \mathbf{R}$.

(a) Given $1 \le p < \infty$, we define $L^p(E)$ to be the space of all measurable functions on E that are p-integrable, i.e.,

$$L^p(E) \;=\; \biggl\{f \colon E \to \mathbf{F} \;:\; \int_E |f(t)|^p\,dt < \infty\biggr\}.$$

Then

$$\|f\|_{L^p} \;=\; \biggl(\int_E |f(t)|^p\,dt\biggr)^{1/p}$$

defines a seminorm on $L^p(E)$. It is not a norm because any function f such that $f = 0$ a.e. will satisfy $\|f\|_{L^p} = 0$ even though f need not be identically zero. Therefore, we usually "identify" any two functions that differ only on a set of measure zero. Regarding any two such functions as defining the same element of $L^p(E)$, we have that $\|\cdot\|_{L^p}$ is a norm. In a more technical language, the elements of $L^p(E)$ are actually equivalence classes of functions that are equal a.e., and the norm of such an equivalence class is the norm of any representative of the class.

(b) In the opening section on General Notation, we declared that a measurable function f is *essentially bounded* if there exists a real number M such that $|f(t)| \le M$ almost everywhere. We define $L^\infty(E)$ to be the space of all essentially bounded functions on E:

$$L^\infty(E) \;=\; \bigl\{f \colon E \to \mathbf{F} \;:\; f \text{ is essentially bounded on } E\bigr\}.$$

Again identifying functions that are equal a.e., $L^\infty(E)$ is a Banach space with respect to the norm

$$\|f\|_{L^\infty} \;=\; \operatorname*{ess\,sup}_{t\in E} |f(t)| \;=\; \inf\bigl\{M \ge 0 : |f(t)| \le M \text{ a.e.}\bigr\}. \qquad \diamondsuit \tag{1.4}$$

We will see several more examples of normed spaces and Banach spaces in Section 1.3.

Exercises

1.1. Show that if $\|\cdot\|$ is a norm on the scalar field $\mathbf{F}$, then there exists a positive number $\lambda > 0$ such that $\|x\| = \lambda\,|x|$, where $|x|$ is the absolute value of x.

1.2. Given a normed linear space X, prove the following facts.

(a) Every convergent sequence in X is Cauchy, and the limit of a convergent sequence is unique.

(b) Every Cauchy sequence in X is bounded.

(c) Reverse Triangle Inequality: $\bigl|\|x\| - \|y\|\bigr| \le \|x - y\|$ for all $x,\, y \in X$.

(d) Continuity of the norm: $x_n \to x \implies \|x_n\| \to \|x\|$.

(e) Continuity of vector addition:

$$x_n \to x \text{ and } y_n \to y \implies x_n + y_n \to x + y.$$

(f) Continuity of scalar multiplication:

$$x_n \to x \text{ and } c_n \to c \implies c_n x_n \to cx.$$

(g) Convexity of open balls: If $x,\, y \in B_r(z)$ then

$$\theta x + (1-\theta)y \in B_r(z) \quad \text{for all } 0 \le \theta \le 1.$$

1.3. (a) Assuming that the functions $|\cdot|_p$ on $\mathbf{F}^d$ given in equation (1.1) are norms on $\mathbf{F}^d$, show that any two of these norms are equivalent.

(b) Assuming that $\mathbf{F}$ is complete with respect to absolute value, show that $\mathbf{F}^d$ is complete with respect to any one of the norms $|\cdot|_p$.

1.4. Let V be a finite-dimensional vector space. Prove that the function $\|\cdot\|_1$ on V defined in Example 1.6 is a norm, and that V is complete with respect to this norm.

1.5. Show that if vectors x_n in a normed space X satisfy $\|x_{n+1} - x_n\| < 2^{-n}$ for every $n \in \mathbf{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

1.6. Let $\{x_n\}$ be a sequence in a normed space X, and let $x \in X$ be fixed. Suppose that every subsequence $\{y_n\}$ of $\{x_n\}$ has a subsequence $\{z_n\}$ of $\{y_n\}$ such that $z_n \to x$. Show that $x_n \to x$.

1.7. Let X be a complex Banach space. Let $X_{\mathbf{R}} = X$ as a set, but consider $X_{\mathbf{R}}$ as a vector space over the real field. That is, vector addition in $X_{\mathbf{R}}$ is defined just as in X, but scalar multiplication in $X_{\mathbf{R}}$ is restricted to multiplication by real scalars. Let $\|\cdot\|_{\mathbf{R}} = \|\cdot\|$, and show that $(X_{\mathbf{R}}, \|\cdot\|_{\mathbf{R}})$ is a real Banach space.

1.8. We say that a set X (not necessarily a vector space) is a *metric space* if for each $x,\, y \in X$ there exists a real number $\mathrm{d}(x,y)$ such that for all $x,\, y,\, z \in X$ we have:

i. i. $\mathrm{d}(x,y) \geq 0$,
ii. $\mathrm{d}(x,y) = 0$ if and only if $x = y$,
iii. $\mathrm{d}(x,y) = \mathrm{d}(y,x)$, and
iv. the *Triangle Inequality*: $\mathrm{d}(x,y) \leq \mathrm{d}(x,z) + \mathrm{d}(y,z)$.

In this case we call d a *metric* on X, and we refer to $\mathrm{d}(x,y)$ as the *distance* between x and y.

(a) Show that if X is a normed space, then $\mathrm{d}(x,y) = \|x-y\|$ is a metric on X.

(b) Make a definition of convergent and Cauchy sequences in a metric space, and show that every convergent sequence is Cauchy.

(c) Define a metric space to be *complete* if every Cauchy sequence in X converges to an element of X. Let $X = \mathbf{Q}$ and set $\mathrm{d}(x,y) = |x-y|$, the ordinary absolute value of the difference of x and y. Show that d is a metric on $\mathbf{Q}$, but $\mathbf{Q}$ is incomplete with respect to this metric.

1.9. Fix $0 < p < 1$, and define ℓ^p by equation (1.2) and $\|\cdot\|_{\ell^p}$ by equation (1.3).

(a) Show that $\|\cdot\|_{\ell^p}$ fails the Triangle Inequality and hence is not a norm on ℓ^p.

(b) Show that $\|x+y\|_{\ell^p}^p \leq \|x\|_{\ell^p}^p + \|y\|_{\ell^p}^p$, and use this to show that ℓ^p is a vector space and $\mathrm{d}(x,y) = \|x-y\|_{\ell^p}^p$ is a metric on ℓ^p.

(c) Let $B = \{x \in \ell^p : \mathrm{d}(x,0) < 1\}$ be the "open unit ball" with respect to the metric d, and show that B is not convex. Use this to show that there is no norm $\|\cdot\|$ on ℓ^p such that $\mathrm{d}(x,y) = \|x-y\|$.

1.2 Hölder's and Minkowski's Inequalities

In this section we will prove that ℓ^p is a Banach space. The proof of the completeness of $L^p(E)$ is similar in spirit, but is somewhat more technical as various notions from measure theory are required, and will be omitted.

For $p = 1$ and $p = \infty$ it is easy to see that the function $\|\cdot\|_{\ell^p}$ defined in Example 1.9 satisfies the Triangle Inequality. It is not nearly so obvious that the Triangle Inequality holds when $1 < p < \infty$. The next theorem gives a fundamental inequality on the norm of a product of two functions or sequences, and we will use this inequality to prove the Triangle Inequality on ℓ^p for $1 < p < \infty$. For this result, the following notion of the dual index will be useful.

Notation 1.11. Given $1 \le p \le \infty$, its *dual index* is the number $1 \le p' \le \infty$ satisfying

$$\frac{1}{p} + \frac{1}{p'} \;=\; 1,$$

where we use the conventions that $1/0 = \infty$ and $1/\infty = 0$. ♢

For example, $1' = \infty,\ 2' = 2,\ 3' = 4/3$, and $\infty' = 1$. Explicitly,

$$p' = \frac{p}{p-1},$$

and we have $(p')' = p$.

Theorem 1.12 (Hölder's Inequality). *Fix* $1 \le p \le \infty$.

(a) *If* $x = (x_k) \in \ell^p$ *and* $y = (y_k) \in \ell^{p'}$, *then* $(x_k y_k) \in \ell^1$ *and*

$$\|(x_k y_k)\|_{\ell^1} \;\le\; \|(x_k)\|_{\ell^p}\, \|(y_k)\|_{\ell^{p'}}.$$

For $1 < p < \infty$ *this is equivalent to the statement*

$$\sum_k |x_k y_k| \;\le\; \biggl(\sum_k |x_k|^p\biggr)^{1/p} \biggl(\sum_k |y_k|^{p'}\biggr)^{1/p'}.$$

(b) *If* $f \in L^p(E)$ *and* $g \in L^{p'}(E)$, *then* $fg \in L^1(E)$ *and*

$$\|fg\|_{L^1} \;\le\; \|f\|_{L^p}\, \|g\|_{L^{p'}}.$$

For $1 < p < \infty$ *this is equivalent to the statement*

$$\int_E |f(t)\, g(t)|\, dt \;\le\; \biggl(\int_E |f(t)|^p\, dt\biggr)^{1/p} \biggl(\int_E |g(t)|^{p'}\, dt\biggr)^{1/p'}.$$

Proof. We will concentrate on the ℓ^p spaces, as the proof for $L^p(E)$ is similar. The cases $p = 1$ and $p = \infty$ are straightforward, so assume that $1 < p < \infty$.

Suppose that $x = (x_k) \in \ell^p$ and $y = (y_k) \in \ell^{p'}$ satisfy $\|x\|_{\ell^p} = 1 = \|y\|_{\ell^{p'}}$. By Exercise 1.10, we have the inequality

$$|x_k y_k| \;\le\; \frac{|x_k|^p}{p} + \frac{|y_k|^{p'}}{p'}, \qquad k \in \mathbf{N}.$$

Consequently,

$$\begin{aligned} \|(x_k y_k)\|_{\ell^1} \;=\; \sum_k |x_k y_k| \;&\le\; \sum_k \left(\frac{|x_k|^p}{p} + \frac{|y_k|^{p'}}{p'} \right) \\ &=\; \frac{\|(x_k)\|_{\ell^p}^p}{p} + \frac{\|(y_k)\|_{\ell^{p'}}^{p'}}{p'} \\ &=\; \frac{1}{p} + \frac{1}{p'} \;=\; 1. \end{aligned} \tag{1.5}$$

Given arbitrary $x \in \ell^p$ and $y \in \ell^{p'}$, let $s = \|x\|_{\ell^p}$ and $t = \|y\|_{\ell^{p'}}$. Then $x/s = (x_k/s) \in \ell^p$ and $y/t = (y_k/t) \in \ell^{p'}$ are each unit vectors, i.e., $\|x/s\|_{\ell^p} = 1 = \|y/t\|_{\ell^{p'}}$. The result then follows by applying equation (1.5) to x/s and y/t. □

Note that if $p = 2$ then the dual index is $p' = 2$ as well. Therefore, we have the following special cases of Hölder's inequality, usually referred to as the *Cauchy–Schwarz* or *Cauchy–Bunyakovski–Schwarz* inequalities:

$$\|(x_k y_k)\|_{\ell^1} \;\le\; \|(x_k)\|_{\ell^2}\, \|(y_k)\|_{\ell^2} \quad \text{and} \quad \|fg\|_{L^1} \;\le\; \|f\|_{L^2}\, \|g\|_{L^2}. \tag{1.6}$$

ℓ^2 and $L^2(E)$ are specific examples of *Hilbert spaces*, which are discussed in more detail in Section 1.5. In particular, Theorem 1.37 will present a generalization of the Cauchy–Bunyakovski–Schwarz inequalities that is valid in any Hilbert space.

Now we show that $\|\cdot\|_{\ell^p}$ is a norm on ℓ^p. Although we will not prove it, a similar argument shows that $\|\cdot\|_{L^p}$ is a norm on $L^p(E)$. The Triangle Inequality on ℓ^p or L^p is often called *Minkowski's Inequality.*

Theorem 1.13 (Minkowski's Inequality). *If $1 \le p \le \infty$, then $\|\cdot\|_{\ell^p}$ is a norm on ℓ^p and $\|\cdot\|_{L^p}$ is a norm on $L^p(E)$.*

Proof. The cases $p = 1$ and $p = \infty$ are straightforward, so consider $1 < p < \infty$. All of the properties of a norm are clear except for the Triangle Inequality. To prove this, fix $x = (x_k)$ and $y = (y_k)$ in ℓ^p. Then we have

$$\begin{aligned} \|x+y\|_{\ell^p}^p \;&=\; \sum_k |x_k + y_k|^{p-1}\, |x_k + y_k| \\ &\le\; \sum_k |x_k + y_k|^{p-1}\, |x_k| \;+\; \sum_k |x_k + y_k|^{p-1}\, |y_k| \\ &\le\; \left(\sum_k \big(|x_k + y_k|^{p-1}\big)^{p'} \right)^{1/p'} \left(\sum_k |x_k|^p \right)^{1/p} \\ &\qquad +\; \left(\sum_k \big(|x_k + y_k|^{p-1}\big)^{p'} \right)^{1/p'} \left(\sum_k |y_k|^p \right)^{1/p} \end{aligned}$$

$$= \left(\sum_k |x_k + y_k|^p\right)^{(p-1)/p} \|x\|_{\ell^p} + \left(\sum_k |x_k + y_k|^p\right)^{(p-1)/p} \|y\|_{\ell^p}$$

$$= \|x+y\|_{\ell^p}^{p-1} \|x\|_{\ell^p} + \|x+y\|_{\ell^p}^{p-1} \|y\|_{\ell^p},$$

where we have applied Hölder's Inequality with exponents p' and p, and used the fact that $p' = p/(p-1)$. Dividing both sides by $\|x+y\|_{\ell^p}^{p-1}$, we obtain $\|x+y\|_{\ell^p} \le \|x\|_{\ell^p} + \|y\|_{\ell^p}$. □

Finally, we show that ℓ^p is complete and therefore is a Banach space. The argument for $L^p(E)$ is similar in spirit but is technically more complicated, and will be omitted (see [Fol99] or [WZ77]).

Theorem 1.14. *If $1 \le p \le \infty$, then ℓ^p is a Banach space with respect to the norm $\|\cdot\|_{\ell^p}$, and $L^p(E)$ is a Banach space with respect to the norm $\|\cdot\|_{L^p}$.*

Proof. Fix $1 \le p < \infty$ (the case $p = \infty$ is similar), and suppose that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in ℓ^p. Each x_n is a vector in ℓ^p, so let us write the components of x_n as

$$x_n = \big(x_n(1), x_n(2), \dots\big).$$

Then for each fixed index $k \in \mathbf{N}$ we have $|x_m(k) - x_n(k)| \le \|x_m - x_n\|_{\ell^p}$. Hence $\big(x_n(k)\big)_{n\in\mathbf{N}}$ is a Cauchy sequence of scalars, and therefore must converge since $\mathbf{F}$ is complete. Define $x(k) = \lim_{n\to\infty} x_n(k)$. Then x_n *converges componentwise* to $x = \big(x(1), x(2), \dots\big)$, i.e.,

$$\forall\, k \in \mathbf{N}, \quad x(k) = \lim_{n\to\infty} x_n(k).$$

We need to show that x_n converges to x *in the norm of* ℓ^p.

Choose any $\varepsilon > 0$. Then, by the definition of a Cauchy sequence, there exists an N such that $\|x_m - x_n\|_{\ell^p} < \varepsilon$ for all m, $n > N$. Fix any particular $n > N$. For each $M > 0$ we have

$$\sum_{k=1}^{M} |x(k) - x_n(k)|^p = \lim_{m\to\infty} \sum_{k=1}^{M} |x_m(k) - x_n(k)|^p \le \lim_{m\to\infty} \|x_m - x_n\|_{\ell^p}^p \le \varepsilon^p.$$

Since this is true for every M, we conclude that

$$\|x - x_n\|_{\ell^p}^p = \sum_{k=1}^{\infty} |x(k) - x_n(k)|^p = \lim_{M\to\infty} \sum_{k=1}^{M} |x(k) - x_n(k)|^p \le \varepsilon^p. \quad (1.7)$$

Consequently,

$$\|x\|_{\ell^p} = \|x - x_n + x_n\|_{\ell^p} \le \|x - x_n\|_{\ell^p} + \|x_n\|_{\ell^p} < \infty,$$

so $x \in \ell^p$. Further, since equation (1.7) holds for all $n > N$, we have that $\lim_{n\to\infty} \|x - x_n\|_{\ell^p} = 0$, i.e., $x_n \to x$ in ℓ^p. Therefore ℓ^p is complete. □

Theorems 1.13 and 1.14 carry over with minimal changes to show that $\mathbf{F}^d$ is a Banach space with respect to any of the norms $|\cdot|_p$ defined in Example 1.5.

Exercises

1.10. (a) Show that if $0 < \theta < 1$, then $t^\theta \le \theta t + (1-\theta)$ for $t \ge 0$, with equality if and only if $t = 1$.

(b) Suppose that $1 < p < \infty$ and $a, b \ge 0$. Apply part (a) with $t = a^p b^{-p'}$ and $\theta = 1/p$ to show that $ab \le a^p/p + b^{p'}/p'$, with equality if and only if $b = a^{p-1}$.

1.11. Show that equality holds in Hölder's Inequality for sequences (part (a) of Theorem 1.12) if and only if there exist scalars α, β, not both zero, such that $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ for each $k \in I$.

1.12. Show that if $1 \le p < q \le \infty$, then $\ell^p \subsetneq \ell^q$ and $\|x\|_{\ell^q} \le \|x\|_{\ell^p}$ for all $x \in \ell^p$.

1.13. Let $E \subseteq \mathbf{R}$ be measurable with $|E| < \infty$. Show that if $1 \le p < q \le \infty$, then $L^q(E) \subsetneq L^p(E)$ and $\|f\|_{L^p} \le |E|^{\frac{1}{p}-\frac{1}{q}}\,\|f\|_{L^q}$ for all $f \in L^p(E)$.

1.14. Show that if $x \in \ell^q$ for some finite q, then $\|x\|_{\ell^p} \to \|x\|_{\ell^\infty}$ as $p \to \infty$, but this can fail if $x \notin \ell^q$ for any finite q.

1.15. Given an arbitrary index set I, define $\ell^\infty(I)$ to be the space of all bounded sequences $x = (x_i)_{i\in I}$ indexed by I, with $\|x\|_\infty = \sup_{i\in I} |x_i|$. For $1 \le p < \infty$ let $\ell^p(I)$ consist of all sequences $x = (x_i)_{i\in I}$ *with at most countably many nonzero components* such that $\|x\|_{\ell^p}^p = \sum |x_i|^p < \infty$. Show that each of these spaces $\ell^p(I)$ is a Banach space with respect to $\|\cdot\|_{\ell^p}$. What is $\ell^p(I)$ if $I = \{1, \dots, d\}$?

1.16. Let X, Y be normed linear spaces. Given $1 \le p < \infty$, $x \in X$, and $y \in Y$, define $\|(x,y)\|_p = \bigl(\|x\|_X^p + \|y\|_Y^p\bigr)^{1/p}$ and $\|(x,y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\}$.

(a) Prove that $\|\cdot\|_p$ is a norm on the Cartesian product $X \times Y$, and $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent norms on $X \times Y$ for any $1 \le p, q \le \infty$.

(b) Show that if X and Y are Banach spaces, then $X \times Y$ is a Banach space with respect to $\|\cdot\|_p$.

1.3 Basic Properties of Banach Spaces

In this section we will give some definitions and facts that hold for normed spaces and Banach spaces.

Definition 1.15. Let X be a normed linear space.

(a) Recall from Definition 1.1 that if $x \in X$ and $r > 0$, then the *open ball in X centered at x with radius r* is $B_r(x) = \{y \in X : \|x - y\| < r\}$.

(b) A subset $U \subseteq X$ is *open* if for each $x \in U$ there exists an $r > 0$ such that $B_r(x) \subseteq U$.

(c) A subset $E \subseteq X$ is *closed* if $X \backslash E$ is open.

(d) Let $E \subseteq X$. Then $x \in X$ is a *limit point* of E if there exist $x_n \in E$ with all $x_n \neq x$ such that $x_n \to x$.

(e) The *closure* of a subset $E \subseteq X$ is the smallest closed set $\overline{E}$ that contains E, i.e., $\overline{E} = \cap \bigl\{F : F \text{ is closed and } E \subseteq F\bigr\}$.

(f) A subset $E \subseteq X$ is *dense* in X if $\overline{E} = X$. $\diamondsuit$

The following lemma gives useful equivalent reformulations of some of the notions defined above. Exercise 1.17 asks for proof of Lemma 1.16.

Lemma 1.16. *Given a Banach space X and given $E \subseteq X$, the following statements hold.*

(a) *E is closed if and only if it contains all of its limit points.*

(b) $\overline{E} = E \cup \{x \in X : x \text{ is a limit point of } E\}$. *Consequently, E is closed if and only if $\overline{E} = E$.*

(c) *E is dense in X if and only if every $x \in X$ is a limit point of E.* $\diamondsuit$

Once we have a space X in hand that we know is a Banach space with respect to a norm $\|\cdot\|$, we often need to know if a given subspace S of X is also a Banach space with respect to this same norm. The next lemma (whose proof is Exercise 1.18) gives a convenient characterization of those subspaces that are complete with respect to the norm on X.

Lemma 1.17. *Let S be a subspace of a Banach space X. Then S is a Banach space with respect to the norm on X if and only if S is a closed subset of X.* $\diamondsuit$

We can use Lemma 1.17 to give some additional examples of normed spaces that are or are not Banach spaces.

Example 1.18. (a) Define

$$c \;=\; c(\mathbf{N}) \;=\; \bigl\{a = (a_k) : \lim_{k\to\infty} a_k \text{ exists}\bigr\},$$

$$c_0 \;=\; c_0(\mathbf{N}) \;=\; \bigl\{a = (a_k) : \lim_{k\to\infty} a_k = 0\bigr\}.$$

Exercise 1.20 asks for a proof that c and c_0 are closed subspaces of ℓ^∞. Consequently, Lemma 1.17 implies that c and c_0 are Banach spaces with respect to the norm $\|\cdot\|_{\ell^\infty}$.

(b) Now consider

$$c_{00} \;=\; c_{00}(\mathbf{N}) \;=\; \bigl\{a = (a_k) : \text{only finitely many } a_k \text{ are nonzero}\bigr\}.$$

Even though the elements of c_{00} are infinite sequences, since only finitely many components are nonzero, they are often called "finite sequences." Note that c_{00} is a subspace of c_0, c, and ℓ^∞. By Exercise 1.20, c_{00} is a proper, dense subset of c_0. Consequently, $\overline{c_{00}} = c_0 \neq c_{00}$, so c_{00} is not a closed subspace of c_0 (or of c or ℓ^∞), and therefore it is not a Banach space with respect to $\|\cdot\|_{\ell^\infty}$.

A specific example of a Cauchy sequence in c_{00} that does not have a limit in c_{00} is the sequence $\{x_n\}$, where x_n is the vector in ℓ^p given by

$$x_n \;=\; \bigl(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}, 0, 0, \dots\bigr).$$

The vectors x_n do converge in ℓ^∞ norm to the vector $x = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, but *$x$ does not belong to* c_{00}. While $\{x_n\}$ converges in c_0 and in ℓ^∞, it does not converge in c_{00}. $\diamondsuit$

Although c_{00} is not a Banach space with respect to the sup-norm, it is a proper, dense subspace of c_0, which is a Banach space. More generally, given any normed linear space X that is not complete, there exists a unique Banach space $\widetilde{X}$ such that X is a proper, dense subspace of $\widetilde{X}$ and the norm on $\widetilde{X}$ extends the norm on X (see Exercise 1.25).

Following are some examples of normed spaces whose elements are continuous functions. More examples appear in the Exercises.

Example 1.19. (a) Define

$$C(\mathbf{R}) \;=\; \bigl\{f \colon \mathbf{R} \to \mathbf{F} \;:\; f \text{ is continuous on } \mathbf{R}\bigr\}.$$

While there is no convenient norm on $C(\mathbf{R})$, Exercise 1.21 shows that the subspace

$$C_b(\mathbf{R}) \;=\; \bigl\{f \in C(\mathbf{R}) : f \text{ is bounded}\bigr\}$$

is a Banach space with respect to the *sup-norm* or *uniform norm*

$$\|f\|_\infty \;=\; \sup_{t \in \mathbf{R}} |f(t)|.$$

(b) By Exercise 1.21,

$$C_0(\mathbf{R}) \;=\; \Bigl\{f \in C_b(\mathbf{R}) : \lim_{|t| \to \infty} f(t) = 0\Bigr\}$$

is a closed subspace of $C_b(\mathbf{R})$ with respect to the uniform norm. Therefore $C_0(\mathbf{R})$ is a Banach space with respect to $\|\cdot\|_\infty$.

(c) Recall that a continuous function $f \colon \mathbf{R} \to \mathbf{F}$ has *compact support* if $f(t) = 0$ for all t outside of some finite interval. The space

$$C_c(\mathbf{R}) \;=\; \bigl\{f \in C(\mathbf{R}) : f \text{ has compact support}\bigr\}$$

is a subspace of $C_0(\mathbf{R})$ and $C_b(\mathbf{R})$. However, $C_c(\mathbf{R})$ is a dense but proper subset of $C_0(\mathbf{R})$ with respect to the uniform norm (Exercise 1.21). Hence $C_c(\mathbf{R})$ is not closed and therefore is not complete with respect to $\|\cdot\|_\infty$.

(d) We can similarly define spaces of functions that are continuous on domains other than $\mathbf{R}$. The most important example for us is the space of functions that are continuous on a closed finite interval $[a,b]$, which we write as follows:

$$C[a,b] \;=\; \bigl\{f\colon [a,b] \to \mathbf{F} \;:\; f \text{ is continuous on } [a,b]\bigr\}.$$

Every continuous function on $[a,b]$ is bounded, and $C[a,b]$ is a Banach space with respect to the uniform norm. $\diamondsuit$

Remark 1.20. For a continuous function f, the supremum of $|f(x)|$ coincides with its essential supremum, i.e., $\|f\|_\infty = \|f\|_{L^\infty}$ (see Exercise 1.19). Therefore, if we identify a continuous function f with the equivalence class of all functions that equal f almost everywhere, then we can regard $C_b(E)$ as being a subspace of $L^\infty(E)$. In this sense of identification, $C_b(\mathbf{R})$ and $C_0(\mathbf{R})$ are closed subspaces of $L^\infty(E)$, but $C_c(\mathbf{R})$ is not a closed subspace with respect to $\|\cdot\|_{L^\infty}$. $\diamondsuit$

When considering a space of bounded continuous functions by itself, we usually assume that the norm on the space is the uniform norm. For example, Example 1.29 below will present the Weierstrass Approximation Theorem, which deals with $C[a,b]$ under the uniform norm. However, if we are thinking of a space of continuous functions as being a subspace of a larger space X, then we use the norm on X unless specifically stated otherwise. For example, the next lemma considers $C[a,b]$ as a subspace of $L^p[a,b]$, and we therefore implicitly assume in this result that the norm on $C[a,b]$ is $\|\cdot\|_{L^p}$. For proof of Lemma 1.21, we refer to [Fol99].

Lemma 1.21. (a) $C[a,b]$ *is dense in* $L^p[a,b]$ *for each* $1 \le p < \infty$.
(b) $C_c(\mathbf{R})$ *is dense in* $L^p(\mathbf{R})$ *for each* $1 \le p < \infty$. $\diamondsuit$

Next we prove directly that all finite-dimensional subspaces of a normed space are closed.

Theorem 1.22. *If M is a finite-dimensional subspace of a normed linear space X, then M is closed.*

Proof. Suppose that $x_n \in M$ and $x_n \to y \in X$. If $y \notin M$, define

$$M_1 \;=\; M \;+\; \operatorname{span}\{y\} \;=\; \{m + cy : m \in M,\, c \in \mathbf{F}\}.$$

Since $y \notin M$, every vector $x \in M_1$ has a unique representation of the form $x = m_x + c_x y$ with $m_x \in M$ and $c_x \in \mathbf{F}$. Therefore we can define

$$\|x\|_{M_1} = \|m_x\| + |c_x|,$$

where $\|\cdot\|$ is the norm on X. This forms a norm on M_1.

Since M_1 is finite dimensional, Theorem 1.8 implies that all norms on M_1 are equivalent. Hence there exist constants A, $B > 0$ such that

$$A\,\|x\| \le \|x\|_{M_1} \le B\,\|x\|, \qquad x \in M_1.$$

Now, since $x_n \in M$, the representation of $y - x_n$ as a vector in M_1 takes $m_z = -x_n$ and $c_z = 1$. Therefore

$$1 \le \|x_n\| + 1 = \|y - x_n\|_{M_1} \le B\,\|y - x_n\| \to 0 \quad \text{as } n \to \infty.$$

This is a contradiction, so we must have $y \in M$. Hence M is closed. □

The next definition provides one way to distinguish "large" Banach spaces from "small" ones. Specifically, a "small" Banach space is one that contains a *countable* dense subset; we call such a space separable. For example, $\mathbf{Q}$ is a countable dense subset of $\mathbf{R}$, so $\mathbf{R}$ is separable, and likewise $\mathbf{F}^d$ is separable for each $d \in \mathbf{N}$.

Definition 1.23. A normed linear space X is *separable* if it contains a countable dense subset. ♢

Example 1.24. (a) If $1 \le p < \infty$ then ℓ^p is separable, but ℓ^∞ is not separable (see Example 1.31 and Exercise 1.28). It is likewise true that $L^p(E)$ is separable for $1 \le p < \infty$ and $L^\infty(E)$ is not separable (unless $|E| = 0$), although the proof of these facts requires some knowledge of measure theory and will be omitted.

(b) In Example 1.29 we will see that the Weierstrass Approximation Theorem implies that the Banach space $C[a, b]$ is separable (with respect to the uniform norm). This can then be used to show that $C_0(\mathbf{R})$ is separable as well (Exercise 1.27). ♢

Exercises

1.17. Prove Lemma 1.16.

1.18. Prove Lemma 1.17.

1.19. Show that if $f \in C_b(\mathbf{R})$, then $\operatorname{ess\,sup}_{t\in\mathbf{R}} |f(t)| = \sup_{t\in\mathbf{R}} |f(t)|$, and consequently $\|f\|_{L^\infty} = \|f\|_\infty$.

1.20. (a) Show that c and c_0 are closed subspaces of ℓ^∞.

(b) Show that c_{00} is a proper, dense subspace of c_0, and hence is not closed with respect to the norm $\|\cdot\|_{\ell^\infty}$.

(c) Let $\{\delta_n\}$ denote the sequence of standard basis vectors (see Example 1.30). Given $x = (x_n) \in c_0$, show that $x = \sum x_n \delta_n$, where the series converges with respect to the norm $\|\cdot\|_{\ell^\infty}$. Show further that the scalars x_n in this representation are unique.

1.21. (a) Show that $C_b(\mathbf{R})$ is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$.

(b) Show that $C_0(\mathbf{R})$ is a closed subspace of $C_b(\mathbf{R})$.

(c) Show that $C_c(\mathbf{R})$ is a proper, dense subspace of $C_0(\mathbf{R})$, and hence is not closed with respect to $\|\cdot\|_\infty$.

(d) Let $C(\mathbf{T})$ be the set of all continuous functions $f \in C(\mathbf{R})$ that are 1-periodic, i.e., $f(t+1) = f(t)$ for every $t \in \mathbf{R}$. Show that $C(\mathbf{T})$ is a closed subspace of $C_b(\mathbf{R})$.

1.22. Let $C_b^m(\mathbf{R})$ be the space of all m-times differentiable functions on $\mathbf{R}$ each of whose derivatives is bounded and continuous, i.e.,

$$C_b^m(\mathbf{R}) = \bigl\{f \in C_b(\mathbf{R}) : f, f', \dots, f^{(m)} \in C_b(\mathbf{R})\bigr\}.$$

(a) Show that $C_b^m(\mathbf{R})$ is a Banach space with respect to the norm

$$\|f\|_{C_b^m} = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(m)}\|_\infty,$$

and

$$C_0^m(\mathbf{R}) = \bigl\{f \in C_0(\mathbf{R}) \,:\, f, f', \dots, f^{(m)} \in C_0(\mathbf{R})\bigr\}$$

is a subspace of $C_b^m(\mathbf{R})$ that is also a Banach space with respect to the same norm.

(b) Now we change the norm on $C_b^1(\mathbf{R})$. Show that $(C_b^1(\mathbf{R}), \|\cdot\|_\infty)$ is a normed space, but is not complete.

1.23. We say that a function $f\colon \mathbf{R} \to \mathbf{F}$ is *Hölder continuous* with exponent $\alpha > 0$ if there exists a constant $K > 0$ such that

$$\forall\, x, y \in \mathbf{R}, \quad |f(x) - f(y)| \le K\,|x-y|^\alpha.$$

A function that is Hölder continuous with exponent $\alpha = 1$ is said to be *Lipschitz*.

(a) Show that if f is Hölder continuous for some $\alpha > 1$, then f is constant.

(b) Show that if f is differentiable on $\mathbf{R}$ and f' is bounded, then f is Lipschitz. Find a function g that is Lipschitz but is not differentiable at every point.

(c) Given $0 < \alpha < 1$, define

$$C^\alpha(\mathbf{R}) = \bigl\{f \in C(\mathbf{R}) : f \text{ is Hölder continuous with exponent } \alpha\bigr\}.$$

Show that $C^\alpha(\mathbf{R})$ is a Banach space with respect to the norm

$$\|f\|_{C^\alpha} = |f(0)| + \sup_{x \ne y} \frac{f(x) - f(y)}{|x-y|^\alpha}.$$

1.24. Consider the two functions φ_1, φ_2 pictured in Figure 1.1. The function φ_1 takes the constant value $1/2$ on the interval $(1/3, 2/3)$ that is removed in the first stage of the construction of the classical Cantor middle-thirds set, and is linear on the remaining intervals. The function φ_2 also takes the same constant $1/2$ on the interval $(1/3, 2/3)$ but additionally is constant with values $1/4$ and $3/4$ on the two intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ that are removed in the second stage of the construction of the Cantor set. Continue this process, defining $\varphi_3, \varphi_4, \ldots$, and prove the following facts.

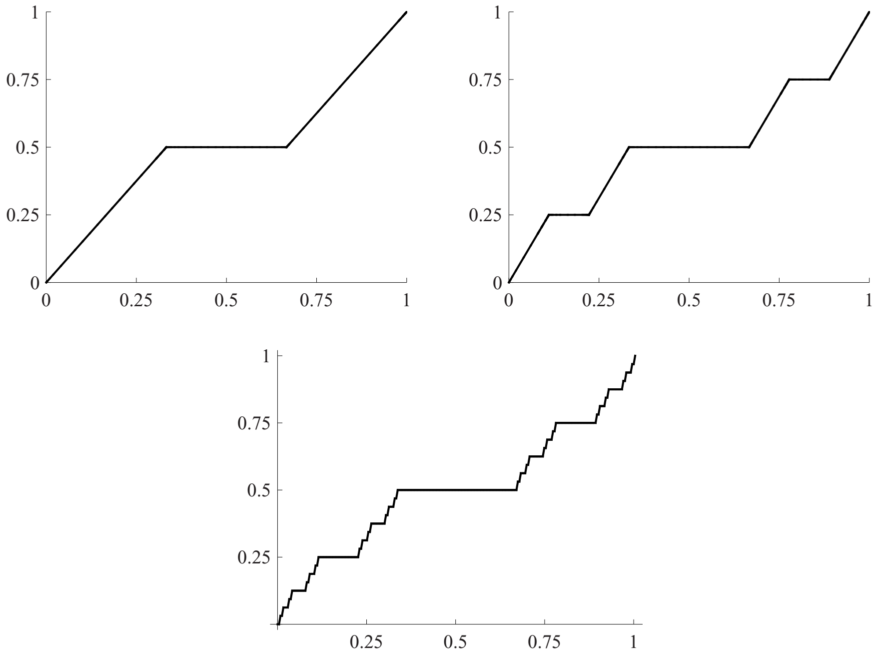

Fig. 1.1. Top left: The function φ_1. Top right: The function φ_2. Bottom: The Devil's staircase (Cantor–Lebesgue function).

(a) Each φ_k is monotone increasing on $[0,1]$, and $|\varphi_{k+1}(t) - \varphi_k(t)| < 2^{-k}$ for every $t \in [0,1]$.

(b) $\varphi(t) = \lim_{k\to\infty} \varphi_k(t)$ converges uniformly on $[0,1]$. The limit function φ is called the *Cantor–Lebesgue function* or, more picturesquely, the *Devil's staircase*.

(c) The Cantor–Lebesgue function is Hölder continuous on the interval $[0,1]$ precisely for exponents α in the range $0 < \alpha \le \log_3 2 \approx 0.6309\ldots$. In particular, φ is continuous and monotone increasing on $[0,1]$ but it is not Lipschitz. Even so, φ is differentiable for a.e. $t \in [0,1]$, and $\varphi'(t) = 0$ a.e.

1.25. Let X be a normed linear space that is not complete. Let $\mathcal{C}$ be the set of all Cauchy sequences in X, and define a relation $\sim$ on $\mathcal{C}$ by declaring that $\{x_n\} \sim \{y_n\}$ if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

(a) Show that $\sim$ is an equivalence relation on $\mathcal{C}$.

(b) Let $[x_n] = \big\{\{y_n\} : \{y_n\} \sim \{x_n\}\big\}$ denote the equivalence class of $\{x_n\}$ under the relation $\sim$. Let $\widetilde{X}$ be the set of all equivalence classes $[x_n]$. Define $\big\|[x_n]\big\|_{\widetilde{X}} = \lim_{n\to\infty} \|x_n\|$. Prove that $\|\cdot\|_{\widetilde{X}}$ is a well-defined norm on $\widetilde{X}$.

(c) Given $x \in X$, let $[x]$ denote the equivalence class of the Cauchy sequence $\{x, x, x, \dots\}$. Show that $T\colon x \mapsto [x]$ is an isometric map of X into $\widetilde{X}$, where *isometric* means that $\|Tx\|_{\widetilde{X}} = \|x\|$ for every x (see Definition 1.60). Show also that $T(X)$ is a dense subspace of $\widetilde{X}$ (so, in the sense of identifying of X with $T(X)$, we can consider X to be a subspace of $\widetilde{X}$).

(d) Show that $\widetilde{X}$ is a Banach space with respect to $\|\cdot\|_{\widetilde{X}}$. We call $\widetilde{X}$ the *completion* of X.

(e) Prove that $\widetilde{X}$ is unique in the sense that if Y is a Banach space and $U\colon X \to Y$ is a linear isometry such that $U(X)$ is dense in Y, then there exists a linear isometric bijection $V\colon Y \to \widetilde{X}$.

1.4 Linear Combinations, Sequences, Series, and Complete Sets

In this section we review some definitions and concepts related to sequences, linear combinations, and infinite series in normed spaces.

Definition 1.25. Let S be a subset of a normed linear space X.

(a) S is *finitely linearly independent*, or simply *independent* for short, if for every choice of finitely many distinct vectors $x_1, \dots, x_N \in S$ and scalars $c_1, \dots, c_N \in \mathbf{F}$ we have

$$\sum_{n=1}^{N} c_n x_n \;=\; 0 \qquad \Longrightarrow \qquad c_1 = \cdots = c_N = 0.$$

(b) The *finite linear span*, or simply the *span*, of S is the set of all finite linear combinations of elements of S, i.e.,

$$\operatorname{span}(S) \;=\; \left\{\sum_{n=1}^{N} c_n x_n : N \in \mathbf{N},\, x_1, \dots, x_N \in S,\, c_1, \dots, c_N \in \mathbf{F}\right\}.$$

If $S = \{x_n\}$ is countable, then we often write $\operatorname{span}\{x_n\}$ instead of $\operatorname{span}(\{x_n\})$:

$$\operatorname{span}\{x_n\} = \left\{ \sum_{n=1}^{N} c_n x_n : N \in \mathbf{N} \text{ and } c_1, \dots, c_N \in \mathbf{F} \right\}.$$

(c) The *closed linear span*, or simply the *closed span*, of S is the closure in X of $\operatorname{span}(S)$, and is denoted $\overline{\operatorname{span}}(S)$. If $S = \{x_n\}$ then we write $\overline{\operatorname{span}}\{x_n\} = \overline{\operatorname{span}}(\{x_n\})$.

(d) $\{x_n\}$ is *complete* (or *total* or *fundamental*) in X if $\overline{\operatorname{span}}(S) = X$, i.e., if $\operatorname{span}(S)$ is dense in X. $\diamondsuit$

Later we will see an equivalent characterization of complete sequences in Banach spaces (see Corollary 2.5).

Remark 1.26. Unfortunately, the term "complete" is heavily overused in mathematics, and indeed we have now introduced two distinct uses for it. First, a *normed linear space* X is complete if every Cauchy sequence in X is convergent. Second, a *sequence* $\{x_n\}$ in a normed linear space X is complete if $\operatorname{span}\{x_n\}$ is dense in X. Which of these two distinct uses is meant should be clear from context. $\diamondsuit$

Only separable normed spaces can contain a countable complete sequence.

Theorem 1.27. *Let X be a normed space. If there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ in X that is complete, then X is separable.*

Proof. Suppose that $\{x_n\}$ is a complete sequence in X, and let

$$S = \left\{ \sum_{n=1}^{N} r_n x_n : N > 0,\ r_n \text{ is rational} \right\},$$

where if $\mathbf{F} = \mathbf{C}$ then "rational" means that both the real and imaginary parts are rational. Then S is countable, and we claim it is dense in X. Without loss of generality, we may assume that every x_n is nonzero.

Choose any $x \in X$. Since $\operatorname{span}(S)$ is dense in X, there exists a vector

$$y = \sum_{n=1}^{N} c_n x_n$$

such that $\|x - y\| < \varepsilon$. For each n, choose a rational scalar r_n such that

$$|c_n - r_n| < \frac{\varepsilon}{N\,\|x_n\|},$$

and set $z = \sum_{n=1}^{N} r_n x_n$. Then $z \in S$ and

$$\|y - z\| \le \sum_{n=1}^{N} |c_n - r_n|\,\|x_n\| < \sum_{n=1}^{N} \frac{\varepsilon}{N\,\|x_n\|}\,\|x_n\| = \varepsilon.$$

Hence $\|x - z\| < 2\varepsilon$, so S is dense in X. $\square$

It is very important to distinguish between elements of the closed span and vectors that can be written in the form $x = \sum_{n=1}^{\infty} c_n x_n$. Before elaborating on this, we give the definition of an infinite series in a normed space. We will explore infinite series in depth in Chapter 3.

Definition 1.28 (Convergent Series). Let $\{x_n\}$ be a sequence in a normed linear space X. Then the series $\sum_{n=1}^{\infty} x_n$ *converges and equals* $x \in X$ if the *partial sums* $s_N = \sum_{n=1}^{N} x_n$ converge to x, i.e., if

$$\lim_{N\to\infty} \|x - s_N\| \;=\; \lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\| \;=\; 0. \qquad \diamondsuit$$

We emphasize that the definition of the closed span does *not* say that

$$\overline{\operatorname{span}}\{x_n\} \;=\; \left\{ \sum_{n=1}^{\infty} c_n x_n : c_n \in \mathbf{F} \right\} \qquad \leftarrow \text{This need not hold!}$$

In particular it is not true that an arbitrary element of $\overline{\operatorname{span}}\{x_n\}$ can always be written as $x = \sum_{n=1}^{\infty} c_n x_n$ for some $c_n \in \mathbf{F}$ (see Example 1.29). Instead,

$$\overline{\operatorname{span}}\{x_n\} = \left\{ x \in X : \exists\, c_{n,N} \in \mathbf{F} \text{ such that } \sum_{n=1}^{N} c_{n,N} x_n \to x \text{ as } N \to \infty \right\}.$$

That is, an element x lies in the closed span of $\{x_n\}$ if and only if there exist $c_{n,N} \in \mathbf{F}$ for $N \in \mathbf{N}$ and $n = 1, \dots, N$ such that

$$\sum_{n=1}^{N} c_{n,N} x_n \to x \qquad \text{as } N \to \infty. \tag{1.8}$$

We emphasize that the scalars $c_{n,N}$ in equation (1.8) can depend on N. In contrast, to say that $x = \sum_{n=1}^{\infty} c_n x_n$ means that

$$\sum_{n=1}^{N} c_n x_n \to x \qquad \text{as } N \to \infty. \tag{1.9}$$

In order for equation (1.9) to hold, the scalars c_n must be *independent* of N. We will explore these issues in more detail in Chapter 3.

Example 1.29. Consider the Banach space $C[a,b]$ under the uniform norm. The *Weierstrass Approximation Theorem* states that if $f \in C[a,b]$ and $\varepsilon > 0$, then there exists polynomial $p(x) = \sum_{k=0}^{n} c_k x^k$ such that $\|f - p\|_\infty < \varepsilon$ (see [BBT97, Cor. 9.65]). This is equivalent to saying that the sequence of monomials $\{x^k\}_{k=0}^{\infty}$ is complete in $C[a,b]$ (which implies by Theorem 1.27 that $C[a,b]$ is separable). However, not every function $f \in C[a,b]$ can be written as $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ with convergence of the series in the uniform norm. A series of this form is called a *power series*, and if it converges at some point x, then it converges absolutely for all points t with $|t| < r$ where $r = |x|$. Moreover, by Exercise 1.29 the function f so defined is infinitely differentiable on $(-r, r)$. Therefore, for example, the function $f(x) = |x - c|$ where $a < c < b$ cannot be written as a power series, even though it belongs to the closed span of $\{x^k\}_{k=0}^{\infty}$. In the language of Chapter 4, while $\{x^k\}_{k=0}^{\infty}$ is complete in $C[a,b]$, it does *not* form a basis for $C[a,b]$. Moreover, $\{x^k\}_{k=0}^{\infty}$ is not a basis even though it is both complete *and* finitely linearly independent.

Although we will not prove it, given $f \in C[a,b]$ it is possible to explicitly construct polynomials p_n such that $\|f - p_n\|_\infty \to 0$ as $n \to \infty$. The *Bernstein polynomials for* f are one example of such a construction, e.g., see [Bar76, Thm. 24.7] for details. ◇

On the other hand, the next example shows that it is possible for the closed span of a sequence to consist of "infinite linear combinations" of the sequence elements.

Example 1.30 (Standard Basis for c_0). For each $n \in \mathbf{N}$, we let δ_n denote the sequence $\delta_n = (\delta_{nk})_{k \in \mathbf{N}} = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the nth component. The finite span of $\{\delta_n\}$ is $\operatorname{span}\{\delta_n\} = c_{00}$. Given $x = (x_n) \in c_0$, set $s_N = \sum_{n=1}^{N} x_n \delta_n$. Then since $x_n \to 0$ we have

$$\lim_{N\to\infty} \|x - s_N\|_{\ell^\infty} \;=\; \lim_{N\to\infty} \sup_{n>N} |x_n| \;=\; \limsup_{n\to\infty} |x_n| \;=\; 0.$$

Hence

$$x \;=\; \sum_{n=1}^{\infty} x_n \delta_n, \tag{1.10}$$

where the series converges with respect to $\|\cdot\|_{\ell^\infty}$, which is the norm of c_0. Further, the scalars x_n in equation (1.10) are unique. Thus every x is a limit of elements of $\operatorname{span}\{\delta_n\}$ so $\{\delta_n\}$ is complete in c_0. However, even more is true. In the language of Chapter 4, the fact that every $x \in c_0$ has a unique representation of the form given in equation (1.10) says that $\{\delta_n\}$ forms a *basis* for c_0. Not only do we know that finite linear combinations of the vectors δ_n are dense in c_0, but we can actually write every element of c_0 as a unique "infinite linear combination" of the δ_n. Hence

$$c_0 \;=\; \overline{\operatorname{span}}\{\delta_n\} \;=\; \biggl\{\sum_{n=1}^{\infty} c_n \delta_n : c_n \in \mathbf{F} \text{ and } \lim_{n\to\infty} c_n = 0\biggr\}.$$

We call $\{\delta_n\}$ the *standard basis for* c_0. ◇

Note that if we use a norm other than $\|\cdot\|_{\ell^\infty}$, then the closed span of $\{\delta_n\}$ might be different.

Example 1.31 (Standard Basis for ℓ^p). If $x = (x_n) \in \ell^p$ where $1 \le p < \infty$ and we set $s_N = \sum_{n=1}^{N} x_n\delta_n$, then

$$\lim_{N\to\infty} \|x - s_N\|_{\ell^p} \;=\; \lim_{N\to\infty} \sum_{n=N+1}^{\infty} |x_n|^p \;=\; 0,$$

so we have $x = \sum x_n\delta_n$ with convergence of this series in ℓ^p-norm. Further, if $x = \sum c_n\delta_n$ for some scalars c_n, then we must have $c_n = x_n$ (why?). In the terminology of Definition 4.3, the sequence $\{\delta_n\}$ is a basis for ℓ^p, which we call the *standard basis* for ℓ^p. Consequently ℓ^p is separable when p is finite, and an explicit countable dense subset is

$$S \;=\; \bigl\{x = (x_1, \dots, x_n, 0, 0, \dots) : n \in \mathbf{N},\, x_n \text{ rational}\bigr\}.$$

In contrast, ℓ^∞ is not separable; see Exercise 1.28. $\diamondsuit$

We introduce the following terminology to distinguish between a sequence that is merely complete in a Banach space X and one that has properties similar to those possessed by the standard basis in c_0 and ℓ^p.

Definition 1.32 (Basis). A sequence $\{x_n\}$ of vectors in a Banach space X is a *basis* for X if every $x \in X$ can be written

$$x \;=\; \sum_{n=1}^{\infty} c_n x_n \tag{1.11}$$

for a unique choice of scalars $c_n \in \mathbf{F}$. $\diamondsuit$

In particular, if $\{x_n\}$ is a basis and we write x as in equation (1.11), then the partial sums $s_N = \sum_{n=1}^{N} c_n x_n$ belong to the finite span of $\{x_n\}$ and converge to x as $N \to \infty$. Hence $\operatorname{span}\{x_n\}$ is dense in X. Therefore every basis is a complete sequence, but Example 1.29 shows us that a complete sequence need not be a basis.

Bases and their relatives will occupy us from Chapter 4 onwards. For now we simply warn the reader not to confuse the meaning of "basis" in the sense of Definition 1.32 with the familiar notion of a "basis" in finite-dimensional linear algebra. A vector space basis (which we call a *Hamel basis* in this volume) is one that spans and is linearly independent using *finite linear combinations* only, whereas Definition 1.32 is worded in terms of "infinite linear combinations." The two notions are *not* equivalent. For more detailed discussions of this issue, see Chapters 4 and 5, and Sections 4.1 and 5.2 in particular.

Exercises

1.26. Suppose that $\sum x_n$ and $\sum y_n$ are convergent series in a normed space X. Show that $\sum (x_n + y_n)$ is convergent and equals $\sum x_n + \sum y_n$.

1.27. Show that $C_0(\mathbf{R})$ is separable.

1.28. (a) Show that if X is a normed linear space and there exists an uncountable set $S \subseteq X$ such that $\|x - y\| = 1$ for every $x \neq y \in S$, then X is not separable.

(b) Let $D = \{x = (x_1, x_2, \dots) \in \ell^\infty : x_k = 0, 1 \text{ for each } k\}$. Show that D is uncountable and that if x, y are two distinct vectors in D, then $\|x-y\|_{\ell^\infty} = 1$. Conclude that ℓ^∞ is not separable.

(c) Show that $L^\infty(\mathbf{R})$ is not separable.

1.29. Let $(c_k)_{k\geq 0}$ be a fixed sequence of real numbers. Show that if the series $\sum_{k=0}^\infty c_k y^k$ converges for some $y \in \mathbf{R}$, then the series $f(x) = \sum_{k=0}^\infty c_k x^k$ converges absolutely for all $|x| < |y|$, and show that this function f is infinitely differentiable for all x with $|x| < |y|$.

1.5 Hilbert Spaces

A *Hilbert space* is a Banach space with additional geometric properties. In particular, the norm of a Hilbert space is obtained from an *inner product* that mimics the properties of the dot product of vectors in $\mathbf{R}^n$ or $\mathbf{C}^n$. Recall that the dot product of u, $v \in \mathbf{C}^n$ is defined by

$$u \cdot v \;=\; u_1\overline{v_1} + \cdots + u_n\overline{v_n}. \tag{1.12}$$

The Euclidean norm $|v| = (|v_1|^2 + \cdots + |v_n|^2)^{1/2}$ is related to the dot product by the equation $|v| = (v \cdot v)^{1/2}$. For $p \neq 2$ the norm $|v|_p = (|v_1|^p + \cdots + |v_n|^p)^{1/p}$ has no such relation to the dot product. In fact, when $p \neq 2$ there is *no* way to define a "generalized dot product" $u \cdot v$ that has the same essential algebraic properties as the usual dot product and which also satisfies $|v|_p = (v \cdot v)^{1/2}$. These "essential algebraic properties" of the dot product are the properties (a)–(d) that appear in the following definition. Note that if $\mathbf{F} = \mathbf{R}$, then the complex conjugate appearing in this definition is superfluous.

Definition 1.33. A vector space H is an *inner product space* if for each x, $y \in H$ there exists a scalar $\langle x, y\rangle \in \mathbf{F}$, called the *inner product* of x and y, so that the following statements hold:

(a) $\langle x, x\rangle$ is real and $\langle x, x\rangle \geq 0$ for each $x \in H$,

(b) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(c) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all x, $y \in H$, and

(d) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all x, y, $z \in H$ and all a, $b \in \mathbf{F}$. $\diamondsuit$

Alternative symbols for inner products include $[\cdot,\cdot]$, $(\cdot,\cdot)$, etc.

Combining parts (c) and (d) of Definition 1.33, it follows by induction that

$$\Big\langle \sum_{n=1}^{N} c_n x_n,\, y \Big\rangle \;=\; \sum_{n=1}^{N} c_n \,\langle x_n, y\rangle \quad \text{and} \quad \Big\langle x,\, \sum_{n=1}^{N} c_n y_n \Big\rangle \;=\; \sum_{n=1}^{N} \overline{c_n}\,\langle x, y_n\rangle.$$

If H is an inner product space, then we will see in Theorem 1.37 that $\|x\| = \langle x,x\rangle^{1/2}$ defines a norm for H, called the *induced norm*. Hence all inner product spaces are normed linear spaces. If H is complete with respect to this induced norm, then H is called a *Hilbert space*. Thus Hilbert spaces are those Banach spaces whose norms can be derived from an inner product.

A given vector space may have many inner products.

Definition 1.34. We say that two inner products $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ for a Hilbert space H are equivalent if the corresponding induced norms $\|x\|^2 = \langle x,x\rangle$ and $|||x|||^2 = (x,x)$ are equivalent in the sense of Definition 1.7. ♢

Now we give some examples of Hilbert spaces.

Example 1.35. (a) $\mathbf{F}^d$ is a Hilbert space with respect to the dot product given in equation (1.12). Also, if A is a positive definite $n \times n$ matrix with entries in $\mathbf{F}$, then

$$\langle x,y\rangle_A \;=\; Ax\cdot y, \qquad x,y \in \mathbf{F}^d, \tag{1.13}$$

defines another inner product on $\mathbf{F}^d$, and $\mathbf{F}^d$ is a Hilbert space with respect to this inner product. Moreover, every inner product on $\mathbf{F}^d$ has the form given in equation (1.13) for some positive definite matrix A, and all such inner products on $\mathbf{F}^d$ are equivalent (see Exercise 1.30).

(b) ℓ^2 is a Hilbert space with respect to the inner product

$$\big\langle (x_k),\,(y_k)\big\rangle \;=\; \sum_{k=1}^{\infty} x_k\,\overline{y_k}, \qquad (x_k),(y_k) \in \ell^2.$$

Note that the Cauchy–Bunyakovski–Schwarz Inequality implies that the series above converges absolutely for each choice of sequences (x_k), $(y_k) \in \ell^2$. Since c_{00} is a subset of ℓ^2, we can use the same rule to define an inner product on c_{00}. Thus c_{00} is an inner product space, but it is not complete with respect to this inner product.

(c) $L^2(E)$ is a Hilbert space with respect to the inner product

$$\langle f,g\rangle \;=\; \int_E f(t)\,\overline{g(t)}\,dt, \qquad f,g \in L^2(E).$$

The fact that the integral above exists is again a consequence of the Cauchy–Bunyakovski–Schwarz Inequality. The same rule defines an inner product on any subspace X of $L^2(E)$, but X will only be complete with respect to that inner product if it is a closed subspace of $L^2(E)$. For example,

$$X = \{f \in L^2(\mathbf{R}) : f = 0 \text{ a.e. on } (-\infty, 0)\}$$

is a closed subspace of $L^2(\mathbf{R})$ and hence is a Hilbert space with respect to the inner product of $L^2(\mathbf{R})$. On the other hand, $C_c(\mathbf{R})$ is a proper dense subspace of $L^2(\mathbf{R})$ and hence is not closed in $L^2(\mathbf{R})$. If we place the inner product from $L^2(\mathbf{R})$ on $C_c(\mathbf{R})$, then $C_c(\mathbf{R})$ is an inner product space, but it is not complete and therefore is not a Hilbert space with respect to that inner product. ◇

Here are a few basic properties of an inner product (see Exercise 1.31). We say that vectors x, y in a Hilbert space are *orthogonal* if $\langle x, y\rangle = 0$, and in this case we often write $x \perp y$.

Lemma 1.36. *Let H be a Hilbert space, and let x, $y \in H$ be given.*

(a) Polar Identity: $\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle) + \|y\|^2$.

(b) Pythagorean Theorem: *If* $\langle x, y\rangle = 0$ *then* $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$.

(c) Parallelogram Law: $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$. ◇

The following result generalizes the Cauchy–Bunyakovski–Schwarz inequality to any Hilbert space H, and shows that the induced norm is indeed a norm on H.

Theorem 1.37. *Let H be an inner product space.*

(a) Cauchy–Bunyakovski–Schwarz Inequality:

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \qquad \text{for all } x, y \in H.$$

(b) $\|x\| = \langle x, x\rangle^{1/2}$ *is a norm on H.*

(c) $\|x\| = \sup\limits_{\|y\|=1} |\langle x, y\rangle|$.

Proof. (a) If $x = 0$ or $y = 0$ then there is nothing to prove, so suppose that both are nonzero. Write $\langle x, y\rangle = \alpha\,|\langle x, y\rangle|$ where $\alpha \in \mathbf{F}$ and $|\alpha| = 1$. Then for $t \in \mathbf{R}$ we have by the Polar Identity that

$$\begin{aligned} 0 \le \|x - \alpha t y\|^2 &= \|x\|^2 - 2\,\mathrm{Re}(\bar{\alpha} t\,\langle x, y\rangle) + t^2\,\|y\|^2 \\ &= \|x\|^2 - 2t\,|\langle x, y\rangle| + t^2\,\|y\|^2. \end{aligned}$$

This is a real-valued quadratic polynomial in the variable t. In order for it to be nonnegative, it can have at most one real root. This requires that the discriminant be at most zero, so $\big(-2\,|\langle x, y\rangle|\big)^2 - 4\,\|x\|^2\,\|y\|^2 \le 0$. The desired inequality then follows upon rearranging.

(b) The only property that is not obvious is the Triangle Inequality. From the Polar Identity and the Cauchy–Bunyakovski–Schwarz Inequality,

$$\begin{aligned} \|x+y\|^2 \;=\; \langle x+y, x+y\rangle \;&=\; \|x\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) + \|y\|^2 \\ &\le\; \|x\|^2 + 2\,|\langle x,y\rangle| + \|y\|^2 \\ &\le\; \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 \\ &=\; \big(\|x\|+\|y\|\big)^2. \end{aligned}$$

(c) We have $\sup_{\|y\|=1} |\langle x,y\rangle| \le \|x\|$ by the Cauchy–Bunyakovski–Schwarz Inequality, and the opposite inequality follows by considering $y = x/\|x\|$. □

We obtain the following useful facts as corollaries of Cauchy–Bunyakovski–Schwarz.

Corollary 1.38. *Let H be a Hilbert space.*

(a) Continuity of the inner product: *If $x_n \to x$ and $y_n \to y$ in H, then $\langle x_n, y_n\rangle \to \langle x,y\rangle$.*

(b) *If the series $x = \sum_{n=1}^\infty x_n$ converges in H, then for any $y \in H$ we have*

$$\langle x,y\rangle \;=\; \Big\langle \sum_{n=1}^\infty x_n,\; y\Big\rangle \;=\; \sum_{n=1}^\infty \langle x_n, y\rangle.$$

Proof. (a) Suppose $x_n \to x$ and $y_n \to y$. Since convergent sequences are bounded, $C = \sup \|x_n\| < \infty$. Therefore

$$\begin{aligned} |\langle x,y\rangle - \langle x_n,y_n\rangle| \;&\le\; |\langle x-x_n, y\rangle| + |\langle x_n, y-y_n\rangle| \\ &\le\; \|x-x_n\|\,\|y\| + \|x_n\|\,\|y-y_n\| \\ &\le\; \|x-x_n\|\,\|y\| + C\,\|y-y_n\| \;\to\; 0 \quad \text{as } n\to\infty. \end{aligned}$$

(b) Suppose that the series $x = \sum_{n=1}^\infty x_n$ converges in H, and let $s_N = \sum_{n=1}^N x_n$ denote the partial sums of this series. Then, by definition, $s_N \to x$ in H. Hence, given $y \in H$ we have

$$\begin{aligned} \sum_{n=1}^\infty \langle x_n, y\rangle \;&=\; \lim_{N\to\infty} \Big(\sum_{n=1}^N \langle x_n, y\rangle\Big) \\ &=\; \lim_{N\to\infty} \Big\langle \sum_{n=1}^N x_n,\; y\Big\rangle \;=\; \lim_{N\to\infty} \langle s_N, y\rangle \;=\; \langle x,y\rangle, \end{aligned}$$

where at the last step we have used the continuity of the inner product. □

Now we give the definition and basic properties of orthogonal projections in a Hilbert space.

Theorem 1.39. *Let H be a Hilbert space, and let M be a closed subspace of H. Given $x \in H$, there exists a unique element $p \in M$ that is closest to x, i.e., $\|x-p\| = \operatorname{dist}(x, M) = \inf\{\|x-m\| : m \in M\}$.*

Proof. Fix $x \in H$, and let $d = \operatorname{dist}(x, M) = \inf\{\|x-m\| : m \in M\}$. Then, by definition, there exist $y_n \in M$ such that $d \le \|x-y_n\| \to d$ as $n \to \infty$. Therefore, if we fix any $\varepsilon > 0$ then we can find an N such that

$$n > N \quad \Longrightarrow \quad d^2 \le \|x-y_n\|^2 \le d^2+\varepsilon^2.$$

By the Parallelogram Law,

$$\|(x-y_n)-(x-y_m)\|^2 + \|(x-y_n)+(x-y_m)\|^2 \;=\; 2\left(\|x-y_n\|^2+\|x-y_m\|^2\right).$$

Hence,

$$\begin{aligned} \left\|\frac{y_m-y_n}{2}\right\|^2 &= \frac{1}{4}\,\|(x-y_n)-(x-y_m)\|^2 \\ &= \frac{\|x-y_n\|^2}{2} + \frac{\|x-y_m\|^2}{2} - \left\|x - \frac{y_m+y_n}{2}\right\|^2. \end{aligned}$$

However, $\frac{y_m+y_n}{2} \in M$ since M is a subspace, so $\|x - \frac{y_m+y_n}{2}\| \ge d$. Also, if m, $n > N$ then $\|x-y_n\|^2$, $\|x-y_m\|^2 \le d^2+\varepsilon^2$. Therefore, for m, $n > N$ we have

$$\left\|\frac{y_m-y_n}{2}\right\|^2 \;\le\; \frac{d^2+\varepsilon^2}{2} + \frac{d^2+\varepsilon^2}{2} - d^2 \;=\; \varepsilon^2.$$

Thus, $\|y_m-y_n\| \le 2\varepsilon$ for all m, $n > N$, which says that the sequence $\{y_n\}$ is Cauchy. Since H is complete, this sequence must converge, so $y_n \to p$ for some $p \in H$. But $y_n \in M$ for all n and M is closed, so we must have $p \in M$. Since $x-y_n \to x-p$, it follows from the continuity of the norm (Exercise 1.2) that

$$\|x-p\| \;=\; \lim_{n\to\infty} \|x-y_n\| \;=\; d,$$

and hence $\|x-p\| \le \|x-y\|$ for every $y \in M$. Thus p is a closest point in M to x, and we leave as an exercise the task of proving that p is unique (Exercise 1.35). □

The proof of Theorem 1.39 carries over without change to show that if K is a closed, convex subset of a Hilbert space, then given any $x \in H$ there is a unique point p in K that is closest to x.

Definition 1.40. Let M be a closed subspace of a Hilbert space H.

(a) Given $x \in H$, the unique vector $p \in M$ that is closest to x is called the *orthogonal projection* of x onto M.

(b) For $x \in H$ let Px denote the vector that is the orthogonal projection of x onto M. Then the mapping $P\colon x \mapsto Px$ is called the *orthogonal projection* of H onto M. ◊

Now we define orthogonal complements, which play an important role in the analysis of Hilbert spaces. The lack of orthogonal projections and orthogonal complements in non-Hilbert spaces is often what makes the analysis of generic Banach spaces so much more difficult than it is for Hilbert spaces.

Definition 1.41 (Orthogonal Complement). Let A be a subset (not necessarily closed or a subspace) of a Hilbert space H. The *orthogonal complement* of A is

$$A^\perp = \bigl\{x \in H : x \perp A\bigr\} = \bigl\{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in A\bigr\}. \qquad \diamondsuit$$

The orthogonal complement of A is always a closed subspace of H, even if A is not (Exercise 1.36).

Lemma 1.42. *If A is a subset of a Hilbert space H, then $A^\perp$ is a closed subspace of H.* $\diamondsuit$

Orthogonal projections can be characterized in terms of orthogonal complements as follows (see Exercise 1.37).

Theorem 1.43. *If M is a closed subspace of a Hilbert space H and $x \in H$, then the following statements are equivalent.*

(a) *$x = p + e$ where p is the orthogonal projection of x onto M.*

(b) *$x = p + e$ where $p \in M$ and $e \in M^\perp$.*

(c) *$x = p + e$ where e is the orthogonal projection of x onto $M^\perp$.* $\diamondsuit$

Consequently, if P is the orthogonal projection of H onto M, then the orthogonal projection of H onto $M^\perp$ is $I - P$.

Here are some properties of orthogonal complements. (Exercise 1.38).

Lemma 1.44. *Let H be a Hilbert space.*

(a) *If M is a closed subspace of H, then $(M^\perp)^\perp = M$.*

(b) *If A is any subset of H, then*

$$A^\perp = \operatorname{span}(A)^\perp = \overline{\operatorname{span}}(A)^\perp \qquad \textit{and} \qquad (A^\perp)^\perp = \overline{\operatorname{span}}(A).$$

(c) *A sequence $\{x_n\}$ in H is complete if and only if the following statement holds:*

$$x \in H \textit{ and } \langle x, x_n\rangle = 0 \textit{ for every } n \quad \Longrightarrow \quad x = 0. \qquad \diamondsuit$$

Definition 1.45 (Orthogonal Direct Sum). Let M, N be closed subspaces of a Hilbert space H.

(a) The *direct sum* of M and N is $M + N = \{x + y : x \in M, y \in N\}$.

(b) We say that M and N are *orthogonal subspaces*, denoted $M \perp N$, if $x \perp y$ for every $x \in M$ and $y \in N$.

(c) If M, N are orthogonal subspaces in H, then we call their direct sum the *orthogonal direct sum* of M and N, and denote it by $M \oplus N$. ♢

The proof of the next lemma is Exercise 1.39.

Lemma 1.46. *Let M, N be closed, orthogonal subspaces of H.*
(a) *$M \oplus N$ is a closed subspace of H.*
(b) *$M \oplus M^\perp = H$.* ♢

Exercises

1.30. An $n \times n$ matrix A is said to be *positive definite* if $Ax \cdot x > 0$ for all $x \in \mathbf{F}^n$ (compare Definition 2.14).

(a) Show that if A is a positive definite $n \times n$ matrix then equation (1.13) defines an inner product on $\mathbf{F}^d$.

(b) Show that if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbf{F}^d$ then there exists some positive definite matrix A such that equation (1.13) holds.

(c) Prove that the inner products on $\mathbf{F}^d$ defined by equation (1.13) are all equivalent.

1.31. Prove Lemma 1.36.

1.32. Show that if $p \neq 2$ then the norm $\|\cdot\|_{\ell^p}$ on ℓ^p is not induced from any inner product on ℓ^p.

1.33. Show that equality holds in the Cauchy–Bunyakovski–Schwarz Inequality if and only if $x = cy$ or $y = cx$ for some scalar $c \in \mathbf{F}$.

1.34. We say that $\langle \cdot, \cdot \rangle$ is a *semi-inner product* on a vector space H if properties (a), (c), and (d) of Definition 1.33 are satisfied. Prove that if $\langle \cdot, \cdot \rangle$ is a semi-inner product then $\|x\| = \langle x, x \rangle^{1/2}$ defines a seminorm on H, and $|\langle x, y \rangle| \le \|x\| \, \|y\|$ for all x, $y \in H$.

1.35. Given a closed subspace M of a Hilbert space H and given $x \in H$, prove that the point in M closest to x is unique.

1.36. Prove Lemma 1.42.

1.37. Prove Theorem 1.43.

1.38. Prove Lemma 1.44.

1.39. Prove Lemma 1.46.

1.40. Let H, K be Hilbert spaces. Show that $H \times K$ is a Hilbert space with respect to the inner product $\big\langle (h_1, k_1), (h_2, k_2) \big\rangle = \langle h_1, h_2 \rangle_H + \langle k_1, k_2 \rangle_K$.

1.41. Prove that the completion $\widetilde{H}$ (Exercise 1.25) of an inner product space H is a Hilbert space with respect to an inner product that extends the inner product on H.

1.6 Orthogonal Sequences in Hilbert Spaces

Two vectors x, y in a Hilbert space are *orthogonal* if $\langle x, y\rangle = 0$. Sequences in a Hilbert space which possess the property that any two distinct elements are orthogonal have a number of useful features, which we consider in this section.

Definition 1.47. Let $\{x_n\}$ be a sequence in a Hilbert space H.

(a) $\{x_n\}$ is an *orthogonal sequence* if $\langle x_m, x_n\rangle = 0$ whenever $m \neq n$.

(b) $\{x_n\}$ is an *orthonormal sequence* if $\langle x_m, x_n\rangle = \delta_{mn}$, i.e., $\{x_n\}$ is orthogonal and $\|x_n\| = 1$ for every n.

(c) We recall from Definition 1.32 that $\{x_n\}$ is a *basis* for H if every $x \in H$ can be written $x = \sum_{n=1}^{\infty} c_n x_n$ for a unique choice of scalars c_n.

(d) An orthonormal sequence $\{x_n\}$ is an *orthonormal basis* if it is both orthonormal and a basis. $\diamondsuit$

The definition of orthogonal and orthonormal sequences in a Hilbert space H can be extended to arbitrary subsets of H. In particular, we say that $S \subseteq H$ is orthogonal if given any $x \neq y \in S$ we have $\langle x, y\rangle = 0$.

While orthogonality only makes sense in an inner product space, the definition of a basis extends without change to Banach spaces. We will explore bases in detail starting in Chapter 4. Here in this section we will concentrate on the specific issue of basis properties of orthonormal sequences in Hilbert spaces. We note that in the Banach space literature and in this volume the word *basis* is reserved for *countable* sequences that satisfy statement (c) of Definition 1.47.

We first recall the Pythagorean Theorem (Lemma 1.36), extended by induction to finite collections of orthogonal vectors.

Lemma 1.48 (Pythagorean Theorem). *If $\{x_1, \dots, x_N\}$ are orthogonal vectors in an inner product space H, then* $\left\|\sum_{n=1}^{N} x_n\right\|^2 = \sum_{n=1}^{N} \|x_n\|^2$. $\diamondsuit$

The following result summarizes some basic results connected to convergence of infinite series of orthonormal vectors.

Theorem 1.49. *If $\{x_n\}$ is an orthonormal sequence in a Hilbert space H, then the following statements hold.*

(a) Bessel's Inequality: $\sum |\langle x, x_n\rangle|^2 \le \|x\|^2$ *for every* $x \in H$.

(b) *If* $x = \sum c_n x_n$ *converges, then* $c_n = \langle x, x_n\rangle$.

(c) $\sum c_n x_n$ *converges* $\iff$ $\sum |c_n|^2 < \infty$.

(d) $x \in \overline{\mathrm{span}}\{x_n\} \iff x = \sum \langle x, x_n\rangle\, x_n$.

(e) *If* $x \in H$, *then* $p = \sum \langle x, x_n\rangle\, x_n$ *is the orthogonal projection of* x *onto* $\overline{\mathrm{span}}\{x_n\}$.

Proof. We will prove some statements, and the rest are assigned as Exercise 1.42.

(a) Choose $x \in H$. For each $N \in \mathbf{N}$ define $y_N = x - \sum_{n=1}^{N} \langle x, x_n \rangle x_n$. If $1 \le m \le N$, then

$$\langle y_N, x_m \rangle \;=\; \langle x, x_m \rangle - \sum_{n=1}^{N} \langle x, x_n \rangle \langle x_n, x_m \rangle \;=\; \langle x, x_m \rangle - \langle x, x_m \rangle \;=\; 0.$$

Thus $y_N \perp x_1, \dots, x_N$. Therefore, by the Pythagorean Theorem,

$$\begin{aligned} \|x\|^2 \;&=\; \Bigl\| y_N + \sum_{n=1}^{N} \langle x, x_n \rangle x_n \Bigr\|^2 \\ &=\; \|y_N\|^2 \;+\; \sum_{n=1}^{N} \|\langle x, x_n \rangle x_n\|^2 \\ &=\; \|y_N\|^2 \;+\; \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 \;\ge\; \sum_{n=1}^{N} |\langle x, x_n \rangle|^2. \end{aligned}$$

Letting $N \to \infty$, we obtain Bessel's Inequality.

(c) Suppose that $\sum_{n=1}^{\infty} |c_n|^2 < \infty$. Set

$$s_N \;=\; \sum_{n=1}^{N} c_n x_n \qquad \text{and} \qquad t_N \;=\; \sum_{n=1}^{N} |c_n|^2.$$

We know that $\{t_N\}_{N \in \mathbf{N}}$ is a convergent (hence Cauchy) sequence of scalars, and we must show that $\{s_N\}_{N \in \mathbf{N}}$ is a convergent sequence of vectors. We have for $N > M$ that

$$\begin{aligned} \|s_N - s_M\|^2 \;&=\; \Bigl\| \sum_{n=M+1}^{N} c_n x_n \Bigr\|^2 \\ &=\; \sum_{n=M+1}^{N} \|c_n x_n\|^2 \;=\; \sum_{n=M+1}^{N} |c_n|^2 \;=\; |t_N - t_M|. \end{aligned}$$

Since $\{t_N\}_{N \in \mathbf{N}}$ is Cauchy, we conclude that $\{s_N\}_{N \in \mathbf{N}}$ is a Cauchy sequence in H and hence converges.

(d) Choose $x \in \overline{\mathrm{span}}\{x_n\}$. By Bessel's Inequality, $\sum |\langle x, x_n \rangle|^2 < \infty$, and therefore by part (c) we know that the series $y = \sum \langle x, x_n \rangle x_n$ converges. Given any particular $m \in \mathbf{N}$, by applying Corollary 1.38(b) we have

$$\begin{aligned} \langle x-y, x_m\rangle &= \langle x, x_m\rangle - \Big\langle \sum_n \langle x, x_n\rangle\, x_n,\ x_m \Big\rangle \\ &= \langle x, x_m\rangle - \sum_n \langle x, x_n\rangle \langle x_n, x_m\rangle \\ &= \langle x, x_m\rangle - \langle x, x_m\rangle \ = \ 0. \end{aligned}$$

Thus $x - y \in \{x_n\}^\perp = \overline{\operatorname{span}}\{x_n\}^\perp$. However, we also have $x - y \in \overline{\operatorname{span}}\{x_n\}$, so $x - y = 0$. $\square$

It is tempting to conclude from Theorem 1.49 that if $\{x_n\}$ is an orthonormal sequence in a Hilbert space H, then *every* $x \in H$ can be written $x = \sum \langle x, x_n\rangle\, x_n$. This, however, is not always the case, for there may not be "enough" vectors in the sequence to span all of H. In particular, if $\{x_n\}$ is not complete, then its closed span is only a proper closed subspace of H and not all of H. For example, a finite sequence of orthonormal vectors $\{x_1, \dots, x_N\}$ can only span a finite-dimensional subspace of an infinite-dimensional Hilbert space, and therefore cannot be complete in an infinite-dimensional space. As another example, if $\{x_n\}$ is an orthonormal sequence in H, then $\{x_{2n}\}$ is also an orthonormal sequence in H. However, x_1 is orthogonal to every x_{2n}, so it follows from Lemma 1.44 that $\{x_{2n}\}$ is incomplete.

The next theorem presents several equivalent conditions which imply that an orthonormal sequence is complete in H.

Theorem 1.50. *If $\{x_n\}$ is an orthonormal sequence in a Hilbert space H, then the following statements are equivalent.*

(a) *$\{x_n\}$ is complete in H.*

(b) *$\{x_n\}$ is a basis for H, i.e., for each $x \in H$ there exists a unique sequence of scalars (c_n) such that $x = \sum c_n x_n$.*

(c) *$x = \sum \langle x, x_n\rangle\, x_n$ for each $x \in H$.*

(d) Plancherel's Equality: *$\|x\|^2 = \sum |\langle x, x_n\rangle|^2$ for all $x \in H$.*

(e) Parseval's Equality: *$\langle x, y\rangle = \sum \langle x, x_n\rangle\, \langle x_n, y\rangle$ for all $x, y \in H$.*

Proof. We will prove some implications, and assign the rest as Exercise 1.42.

(a) $\Rightarrow$ (c). If $\{x_n\}$ is complete, then its closed span is all of H by definition, so by Theorem 1.49(d) we have that $x = \sum \langle x, x_n\rangle\, x_n$ for every $x \in H$.

(d) $\Rightarrow$ (c). Suppose that $\|x\|^2 = \sum |\langle x, x_n\rangle|^2$ for all $x \in H$. Fix $x \in H$, and define $s_N = \sum_{n=1}^N \langle x, x_n\rangle\, x_n$. Then, by the Polar Identity and the Pythagorean Theorem,

$$\begin{aligned}\|x - s_N\|^2 &= \|x\|^2 - 2\,\mathrm{Re}(\langle x, s_N\rangle) + \|s_N\|^2 \\ &= \|x\|^2 - 2\sum_{n=1}^{N} |\langle x, x_n\rangle|^2 + \sum_{n=1}^{N} |\langle x, x_n\rangle|^2 \\ &= \|x\|^2 - \sum_{n=1}^{N} |\langle x, x_n\rangle|^2 \;\to\; 0 \qquad \text{as } N \to \infty.\end{aligned}$$

Hence $x = \sum \langle x, x_n\rangle\, x_n$. □

As the Plancherel and Parseval Equalities are equivalent, these terms are often used interchangeably.

Note that Theorem 1.50 implies that every *complete orthonormal sequence* in a Hilbert space is actually a *basis* for H. This need not be true for nonorthogonal sequences. An example in ℓ^2 is constructed in Exercise 1.46. In fact, that example is complete and finitely linearly independent yet is not a basis for ℓ^2.

Now we give some examples of orthonormal bases.

Example 1.51 (Standard Basis for ℓ^2). By Example 1.31, the standard basis $\{\delta_n\}$ is complete in ℓ^2, and it is clearly orthonormal. Hence it is an orthonormal basis for ℓ^2. ◇

Example 1.52 (The Trigonometric System). Let $H = L^2(\mathbf{T})$ denote the space of complex-valued functions that are 1-periodic on $\mathbf{R}$ and are square integrable on the interval $[0,1]$. The norm and inner product are defined by integrating on the interval $[0,1]$, i.e.,

$$\langle f, g\rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt \qquad \text{and} \qquad \|f\|_{L^2}^2 = \int_0^1 |f(t)|^2\,dt.$$

For each $n \in \mathbf{Z}$, define a function e_n by

$$e_n(t) = e^{2\pi i n t}, \qquad t \in \mathbf{R}.$$

We call $\{e_n\}_{n\in\mathbf{Z}}$ the *trigonometric system*. Given integers $m \neq n$,

$$\begin{aligned}\langle e_m, e_n\rangle &= \int_0^1 e_m(t)\,\overline{e_n(t)}\,dt \\ &= \int_0^1 e^{-2\pi i(m-n)t}\,dt = \frac{e^{-2\pi i(m-n)} - 1}{-2\pi i(m-n)} = 0.\end{aligned}$$

Hence $\{e_n\}_{n\in\mathbf{Z}}$ is an orthogonal sequence in $L^2(\mathbf{T})$, and it is easy to check that it is orthonormal.

It is a more subtle fact that $\{e_n\}_{n\in\mathbf{Z}}$ is complete in $L^2(\mathbf{T})$. One approach to proving this relies on techniques from harmonic analysis, and will be presented in Chapters 13 and 14. Therefore, until we reach those chapters we will simply assume that the trigonometric system is complete in $L^2(\mathbf{T})$. Assuming this completeness, Theorem 1.50 implies that $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$.

If $f \in L^2(\mathbf{T})$ then the expansion $f = \sum_{n\in\mathbf{Z}} \langle f, e_n\rangle\, e_n$ is called the *Fourier series representation* of f, and $\big(\langle f, e_n\rangle\big)_{n\in\mathbf{Z}}$ is the sequence of *Fourier coefficients* of f. The Fourier coefficients are often denoted by

$$\widehat{f}(n) \;=\; \langle f, e_n\rangle = \int_0^1 f(t)\, e^{-2\pi i n t}\, dt, \qquad n \in \mathbf{Z}.$$

The elements of the space $L^2(\mathbf{T})$ are 1-periodic functions on the real line. Sometimes it is more convenient to work with the space $L^2[0,1]$ consisting of complex-valued square integrable functions whose domain is the interval $[0,1]$. All of the statements above apply equally to $L^2[0,1]$, i.e., $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,1]$. In fact, by the periodicity of the exponentials, $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$ where I is any interval in $\mathbf{R}$ of length 1. $\diamondsuit$

Note that we are only guaranteed that the Fourier series of $f \in L^2(\mathbf{T})$ will converge in L^2-norm. Establishing the convergence of Fourier series in other senses can be extremely difficult. We explore some of these issues in Chapter 14, and prove there that the symmetric partial sums $s_N(x) = \sum_{n=-N}^{n} \widehat{f}(n)\, e^{2\pi i n x}$ of the Fourier series of $f \in L^p(\mathbf{T})$ converge in L^p-norm when $1 < p < \infty$. One of the deepest results in harmonic analysis, which we do not prove in this volume, is the *Carleson–Hunt Theorem,* which states that the symmetric partial sums of the Fourier series of a function $f \in L^p(\mathbf{T})$ converge pointwise almost everywhere to f when $1 < p < \infty$ (see Theorem 14.9).

Notation 1.53. Even though we will not prove that the trigonometric system is complete in $L^2(\mathbf{T})$ until Chapters 13 and 14, as stated above we will take this as given in Chapters 1–12 and use the fact that $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$ and $L^2[0,1]$ without further comment. If we prefer to work with real-valued functions,

$$\{1\} \cup \{\sqrt{2}\,\sin 2\pi n t\}_{n\in\mathbf{N}} \cup \{\sqrt{2}\,\cos 2\pi n t\}_{n\in\mathbf{N}}$$

forms an orthonormal basis for $L^2(\mathbf{T})$ or $L^2[0,1]$ when we take $\mathbf{F} = \mathbf{R}$ (see Exercise 1.49). $\diamondsuit$

In light of Example 1.52, if $\{e_n\}$ is an orthonormal basis for an arbitrary Hilbert space H, then the basis representation $x = \sum \langle x, e_n\rangle\, e_n$ is sometimes called the *generalized Fourier series* of $x \in H$, and $\big(\langle x, e_n\rangle\big)$ is called the sequence of *generalized Fourier coefficients* of x.

The next example gives another important orthonormal basis.

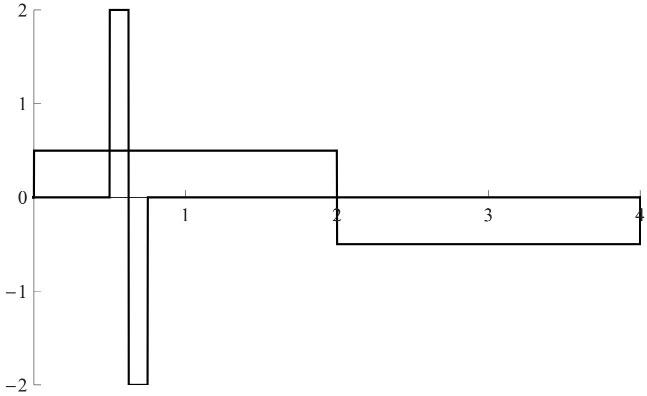

Fig. 1.2. Graphs of $\psi_{-2,0}$ and $\psi_{2,2}$.

Example 1.54 (The Haar System). Let $\chi = \chi_{[0,1)}$ be the box function. The function

$$\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$$

is called the *Haar wavelet*. For integer $n, k \in \mathbf{Z}$, define

$$\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k).$$

The *Haar system* for $L^2(\mathbf{R})$ is

$$\{\chi(t-k)\}_{k\in\mathbf{Z}} \cup \{\psi_{n,k}\}_{n\geq 0,\, k\in\mathbf{Z}}.$$

Direct calculations show that the Haar system is an orthonormal sequence in $L^2(\mathbf{R})$ (see Exercise 1.50 or the "proof by picture" in Figure 1.2), and we will prove that the Haar system is complete in $L^2(\mathbf{R})$.

Suppose that $f \in L^2(\mathbf{R})$ is orthogonal to each element of the Haar system. Considering the integer translates $\chi(t-k)$ of the box function, this implies that

$$\int_k^{k+1} f(t)\,dt = 0, \qquad k \in \mathbf{Z}.$$

Next, since $f \perp \chi$ we have

$$\int_0^{1/2} f(t)\,dt + \int_{1/2}^1 f(t)\,dt = \int_0^1 f(t)\,dt = \langle f, \chi\rangle = 0,$$

and since $f \perp \psi$ we have

$$\int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt = \langle f, \psi\rangle = 0.$$

Adding and subtracting,

$$\int_0^{1/2} f(t)\,dt \;=\; 0 \;=\; \int_{1/2}^1 f(t)\,dt.$$

Continuing in this way, it follows that

$$\int_{I_{n,k}} f(t)\,dt \;=\; 0 \quad \text{for every dyadic interval } I_{n,k} = \Bigl[\frac{k}{2^n}, \frac{k+1}{2^n}\Bigr].$$

Given $t \in \mathbf{R}$, for each $n \in \mathbf{N}$ there exists some dyadic interval $J_n(t) = I_{n,k_n(t)}$ such that $t \in J_n(t)$. The diameter of $J_n(t)$ shrinks rapidly with n, and we have $\cap J_n(t) = \{t\}$. We appeal now to a fundamental result from real analysis, the Lebesgue Differentiation Theorem (Theorem A.30), which implies that for almost every $t \in \mathbf{R}$ we have

$$f(t) \;=\; \lim_{n\to\infty} \frac{1}{|J_n(t)|} \int_{J_n(t)} f(u)\,du.$$

Since $\int_{J_n(t)} f(u)\,du = 0$ for every n, we conclude that $f = 0$ a.e. $\diamondsuit$

Remark 1.55. (a) The sequence $\{\psi_{n,k}\}_{n,k\in\mathbf{Z}}$, containing dilations of the Haar wavelet at all dyadic scales 2^n with $n \in \mathbf{Z}$, also forms an orthonormal basis for $L^2(\mathbf{R})$, and this basis is also referred to as the Haar system for $L^2(\mathbf{R})$. This system is the simplest example of a *wavelet orthonormal basis* for $L^2(\mathbf{R})$. Unfortunately, our proof that the Haar system is an orthonormal basis yields little insight into the elegant construction of general wavelet orthonormal bases, which is the topic of Chapter 12.

(b) The system originally introduced by Haar is

$$\{\chi\} \;\cup\; \{\psi_{n,k}\}_{n\ge 0,\, k=0,\dots,2^n-1},$$

which forms an orthonormal basis for $L^2[0,1]$. An English translation of Haar's 1910 paper [Haa10] can be found in [HW06]. $\diamondsuit$

Suppose that a Hilbert space H has an orthonormal basis $\{e_n\}$. Then

$$S \;=\; \biggl\{\sum_{n=1}^N r_n e_n : N > 0, \;\; \text{rational } r_n \in \mathbf{F}\biggr\} \tag{1.14}$$

is a countable, dense subset of H, so H is separable (see Theorem 1.27). The next result proves the converse, i.e., every separable Hilbert space possesses an orthonormal basis. This proof requires some familiarity with *Zorn's Lemma*, which is an equivalent form of the *Axiom of Choice*.

Theorem 1.56. *Let H be a Hilbert space.*

(a) *H contains a subset T that is both complete and orthonormal.*

(b) *H has an orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$ if and only if H is separable.*

Proof. (a) Let $\mathcal{S}$ denote the set of all orthonormal subsets of H. Inclusion of sets forms a partial order on $\mathcal{S}$.

Suppose that $\mathcal{C} = \{S_i\}_{i \in I}$ is a chain in H, i.e., I is an arbitrary index set and for each $i, j \in I$ we have either $S_i \subseteq S_j$ or $S_j \subseteq S_i$. Define $S = \cup_{i \in I} S_i$. If x, y are two distinct elements of S, then $x \in S_i$ and $y \in S_j$ for some i and j. Since $\mathcal{C}$ is a chain, we must either have $x, y \in S_i$ or $x, y \in S_j$. In any case, $\langle x, y \rangle = 0$ since S_i and S_j are each orthonormal. Thus S is itself orthonormal. Since $S_i \subseteq S$ for every $i \in I$, this tells us that S is an *upper bound* for the chain $\mathcal{C}$.

Zorn's Lemma says that, given a partially ordered set, if every chain has an upper bound, then the set has a maximal element. Therefore, $\mathcal{S}$ must have a maximal element, i.e., there exists some orthonormal set $T \in \mathcal{S}$ which has the property that if $S \in \mathcal{S}$ and S is comparable to T (either $S \subseteq T$ or $T \subseteq S$), then we must have $S \subseteq T$.

We claim now that T is a complete orthonormal subset of H. If T is not complete, then $\overline{\text{span}}(T)$ is a proper subset of H, and hence there exists some nonzero vector $x \in \overline{\text{span}}(T)^{\perp}$. By rescaling, we may assume $\|x\| = 1$. But then $T' = T \cup \{x\}$ is orthonormal and $T \subsetneq T'$, contradicting the fact that T is a maximal element of $\mathcal{S}$. Hence T must be complete.

(b) We already know that if H contains an orthonormal basis then it is separable, so suppose that H is an arbitrary separable Hilbert space. By part (a), H contains a complete orthonormal subset T. By Exercise 1.58, any orthonormal subset of a separable Hilbert space must be countable, so we can write $T = \{e_n\}_{n \in \mathbf{N}}$. Thus T is a complete orthonormal sequence, and therefore it is an orthonormal basis by Theorem 1.50. □

Example 1.57. An example of a nonseparable Hilbert space is the space $\ell^2(\mathbf{R})$ consisting of all sequences $x = (x_i)_{i \in \mathbf{R}}$ indexed by the real line with at most countably many terms nonzero and such that $\sum |x_i|^2 < \infty$. The inner product on this space is $\langle x, y \rangle = \sum x_i \overline{y_i}$ (see Exercise 1.58). ♢

Exercises 3.6 and 3.7 in Chapter 3 deal with orthonormal sets in nonseparable Hilbert spaces.

Exercises

1.42. Prove the remaining parts of Theorems 1.49 and 1.50.

1.43. (a) Let M be a proper, closed subspace of a Hilbert space H. Given $x \in H \backslash M$, let p be the orthogonal projection of x onto M, and show that the vector $y = (x - p)/\|x - p\|$ satisfies $\|y\| = 1$, $y \in M^{\perp}$, and $\text{dist}(y, M) = \inf_{m \in M} \|y - m\| = 1$.

(b) Show that if H is an infinite-dimensional Hilbert space, then there exists an infinite orthonormal sequence $\{e_n\}$ in H, and no subsequence $\{e_{n_k}\}$

is Cauchy. Contrast this with the *Bolzano–Weierstrass Theorem*, which states that every bounded sequence in $\mathbf{F}^d$ has a convergent subsequence.

Remark: In another language, this problem shows that the closed unit ball $D = \{x \in H : \|x\| \le 1\}$ in H is *not compact* when H is infinite dimensional.

1.44. This exercise will extend Exercise 1.43 to an arbitrary normed linear space X.

(a) Prove *F. Riesz's Lemma*: If M is a proper, closed subspace of X and $\varepsilon > 0$, then there exists $x \in X$ with $\|x\| = 1$ such that $\mathrm{dist}(x, M) = \inf_{m \in M} \|x - m\| > 1 - \varepsilon$.

(b) Prove that if X is infinite dimensional then there exists a bounded sequence $\{x_n\}$ in X that has no convergent subsequences (hence the closed unit ball $D = \{x \in X : \|x\| \le 1\}$ is never compact in an infinite-dimensional Banach space).

1.45. Let $\{x_n\}$ be a finitely linearly independent sequence in a Hilbert space H. Show that there exists an orthogonal sequence $\{y_n\}$ in H such that $\mathrm{span}\{y_1, \dots, y_N\} = \mathrm{span}\{x_1, \dots, x_N\}$ for each $N \in \mathbf{N}$ (this is the *Gram–Schmidt orthogonalization procedure*).

1.46. We say that a sequence $\{x_n\}$ in a Banach space X is *ω-dependent* if there exist scalars c_n, not all zero, such that $\sum c_n x_n = 0$, where the series converges in the norm of X. A sequence is *ω-independent* if it is not ω-dependent (we explore ω-independent sequences in more detail in Chapter 5).

(a) Show that if $\{x_n\}$ is a basis for a Hilbert space H then $\{x_n\}$ is ω-independent.

(b) Let $\alpha, \beta \in \mathbf{C}$ be fixed nonzero scalars such that $|\alpha/\beta| > 1$. Let $\{\delta_n\}_{n \in \mathbf{N}}$ be the standard basis for ℓ^2, and define $x_0 = \delta_1$ and $x_n = \alpha\delta_n + \beta\delta_{n+1}$ for $n \in \mathbf{N}$. Prove that $\{x_n\}_{n \ge 0}$ is complete and finitely independent in ℓ^2, but is not ω-independent and therefore is not a basis for ℓ^2.

1.47. Let $\{x_n\}$ be a sequence in a Hilbert space H. Prove that the following two statements are equivalent.

(a) For each $m \in \mathbf{N}$ we have $x_m \notin \overline{\mathrm{span}}\{x_n\}_{n \ne m}$ (such a sequence is said to be *minimal*).

(b) There exists a sequence $\{y_n\}$ in H such that $\langle x_m, y_n \rangle = \delta_{mn}$ for all $m, n \in \mathbf{N}$ (we say that sequences $\{x_n\}$ and $\{y_n\}$ satisfying this condition are *biorthogonal*).

Show further that, in case these hold, the sequence $\{y_n\}$ is unique if and only if $\{x_n\}$ is complete.

Remark: Lemma 5.4 will establish an analogous result for Banach spaces.

1.48. Let $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ be the Haar wavelet. Compute the Fourier coefficients of ψ and apply the Plancherel Equality to show that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

1.49. This exercise provides a real-valued analogue of the orthonormal basis $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ for complex $L^2(\mathbf{T})$ discussed in Example 1.52. Let real $L^2(\mathbf{T})$ consist of all real-valued, 1-periodic functions on $\mathbf{R}$ that are square integrable on $[0,1]$. Assuming the fact that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is complete in complex $L^2(\mathbf{T})$, show that $\{1\} \cup \{\sqrt{2}\,\sin 2\pi nt\}_{n\in\mathbf{N}} \cup \{\sqrt{2}\,\cos 2\pi nt\}_{n\in\mathbf{N}}$ forms an orthonormal basis for real $L^2(\mathbf{T})$ if we take $\mathbf{F} = \mathbf{R}$, and forms an orthonormal basis for complex $L^2(\mathbf{T})$ if we take $\mathbf{F} = \mathbf{C}$.

1.50. (a) Prove that the Haar system discussed in Remark 1.55(a) is an orthonormal basis for $L^2(\mathbf{R})$.

(b) Prove that the Haar system given in Remark 1.55(b) is an orthonormal basis for $L^2[0,1]$.

1.51. Let $\{x_n\}$ be an orthonormal basis for a separable Hilbert space H. Show that if $\sum \|x_n - y_n\|^2 < 1$, then $\{y_n\}$ is complete in H. Show that this conclusion need not hold if $\sum \|x_n - y_n\|^2 = 1$.

1.52. Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H. Given a vector $x \in H$, show that $x \in \overline{\mathrm{span}}\{x_n\}$ if and only if $\|x\|^2 = \sum |\langle x, x_n\rangle|^2$.

1.53. Let M be a closed subspace of a Hilbert space H. If M is finite dimensional, let $\dim(M)$ be the dimension of M, otherwise set $\dim(M) = \infty$. Let P be the orthogonal projection of H onto M. Show that if $\{e_n\}$ is any orthonormal basis for H, then $\sum \|Pe_n\|^2 = \dim(M)$.

1.54. This result is due to Vitali. Let $\{f_n\}$ be an orthonormal sequence in $L^2[a,b]$. Show that $\{f_n\}$ is complete if and only if

$$\sum_{n=1}^{\infty} \left| \int_a^x f_n(t)\,dt \right|^2 = x - a, \qquad x \in [a,b].$$

1.55. Use the Vitali criterion (Exercise 1.54) to prove that the following two statements are equivalent.

(a) The trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is complete in $L^2(\mathbf{T})$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1-\cos 2\pi nx}{\pi^2 n^2} = x - x^2$ for $x \in [0,1]$.

Remark: If we assume *Euler's formula*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

then statement (b) reduces to

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{\pi^2 n^2} = x^2 - x + \frac{1}{6}, \qquad x \in [0,1]. \tag{1.15}$$

Conversely, equation (1.15) implies Euler's formula by taking $x = 0$. For a proof that equation (1.15) holds, see Exercise 13.25.

1.56. This result is due to Dalzell. Let $\{f_n\}$ be an orthonormal sequence in $L^2[a,b]$. Show that $\{f_n\}$ is complete if and only if

$$\sum_{n=1}^{\infty} \int_a^b \left| \int_a^x f_n(t)\,dt \right|^2 = \frac{(b-a)^2}{2}.$$

1.57. This result is due to Boas and Pollard [BP48]. Suppose that $\{f_n\}_{n\in\mathbf{N}}$ is an orthonormal basis for $L^2[a,b]$. Show that there exists a function $m \in L^\infty[a,b]$ such that $\{mf_n\}_{n\geq 2}$ is complete in $L^2[a,b]$.

1.58. (a) Let H be a Hilbert space. Show that if H contains an uncountable orthonormal subset, then H is not separable.

(b) Prove that the space $\ell^2(\mathbf{R})$ defined in Example 1.57 is a Hilbert space. Given $t \in \mathbf{R}$, define $e_t(t) = 1$ and $e_t(i) = 0$ for $i \neq t$. Show that $\{e_t\}_{t\in\mathbf{R}}$ is a complete uncountable orthonormal system for $\ell^2(\mathbf{R})$, and conclude that $\ell^2(\mathbf{R})$ is nonseparable.

1.59. For each $\xi \in \mathbf{R}$, define a function $e_\xi\colon \mathbf{R} \to \mathbf{C}$ by $e_\xi(t) = e^{2\pi i\xi t}$. Let $H = \operatorname{span}\{e_\xi\}_{\xi\in\mathbf{R}}$, i.e., H consists of all finite linear combinations of the functions e_ξ. Show that

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,\overline{g(t)}\,dt, \qquad f, g \in H,$$

defines an inner product on H, and $\{e_\xi\}_{\xi\in\mathbf{R}}$ is an uncountable orthonormal system in H.

Remark: The completion $\widetilde{H}$ of H is an important nonseparable Hilbert space. In particular, it contains the class of *almost periodic functions* [Kat04], and it plays an important role in Wiener's theory of *generalized harmonic analysis* [Wie33]. Since $\operatorname{span}\{e_\xi\}_{\xi\in\mathbf{R}} = H$ and H is dense in $\widetilde{H}$, $\{e_\xi\}_{\xi\in\mathbf{R}}$ is a complete orthonormal system in $\widetilde{H}$.

1.7 Operators

Let X and Y be normed linear spaces. An *operator* is simply another name for a function $L\colon X \to Y$ (although sometimes this terminology is restricted to functions that are linear).

In addition to the basic terminology for functions reviewed in the opening section of General Notation, we introduce the following notation for operators.

Definition 1.58. Let X and Y be normed linear spaces, and let $L\colon X \to Y$ be an operator. We write either Lx or $L(x)$ to denote the action of L on an element $x \in X$.

(a) L is *linear* if

$$\forall\, x, y \in X, \quad \forall\, a, b \in \mathbf{F}, \quad L(ax + by) \;=\; aL(x) + bL(y).$$

(b) L is *antilinear* if

$$\forall\, x, y \in X, \quad \forall\, a, b \in \mathbf{F}, \quad L(ax + by) \;=\; \bar{a}L(x) + \bar{b}L(y).$$

(c) L is *continuous* if $x_n \to x$ in X implies $L(x_n) \to L(x)$ in Y.

(d) The *kernel* or *nullspace* of L is $\ker(L) = \{f \in X : Lx = 0\}$.

(e) The *rank* of L is the dimension of its range: $\operatorname{rank}(L) = \dim(\operatorname{range}(L))$. We say that L is a *finite-rank operator* if its range is finite dimensional.

(f) L is a *functional* if $Y = \mathbf{F}$.

(g) Two operators A, $B\colon X \to X$ *commute* if $AB = BA$. ♢

Continuity can be equivalently stated in the following abstract form (see Exercise 1.60). Indeed, for functions on abstract topological spaces, this is usually taken to be the *definition* of continuity.

Theorem 1.59. *Let X, Y be normed spaces, and let $f\colon X \to Y$ be given. Then f is continuous if and only if $f^{-1}(V)$ is open in X for each open set $V \subseteq Y$.* ♢

A nonzero linear operator $L\colon X \to Y$ cannot map X into a bounded subset of Y, simply because $\|L(cx)\|_Y = |c|\,\|Lx\|_Y$ and $|c|$ can be arbitrarily large. However, we can consider "boundedness" in another way, by examining the relationship between the size of $\|x\|_X$ and $\|Lx\|_Y$. If there is a limit to how large $\|Lx\|_Y$ can be in comparison to $\|x\|_X$, then we say that L is bounded. We quantify this notion and introduce some related terminology in the next definition.

Definition 1.60. Let X and Y be normed linear spaces, and let $L\colon X \to Y$ be a linear operator.

(a) L is *bounded* if there exists a finite $K \ge 0$ such that

$$\forall\, x \in X, \quad \|Lx\|_Y \le K\,\|x\|_X.$$

(b) The *operator norm*, or simply the *norm*, of L is

$$\|L\|_{X \to Y} \;=\; \sup_{\|x\|_X = 1} \|Lx\|_Y.$$

(c) We say that L is *norm-preserving* or *isometric* if $\|Lx\|_Y = \|x\|_X$ for every $x \in X$.

(d) If L is linear, bijective, and isometric, then we call L an *isometric isomorphism.*

(e) We say that X and Y are *isometrically isomorphic*, denoted $X \cong Y$, if there exists an isometric isomorphism $L\colon X \to Y$. ♢

Most of the operators that we will deal with will be linear, but on occasion we will encounter antilinear operators. The properties of and terminology for antilinear operators are similar to those for linear operators. For example, we say that X and Y are *antilinearly isometrically isomorphic* if there exists an antilinear bijective isometry $L\colon X \to Y$. Of course, the distinction between linear and antilinear operators is only an issue over the complex field.

Usually it is clear from context which space a norm is being applied to, and so we usually just write $\|L\|$ for the operator norm of L. With similar implicit notation for the norm on X and Y, we can write the definition of the operator norm as

$$\|L\| \;=\; \sup_{\|x\|=1} \|Lx\|,$$

but it is important to note that there are three different meanings of the symbol $\|\cdot\|$ on the line above: $\|x\|$ is the norm of $x \in X$, $\|Lx\|$ is the norm of $Lx \in Y$, and $\|L\|$ is the operator norm of L. We will see in Theorem 1.67 that the operator norm is a true norm on the vector space $\mathcal{B}(X,Y)$ of all bounded linear operators that map X into Y.

Here are some of the basic properties of linear operators and the operator norm (see Exercise 1.61).

Theorem 1.61. *Let X, Y be normed linear spaces, and let $L\colon X \to Y$ be a linear operator.*

(a) $L(0) = 0$, *and L is injective if and only if* $\ker L = \{0\}$. *In particular, if L is an isometry then it is injective.*

(b) *If L is a bijection then the inverse map $L^{-1}\colon Y \to X$ is also a linear bijection.*

(c) *L is bounded if and only if $\|L\| < \infty$.*

(d) *If L is bounded then*

$$\|Lx\| \le \|L\|\,\|x\|, \qquad x \in X,$$

and $\|L\|$ is the smallest real number K such that $\|Lx\| \le K\|x\|$ for all $x \in X$.

(e) $\|L\| = \sup\limits_{\|x\|\le 1} \|Lx\| = \sup\limits_{x\neq 0} \dfrac{\|Lx\|}{\|x\|}.$ ♢

Example 1.62. Consider a linear operator on a finite-dimensional real vector space, say $L\colon \mathbf{R}^n \to \mathbf{R}^m$. For simplicity, impose the Euclidean norm on both $\mathbf{R}^n$ and $\mathbf{R}^m$. If we let $S = \{x \in \mathbf{R}^n : \|x\| = 1\}$ be the unit sphere in $\mathbf{R}^n$, then $L(S) = \{Lx : \|x\| = 1\}$ is a (possibly degenerate) ellipsoid in $\mathbf{R}^m$. The supremum in the definition of the operator norm of L is achieved in this case, and is the length of a semimajor axis of the ellipsoid $L(S)$. Thus, $\|L\|$ is the "maximum distortion" of the unit sphere under L, illustrated for the case $m = n = 2$ in Figure 1.3. ◇

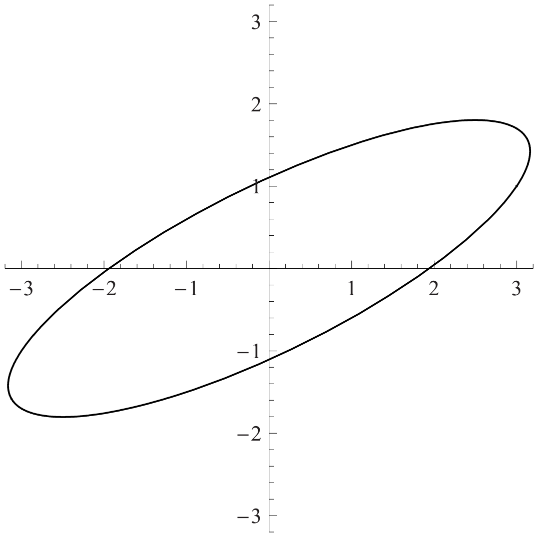

Fig. 1.3. Image of the unit circle under a particular linear operator $L\colon \mathbf{R}^2 \to \mathbf{R}^2$. The operator norm $\|L\|$ of L is the length of a semimajor axis of the ellipse.

A crucial property of linear operators on normed linear spaces is that boundedness and continuity are equivalent.

Theorem 1.63. *If X, Y are normed linear spaces and $L\colon X \to Y$ is a linear operator, then*

$$L \text{ is continuous} \quad \Longleftrightarrow \quad L \text{ is bounded.}$$

Proof. $\Leftarrow$. If L is bounded and $x_n \to x$, then we have

$$\|Lx - Lx_n\| \;=\; \|L(x - x_n)\| \;\le\; \|L\|\,\|x - x_n\| \;\to\; 0 \quad \text{as } n \to \infty,$$

so L is continuous.

$\Rightarrow$. Suppose that L is linear and continuous but unbounded. Then we have $\|L\| = \infty$, so there must exist $x_n \in X$ with $\|x_n\| = 1$ such that $\|Lx_n\| \ge n$. Set $y_n = x_n/n$. Then $\|y_n - 0\| = \|y_n\| = \|x_n\|/n \to 0$, so $y_n \to 0$. Since L is continuous and linear, this implies that $Ly_n \to L0 = 0$. Consequently $\|Ly_n\| \to \|0\| = 0$ by the continuity of the norm (Exercise 1.2). However,

$$\|Ly_n\| \;=\; \frac{1}{n}\,\|Lx_n\| \;\geq\; \frac{1}{n}\cdot n \;=\; 1$$

for all n, which is a contradiction. Hence L must be bounded. $\square$

As a consequence, we use the terms *continuous* and *bounded* interchangeably when speaking of linear operators.

The "metaspace" of all bounded linear operators that map one normed space into another plays an important role in functional analysis.

Definition 1.64. Given normed linear spaces X, Y, we define

$$\mathcal{B}(X,Y) \;=\; \bigl\{L\colon X \to Y \;:\; L \text{ is bounded and linear}\bigr\}.$$

If $X = Y$ then we write $\mathcal{B}(X) = \mathcal{B}(X,X)$. $\diamondsuit$

Example 1.65. Continuing the discussion in Example 1.62, any linear operator $A\colon \mathbf{F}^n \to \mathbf{F}^m$ has the form $x \mapsto Ax$ where A is an $m \times n$ matrix with entries in $\mathbf{F}$. All such operators are bounded, and Exercise 1.70 derives some explicit formulas for the operator norm (which depends on the norm that we choose for $\mathbf{F}^n$ and $\mathbf{F}^m$). We usually identify the operator A with the matrix A. In this sense, $\mathcal{B}(\mathbf{F}^n, \mathbf{F}^m)$ is identified with the set of all $m \times n$ matrices with entries in $\mathbf{F}$. $\diamondsuit$

The following result states that any linear operator mapping a finite-dimensional vector space V into a normed space Y must be bounded (see Exercise 1.68). Recall from Example 1.6 and Theorem 1.8 that every finite-dimensional space has norms, and all of these norms are equivalent.

Theorem 1.66. *If V is a finite-dimensional vector space and Y is a normed linear space, then any linear operator $T\colon V \to Y$ is bounded.* $\diamondsuit$

Be aware that the situation for linear operators with finite-dimensional *ranges* is quite different. In Example 4.2 we will show that if X is an infinite-dimensional normed linear space, then there exists a linear operator $\mu\colon X \to \mathbf{F}$ that is unbounded!

Now we show that the operator norm is a norm on $\mathcal{B}(X,Y)$, and $\mathcal{B}(X,Y)$ is complete whenever Y is complete.

Theorem 1.67. *Let X, Y, and Z be normed spaces.*

(a) *$\mathcal{B}(X,Y)$ is a normed linear space with respect to the operator norm.*

(b) *If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space with respect to the operator norm.*

(c) *The operator norm is* submultiplicative, *i.e., if $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$, then $BA \in \mathcal{B}(X,Z)$ and $\|BA\| \leq \|B\|\,\|A\|$.*

Proof. We sketch the proof of statement (b), and assign the remainder of the proof as Exercise 1.69. The idea is similar to our proof that ℓ^p is complete (see Theorem 1.14). Given a Cauchy sequence $\{A_n\}$, we will use the Cauchyness to construct a candidate limit A that A_n converges to "pointwise." Then we show that A_n actually converges to the candidate limit A *in norm.* "Pointwise convergence" of operators means $A_nx \to Ax$ for each $x \in X$, but by itself this usually does not imply that A_n converges to A *in operator norm.*

Assume that X is normed and Y is Banach, and let $\{A_n\}$ be a sequence of operators in $\mathcal{B}(X,Y)$ that is Cauchy with respect to the operator norm. Given any particular $x \in X$, we have

$$\|A_mx - A_nx\| \;\le\; \|A_m - A_n\|\,\|x\|.$$

Therefore $\{A_nx\}$ is a Cauchy sequence in Y. Since Y is complete, this sequence must converge, say $A_nx \to y \in Y$. Define $Ax = y$. This gives us our candidate limit operator A, and we leave as an exercise the task of showing that A defined in this way is linear and bounded.

It remains to show that $A_n \to A$ in operator norm. Fix any $\varepsilon > 0$. Since $\{A_n\}$ is Cauchy, there exists an N such that

$$m, n > N \quad\implies\quad \|A_m - A_n\| < \frac{\varepsilon}{2}.$$

Choose any $x \in X$ with $\|x\| = 1$. Then since $A_mx \to Ax$, there exists an $m > N$ such that

$$\|Ax - A_mx\| \;<\; \frac{\varepsilon}{2}.$$

Hence for any $n > N$ we have

$$\begin{aligned}\|Ax - A_nx\| &\le \|Ax - A_mx\| + \|A_mx - A_nx\| \\ &\le \|Ax - A_mx\| + \|A_m - A_n\|\,\|x\| \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon.\end{aligned}$$

Taking the supremum over all unit vectors, we conclude that $\|A - A_n\| \le \varepsilon$ for all $n > N$, so $A_n \to A$ in operator norm. □

For operators that map a space into itself, we can define the notion of eigenvalues and eigenvectors.

Definition 1.68 (Eigenvalues and Eigenvectors). Let X be a normed space and $L\colon X \to X$ a linear operator.

(a) A scalar λ is an *eigenvalue* of L if there exists a nonzero vector $x \in X$ such that $Lx = \lambda x$.

(b) A nonzero vector $x \in X$ is an *eigenvector* of L if there exists a scalar λ such that $Lx = \lambda x$. If x is an eigenvector of L corresponding to the eigenvalue λ, then we often say that x is a λ-*eigenvector* of L.

(c) If λ is an eigenvalue of L, then $\ker(L-\lambda I)$ is called the *eigenspace* corresponding to λ, or the λ-*eigenspace* for short. ♢

Eigenvalues and eigenvectors are especially important when dealing with *self-adjoint* operators on Hilbert spaces, see Section 2.4.

We end this section by looking at isometric operators acting on Hilbert spaces. The next result shows that a linear isometry that maps one Hilbert space into another must preserve inner products as well as norms.

Theorem 1.69. *Let H, K be Hilbert spaces, and let $L\colon H \to K$ be a linear mapping. Then L is an isometry if and only if $\langle Lx, Ly\rangle = \langle x, y\rangle$ for all x, $y \in H$.*

Proof. $\Rightarrow$. This follows immediately from the fact that $\|x\|^2 = \langle x, x\rangle$.

$\Leftarrow$. We assume $\mathbf{F} = \mathbf{C}$, as the proof for real scalars is almost identical. Suppose that L is an isometry, and fix x, $y \in H$. Then for any scalar $c \in \mathbf{C}$ we have by the Polar Identity and the fact that L is isometric that

$$\begin{aligned}
\|x\|^2 + 2\,\mathrm{Re}(\bar{c}\,\langle x, y\rangle) + |c|^2\,\|y\|^2 &= \|x + cy\|^2 \\
&= \|Lx + cLy\|^2 \\
&= \|Lx\|^2 + 2\,\mathrm{Re}(\bar{c}\,\langle Lx, Ly\rangle) + |c|^2\,\|Ly\|^2 \\
&= \|x\|^2 + 2\,\mathrm{Re}(\bar{c}\,\langle Lx, Ly\rangle) + |c|^2\,\|y\|^2.
\end{aligned}$$

Thus $\mathrm{Re}(\bar{c}\,\langle Lx, Ly\rangle) = \mathrm{Re}(\bar{c}\,\langle x, y\rangle)$ for every $c \in \mathbf{C}$. Taking $c = 1$ and $c = i$, this implies that $\langle Lx, Ly\rangle = \langle x, y\rangle$. □

We earlier defined an isometric isomorphism to be a bijective isometry between normed spaces. For operators on Hilbert spaces, we call such operators *unitary* (although this word is sometimes reserved for the case $H = K$).

Definition 1.70 (Unitary Operator). If H, K are Hilbert spaces and $L\colon H \to K$ is an isometric isomorphism, then L is called a *unitary operator*, and in this case we say that H and K are *unitarily isomorphic*. ♢

Exercises

1.60. Prove Theorem 1.59.

1.61. Prove Theorem 1.61.

1.62. If X, Y are normed spaces and $L\colon X \to Y$ is continuous, show that $\ker(L)$ is a closed subspace of X.

1.63. Let X be a normed space and suppose $L \in \mathcal{B}(X)$. Show that if λ is an eigenvalue of L, then $|\lambda| \le \|L\|$.

1.64. (a) Define $L\colon \ell^2 \to \ell^2$ by $L(x) = (x_2, x_3, \dots)$. Prove that this *left-shift operator* is bounded, linear, surjective, not injective, is not an isometry, and satisfies $\|L\| = 1$. Find all of the eigenvalues and eigenvectors of L.

(b) Define $R\colon \ell^2 \to \ell^2$ by $R(x) = (0, x_1, x_2, x_3, \dots)$. Prove that this *right-shift operator* is bounded, linear, injective, not surjective, and is an isometry. Find all of the eigenvalues and eigenvectors of R.

(c) Compute LR and RL and show that $LR \neq RL$. Contrast this with the fact that in finite dimensions, if A, $B\colon \mathbf{F}^n \to \mathbf{F}^n$ are linear maps (hence correspond to multiplication by $n \times n$ matrices), then $AB = I$ if and only if $BA = I$.

1.65. Recall from Exercise 1.22 that $C_b(\mathbf{R})$ is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$, and $C_b^1(\mathbf{R})$ is a Banach space with respect to the norm $\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty$.

(a) Define $D\colon C_b^1(\mathbf{R}) \to C_b(\mathbf{R})$ by $Df = f'$, and show that D is a bounded operator.

(b) Let $D\colon C_b^1(\mathbf{R}) \to C_b(\mathbf{R})$ be the same operator as in part (a), but replace the norm on $C_b^1(\mathbf{R})$ by the L^∞-norm. Show that D is now an unbounded operator.

1.66. Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space H, and fix a sequence of scalars $\lambda = (\lambda_n) \in \ell^\infty$. Define

$$M_\lambda x \;=\; \sum_{n=1}^{\infty} \lambda_n \, \langle x, e_n \rangle \, e_n, \qquad x \in H,$$

and prove the following facts.

(a) M_λ is a bounded mapping of H into itself, and its operator norm is $\|M_\lambda\| = \|\lambda\|_{\ell^\infty}$.

(b) Each λ_n is an eigenvalue for M_λ with corresponding eigenvector e_n.

(c) M_λ is injective if and only if $\lambda_n \neq 0$ for every n.

(d) M_λ is surjective if and only if $\inf_n |\lambda_n| > 0$. Further, if $\inf |\lambda_n| = 0$ but $\lambda_n \neq 0$ for every n then $\operatorname{range}(M_\lambda)$ is a dense but proper subspace of H.

1.67. Let $E \subseteq \mathbf{R}$ be Lebesgue measurable, and choose $1 \le p < \infty$ and $m \in L^\infty(\mathbf{R})$. Define $T_m\colon L^p(\mathbf{R}) \to L^p(\mathbf{R})$ by $T_m f = fm$, i.e., pointwise multiplication of f by the function m. Show that T_m is bounded and $\|T_m\| = \|m\|_\infty$.

1.68. Prove Theorem 1.66.

1.69. Fill in the details and finish the proof of Theorem 1.67.

1.70. Let A be an $m \times n$ matrix with entries in $\mathbf{F}$, which we view as a linear operator $A\colon \mathbf{F}^n \to \mathbf{F}^m$. The operator norm of A depends on the choice of norm for $\mathbf{F}^n$ and $\mathbf{F}^m$. Show that if the norm on $\mathbf{F}^n$ and $\mathbf{F}^m$ is $|\cdot|_1$ then

$$\|A\| \;=\; \max_{j=1,\dots,n} \biggl\{\sum_{i=1}^m |a_{ij}|\biggr\},$$

and if the norm on $\mathbf{F}^n$ and $\mathbf{F}^m$ is $|\cdot|_\infty$ then

$$\|A\| \;=\; \max_{i=1,\dots,m} \biggl\{\sum_{j=1}^n |a_{ij}|\biggr\}.$$

1.71. Show that if H, K are separable Hilbert spaces, then H and K are isometrically isomorphic.

1.72. Let Y be a dense subspace of a normed space X, and let Z be a Banach space. Given $L \in \mathcal{B}(Y,Z)$, show that there exists a unique operator $\widetilde{L} \in \mathcal{B}(X,Z)$ whose restriction to Y is L. Prove that $\|\widetilde{L}\| = \|L\|$.

1.8 Bounded Linear Functionals and the Dual Space

The space of bounded linear functionals on a normed space is especially important in functional analysis.

Definition 1.71 (Dual Space). Given a normed linear space X, the space of all bounded linear functionals on X is the *dual space* of X, and is denoted by

$$X^* \;=\; \mathcal{B}(X,\mathbf{F}) \;=\; \bigl\{L\colon X \to \mathbf{F} \,:\, L \text{ is bounded and linear}\bigr\}. \qquad \diamondsuit$$

Since $\mathbf{F}$ is complete, Theorem 1.67 implies that the dual space X^* of a normed space X is complete, even if X is not.

Notation 1.72 (Bilinear Form Notation). We often use Greek letters such as λ, μ, Λ to denote continuous linear functionals. It is also convenient to use the symbol x^* to denote a typical element of X^*. When using this notation it is important to note that x^* is simply a functional on X, and is not somehow determined from some specific element $x \in X$. That is, x^* is a mapping from X to $\mathbf{F}$, and the value of x^* at an arbitrary point $x \in X$ is $x^*(x)$.

Given a linear functional x^*, we often denote the action of x^* on an element x in its domain by

$$\langle x, x^*\rangle \;=\; x^*(x). \tag{1.16}$$

Note that this notation does not represent an inner product, but rather stands for the value of the functional x^* evaluated at the point x.

Using this notation, the linearity of x^* is expressed by the statement

$$\forall\, x, y \in X, \quad \forall\, a, b \in \mathbf{F}, \quad \langle ax + by, x^* \rangle \;=\; a\langle x, x^* \rangle + b\langle y, x^* \rangle.$$

Similarly, the continuity of x^* is expressed in this notation by the statement

$$x_n \to x \quad \Longrightarrow \quad \langle x_n, x^* \rangle \to \langle x, x^* \rangle.$$

Since the norm on the scalar field $\mathbf{F}$ is simply the absolute value, the operator norm of a linear functional x^* is given in this notation by the formula

$$\|x^*\| \;=\; \sup_{\|x\|_X = 1} |\langle x, x^* \rangle|.$$

Sometimes we write $\|x^*\|_{X^*}$ to emphasize that this is the norm of x^* as an element of X^*.

We refer to the notation $\langle \cdot, \cdot \rangle$ given in equation (1.16) as a *bilinear form notation*, because not only is it linear as a function of x, but it is also linear as a function of x^*. That is, with $x \in X$ fixed we have

$$\forall\, x^*, y^* \in X^*, \quad \forall\, a, b \in \mathbf{F}, \quad \langle x, ax^* + by^* \rangle \;=\; a\langle x, x^* \rangle + b\langle x, y^* \rangle.$$

This bilinearity is quite convenient for the purposes of Banach space theory, although it does create certain notational ambiguities that we will address in Notation 1.74. $\diamondsuit$

It is often difficult to explicitly characterize the dual space X^* of a given Banach space X. However, it is possible to characterize the dual spaces of some particular Banach spaces. For example, fix $1 \le p \le \infty$ and consider the Banach space ℓ^p. Let p' be the dual index to p as defined in Notation 1.11, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Given $y \in \ell^{p'}$, define $\mu_y \colon \ell^p \to \mathbf{F}$ as follows (we show both the standard notation $\mu_y(x)$ and the bilinear form notation $\langle x, \mu_y \rangle$):

$$\mu_y(x) \;=\; \langle x, \mu_y \rangle \;=\; \sum_n x_n y_n, \qquad x \in \ell^p. \tag{1.17}$$

The series in equation (1.17) converges by Hölder's Inequality, and we have

$$|\langle x, \mu_y \rangle| \;\le\; \|x\|_{\ell^p}\, \|y\|_{\ell^{p'}}.$$

Taking the supremum over all unit vectors $x \in X$, we conclude that $\|\mu_y\| \le \|y\|_{\ell^{p'}} < \infty$. Hence μ_y is a bounded linear functional on ℓ^p. We will shortly prove that we actually have $\|\mu_y\| = \|y\|_{\ell^{p'}}$. Consequently, each vector $y \in \ell^{p'}$ determines a continuous linear functional $\mu_y \in (\ell^p)^*$ whose operator norm is exactly the norm of y. Therefore the mapping $y \mapsto \mu_y$ is an isometric map of $\ell^{p'}$ into $(\ell^p)^*$. The next theorem will show that if $1 \le p < \infty$ then every continuous linear functional on ℓ^p has this form. That is, $y \mapsto \mu_y$ is an isometric isomorphism of $\ell^{p'}$ onto $(\ell^p)^*$ when p is finite. Therefore we often write $\ell^{p'} = (\ell^p)^*$ and $y = \mu_y$, although these are technically identifications rather than true equalities. For $p = \infty$ the mapping $y \mapsto \mu_y$ is an isometry but is not surjective (see Exercise 2.8). Therefore, identifying y with μ_y, we have $\ell_1 \subsetneq (\ell^\infty)^*$.

Now we make these statements precise.

Theorem 1.73 (Dual Space of ℓ^p). *Fix $1 \le p \le \infty$. For each $y \in \ell^{p'}$, let μ_y be as in equation* (1.17). *Then the mapping $T\colon \ell^{p'} \to (\ell^p)^*$ given by $T(y) = \mu_y$ is a linear isometry of $\ell^{p'}$ into $(\ell^p)^*$, and it is an isometric isomorphism if $p < \infty$.*

Proof. We will consider the case $1 < p < \infty$; the cases $p = 1$ and $p = \infty$ are similar (see Exercise 1.73). We have already seen that T maps $\ell^{p'}$ into $(\ell^p)^*$ with $\|\mu_y\| \le \|y\|_{\ell^{p'}}$ for each $y \in \ell^{p'}$, so it remains to show that T is isometric and surjective.

To show that T is isometric, we must show that $\|\mu_y\| = \|y\|_{\ell^{p'}}$ for each $y \in \ell^{p'}$. If $y = 0$ then $\|\mu_y\| = 0 = \|y\|_{\ell^p}$, so assume that $y \neq 0$. If we can find a particular unit vector $x \in \ell^p$ such that $|\langle x, \mu_y\rangle| = \|y\|_{\ell^{p'}}$, then we will have

$$\|\mu_y\| \;=\; \sup_{\|z\|_{\ell^p}=1} |\langle z, \mu_y\rangle| \;\ge\; |\langle x, \mu_y\rangle| \;=\; \|y\|_{\ell^{p'}},$$

which is the inequality we need to conclude that T is isometric. To create this vector x, let $\alpha_k \in \mathbf{F}$ be the scalar of unit modulus such that $\alpha_k y_k = |y_k|$, and define $x = (x_k)$ by

$$x_k \;=\; \frac{\alpha_k\, |y_k|^{p'-1}}{\|y\|_{\ell^{p'}}^{p'-1}}, \qquad k \in \mathbf{N}.$$

Using the fact that $(p'-1)\,p = p'$, we have

$$\|x\|_{\ell^p}^p \;=\; \sum_k \left(\frac{|y_k|^{p'-1}}{\|y\|_{\ell^{p'}}^{p'-1}}\right)^p \;=\; \sum_k \frac{|y_k|^{p'}}{\|y\|_{\ell^{p'}}^{p'}} \;=\; 1,$$

so x is indeed a unit vector. Also,

$$|\langle x, \mu_y\rangle| \;=\; \sum_k x_k y_k \;=\; \sum_k \frac{\alpha_k\, |y_k|^{p'-1}}{\|y\|_{\ell^{p'}}^{p'-1}}\, y_k \;=\; \frac{\|y\|_{p'}^{p'}}{\|y\|_{\ell^{p'}}^{p'-1}} \;=\; \|y\|_{\ell^{p'}}.$$

This shows that $\|\mu_y\| \ge \|y\|_{\ell^{p'}}$, and therefore T is an isometry.

To show that T is surjective, choose any bounded linear functional $\mu \in (\ell^p)^*$. Let $\{\delta_n\}$ be the standard basis for ℓ^p. For each $k \in \mathbf{N}$ let $y_k = \langle \delta_k, \mu\rangle$, and set $y = (y_k)$. Given any $x = (x_k) \in \ell^p$, we have $x = \sum x_k \delta_k$, where the series converges in the norm of ℓ^p. Since μ is continuous on ℓ^p, it therefore follows from Exercise 1.76 that

$$\langle x, \mu\rangle \;=\; \left\langle \sum_k x_k \delta_k,\; \mu \right\rangle \;=\; \sum_k x_k \,\langle \delta_k, \mu\rangle \;=\; \sum_k x_k y_k.$$

Now we will appeal to Theorem 2.24. We have not proved that theorem yet, but we will give its proof later, and its proof does not depend on what we are doing here so we are not reasoning circularly. Theorem 2.24 tells us that if $\sum x_k y_k$ converges for each $x \in \ell^p$ then we must have $y \in \ell^{p'}$. Consequently $\mu = \mu_y = T(y)$, so T is surjective. □

It is likewise true that $L^p(E)^* = L^{p'}(E)$ for $1 \le p < \infty$, and $L^1(E) \subsetneq L^\infty(E)^*$. More precisely, each $g \in L^{p'}(E)$ determines a continuous linear functional $\mu_g \in L^p(E)^*$ by the formula

$$\mu_g(f) \;=\; \langle f, \mu_g \rangle \;=\; \int_E f(t)\, g(t)\, dt, \qquad f \in L^p(E).$$

Indeed, μ_g is bounded because we have by Hölder's Inequality that

$$|\langle f, \mu_g \rangle| \;\le\; \int_E |f(t)|\, |g(t)|\, dt \;\le\; \|f\|_{L^p}\, \|g\|_{L^{p'}}.$$

This tells us that $\|\mu_g\| \le \|g\|_{L^{p'}}$, and it can be shown that equality holds. Therefore the mapping $g \mapsto \mu_g$ is an injective embedding of $L^{p'}(E)$ into $L^p(E)^*$, and it is surjective if p is finite [Fol99].

Another example of a function space whose dual has a nice characterization is $C_0(\mathbf{R})$. Although there are several results that go by the name "Riesz Representation Theorem," one version asserts that $C_0(\mathbf{R})^* \cong M_b(\mathbf{R})$, the space of bounded Radon measures on $\mathbf{R}$ [Fol99, Thm. 7.17]. The discrete version of this result takes the form $c_0{}^* \cong \ell^1$. This fact is easier to prove than the identification $C_0(\mathbf{R})^* \cong M_b(\mathbf{R})$, and is assigned as Exercise 1.75.

Notation 1.74. Consider again the identification of $\ell^{p'}$ with $(\ell^p)^*$ derived in Theorem 1.73. For simplicity of discussion, we restrict our attention to the case $1 \le p < \infty$, where we have $(\ell^p)^* = \ell^{p'}$.

Given $y \in \ell^{p'}$, if we follow the practice of identifying y with the functional $\mu_y \in (\ell^p)^*$ that it determines, and consider the bilinear form notation introduced in Notation 1.72, then we can write the action of y as a linear functional on an element $x \in \ell^p$ in any of these ways:

$$\langle x, y \rangle \;=\; \langle x, \mu_y \rangle \;=\; \mu_y(x) \;=\; \sum_k x_k y_k, \qquad x \in \ell^p.$$

Our preferred notation for the rest of this volume will be $\langle x, y \rangle$.

Unfortunately, this does create a notational ambiguity, because if $p = 2$ then $p' = 2$ and we have already used the notation $\langle \cdot, \cdot \rangle$ to denote the inner product on ℓ^2. Given $x,\, y \in \ell^2$, we now have two conflicting meanings for the notation $\langle x, y \rangle$. On the one hand, $\langle x, y \rangle$ denotes the inner product of x with y, which is

$$\langle x, y \rangle \;=\; \sum_k x_k \overline{y_k}. \tag{1.18}$$

On the other hand, since $y \in \ell^2$ it determines a linear functional μ_y on ℓ^2, and $\langle x, y\rangle$ denotes the action of this linear functional on x, which is

$$\langle x, y\rangle \;=\; \sum_k x_k y_k. \tag{1.19}$$

This ambiguity is usually not a problem in practice. If we are dealing with a generic Banach space X, then we assume that the notation $\langle x, x^*\rangle$ represents the bilinear form notation for a bounded linear functional $x^* \in X^*$ acting on an element $x \in X$. However, if we know that H is a Hilbert space, then $\langle x, y\rangle$ will denote the inner product of $x,\, y \in H$.

The main advantage of the bilinear form notation $\langle x, x^*\rangle$ is that it is linear both as a function of x and x^*, and for most purposes in Banach space theory this is quite convenient. In contrast, an inner product $\langle x, y\rangle$ is linear in x but *antilinear* in y, because $\langle x, ay+bz\rangle = \bar{a}\langle x, y\rangle + \bar{b}\langle x, z\rangle$. In some areas, especially harmonic analysis and the theory of distributions, it is more convenient to use a functional notation $\langle x, x^*\rangle$ that directly extends the inner product notation in the sense that it is linear as a function of $x \in X$ but antilinear as a function of $x^* \in X^*$ (indeed, this is the viewpoint taken in [Heil]). ◇

As a special case of Theorem 1.73, $(\ell^2)^*$ can be isometrically identified with ℓ^2. The next result states that if H is any Hilbert space, then H^* can be isometrically identified with H. In particular, any continuous linear functional on H is formed by taking the inner product with some unique element of H. However, since we are dealing with an inner product, which by definition is antilinear in the second variable, this identification of H with H^* is antilinear rather than linear. This result is one of several that are known as the *Riesz Representation Theorem*.

Theorem 1.75 (Riesz Representation Theorem). *Let H be a Hilbert space, and for each $y \in H$ define $\mu_y \colon H \to \mathbf{F}$ by*

$$\mu_y(x) \;=\; \langle x, y\rangle, \qquad x \in H.$$

Then the following statements hold.

(a) *$\mu_y \in H^*$ and $\|\mu_y\| = \|y\|$ for each $y \in H$.*

(b) *The mapping $y \mapsto \mu_y$ is an antilinear bijective isometry of H onto H^*.*

Proof. Let $T(y) = \mu_y$. To see that T is antilinear, fix $y,\, z \in H$ and $a,\, b \in \mathbf{F}$. Then for $x \in H$ we have that

$$\mu_{ay+bz}(x) \;=\; \langle x, ay+bz\rangle \;=\; \bar{a}\langle x, y\rangle + \bar{b}\langle x, z\rangle \;=\; \bar{a}\mu_y(x) + \bar{b}\mu_z(x),$$

which means that

$$T(ax+by) \;=\; \mu_{ay+bz} \;=\; \bar{a}\mu_y + \bar{b}\mu_z \;=\; \bar{a}T(y) + \bar{b}T(z).$$

We assign the rest of the proof as Exercise 1.77. □

As before, we usually "identify" the element $y \in H$ with the functional $\mu_y \in H^*$. Hence we write simply $y = \mu_y$ and say that y "is" a linear functional on H, when we actually mean that y determines the functional $\mu_y(x) = \langle x, y \rangle$. In the same way, we identify H with H^* and write $H = H^*$. In this sense, all Hilbert spaces are self-dual. Again, there is a possible source of confusion deriving from the fact that the identification of H with H^* given by $y \mapsto \mu_y$ is antilinear, in contrast to the linear identification made in Theorem 1.73, but this is usually not a problem in practice.

Exercises

1.73. Complete the proof of Theorem 1.73 for $p = 1$ and $p = \infty$.

1.74. Let X be a normed space. Given $\mu \in X^*$, $\mu \neq 0$, show that if $z \notin \ker(\mu)$, then every $x \in X$ can be written uniquely as $x = y + cz$ where $y \in \ker(\mu)$ and $c \in \mathbf{F}$.

1.75. Given $y = (y_k) \in \ell^1$, define $\mu_y \colon c_0 \to \mathbf{F}$ by $\langle x, \mu_y \rangle = \sum x_k y_k$. Show that $y \mapsto \mu_y$ is an isometric isomorphism of ℓ^1 onto ${c_0}^*$. Thus ${c_0}^* \cong \ell^1$.

1.76. Let X be a normed space, and suppose that the series $x = \sum x_n$ converges in X. Show that if $\mu \in X^*$, then $\sum \langle x_n, \mu \rangle$ is a convergent series of scalars, and $\langle x, \mu \rangle = \sum \langle x_n, \mu \rangle$.

1.77. Prove Theorem 1.75.

1.78. Fix $1 \le p \le \infty$, and let $y = (y_n)$ be a fixed sequence of scalars. Use Theorem 1.73 to show that there exists a constant C such that $|\langle x, y \rangle| \le C \, \|x\|_{\ell^p}$ for all finite sequences $x \in c_{00}$ if and only if $y \in \ell^{p'}$.

1.79. Let μ be a linear functional on a normed space X. Prove that

$$\mu \text{ is bounded} \quad \Longleftrightarrow \quad \ker(\mu) \text{ is closed.}$$

1.80. Let X, Y be normed spaces. Then, by Exercise 1.16, the Cartesian product $X \times Y$ is a normed space with respect to the norm $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$. Likewise, $X^* \times Y^*$ is normed with respect to $\|(x^*, y^*)\|_1 = \|x^*\| + \|y^*\|$. Show that, with respect to these choices of norms, $(X \times Y)^*$ is isometrically isomorphic to $X^* \times Y^*$.

2

Functional Analysis

In this chapter we survey the main theorems of functional analysis that deal with Banach spaces, including the Hahn–Banach, Baire Category, Uniform Boundedness, Open Mapping, and Closed Graph Theorems. References for additional information on this material (and sources for many of the proofs that we give) include the texts by Conway [Con90], Folland [Fol99], and Rudin [Rud91].

2.1 The Hahn–Banach Theorem and Its Implications

Orthogonality played an essential role in many of the proofs for Hilbert spaces that appeared in Sections 1.5 and 1.6. The analysis of general Banach spaces is much more difficult because there need not be any notion of orthogonality in a Banach space. The Hahn–Banach Theorem is a fundamental result for Banach spaces that allows us to do some things in Banach spaces that at first glance seem to be impossible without having the tools that orthogonality provides.

The abstract form of the Hahn–Banach Theorem is a statement about extension of linear functionals. We state a form that applies to both real and complex vector spaces.

Theorem 2.1 (Hahn–Banach Theorem). *Let X be a vector space over $\mathbf{F}$ and let ρ be a seminorm on X. If M is a subspace of X and $\lambda\colon M \to \mathbf{F}$ is a linear functional on M satisfying*

$$|\langle x, \lambda\rangle| \;\le\; \rho(x), \qquad x \in M,$$

then there exists a linear functional $\Lambda\colon X \to \mathbf{F}$ such that

$$\Lambda|_M \;=\; \lambda \qquad \textit{and} \qquad |\langle x, \Lambda\rangle| \;\le\; \rho(x), \quad x \in X. \qquad \diamondsuit$$

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The proof of the Hahn–Banach Theorem takes some preparation, and therefore we will omit it (see [Con90] for a proof). The most important point to note is that the extension Λ obeys the same bound that is satisfied by λ, but does so on the *entire* space X and not just on the subspace M.

In practice, it is usually not the Hahn–Banach Theorem itself but rather one of its many corollaries that is applied. Therefore we will concentrate in this section on these implications. Since these corollaries are so important, when invoking any one of them it is customary to write "by the Hahn–Banach Theorem" instead of "by a corollary to the Hahn–Banach Theorem."

Our first corollary states that any bounded linear functional on a subspace M of a normed space X has an extension to the entire space whose operator norm on X equals the operator norm on M. This is easy to prove when the space is a Hilbert space (see Exercise 2.1), but it is far from obvious that such an extension should be possible on non-inner product spaces.

Corollary 2.2 (Hahn–Banach). *Let X be a normed linear space and M a subspace of X. If $\lambda \in M^*$, then there exists $\Lambda \in X^*$ such that*

$$\Lambda|_M = \lambda \qquad \textit{and} \qquad \|\Lambda\|_{X^*} = \|\lambda\|_{M^*}.$$

Proof. Set $\rho(x) = \|\lambda\|_{M^*}\, \|x\|_X$ for $x \in X$. Note that ρ is defined on all of X, and is a seminorm on X (in fact, it is a norm if $\lambda \neq 0$). Further,

$$\forall\, x \in M, \quad |\langle x, \lambda\rangle| \;\le\; \|x\|_X\, \|\lambda\|_{M^*} \;=\; \rho(x).$$

Hence Theorem 2.1 implies that there exists a linear functional $\Lambda\colon X \to \mathbf{F}$ such that $\Lambda|_M = \lambda$ (which implies $\|\Lambda\|_{X^*} \ge \|\lambda\|_{M^*}$) and

$$\forall\, x \in X, \quad |\langle x, \Lambda\rangle| \;\le\; \rho(x) \;=\; \|\lambda\|_{M^*}\, \|x\|_X,$$

which implies that $\|\Lambda\|_{X^*} \le \|\lambda\|_{M^*}$. $\square$

Given a normed space X and given $x^* \in X^*$, the operator norm of x^* is

$$\|x^*\|_{X^*} \;=\; \sup_{x \in X,\, \|x\|_X = 1} |\langle x, x^*\rangle|.$$

Thus, we obtain the operator norm of x^* on X^* by "looking back" at its action on X. The next corollary provides a complementary viewpoint: The norm of $x \in X$ can be obtained by "looking forward" to its action on X^*. Again, this is easy to prove directly for Hilbert spaces (see Theorem 1.37), but is a much more subtle fact for generic Banach spaces.

Corollary 2.3 (Hahn–Banach). *Let X be a Banach space. Then for each $x \in X$ we have*

$$\|x\|_X \;=\; \sup_{x^* \in X^*,\, \|x^*\|_{X^*} = 1} |\langle x, x^*\rangle|. \tag{2.1}$$

Further, the supremum is achieved.

Proof. Fix $x \in X$, and let α denote the supremum on the right-hand side of equation (2.1). Since $|\langle x, x^*\rangle| \le \|x\|_X \, \|x^*\|_{X^*}$, we have $\alpha \le \|x\|_X$.

Let $M = \operatorname{span}\{x\}$, and define $\lambda\colon M \to \mathbf{F}$ by $\langle cx, \lambda\rangle = c\,\|x\|_X$. Then $\lambda \in M^*$ and $\|\lambda\|_{M^*} = 1$. Corollary 2.2 therefore implies that there exists some $\Lambda \in X^*$ with $\Lambda|_M = \lambda$ and $\|\Lambda\|_{X^*} = \|\lambda\|_{M^*} = 1$. In particular, since $x \in M$, we have $\alpha \ge |\langle x, \Lambda\rangle| = |\langle x, \lambda\rangle| = \|x\|_X$, and therefore the supremum in equation (2.1) is achieved. □

Now we can give one of the most powerful and often-used implications of the Hahn–Banach Theorem. It states that we can find a bounded linear functional that separates a point from a closed subspace of a normed space. This is easy to prove constructively for the case of a Hilbert space (see Exercise 2.2), but it is quite amazing that we can do this in arbitrary normed spaces.

Corollary 2.4 (Hahn–Banach). *Let X be a normed linear space. Suppose that:*

(a) *M is a closed subspace of X,*

(b) *$x_0 \in X \backslash M$, and*

(c) *$d = \operatorname{dist}(x_0, M) = \inf\bigl\{\|x_0 - m\| : m \in M\bigr\}$.*

Then there exists $\Lambda \in X^*$ such that

$$\langle x_0, \Lambda\rangle = 1, \qquad \Lambda|_M = 0, \qquad \textit{and} \qquad \|\Lambda\|_{X^*} = \frac{1}{d}.$$

Proof. Note that $d > 0$ since M is closed. Define $M_1 = \operatorname{span}\{M, x_0\}$. Then each $x \in M_1$ can be written as $x = m_x + t_x x_0$ for some $m_x \in M$ and $t_x \in \mathbf{F}$, and since $x_0 \notin M$, this representation is unique (verify!). Define $\lambda\colon M_1 \to \mathbf{F}$ by $\langle x, \lambda\rangle = t_x$. Then λ is linear, $\lambda|_M = 0$, and $\langle x_0, \lambda\rangle = 1$.

If $x \in M_1$ and $t_x \ne 0$, then we have $m_x/t_x \in M$, so

$$\|x\| \;=\; \|t_x x_0 + m_x\|_X \;=\; |t_x| \left\| x_0 - \Bigl(\frac{-m_x}{t_x}\Bigr) \right\|_X \;\ge\; |t_x|\, d.$$

If $t_x = 0$ (so $x \in M$), this is still true. Hence, $|\langle x, \lambda\rangle| = |t_x| \le \|x\|_X/d$ for all $x \in M_1$. Therefore λ is continuous on M_1, and $\|\lambda\|_{M_1^*} \le 1/d$.

On the other hand, there exist vectors $m_n \in M$ such that $\|x_0 - m_n\|_X \to d$. Since λ vanishes on M, we therefore have

$$1 \;=\; \langle x_0, \lambda\rangle \;=\; \langle x_0 - m_n, \lambda\rangle \;\le\; \|x_0 - m_n\|_X \, \|\lambda\|_{M_1^*} \;\to\; d\, \|\lambda\|_{M_1^*}.$$

Therefore $\|\lambda\|_{M_1^*} \ge 1/d$.

Applying Corollary 2.2, there exists a $\Lambda \in X^*$ such that $\Lambda|_{M_1} = \lambda$ and $\|\Lambda\|_{X^*} = \|\lambda\|_{M_1^*}$. This functional Λ has all of the required properties. □

Unlike the preceding corollaries, the next corollary is usually not given a special name, but we will have occasion to use it often (compare Lemma 1.44 for the case of Hilbert spaces).

Corollary 2.5. *Let X be a Banach space. Then $\{x_n\} \subseteq X$ is complete if and only if the following statement holds:*

$$x^* \in X^* \text{ and } \langle x_n, x^* \rangle = 0 \text{ for every } n \quad \Longrightarrow \quad x^* = 0.$$

Proof. $\Rightarrow$. Suppose that $\{x_n\}$ is complete, i.e., $\overline{\text{span}}\{x_n\} = X$. Suppose that $x^* \in X^*$ satisfies $\langle x_n, x^* \rangle = 0$ for every n. Since x^* is linear, we therefore have $\langle x, x^* \rangle = 0$ for every $x = \sum_{n=1}^{N} c_n x_n \in \text{span}\{x_n\}$. However, x^* is continuous, so this implies $\langle x, x^* \rangle = 0$ for every $x \in \overline{\text{span}}\{x_n\} = X$. Hence x^* is the zero functional.

$\Leftarrow$. Suppose that the only $x^* \in X^*$ satisfying $\langle x_n, x^* \rangle = 0$ for every n is $x^* = 0$. Define $Z = \overline{\text{span}}\{x_n\}$, and suppose that $Z \neq X$. Then we can find an element $y \in X$ such that $y \notin Z$. Since Z is a closed subset of X, we therefore have $d = \text{dist}(y, Z) > 0$. By the Hahn–Banach Theorem (Corollary 2.4), there exists a functional $\Lambda \in X^*$ satisfying $\langle y, \Lambda \rangle = 1 \neq 0$ and $\langle z, \Lambda \rangle = 0$ for every $z \in Z$. However, this implies that $\langle x_n, \Lambda \rangle = 0$ for every n. By hypothesis, Λ must then be the zero functional, contradicting the fact that $\langle y, \Lambda \rangle \neq 0$. Hence, we must have $Z = X$, so $\{x_n\}$ is complete in X. $\square$

Exercises

2.1. Let M be a subspace of a Hilbert space H and fix $\lambda \in M$. Show directly that there exists some $\Lambda \in H$ such that $\langle x, \Lambda \rangle = \langle x, \lambda \rangle$ for all $x \in M$ and $\|\Lambda\| = \|\lambda\|$.

2.2. Suppose that M is a closed subspace of a Hilbert space H, $x_0 \in H \backslash M$, and $d = \text{dist}(x_0, M)$. Show directly that there exists a $\mu \in H$ such that $\langle x_0, \mu \rangle = 1$, $\langle x, \mu \rangle = 0$ for all $x \in M$, and $\|\mu\| = 1/d$.

2.3. Let X be a normed space. Show that if X^* is separable then X is separable, but the converse can fail.

2.4. The Weierstrass Approximation Theorem implies that $\{x^k\}_{k \geq 0}$ is complete in $C[0,1]$. Show that $\{x^{2k}\}_{k \geq 0}$ is also complete in $C[0,1]$.

2.5. Given a subset A of a normed space X, define its *orthogonal complement* $A^\perp \subseteq X^*$ by

$$A^\perp = \{\mu \in X^* : \langle x, \mu \rangle = 0 \text{ for all } x \in A\}.$$

Prove that $A^\perp$ is a closed subspace of X^*, and explain how this relates to Corollary 2.5.

2.6. Let S be a subspace of a normed space X, and show that its closure $\overline{S}$ is given by

$$\overline{S} = \bigcap \{\ker(\mu) : \mu \in X^* \text{ and } S \subseteq \ker(\mu)\}.$$

2.2 Reflexivity

Given a normed space X, its dual space X^* is a Banach space, so we can consider the dual of the dual space, which we denote by X^{**}. The next result shows that there is a natural isometry that maps X into X^{**}.

Theorem 2.6. *Let X be a normed linear space. Given $x \in X$, define $\pi(x) : X^* \to \mathbf{F}$ by*

$$\big\langle x^*, \pi(x) \big\rangle \;=\; \langle x, x^* \rangle, \qquad x^* \in X^*.$$

Then $\pi(x)$ is a bounded linear functional on X^*, and has operator norm

$$\|\pi(x)\|_{X^{**}} \;=\; \|x\|_X.$$

Consequently, the mapping

$$\begin{aligned} \pi \colon X &\to X^{**} \\ x &\mapsto \pi(x), \end{aligned}$$

is a linear isometry of X into X^{**}.

Proof. By definition of the operator norm,

$$\|\pi(x)\|_{X^{**}} \;=\; \sup_{x^* \in X^*,\, \|x^*\|_{X^*}=1} |\langle x^*, \pi(x) \rangle|.$$

On the other hand, by the Hahn–Banach Theorem in the form of Corollary 2.3,

$$\|x\| \;=\; \sup_{x^* \in X^*,\, \|x^*\|_{X^*}=1} |\langle x, x^* \rangle|.$$

Since $\langle x, x^* \rangle = \langle x^*, \pi(x) \rangle$, the result follows. □

Definition 2.7 (Natural Embedding of X into X^{**}). Let X be a normed space.

(a) The mapping $\pi \colon X \to X^{**}$ defined in Theorem 2.6 is called the *natural embedding* or the *canonical embedding* of X into X^{**}.

(b) If the natural embedding of X into X^{**} is surjective, then we say that X is *reflexive*. ◇

Note that in order for X to be called reflexive, the natural embedding must be a surjective isometry. There exist Banach spaces X such that X is isometrically isomorphic to X^{**} even though X is not reflexive [Jam51].

By the Riesz Representation Theorem (Theorem 1.75), every Hilbert space is reflexive.

Exercise 2.7 asks for a proof that ℓ^p is reflexive for each $1 < p < \infty$. However, ℓ^1 and ℓ^∞ are not reflexive. Another nonreflexive example is the space c_0, since by Exercise 1.75 we have ${c_0}^{**} \cong (\ell^1)^* \cong \ell^\infty$. The space c_0 is one of the few easily exhibited nonreflexive separable spaces whose dual is separable.

It is likewise true that $L^p(E)$ is reflexive when $1 < p < \infty$, but not for $p = 1$ or $p = \infty$.

Exercises

2.7. Show that ℓ^p is reflexive for each $1 < p < \infty$.

2.8. Let X be a Banach space. Show that if X is separable but X^* is not, then X is not reflexive. Use this to show that ℓ^1 is a proper subspace of $(\ell^\infty)^*$.

2.3 Adjoints of Operators on Banach Spaces

The duality between Banach spaces and their dual spaces allows us to define the "dual" of a bounded linear operator on Banach spaces.

Let X and Y be Banach spaces, and let $T\colon X \to Y$ be a bounded linear operator. Fix $\nu \in Y^*$, and define a functional $\mu\colon X \to \mathbf{F}$ by

$$\langle x, \mu \rangle \;=\; \langle Tx, \nu \rangle, \qquad x \in X.$$

That is, $\mu = \nu \circ T$. Then μ is linear since T and ν are linear. Further,

$$|\langle x, \mu \rangle| \;=\; |\langle Tx, \nu \rangle| \;\le\; \|Tx\|_Y \, \|\nu\|_{Y^*} \;\le\; \|T\| \, \|x\|_X \, \|\nu\|_{Y^*},$$

so

$$\|\mu\|_{X^*} \;=\; \sup_{\|x\|_X = 1} |\langle x, \mu \rangle| \;\le\; \|T\| \, \|\nu\|_{Y^*} \;<\; \infty. \tag{2.2}$$

Hence μ is bounded, so $\mu \in X^*$. Thus, for each $\nu \in Y^*$ we have defined a functional $\mu \in X^*$, so we can define an operator $T^*\colon Y^* \to X^*$ by setting $T^*\nu = \mu$. This mapping T^* is linear, and by equation (2.2) we have

$$\|T^*\nu\|_{X^*} \;=\; \|\mu\|_{X^*} \;\le\; \|T\| \, \|\nu\|_{Y^*}.$$

Taking the supremum over all unit vectors $\nu \in Y^*$, we conclude that T^* is bounded and $\|T^*\| \le \|T\|$.

We can use the Hahn–Banach Theorem to show that $\|T^*\| = \|T\|$. Choose any $x \in X$ with $\|x\|_X = 1$. By Corollary 2.3,

$$\|Tx\|_Y \;=\; \sup_{\|\nu\|_{Y^*} = 1} |\langle Tx, \nu \rangle|, \tag{2.3}$$

and this supremum is achieved. Let $\nu \in Y^*$ be any particular functional with unit norm that achieves the supremum in equation (2.3). Then we have

$$\begin{aligned} \|Tx\|_Y \;=\; |\langle Tx, \nu \rangle| \;&=\; |\langle x, T^*\nu \rangle| \\ &\le\; \|x\|_X \, \|T^*\nu\|_{X^*} \\ &\le\; \|x\|_X \, \|T^*\| \, \|\nu\|_{Y^*} \\ &=\; \|x\|_X \, \|T^*\|. \end{aligned}$$

Since this is true for every unit vector $x \in X$, we conclude that $\|T\| \le \|T^*\|$.

In summary, given $T \in \mathcal{B}(X,Y)$, we have constructed an operator $T^* \in \mathcal{B}(Y^*, X^*)$ that satisfies

$$\forall\, x \in X, \quad \forall\, \nu \in Y^*, \quad \big\langle Tx, \nu \big\rangle \;=\; \big\langle x, T^*\nu \big\rangle. \tag{2.4}$$

According to Exercise 2.9, there is a unique such operator, and we call it the adjoint of T.

Definition 2.8 (Adjoint). Given $T \in \mathcal{B}(X,Y)$, the unique operator $T^* \in \mathcal{B}(Y^*, X^*)$ satisfying equation (2.4) is called the *adjoint* of T. ◇

Example 2.9. Let $E \subseteq \mathbf{R}$ be Lebesgue measurable, choose $1 \le p < \infty$, and fix $m \in L^\infty(\mathbf{R})$. Let $T_m \colon L^p(\mathbf{R}) \to L^p(\mathbf{R})$ be the operation of pointwise multiplication of f by m, i.e., $T_m f = fm$ for $f \in L^p(\mathbf{R})$. Exercise 1.67 shows that T_m is bounded and has operator norm $\|T_m\| = \|m\|_{L^\infty}$. Therefore, $T_m^* \colon L^{p'}(E) \to L^{p'}(E)$ is the unique operator that satisfies

$$\langle f, T_m^* g\rangle \;=\; \langle T_m f, g\rangle \;=\; \langle fm, g\rangle \;=\; \int_E f(t)\, m(t)\, g(t)\, dt \;=\; \langle f, gm\rangle$$

for $f \in L^p(E)$ and $g \in L^p(E)^* = L^{p'}(E)$. Therefore $T_m^* g = gm$, so T_m^* is also multiplication by the function m. Technically, however, T_m and T_m^* are not the same operator, since T_m maps $L^p(E)$ into itself, while T_m^* maps $L^{p'}(E)$ into itself. ◇

Exercises

2.9. Let X, Y be Banach spaces. Given $T \in \mathcal{B}(X,Y)$, show that there is a unique operator $T^* \in \mathcal{B}(Y^*, X^*)$ that satisfies equation (2.4).

2.10. Let X be a Banach space. Given $\mu \in X^* = \mathcal{B}(X, \mathbf{F})$, explicitly describe its adjoint μ^*.

2.11. Let M be a closed subspace M of a normed space X, and fix $L \in \mathcal{B}(X)$. We say that M is *invariant* under L if $L(M) \subseteq M$. Show that if $M \subseteq X$ is invariant under L, then $M^\perp$ is invariant under L^*, where $M^\perp$ is the orthogonal complement defined in Exercise 2.5.

2.12. Suppose that M is a closed subspace of a Banach space X. Let $\epsilon \colon M \to X$ be the embedding map, i.e., $\epsilon(x) = x$ for $x \in M$. Show that $\epsilon^* \colon X^* \to M^*$ is the restriction map, i.e., if $\mu \in X^*$, then $\epsilon^*\mu = \mu|_M$.

2.4 Adjoints of Operators on Hilbert Spaces

Since Hilbert spaces are Banach spaces, if H, K are Hilbert spaces and $T \in \mathcal{B}(H,K)$, then there exists a unique adjoint operator $T^* \in \mathcal{B}(K^*,H^*)$. However, since Hilbert spaces are self-dual, we can regard the adjoint as belonging to $\mathcal{B}(K,H)$. In particular, if $K = H$ then T and T^* both belong to $\mathcal{B}(H)$. This makes adjoints of operators on Hilbert spaces quite special, and so we study them in more detail in this section.

Because of the conflict between our bilinear form notation $\langle x, x^* \rangle$ for functionals x^* acting on elements x and the inner product $\langle x, y \rangle$, which is antilinear as a function of y, the definition of adjoints on Hilbert spaces differs slightly from the definition on Banach spaces. We defined the adjoint using the bilinear form notation, but when dealing with a space that we know is a Hilbert space, it is usually more convenient to employ that space's inner product. Therefore, we define the adjoint of an operator on a Hilbert space as follows.

Definition 2.10 (Adjoint). Let H and K be Hilbert spaces. Let $\langle \cdot, \cdot \rangle_H$ denote the inner product on H, and $\langle \cdot, \cdot \rangle_K$ the inner product on K. If $A \in \mathcal{B}(H,K)$, then the *adjoint* of A is the unique operator $A^* \in \mathcal{B}(K,H)$ satisfying

$$\forall\, x \in H, \quad \forall\, y \in K, \quad \langle Ax, y \rangle_K \;=\; \langle x, A^*y \rangle_H. \qquad \diamondsuit$$

Comparing Definitions 2.8 and 2.10, we see that there is an ambiguity in the definition of an adjoint. We use the convention that if X, Y are Banach spaces then the adjoint of $T \in \mathcal{B}(X,Y)$ is defined by Definition 2.8, while if we know that H, K are Hilbert spaces then the adjoint of $A \in \mathcal{B}(H,K)$ is defined by Definition 2.10.

Example 2.11. Consider again the mapping T_m discussed in Example 2.9, but now consider the particular case $p = 2$. Since $L^2(E)$ is a Hilbert space, we define T_m^* to be the unique operator that, for f, $g \in L^2(E)$, satisfies

$$\begin{aligned} \langle f, T_m^* g \rangle \;=\; \langle T_m f, g \rangle \;=\; \langle fm, g \rangle \;&=\; \int_E f(t)\, m(t)\, \overline{g(t)}\, dt \\ &=\; \int_E f(t)\, \overline{g(t)\, \overline{m(t)}}\, dt \;=\; \langle f, g\overline{m} \rangle. \end{aligned}$$

Therefore $T_m^* g = g\overline{m}$, i.e., T_m^* is multiplication by the function $\overline{m}$. $\diamondsuit$

Thus, we see that Definitions 2.8 and 2.10 differ in how they define the adjoint. Fortunately, this is not a significant problem in practice.

Example 2.12. Consider the finite-dimensional Hilbert spaces $H = \mathbf{C}^n$ and $K = \mathbf{C}^m$. A linear operator $A\colon \mathbf{C}^n \to \mathbf{C}^m$ is given by multiplication by an $m \times n$ matrix A, which we identify with the operator A. The Hilbert space adjoint of A corresponds to multiplication by the *conjugate transpose* or *Hermitian* matrix $A^* = \overline{A^{\mathrm{T}}}$, while the Banach space adjoint corresponds to multiplication by the transpose matrix A^{T} (see Exercise 2.13). $\diamondsuit$

The next result summarizes some of the properties of adjoints on Hilbert spaces (see Exercise 2.16).

Theorem 2.13. *Let H, K, L be Hilbert spaces, and fix $A \in \mathcal{B}(H,K)$ and $B \in \mathcal{B}(K,L)$.*

(a) $(A^*)^* = A$.

(b) $(BA)^* = A^*B^*$.

(c) $\ker(A) = \mathrm{range}(A^*)^\perp$.

(d) $\ker(A)^\perp = \overline{\mathrm{range}(A^*)}$.

(e) *A is injective if and only if* $\mathrm{range}(A^*)$ *is dense in H.*

(f) $\|A\| = \|A^*\| = \|A^*A\|^{1/2} = \|AA^*\|^{1/2}$. ♢

We now make some definitions specifically for the case of adjoints of operators that map a Hilbert space into itself.

Definition 2.14. Let H be a Hilbert space, and let $A, B\colon H \to H$ be bounded linear operators.

(a) A is *self-adjoint* or *Hermitian* if $A = A^*$. By definition,

$$A \text{ is self-adjoint} \quad \Longleftrightarrow \quad \forall\, x, y \in H, \quad \langle Ax, y\rangle \;=\; \langle x, Ay\rangle.$$

(b) A is *positive*, denoted $A \ge 0$, if A is self-adjoint and $\langle Ax, x\rangle$ is real with $\langle Ax, x\rangle \ge 0$ for every $x \in H$.

(c) A is *positive definite* or *strictly positive*, denoted $A > 0$, if A is self-adjoint and $\langle Ax, x\rangle$ is real with $\langle Ax, x\rangle > 0$ for every $x \ne 0$.

(d) We write $A \ge B$ if $A - B \ge 0$, and $A > B$ if $A - B > 0$. ♢

We will need the following results for self-adjoint and positive operators.

Theorem 2.15. *If $A \in \mathcal{B}(H)$ is self-adjoint, then*

$$\|A\| \;=\; \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

Proof. Let us take $\mathbf{F} = \mathbf{C}$; the proof for real scalars is similar. Set $M = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. By the Cauchy–Bunyakovski–Schwarz Inequality and the definition of operator norm, we have $M \le \|A\|$.

Choose any unit vectors x, $y \in H$. Then, by expanding the inner products, canceling terms, and using the fact that $A = A^*$, we see that

$$\begin{aligned}
\big\langle A(x+y),\, x+y\big\rangle - \big\langle A(x-y),\, x-y\big\rangle \;&=\; 2\,\langle Ax, y\rangle + 2\,\langle Ay, x\rangle \\
&=\; 2\,\langle Ax, y\rangle + 2\,\langle y, Ax\rangle \\
&=\; 4\,\mathrm{Re}(\langle Ax, y\rangle).
\end{aligned}$$

Applying the definition of M and using the Parallelogram Law, it follows that

$$\begin{aligned} 4\,\mathrm{Re}(\langle Ax, y\rangle) &\le |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle| \\ &\le M\,\|x+y\|^2 + M\,\|x-y\|^2 \\ &= 2M\left(\|x\|^2 + \|y\|^2\right) = 4M. \end{aligned}$$

That is, $\mathrm{Re}(\langle Ax, y\rangle) \le M$ for every choice of unit vectors x and y. Write $|\langle Ax, y\rangle| = \alpha\,\langle Ax, y\rangle$ where $\alpha \in \mathbf{C}$ satisfies $|\alpha| = 1$. Then $\bar{\alpha}y$ is another unit vector, so

$$|\langle Ax, y\rangle| = \alpha\langle Ax, y\rangle = \langle Ax, \bar{\alpha}y\rangle \le M.$$

Using Lemma 1.36(c), we therefore have

$$\|Ax\| = \sup_{\|y\|=1} |\langle Ax, y\rangle| \le M.$$

Since this is true for every unit vector x, we conclude that $\|A\| \le M$. □

As a corollary, we obtain the following useful fact for self-adjoint operators.

Corollary 2.16. *Let H be a Hilbert space. If $A \in \mathcal{B}(H)$ is self-adjoint and $\langle Ax, x\rangle = 0$ for every $x \in H$, then $A = 0$.* ◊

Although we will not prove it, it can be shown that if H is a *complex* Hilbert space, then $A \in \mathcal{B}(H)$ is self-adjoint if and only if $\langle Ax, x\rangle$ is real for every $x \in H$. Hence for complex Hilbert spaces, the hypothesis in Corollary 2.16 that A is self-adjoint is redundant.

We end this section by proving that every positive operator A on a Hilbert space has a square root. That is, there exists a positive operator S such that $S^2 = A$. The idea of the proof is that if a is a real number with $0 < a < 1$ and if $(1-t)^2 = a$, then $t = \frac{1}{2}(1-a) + \frac{1}{2}t^2$ and the iteration $t_{n+1} = \frac{1}{2}(1-a) + \frac{1}{2}t_n^2$ converges to t. We make an operator analogue of this recursion. To prove convergence, we need the following lemma, which will be useful to us again in Chapter 8.

Lemma 2.17. *If $T\colon H \to H$ is a positive operator on a Hilbert space H, then*

$$\forall\, x, y \in H, \quad |\langle Tx, y\rangle|^2 \le \langle Tx, x\rangle\,\langle Ty, y\rangle.$$

Proof. By definition of a positive operator, $\langle Tx, x\rangle \ge 0$ for every $x \in H$. Therefore $(x, y) = \langle Tx, y\rangle$ defines a semi-inner product on H, and $|||x||| = (x, x)^{1/2}$ is a seminorm on H. In general, $(\cdot, \cdot)$ need not be an inner product (this happens if and only if T is positive definite). Still, the Cauchy–Bunyakovski–Schwarz Inequality holds for semi-inner products by Exercise 1.34, so we have

$$|\langle Tx, y\rangle|^2 = |(x, y)|^2 \le |||x|||^2\, |||y|||^2 = (x, x)\,(y, y) = \langle Tx, x\rangle\,\langle Ty, y\rangle. \quad \square$$

Theorem 2.18. *If $A \in \mathcal{B}(H)$ is a positive operator on a Hilbert space H, then there exists a positive operator $A^{1/2} \in \mathcal{B}(H)$ such that $A^{1/2}A^{1/2} = A$. Moreover, $A^{1/2}$ commutes with A and with all operators that commute with A.*

Proof. We present some parts of the proof and assign the remainder as Exercise 2.23.

Suppose that $A \geq 0$. The result is trivial if A is the zero operator, so assume $A \neq 0$. Let $c = \|A\|^{-1}$. Then for every x we have

$$\langle cAx, x\rangle \;\leq\; |c|\,\|Ax\|\,\|x\| \;\leq\; |c|\,\|A\|\,\|x\|^2 \;=\; \|x\|^2 \;=\; \langle Ix, x\rangle,$$

which in operator notation says that $cA \leq I$. Since A has a square root if and only if cA has a square root, we can simply replace A by cA. That is, it suffices to prove the result under the assumptions that $A \geq 0$, $A \leq I$, and $\|A\| = 1$.

Let $B = I - A$. Set $T_0 = 0$, $T_1 = \frac{1}{2}B$, and

$$T_{n+1} \;=\; \frac{1}{2}\,(B + T_n^2), \qquad n \geq 2.$$

Each T_n is a polynomial in B, and therefore commutes with T_m and with every operator that commutes with B. The polynomial defining T_n has only nonnegative coefficients, so $T_n \geq 0$. Further, $T_{n+1} - T_n$ is also a polynomial in B *with all nonnegative coefficients*. Consequently, $T_n - T_m \geq 0$ for all $n \geq m \geq 0$.

By induction, $\|T_n\| \leq 1$ for every n. Therefore, if we fix $x \in H$ then the sequence $\big(\langle T_n x, x\rangle\big)$ is a bounded, increasing sequence of nonnegative real scalars. Hence this sequence must converge, and so is Cauchy. Now, if $n \geq m$ then by using Theorem 1.37(c) and Lemma 2.17 we compute that

$$\begin{aligned}
\|T_n x - T_m x\|^2 &= \sup_{\|y\|=1} |\langle (T_n - T_m)x, y\rangle|^2 \\
&\leq \sup_{\|y\|=1} |\langle (T_n - T_m)x, x\rangle|\,|\langle (T_n - T_m)y, y\rangle| \\
&\leq \sup_{\|y\|=1} |\langle T_n x, x\rangle - \langle T_m x, x\rangle|\,\|T_n - T_m\|\,\|y\|^2 \\
&\leq 2\,|\langle T_n x, x\rangle - \langle T_m x, x\rangle|.
\end{aligned}$$

Since $\big(\langle T_n x, x\rangle\big)$ is a Cauchy sequence of scalars, we conclude that $\{T_n x\}$ is a Cauchy sequence of vectors in H. Therefore $\{T_n x\}_{n \in \mathbf{N}}$ converges in H, and we define Tx to be the limit of this sequence. This operator T is bounded, linear, and positive, and it commutes with B and with every operator that commutes with B. Further, $T = \frac{1}{2}\,(B+T^2)$. Consequently, the operator $S = I-T$ satisfies $S^2 = A$, and S is positive since $\|T\| \leq 1$. □

In fact, the square root $A^{1/2}$ is unique; see Exercise 2.24.

Exercises

2.13. Let $A: \mathbf{C}^n \to \mathbf{C}^m$ be a linear operator, which we identify with its $m \times n$ matrix representation. Show that the adjoint of A in the Hilbert space sense (Definition 2.10) is the conjugate transpose matrix $A^* = \overline{A^{\mathrm{T}}}$, while the adjoint of A in the Banach space sense (Definition 2.8) is the transpose matrix A^{T}.

2.14. Let L, R be the left- and right-shift operators on ℓ^2 defined in Exercise 1.64. Show that $R = L^*$.

2.15. Fix $\lambda \in \ell^\infty$, and let M_λ be the multiplication operator defined in Exercise 1.66. Find M_λ^*, and determine when M_λ is self-adjoint, positive, or positive definite.

2.16. Prove Theorem 2.13.

2.17. Let M be a closed subspace of a Hilbert space H, and let $P \in \mathcal{B}(H)$ be given. Show that P is the orthogonal projection of H onto M if and only if $P^2 = P$, $P^* = P$, and $\mathrm{range}(P) = M$.

2.18. Let H be a Hilbert space and suppose that $A, B \in \mathcal{B}(H)$ are self-adjoint. Show that ABA, and BAB are self-adjoint, but AB is self-adjoint if and only if $AB = BA$. Exhibit self-adjoint operators A, B that do not commute.

2.19. Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be fixed.

(a) Show that if A is self-adjoint then all eigenvalues of A are real, and eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

(b) Show that if A is a positive operator then all eigenvalues of A are real and nonnegative.

(c) Show that if A is a positive definite operator then all eigenvalues of A are real and strictly positive.

2.20. Let H, K be Hilbert spaces. Show that if $A \in \mathcal{B}(H,K)$, then $A^*A \in \mathcal{B}(H)$ and $AA^* \in \mathcal{B}(K)$ are positive operators.

2.21. Let H be a Hilbert space. Given $A \in \mathcal{B}(H)$, show that $\ker(A) = \ker(A^*A)$ and $\overline{\mathrm{range}(A^*A)} = \overline{\mathrm{range}(A^*)}$.

2.22. Let H, K be Hilbert spaces, and fix $U \in \mathcal{B}(H,K)$. Show that U is unitary if and only if U is a bijection and $U^{-1} = U^*$.

2.23. Fill in the details in the proof of Theorem 2.18.

2.24. Let A be a positive operator on a Hilbert space H.

(a) Show that $\langle Ax, x\rangle = 0$ if and only if $Ax = 0$.

(b) Show that the operator $A^{1/2}$ constructed in Theorem 2.18 is unique, i.e., there is only one positive operator S satisfying $S^2 = A$.

2.5 The Baire Category Theorem

Just as it is not possible to write the Euclidean plane $\mathbf{R}^2$ as the union of *countably many* straight lines, the Baire Category Theorem states that a complete metric space cannot be written as a countable union of "nowhere dense" sets. Since we are mainly interested in Banach spaces in this volume, we will prove this theorem in the setting of complete normed spaces, but the proof carries over without change to complete metric spaces.

Definition 2.19 (Nowhere Dense Sets). Let X be a Banach space, and let $E \subseteq X$ be given.

(a) E is *nowhere dense* or *rare* if $X \backslash \overline{E}$ is dense in X.

(b) E is *meager* or *first category* if it can be written as a countable union of nowhere dense sets.

(c) E is *nonmeager* or *second category* if it is not meager. ◊

We can restate the meaning of nowhere dense sets as follows (see Exercise 2.25).

Lemma 2.20. *Let E be a nonempty subset of a Banach space X. Then E is nowhere dense if and only if $\overline{E}$ contains no nonempty open subsets.* ◊

The set of rationals $\mathbf{Q}$ is not a nowhere dense subset of $\mathbf{R}$, but it is meager in $\mathbf{R}$. Although it is not a real vector space and hence not a normed space, $\mathbf{Q}$ under the metric $\mathrm{d}(x, y) = |x - y|$ is an example of an incomplete metric space that is a meager subset of itself.

Now we prove the Baire Category Theorem.

Theorem 2.21 (Baire Category Theorem). *Every Banach space X is a nonmeager subset of itself. Consequently, if*

$$X = \bigcup_{n=1}^{\infty} E_n$$

where each E_n is a closed subset of X, then at least one E_n contains a nonempty open subset.

Proof. Suppose that $X = \cup E_n$ where each E_n is nowhere dense. Then, by definition, $U_n = X \backslash \overline{E_n}$ is dense, and it is open since $\overline{E_n}$ is closed.

Choose $x_1 \in U_1$ and let $r_1 > 0$ be such that $B_1 = B_{r_1}(x_1) \subseteq U_1$. Then since U_2 is dense, there exists a point $x_2 \in U_2 \cap B_1$. Since U_2 and B_1 are both open, there exists some $r_2 > 0$ such that $B_2 = B_{r_2}(x_2) \subseteq U_2 \cap B_1$. Without loss of generality, we can take r_2 small enough that we have both $r_2 < r_1/2$ and $\overline{B_2} \subseteq B_1$. Continuing in this way we obtain points $x_n \in U_n$ and open balls $B_n = B_{r_n}(x_n) \subseteq U_n$ such that

$$r_n < \frac{r_{n-1}}{2} \qquad \text{and} \qquad \overline{B_n} \subseteq B_{n-1}.$$

In particular, $r_n \to 0$ and the balls B_n are nested.

Fix $\varepsilon > 0$, and let N be large enough so that $r_N < \varepsilon/2$. If $m, n > N$, then we have $x_m, x_n \in B_N$. Hence $\|x_m - x_n\| < 2r_N < \varepsilon$. Thus $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy, and therefore there exists some $x \in X$ such that $x_n \to x$.

Now fix any $N > 0$. Then, since the B_n are nested, we have $x_n \in B_{N+1}$ for all $n > N$. As $x_n \to x$, this implies that $x \in \overline{B_{N+1}} \subseteq B_N$. This is true for every N, so

$$x \in \bigcap_{n=1}^{\infty} B_n \subseteq \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (X \backslash \overline{E_n}).$$

But then $x \notin \cup E_n$, which is a contradiction. □

Exercises

2.25. Prove Lemma 2.20.

2.26. Show that $C_c(\mathbf{R})$ is a meager subset of $C_0(\mathbf{R})$.

2.27. Suppose that f is an infinitely differentiable function on $\mathbf{R}$ such that for each $t \in \mathbf{R}$ there exists some integer $n_t \geq 0$ so that $f^{(n_t)}(t) = 0$. Prove that there exists some open interval (a, b) and some polynomial p such that $f(t) = p(t)$ for all $t \in (a, b)$.

2.28. Let D be the subset of $C[0,1]$ consisting of all functions $f \in C[0,1]$ that have a right-hand derivative at at least one point in $[0,1]$. Show that D is meager in $C[0,1]$, and conclude that there are functions in $C[0,1]$ that are not differentiable at any point.

2.6 The Uniform Boundedness Principle

The Uniform Boundedness Principle states that a family of bounded linear operators on a Banach space that are uniformly bounded *at each individual point* must actually be *uniformly bounded in operator norm.*

Theorem 2.22 (Uniform Boundedness Principle). *Let X be a Banach space and Y a normed linear space. If $\{A_i\}_{i\in I}$ is any collection of operators in $\mathcal{B}(X,Y)$ such that*

$$\forall\, x \in X, \quad \sup_{i\in I} \|A_i x\| < \infty,$$

then

$$\sup_{i\in I} \|A_i\| < \infty.$$

Proof. Set

$$E_n = \Big\{x \in X : \sup_{i \in I} \|A_i x\| \le n\Big\}.$$

Then $X = \cup E_n$ by hypothesis, and since each A_i is continuous it follows that E_n is closed. Consequently, the Baire Category Theorem implies that some E_n must contain an open ball, say $B_r(x_0) \subseteq E_n$.

Given any nonzero $x \in X$, if we set $y = x_0 + sx$ with $s = \frac{r}{2\|x\|}$ then we have $y \in B_r(x_0) \subseteq E_n$, and therefore

$$\|A_i x\| = \Big\|A_i\Big(\frac{y - x_0}{s}\Big)\Big\| \le \frac{1}{s}\big(\|A_i y\| + \|A_i x_0\|\big) \le \frac{2\,\|x\|}{r}\,2n = \frac{4n}{r}\,\|x\|.$$

Consequently, $\|A_i\| \le 4n/r$, which is a constant independent of i. □

The following special case of the Uniform Boundedness Principle is often useful (sometimes the names "Uniform Boundedness Principle" and "Banach–Steinhaus Theorem" are used interchangeably). The proof of Theorem 2.23 is assigned as Exercise 2.29.

Theorem 2.23 (Banach–Steinhaus Theorem). *Let X and Y be Banach spaces. If $A_n \in \mathcal{B}(X,Y)$ for $n \in \mathbf{N}$ and $Ax = \lim_{n\to\infty} A_n x$ exists for each $x \in X$, then $A \in \mathcal{B}(X,Y)$ and $\|A\| \le \sup_n \|A_n\| < \infty$.* ♢

Note that the hypotheses of the Banach–Steinhaus Theorem do *not* imply that $A_n \to A$ in operator norm. A counterexample is given in Exercise 2.30.

As an application of the Banach–Steinhaus Theorem, we prove a fact that was used earlier to show that the dual space ℓ^p is (isomorphic to) $\ell^{p'}$ when p is finite (see Theorem 1.73).

Theorem 2.24. *Fix $1 \le p \le \infty$ and any sequence of scalars $y = (y_k)$. Then $\sum x_k y_k$ converges for all $x \in \ell^p$ if and only if $y \in \ell^{p'}$. Furthermore, in this case $T_y x = (x_k y_k)$ defines a bounded linear map of ℓ^p into ℓ^1, and*

$$\sum_k |x_k y_k| = \|T_y x\|_{\ell^1} \le \|x\|_{\ell^p}\,\|y\|_{\ell^{p'}}, \qquad x \in \ell^p.$$

Proof. We will prove the case $1 < p < \infty$ (the cases $p = 1$ and $p = \infty$ are Exercise 2.32). Assume that $\sum x_k y_k$ converges for all $x \in \ell^p$. Define functionals $T_N,\, T\colon \ell^p \to \mathbf{F}$ by

$$Tx = \sum_{k=1}^{\infty} x_k y_k \qquad \text{and} \qquad T_N x = \sum_{k=1}^{N} x_k y_k.$$

Clearly T_N is linear, and for $x \in \ell^p$ we have

$$|T_N x| \le \bigg(\sum_{k=1}^{N} |x_k|^p\bigg)^{1/p} \bigg(\sum_{k=1}^{N} |y_k|^{p'}\bigg)^{1/p'} \le C_N\,\|x\|_{\ell^p},$$

where $C_N = \left(\sum_{k=1}^N |y_k|^{p'}\right)^{1/p'}$ is a finite constant independent of x (though not independent of N). Therefore $T_N \in \mathcal{B}(\ell^p, \mathbf{F}) = (\ell^p)^*$ for each N.

By hypothesis, $T_N x \to Tx$ as $N \to \infty$ for each $x \in \ell^p$. The Banach–Steinhaus Theorem therefore implies that $T \in \mathcal{B}(\ell^p, \mathbf{F}) = (\ell^p)^*$ and $\|T\| \le C = \sup \|T_N\| < \infty$.

At this point, if we accept the fact that $(\ell^p)^* = \ell^{p'}$ then we can argue as follows. Since $T \in (\ell^p)^*$ there must exist some $z \in \ell^{p'}$ such that $Tx = \langle x, z\rangle = \sum x_k z_k$ for all $x \in \ell^p$. Letting $\{\delta_k\}$ denote the standard basis vectors on ℓ^p, we have $y_k = T\delta_k = z_k$ for every k, so $y = z \in \ell^{p'}$.

However, since the current theorem was used in the proof that $(\ell^p)^* = \ell^{p'}$, in order to avoid circularity we need to give a direct proof that y belongs to $\ell^{p'}$. To do this, set

$$x_N = \left(\alpha_1 |y_1|^{p'-1}, \dots, \alpha_N |y_N|^{p'-1}, 0, 0, \dots\right) \in \ell^p,$$

where α_k is a scalar of unit modulus such that $\alpha_k y_k = |y_k|$. Then we have from the definition of T that

$$|Tx_N| = \sum_{k=1}^N \alpha_k |y_k|^{p'-1} y_k = \sum_{k=1}^N |y_k|^{p'},$$

while from $\|T\| \le C$ we obtain

$$|Tx_N| \le C \|x_N\|_{\ell^p} = C \left(\sum_{k=1}^N |y_k|^{(p'-1)p}\right)^{1/p} = C \left(\sum_{k=1}^N |y_k|^{p'}\right)^{1/p}.$$

Combining the two preceding equations, dividing through by $\left(\sum_{k=1}^N |y_k|^{p'}\right)^{1/p}$, and noting that $1 - \frac{1}{p} = \frac{1}{p'}$, this implies that

$$\left(\sum_{k=1}^N |y_k|^{p'}\right)^{1/p'} = \left(\sum_{k=1}^N |y_k|^{p'}\right)^{1-\frac{1}{p}} \le C.$$

Letting $N \to \infty$, we see that $\|y\|_{\ell^{p'}} \le C$. $\square$

Exercises

2.29. Prove Theorem 2.23.

2.30. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H, and let P_N be the orthogonal projection of H onto $\operatorname{span}\{e_1, \dots, e_N\}$. Show that $P_N x \to x$ for every $x \in H$, but $\|I - P_N\| \not\to 0$ as $N \to \infty$.

2.31. Let X, Y be Banach spaces. Suppose $A_n \in \mathcal{B}(X,Y)$ for $n \in \mathbf{N}$ and $Ax = \lim_{n\to\infty} A_n x$ exists for each x in a dense subspace S of X.

(a) Show that if $\sup_n \|A_n\| < \infty$ then A extends to a bounded map on X, and $Ax = \lim_{n\to\infty} A_n x$ for all $x \in X$.

(b) Give an example that shows that the hypothesis $\sup_n \|A_n\| < \infty$ in part (a) is necessary.

2.32. Prove Theorem 2.24 for the cases $p = 1$ and $p = \infty$.

2.33. (a) Let X be a Banach space. Show that $S \subseteq X^*$ is bounded if and only if $\sup\big\{|\langle x, x^*\rangle| : x^* \in S\big\} < \infty$ for each $x \in X$.

(b) Let X be a normed linear space. Show that $S \subseteq X$ is bounded if and only if $\sup\big\{|\langle x, x^*\rangle| : x \in S\big\} < \infty$ for each $x^* \in X^*$.

2.34. Fix $1 \le p,\ q \le \infty$. Let $A = [a_{ij}]_{i,j\in\mathbf{N}}$ be an infinite matrix and set $a_i = (a_{ij})_{j\in\mathbf{N}}$ for each $i \in \mathbf{N}$. Suppose that

(a) $(Ax)_i = \langle x, a_i\rangle = \sum_j a_{ij}x_j$ converges for each $x \in \ell^p$ and $i \in \mathbf{N}$, and

(b) $Ax = \big((Ax)_i\big)_{i\in\mathbf{N}} = \big(\langle x, a_i\rangle\big)_{i\in\mathbf{N}} \in \ell^q$ for each $x \in \ell^p$.

Identifying the matrix A with the map $x \mapsto Ax$, prove that $A \in \mathcal{B}(\ell^p, \ell^q)$.

2.7 The Open Mapping Theorem

By Theorem 1.59, a function $f\colon X \to Y$ is continuous if the inverse image under f of any open subset of Y is open in X. It is often important to consider direct images of open sets as well.

Definition 2.25 (Open Mapping). Let X, Y be normed linear spaces. A function $A\colon X \to Y$ is an *open mapping* if

$$U \text{ is open in } X \implies A(U) \text{ is open in } Y. \qquad \diamondsuit$$

In general, a continuous function need not be an open mapping. For example, $f(x) = \sin x$ is a continuous mapping of the real line into itself, but f maps the open interval $(0, 2\pi)$ onto the closed interval $[-1, 1]$.

The Open Mapping Theorem asserts that any continuous linear surjection of one Banach space onto another must be an open mapping. The key to the proof is the following lemma. For clarity, we will write $B_r^X(x)$ and $B_r^Y(y)$ to distinguish open balls in X from open balls in Y.

Lemma 2.26. *Let X, Y be Banach spaces and fix $A \in \mathcal{B}(X,Y)$. If $\overline{A(B_1^X(0))}$ contains an open ball in Y, then $A(B_1^X(0))$ contains an open ball $B_r^Y(0)$ for some $r > 0$.*

Proof. Suppose that $\overline{A(B_1^X(0))}$ contains some open ball $B_s^Y(z)$. We claim that if we set $r = s/2$, then

$$B_r^Y(0) \subseteq \overline{A(B_1^X(0))}. \tag{2.5}$$

To see this, fix $x \in B_r^Y(0)$, i.e., $\|x\|_X < r = s/2$. Then $2x + z \in B_s^Y(z) \subseteq \overline{A(B_1^X(0))}$. Hence there exist vectors $y_n \in X$ with $\|y_n\|_X < 1$ such that $Ay_n \to 2x + z$. Also, $z \in B_s^Y(z) \subseteq \overline{A(B_1^X(0))}$, so there exist vectors $z_n \in X$ with $\|z_n\|_X < 1$ such that $Az_n \to z$. Then $w_n = (y_n - z_n)/2 \in B_1^X(0)$, and

$$Aw_n = \frac{Ay_n - Az_n}{2} \to \frac{(2x+z) - z}{2} = x \quad \text{as } n \to \infty.$$

Hence $x \in \overline{A(B_1^X(0))}$, so equation (2.5) holds.

Now we will show that we actually have $B_{r/2}^Y(0) \subseteq A(B_1^X(0))$. To see this, suppose that $y \in B_{r/2}^Y(0)$. Rescaling equation (2.5), we have $y \in \overline{A(B_{1/2}^X(0))}$, so there exists some $x_1 \in X$ with $\|x_1\| < 1/2$ such that $\|y - Ax_1\| < r/4$. Then $y - Ax_1 \in B_{r/4}^Y(0) \subseteq \overline{A(B_{1/4}^X(0))}$, so there exists some $x_2 \in X$ with $\|x_2\| < 1/4$ such that $\|(y - Ax_1) - Ax_2\| < r/8$. Continuing in this way, we obtain vectors $x_n \in X$ with $\|x_n\| < 2^{-n}$ such that

$$\|y - Az_n\| < \frac{r}{2^{n+1}},$$

where $z_n = \sum_{k=1}^n x_k$. Hence $Az_n \to y$. However, $\{z_n\}_{n\in\mathbf{N}}$ is Cauchy in X, so $z_n \to z$ for some $z \in X$. Since A is continuous, it follows that $y = Az$. Since $\|y\| < 1$, we therefore have $y \in A(B_1^X(0))$. □

Theorem 2.27 (Open Mapping Theorem). *If X, Y are Banach spaces and $A\colon X \to Y$ is a continuous linear surjection, then A is an open mapping.*

Proof. Since A is surjective, we have

$$Y = \bigcup_{k=1}^{\infty} \overline{A(B_k^X(0))}.$$

The Baire Category Theorem implies that some set $\overline{A(B_k^X(0))}$ must contain an open ball. Therefore, by Lemma 2.26, there is some $r > 0$ such that

$$B_r^Y(0) \subseteq A(B_1^X(0)). \tag{2.6}$$

Now suppose that $U \subseteq X$ is open and $y \in A(U)$. Then $y = Ax$ for some $x \in U$, so $B_s^X(x) \subseteq U$ for some $s > 0$. Rescaling equation (2.6), we have $B_t^Y(0) \subseteq A(B_s^X(0))$ for some $t > 0$. Therefore

$$B_t^Y(y) = B_t^Y(0) + Ax \subseteq A(B_s^X(0) + x) = A(B_s^X(x)) \subseteq A(U),$$

so $A(U)$ is open. □

The hypotheses in the Open Mapping Theorem that X and Y are both complete is necessary; see [Con90].

Exercises

2.35. Let X and Y be Banach spaces. Show that $A \in \mathcal{B}(X,Y)$ is surjective if and only if range(A) is not meager in Y.

2.8 Topological Isomorphisms

Topological isomorphisms will play an important role in the remainder of this volume.

Definition 2.28. Let X, Y be normed linear spaces.

(a) A linear operator $T\colon X \to Y$ is a *topological isomorphism* if T is a bijection and both T and T^{-1} are continuous.

(b) We say that X and Y are *topologically isomorphic* if there exists a topological isomorphism $T\colon X \to Y$. ◇

Every isometric isomorphism is a topological isomorphism, but the converse need not hold (see Exercise 2.39).

For the case of linear operators on Banach spaces, we have the following useful consequence of the Open Mapping Theorem.

Theorem 2.29 (Inverse Mapping Theorem). *If X, Y are Banach spaces and $T\colon X \to Y$ is a continuous linear bijection, then $T^{-1}\colon Y \to X$ is continuous. Consequently T is a topological isomorphism.*

Proof. The Open Mapping Theorem implies that T is an open mapping, so if $U \subseteq X$ is open then $T(U)$ is an open subset of Y. However, since T is a bijection we have $(T^{-1})^{-1}(U) = T(U)$. Hence the inverse image under T^{-1} of any open set is open, which implies by Theorem 1.59 that T^{-1} is continuous. □

The next result is a typical application of the Inverse Mapping Theorem.

Theorem 2.30. *Suppose X is a vector space that is complete with respect to each of two norms $\|\cdot\|$ and $|||\cdot|||$. If there exists $C > 0$ such that $\|x\| \le C\,|||x|||$ for all $x \in X$, then $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on X.*

Proof. The hypotheses imply that the identity map $I\colon (X, |||\cdot|||) \to (X, \|\cdot\|)$ is a bounded bijection, so, by the Inverse Mapping Theorem, the inverse map $I^{-1}\colon (X, \|\cdot\|) \to (X, |||\cdot|||)$ is a bounded bijection. Hence there is some $c > 0$ such that

$$|||x||| \;=\; |||I^{-1}(x)||| \;\le\; c\,\|x\|,$$

so the two norms are equivalent. □

The next theorem, whose proof is Exercise 2.42, states that the adjoint of a topological isomorphism is itself a topological isomorphism.

Theorem 2.31. *Let X, Y be Banach spaces. If $T\colon X \to Y$ is a topological isomorphism, then its adjoint $T^*\colon Y^* \to X^*$ is a topological isomorphism, and if T is an isometric isomorphism then so is T^*.* $\diamondsuit$

We will use the Inverse Mapping Theorem to derive two results for operators on Hilbert spaces. The following theorem shows that a bounded linear operator has closed range if and only if its adjoint has closed range (this also holds for operators on Banach space, see [Rud91, Thm. 4.14]).

Theorem 2.32. *Fix $A \in \mathcal{B}(H,K)$, where H and K are Hilbert spaces. Then*

$$\mathrm{range}(A) \text{ is closed} \quad \Longleftrightarrow \quad \mathrm{range}(A^*) \text{ is closed.}$$

Proof. $\Leftarrow$. Suppose that $\mathrm{range}(A^*)$ is closed, and let $M = \overline{\mathrm{range}(A)}$. Define $T \in \mathcal{B}(H,M)$ by $Tx = Ax$ for $x \in H$. Since $\mathrm{range}(T)$ is dense in M, Theorem 2.13 implies that $T^*\colon M \to H$ is injective. Given $y \in K$, write $y = m+e$ where $m \in M$ and $e \in M^\perp$. Since $\ker(A^*) = \mathrm{range}(A)^\perp = M^\perp$, for any $x \in H$ we have

$$\langle x, A^*y\rangle \;=\; \langle x, A^*m\rangle \;=\; \langle Ax, m\rangle \;=\; \langle Tx, m\rangle \;=\; \langle x, T^*m\rangle.$$

Hence $A^*y = A^*m = T^*m$, and it follows from this that $\mathrm{range}(T^*) = \mathrm{range}(A^*)$, which is closed. Now set $N = \mathrm{range}(T^*)$ and define $U \in \mathcal{B}(M,N)$ by $Uy = T^*y$ for $y \in M$. Then U is a continuous bijection, so it is a topological isomorphism by the Inverse Mapping Theorem. Theorem 2.31 therefore implies that $U^* \in \mathcal{B}(N,M)$ is a topological isomorphism. In particular, $\mathrm{range}(U^*) = M$ is closed.

Fix $y \in M$, so $y = U^*x$ for some $x \in N$. Let z be any vector in K, and let p be its orthogonal projection onto M. Then, since Ax, U^*x, and y all belong to M,

$$\begin{aligned}
\langle y, z\rangle &= \langle U^*x, z\rangle \\
&= \langle U^*x, p\rangle \\
&= \langle x, Up\rangle \\
&= \langle x, T^*p\rangle \\
&= \langle Tx, p\rangle \\
&= \langle Ax, p\rangle \;=\; \langle Ax, z\rangle.
\end{aligned}$$

Therefore $y = Ax$, so $M \subseteq \mathrm{range}(A)$ and hence $\mathrm{range}(A) = M$ is closed.

$\Rightarrow$. Since $(A^*)^* = A$, this follows from the previous case. $\square$

Our next application of the Inverse Mapping Theorem constructs a "pseudoinverse" of a bounded operator A that has closed range. Although A need not be injective, the pseudoinverse $A^\dagger$ acts as a right-inverse of A, at least when we restrict the domain of $A^\dagger$ to $\mathrm{range}(A)$.

Theorem 2.33. *Let H and K be Hilbert spaces. Assume that $A \in \mathcal{B}(H,K)$ has closed range, and let P be the orthogonal projection of K onto* range(A). *Then the mapping $B\colon \ker(A)^\perp \to \mathrm{range}(A)$ defined by $Bx = Ax$ for $x \in \ker(A)^\perp$ is a topological isomorphism, and $A^\dagger = B^{-1}P \in \mathcal{B}(K,H)$ satisfies the following:*

(a) *$AA^\dagger y = y$ for every $y \in \mathrm{range}(A)$,*

(b) *$AA^\dagger$ is the orthogonal projection of K onto* range(A), *and*

(c) *$A^\dagger A$ is the orthogonal projection of H onto* range(A^*).

Proof. The mapping B is bounded and linear since it is a restriction of the bounded mapping A. Further, the fact that $H = \ker(A) \oplus \ker(A)^\perp$ implies that B is a bijection of $\ker(A)^\perp$ onto range(A). Applying the Inverse Mapping Theorem, we conclude that $B\colon \ker(A)^\perp \to \mathrm{range}(A)$ is a topological isomorphism. Hence $B^{-1}\colon \mathrm{range}(A) \to \ker(A)^\perp$ is a topological isomorphism, and therefore $A^\dagger = B^{-1}P$ is bounded. We assign the proof of statements (a)–(c) as Exercise 2.43. □

Definition 2.34 (Pseudoinverse). Given $A \in \mathcal{B}(H,K)$, the operator $A^\dagger$ constructed in Theorem 2.33 is called the *Moore–Penrose pseudoinverse*, or simply the *pseudoinverse*, of A. ♢

Exercise 2.44 gives an equivalent characterization of the pseudoinverse.

Exercises

2.36. Show that if $T\colon X \to Y$ is a topological isomorphism of a normed space X onto a normed space Y, then a sequence $\{x_n\}$ is complete in X if and only if $\{Tx_n\}$ is complete in Y.

2.37. Let X and Y be normed linear spaces. Show that if $T\colon X \to Y$ is a topological isomorphism, then $\|T^{-1}\|^{-1}\,\|x\| \le \|Tx\| \le \|T\|\,\|x\|$ for all $x \in X$.

2.38. Let X be a Banach space and Y a normed linear space. Suppose that $L\colon X \to Y$ is bounded and linear. Prove that the following two statements are equivalent.

(a) There exists $c > 0$ such that $\|Lx\| \ge c\|x\|$ for all $x \in X$.

(b) L is injective and range(L) is closed.

Show further that, in case these hold, $L\colon X \to \mathrm{range}(L)$ is a topological isomorphism.

2.39. Given a sequence of scalars $\lambda = (\lambda_k)$, define a mapping T_λ on sequences $x = (x_k)$ by $T_\lambda x = (\lambda_k x_k)$. Prove the following statements.

(a) T_λ is a bounded map of ℓ^2 into itself if and only if $\lambda \in \ell^\infty$.

(b) T_λ is a topological isomorphism of ℓ^2 onto itself if and only if $0 < \inf |\lambda_k| \le \sup |\lambda_k| < \infty$.

(c) T_λ is an isometric isomorphism of ℓ^2 onto itself if and only if $|\lambda_k| = 1$ for every n.

2.40. Let X be a Banach space. Given $T \in \mathcal{B}(X)$, define $T^0 = I$. Show that if $\|T\| < 1$, then $I - T$ is a topological isomorphism of X onto itself and $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$, where the series converges in operator norm (this is called a *Neumann series* for $(I-T)^{-1}$).

2.41. Show that if X is a Banach space, Y is a normed linear space, and $T\colon X \to Y$ is a topological isomorphism, then Y is a Banach space.

2.42. Prove Theorem 2.31.

2.43. Show that the operator $A^\dagger = B^{-1}P$ defined in Theorem 2.33 satisfies statements (a)–(c) of that theorem.

2.44. Assume that $A \in \mathcal{B}(H,K)$ has closed range, and let $A^\dagger$ be its pseudoinverse. Prove the following statements.

(a) $\ker(A^\dagger) = \operatorname{range}(A)^\perp$.

(b) $\operatorname{range}(A^\dagger) = \ker(A)^\perp$.

(c) $AA^\dagger y = y$ for all $y \in \operatorname{range}(A)$.

(d) $A^\dagger$ is the unique operator in $\mathcal{B}(K,H)$ that satisfies statements (a)–(c) above.

2.45. Let H be a Hilbert space. Given a positive definite operator $A \in \mathcal{B}(H)$, prove the following statements.

(a) A is injective and has dense range.

(b) A is a topological isomorphism of H onto itself if and only if it is surjective. Show by example that a positive definite operator need not be a topological isomorphism.

(c) If A is a surjective positive definite operator, then $(x,y) = \langle Ax, y\rangle$ defines an inner product that is equivalent to the original inner product $\langle \cdot, \cdot \rangle$ on H.

2.9 The Closed Graph Theorem

The Closed Graph Theorem provides a convenient means of testing whether a linear operator on Banach spaces is continuous.

Theorem 2.35 (Closed Graph Theorem). *Let X and Y be Banach spaces. If $T\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *T is continuous.*

(b) *If $x_n \to x$ in X and $Tx_n \to y$ in Y, then $y = Tx$.*

Proof. (a) $\Rightarrow$ (b). This follows immediately from the definition of continuity.

(b) $\Rightarrow$ (a). Assume that statement (b) holds. Define

$$|||x||| \;=\; \|x\|_X + \|Tx\|_Y, \qquad x \in X.$$

Now we appeal to Exercise 2.46, which states that $|||\cdot|||$ is a norm on X and X is complete with respect to this norm.

Since $\|x\|_X \le |||x|||$ for $x \in X$ and X is complete with respect to both norms, it follows from Theorem 2.30 that there exists a constant $C > 0$ such that $|||x||| \le C\,\|x\|_X$ for $x \in X$. Consequently, $\|Tx\|_Y \le |||x||| \le C\,\|x\|_X$, so T is bounded. □

The name of the Closed Graph Theorem comes from the fact that hypothesis (b) in Theorem 2.35 can be equivalently formulated as follows: The graph of T, $\operatorname{graph}(T) = \big\{(f, Tf) : f \in X\big\}$, is a closed subset of the product space $X \times Y$.

Exercise 2.49 shows that the hypothesis in the Closed Graph Theorem that X is complete is necessary, and it can be shown that it is also necessary that Y be complete.

Exercises

2.46. Prove the claim in Theorem 2.35 that $|||\cdot|||$ is a norm on X and X is complete with respect to this norm.

2.47. Use the Closed Graph Theorem to give another proof of Theorem 2.24.

2.48. Use the Closed Graph Theorem to give another proof of Exercise 2.34.

2.49. Let $C_b(\mathbf{R})$ and $C_b^1(\mathbf{R})$ be as in Exercise 1.22, and assume that the norm on both of these spaces is the uniform norm. In this case $C_b(\mathbf{R})$ is complete, but $C_b^1(\mathbf{R})$ is not. Show that the differentiation operator $D\colon C_b^1(\mathbf{R}) \to C_b(\mathbf{R})$ given by $Df = f'$ is unbounded, but has a closed graph, i.e., if $f_n \to f$ uniformly and $f_n' \to g$ uniformly then $f' = g$.

2.10 Weak Convergence

In this section we discuss some types of "weak convergence" that we will occasionally make use of (see especially Section 4.7). Part (a) of the following definition recalls the usual notion of convergence as given in Definition 1.2, and parts (b) and (c) introduce some new types of convergence.

Definition 2.36. Let X be a Banach space.

(a) We say that a sequence $\{x_n\}$ of elements of X *converges* to $x \in X$ if $\lim_{n\to\infty} \|x - x_n\| = 0$. For emphasis, we sometimes refer to this type of convergence as *strong convergence* or *norm convergence*. We denote norm convergence by $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

(b) A sequence $\{x_n\}$ of elements of X *converges weakly* to $x \in X$ if

$$\forall\, x^* \in X^*, \quad \lim_{n\to\infty} \langle x_n, x^* \rangle \;=\; \langle x, x^* \rangle.$$

We denote weak convergence by $x_n \xrightarrow{\text{w}} x$.

(c) A sequence $\{x_n^*\}$ of functionals in X^* *converges weak** to $x^* \in X^*$ if

$$\forall\, x \in X, \quad \lim_{n\to\infty} \langle x, x_n^* \rangle \;=\; \langle x, x^* \rangle.$$

We denote weak* convergence by $x_n^* \xrightarrow{\text{w}^*} x^*$. ♢

Note that weak* convergence only applies to convergence of functionals in a dual space X^*. However, since X^* is a Banach space, we can consider strong or weak convergence of functionals in X^* as well as weak* convergence. By definition, strong (norm), weak, and weak* convergence of a sequence $\{x_n^*\}$ in X^* mean:

$$x_n^* \to x^* \iff \lim_{n\to\infty} \|x^* - x_n^*\| \;=\; 0,$$

$$x_n^* \xrightarrow{\text{w}} x^* \iff \forall\, x^{**} \in X^{**}, \quad \lim_{n\to\infty} \langle x_n^*, x^{**} \rangle \;=\; \langle x^*, x^{**} \rangle,$$

$$x_n^* \xrightarrow{\text{w}^*} x^* \iff \forall\, x \in X, \quad \lim_{n\to\infty} \langle x, x_n^* \rangle \;=\; \langle x, x^* \rangle.$$

If X is reflexive then $X = X^{**}$, and therefore $x_n^* \xrightarrow{\text{w}} x^*$ if and only if $x_n^* \xrightarrow{\text{w}^*} x^*$. For general Banach spaces, we have the following implications.

Lemma 2.37. *Let X be a Banach space, and let $x_n,\, x \in X$ and $x_n^*,\, x^* \in X^*$ be given.*

(a) *Strong convergence in X implies weak convergence in X:*

$$x_n \to x \implies x_n \xrightarrow{\text{w}} x.$$

(b) *Weak convergence in X^* implies weak* convergence in X^*:*

$$x_n^* \xrightarrow{\text{w}} x^* \implies x_n^* \xrightarrow{\text{w}^*} x^*.$$

Proof. (a) Suppose that $x_n \to x$ strongly, and fix any $x^* \in X^*$. Since x^* is continuous we have $\lim_{n\to\infty} \langle x_n, x^* \rangle = \langle x, x^* \rangle$, so $x_n \xrightarrow{\text{w}} x$.

(b) Suppose that $x_n^*, x^* \in X^*$ and $x_n^* \xrightarrow{\text{w}} x^*$. Given $x \in X$ we have $\pi(x) \in X^{**}$, where $\pi\colon X \to X^{**}$ is the natural embedding of X into X^{**}. By definition of weak convergence, $\lim_{n\to\infty} \langle x_n^*, x^{**}\rangle = \langle x^*, x^{**}\rangle$ for every $x^{**} \in X^{**}$. Taking $x^{**} = \pi(x)$ in particular, we have

$$\lim_{n\to\infty} \langle x, x_n^*\rangle \;=\; \lim_{n\to\infty} \bigl\langle x_n^*, \pi(x)\bigr\rangle \;=\; \bigl\langle x^*, \pi(x)\bigr\rangle \;=\; \langle x, x^*\rangle.$$

Thus $x_n^* \xrightarrow{\text{w}^*} x^*$. □

It is easy to see that strongly convergent sequences are norm-bounded above. It is a more subtle fact that the same is true of weakly convergent sequences.

Theorem 2.38. *Let X be a Banach space.*

(a) *If $\{x_n\} \subseteq X$ and $x_n \xrightarrow{\text{w}} x$ in X, then x is unique and* $\sup \|x_n\|_X < \infty$.

(b) *If $\{x_n^*\} \subseteq X^*$ and $x_n^* \xrightarrow{\text{w}^*} x^*$ in X^*, then x^* is unique and* $\sup \|x_n^*\|_{X^*} < \infty$.

Proof. We prove statement (a) and assign statement (b) as Exercise 2.50.

Suppose that $x_n \xrightarrow{\text{w}} x$. If we also had $x_n \xrightarrow{\text{w}} y$, then for each $x^* \in X^*$ we would have

$$\langle x - y, x^*\rangle \;=\; \langle x, x^*\rangle - \langle y, x^*\rangle \;=\; \lim_{n\to\infty} \langle x_n, x^*\rangle - \lim_{n\to\infty} \langle x_n, x^*\rangle \;=\; 0.$$

The Hahn–Banach Theorem (Corollary 2.3) therefore implies that $x = y$.

For each $x \in X$, let $\pi(x)$ be the image of x in X^{**} under the natural embedding of X into X^{**}. Then for each $x^* \in X^*$,

$$\lim_{n\to\infty} \bigl\langle x^*, \pi(x_n)\bigr\rangle \;=\; \lim_{n\to\infty} \langle x_n, x^*\rangle \;=\; \langle x, x^*\rangle.$$

Since convergent sequences of scalars are bounded, we therefore have

$$\forall\, x^* \in X^*, \quad \sup_n \bigl|\bigl\langle x^*, \pi(x_n)\bigr\rangle\bigr| < \infty.$$

Hence, by the Uniform Boundedness Principle, $\sup \|\pi(x_n)\|_{X^{**}} < \infty$. Since $\|\pi(x_n)\|_{X^{**}} = \|x_n\|_X$ (Theorem 2.6), we conclude that $\{x_n\}$ is bounded in X. □

Strong, weak, and weak* convergence can all be defined in terms of topologies on X or X^*. For example, the strong topology is induced from the norm $\|\cdot\|$ on X. The weak topology on X is induced from the family of seminorms $\rho_{x^*}(x) = |\langle x, x^*\rangle|$ with x^* ranging through X^*. The weak* topology on X^* is induced from the family of seminorms $\rho_x(x^*) = |\langle x, x^*\rangle|$ with x ranging through X. One difference between these latter two topologies and the strong topology is that, because the weak and weak* topologies are not defined by a norm, in order to rigorously relate topological concepts to limit concepts we must use *nets* instead of ordinary sequences indexed by the natural numbers. For example, a set $E \subseteq X$ is weakly closed if its complement is an open set in the weak topology, and this is equivalent to the requirement that E contains all of its weak limit points (compare Lemma 1.16). However, the definition of a weak limit is a point $x \in X$ for which there exists a *net* $\{x_i\}_{i \in I}$ such that x_i converges to x in the appropriate net sense (see the discussion in Section 3.2).

We will not pursue the connection between weak or weak* convergence and topologies in this volume, but we sketch the proof of one result in order to give a brief (albeit incomplete) illustration of these ideas.

Theorem 2.39. *Let M be a subspace of a normed space X. If M is strongly closed (i.e., closed with respect to the norm topology), then it is weakly closed (i.e., closed with respect to the weak topology).*

Proof. If $M = X$ then we are done, so suppose that M is strongly closed and there exists some vector $x \notin M$. Then, by the Hahn–Banach Theorem (Corollary 2.4), there exists an $x^* \in X^*$ such that $x^*|_M = 0$ and $\langle x, x^* \rangle = 1$.

By definition, X^* is the set of all *strongly continuous* linear functionals on X, so we know that the functional x^* is strongly continuous. On the other hand, if $x_n \xrightarrow{\text{w}} x$ then, by definition of weak convergence, $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$. Hence, simply by definition, each element of X^* is *weakly continuous* (technically, we should justify this by using nets instead of sequences, but the idea is the same).

Just as in Theorem 1.59, weak continuity of x^* is equivalent to the fact that the inverse image of any open set in the codomain of x^* (which is $\mathbf{F}$) is weakly open in X. Therefore, since $\mathbf{F}\backslash\{0\}$ is an open subset of $\mathbf{F}$, the set $U = (x^*)^{-1}(\mathbf{F}\backslash\{0\}) \subseteq X$ is open in the weak topology. Since x^* maps every element of M to zero, no element of M is contained in U, i.e., $U \subseteq X\backslash M$. Further, $x \in U$ since $\langle x, x^* \rangle \neq 0$. Thus, given an arbitrary element $x \in X\backslash M$, we have found a weakly open set U such that $x \in U \subseteq X\backslash M$. Therefore $X\backslash M$ is open in the weak topology, which says that M is closed in the weak topology. □

The converse of Theorem 2.39 is true as well, i.e., every weakly closed subspace is strongly closed. In fact, since strong convergence always implies weak convergence, every strong limit point of an arbitrary set is a weak limit point. Therefore, if a set is weakly closed then it contains all of its weak limit points and hence contains all of its strong limit points. Thus every weakly closed set is strongly closed. By taking complements, every weakly open set is strongly open, so the weak topology is a subset of the strong topology. However, the strong and weak topologies are distinct in infinite-dimensional spaces, so in general it is not true that every strongly closed *set* is weakly closed—this is why Theorem 2.39 is interesting!

The strong, weak, and weak* topologies are only three specific examples of topologies on a Banach space X or X^*. There are many other topologies that are useful in specific applications. Additionally, there are many other useful vector spaces that are not Banach spaces, but for which topologies can still be defined. We shall not deal with such *topological vector spaces*, but instead refer to texts such as [Con90] for details.

Exercises

2.50. Prove part (b) of Theorem 2.38.

2.51. In this exercise we will denote the components of $x \in \ell^p$ by $x = \big(x(k)\big)$.

(a) Given $1 < p < \infty$ and $x_n,\, y \in \ell^p$, show that $x_n \xrightarrow{\text{w}} y$ in ℓ^p if and only if $\sup \|x_n\|_{\ell^p} < \infty$ and x_n *converges componentwise* to y, i.e., $\lim_{n\to\infty} x_n(k) = y(k)$ for each $k \in \mathbf{N}$. Does either implication remain valid if $p = 1$?

(b) Given $1 \le p \le \infty$ and $x_n,\, y \in \ell^p$, show that $x_n \xrightarrow{\text{w*}} y$ in ℓ^p if and only if x_n converges componentwise to y and $\sup \|x_n\|_{\ell^p} < \infty$ (recall that $\ell^1 \cong {c_0}^*$ and $\ell^{p'} \cong (\ell^p)^*$ for $1 \le p < \infty$).

2.52. Show that if $\{x_n\}$ is an orthonormal sequence in a Hilbert space H, then $x_n \xrightarrow{\text{w}} 0$.

Part II

Bases and Frames

3

Unconditional Convergence of Series in Banach and Hilbert Spaces

In any real or complex vector space X we can always form *finite* linear combinations $\sum_{n=1}^{N} c_n x_n$ of elements of X. However, we cannot form infinite series or "infinite linear combinations" unless we have some notion of what it means to converge in X. This is because an infinite series $\sum_{n=1}^{\infty} x_n$ is, by definition, the *limit* of the partial sums $\sum_{n=1}^{N} x_n$. Fortunately, we are interested in *normed* vector spaces. A normed space has a natural notion of convergence, and therefore we can consider infinite series and "infinite linear combinations" in these spaces.

Of course, even if X is a normed space, given arbitrary vectors $x_n \in X$ the infinite series $\sum_{n=1}^{\infty} x_n$ need not converge. If it does converge then there are additional issues about the convergence that we need to consider. These often have to do with some aspect of the "stability" of the convergence. One such stability requirement that we will focus on in great detail in this chapter is unconditional convergence, which is convergence *independent of the ordering* of the terms in the series. Another example is absolute convergence, which is *convergence of the norms* of the terms in the infinite series. Although we learn in Calculus class that these two requirements are equivalent for series of real numbers, we will see that they are not equivalent in infinite-dimensional normed spaces. Absolute convergence always implies unconditional convergence, but unconditional convergence need not imply absolute convergence in general.

These more restrictive notions of convergence will be very important to us in later chapters when we study bases and related systems in Banach and Hilbert spaces. A basis is a countable subset $\{x_n\}$ such that every vector x in our space has a unique representation of the form $x = \sum_{n=1}^{\infty} c_n x_n$. In practice, we often need to know if these representations converge unconditionally or absolutely, or are stable in other senses. We will explore bases, unconditional bases, and other systems in the later chapters of this volume. In this chapter we focus on the meaning of convergence, absolute convergence, and especially unconditional convergence of series.

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3.1 Convergence, Absolute Convergence, and Unconditional Convergence of Series

Convergent series in normed spaces were introduced in Definition 1.28. We recall that definition now, and also define Cauchy series.

Definition 3.1. Let $\{x_n\}$ be a sequence in a normed linear space X.

(a) The series $\sum_{n=1}^{\infty} x_n$ is *convergent* in X and equals $x \in X$ if the partial sums $s_N = \sum_{n=1}^{N} x_n$ converge to x in the norm of X, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N_0 > 0, \quad \forall\, N \geq N_0, \quad \|x - s_N\| \;=\; \left\| x - \sum_{n=1}^{N} x_n \right\| \;<\; \varepsilon.$$

(b) The series $\sum_{n=1}^{\infty} x_n$ is *Cauchy* in X if the sequence $\{s_N\}$ of partial sums is a Cauchy sequence in X, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N_0 > 0, \quad \forall\, N > M \geq N_0,$$

$$\|s_N - s_M\| \;=\; \left\| \sum_{n=M+1}^{N} x_n \right\| \;<\; \varepsilon. \qquad \diamondsuit$$

As noted in the opening section on General Notation, we often write $\sum x_n$ as an abbreviation for $\sum_{n=1}^{\infty} x_n$, although for clarity we sometimes write out the latter expression explicitly.

We will deal almost exclusively with Banach spaces in this volume. By definition, given a Banach space X, a series $\sum x_n$ converges in X if and only if it is a Cauchy series in X.

Here are some more restrictive types of convergence of series.

Definition 3.2. Let $\{x_n\}$ be a sequence in a Banach space X.

(a) The series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges in X for every permutation σ of $\mathbf{N}$.

(b) The series $\sum_{n=1}^{\infty} x_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. $\diamondsuit$

Although Definition 3.2(a) does not require that $\sum x_{\sigma(n)}$ converge to the same value for every permutation σ, we will see in Corollary 3.11 that if a series is unconditionally convergent then $\sum x_{\sigma(n)}$ is independent of σ.

If a series $\sum_{n=1}^{\infty} x_n$ converges but does not converge unconditionally, we say that it is *conditionally convergent*.

The next lemma shows that if (c_n) is a sequence of real or complex scalars, then $\sum c_n$ converges unconditionally if and only if it converges absolutely. Lemma 3.5 will show us that absolute convergence implies unconditional convergence in any Banach space. However, Example 3.4 shows that the converse fails in any infinite-dimensional Hilbert space. In Section 3.6 we will show that unconditional convergence is equivalent to absolute convergence only in finite-dimensional Banach spaces.

Lemma 3.3. *If (c_n) is a sequence of real or complex scalars, then*

$$\sum_n c_n \text{ converges absolutely} \iff \sum_n c_n \text{ converges unconditionally.}$$

Proof. $\Rightarrow$. Suppose that $\sum |c_n| < \infty$, and choose any $\varepsilon > 0$. Then $\sum |c_n|$ is Cauchy, so there exists some $N_0 > 0$ such that $\sum_{n=M+1}^{N} |c_n| < \varepsilon$ for all $N > M \geq N_0$. Let σ be any permutation of $\mathbf{N}$, and let

$$N_1 = \max\bigl\{\sigma^{-1}(1), \dots, \sigma^{-1}(N_0)\bigr\}.$$

Suppose that $N > M \geq N_1$ and $M+1 \leq n \leq N$. Then $n > N_1$, so $n \neq \sigma^{-1}(1), \dots, \sigma^{-1}(N_0)$. Hence $\sigma(n) \neq 1, \dots, N_0$, so $\sigma(n) > N_0$. In particular, $K = \min\bigl\{\sigma(M+1), \dots, \sigma(N)\bigr\} > N_0$ and $L = \max\bigl\{\sigma(M+1), \dots, \sigma(N)\bigr\} \geq K$, so

$$\biggl|\sum_{n=M+1}^{N} c_{\sigma(n)}\biggr| \leq \sum_{n=M+1}^{N} |c_{\sigma(n)}| \leq \sum_{n=K}^{L} |c_n| < \varepsilon.$$

Hence $\sum c_{\sigma(n)}$ is a Cauchy series of scalars, and therefore must converge.

$\Leftarrow$. Suppose first that $\sum c_n$ is a sequence of *real* scalars that does not converge absolutely. Let (p_n) be the sequence of nonnegative terms of (c_n) in order, and let (q_n) be the sequence of negative terms of (c_n) in order (where either (p_n) or (q_n) may be a finite sequence). If $\sum p_n$ and $\sum q_n$ both converge, then $\sum |c_n|$ converges and equals $\sum p_n - \sum q_n$, which is a contradiction. Hence at least one of $\sum p_n$ or $\sum q_n$ must diverge.

Suppose that $\sum p_n$ diverges. Since $p_n \geq 0$ for every n, there must exist an $m_1 > 0$ such that

$$p_1 + \cdots + p_{m_1} > 1.$$

Then, there must exist an $m_2 > m_1$ such that

$$p_1 + \cdots + p_{m_1} - q_1 + p_{m_1+1} + \cdots + p_{m_2} > 2.$$

Continuing in this way, we see that

$$p_1 + \cdots + p_{m_1} - q_1 + p_{m_1+1} + \cdots + p_{m_2} - q_2 + \cdots$$

is a rearrangement of $\sum c_n$ that diverges. Hence $\sum c_n$ cannot converge unconditionally. A similar proof applies if $\sum q_n$ diverges.

Thus we have shown, by a contrapositive argument, that if $\sum c_n$ is a series of real scalars that converges unconditionally, then it must converge absolutely. Suppose that $\sum c_n$ is a series of complex scalars that converges unconditionally. Write $c_n = a_n + ib_n$ with $a_n,\, b_n \in \mathbf{R}$, and let σ be any permutation of $\mathbf{N}$. Then $c = \sum c_{\sigma(n)}$ converges, and we can write $c = a + ib$ with $a,\, b \in \mathbf{R}$. Since $\bigl|a - \sum_{n=1}^{N} a_{\sigma(n)}\bigr| \leq \bigl|c - \sum_{n=1}^{N} c_{\sigma(n)}\bigr|$, it follows that $a = \sum a_{\sigma(n)}$ converges. This is true for every permutation σ, so $\sum a_n$ converges unconditionally, and therefore must converge absolutely since it is a series of real scalars. Similarly, $\sum b_n$ converges absolutely, and finally

$$\sum |c_n| \;=\; \sum |a_n + ib_n| \;\le\; \sum |a_n| + \sum |b_n| \;<\; \infty. \quad \square$$

Exercise 3.2 shows that if $\sum c_n$ is a sequence of real scalars that converges conditionally, then there exist permutations σ of $\mathbf{N}$ such that the series $\sum c_{\sigma(n)}$ diverges to ∞, diverges to $-\infty$, converges to any given finite real value, or oscillates without converging.

Example 3.4. The alternating harmonic series $\sum (-1)^n/n$ converges (in fact, it converges to $\ln(1/2)$, the natural logarithm of $1/2$). However, it does not converge absolutely, so it cannot converge unconditionally. $\diamondsuit$

We show now that absolute convergence always implies unconditional convergence in any Banach space.

Lemma 3.5. *Let $\{x_n\}$ be a sequence in a Banach space X. If $\sum x_n$ converges absolutely then it converges unconditionally.*

Proof. Assume that $\sum \|x_n\| < \infty$. If $M < N$, then

$$\biggl\| \sum_{n=M+1}^{N} x_n \biggr\| \;\le\; \sum_{n=M+1}^{N} \|x_n\|.$$

Since $\sum \|x_n\|$ is a Cauchy series of real numbers, it follows that $\sum x_n$ is a Cauchy series in X and therefore converges. We can repeat this argument for any permutation σ of $\mathbf{N}$ since we always have $\sum \|x_{\sigma(n)}\| < \infty$ by Lemma 3.3. Therefore $\sum x_n$ is unconditionally convergent. $\square$

However, unconditional convergence does not imply absolute convergence in general.

Example 3.6. Let $\{e_n\}$ be an infinite orthonormal sequence in an infinite-dimensional Hilbert space H. Then, by Exercise 3.1, $\sum c_n e_n$ converges if and only if it converges unconditionally, and this happens precisely for $(c_n) \in \ell^2$.

On the other hand, since $\|e_n\| = 1$, the series $\sum c_n e_n$ converges absolutely if and only if $\sum |c_n| < \infty$. Hence absolute convergence holds exactly for $(c_n) \in \ell^1$. Since ℓ^1 is a proper subset of ℓ^2, there are series $\sum c_n e_n$ which converge unconditionally but not absolutely. In particular, this is the case for the series $\sum e_n/n$. $\diamondsuit$

Note that in Example 3.6 we were able to completely characterize the collection of coefficients (c_n) such that $\sum c_n e_n$ converges, because we knew that $\{e_n\}$ was an orthonormal sequence in a Hilbert space. For arbitrary sequences $\{x_n\}$ in Hilbert or Banach spaces, it is usually much more difficult to characterize explicitly those coefficients (c_n) such that $\sum c_n x_n$ converges or converges unconditionally.

In Section 3.6 we will see that absolute convergence is equivalent to unconditional convergence only for finite-dimensional vector spaces.

Exercises

3.1. Given an orthonormal sequence $\{e_n\}$ in a Hilbert space H, prove that the following statements are equivalent.

(a) $\sum c_n e_n$ converges.

(b) $\sum c_n e_n$ converges unconditionally.

(c) $\sum |c_n|^2 < \infty$.

3.2. Assume that $\sum c_n$ is a conditionally convergent series of real scalars, i.e., the series converges but does not converge unconditionally.

(a) Let (p_n) be the sequence of nonnegative terms of (c_n) in order, and let (q_n) be the sequence of negative terms of (c_n) in order. Show that $\sum p_n$ and $\sum q_n$ must both diverge.

(b) Given $x \in \mathbf{R}$, show there exists a permutation σ of $\mathbf{N}$ such that $\sum c_{\sigma(n)}$ converges and equals x.

(c) Show that there exists a permutation σ of $\mathbf{N}$ such that $\sum c_{\sigma(n)}$ diverges to ∞, i.e., $\lim_{N\to\infty} \sum_{n=1}^{N} c_{\sigma(n)} = \infty$, and another permutation τ such that $\sum c_{\tau(n)}$ diverges to $-\infty$.

(d) Show that there exists a permutation σ of $\mathbf{N}$ such that $\sum c_{\sigma(n)}$ does not converge and does not diverge to ∞ or $-\infty$.

3.3. Let X be a normed space. Prove that the following two statements are equivalent.

(a) X is a Banach space.

(b) Every absolutely convergent series in X converges in X. That is, if (x_n) is a sequence in X and $\sum \|x_n\| < \infty$, then the series $\sum x_n$ converges in X.

3.4. Given vectors x_{mn} in a Banach space X such that $\sum_m \sum_n \|x_{mn}\| < \infty$, show that for any bijection $\sigma \colon \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ the following series exist and are equal:

$$\sum_m \biggl(\sum_n x_{mn}\biggr) = \sum_n \biggl(\sum_m x_{mn}\biggr) = \sum_k x_{\sigma(k)}.$$

Compare Theorem A.34, which formulates a similar result for series of scalars.

3.5. Let X be a Banach space, and fix $A \in \mathcal{B}(X)$. For $n > 0$ let A^n denote the usual nth power of A ($A^n = A \cdots A$, n times), and define $A^0 = I$ (the identity map on X).

(a) Given $x \in X$, show that the series $e^A(x) = \sum_{k=0}^{\infty} \frac{A^k x}{k!}$ converges absolutely in X, and show that e^A is a linear operator on X.

(b) Prove that the series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges absolutely in $\mathcal{B}(X)$, and equals the operator e^A defined in part (a). Conclude that $e^A \in \mathcal{B}(X)$ and $\|e^A\| \le e^{\|A\|}$.

(c) Prove that if $A, B \in \mathcal{B}(X)$ and $AB = BA$, then $e^A e^B = e^{A+B} = e^B e^A$.

(d) Let H be a Hilbert space. Show that if $A \in \mathcal{B}(H)$ is self-adjoint, then e^{iA} is unitary.

3.2 Convergence with Respect to the Directed Set of Finite Subsets of N

In Section 3.3 we will give several equivalent reformulations of unconditional convergence. One of these will be in terms of convergence with respect to the net determined by the finite subsets of $\mathbf{N}$. Nets are generalizations of sequences indexed by the natural numbers, and they are instrumental in formulating the notion of convergence in abstract topological spaces. We briefly review nets in this section, especially the net of finite subsets of $\mathbf{N}$.

Definition 3.7 (Directed Sets, Nets). A *directed set* is a set I together with a relation $\le$ on I such that:

(a) $\le$ is reflexive: $i \le i$ for all $i \in I$,

(b) $\le$ is transitive: $i \le j$ and $j \le k$ implies $i \le k$, and

(c) for any $i, j \in I$, there exists some $k \in I$ such that $i \le k$ and $j \le k$.

A *net* in a set X is a sequence $\{x_i\}_{i \in I}$ of elements of X indexed by a directed set I. ◇

The set of natural numbers $I = \mathbf{N}$ under the usual ordering is one example of a directed set, and hence every ordinary sequence indexed by the natural numbers is a net. Another typical example is $I = \mathcal{P}(X)$, the power set of X, ordered by inclusion, i.e., $U \le V$ if and only if $U \subseteq V$, and there are many variations on this theme.

We will not need to deal with abstract topological spaces (and hence will not even define them), but even without knowing what every term means it is interesting to see the definition of convergence of a net—it is strikingly similar in spirit to the definition of convergence of a sequence. A directed set takes the place of the natural numbers $\mathbf{N}$ and an open set U takes the place of $\varepsilon > 0$ (which really means the open ball of radius ε), but otherwise the two definitions are quite similar.

Definition 3.8 (Convergence of a Net). Let X be a topological space, let $\{x_i\}_{i \in I}$ be a net in X, and let $x \in X$ be given. Then we say that $\{x_i\}_{i \in I}$ *converges* to x if for any open set U containing x there exists an $i_0 \in I$ such that

$$i \ge i_0 \implies x_i \in U. \qquad \diamondsuit$$

Most of the topological spaces that we will encounter are normed linear spaces (and in fact are usually Banach spaces). Since open sets in normed linear spaces are defined in terms of open balls, when implementing Definition 3.8 in these spaces it suffices to consider the open balls $B_\varepsilon(x)$ centered at x instead of arbitrary open sets U that contain x.

The directed set I of interest to us here is the set of all *finite* subsets of $\mathbf{N}$:

$$I = \bigl\{F \subseteq \mathbf{N} : F \text{ is finite}\bigr\},$$

ordered by inclusion. Given a formal series $\sum x_n$ (i.e., the vectors x_n are arbitrary, and there is no requirement that the series converges in any sense), the associated net of all possible finite partial sums of this series is

$$\biggl\{\sum_{n\in F} x_n\biggr\}_{F\in I} = \biggl\{\sum_{n\in F} x_n : F \subseteq \mathbf{N}, F \text{ finite}\biggr\}.$$

Restating Definition 3.8 for the specific case of this net gives us the following definition.

Definition 3.9. Let X be a Banach space and let x_n, $x \in X$ be given. Then the series $\sum x_n$ *converges to x with respect to the directed set of finite subsets of* $\mathbf{N}$ if

$$\forall\, \varepsilon > 0, \quad \exists \text{ finite } F_0 \subseteq \mathbf{N}, \quad \forall \text{ finite } F \supseteq F_0, \quad \biggl\|x - \sum_{n\in F} x_n\biggr\| < \varepsilon.$$

In this case, we write $x = \lim_F \sum_{n\in F} x_n$. ◇

Often we abuse terminology slightly and say that a series $\sum x_n$ converges with respect to the net of finite subsets of $\mathbf{N}$, rather than the directed set of finite subsets of $\mathbf{N}$.

Convergence of $\sum x_n$ with respect to the net of finite subsets of $\mathbf{N}$ implies convergence of the series in the sense of Definition 3.1. To see this, suppose that $x = \lim_F \sum_{n\in F} x_n$ exists. Fix $\varepsilon > 0$, and let F_0 be the corresponding finite subset of $\mathbf{N}$ given by Definition 3.9. Let N_0 be the largest integer in F_0. Then for any $N > N_0$ we have $\{1, \dots, N\} \supseteq F_0$, so $\|x - \sum_{n=1}^N x_n\| < \varepsilon$. Thus the partial sums $s_N = \sum_{n=1}^N x_n$ converge to x, which precisely says that $\sum x_n$ converges and equals x. In fact, we will see in the next section that $\lim_F \sum_{n\in F} x_n$ exists if and only if the series $\sum x_n$ converges unconditionally.

Exercises

3.6. Let H be a (possibly nonseparable) Hilbert space and let I be a (possibly uncountable) index set. Show that if $\{x_i\}_{i\in I}$ is an orthonormal set in H, then the following statements hold.

(a) If $x \in H$ then $\langle x, x_i \rangle \neq 0$ for at most countably many $i \in I$.

(b) For each $x \in H$, $\sum_{i \in I} |\langle x, x_i \rangle|^2 \leq \|x\|^2$.

(c) For each $x \in H$, the series $p = \sum_{i \in I} \langle x, x_i \rangle \, x_i$ converges with respect to the net of finite subsets of I, and p is the orthogonal projection of x onto $\overline{\mathrm{span}}\{x_i\}_{i \in I}$.

3.7. Let H be a (possibly nonseparable) Hilbert space and let I be an index set. Given an orthonormal set $\{x_i\}_{i \in I}$ in H, prove that the following statements are equivalent.

(a) $\{x_i\}_{i \in I}$ is complete.

(b) For each $x \in H$ we have $x = \sum_{i \in I} \langle x, x_i \rangle \, x_i$, where the series converges with respect to the net of finite subsets of I.

(c) For each $x \in H$, $\|x\|^2 = \sum_{i \in I} |\langle x, x_i \rangle|^2$.

3.3 Equivalent Characterizations of Unconditional Convergence

The following theorem is one of the main results of this chapter. It provides several equivalent formulations of unconditional convergence in Banach spaces.

Theorem 3.10. *Given a sequence $\{x_n\}$ in a Banach space X, the following statements are equivalent.*

(a) $\sum x_n$ *converges unconditionally.*

(b) $\lim_F \sum_{n \in F} x_n$ *exists.*

(c) *For every $\varepsilon > 0$ there exists an $N > 0$ such that*

$$\forall \textit{ finite } F \subseteq \mathbf{N}, \quad \min(F) > N \implies \Big\| \sum_{n \in F} x_n \Big\| < \varepsilon.$$

(d) $\sum x_{n_j}$ *converges for every increasing sequence* $0 < n_1 < n_2 < \cdots$.

(e) $\sum \varepsilon_n x_n$ *converges for every choice of signs* $\varepsilon_n = \pm 1$.

(f) $\sum \lambda_n x_n$ *converges for every bounded sequence of scalars* (λ_n).

(g) $\sum |\langle x_n, x^* \rangle|$ *converges uniformly with respect to the unit ball in* X^*, *i.e.,*

$$\lim_{N \to \infty} \sup \left\{ \sum_{n=N}^{\infty} |\langle x_n, x^* \rangle| \; : \; x^* \in X^*, \; \|x^*\| \leq 1 \right\} \;=\; 0.$$

Proof. To illustrate some of the variety of techniques that can be employed, we will prove more implications than are strictly necessary.

(a) $\Rightarrow$ (b). Suppose that $x = \sum x_n$ is unconditionally convergent, but $\lim_F \sum_{n \in F} x_n$ does not exist. Then there is some $\varepsilon > 0$ such that

$$\forall \text{ finite } F_0, \quad \exists \text{ finite } F \supseteq F_0 \text{ such that } \Big\| x - \sum_{n \in F} x_n \Big\| \geq \varepsilon. \tag{3.1}$$

Since $\sum x_n$ converges, there is an integer $M_1 > 0$ such that

$$\forall N \geq M_1, \quad \Big\| x - \sum_{n=1}^{N} x_n \Big\| < \frac{\varepsilon}{2}.$$

Define $F_1 = \{1, \dots, M_1\}$. Then, by equation (3.1), there is a finite $G_1 \supseteq F_1$ such that $\|x - \sum_{n \in G_1} x_n\| \geq \varepsilon$. Let M_2 be the largest integer in G_1 and let $F_2 = \{1, \dots, M_2\}$. Continuing in this way, we obtain a sequence of finite sets $F_1 \subseteq G_1 \subseteq F_2 \subseteq G_2 \subseteq \cdots$ such that

$$\Big\| x - \sum_{n \in F_N} x_n \Big\| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \Big\| x - \sum_{n \in G_N} x_n \Big\| \geq \varepsilon.$$

Hence

$$\begin{aligned}
\Big\| \sum_{n \in G_N \setminus F_N} x_n \Big\| &= \Big\| \sum_{n \in G_N} x_n - \sum_{n \in F_N} x_n \Big\| \\
&\geq \Big\| x - \sum_{n \in G_N} x_n \Big\| - \Big\| x - \sum_{n \in F_N} x_n \Big\| \\
&\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, F_N must be a proper subset of G_N. Let σ be any permutation of $\mathbf{N}$ obtained by enumerating in turn the elements of F_1, then $G_1 \setminus F_1$, then $F_2 \setminus G_1$, then $G_2 \setminus F_2$, etc. Then for each N we have

$$\Big\| \sum_{n=|F_N|+1}^{|G_N|} x_{\sigma(n)} \Big\| = \Big\| \sum_{n \in G_N \setminus F_N} x_n \Big\| \geq \frac{\varepsilon}{2}.$$

Since $|F_N|, |G_N| \to \infty$ as N increases, we see that $\sum x_{\sigma(n)}$ is not Cauchy and hence cannot converge, which is a contradiction.

(b) $\Rightarrow$ (c). Suppose that $x = \lim_F \sum_{n \in F} x_n$ exists, and choose $\varepsilon > 0$. By definition, there must be a finite set $F_0 \subseteq \mathbf{N}$ such that

$$\forall \text{ finite } F \supseteq F_0, \quad \Big\| x - \sum_{n \in F} x_n \Big\| \ < \ \frac{\varepsilon}{2}.$$

Let $N = \max(F_0)$, and suppose that G is any finite subset of $\mathbf{N}$ with $\min(G) > N$. Then since $F_0 \cap G = \emptyset$,

$$\begin{aligned}
\Big\| \sum_{n \in G} x_n \Big\| &= \Big\| \Big(x - \sum_{n \in F_0} x_n \Big) - \Big(x - \sum_{n \in F_0 \cup G} x_n \Big) \Big\| \\
&\le \Big\| x - \sum_{n \in F_0} x_n \Big\| + \Big\| x - \sum_{n \in F_0 \cup G} x_n \Big\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore statement (c) holds.

(c) $\Rightarrow$ (a). Assume that statement (c) holds, and let σ be any permutation of $\mathbf{N}$. We need to show that $\sum x_{\sigma(n)}$ is Cauchy. So, choose $\varepsilon > 0$ and let N be the number whose existence is implied by statement (c). Define

$$N_0 \ = \ \max\big\{\sigma^{-1}(1), \dots, \sigma^{-1}(N)\big\}.$$

Assume that $L > K \ge N_0$, and set $F = \{\sigma(K+1), \dots, \sigma(L)\}$. If $k \ge K+1$ then $k > N_0$, so $k \ne \sigma^{-1}(1), \dots, \sigma^{-1}(N)$ and therefore $\sigma(k) \ne 1, \dots, N$. Hence

$$\min(F) \ = \ \min\big\{\sigma(K+1), \dots, \sigma(L)\big\} \ > \ N.$$

Hypothesis (c) therefore implies that

$$\Big\| \sum_{n=K+1}^{L} x_{\sigma(n)} \Big\| \ = \ \Big\| \sum_{n \in F} x_n \Big\| \ < \ \varepsilon.$$

Thus $\sum x_{\sigma(n)}$ is Cauchy and therefore must converge.

(c) $\Rightarrow$ (d). Assume that statement (c) holds, and let $0 < n_1 < n_2 < \cdots$ be any increasing set of integers. We will show that $\sum x_{n_i}$ is Cauchy, hence convergent. Given $\varepsilon > 0$ let N be the number whose existence is implied by statement (c). Let j be such that $n_j > N$. If $\ell > k \ge j$ then

$$\min\{n_{k+1}, \dots, n_\ell\} \ \ge \ n_j \ > \ N,$$

so statement (c) implies that $\big\| \sum_{i=k+1}^{\ell} x_{n_i} \big\| < \varepsilon$, as desired.

(c) $\Rightarrow$ (g). Assume that statement (c) holds, and choose $\varepsilon > 0$. Let N be the integer whose existence is guaranteed by statement (c). Given $L \ge K > N$ and any $x^* \in X^*$ with $\|x^*\| \le 1$, define

$$F^+ = \{n \in \mathbf{N} : K \le n \le L \text{ and } \mathrm{Re}(\langle x_n, x^* \rangle) \ge 0\},$$
$$F^- = \{n \in \mathbf{N} : K \le n \le L \text{ and } \mathrm{Re}(\langle x_n, x^* \rangle) < 0\}.$$

Note that $\min(F^+) \ge K > N$, so

$$\begin{aligned} \sum_{n\in F^+} |\mathrm{Re}(\langle x_n, x^*\rangle)| &= \mathrm{Re}\biggl(\sum_{n\in F^+} \langle x_n, x^*\rangle\biggr) \\ &= \mathrm{Re}\biggl(\biggl\langle \sum_{n\in F^+} x_n,\ x^* \biggr\rangle\biggr) \\ &\le \biggl|\biggl\langle \sum_{n\in F^+} x_n,\ x^* \biggr\rangle\biggr| \\ &\le \|x^*\| \, \biggl\| \sum_{n\in F^+} x_n \biggr\| < \varepsilon. \end{aligned}$$

A similar inequality holds for F^-, so $\sum_{n=K}^{L} |\mathrm{Re}(\langle x_n, x^*\rangle)| < 2\varepsilon$. If $F = \mathbf{C}$ then we apply a similar argument to the imaginary parts, and in any case obtain $\sum_{n=K}^{L} |\langle x_n, x^*\rangle| < 4\varepsilon$. Letting $L \to \infty$, we conclude that

$$K > N \quad \Longrightarrow \quad \sup\biggl\{ \sum_{n=K}^{\infty} |\langle x_n, x^*\rangle| : x^* \in X^*,\ \|x^*\| \le 1 \biggr\} \le 4\varepsilon,$$

from which statement (g) follows.

(d) $\Rightarrow$ (c) and (a) $\Rightarrow$ (c). Assume that statement (c) does not hold. Then there exists an $\varepsilon > 0$ such that for each $N \in \mathbf{N}$ there is some finite set of integers F_N such that $\min(F_N) > N$ yet $\bigl\|\sum_{n\in F_N} x_n\bigr\| \ge \varepsilon$.

Let $G_1 = F_1$ and $N_1 = \max(G_1)$. Then let $G_2 = F_{N_1}$ and $N_2 = \max(G_2)$. Continuing in this way, we obtain a sequence of finite sets G_K such that for each K,

$$\max(G_K) < \min(G_{K+1}) \qquad \text{and} \qquad \biggl\| \sum_{n\in G_K} x_n \biggr\| \ge \varepsilon. \tag{3.2}$$

Now let $0 < n_1 < n_2 < \cdots$ be a complete list of the elements of $\bigcup G_K$. It is clear then from equation (3.2) that $\sum x_{n_j}$ is not Cauchy, hence not convergent, so statement (d) does not hold.

Next let σ be any permutation of $\mathbf{N}$ obtained by enumerating in turn the elements of

$$G_1,\ \{1, \dots, \max(G_1)\}\backslash G_1,\ G_2,\ \{\max(G_1)+1, \dots, \max(G_2)\}\backslash G_2,\ G_3,\ \dots$$

As this is a complete listing of $\mathbf{N}$, it follows from equation (3.2) that $\sum x_{\sigma(n)}$ is not Cauchy, so statement (a) does not hold either.

(d) ⇒ (e). Assume that statement (d) holds and let (ε_n) be any sequence of signs $\varepsilon_n = \pm 1$. Define

$$F^+ = \{n : \varepsilon_n = 1\} \qquad \text{and} \qquad F^- = \{n : \varepsilon_n = -1\}.$$

Let $F^+ = \{n_j^+\}$ and $F^- = \{n_j^-\}$ be enumerations of F^+ and F^- in increasing order. By hypothesis, both $\sum x_{n_j^+}$ and $\sum x_{n_j^-}$ converge, whence $\sum \varepsilon_n x_n = \sum x_{n_j^+} - \sum x_{n_j^-}$ converges as well. Therefore statement (e) holds.

(e) ⇒ (d). Suppose that statement (e) holds, and let $0 < n_1 < n_2 < \cdots$ be an increasing sequence of integers. Define $\varepsilon_n = 1$ for all n, and set

$$\gamma_n = \begin{cases} 1, & \text{if } n = n_j \text{ for some } j, \\ -1, & \text{if } n \neq n_j \text{ for any } j. \end{cases}$$

By hypothesis, both $\sum \varepsilon_n x_n$ and $\sum \gamma_n x_n$ converge, so

$$\sum_j x_{n_j} = \frac{1}{2}\biggl(\sum_n \varepsilon_n x_n + \sum_n \gamma_n x_n\biggr)$$

converges as well. Therefore statement (d) holds.

(g) ⇒ (f). Suppose that statement (g) holds, and let (λ_n) be any sequence of scalars with $|\lambda_n| \le 1$. Given $\varepsilon > 0$, by hypothesis there exists a number N_0 such that

$$\forall\, K \ge N_0, \quad \sup\biggl\{\sum_{n=K}^{\infty} |\langle x_n, x^*\rangle| : x^* \in X^*,\ \|x^*\| \le 1\biggr\} < \varepsilon.$$

Suppose that $N > M \ge N_0$. By the Hahn–Banach theorem (Corollary 2.3), we can find a functional $x^* \in X^*$ such that $\|x^*\| = 1$ and

$$\biggl\langle \sum_{n=M+1}^{N} \lambda_n x_n,\ x^* \biggr\rangle = \biggl\| \sum_{n=M+1}^{N} \lambda_n x_n \biggr\|.$$

Then

$$\begin{aligned} \biggl\| \sum_{n=M+1}^{N} \lambda_n x_n \biggr\| &= \sum_{n=M+1}^{N} \lambda_n \langle x_n, x^*\rangle \\ &\le \sum_{n=M+1}^{N} |\lambda_n|\, |\langle x_n, x^*\rangle| \\ &\le \sum_{n=M+1}^{\infty} |\langle x_n, x^*\rangle| < \varepsilon. \end{aligned}$$

Hence $\sum \lambda_n x_n$ is Cauchy, and therefore must converge. Thus statement (f) holds.

(f) $\Rightarrow$ (e). This implication is trivial. $\square$

Now we can show that the value of an unconditionally convergent series is independent of the choice of permutation σ.

Corollary 3.11. *Let $\{x_n\}$ be a sequence in a Banach space X. If the series $\sum x_n$ converges unconditionally, then $\sum x_{\sigma(n)} = \sum x_n$ for every permutation σ of* $\mathbf{N}$.

Proof. Suppose that $\sum x_n$ is unconditionally convergent. Then by Theorem 3.10 we know that $x = \lim_F \sum_{n\in F} x_n$ exists. Let σ be any permutation of $\mathbf{N}$, and choose $\varepsilon > 0$. By Definition 3.9, there is a finite set $F_0 \subseteq N$ such that

$$\forall \text{ finite } F \supseteq F_0, \qquad \Big\| x - \sum_{n\in F} x_n \Big\| \;<\; \varepsilon. \tag{3.3}$$

Let N_0 be large enough so that $F_0 \subseteq \{\sigma(1), \dots, \sigma(N_0)\}$. Choose any $N \geq N_0$, and define $F = \{\sigma(1), \dots, \sigma(N)\}$. Then $F \supseteq F_0$, so by equation (3.3),

$$\Big\| x - \sum_{n=1}^{N} x_{\sigma(n)} \Big\| \;=\; \Big\| x - \sum_{n\in F} x_n \Big\| \;<\; \varepsilon.$$

Hence $x = \sum x_{\sigma(n)}$, with x independent of σ. $\square$

We conclude this section with some comments on Theorem 3.10. Intuitively, we expect that if a series $\sum x_n$ converges and we make the terms "smaller" then the series should still converge. Yet Theorem 3.10 implies that if a series $\sum x_n$ is conditionally convergent, then there will exist some scalars $|\lambda_n| \leq 1$ such that $\sum \lambda_n x_n$ no longer converges. In some sense, conditional convergence requires "miraculous cancellations," and if we multiply the terms by scalars λ_n then we can remove these cancellations and hence the convergence. For example, the alternating harmonic series $\sum (-1)^n/n$ converges, but if we multiply each term by $(-1)^n$ then we remove the cancellations that allow it to converge. Even if we require $|\lambda_n| < 1$, the series $\sum (-1)^n \lambda_n/n$ need not converge, e.g., consider $\lambda_n = (-1)^n/\ln(n+1)$.

Exercises

3.8. Let $\{x_n\}$ be a sequence in a Banach space X. Fix $1 \leq p \leq \infty$, and suppose that $T(\mu) = \big(\langle x_n, \mu\rangle\big) \in \ell^{p'}$ for every $\mu \in X^*$. Prove the following statements about T, which is called the *analysis operator* associated with $\{x_n\}$. These facts will be useful to us in Chapters 7 and 8.

(a) $T\colon X^* \to \ell^{p'}$ is bounded and linear.

(b) If $1 \le p < \infty$ then the series $\sum c_n x_n$ converges unconditionally for each $(c_n) \in \ell^p$, and the *synthesis operator* $Uc = \sum c_n x_n$ is a bounded map of ℓ^p into X.

(c) If $p = \infty$ and $(c_n) \in \ell^1$, then the series $\sum c_n x_n$ converges weakly, i.e., $\sum c_n \langle x_n, \mu \rangle$ converges for each $\mu \in X^*$. However, $\sum c_n x_n$ need not converge in the norm of X.

(d) If $1 \le p < \infty$ then $T = U^*$.

(e) If $1 < p < \infty$ and X is reflexive, then $U = T^*$.

3.4 Further Results on Unconditional Convergence

We will show in Theorem 3.15 of this section that each of the numbers $\mathcal{R}$, $\mathcal{R}_{\mathcal{E}}$, and $\mathcal{R}_{\Lambda}$ that we define next must be finite when $\sum x_n$ converges unconditionally. However, Exercise 3.9 demonstrates that the converse fails in general, i.e., finiteness of $\mathcal{R}$, $\mathcal{R}_{\mathcal{E}}$, $\mathcal{R}_{\Lambda}$ does not imply that $\sum x_n$ converges unconditionally. Still, this gives us valuable information about unconditionally convergent series that we will make use of in Section 3.5 and Chapter 6.

Notation 3.12. Given a sequence $\{x_n\}$ in a Banach space X, we associate the following numbers (which exist in the extended real sense):

$$\mathcal{R} = \sup\left\{\left\|\sum_{n\in F} x_n\right\| : \text{all finite } F \subseteq N\right\},$$

$$\mathcal{R}_{\mathcal{E}} = \sup\left\{\left\|\sum_{n\in F} \varepsilon_n x_n\right\| : \text{all finite } F \subseteq N \text{ and } \mathcal{E} = (\varepsilon_n) \text{ with } \varepsilon_n = \pm 1\right\},$$

$$\mathcal{R}_{\Lambda} = \sup\left\{\left\|\sum_{n\in F} \lambda_n x_n\right\| : \text{all finite } F \subseteq N \text{ and } \Lambda = (\lambda_n) \text{ with } |\lambda_n| \le 1\right\}. \qquad \diamondsuit$$

Note that we always have $0 \le \mathcal{R} \le \mathcal{R}_{\mathcal{E}} \le \mathcal{R}_{\Lambda} \le +\infty$. We will see in Theorem 3.14 that that any one of $\mathcal{R}$, $\mathcal{R}_{\mathcal{E}}$, $\mathcal{R}_{\Lambda}$ is finite if and only if they are all finite.

The following standard result is due to Carathéodory, e.g., see [Sin70, p. 467].

Theorem 3.13. *Given real numbers $\lambda_1, \dots, \lambda_N$ each with $|\lambda_n| \le 1$, there exist real numbers $c_k \ge 0$ and signs $\varepsilon_k^n = \pm 1$ for $k = 1, \dots, N+1$ and $n = 1, \dots, N$ such that*

$$\sum_{k=1}^{N+1} c_k = 1 \qquad \textit{and} \qquad \sum_{k=1}^{N+1} \varepsilon_k^n\, c_k = \lambda_n \quad \textit{for } n = 1, \dots, N. \qquad \diamondsuit$$

Theorem 3.14. *Let X be a Banach space. Given a sequence $\{x_n\}$ in X, the following relations hold in the extended real sense:*

(a) $\mathcal{R} \le \mathcal{R}_{\mathcal{E}} \le 2\mathcal{R}$,

(b) $\mathcal{R}_{\mathcal{E}} = \mathcal{R}_{\Lambda}$ *if* $\mathbf{F} = \mathbf{R}$,

(c) $\mathcal{R}_{\mathcal{E}} \le \mathcal{R}_{\Lambda} \le 2\mathcal{R}_{\mathcal{E}}$ *if* $\mathbf{F} = \mathbf{C}$.

As a consequence, any one of $\mathcal{R}$, $\mathcal{R}_{\mathcal{E}}$, $\mathcal{R}_{\Lambda}$ is finite if and only if the other two are finite.

Proof. (a) We have $\mathcal{R} \le \mathcal{R}_{\mathcal{E}}$ by definition. Given any finite set $F \subseteq \mathbf{N}$ and any sequence of signs $\varepsilon_n = \pm 1$, define

$$F^+ = \{n : \varepsilon_n = 1\} \qquad \text{and} \qquad F^- = \{n : \varepsilon_n = -1\}.$$

Then

$$\Big\|\sum_{n\in F} \varepsilon_n x_n\Big\| = \Big\|\sum_{n\in F^+} x_n - \sum_{n\in F^-} x_n\Big\| \le \Big\|\sum_{n\in F^+} x_n\Big\| + \Big\|\sum_{n\in F^-} x_n\Big\| \le 2\mathcal{R}.$$

Taking suprema, we obtain $\mathcal{R}_{\mathcal{E}} \le 2\mathcal{R}$.

(b) Choose any finite $F \subseteq \mathbf{N}$ and any sequence $\Lambda = (\lambda_n)$ of real scalars such that $|\lambda_n| \le 1$ for every n. Let N be the cardinality of F. Since the λ_n are real, it follows from Theorem 3.13 that there exist real numbers $c_k \ge 0$ and signs $\varepsilon_k^n = \pm 1$, where the indices range over $k = 1, \dots, N+1$ and $n \in F$, such that

$$\sum_{k=1}^{N+1} c_k = 1 \qquad \text{and} \qquad \sum_{k=1}^{N+1} \varepsilon_k^n c_k = \lambda_n \quad \text{for } n \in F.$$

Therefore,

$$\begin{aligned}\Big\|\sum_{n\in F} \lambda_n x_n\Big\| &= \Big\|\sum_{n\in F}\sum_{k=1}^{N+1} \varepsilon_k^n c_k x_n\Big\| \\ &\le \sum_{k=1}^{N+1} c_k \Big\|\sum_{n\in F} \varepsilon_k^n x_n\Big\| \\ &\le \sum_{k=1}^{N+1} c_k \mathcal{R}_{\mathcal{E}} = \mathcal{R}_{\mathcal{E}}.\end{aligned}$$

Taking suprema, we obtain $\mathcal{R}_{\Lambda} \le \mathcal{R}_{\mathcal{E}}$.

(c) Choose any finite $F \subseteq \mathbf{N}$ and any sequence $\Lambda = (\lambda_n)$ of complex scalars such that $|\lambda_n| \le 1$ for every n. Write $\lambda_n = \alpha_n + i\beta_n$ with α_n, β_n real. Then, as in the proof of part (b), we obtain

$$\Big\|\sum_{n\in F}\alpha_n x_n\Big\| \;\le\; \mathcal{R}_{\mathcal{E}} \qquad\text{and}\qquad \Big\|\sum_{n\in F}\beta_n x_n\Big\| \;\le\; \mathcal{R}_{\mathcal{E}}.$$

Therefore $\|\sum_{n\in F}\lambda_n x_n\| \le 2\mathcal{R}_{\mathcal{E}}$, from which it follows that $\mathcal{R}_{\Lambda} \le 2\mathcal{R}_{\mathcal{E}}$.

Alternative proof of (b) *and* (c). We will give another proof of statements (b) and (c) that uses the Hahn–Banach Theorem instead of Carathéodory's Theorem.

Assume first that the scalar field is real. Let $F \subseteq \mathbf{N}$ be finite, and let $\Lambda = (\lambda_n)$ be any sequence of real scalars such that $|\lambda_n| \le 1$ for each n. By the Hahn–Banach theorem (Corollary 2.3), there exists an $x^* \in X^*$ such that

$$\|x^*\| \;=\; 1 \qquad\text{and}\qquad \Big\langle\sum_{n\in F}\lambda_n x_n,\; x^*\Big\rangle \;=\; \Big\|\sum_{n\in F}\lambda_n x_n\Big\|.$$

Since x^* is a real-valued functional, $\langle x_n, x^*\rangle$ is real for every n. Define

$$\varepsilon_n \;=\; \begin{cases} 1, & \text{if } \langle x_n, x^*\rangle \ge 0,\\ -1, & \text{if } \langle x_n, x^*\rangle < 0.\end{cases}$$

Then

$$\begin{aligned}\Big\|\sum_{n\in F}\lambda_n x_n\Big\| &= \sum_{n\in F}\lambda_n\,\langle x_n, x^*\rangle\\ &\le \sum_{n\in F}|\lambda_n\,\langle x_n, x^*\rangle|\\ &\le \sum_{n\in F}|\langle x_n, x^*\rangle|\\ &= \sum_{n\in F}\varepsilon_n\,\langle x_n, x^*\rangle\\ &= \Big\langle\sum_{n\in F}\varepsilon_n x_n,\; x^*\Big\rangle\\ &\le \|x^*\|\;\Big\|\sum_{n\in F}\varepsilon_n x_n\Big\| \;=\; \Big\|\sum_{n\in F}\varepsilon_n x_n\Big\|.\end{aligned}$$

Taking suprema, we obtain $\mathcal{R}_{\Lambda} \le \mathcal{R}_{\mathcal{E}}$, as desired.

For the complex case, we split into real and imaginary parts as before, i.e., we choose any finite $F \subseteq \mathbf{N}$ and any sequence $\Lambda = (\lambda_n)$ of complex scalars such that $|\lambda_n| \le 1$ for every n, and we write $\lambda_n = \alpha_n + i\beta_n$ with α_n, β_n real. The trouble now is finding a *real-valued* functional x^* with the desired properties. We accomplish this by considering X as a Banach space over the real field instead of the complex field. That is, we let $X_{\mathbf{R}} = X$ as a set and define $\|\cdot\|_{X_\mathbf{R}} = \|\cdot\|$, but we take $\mathbf{F} = \mathbf{R}$. Then $(X_\mathbf{R}, \|\cdot\|_{X_\mathbf{R}})$ is a real Banach space by Exercise 1.7, so by Corollary 2.3 applied to $X_\mathbf{R}$ there is an $x^* \in X_\mathbf{R}^*$ such that $\|x^*\| = 1$ and $\big\langle \sum_{n\in F} \alpha_n x_n,\, x^* \big\rangle = \big\|\sum_{n\in F} \alpha_n x_n\big\|$. Then, as in part (b), we obtain $\big\|\sum_{n\in F} \alpha_n x_n\big\| \le \big\|\sum_{n\in F} \varepsilon_n x_n\big\|$, and a similar argument applies to the imaginary parts. □

Theorem 3.15. *If $\sum x_n$ converges unconditionally then $\mathcal{R}$, $\mathcal{R}_\mathcal{E}$, and $\mathcal{R}_\Lambda$ are all finite.*

Proof. By Proposition 3.14, we need only show that any one of $\mathcal{R}$, $\mathcal{R}_\mathcal{E}$, or $\mathcal{R}_\Lambda$ is finite. However, since the arguments have different flavors, we will give separate proofs of the finiteness of $\mathcal{R}$ and $\mathcal{R}_\Lambda$.

Proof that $\mathcal{R} < \infty$. Assume that $\sum x_n$ converges unconditionally. Then, by Theorem 3.10(c), we can find an $N > 0$ such that

$$\forall \text{ finite } G \subseteq \mathbf{N}, \quad \min(G) \,>\, N \implies \Big\|\sum_{n\in G} x_n\Big\| \,<\, 1.$$

Define $F_0 = \{1, \dots, N\}$ and set

$$M \;=\; \max_{F\subseteq F_0} \Big\|\sum_{n\in F} x_n\Big\|.$$

Note that $M < \infty$ since F_0 is finite.

Now choose any finite $F \subseteq \mathbf{N}$, and write $F = (F \cap F_0) \cup (F \backslash F_0)$. Then

$$\Big\|\sum_{n\in F} x_n\Big\| \;\le\; \Big\|\sum_{n\in F\cap F_0} x_n\Big\| \;+\; \Big\|\sum_{n\in F\backslash F_0} x_n\Big\| \;\le\; M+1.$$

Hence $\mathcal{R} \le M + 1 < \infty$, as desired.

Proof that $\mathcal{R}_\Lambda < \infty$. Assume that $\sum x_n$ converges unconditionally. For each finite $F \subseteq \mathbf{N}$ and each sequence $\Lambda = (\lambda_n)$ satisfying $|\lambda_n| \le 1$ for all n, define a functional $T_{F,\Lambda} \colon X^* \to \mathbf{F}$ by

$$T_{F,\Lambda}(x^*) \;=\; \Big\langle \sum_{n\in F} \lambda_n x_n,\; x^* \Big\rangle.$$

Then, by definition of the operator norm and by the Hahn–Banach Theorem (Theorem 2.3), we have

$$\|T_{F,\Lambda}\| \;=\; \sup_{\|x^*\|=1} |T_{F,\Lambda}(x^*)| \;=\; \sup_{\|x^*\|=1} \Big|\Big\langle \sum_{n\in F} \lambda_n x_n,\; x^* \Big\rangle\Big| \;=\; \Big\|\sum_{n\in F} \lambda_n x_n\Big\|.$$

Therefore, $\mathcal{R}_\Lambda$ is realized by the formula

$$\mathcal{R}_\Lambda = \sup_{F,\Lambda} \|T_{F,\Lambda}\|,$$

where the supremum is over finite $F \subseteq \mathbf{N}$ and bounded sequences of scalars $\Lambda = (\lambda_n)$.

Now let $x^* \in X^*$ be fixed. Then, by the continuity of x^* and the unconditional convergence of $\sum x_n$, we have that $\sum \langle x_{\sigma(n)}, x^* \rangle = \left\langle \sum x_{\sigma(n)},\, x^* \right\rangle$ exists for every permutation σ of $\mathbf{N}$. Therefore, the series $\sum \langle x_n, x^* \rangle$ converges unconditionally. However, the terms $\langle x_n, x^* \rangle$ in this series are scalars, and unconditional convergence of a series of scalars is equivalent to absolute convergence (Lemma 3.3). Therefore,

$$\begin{aligned} |T_{F,\Lambda}(x^*)| &= \left| \left\langle \sum_{n\in F} \lambda_n x_n,\ x^* \right\rangle \right| \\ &\le \sum_{n\in F} |\lambda_n|\, |\langle x_n, x^* \rangle| \\ &\le \sum_{n\in F} |\langle x_n, x^* \rangle|, \end{aligned}$$

and hence

$$\sup_{F,\Lambda} |T_{F,\Lambda}(x^*)| \le \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle| < \infty.$$

The Uniform Boundedness Principle (Theorem 2.22) therefore implies that $\mathcal{R}_\Lambda = \sup_{F,\Lambda} \|T_{F,\Lambda}\| < \infty$. □

Exercise 3.9 shows that the converse of Theorem 3.15 is false in general, i.e., finiteness of $\mathcal{R}$, $\mathcal{R}_\mathcal{E}$, or $\mathcal{R}_\Lambda$ need not imply that the series $\sum x_n$ converges unconditionally, or even that the series converges at all.

Exercises

3.9. (a) Let $X = \ell^\infty$, and let $\{\delta_n\}$ be the sequence of standard basis vectors (which is contained in ℓ^∞ but does not form a basis for it). Show that $\mathcal{R} = \mathcal{R}_\mathcal{E} = \mathcal{R}_\Lambda = 1$, but $\sum \delta_n$ does not converge in ℓ^∞.

(b) Exhibit a sequence $\{x_n\}$ in a separable Banach space X such that $\mathcal{R}$, $\mathcal{R}_\Lambda$, and $\mathcal{R}_\mathcal{E}$ are finite but $\sum x_n$ does not converge.

3.5 Unconditional Convergence of Series in Hilbert Spaces

In this section we will derive a necessary condition for the unconditional convergence of series in Hilbert spaces, and also extend this to the Banach spaces $L^p(E)$ with $1 \le p \le 2$.

The following result was first obtained in [Orl33].

Theorem 3.16 (Orlicz's Theorem). *If $\{x_n\}$ is a sequence in a Hilbert space H, then*

$$\sum_{n=1}^{\infty} x_n \text{ converges unconditionally} \quad\Longrightarrow\quad \sum_{n=1}^{\infty} \|x_n\|^2 < \infty. \qquad \diamondsuit$$

Orlicz's Theorem does not extend to Banach spaces in general (but see Theorem 3.27 for some specific Banach spaces in which it does hold). Further, the following example shows that the converse of Theorem 3.16 is false in general, even in Hilbert spaces.

Example 3.17. Let H be a Hilbert space, and fix any $x \in H$ with $\|x\| = 1$. Then $\bigl\|\sum_{n=M+1}^{N} c_n x\bigr\| = \bigl|\sum_{n=M+1}^{N} c_n\bigr|$, so $\sum c_n x$ converges in H if and only if $\sum c_n$ converges as a series of scalars. Likewise, $\sum c_n x$ converges unconditionally if and only if $\sum c_n$ converges unconditionally. Therefore, if $(c_n) \in \ell^2$ is such that $\sum c_n$ converges conditionally, then $\sum c_n x$ converges conditionally even though $\sum \|c_n x\|^2 = \sum |c_n|^2 < \infty$. For example, this is the case for $c_n = (-1)^n/n$. $\diamondsuit$

We will give three proofs of Orlicz's Theorem. The first is simpler, but the second and third give improved bounds on the value of $\sum \|x_n\|^2$, and each has a different flavor. We will use the numbers $\mathcal{R}$, $\mathcal{R}_{\mathcal{E}}$, and $\mathcal{R}_{\Lambda}$ introduced in Notation 3.12. By Theorem 3.15, if $\sum x_n$ converges unconditionally, then $\mathcal{R}$, $\mathcal{R}_{\mathcal{E}}$, and $\mathcal{R}_{\Lambda}$ are all finite.

The first proof requires the following simple lemma.

Lemma 3.18. *Let H be a Hilbert space, and suppose $x_1, \dots, x_N \in H$. Then there exist scalars $\lambda_1, \dots, \lambda_N$, each with $|\lambda_n| \le 1$, such that*

$$\sum_{n=1}^{N} \|x_n\|^2 \;\le\; \biggl\|\sum_{n=1}^{N} \lambda_n x_n\biggr\|^2.$$

Proof. This is clear for $N = 1$. For $N = 2$, define $\lambda_1 = 1$ and let λ_2 be the scalar such that $|\lambda_2| = 1$ and $\overline{\lambda_2}\,\langle x_1, x_2\rangle = |\langle x_1, x_2\rangle|$. Then

$$\begin{aligned} \|\lambda_1 x_1 + \lambda_2 x_2\|^2 \;&=\; \|x_1\|^2 \;+\; 2\,\mathrm{Re}\bigl(\lambda_1 \overline{\lambda_2}\,\langle x_1, x_2\rangle\bigr) \;+\; \|x_2\|^2 \\ &=\; \|x_1\|^2 \;+\; 2\,|\langle x_1, x_2\rangle| \;+\; \|x_2\|^2 \\ &\ge\; \|x_1\|^2 \;+\; \|x_2\|^2. \end{aligned}$$

The full result then follows by induction. $\square$

We can now give our first proof of Orlicz's Theorem.

Theorem 3.19. *If $\{x_n\}$ is a sequence in a Hilbert space H, then*

$$\sum_{n=1}^{\infty} \|x_n\|^2 \;\le\; \mathcal{R}_\Lambda{}^2.$$

In particular, if $\sum x_n$ *converges unconditionally, then both of these quantities are finite.*

Proof. Fix any $N > 0$. Then by Lemma 3.18, we can find scalars λ_n with $|\lambda_n| \le 1$ such that

$$\sum_{n=1}^{N} \|x_n\|^2 \;\le\; \biggl\|\sum_{n=1}^{N} \lambda_n x_n\biggr\|^2 \;\le\; \mathcal{R}_\Lambda{}^2.$$

Letting $N \to \infty$ therefore gives the result. □

The second proof uses the following lemma.

Lemma 3.20. *If* $x_1, \dots, x_N$ *are elements of a Hilbert space* H, *then*

$$\text{Average}\biggl\{\biggl\|\sum_{n=1}^{N} \varepsilon_n x_n\biggr\|^2 : \textit{all } \varepsilon_n = \pm 1\biggr\} \;=\; \sum_{n=1}^{N} \|x_n\|^2. \tag{3.4}$$

Proof. For each N, define $\mathcal{S}_N = \{(\varepsilon_1, \dots, \varepsilon_N) : \text{all } \varepsilon_n = \pm 1\}$, and note that $|\mathcal{S}_N| = 2^N$. We will proceed by induction on N. For $N = 1$ we have

$$\text{Average}\biggl\{\biggl\|\sum_{n=1}^{1} \varepsilon_n x_n\biggr\|^2 : (\varepsilon_n) \in \mathcal{S}_1\biggr\} \;=\; \frac{1}{2}\bigl(\|x_1\|^2 \;+\; \|-x_1\|^2\bigr) \;=\; \|x_1\|^2.$$

Therefore equation (3.4) holds when $N = 1$.

Suppose now that equation (3.4) holds for some $N \ge 1$. Using the Parallelogram Law (Lemma 1.36), we compute that

$$\begin{aligned}
&\text{Average}\biggl\{\biggl\|\sum_{n=1}^{N+1} \varepsilon_n x_n\biggr\|^2 : (\varepsilon_n) \in \mathcal{S}_{N+1}\biggr\} \\
&\qquad = \frac{1}{2^{N+1}} \sum_{(\varepsilon_n)\in\mathcal{S}_{N+1}} \biggl\|\sum_{n=1}^{N+1} \varepsilon_n x_n\biggr\|^2 \\
&\qquad = \frac{1}{2^{N+1}} \sum_{(\varepsilon_n)\in\mathcal{S}_{N}} \sum_{\varepsilon_{N+1}=\pm 1} \biggl\|\sum_{n=1}^{N+1} \varepsilon_n x_n\biggr\|^2 \\
&\qquad = \frac{1}{2^{N+1}} \sum_{(\varepsilon_n)\in\mathcal{S}_{N}} \Biggl(\biggl\|\sum_{n=1}^{N} \varepsilon_n x_n \;+\; x_{N+1}\biggr\|^2 \;+\; \biggl\|\sum_{n=1}^{N} \varepsilon_n x_n \;-\; x_{N+1}\biggr\|^2\Biggr) \\
&\qquad = \frac{1}{2^{N+1}} \sum_{(\varepsilon_n)\in\mathcal{S}_{N}} 2\Biggl(\biggl\|\sum_{n=1}^{N} \varepsilon_n x_n\biggr\|^2 \;+\; \|x_{N+1}\|^2\Biggr)
\end{aligned}$$

$$= \frac{1}{2^N} \sum_{(\varepsilon_n)\in\mathcal{S}_N} \left\|\sum_{n=1}^{N} \varepsilon_n x_n\right\|^2 + \frac{1}{2^N} \sum_{(\varepsilon_n)\in\mathcal{S}_N} \|x_{N+1}\|^2$$

$$= \left(\sum_{n=1}^{N} \|x_n\|^2\right) + \|x_{N+1}\|^2,$$

the last equality following from the induction hypothesis. Thus equation (3.4) holds for $N+1$ as well. □

We can now give a second proof of Orlicz's Theorem. Since $\mathcal{R}_{\mathcal{E}} \le \mathcal{R}_{\Lambda}$, the bound on the value of $\sum \|x_n\|^2$ in the following result is sharper in general than the corresponding bound in Theorem 3.19.

Theorem 3.21. *If $\{x_n\}$ is a sequence in a Hilbert space H then*

$$\sum \|x_n\|^2 \le {\mathcal{R}_{\mathcal{E}}}^2.$$

In particular, if $\sum x_n$ converges unconditionally then both of these quantities are finite.

Proof. Fix any $N > 0$. Then by Lemma 3.20,

$$\sum_{n=1}^{N} \|x_n\|^2 = \text{Average}\left\{\left\|\sum_{n=1}^{N} \varepsilon_n x_n\right\|^2 : \text{all } \varepsilon_n = \pm 1\right\}$$

$$\le \text{Average}\{{\mathcal{R}_{\mathcal{E}}}^2 : \text{all } \varepsilon_n = \pm 1\} = {\mathcal{R}_{\mathcal{E}}}^2.$$

Letting $N \to \infty$ therefore gives the result. □

Our final proof uses the Rademacher system, which is a sequence of orthonormal functions in $L^2[0,1]$, to derive Orlicz's Theorem for the special case $H = L^2(E)$. Since all separable Hilbert spaces are isometrically isomorphic (Exercise 1.71), this proves Orlicz's Theorem for all separable Hilbert spaces.

Definition 3.22. The *Rademacher system* is the sequence $\{R_n\}_{n=0}^{\infty}$ in $L^2[0,1]$ defined by

$$R_n(t) = \operatorname{sign}(\sin 2^n \pi t) = \begin{cases} 1, & t \in \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{2k}{2^n}, \frac{2k+1}{2^n}\right), \\ 0, & t = \frac{k}{2^n},\ k = 0, \dots, 2^n, \\ -1, & t \in \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right). \end{cases} \qquad \diamondsuit$$

The first four Rademacher functions are pictured in Figure 3.1.

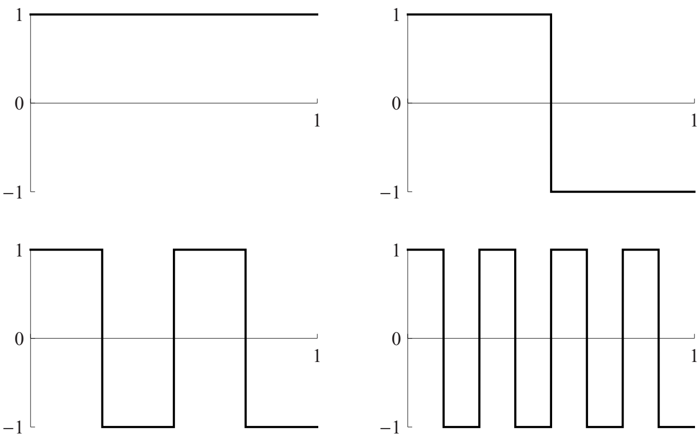

Fig. 3.1. Graphs of R_0, R_1 (top), and R_2, R_3 (bottom).

Theorem 3.23. *The Rademacher system* $\{R_n\}_{n=0}^{\infty}$ *is an orthonormal sequence in* $L^2[0,1]$*, but it is not complete.*

Proof. Since $|R_n(t)| = 1$ almost everywhere on $[0,1]$ we have $\|R_n\|_2 = 1$. Thus, Rademacher functions are normalized. To show the orthogonality, define

$$S_n^+ = \{t \in [0,1] : R_n(t) > 0\} \qquad \text{and} \qquad S_n^- = \{t \in [0,1] : R_n(t) < 0\}.$$

If $m \neq n$ then

$$\begin{aligned} \langle R_m, R_n \rangle &= |S_m^+ \cap S_n^+| - |S_m^+ \cap S_n^-| - |S_m^- \cap S_n^+| + |S_m^- \cap S_n^-| \\ &= \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 0. \end{aligned}$$

Thus $\{R_n\}_{n=0}^{\infty}$ is an orthonormal sequence in $L^2[0,1]$.

Finally, consider the function $w(t) = R_1(t)\,R_2(t)$, pictured in Figure 3.2. Reasoning similar to the above shows that $\langle w, R_n \rangle = 0$ for every $n \geq 0$. Hence $\{R_n\}_{n=0}^{\infty}$ is incomplete in $L^2[0,1]$. □

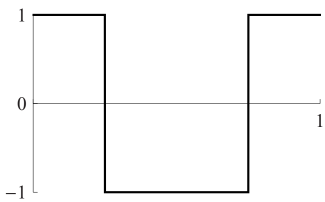

Fig. 3.2. Graph of $w(t) = R_1(t)\,R_2(t)$.

Although the Rademacher system is not complete, it is the starting point for the construction of the *Walsh system*, which is a complete orthonormal basis for $L^2[0,1]$. Elements of the Walsh system are formed by taking finite products of Rademacher functions. The graphs of the Rademacher and Walsh systems suggest a close relationship to the *Haar system* (see Example 1.54), which is the simplest wavelet orthonormal basis for $L^2(\mathbf{R})$. Indeed, the Walsh system is the system of *wavelet packets* corresponding to the Haar system. We will discuss wavelet bases in Chapter 12, and for more details on wavelet packets we refer to [Wic94].

We can now give our final proof of Orlicz's Theorem. This proof does require some knowledge of Lebesgue measure and integration.

Theorem 3.24. *Let $E \subseteq \mathbf{R}$ be measurable. If $\{f_n\}$ is a sequence of functions in $L^2(E)$ then*

$$\sum_n \|f_n\|_{L^2}^2 \;\le\; \mathcal{R}_{\mathcal{E}}{}^2.$$

In particular, if $\sum f_n$ converges unconditionally then both of these quantities are finite.

Proof. By definition, a vector f_n in $L^2(E)$ is actually an equivalence class of functions that are equal almost everywhere, so we must be careful when speaking about the value of f_n at a point. For this proof, we fix any particular representative of f_n. Since f_n is square integrable, we can take this representative to be defined and finite at all points of E.

Let $\{R_n\}_{n=0}^{\infty}$ be the Rademacher system (Definition 3.22). Since $\{R_n\}$ is an orthonormal system, the Plancherel Equality (Theorem 1.50) implies that

$$\forall\, x \in E, \qquad \biggl\|\sum_{n=1}^{N} f_n(x)\, R_n\biggr\|_{L^2[0,1]}^2 \;=\; \sum_{n=1}^{N} |f_n(x)|^2.$$

Moreover, since $R_n(t) = \pm 1$ for a.e. t, we have from the definition of $\mathcal{R}_{\mathcal{E}}$ that

$$\biggl\|\sum_{n=1}^{N} R_n(t)\, f_n\biggr\|_{L^2(E)} \;\le\; \mathcal{R}_{\mathcal{E}} \quad \text{for a.e. } t \in [0,1]. \tag{3.5}$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{N} \|f_n\|_{L^2(E)}^2 \;&=\; \int_E \sum_{n=1}^{N} |f_n(x)|^2\, dx \\
&=\; \int_E \biggl\|\sum_{n=1}^{N} f_n(x)\, R_n\biggr\|_{L^2[0,1]}^2 dx \\
&=\; \int_E \int_0^1 \biggl|\sum_{n=1}^{N} f_n(x)\, R_n(t)\biggr|^2 dt\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_E \left| \sum_{n=1}^N f_n(x)\, R_n(t) \right|^2 dx\, dt \qquad \text{(by Tonelli's Theorem)} \\
&= \int_0^1 \left\| \sum_{n=1}^N R_n(t)\, f_n \right\|_{L^2(E)}^2 dt \\
&\le \int_0^1 \mathcal{R}_{\mathcal{E}}{}^2\, dt \\
&= \mathcal{R}_{\mathcal{E}}{}^2,
\end{aligned}
$$

where Tonelli's Theorem allows us to interchange the order of integration at the point indicated because of the fact that the integrands are nonnegative. Letting $N \to \infty$ therefore gives the result. □

We used the Rademacher system in the proof of Theorem 3.24 because it is the orthonormal sequence originally used by Orlicz in his proof. However, any orthonormal sequence $\{e_n\}$ in $L^2[0,1]$ whose elements are uniformly bounded in L^∞-norm would do just as well. For, if $\|e_n\|_{L^\infty} \le M$ for all n then we can replace equation (3.5) with $\bigl\|\sum_{n=1}^N e_n(t)\, f_n\bigr\|_{L^2} \le M\mathcal{R}_\Lambda$ a.e. The remainder of the proof then carries through with R_n replaced by e_n, except that the final conclusion becomes $\sum_{n=1}^N \|e_n\|_{L^2}^2 \le (M\mathcal{R}_\Lambda)^2$.

For example, we could use the trigonometric system $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ to prove Theorem 3.24, in which case we can take $M = 1$. However, we cannot use the Haar system to prove Theorem 3.24, because it is not uniformly bounded in L^∞-norm. A typical element of the Haar system is $\psi_{nk}(t) = 2^{n/2}\psi(2^n t - k)$, where $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, so although ψ_{nk} is a unit vector in L^2-norm, with respect to L^∞-norm we have $\|\psi_{nk}\|_\infty = 2^{n/2}$.

Although the Rademacher functions did not play a special role in the proof of Theorem 3.24, we will use their specific structure to extend Theorem 3.24 to $L^p(E)$ with $1 \le p \le 2$. The key is an estimate known as *Khinchine's Inequalities* (or *Khintchine's Inequalities*). To prove this estimate we will need the *Multinomial Theorem*, which is the generalization of the Binomial Theorem to expressions containing more than two terms. Specifically, the Multinomial Theorem states that

$$
(a_1 + \cdots + a_N)^m \;=\; \sum_{\substack{j_1+\cdots+j_N=m,\\ j_n\ge 0}} \binom{m}{j_1,\dots,j_N}\, a_1^{j_1}\cdots a_N^{j_N}
$$

where

$$
\binom{m}{j_1,\dots,j_N} \;=\; \frac{m!}{j_1!\cdots j_N!},
$$

which is known as a *multinomial coefficient*.

We will also need the following inequality, whose proof is Exercise 1.13:

$$\|\cdot\|_{L^p[0,1]} \;\le\; \|\cdot\|_{L^q[0,1]}, \qquad 1 \le p \le q \le \infty. \tag{3.6}$$

Note that the following result applies to *real scalars* only.

Theorem 3.25 (Khinchine's Inequalities). *For each $1 \le p < \infty$ there exist constants $k_p, K_p > 0$ such that for every $N \in \mathbf{N}$ and* real *scalars $c_1, \dots, c_N$,*

$$k_p \left(\sum_{n=1}^N c_n^2\right)^{1/2} \;\le\; \left\|\sum_{n=1}^N c_n R_n\right\|_{L^p[0,1]} \;\le\; K_p \left(\sum_{n=1}^N c_n^2\right)^{1/2}. \tag{3.7}$$

Proof. Step 1. We begin with a generic calculation. Suppose that $j_1, \dots, j_N$ are nonnegative integers. If every j_n is even then $R_1(t)^{j_1} \cdots R_N(t)^{j_N} = 1$ for almost every t, and hence $\int_0^1 R_1(t)^{j_1} \cdots R_N(t)^{j_N}\,dt = 1$ in this case. However, if any j_n is odd then $\int_0^1 R_1(t)^{j_1} \cdots R_N(t)^{j_N}\,dt = 0$.

Step 2. Suppose that p is an even integer, say $p = 2m$ where $m \in \mathbf{N}$. Fix real scalars $c_1, \dots, c_N$ and set

$$I \;=\; \left\|\sum_{n=1}^N c_n R_n\right\|_{L^{2m}[0,1]}^{2m}.$$

Using the Multinomial Theorem and applying Step 1, we compute that

$$\begin{aligned}
I \;&=\; \int_0^1 \left(\sum_{n=1}^N c_n R_n(t)\right)^{2m} dt \\
&=\; \sum_{\substack{j_1+\cdots+j_N=2m,\\ j_n\ge 0}} \binom{2m}{j_1, \dots, j_N} c_1^{j_1} \cdots c_N^{j_N} \int_0^1 R_1(t)^{j_1} \cdots R_N(t)^{j_N}\,dt \\
&=\; \sum_{\substack{2j_1+\cdots+2j_N=2m,\\ j_n\ge 0}} \binom{2m}{2j_1, \dots, 2j_N} c_1^{2j_1} \cdots c_N^{2j_N}.
\end{aligned} \tag{3.8}$$

Now we estimate the multinomial coefficient appearing on the line above:

$$\begin{aligned}
\binom{2m}{2j_1, \dots, 2j_N} \;&=\; \frac{(2m)!}{(2j_1)! \cdots (2j_N)!} \\
&=\; \frac{(2m)(2m-1)\cdots(m+1)}{(2j_1)(2j_1-1)\cdots(j_1+1)\cdots(2j_N)(2j_N-1)\cdots(j_N+1)}\,\frac{m!}{j_1!\cdots j_N!} \\
&\le\; \frac{(2m)(2m)\cdots(2m)}{(2)(2)\cdots(2)\cdots(2)(2)\cdots(2)} \binom{m}{j_1, \dots, j_N}
\end{aligned}$$

$$= \frac{2^m\, m^m}{2^{j_1}\cdots 2^{j_N}} \binom{m}{j_1,\dots,j_N}$$
$$= m^m \binom{m}{j_1,\dots,j_N}.$$

Therefore we can continue equation (3.8) as follows:

$$I \le \sum_{\substack{j_1+\cdots+j_N=m,\\ j_n\ge 0}} m^m \binom{m}{j_1,\dots,j_N} c_1^{2j_1}\cdots c_N^{2j_N} = m^m \biggl(\sum_{n=1}^N c_n^2\biggr)^m.$$

Taking $(2m)$th roots,

$$\biggl\|\sum_{n=1}^N c_n R_n\biggr\|_{L^{2m}[0,1]} = I^{1/(2m)} \le m^{1/2} \biggl(\sum_{n=1}^N c_n^2\biggr)^{1/2},$$

so the upper inequality in equation (3.7) holds with $K_{2m} = m^{1/2}$.

Step 3. Now suppose that p is any index in the range $2 \le p < \infty$. Let $m \in \mathbf{N}$ be the integer such that $2m-2 < p \le 2m$. Then by combining Step 2 with equation (3.6), we have

$$\biggl\|\sum_{n=1}^N c_n R_n\biggr\|_{L^p[0,1]} \le \biggl\|\sum_{n=1}^N c_n R_n\biggr\|_{L^{2m}[0,1]} \le m^{1/2} \biggl(\sum_{n=1}^N c_n^2\biggr)^{1/2}.$$

Therefore the upper inequality in equation (3.7) holds with $K_p = m^{1/2}$. Also, since $p \ge 2$ the lower inequality follows from the orthonormality of the Rademacher functions and equation (3.6):

$$\biggl(\sum_{n=1}^N c_n^2\biggr)^{1/2} = \biggl\|\sum_{n=1}^N c_n R_n\biggr\|_{L^2[0,1]} \le \biggl\|\sum_{n=1}^N c_n R_n\biggr\|_{L^p[0,1]}.$$

Thus we can take $k_p = 1$ in equation (3.7) when $2 \le p < \infty$.

Step 4. Finally, suppose that $1 \le p < 2$. Let $f = \sum_{n=1}^N c_n R_n$. Applying Step 3 using the index 4, we have

$$\|f\|_{L^4[0,1]} \le K_4 \biggl(\sum_{n=1}^N c_n^2\biggr)^{1/2}. \tag{3.9}$$

Then, by using Hölder's Inequality with index $3/2$ and dual index $(3/2)' = 3$, we compute that

$$\sum_{n=1}^N c_n^2 = \|f\|_{L^2[0,1]}^2 = \int_0^1 |f(t)|^{2/3}\, |f(t)|^{4/3}\, dt$$
$$\le \biggl(\int_0^1 \bigl(|f(t)|^{2/3}\bigr)^{3/2} dt\biggr)^{2/3} \biggl(\int_0^1 \bigl(|f(t)|^{4/3}\bigr)^{3} dt\biggr)^{1/3}$$

$$
\begin{aligned}
&= \left(\int_0^1 |f(t)|\,dt\right)^{2/3} \left(\int_0^1 |f(t)|^4\,dt\right)^{1/3} \\
&= \|f\|_{L^1[0,1]}^{2/3}\, \|f\|_{L^4[0,1]}^{4/3} \\
&\le \|f\|_{L^p[0,1]}^{2/3}\, K_4^{4/3} \left(\sum_{n=1}^N c_n^2\right)^{2/3},
\end{aligned}
$$

where the final inequality comes from applying equations (3.6) and (3.9). Rearranging, we find that

$$\left(\sum_{n=1}^N c_n^2\right)^{1/2} \le \|f\|_{L^p[0,1]}\, K_4^2 = K_4^2 \left\|\sum_{n=1}^N c_n R_n\right\|_{L^p[0,1]}.$$

Hence the lower estimate in equation (3.7) holds with $k_p = K_4^{-2}$. Combining equation (3.6) with the orthonormality of the Rademacher functions we see that

$$\left\|\sum_{n=1}^N c_n R_n\right\|_{L^p[0,1]} \le \left\|\sum_{n=1}^N c_n R_n\right\|_{L^2[0,1]} = \left(\sum_{n=1}^N c_n^2\right)^{1/2}.$$

Hence the upper inequality in equation (3.7) holds with $K_p = 1$. $\square$

We will use Khinchine's Inequalities to prove the following result.

Lemma 3.26. *Fix $1 \le p < \infty$ and a measurable set $E \subseteq \mathbf{R}$. If $\{f_n\}$ is a sequence of functions in $L^p(E)$ and $\sum f_n$ converges unconditionally, then*

$$\int_E \left(\sum_{n=1}^\infty |f_n(x)|^2\right)^{p/2} dx < \infty.$$

Proof. Because Khinchine's Inequalities apply to real scalars, we first assume that the scalar field is $\mathbf{F} = \mathbf{R}$. Just as in equation (3.5), since $\sum f_n$ converges unconditionally and $R_n(t) = \pm 1$ for almost every t, we have

$$\int_E \left|\sum_{n=1}^N f_n(x)\, R_n(t)\right|^p dx = \left\|\sum_{n=1}^N R_n(t)\, f_n\right\|_{L^p(E)}^p \le \mathcal{R}_{\mathcal{E}}{}^p \quad \text{for a.e. } t \in [0,1].$$

Using Khinchine's Inequalities, we therefore have that

$$
\begin{aligned}
\int_E \left(\sum_{n=1}^N |f_n(x)|^2\right)^{p/2} dx &\le k_p^{-p} \int_E \left\|\sum_{n=1}^N f_n(x)\, R_n\right\|_{L^p[0,1]}^p dx \\
&= k_p^{-p} \int_E \int_0^1 \left|\sum_{n=1}^N f_n(x)\, R_n(t)\right|^p dt\, dx
\end{aligned}
$$

$$= k_p^{-p} \int_0^1 \int_E \left| \sum_{n=1}^{N} f_n(x)\, R_n(t) \right|^p dx\, dt \qquad (3.10)$$
$$\le k_p^{-p} \int_0^1 \mathcal{R}_{\mathcal{E}}{}^p\, dt$$
$$= k_p^{-p}\, \mathcal{R}_{\mathcal{E}}{}^p.$$

Tonelli's Theorem allows us to interchange in the order of integration at equation (3.10) because of the fact that the integrand is nonnegative. Since $\sum_{n=1}^{N} |f_n(x)|^2$ increases with N, the Monotone Convergence Theorem implies that

$$\int_E \left(\sum_{n=1}^{\infty} |f_n(x)|^2 \right)^{p/2} dx \;=\; \lim_{N\to\infty} \int_E \left(\sum_{n=1}^{N} |f_n(x)|^2 \right)^{p/2} dx$$
$$\le\; k_p^{-p}\, \mathcal{R}_{\mathcal{E}}{}^p \;<\; \infty.$$

We assign the proof for the case $\mathbf{F} = \mathbf{C}$ as Exercise 3.11. □

Now we can prove Orlicz's Theorem on unconditional convergence of series in $L^p(E)$.

Theorem 3.27 (Orlicz's Theorem). *Choose $1 \le p \le 2$ and a measurable set $E \subseteq \mathbf{R}$. If $f_n \in L^p(E)$ and $\sum f_n$ converges unconditionally in $L^p(E)$, then*

$$\sum_n \|f_n\|_{L^p}^2 \;<\; \infty.$$

Proof. As in many proofs in analysis, the key is to apply Hölder's Inequality using a clever choice of indices. Since $1 \le p \le 2$, we have $1 \le \frac{2}{p} \le 2$. The dual index to $2/p$ is the number $q = (2/p)'$ that satisfies $\frac{1}{2/p} + \frac{1}{q} = 1$. Explicitly,

$$q \;=\; (2/p)' \;=\; \frac{2/p}{2/p - 1} \;=\; \frac{2}{2-p}.$$

Therefore, if we choose any finite sequence $d = (d_1, \dots, d_N, 0, 0, \dots) \in c_{00}$, then we can apply Hölder's Inequality to obtain

$$\sum_{n=1}^{N} |f_n(x)|^p\, |d_n| \;\le\; \left(\sum_{n=1}^{N} \Bigl(|f_n(x)|^p \Bigr)^{2/p} \right)^{p/2} \left(\sum_{n=1}^{N} |d_n|^{(2/p)'} \right)^{1/(2/p)'}$$
$$=\; \left(\sum_{n=1}^{N} |f_n(x)|^2 \right)^{p/2} \left(\sum_{n=1}^{N} |d_n|^q \right)^{1/q}.$$

Let $c_n = \|f_n\|_{L^p}^p$ and set $c = (c_n)$. Then

$$\begin{aligned}
|\langle c, d\rangle| &\le \sum_{n=1}^{N} \|f_n\|_{L^p}^p \, |d_n| \\
&= \sum_{n=1}^{N} \int_E |f_n(x)|^p \, |d_n| \, dx \\
&\le \int_E \biggl(\sum_{n=1}^{N} |f_n(x)|^2\biggr)^{p/2} \biggl(\sum_{n=1}^{N} |d_n|^q\biggr)^{1/q} dx \\
&= C \biggl(\sum_{n=1}^{N} |d_n|^q\biggr)^{1/q} \\
&= C\, \|d\|_q,
\end{aligned}$$

where $C = \int_E \bigl(\sum_{n=1}^N |f_n(x)|^2\bigr)^{p/2} dx$ is finite by Lemma 3.26. Exercise 1.78 therefore implies that $c \in \ell^{q'}$, where q' is the dual index to q. Since q is the dual index to $2/p$, we conclude that $q' = 2/p$, and therefore $\sum \|f_n\|_{L^p}^2 = \|c\|_{q'}^{q'} < \infty$. □

Suppose that $\{f_n\}$ is an orthonormal basis for $L^2(E)$. Then we know exactly when $\sum c_n f_n$ converges in $L^2(E)$. Specifically, $\sum c_n f_n$ converges in $L^2(E)$ if and only if $\sum |c_n|^2 < \infty$, and in this case $\sum c_n f_n$ converges unconditionally (Exercise 3.1). What happens if we change the norm? If E has finite measure and $1 \le p \le 2$, then $L^2(E) \subseteq L^p(E)$ by Exercise 1.13, so $\{f_n\} \subseteq L^p(E)$. Can we determine exactly when $\sum c_n f_n$ converges unconditionally in L^p-norm? This is a very difficult question in general, but the following consequence of Orlicz's Theorem gives us a necessary condition for unconditional convergence. We assign the proof of this result as Exercise 3.12.

Theorem 3.28. *Fix $1 \le p \le 2$, and let $E \subseteq \mathbf{R}$ be a measurable set with $|E| < \infty$. Let $\{f_n\}$ be an orthonormal basis for $L^2(E)$ such that*

$$A = \inf_n \|f_n\|_{L^p} > 0.$$

If $f = \sum_n c_n f_n$ converges unconditionally in $L^p(E)$, then $\sum |c_n|^2 < \infty$ and $f \in L^2(E)$. ♢

Exercises

3.10. Suppose that $\{x_n\}$ is a sequence in a Hilbert space H that is bounded above and below in norm. Show that if $\sum c_n x_n$ converges unconditionally, then $(c_n) \in \ell^2$. Is the converse true?

3.11. Prove Lemma 3.26 for the case of complex scalars.

3.12. Prove Theorem 3.28.

3.13. Prove the following version of Orlicz's Theorem for the case $2 < p < \infty$: If $\sum f_n$ converges unconditionally in $L^p(E)$ then $\sum \|f_n\|_{L^p}^p < \infty$.

3.6 The Dvoretzky–Rogers Theorem

Example 3.6 showed that in any infinite-dimensional Hilbert space, there exists a series $\sum x_n$ that converges unconditionally but not absolutely. In fact, we can do this with vectors x_n that satisfy $\sum \|x_n\|^2 < \infty$. The Dvoretzky–Rogers Theorem will show us that in any infinite-dimensional Banach space there exists an unconditionally convergent series $\sum x_n$ such that $\sum \|x_n\|^2 < \infty$ but $\sum \|x_n\| = \infty$. As a consequence, unconditional and absolute convergence are only equivalent in finite-dimensional spaces.

There are several different proofs of the Dvoretzky–Rogers Theorem. The original proof [DR50] is quite geometric, and an account in that spirit can be found in [Mar69]. A proof based on p-summing operators appears in [Die84]. We follow another argument from [LT77].

Our first lemma states that in any finite-dimensional normed space we can find a basis consisting of unit vectors whose biorthogonal system also consists of unit vectors.

Definition 3.29. Let X be a finite-dimensional normed space of dimension n, and fix vectors $x_1, \dots, x_n \in X$ and functionals $a_1, \dots, a_n \in X^*$. If $\|x_k\| = 1 = \|a_k\|$ and $\langle x_j, a_k \rangle = \delta_{jk}$ for $1 \le j, k \le n$ then we call $\big(\{x_k\}_{k=1}^n, \{a_k\}_{k=1}^n\big)$ an *Auerbach system* for X. ♢

In the language of Chapter 4, an Auerbach system is a normalized basis for X whose dual basis is also normalized.

Lemma 3.30. *Every finite-dimensional normed space X has an Auerbach system* $\big(\{x_k\}_{k=1}^n, \{a_k\}_{k=1}^n\big)$.

Proof. Since X is a finite-dimensional vector space, it has a basis $\{y_1, \dots, y_n\}$. By rescaling, we can assume that $\|y_k\| = 1$ for every k. Each $x \in X$ can be written uniquely as

$$x = \sum_{k=1}^{n} c_k(x)\, y_k, \qquad x \in X,$$

and furthermore each c_k is a continuous functional since it is a linear map on a finite-dimensional normed space. Therefore we adopt our preferred notation for continuous linear functionals and write $\langle x, c_k \rangle$ instead of $c_k(x)$ in this proof. Let

$$S = \big\{x \in X : \|x\| = 1\big\}$$

be the unit sphere in X. This is a compact set since X is finite dimensional. Define a (nonlinear) functional $d\colon X^n \to \mathbf{R}$ by

$$d(z_1, \dots, z_n) \;=\; \det\bigl[\langle z_j, c_k\rangle\bigr]_{j,k=1,\dots,n}, \qquad z_1, \dots, z_n \in X.$$

Since each c_k is continuous on X and d is a polynomial function of the c_k, it follows that d is continuous on X^n (see Exercise 1.16 for a discussion of the product space X^n). As S^n is a compact subset of X^n, the continuous function $|d|$ must achieve a maximum on S^n, say at $(x_1, \dots, x_n) \in S^n$. Note that since $\langle y_j, c_k\rangle = \delta_{jk}$ we have $d(y_1, \dots, y_n) = 1$, and therefore

$$|d(x_1, \dots, x_n)| \;\ge\; |d(y_1, \dots, y_n)| \;=\; 1.$$

Now define continuous linear functionals a_k on X by

$$\langle x, a_k\rangle \;=\; \frac{d(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)}{d(x_1, \dots, x_n)}, \qquad x \in X.$$

Then

$$\langle x_k, a_k\rangle \;=\; \frac{d(x_1, \dots, x_n)}{d(x_1, \dots, x_n)} \;=\; 1.$$

On the other hand, if $j \ne k$ then $d(x_1, \dots, x_{k-1}, x_j, x_{k+1}, \dots, x_n)$ is the determinant of a matrix with two identical rows, so $\langle x_j, a_k\rangle = 0$. Finally, if x is a unit vector then

$$|\langle x, a_k\rangle| \;=\; \frac{|d(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n)|}{|d(x_1, \dots, x_n)|} \;\le\; \frac{|d(x_1, \dots, x_n)|}{|d(x_1, \dots, x_n)|} \;=\; 1,$$

while for the unit vector x_k we have $\langle x_k, a_k\rangle = 1$. Thus $\|a_k\| = 1$. □

The next lemma is the key to the proof of the Dvoretzky–Rogers Theorem. Although its proof is long, it is well worth reading as it has a kind of "surprise ending."

Lemma 3.31. *Let X be a finite-dimensional normed space with $\dim(X) = n^2$. Then there exists a subspace Y of X and an inner product $(\cdot,\cdot)$ on Y such that:*

(a) $\dim(Y) = n$,

(b) *the norm $|||\cdot|||$ on Y induced from $(\cdot,\cdot)$ satisfies $\|y\| \le |||y|||$ for all $y \in Y$,*

(c) *there exists a basis $\{y_1, \dots, y_n\}$ for Y that is orthonormal with respect to $(\cdot,\cdot)$ and satisfies $\|y_k\| \ge 1/8$ for $k = 1, \dots, n$.*

Proof. Let $\bigl(\{x_k\}_{k=1}^{n^2}, \{a_k\}_{k=1}^{n^2}\bigr)$ be an Auerbach system for X. Define an inner product $(\cdot,\cdot)_1$ on X by

$$(x, y)_1 \;=\; n^2 \sum_{k=1}^{n^2} \langle x, a_k\rangle\, \overline{\langle y, a_k\rangle}.$$

The corresponding induced norm is

$$|||x|||_1 = n\biggl(\sum_{k=1}^{n^2} |\langle x, a_k\rangle|^2\biggr)^{1/2}.$$

Since $\|x_k\| = 1 = \|a_k\|$, for any $x \in X$ we have

$$\begin{aligned}
\frac{1}{n^2}\,|||x|||_1 &= \frac{1}{n}\biggl(\sum_{k=1}^{n^2} |\langle x, a_k\rangle|^2\biggr)^{1/2} \\
&\le \frac{1}{n}\biggl(\sum_{k=1}^{n^2} \|x\|^2\,\|a_k\|^2\biggr)^{1/2} \\
&= \|x\| \\
&= \biggl\|\sum_{k=1}^{n^2} \langle x, a_k\rangle\, x_k\biggr\| \\
&\le \sum_{k=1}^{n^2} |\langle x, a_k\rangle|\,\|x_k\| \\
&\le n\biggl(\sum_{k=1}^{n^2} |\langle x, a_k\rangle|^2\biggr)^{1/2} \\
&= |||x|||_1.
\end{aligned}$$

Thus

$$\frac{1}{n^2}\,|||x|||_1 \le \|x\| \le |||x|||_1, \qquad \text{all } x \in X. \tag{3.11}$$

If $1 \le n^2 \le 8$, then

$$\frac{1}{8}\,|||x|||_1 \le \|x\| \le |||x|||_1, \qquad \text{all } x \in X.$$

In this case the proof is finished—we simply let Y be any n-dimensional subspace of X, set $(\cdot,\cdot) = (\cdot,\cdot)_1$ and $|||\cdot||| = |||\cdot|||_1$, and choose any basis $\{y_1, \dots, y_n\}$ for Y that is orthonormal with respect to $(\cdot,\cdot)$. Therefore our task is to deal with the case $n^2 > 8$.

So, fix $n^2 > 8$, and suppose that it is the case that every subspace Y of X with $\dim(Y) > \dim(X)/2 = n^2/2$ contains a vector y such that $|||y|||_1 = 1$ and $\|y\| \ge 1/8$. Since $Y = X$ is such a subspace, we can let y_1 be a vector in X such that $|||y_1|||_1 = 1$ and $\|y_1\| \ge 1/8$. Taking the orthogonal complement with respect to the inner product $(\cdot,\cdot)_1$, the subspace $Y = \operatorname{span}\{y_1\}^\perp$ has dimension $n^2 - 1 > n^2/2$, so there exists some vector $y_2 \in \operatorname{span}\{y_1\}^\perp$ such that $|||y_2|||_1 = 1$ and $\|y_2\| \ge 1/8$. Continuing in this way, we can construct at least $\frac{n^2}{2} - 1$ vectors y_k that are orthonormal with respect to $(\cdot, \cdot)_1$ and satisfy $|||y_k|||_1 = 1$ and $\|y_k\| \geq 1/8$ for each k. This gives us $\frac{n^2}{2} - 1 \geq n$ vectors with the properties we are seeking, so the proof is finished in this case.

The other possibility is that there exists some subspace X_2 of X with $\dim(X_2) > \dim(X)/2 = n^2/2$ such that $\|y\| < 1/8$ for all $y \in X_2$ with $|||y|||_1 = 1$. In this case, define

$$(\cdot, \cdot)_2 = \frac{1}{8^2}(\cdot, \cdot)_1 \qquad \text{and} \qquad |||\cdot|||_2 = \frac{1}{8}|||\cdot|||_1.$$

Despite the fact that we have simply scaled the norm and inner product, our hypotheses on X_2 combined with equation (3.11) give us the following norm equivalence:

$$\frac{8}{n^2}\,|||y|||_2 = \frac{1}{n^2}\,|||y|||_1 \leq \|y\| < \frac{1}{8}\,|||y|||_1 = |||y|||_2, \qquad y \in X_2.$$

If it is the case that $8 < n^2 \leq 8^2$, then

$$\frac{1}{8}\,|||y|||_2 \leq \frac{8}{n^2}\,|||y|||_2 \leq \|y\| < |||y|||_2, \qquad \text{all } y \in X_2.$$

Since $\dim(X_2) > n^2/2 \geq n$, we are done with this case by letting Y be any n-dimensional subspace of X_2 and taking $(\cdot, \cdot) = (\cdot, \cdot)_2$ and $|||\cdot||| = |||\cdot|||_2$.

Hence we are reduced to the case $n^2 > 8^2$. There are two possibilities. One is that every subspace Y of X_2 with $\dim(Y) > \dim(X_2)/2$ contains a vector y such that $|||y|||_2 = 1$ and $\|y\| \geq 1/8$. In this case, since $\dim(X_2)/2 > n^2/4$ we can find at least $\frac{n^2}{4} - 1$ vectors y_k that are orthonormal with respect to $(\cdot, \cdot)$ and satisfy $|||y_k|||_2 = 1$ and $\|y_k\| \geq 1/8$. Since $n^2 > 8^2$ we have $\frac{n^2}{4} - 1 \geq n$, so the proof is complete in this case.

The other possibility is that there exists a subspace X_3 of X_2 with $\dim(X_3) > \dim(X_2)/2 > n^2/4$ such that $\|y\| < 1/8$ for all $y \in X_3$ with $|||y|||_2 = 1$. In this case we set $(\cdot, \cdot)_3 = \frac{1}{8^2}(\cdot, \cdot)_2$ and $|||\cdot|||_3 = \frac{1}{8}|||\cdot|||_1$ and proceed as before, eventually settling the proof for the case $8^2 < n^2 \leq 8^3$. If $n^2 > 8^3$ we continue on, but since n is fixed, this procedure must end after finitely many steps! □

Corollary 3.32. *If X is a finite-dimensional normed space with* $\dim(X) = n^2$, *then there exist unit vectors $x_1, \dots, x_n \in X$ such that*

$$\left\|\sum_{k=1}^{n} c_k x_k\right\| \leq 8\left(\sum_{k=1}^{n} |c_k|^2\right)^{1/2}, \qquad c_1, \dots, c_n \in \mathbf{F}.$$

Proof. Let $y_1, \dots, y_n$ be the vectors whose existence is implied by Lemma 3.31, and set $x_k = y_k/\|y_k\|$. Using the notation of that lemma, since $\{y_1, \dots, y_n\}$ is orthonormal with respect to $(\cdot, \cdot)$ and since $\|y_k\| \geq 1/8$, we have

$$\begin{aligned}\left\|\sum_{k=1}^{n} c_k x_k\right\| &\le \left\|\left\|\sum_{k=1}^{n} c_k x_k\right\|\right\| \\ &= \left\|\left\|\sum_{k=1}^{n} c_k \frac{y_k}{\|y_k\|}\right\|\right\| \\ &\le 8 \left\|\left\|\sum_{k=1}^{n} c_k y_k\right\|\right\| \\ &= 8\left(\sum_{k=1}^{n} |c_k|^2\right)^{1/2},\end{aligned}$$

the last equality following from orthonormality. □

As a consequence we can prove that if absolute and unconditional convergence are equivalent in a Banach space X, then X must be finite dimensional. The converse implication, that absolute and unconditional convergence are equivalent in finite-dimensional spaces, is Exercise 3.14.

Theorem 3.33 (Dvoretzky–Rogers Theorem). *Let X be an infinite-dimensional Banach space. If $(c_n) \in \ell^2$, then there exist unit vectors $x_n \in X$ such that the series $\sum c_n x_n$ converges unconditionally in X. Consequently, there exist series that converge unconditionally but not absolutely in X.*

Proof. Choose an increasing sequence of integers (n_k) such that

$$\sum_{j=n_k}^{\infty} |c_j|^2 \le 2^{-2k}, \qquad k \in \mathbf{N}.$$

Since X is infinite dimensional, given any $k \in \mathbf{N}$ there exist subspaces of X of dimension $(n_{k+1} - n_k)^2$. Corollary 3.32 therefore implies that there exist unit vectors $x_{n_k}, \dots, x_{n_{k+1}-1}$ such that for any scalars λ_k with $|\lambda_k| \le 1$ we have

$$\left\|\sum_{j=n_k}^{n_{k+1}-1} \lambda_j c_j x_j\right\| \le 8\left(\sum_{j=n_k}^{n_{k+1}-1} |\lambda_j c_j|^2\right)^{1/2} \le 8 \cdot 2^{-k}. \tag{3.12}$$

To finish the proof we just have to show that the series $\sum \lambda_n c_n x_n$ converges, since Theorem 3.10 then implies that $\sum c_n x_n$ converges unconditionally.

So, fix $\varepsilon > 0$, and choose K large enough that

$$8 \sum_{k=K}^{\infty} 2^{-k} < \varepsilon.$$

Fix any $N > M > n_K$, and set $\mu_n = 1$ for $n = M, \dots, N$ and $\mu_n = 0$ otherwise. Appealing then to equation (3.12), we compute that

$$\begin{aligned}
\biggl\|\sum_{n=M}^{N} \lambda_n c_n x_n\biggr\| &= \biggl\|\sum_{k=K}^{\infty} \sum_{j=n_k}^{n_{k+1}-1} \mu_j \lambda_j c_j x_j\biggr\| \\
&\le \sum_{k=K}^{\infty} \biggl\|\sum_{j=n_k}^{n_{k+1}-1} \mu_j \lambda_j c_j x_j\biggr\| \\
&\le \sum_{k=K}^{\infty} 8 \biggl(\sum_{j=n_k}^{n_{k+1}-1} |\mu_j \lambda_j c_j|^2\biggr)^{1/2} \\
&= 8 \sum_{k=K}^{\infty} 2^{-k} \\
&< \varepsilon.
\end{aligned}$$

Therefore $\sum \lambda_n c_n x_n$ is Cauchy, and hence converges. □

The space $C[0,1]$ is infinite dimensional, so unconditional convergence does not imply absolute convergence in this space. That is, there exist series $\sum f_n$ in $C[0,1]$ that converge unconditionally with respect to $\|\cdot\|_\infty$, yet $\sum \|f_n\|_\infty = \infty$. Still, there is an interesting characterization of unconditional convergence in this space, due to Sierpiński [Sie10] (see [Sem82, Prop. 1.5.7]).

Theorem 3.34. *Let $f_n \in C[0,1]$ be given. Then, with respect to the norm of $C[0,1]$,*

$$\sum_n f_n \text{ converges unconditionally} \quad \Longleftrightarrow \quad \sum_n |f_n| \text{ converges.}$$

Proof. $\Leftarrow$. Suppose that the series $\sum |f_n|$ converges with respect to the uniform norm. Then $\sum |f_n|$ is a Cauchy series, so given $\varepsilon > 0$ there exists some N_0 such that

$$\forall\, N > M \ge N_0, \qquad \biggl\|\sum_{n=M}^{N} |f_n|\biggr\|_\infty < \varepsilon.$$

Let F be any finite subset of $\mathbf{N}$ with $\min(F) > N_0$, and set $M = \min(F)$ and $N = \max(F)$. Since $\bigl|\sum_{n\in F} f_n(t)\bigr| \le \sum_{n\in F} |f_n(t)|,$ we have

$$\biggl\|\sum_{n\in F} f_n\biggr\|_\infty = \biggl\|\,\biggl|\sum_{n\in F} f_n\biggr|\,\biggr\|_\infty \le \biggl\|\sum_{n\in F} |f_n|\biggr\|_\infty \le \biggl\|\sum_{n=M}^{N} |f_n|\biggr\|_\infty < \varepsilon.$$

Criterion (c) from Theorem 3.10 therefore tells us that $\sum f_n$ converges unconditionally with respect to $\|\cdot\|_\infty$.

$\Rightarrow$. Suppose that the series $\sum |f_n|$ does not converge with respect to $\|\cdot\|_\infty$. Then there exists an $\varepsilon > 0$ and an increasing sequence $M_1 < N_1 < M_2 < N_2 < M_3 < \cdots$ such that

$$\Big\| \sum_{n=M_k}^{N_k} |f_n| \Big\|_\infty \;\ge\; \varepsilon, \qquad k \in \mathbf{N}.$$

Hence for each k there exists some point t_k such that

$$\sum_{n=M_k}^{N_k} |f_n(t_k)| \;\ge\; \varepsilon, \qquad k \in \mathbf{N}.$$

For each k and all n in the range $M_k \le n \le N_k$, let $\lambda_n \in \mathbf{F}$ be a scalar of unit modulus such that $\lambda_n f_n(t_k) = |f_n(t_k)|$. For all other n, set $\lambda_n = 0$. Then (λ_n) is a bounded sequence of scalars, but for each k we have

$$\Big\| \sum_{n=M_k}^{N_k} \lambda_n f_n \Big\|_\infty \;\ge\; \Big| \sum_{n=M_k}^{N_k} \lambda_n f_n(t_k) \Big| \;=\; \sum_{n=M_k}^{N_k} |f_n(t_k)| \;\ge\; \varepsilon.$$

Hence the series $\sum \lambda_n f_n$ is not Cauchy in $C[0,1]$, so criterion (f) from Theorem 3.10 implies that $\sum f_n$ does not converge unconditionally in $C[0,1]$. □

Obviously, the interaction between the absolute value of f and the norm of $C[0,1]$ plays an important role in this proof. To expand on this, let the scalar field be real. Then $C[0,1]$ has a natural partial ordering $\le$ defined by

$$f \le g \quad \Longleftrightarrow \quad f(t) \le g(t) \text{ for all } t \in [0,1]. \tag{3.13}$$

With respect to this partial order, $C[0,1]$ is a Banach lattice in the following sense.

Definition 3.35 (Banach Lattice). Let X be a real Banach space, and let $\le$ be a partial order on X. Then X is a *Banach lattice* if the following statements hold.

(a) $x \le y \implies x + z \le y + z$ for all $x,\ y,\ z \in X$.

(b) $x \ge 0$ and $a \ge 0 \implies ax \ge 0$ for all $x \in X$ and $a \in \mathbf{R}$.

(c) Every pair of elements $x,\ y \in X$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

(d) $|x| \le |y| \implies \|x\| \le \|y\|$ for all $x,\ y \in X$, where $|x|$ is defined by

$$|x| \;=\; x \vee (-x). \qquad \diamondsuit$$

In this definition, a partial order is a relation that is reflexive, symmetric, and transitive, and $x \vee y$ is a least upper bound for x and y if $x,\ y \le x \vee y$, and $x,\ y \le z$ implies $x \vee y \le z$.

For a discussion of a generalization of Theorem 3.34 to Banach lattices, see [DW02], and for general details on bases and Banach lattices, see [LT79].

Exercises

3.14. Let X be a finite-dimensional normed space. Show that a series $\sum x_n$ in X is unconditionally convergent if and only if it is absolutely convergent.

3.15. Let $\mathbf{F} = \mathbf{R}$. Show that $C[0,1]$ is a Banach lattice with respect to the ordering $\le$ given in equation (3.13). Also show that $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$, and $|f|$ is the ordinary absolute value of f.

Remark: Another example of a Banach lattice is given in Exercise 6.6.

4

Bases in Banach Spaces

Now we reach the centerpiece of this volume, which is the theory of bases in Banach spaces. Since every Banach space is a vector space, it has a basis in the ordinary vector space sense, i.e., a set that spans and is linearly independent. However, this definition of basis restricts us to using only *finite* linear combinations of vectors, while in any normed space it makes sense to deal with infinite series. Restricting to finite linear combinations when working in an infinite-dimensional space is simply too restrictive for most purposes. Moreover, the proof that a vector space basis exists is nonconstructive in general, as it relies on the Axiom of Choice. Hence we need a new notion of basis that is appropriate for infinite-dimensional Banach spaces, and that is the main topic of this chapter.

In Section 4.1 we will review the existence, properties, and disadvantages of vector space bases, which we will call *Hamel bases* in order to distinguish them from the more interesting bases that we will consider for Banach spaces. Then in Sections 4.2–4.7 we define and study bases for Banach spaces and give some important specific examples of bases. The final section of this chapter is optional, and discusses some generalization of bases to the weak and weak* topologies on Banach spaces.

4.1 Hamel Bases

We begin with Hamel bases, which are the bases that we are familiar with from linear algebra.

Definition 4.1 (Hamel Basis). Let V be a vector space. A sequence of vectors $\{x_i\}_{i\in I}$ is a *Hamel basis* for V if

(a) the finite linear span of $\{x_i\}_{i\in I}$ is V, i.e., $\text{span}\{x_i\}_{i\in I} = V$, and

(b) $\{x_i\}_{i\in I}$ is finitely linear independent. $\diamond$

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Note that we do not require the index set I of a Hamel basis to be countable. Equivalent formulations of Definition 4.1 are that $\{x_i\}_{i\in I}$ is a Hamel basis for V if and only if every nonzero vector $x \in V$ can be written as $x = \sum_{k=1}^{N} c_k x_{i_k}$ for a unique choice of indices $i_1, \dots, i_N \in I$ and unique nonzero scalars $c_1, \dots, c_N$, or that every $x \in V$ can be written uniquely as $x = \sum_{i\in I} a_i(x)\, x_i$ for some unique choice of scalars $a_i(x)$ of which at most finitely many are nonzero.

In finite-dimensional linear algebra, a Hamel basis is usually just called a "basis." However, when dealing with Banach spaces, the term "basis" is usually reserved for a different concept, one that we will explore in depth in the following sections. In this volume, the unqualified term "basis" will always refer to the Banach space definition of basis that appears in Definition 4.3.

An argument based on the Axiom of Choice in the form of Zorn's Lemma, similar to the one used in the proof of Theorem 1.56, shows that every vector space has a Hamel basis (see Exercise 4.1). It can be shown that all Hamel bases for a given vector space have exactly the same cardinality, and that cardinality is called the *dimension* of the space.

Hamel bases are extremely useful in finite-dimensional vector spaces and in vector spaces with countably infinite dimension. For example, the set of monomials $\{x^k\}_{k\ge 0}$ is a Hamel basis for the vector space of polynomials $\mathcal{P}$. However, for a generic vector space V we usually only know that a Hamel basis for V exists because of the Axiom of Choice. In fact, it is known that the statement "Every vector space has a Hamel basis" is one of many equivalent formulations of the Axiom of Choice.

Unfortunately, a Hamel basis for an infinite-dimensional Banach space must be uncountable (see Exercise 4.2). Besides the fact that there is usually no way to constructively exhibit such a basis, an uncountable Hamel basis is generally too unwieldy to be of much use. Therefore, in the next section we will introduce a definition of a basis for a Banach space that allows the use of "infinite linear combinations," rather than just the finite linear combinations to which Hamel bases are restricted.

One interesting thing that we can use Hamel bases for is to show that if X is an infinite-dimensional Banach space, then there exist linear functionals on X that are not continuous.

Example 4.2. Let X be an infinite-dimensional Banach space, and let $\{x_i\}_{i\in I}$ be a Hamel basis for X. By dividing each vector by its norm, we can assume that $\|x_i\| = 1$ for every $i \in I$. Let $J_0 = \{j_1, j_2, \dots\}$ be any countable subsequence of I. Define a scalar-valued function μ on $\{x_i\}_{i\in I}$ by setting $\mu(x_{j_n}) = n$ for $n \in \mathbf{N}$ and $\mu(x_i) = 0$ for $i \in I\backslash J_0$. Then extend μ linearly to all of X: Each nonzero vector $x \in X$ has a unique representation as $x = \sum_{k=1}^{N} c_k x_{i_k}$ for some $i_1, \dots, i_N \in I$ and nonzero scalars $c_1, \dots, c_N$, so we define $\mu(x) = \sum_{k=1}^{N} c_k\, \mu(x_{i_k})$. We also set $\mu(0) = 0$. Then μ is a linear functional on X, but since $\|x_{j_n}\| = 1$ yet $|\mu(x_{j_n})| = n$, the functional μ is unbounded. ◇

Exercises

4.1. Use Zorn's Lemma to show that if V is a vector space then there exists a sequence $\{x_i\}_{i\in I}$ that is a Hamel basis for V.

Remark: Although we focus in this volume on real and complex vector spaces, the argument of this exercise applies to any vector space V over any field $\mathbf{F}$.

4.2. Let X be an infinite-dimensional Banach space, and prove the following statements.

(a) Any Hamel basis for X must be uncountable.

(b) Any infinite-dimensional subspace of X that has a countable Hamel basis is a meager subset of X, and cannot be a closed subspace of X.

(c) By Exercise 2.26, $C_c(\mathbf{R})$ is a meager, dense subspace of $C_0(\mathbf{R})$. Show that $C_c(\mathbf{R})$ does not have a countable Hamel basis.

4.3. Let $\{x_i\}_{i\in I}$ be a Hamel basis for an infinite-dimensional Banach space X. Then each $x \in X$ can be written uniquely as $x = \sum_{i\in I} a_i(x)\, x_i$ where at most finitely many of the scalars $a_i(x)$ are nonzero. Each a_i is a linear functional on X, and we call $\{a_i\}_{i\in I}$ the sequence of *coefficient functionals* associated with the Hamel basis $\{x_i\}_{i\in I}$. This exercise addresses the question of whether these coefficient functionals can be continuous.

(a) Show by example that it is possible for some particular functional a_i to be continuous.

(b) Show that $a_i(x_j) = \delta_{ij}$ for $i, j \in I$, where δ_{ij} is the Kronecker delta.

(c) Let $J = \{i \in I : a_i \text{ is continuous}\}$. Show that $\sup_{j\in J} \|a_j\| < \infty$.

(d) Show that at most finitely many functionals a_i can be continuous, i.e., J is finite.

(e) Give an example of an infinite-dimensional normed linear space that has a Hamel basis $\{x_i\}_{i\in I}$ such that each of the associated coefficient functionals a_i for $i \in I$ is continuous.

4.4. Let μ be an unbounded linear functional on an infinite-dimensional Banach space X (see Example 4.2). Then by Exercise 1.16, $X_1 = X \times \mathbf{F}$ is a Banach space with respect to the norm $\|(x,c)\|_{X_1} = \|x\|_X + |c|$. Set $Y = \operatorname{graph}(\mu) = \{(x, \mu(x)) : x \in X\}$, and define $\|(x,\mu(x))\|_Y = \|x\|_X$.

(a) Show that $(Y, \|\cdot\|_Y)$ is a Banach space.

(b) Show that even though $Y \subseteq X_1$, the normed space $(Y, \|\cdot\|_Y)$ is not continuously embedded into $(X_1, \|\cdot\|_{X_1})$, i.e., the mapping $I\colon (Y, \|\cdot\|_Y) \to (X_1, \|\cdot\|_{X_1})$ given by $I(z) = z$ is not continuous.

4.5. Since the set $\mathbf{Q}$ of rational numbers is a field, we can consider the vector space $\mathbf{R}$ over the field $\mathbf{Q}$. By Exercise 4.1, there exists a Hamel basis $\{x_i\}_{i\in I}$ for $\mathbf{R}$ over $\mathbf{Q}$. Hence every nonzero number $x \in \mathbf{R}$ can be written uniquely as $x = \sum_{k=1}^{N} c_k x_{i_k}$ for some $i_1, \dots, i_N \in I$ and nonzero *rational* scalars $c_1, \dots, c_N$. Use this to show that there exists a function $f \colon \mathbf{R} \to \mathbf{R}$ that satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$ but does *not* satisfy $f(cx) = cf(x)$ for all $c, x \in \mathbf{R}$. Thus f is not linear, even though f respects addition.

4.2 Bases

As we have seen, every vector space has a Hamel basis that is *finitely* linearly independent and whose *finite* linear span is the entire space. However, when we deal with normed spaces there are many good reasons why we do not want to restrict ourselves to just *finite* linear combinations. Since we have a notion of convergence in a normed space, we can create infinite series. Therefore, we introduce the following notion of a basis in a Banach space.

Definition 4.3 (Basis). A countable sequence $\{x_n\}$ in a Banach space X is a *basis* for X if

$$\forall\, x \in X, \quad \exists \text{ unique scalars } a_n(x) \text{ such that } x = \sum_n a_n(x)\, x_n. \tag{4.1}$$

We call the series in equation (4.1) the *basis expansion* or the *basis representation* of x with respect to $\{x_n\}$. $\diamondsuit$

Some remarks and observations about this definition are in order.

Remark 4.4. (a) We briefly introduced and discussed bases in Section 1.6. There we were mostly interested in the question of the basis properties of orthonormal sequences in Hilbert spaces, while here we are considering the more general (and difficult) setting of Banach spaces.

(b) We restrict ourselves to Banach spaces in order to avoid pathologies from having Cauchy series that do not converge. Since every normed space has a unique completion that is a Banach space (Exercise 1.25), this is not a significant imposition.

(c) If $\{x_n\}$ is a basis, then since the representation of each $x \in X$ as $x = \sum a_n(x)\, x_n$ is unique, we must have $x_n \neq 0$ for every n. Consequently, the sequence $\{x_n/\|x_n\|\}$ is a basis for X consisting of unit vectors (in Definition 4.5 we will call such a basis *normalized*).

(d) The definition of basis requires that $\{x_n\}$ be a *countable* sequence. Sometimes, as in Exercises 3.6 and 3.7, it is possible to deal with uncountable systems that have basis-like properties, but to avoid confusion we will not call such systems bases.

(e) The definition of basis requires that the basis series expansions $x = \sum a_n(x)\, x_n$ converge *in norm*. We could consider other notions of convergence, e.g., weak or weak* convergence of the series. We will consider these types of generalizations in Section 4.7.

(f) If $\{x_n\}$ is a basis for X then $\{x_n\}$ is a countable complete sequence in X. Consequently, the set of all finite linear combinations $\sum_{n=1}^{N} c_n x_n$ with rational c_n forms a countable, dense subset of X (see Theorem 1.27), so X is separable. The question of whether every separable Banach space possesses a basis was a longstanding problem known as the *Basis Problem*. It was shown by Enflo [Enf73a] that there exist separable, reflexive Banach spaces which do not possess any bases! ♢

Some types of bases that have useful extra properties are identified in the next definition.

Definition 4.5. Let $\{x_n\}$ be a basis for a Banach space X.

(a) $\{x_n\}$ is an *unconditional basis* if the series in equation (4.1) converge unconditionally for each $x \in X$. A basis that is not an unconditional basis is called a *conditional basis.*

(b) $\{x_n\}$ is an *absolutely convergent basis* if the series in equation (4.1) converge absolutely for each $x \in X$.

(c) $\{x_n\}$ is a *bounded basis* if $\{x_n\}$ is norm-bounded both above and below, i.e., if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

(d) $\{x_n\}$ is a *normalized basis* if $\|x_n\| = 1$ for every $n \in \mathbf{N}$. ♢

Absolutely convergent bases have a simple characterization: A Banach space X has an absolutely convergent basis if and only if X is topologically isomorphic to ℓ^1 (see Exercise 4.14). Unconditional bases are much more interesting and will be studied in detail in Chapter 6. A great advantage of unconditional bases is that the ordering of the index set is irrelevant. Hence any countable set can be used as the index set of an unconditional basis, whereas if we have a conditional basis indexed by a countable set other than $\mathbf{N}$ then we must specify the ordering of the index set.

Sometimes we need to deal with sequences that are bases for a closed subspace of X rather than the entire space. We use the following terminology for such sequences.

Definition 4.6 (Basic Sequence). Let X be a Banach space. A sequence $\{x_n\}$ in X is a *basic sequence* in X if it is a basis for $\overline{\text{span}}\{x_n\}$. ♢

Most of the terminology for bases carries over to basic sequences. For example, $\{x_n\}$ is an unconditional basic sequence if it is an unconditional basis for $\overline{\text{span}}\{x_n\}$, etc.

Exercises

4.6. Let $\{x_n\}$ be a basis for a Banach space X, and let (λ_n) be a sequence of nonzero scalars.

(a) Show that $\{\lambda_n x_n\}$ is a basis for X. In particular, $\{x_n/\|x_n\|\}$ is a normalized basis for X.

(b) If $\{x_n\}$ is an unconditional basis, will $\{\lambda_n x_n\}$ be an unconditional basis?

(c) If $\{x_n\}$ is an absolutely convergent basis, will $\{\lambda_n x_n\}$ be an absolutely convergent basis?

4.7. Let H be a separable Hilbert space.

(a) Show that an orthonormal basis for H is a normalized unconditional basis for H.

(b) Give an example of a basis $\{x_n\}$ for H that is not an orthogonal sequence.

(c) Give an example of a basis $\{x_n\}$ for H that contains no orthogonal subsequences (equivalently, $\langle x_m, x_n\rangle \neq 0$ for all $m \neq n$).

4.8. (a) Show that the standard basis $\{\delta_n\}_{n\in\mathbf{N}}$ is a normalized unconditional basis for ℓ^p for each $1 \le p < \infty$, and is also a normalized unconditional basis for c_0.

(b) By Exercise 1.20, $c = \bigl\{x = (x_n) \in \ell^\infty : \lim_{n\to\infty} x_n \text{ exists}\bigr\}$ is a closed subspace of ℓ^∞, and c_0 is a proper closed subspace of c. Find a vector $\delta_0 \in c$ such that $\{\delta_n\}_{n\ge 0}$ is a normalized unconditional basis for c.

(c) Show that c^* is isometrically isomorphic to ℓ^1. Compare Exercise 1.75, which shows that we also have ${c_0}^* \cong \ell^1$, and Exercise 4.22, which shows that c and c_0 are topologically isomorphic but not isometrically isomorphic.

4.9. For each $n \in \mathbf{N}$, define $y_n = (1, \dots, 1, 0, 0, \dots)$, where the 1 is repeated n times. Show that $\{y_n\}$ is a normalized conditional basis for c_0.

4.10. For each $n \in \mathbf{N}$, define $z_n = (0, \dots, 0, 1, 1, \dots)$, where the 0 is repeated $n-1$ times. Show that $\{z_n\}$ is a normalized conditional basis for c (this is called the *summing basis* for c).

4.3 Schauder Bases

Suppose that $\{x_n\}$ is a basis for a Banach space X. Then the requirement that the basis expansions in equation (4.1) are unique implies that the coefficients $a_n(x)$ are linear functions of x, and the sequence $\{a_n\}$ is uniquely determined by $\{x_n\}$.

Definition 4.7 (Coefficient Functionals). Given a basis $\{x_n\}$ for a Banach space X, the sequence of linear functionals $\{a_n\}$ defined by equation (4.1) is called the *associated sequence of coefficient functionals*, or simply the *coefficient functionals*, for $\{x_n\}$. $\diamondsuit$

Intuitively, we expect linear functionals to be the "simplest" possible functions on a vector space, and it is tempting to believe that anything as simple as a linear functional must be continuous. Unfortunately, Example 4.2 has already demonstrated that if we accept the Axiom of Choice then there exist unbounded linear functionals on every infinite-dimensional normed space. Hence the first question that we should ask about a basis is whether the associated coefficient functionals are continuous.

Definition 4.8 (Schauder Basis). Let $\{x_n\}$ be a basis for a Banach space X, and let $\{a_n\}$ be the associated coefficient functionals. Then we say that $\{x_n\}$ is a *Schauder basis* for X if each coefficient functional a_n is continuous. $\diamondsuit$

Thus, a basis is a Schauder basis if $a_n \in X^*$ for every n. In Theorem 4.13 we will prove the nontrivial fact that *every basis for a Banach space is a Schauder basis.*

Here are some specific examples of bases for which we already have explicit expressions for the coefficient functionals. In each of these cases we can see directly that the coefficient functionals are continuous.

Example 4.9. (a) Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space H. The basis representation of $x \in H$ is $x = \sum \langle x, e_n \rangle \, e_n$, so the coefficient functionals are $a_n(x) = \langle x, e_n \rangle$, which are continuous on H. Note that, in the sense of the identification of H^* with H given by the Riesz Representation Theorem (Theorem 1.75), we have $a_n = e_n \in H = H^*$ for each n.

(b) Let $\{\delta_n\}$ be the sequence of standard basis vectors introduced in Examples 1.30 and 1.51. Then $\{\delta_n\}$ is a basis for ℓ^p for each $1 \le p < \infty$, called the *standard basis* for ℓ^p. The basis representation of $x = (x_n) \in \ell^p$ is $x = \sum x_n \delta_n$, so the coefficient functionals are given by $a_n(x) = x_n$. These functionals are continuous on ℓ^p, so $a_n \in (\ell^p)^*$. By Theorem 1.73, $(\ell^p)^*$ is identified with $\ell^{p'}$ in the sense that every continuous linear functional μ on ℓ^p has the form $\mu(x) = \langle x, y \rangle = \sum x_n y_n$ for some unique $y \in \ell^{p'}$. For the functional a_n we have $a_n(x) = x_n = \langle x, \delta_n \rangle$, so we usually identify a_n with δ_n and write $a_n = \delta_n$. Thus the sequence of coefficient functionals associated with the basis $\{\delta_n\}$ is $\{\delta_n\}$. In this example, we have the interesting fact that δ_n belongs both to ℓ^p and $(\ell^p)^* = \ell^{p'}$.

For $p = \infty$, the sequence $\{\delta_n\}$ is a basis for c_0 (see Example 1.30), and the sequence of coefficient functionals is again $\{\delta_n\}$, which is contained in ${c_0}^* = \ell^1$. We call $\{\delta_n\}$ the *standard basis* for c_0.

(c) By Exercise 4.9, if we set $y_n = (1, \dots, 1, 0, 0, \dots)$ then $\{y_n\}$ is a conditional basis for c_0. The coefficient functionals are given by $a_n(x) =$

$x_n - x_{n+1} = \langle x, \delta_n \rangle - \langle x, \delta_{n+1} \rangle$ for $x = (x_n) \in c_0$. Hence the coefficient functionals are continuous. Further, in the sense of identification, we have $a_n = \delta_n - \delta_{n+1} \in \ell^1 = c_0{}^*$. $\diamondsuit$

Once we have shown that the coefficient functionals for a basis are necessarily continuous, we will adopt the bilinear form notation discussed in Notation 1.72 and write $\langle x, a_n \rangle$ instead of $a_n(x)$. However, before attempting to prove the continuity of the coefficient functionals, we need to develop some basic facts. The first thing to observe is that if $\{x_n\}$ is a basis and we fix $m \in \mathbf{N}$, then we have two ways to write x_m:

$$x_m = \sum_n a_n(x_m)\, x_n \qquad \text{and} \qquad x_m = \sum_n \delta_{mn}\, x_n. \tag{4.2}$$

Therefore, by uniqueness of the basis representation we must have $a_n(x_m) = \delta_{mn}$ for every m and n.

Definition 4.10 (Biorthogonal Systems). Given a Banach space X and given sequences $\{x_n\} \subseteq X$ and $\{a_n\} \subseteq X^*$, we say that $\{a_n\}$ is *biorthogonal* to $\{x_n\}$ if $\langle x_m, a_n \rangle = \delta_{mn}$ for every m, $n \in \mathbf{N}$. We call $\{a_n\}$ a *biorthogonal system* or a *dual system* to $\{x_n\}$. $\diamondsuit$

We have not yet proved that $\{a_n\}$ is contained in X^*, but we will do so in Theorem 4.13. Once this is proved, we can use the terminology of Definition 4.10 and say that a basis $\{x_n\}$ and its sequence of coefficient functions $\{a_n\}$ are biorthogonal systems. We will study general biorthogonal systems in more detail in Chapter 5.

The following partial sum operators will be of fundamental importance in our analysis.

Definition 4.11 (Partial Sum Operators). Let $\{x_n\}$ be a basis for a Banach space X, with coefficient functionals $\{a_n\}$. The *partial sum operators* or *natural projections* associated with $\{x_n\}$ are the mappings $S_N \colon X \to X$ defined by

$$S_N x = \sum_{n=1}^{N} a_n(x)\, x_n, \qquad x \in X. \qquad \diamondsuit$$

The partial sum operator S_N is linear since the functionals a_n are linear. We claim that

$$a_n \text{ is continuous for each } n \quad \Longleftrightarrow \quad S_N \text{ is continuous for each } N.$$

Certainly if each a_n is continuous then each S_N is continuous. To see the converse implication, given $N \geq 2$ write

$$a_N(x)\, x_N = \sum_{n=1}^{N} a_n(x)\, x_n - \sum_{n=1}^{N-1} a_n(x)\, x_n = S_N x - S_{N-1} x. \tag{4.3}$$

Therefore, if each S_N is continuous, then each a_n is continuous as well. We focus now on the partial sum operators.

Note that if $\{x_n\}$ is a basis, then

$$x \;=\; \sum_{n=1}^{\infty} a_n(x)\, x_n \;=\; \lim_{N\to\infty} S_N x.$$

Since convergent sequences are bounded, we have $\sup_N \|S_N x\| < \infty$ for each $x \in X$. If only we knew that the S_N were bounded, then we could apply the Uniform Boundedness Principle to conclude that $\mathcal{C} = \sup_N \|S_N\| < \infty$. We will be able to do this eventually, and this number $\mathcal{C}$ will be called the *basis constant* for $\{x_n\}$, but for now we must be very careful not to implicitly assume that either the coefficient functionals or the partial sum operators are continuous.

The next theorem is the key tool in our analysis. It states that if $\{x_n\}$ is a basis, then it is possible to endow the space Y of all sequences (c_n) such that $\sum c_n x_n$ converges with a norm so that it becomes a Banach space topologically isomorphic to X. In general it is difficult or impossible to describe this space Y explicitly, but the only fact we really need right now is that Y is isomorphic to X. One situation where the space Y is easily characterized was discussed in Chapter 1: If $\{x_n\}$ is an orthonormal basis for a Hilbert space H, then $\sum c_n x_n$ converges if and only if $(c_n) \in \ell^2$.

Recall that a *topological isomorphism* between normed spaces X and Y is a linear bijection $T\colon X \to Y$ such that T and T^{-1} are both continuous (Definition 2.28). By the Inverse Mapping Theorem, if T is a continuous linear bijection of a Banach space X onto another Banach space Y, then T^{-1} is automatically continuous and therefore T is a topological isomorphism.

Theorem 4.12. *Let $\{x_n\}$ be a sequence in a Banach space X, and assume that $x_n \ne 0$ for every n. Let*

$$Y \;=\; \biggl\{(c_n) : \sum c_n x_n \text{ converges in } X\biggr\},$$

and set

$$\|(c_n)\|_Y \;=\; \sup_N \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|.$$

Then the following statements hold.

(a) *Y is a Banach space.*

(b) *If $\{x_n\}$ is a basis for X, then Y is topologically isomorphic to X via the* synthesis mapping $T\colon (c_n) \mapsto \sum c_n x_n$.

Proof. (a) It is clear that Y is a vector space. If $(c_n) \in Y$ then $\sum c_n x_n = \lim_{N\to\infty} \sum_{n=1}^{N} c_n x_n$ converges. Since convergent sequences are bounded, we therefore have $\|(c_n)\|_Y < \infty$ for each $(c_n) \in Y$. Thus $\|\cdot\|_Y$ is well defined. It is easy to see that

$$\|(c_n) + (d_n)\|_Y \;\le\; \|(c_n)\|_Y + \|(d_n)\|_Y \qquad \text{and} \qquad \|t\,(c_n)\|_Y \;=\; |t|\,\|(c_n)\|_Y,$$

so $\|\cdot\|_Y$ is at least a seminorm on Y. Suppose that $\|(c_n)\|_Y = 0$. Then $\big\|\sum_{n=1}^N c_n x_n\big\| = 0$ for every N. In particular, $\|c_1 x_1\| = 0$, so we must have $c_1 = 0$ since $x_1 \ne 0$. But then $\|c_2 x_2\| = \big\|\sum_{n=1}^2 c_n x_n\big\| = 0$, so $c_2 = 0$, etc. Hence $\|\cdot\|_Y$ is a norm on Y.

Now we must show that Y is complete with respect to this norm. Suppose that $A_N = \big(c_N(n)\big)_{n\in\mathbf{N}} \in Y$ and $\{A_N\}_{N\in\mathbf{N}}$ is a Cauchy sequence in Y. Then for each fixed $n \ge 2$ and $M,\, N \in \mathbf{N}$,

$$\begin{aligned}
&|c_M(n) - c_N(n)|\,\|x_n\| \\
&\qquad = \big\|\big(c_M(n) - c_N(n)\big)\,x_n\big\| \\
&\qquad \le \bigg\|\sum_{k=1}^{n}\big(c_M(k) - c_N(k)\big)\,x_k\bigg\| + \bigg\|\sum_{k=1}^{n-1}\big(c_M(k) - c_N(k)\big)\,x_k\bigg\| \\
&\qquad \le 2\,\|A_M - A_N\|_Y.
\end{aligned}$$

Also, for $n = 1$ we have $|c_M(n) - c_N(n)|\,\|x_n\| \le \|A_M - A_N\|_Y$. Since $\{A_N\}$ is Cauchy and $x_n \ne 0$, we conclude that $\big(c_N(n)\big)_{N\in\mathbf{N}}$ is a Cauchy sequence of scalars, and therefore must converge to some scalar $c(n)$ as $N \to \infty$. Our goal is to show that $A_N \to A = \big(c(n)\big)_{n\in\mathbf{N}}$ in the norm of Y as $N \to \infty$.

Choose any $\varepsilon > 0$. Then since $\{A_N\}$ is Cauchy in Y, there exists an integer $N_0 > 0$ such that

$$\forall\, M, N \ge N_0, \quad \|A_M - A_N\|_Y \;=\; \sup_L \bigg\|\sum_{n=1}^{L}\big(c_M(n) - c_N(n)\big)\,x_n\bigg\| \;<\; \varepsilon.$$

Fix $L > 0$, and define

$$y_{M,N} \;=\; \sum_{n=1}^{L}\big(c_M(n) - c_N(n)\big)\,x_n \qquad \text{and} \qquad y_N \;=\; \sum_{n=1}^{L}\big(c(n) - c_N(n)\big)\,x_n.$$

Note that $\|y_{M,N}\| < \varepsilon$ for each $M,\, N \ge N_0$. Also, keeping L fixed, we have

$$\begin{aligned}
\|y_{M,N} - y_N\| \;&=\; \bigg\|\sum_{n=1}^{L}\big(c(n) - c_M(n)\big)\,x_n\bigg\| \\
&\le\; \sum_{n=1}^{L}|c(n) - c_M(n)|\,\|x_n\| \\
&\to\; 0 \quad \text{as } M \to \infty.
\end{aligned}$$

Thus $y_{M,N} \to y_N$ as $M \to \infty$. Consequently, for all $N \geq N_0$ we have

$$\|y_N\| \;=\; \lim_{M\to\infty} \|y_{M,N}\| \;\leq\; \varepsilon.$$

Substituting the definition of y_N and taking the supremum over L, we obtain

$$\forall\, N \geq N_0, \quad \sup_L \biggl\|\sum_{n=1}^{L} \bigl(c(n) - c_N(n)\bigr)\, x_n\biggr\| \;\leq\; \varepsilon. \tag{4.4}$$

Now, $\bigl(c_{N_0}(n)\bigr)_{n\in\mathbf{N}} \in Y$, so the series $\sum_n c_{N_0}(n)\, x_n$ converges by definition of Y. Hence, there is an $M_0 > 0$ such that

$$\forall\, N > M \geq M_0, \quad \biggl\|\sum_{n=M+1}^{N} c_{N_0}(n)\, x_n\biggr\| \;<\; \varepsilon.$$

Therefore, if $N > M \geq M_0, N_0$ then

$$\begin{aligned}
&\biggl\|\sum_{n=M+1}^{N} c(n)x_n\biggr\| \\
&\quad= \biggl\|\sum_{n=1}^{N} \bigl(c(n) - c_{N_0}(n)\bigr)\, x_n \;-\; \sum_{n=1}^{M} \bigl(c(n) - c_{N_0}(n)\bigr)\, x_n \;+\; \sum_{n=M+1}^{N} c_{N_0}(n)\, x_n\biggr\| \\
&\quad\leq \biggl\|\sum_{n=1}^{N} \bigl(c(n) - c_{N_0}(n)\bigr)\, x_n\biggr\| \;+\; \biggl\|\sum_{n=1}^{M} \bigl(c(n) - c_{N_0}(n)\bigr)\, x_n\biggr\| \\
&\qquad\qquad + \biggl\|\sum_{n=M+1}^{N} c_{N_0}(n)\, x_n\biggr\| \\
&\qquad \leq\; \varepsilon \;+\; \varepsilon \;+\; \varepsilon \;=\; 3\varepsilon.
\end{aligned}$$

Therefore $\sum c(n)x_n$ converges in X, so $A = (c(n)) \in Y$. Finally, by equation (4.4), we have that $A_N \to A$ in the norm of Y, so Y is complete.

(b) Now we assume that $\{x_n\}$ is a basis for X. Then $T(c_n) = \sum c_n x_n$ maps Y into X by definition of Y. Further, T is clearly linear, and it is bijective because $\{x_n\}$ is a basis. If $(c_n) \in Y$, then

$$\|T(c_n)\| \;=\; \biggl\|\sum_{n=1}^{\infty} c_n x_n\biggr\| \;=\; \lim_{N\to\infty} \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\| \;\leq\; \sup_N \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\| \;=\; \|(c_n)\|_Y,$$

so T is bounded. The Inverse Mapping Theorem therefore implies that T is a topological isomorphism of Y onto X. $\square$

As a consequence, we obtain our goal that the partial sum operators are continuous, which also implies that the coefficient functionals are continuous. This fact is really a corollary of Theorem 4.12, but since it is so important we will designate it a theorem.

Theorem 4.13. *Let $\{x_n\}$ be a basis for a Banach space X, with coefficient functionals $\{a_n\}$. Let Y be as in Theorem 4.12, so $T(c_n) = \sum c_n x_n$ is a topological isomorphism of Y onto X. Then the following statements hold.*

(a) *The partial sum operators S_N are bounded, and $\|S_N\| \le \|T^{-1}\|$ for each $N \in \mathbf{N}$.*

(b) $\mathcal{C} = \sup_N \|S_N\| < \infty$.

(c) $|||x||| = \sup_N \|S_N x\|$ *forms a norm on X that is equivalent to the initial norm $\|\cdot\|$, and we have $\|\cdot\| \le |||\cdot||| \le \mathcal{C}\,\|\cdot\|$.*

(d) *The coefficient functionals a_n are continuous linear functionals on X that satisfy*

$$1 \;\le\; \|a_n\|\,\|x_n\| \;\le\; 2\mathcal{C}, \qquad n \in \mathbf{N}.$$

(e) *$\{x_n\}$ is a Schauder basis for X, and $\{a_n\}$ is the unique sequence in X^* that is biorthogonal to $\{x_n\}$.*

Proof. (a) Fix any $x \in X$. Then we have by definition that $x = \sum a_n(x)\, x_n$. The scalars $a_n(x)$ are unique, so T^{-1} is given by $T^{-1}x = \big(a_n(x)\big)$. Hence

$$\begin{aligned}\sup_N \|S_N x\| \;&=\; \sup_N \bigg\|\sum_{n=1}^{N} a_n(x)\, x_n\bigg\| \\ &=\; \big\|\big(a_n(x)\big)\big\|_Y \;=\; \|T^{-1}x\|_Y \;\le\; \|T^{-1}\|\,\|x\|.\end{aligned}$$

Therefore S_N is bounded, and its operator norm satisfies $\|S_N\| \le \|T^{-1}\|$.

(b) From part (a), we have $\mathcal{C} = \sup_N \|S_N\| \le \|T^{-1}\| < \infty$.

(c) It is easy to see that $|||\cdot|||$ has at least the properties of a seminorm. Given $x \in X$ we have

$$|||x||| \;=\; \sup_N \|S_N x\| \;\le\; \sup_N \|S_N\|\,\|x\| \;=\; \mathcal{C}\,\|x\|.$$

Also, since $S_N x \to x$ in the norm of X,

$$\|x\| \;=\; \lim_{N\to\infty} \|S_N x\| \;\le\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows from these two estimates that $|||\cdot|||$ is a norm, and that it is equivalent to $\|\cdot\|$.

(d) As in equation (4.3), we have for $n \ge 2$ that $a_n(x)\, x_n = S_n x - S_{n-1} x$. Hence

$$|a_n(x)|\,\|x_n\| \;=\; \|a_n(x)\,x_n\| \;\le\; \|S_nx\| \;+\; \|S_{n-1}x\| \;\le\; 2\mathcal{C}\,\|x\|.$$

Since each x_n is nonzero, we conclude that $\|a_n\| \le 2\mathcal{C}/\|x_n\| < \infty$. Since $a_1(x)\,x_1 = S_1x$, the same estimate is also valid for $n = 1$. Consequently, each a_n is bounded and $\|a_n\|\,\|x_n\| \le 2\mathcal{C}$ for each n. As in the discussion following equation (4.2), by uniqueness we must have $a_m(x_n) = \delta_{mn}$, so $\{x_n\}$ and $\{a_n\}$ are biorthogonal, and therefore $1 = a_n(x_n) \le \|a_n\|\,\|x_n\|$.

(e) Since the coefficient functionals are continuous, $\{x_n\}$ is a Schauder basis, and we have observed that $\{a_n\}$ is a biorthogonal sequence in X^*. This biorthogonal system is unique because of the fact that $\{x_n\}$ is complete (for explicit proof, see Lemma 5.4). □

Because of Theorem 4.13, the words "basis" and "Schauder basis" are often used interchangeably. Further, the associated sequence of coefficient functionals $\{a_n\} \subseteq X^*$ is synonymously referred to as the biorthogonal system or the dual system to $\{x_n\}$.

The number $\mathcal{C}$ appearing in Theorem 4.13 is important enough to be dignified with a name of its own.

Definition 4.14 (Basis Constant). If $\{x_n\}$ is a basis for a Banach space X, then its *basis constant* is the finite number $\mathcal{C} = \sup_N \|S_N\|$. The basis constant always lies in the range $1 \le \mathcal{C} < \infty$. If the basis constant is $\mathcal{C} = 1$, then the basis is said to be *monotone*. ◇

The basis constant does depend on the choice of norm. Unless otherwise specified, the basis constant is always taken with respect to the original norm on X. Changing to an equivalent norm for X will not change the fact that $\{x_n\}$ is a basis, but it can change the basis constant for $\{x_n\}$. In particular, we show next that the basis constant with respect to the equivalent norm $|||\cdot|||$ is always 1.

Theorem 4.15. *Every basis $\{x_n\}$ is monotone with respect to the equivalent norm $|||x||| = \sup_N \|S_Nx\|$.*

Proof. Note that the composition of the partial sum operators S_M and S_N satisfies the rule

$$S_M\,S_N \;=\; S_{\min\{M,N\}}.$$

Therefore,

$$|||S_Nx||| \;=\; \sup_M \|S_MS_Nx\| \;=\; \sup\bigl\{\|S_1x\|,\dots,\|S_Nx\|\bigr\}$$

and

$$\sup_N |||S_Nx||| \;=\; \sup_N \|S_Nx\| \;=\; |||x|||.$$

It follows from this that $\sup_N |||S_N||| = 1$. □

Remark 4.16. Instead of changing the norm on X, suppose that we consider all possible bases for X with respect to a fixed norm $\|\cdot\|$. Must it be the case that at least one of these bases is monotone? This was shown by Gurariĭ to be false [Gur65]. Further, it was shown by Enflo [Enf73b] that there exists a Banach space $(X, \|\cdot\|)$ such that $\inf\{\mathcal{C}_{\mathcal{B}} : \text{all bases } \mathcal{B} \text{ for } X\} > 1$, where $\mathcal{C}_{\mathcal{B}}$ denotes the basis constant of a basis $\mathcal{B}$ for X with respect to the fixed norm $\|\cdot\|$ for X. $\diamondsuit$

Since we now know that the coefficient functionals a_n for a basis are elements of X^*, we will follow Notation 1.72 and use the notation $a_n(x) = \langle x, a_n \rangle$ interchangeably. In fact, from this point onward our preferred notation will be $\langle x, a_n \rangle$, although on occasion it is more convenient to write $a_n(x)$.

Remark 4.17. Some particular special cases deserve mention.

(a) If $\{x_n\}$ is a basis for a Hilbert space H, then the associated coefficient functionals belong to H^*, which is isometrically isomorphic to H by the Riesz Representation Theorem. As usual, we therefore identify the coefficient functional a_n with the element of H that determines this functional. Thus the dual system $\{a_n\}$ is the sequence of vectors in H such that every $x \in H$ can be written uniquely as $x = \sum \langle x, a_n \rangle\, x_n$.

(b) Similarly, in other cases where we have an explicit identification of the dual space X^*, we follow the usual notational conventions. For example, if $\{x_n\}$ is a basis for ℓ^p with $1 \le p < \infty$, then each coefficient functional a_n is determined by an element of $\ell^{p'}$, and we identify the functional a_n with this vector in $\ell^{p'}$. $\diamondsuit$

As we have seen, the sequence of coefficient functionals $\{a_n\}$ is contained in the dual space X^*, which is itself a Banach space. It is therefore natural to ask what kind of properties this sequence has, especially whether it is a basis for X^*. In general, the answer to this is no. For example, the standard basis $\{\delta_n\}$ is a basis for $X = \ell^1$, and its sequence of coefficient functionals is again $\{\delta_n\}$, which is contained in $X^* = \ell^\infty$ but is not a basis for ℓ^∞. Instead, with respect to the ℓ^∞-norm, $\{\delta_n\}$ is a basis for the space c_0. We will prove in Section 5.6 that this example is typical: If $\{x_n\}$ is a basis for a Banach space X then its sequence of coefficient functionals $\{a_n\}$ is a basis for $\overline{\text{span}}\{a_n\}$, which in general may be a proper subspace of X^*. However, we will also see that if X is reflexive then $\{a_n\}$ is complete in X^* and hence $\{a_n\}$ is a basis for X^* (see Corollary 5.22).

Exercises

4.11. Show that the standard basis $\{\delta_n\}_{n \in \mathbf{N}}$ is a monotone basis for ℓ^p for each index $1 \le p < \infty$, and is also a monotone basis for c_0. Is the basis $\{\delta_n\}_{n \ge 0}$ for c given in Exercise 4.8 monotone?

4.12. Show that any subsequence of a basis is a basic sequence.

4.13. Suppose that $\{x_n\}$ is a Schauder basis for a Hilbert space H, and its biorthogonal system is $\{y_n\}$. Show that if $\|x_n\| = \|y_n\| = 1$ for every n, then $\{x_n\}$ is an orthonormal basis for H.

4.14. Show that if $\{x_n\}$ is an absolutely convergent basis for a Banach space X, then

$$Ty \;=\; \sum_{n=1}^{\infty} \frac{y_n}{\|x_n\|}\, x_n, \qquad y = (y_n) \in \ell^1,$$

is a topological isomorphism of ℓ^1 onto X. Conversely, show that every Banach space topologically isomorphic to ℓ^1 has an absolutely convergent basis.

4.15. This exercise will give an alternative approach to proving Theorem 4.13. Let $\{x_n\}$ be a basis for a Banach space X, and set $|||x||| \;=\; \sup_N \|S_N x\|$, where the S_N are the partial sum operators.

(a) Show that $|||\cdot|||$ is a norm on X, and $\|x\| \le |||x|||$ for all $x \in X$.

(b) Suppose that $\{y_n\}$ is a Cauchy sequence in X with respect to $|||\cdot|||$. With N fixed, show that $\{S_N y_n\}_{n\in\mathbf{N}}$ is Cauchy with respect to $\|\cdot\|$. Let z_N be such that $\|z_N - S_N y_n\| \to 0$ as $n \to \infty$, and observe that $z_N \in \operatorname{span}\{x_1, \dots, x_N\}$.

(c) Show that for each $N \in \mathbf{N}$ we have $\lim_{n\to\infty} \bigl(\sup_N \|z_N - S_N y_n\|\bigr) = 0$, and use this to show that $\{z_N\}_{N\in\mathbf{N}}$ is Cauchy with respect to $\|\cdot\|$. Let $y \in X$ be the element such that $\|y - z_N\| \to 0$.

(d) Show that $S_N(z_{N+1}) = z_N$, and use this to show that $z_N = \sum_{n=1}^{N} c_n x_n$ where c_n is independent of N.

(e) Show that $y = \sum_{n=1}^{\infty} c_n x_n$, and hence $z_N = S_N y$. Use this to show that $|||y - y_n||| \to 0$, and conclude that X is complete with respect to $|||\cdot|||$.

(f) Show that $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on X, and use this to show that $\mathcal{C} = \sup_N \|S_N\| < \infty$. Conclude that $\{x_n\}$ is a Schauder basis for X.

4.16. We say that a Banach space X has the *approximation property* if the identity operator on X can be uniformly approximated on every compact subset of X by operators with finite rank. That is, given a compact set $K \subseteq X$, there must exist continuous finite rank operators T_N such that

$$\lim_{N\to\infty} \left(\sup_{x\in K} \|x - T_N x\| \right) \;=\; 0.$$

(a) Show that if X has a basis, then X has the approximation property.

(b) Suppose that Y is a Banach space that has the approximation property and X is an arbitrary Banach space. Show that if $T\colon X \to Y$ is a compact operator, then there exist continuous finite-rank operators $T_N\colon X \to Y$ such that $\|T - T_N\| \to 0$ as $N \to \infty$.

Remark: Compact sets and compact operators are reviewed in Appendix B.

4.17. Let X be a Banach space.

(a) Show that if $\{x_n\}$ is a basis for X, then the only possible vector $y \in X$ such that $x_n \xrightarrow{w} y$ is $y = 0$ (weak convergence is defined in Section 2.10).

(b) Use Theorem 2.39 to show that part (a) remains valid if we only assume that $\{x_n\}$ is a basic sequence in X.

(c) Give an example of a basis $\{x_n\}$ for a Banach space X such that $x_n \xrightarrow{w} 0$.

(d) Give an example of a basis $\{x_n\}$ for a Banach space X such that $\{x_n\}$ does not converge weakly to any vector in X.

4.4 Equivalent Bases

In this section we prove several results related to the invariance of bases under topological isomorphisms. We begin with the easy fact that bases are preserved by topological isomorphisms.

Lemma 4.18. *Let X, Y be Banach spaces. If $\{x_n\}$ is a basis for X and $T\colon X \to Y$ is a topological isomorphism, then $\{Tx_n\}$ is a basis for Y.*

Proof. If y is any element of Y then $T^{-1}y \in X$, so there are unique scalars (c_n) such that $T^{-1}y = \sum c_n x_n$. Since T is continuous, this implies that $y = T(T^{-1}y) = \sum c_n T x_n$. Suppose $y = \sum b_n T x_n$ is another representation of y. Then since T^{-1} is continuous, we have $T^{-1}y = \sum b_n x_n$, and hence $b_n = c_n$ for each n since $\{x_n\}$ is a basis for X. Thus $\{Tx_n\}$ is a basis for Y. □

This motivates the following definition.

Definition 4.19. Let X and Y be Banach spaces. A basis $\{x_n\}$ for X is *equivalent* to a basis $\{y_n\}$ for Y if there exists a topological isomorphism $T\colon X \to Y$ such that $Tx_n = y_n$ for all n. If $X = Y$ then we write $\{x_n\} \sim \{y_n\}$ to mean that $\{x_n\}$ and $\{y_n\}$ are equivalent bases for X. ♢

Note that $\sim$ is an equivalence relation on the set of all bases for a Banach space X.

We can characterize equivalent bases in terms of convergence of series.

Theorem 4.20. *Let X and Y be Banach spaces. If $\{x_n\}$ is a basis for X and $\{y_n\}$ is a basis for Y, then the following two statements are equivalent.*

(a) *$\{x_n\}$ is equivalent to $\{y_n\}$.*

(b) *$\sum c_n x_n$ converges in X if and only if $\sum c_n y_n$ converges in Y.*

Proof. (a) ⇒ (b). This is Exercise 4.18.

(b) ⇒ (a). Suppose that statement (b) holds. Let $\{a_n\} \subseteq X^*$ be the coefficient functionals for the basis $\{x_n\}$, and let $\{b_n\} \subseteq Y^*$ be the coefficient functionals for the basis $\{y_n\}$. Suppose that $x \in X$ is given. Then $x = \sum \langle x, a_n \rangle x_n$

converges in X, so $Tx = \sum \langle x, a_n \rangle y_n$ converges in Y. The fact that the expansion $x = \sum \langle x, a_n \rangle x_n$ is unique ensures that T is well defined, and it is clear that T is linear.

If $Tx = 0$ then

$$\sum 0y_n = 0 = Tx = \sum \langle x, a_n \rangle y_n,$$

and therefore $\langle x, a_n \rangle = 0$ for every n since $\{y_n\}$ is a basis. Hence $x = \sum \langle x, a_n \rangle x_n = 0$, so T is injective.

Next, choose any element $y \in Y$. Then the series $y = \sum \langle y, b_n \rangle y_n$ converges in Y, so $x = \sum \langle y, b_n \rangle x_n$ converges in X. Since $x = \sum \langle x, a_n \rangle x_n$ and $\{x_n\}$ is a basis, this forces $\langle y, b_n \rangle = \langle x, a_n \rangle$ for every n. Hence $Tx = y$, so T is surjective.

It remains to show that T is continuous. For each N, define $T_N \colon X \to Y$ by $T_N x = \sum_{n=1}^{N} \langle x, a_n \rangle y_n$. Since each functional a_n is continuous, each operator T_N is continuous. Since $T_N x \to Tx$, it follows from the Banach–Steinhaus Theorem (Theorem 2.23) that T is bounded. □

Example 4.21. If $\{e_n\}$ and $\{f_n\}$ are two orthonormal bases for a Hilbert space H, then we know from Theorem 1.49 that

$$\sum_n c_n e_n \text{ converges} \quad \Longleftrightarrow \quad \sum_n |c_n|^2 < \infty \quad \Longleftrightarrow \quad \sum_n c_n f_n \text{ converges.}$$

Hence $\{e_n\} \sim \{f_n\}$ by Theorem 4.20. Thus, all orthonormal bases for H are equivalent. ♢

More generally, we will see in Section 7.2 that all bounded unconditional bases in a Hilbert space are equivalent. In particular, since every orthonormal basis is a bounded unconditional basis, every bounded unconditional basis in a Hilbert space is equivalent to an orthonormal basis.

The situation for general bases is much more complicated, even for Hilbert spaces. In particular, it is known that if X is an infinite-dimensional Banach space that has a basis, then there exist uncountably many nonequivalent normalized conditional bases for X [Sin70, Thm. 23.3].

Exercises

4.18. Prove the implication (a) ⇒ (b) in Theorem 4.20.

4.19. Suppose that $\{x_n\}$ is a basis for a Banach space X, $\{y_n\}$ is a sequence in a normed space Y, and there exists a topological isomorphism $T \colon X \to Y$ such that $Tx_n = y_n$ for every n. Show that Y is a Banach space, $\{y_n\}$ is a basis for Y, and $\{x_n\}$ is equivalent to $\{y_n\}$.

4.20. Suppose $\{x_n\}$ is a basis for a Banach space X that is equivalent to a basis $\{y_n\}$ for a Banach space Y. Show that $\{x_n\}$ is a bounded basis, unconditional basis, or absolutely convergent basis for X if and only if the same is true of the basis $\{y_n\}$ for Y.

4.21. Let $\{x_n\}$ be a basis for a Banach space X and $\{y_n\}$ a basis for a Banach space Y. Show that $\{x_n\}$ is equivalent to $\{y_n\}$ if and only if there exist constants C_1, $C_2 > 0$ such that for all $N \in \mathbf{N}$ and $c_1, \dots, c_N \in \mathbf{F}$ we have

$$C_1 \left\| \sum_{n=1}^{N} c_n y_n \right\| \;\le\; \left\| \sum_{n=1}^{N} c_n x_n \right\| \;\le\; C_2 \left\| \sum_{n=1}^{N} c_n y_n \right\|.$$

4.22. (a) Let $\{\delta_n\}$ be the standard basis for c_0. By Exercise 4.8, if we set $\delta_0 = (1, 1, \dots)$, then $\{\delta_n\}_{n \ge 0}$ is a basis for c. Show that c and c_0 are topologically isomorphic, and these two bases are equivalent.

(b) Show that if $x \in c_0$ and $\|x\|_\infty = 1$, then there exist $y \ne z \in c_0$ with $\|y\|_\infty = \|z\|_\infty = 1$ such that $x = (y+z)/2$. Show that the analogous statement for c can fail.

(c) Show that c is not isometrically isomorphic to c_0 (even so, note that their dual spaces c^* and $c_0{}^*$ are each isometrically isomorphic to ℓ^1 by Exercises 1.75 and 4.8).

4.5 Schauder's Basis for $C[0,1]$

In this section we will give Schauder's original construction from [Sch27] of a basis for the space $C[0,1]$ of continuous functions on $[0,1]$. That paper introduced the notion of what we now call Schauder bases.

Definition 4.22 (The Schauder System). The *Schauder system* in $C[0,1]$ is

$$\{\chi, \ell\} \;\cup\; \{s_{n,k}\}_{n \ge 0,\, k=0,\dots,2^n-1},$$

where $\chi = \chi_{[0,1]}$, $\ell(t) = t$, and $s_{n,k}$ is the continuous function given by

$$s_{n,k}(t) \;=\; \begin{cases} 1, & t = \frac{k+1/2}{2^n}, \\ \text{linear}, & \text{on } \left[\frac{k}{2^n}, \frac{k+1/2}{2^n}\right] \text{ and on } \left[\frac{k+1/2}{2^n}, \frac{k+1}{2^n}\right], \\ 0, & \text{otherwise}; \end{cases}$$

see the illustration in Figure 4.1. ◇

Equivalently, if we let

$$W(t) \;=\; s_{0,0}(t) \;=\; \max\{1 - |2t-1|, 0\}$$

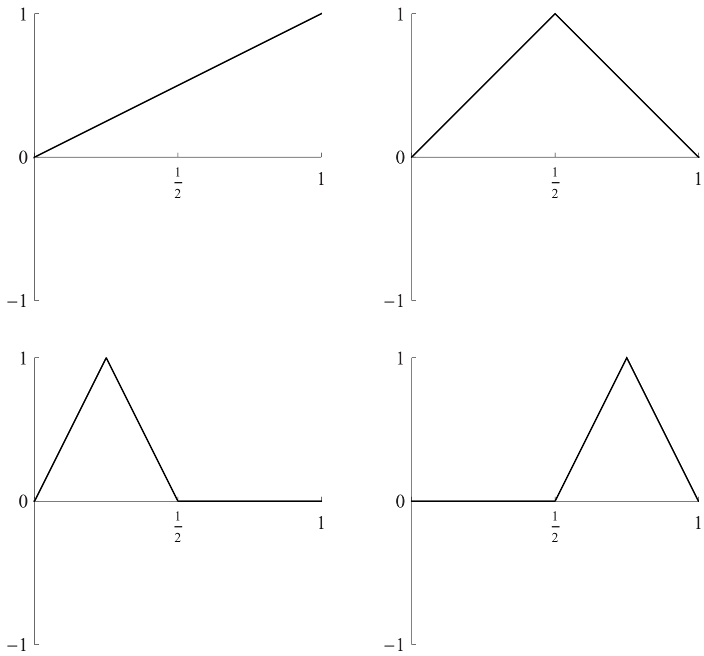

Fig. 4.1. Some elements of the Schauder system: ℓ, $s_{0,0}$ (top), and $s_{1,0}$, $s_{1,1}$ (bottom).

be the "hat function" or "tent function" supported on $[0,1]$, then $s_{n,k}$ is the dilated and translated hat function

$$s_{n,k}(t) \;=\; W(2^n t - k),$$

which is supported on the interval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$. Although we will not need this fact, if we recall the Haar system defined in Example 1.54, it is interesting to observe that, except for a scaling factor, $s_{n,k}$ is an antiderivative of the Haar function $h_{n,k}$.

Fix any $f \in C[0,1]$. Then we can choose scalars a, b such that the function $g = f - a\chi - b\ell$ satisfies $g(0) = g(1) = 0$. Our first goal is to show that there exist scalars $c_{n,k}$ such that

$$g \;=\; \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k} s_{n,k},$$

with uniform convergence of the series.

Set

$$h_0 \;=\; g(\tfrac{1}{2})\, s_{0,0}.$$

Then h_0 is a continuous function with a piecewise linear graph that agrees with g at the points 0, $\frac{1}{2}$, 1. Let $g_0 = g - h_0$, and note that g_0 vanishes at the points 0, $\frac{1}{2}$, 1. Now define

$$h_1 \;=\; g_0(\tfrac{1}{4})\, s_{1,0} + g_0(\tfrac{3}{4})\, s_{1,1}.$$

Then h_1 is a continuous function with a piecewise linear graph that agrees with g_0 at the points 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1. Consequently, $h_0 + h_1$ is continuous with a piecewise linear graph and agrees with g at 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1. Continuing in this way we inductively construct functions

$$h_n \;=\; \sum_{k=0}^{2^n-1} c_{n,k}\, s_{n,k}$$

such that $k_n = h_0 + \cdots + h_n$ is a linear approximation to g on dyadic subintervals of the form $\bigl[\frac{j}{2^{n+1}}, \frac{j+1}{2^{n+1}}\bigr]$. Since g is uniformly continuous, it follows that k_n converges uniformly to g. Hence $g = \lim_{n\to\infty} k_n = \sum_{n=1}^{\infty} h_n$, so

$$f \;=\; a\chi + b\ell + g \;=\; a\chi + b\ell + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k} s_{n,k}, \tag{4.5}$$

where the series converges in the uniform norm. Further, this representation is unique (Exercise 4.23), so it follows that the Schauder system is a basis for $C[0,1]$.

We will show in Section 6.3 that the Schauder system is a conditional basis for $C[0,1]$. In fact, it can be shown that $C[0,1]$ does not contain any unconditional bases. Another space that contains no unconditional bases is $L^1[0,1]$, cf. [LT77], [Sin70].

The *Franklin system* [Fra28] is the orthonormal basis for $L^2[0,1]$ obtained by applying the Gram–Schmidt orthogonalization procedure to the Schauder system. The Franklin system is an unconditional basis for $L^p[0,1]$ for each $1 < p < \infty$.

Exercises

4.23. Prove that the representation of functions $f \in C[0,1]$ given in equation (4.5) is unique.

4.6 The Trigonometric System

In this section we briefly discuss the basis that is—at least from the viewpoint of a harmonic analyst—perhaps the most important of all. This is the trigonometric system $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$. Unfortunately, we cannot prove the basis properties of this system using the tools that we have developed so far. Therefore, we summarize some facts about the trigonometric system in this section, and return to a more detailed study in Chapters 13 and 14.

First, we note a technical detail.

Notation 4.23. Because the functions $e^{2\pi int}$ are 1-periodic on $\mathbf{R}$, we often restrict their domain to $[0,1]$ or another interval of length 1, and regard them as being elements of $L^p[0,1]$. However, in some circumstances, especially when dealing with continuity, it is more convenient to deal with spaces of 1-periodic functions on $\mathbf{R}$ instead of functions on $[0,1]$. For example, taking $p = \infty$, uniform limits of 1-periodic functions are 1-periodic, so $\overline{\text{span}}\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is contained in

$$C(\mathbf{T}) \;=\; \bigl\{f \in C(\mathbf{R}) : f \text{ is 1-periodic}\bigr\}.$$

Restricting functions to the domain $[0,1]$, we can identify $C(\mathbf{T})$ with

$$C_{\text{per}}[0,1] \;=\; \bigl\{f \in C[0,1] : f(0) = f(1)\bigr\}.$$

Since $C_{\text{per}}[0,1]$ is a proper, closed subspace of $C[0,1]$, the trigonometric system cannot be complete in $C[0,1]$. Instead, we will see that the trigonometric system is complete in $C(\mathbf{T})$ and in $C_{\text{per}}[0,1]$, so these are the appropriate spaces of continuous functions to consider when dealing with $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$. For p finite we define

$$L^p(\mathbf{T}) \;=\; \biggl\{f\colon \mathbf{R} \to \mathbf{C} \;:\; f \text{ is 1-periodic and } \int_0^1 |f(t)|^p\,dt < \infty\biggr\},$$

but because functions in L^p are only defined almost everywhere, the distinction between $L^p(\mathbf{T})$ and $L^p[0,1]$ is usually irrelevant. Hence we often use the symbols $L^p(\mathbf{T})$ and $L^p[0,1]$ interchangeably, although technically they are only equivalent in the sense of identification. For $p = \infty$, the space $L^\infty(\mathbf{T})$ consists of the 1-periodic essentially bounded functions, which we identify with $L^\infty[0,1]$.

Another equivalent formulation of $L^p(\mathbf{T})$ is to let $\mathbf{T}$ be the interval $[0,1)$, and endow this set with an additive operation under which it is a group. The appropriate operation is addition modulo 1, which is defined by $(x+y) \bmod 1 = \text{frac}(x+y)$, the fractional part of $x+y$. For example, $(\frac{1}{2}+\frac{3}{4}) \bmod 1 = \frac{1}{4}$. In this formulation we really are identifying $\mathbf{T}$ with the quotient group $\mathbf{R}/\mathbf{Z}$. However we think of $\mathbf{T}$, in both the group and the topological sense it is isomorphic to the circle group $\{e^{i\theta} : \theta \in \mathbf{R}\}$ under multiplication of complex scalars, $e^{i\theta}e^{i\eta} = e^{i(\theta+\eta)}$. The circle is the 1-dimensional torus, hence the letter $\mathbf{T}$ for this group. $\diamondsuit$

In summary, when dealing with the trigonometric system, or in other situations where we implicitly regard functions on $[0,1)$ as being extended 1-periodically to the real line, we use the function spaces $L^p(\mathbf{T})$ and $C(\mathbf{T})$, while in other situations we consider $L^p[0,1]$ and $C[0,1]$. For example, $C[0,1]$ is the appropriate setting for the Schauder system (Section 4.3), and $L^p[0,1]$ is the appropriate setting for the Haar system (Section 5.5).

Now that we have defined the appropriate function spaces, we consider the properties of the trigonometric system in these spaces. The easiest fact to observe is that the trigonometric system is an orthonormal sequence in $L^2(\mathbf{T})$. We proved this in Example 1.52, and stated there that it is also true that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is complete in $L^2(\mathbf{T})$ and hence is an orthonormal basis for that space. We will prove the completeness and basis properties of the trigonometric system in Chapters 13 and 14. For convenience, we summarize them in the following definition and theorem.

Definition 4.24 (Fourier Coefficients). Given $f \in L^1(\mathbf{T})$, we define its *Fourier coefficients* to be

$$\widehat{f}(n) \;=\; \langle f, e^{2\pi int}\rangle \;=\; \int_0^1 f(t)\, e^{-2\pi int}\, dt, \qquad n \in \mathbf{Z},$$

and we set

$$\widehat{f} \;=\; \bigl(\widehat{f}(n)\bigr)_{n\in\mathbf{Z}}.$$

We often refer to the sequence $\widehat{f}$ as the *Fourier transform* of f. ♢

Theorem 4.25 (The Trigonometric System in $L^p(\mathbf{T})$ and $C(\mathbf{T})$).

(a) $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ *is an orthonormal basis for* $L^2(\mathbf{T})$. *Hence each* $f \in L^2(\mathbf{T})$ *can be written uniquely as*

$$f(t) \;=\; \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi int},$$

where the series converges unconditionally in L^2*-norm, and we have*

$$\|f\|_{L^2}^2 \;=\; \sum_{n\in\mathbf{Z}} |\widehat{f}(n)|^2.$$

(b) $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ *is a conditional basis for* $L^p(\mathbf{T})$ *for each* $1 < p < 2$ *and* $2 < p < \infty$ *with respect to the ordering*

$$\mathbf{Z} \;=\; \{0, -1, 1, -2, 2, \dots\} \;=\; \{k_1, k_2, \dots\}.$$

In particular, for these p each $f \in L^p(\mathbf{T})$ *can be written uniquely as*

$$f(t) \;=\; \sum_{n=1}^{\infty} \widehat{f}(k_n)\, e^{2\pi i k_n t},$$

where the series converges in L^p*-norm, but the series is conditionally convergent for some* $f \in L^p(\mathbf{T})$.

(c) $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ *is complete but not a basis for* $L^1(\mathbf{T})$ *or* $C(\mathbf{T})$. *Even so, each function* $f \in L^1(\mathbf{T})$ *is uniquely determined by its Fourier transform* $\widehat{f}$, *i.e., if* $f \in L^1(\mathbf{T})$ *then*

$$\widehat{f}(n) = 0 \text{ for all } n \in \mathbf{Z} \quad\Longleftrightarrow\quad f = 0. \qquad ♢$$

Exercises

4.24. (a) Prove that $C(\mathbf{T})$ is a Banach space with respect to the uniform norm.

(b) Show that $C_{\mathrm{per}}[0,1]$ is a proper closed subspace of $C[0,1]$, and that $C_{\mathrm{per}}[0,1]$ is isometrically isomorphic to $C(\mathbf{T})$.

(c) Show that $L^p(\mathbf{T})$ is isometrically isomorphic to $L^p[0,1]$ for each $1 \le p \le \infty$.

(d) Show that if $1 \le p < q \le \infty$, then $L^q(\mathbf{T}) \subsetneq L^p(\mathbf{T})$.

4.7 Weak and Weak* Bases in Banach Spaces

To this point we have considered sequences that are bases with respect to the norm topology on a Banach space X. In this section we will briefly survey the natural generalization of bases to the weak or weak* topologies. We will content ourselves with these topologies only, although it is certainly possible to generalize the notion of basis further to the setting of abstract topological vector spaces. We refer to [Mar69] and related sources for such generalizations.

The weak and weak* topologies are reviewed in Section 2.10. As discussed there, the norm topology on a Banach space is often referred to as the *strong topology*, and convergence in norm is often called *strong convergence*.

Definition 4.26. Let X be a Banach space, and let $\{x_n\}$ be a sequence in X.

(a) We recall the definition of a basis from Definition 4.3: $\{x_n\}$ is a *basis* for X if for each $x \in X$ there exist unique scalars $a_n(x)$ such that $x = \sum a_n(x)\, x_n$, with convergence of this series in the strong topology, i.e.,

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} a_n(x)\, x_n \right\| \;=\; 0.$$

In this section, in order to emphasize the type of convergence required, we will usually refer to a basis as a *strong basis* or a *norm basis*. By Theorem 4.13, each coefficient functional a_m associated with a strong basis is strongly continuous, i.e., $\|y - y_n\| \to 0$ implies $\lim_{n\to\infty} a_m(y_n) = a_m(y)$ for each $m \in \mathbf{N}$. Hence every strong basis is a *strong Schauder basis*, and therefore we usually write $\langle x, a_n\rangle$ instead of $a_n(x)$.

(b) $\{x_n\}$ is a *weak basis* for X if for each $x \in X$ there exist unique scalars $a_n(x)$ such that $x = \sum a_n(x)\, x_n$, with convergence of this series in the weak topology, i.e.,

$$\forall\, x^* \in X^*, \quad \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} a_n(x)\, x_n,\; x^* \right\rangle \;=\; \langle x, x^*\rangle. \tag{4.6}$$

A weak basis is a *weak Schauder basis* if each coefficient functional a_m is weakly continuous on X, i.e., if $y_n \xrightarrow{\text{w}} y$ in X implies $\lim_{n\to\infty} a_m(y_n) = a_m(y)$ for each $m \in \mathbf{N}$. We refer to $\{a_n\}$ as the sequence of *coefficient functionals* associated to $\{x_n\}$.

(c) A sequence $\{x_n^*\}$ of functionals in X^* is a *weak* basis* for X^* if for each $x^* \in X^*$ there exist unique scalars $a_n^*(x^*)$ such that $x^* = \sum a_n^*(x^*)\, x_n^*$, with convergence of this series in the weak* topology, i.e.,

$$\forall\, x \in X, \quad \lim_{N\to\infty} \Bigl\langle x,\, \sum_{n=1}^{N} a_n^*(x^*)\, x_n^* \Bigr\rangle \;=\; \langle x, x^* \rangle.$$

A weak* basis is a *weak* Schauder basis* if each coefficient functional a_m^* is weak* continuous on X^*, i.e., if $y_n^* \xrightarrow{\text{w}^*} y^*$ in X^* implies $\lim_{n\to\infty} a_m^*(y_n^*) = a_m^*(y^*)$ for each $m \in \mathbf{N}$. We refer to $\{a_n^*\}$ as the sequence of *coefficient functionals* associated to $\{x_n^*\}$. ♢

All strong bases are weak bases (see Exercise 4.25).

Theorem 4.27. *Let X be a Banach space. If $\{x_n\}$ is a strong basis for X, then $\{x_n\}$ is a weak basis for X. Further, in this case $\{x_n\}$ is a weak Schauder basis for X with coefficient functionals that are strongly continuous on X.* ♢

Surprisingly, the converse is also true: Every weak basis for a Banach space X is a strong basis. We will prove this in Theorem 4.30, after establishing some properties of weak bases.

We let the partial sum operators for a weak basis $\{x_n\}$ be defined in the usual way, i.e., $S_N x = \sum_{n=1}^{N} a_n(x)\, x_n$ (compare Definition 4.11). The following result, whose proof we assign as Exercise 4.26, is the analogue for weak bases of Theorem 4.12.

Theorem 4.28. *Let $\{x_n\}$ be a sequence in a Banach space X, and assume that $x_n \neq 0$ for every n. Define $Y = \bigl\{(c_n) : \sum c_n x_n$ converges weakly in $X\bigr\}$, and set*

$$\|(c_n)\|_Y \;=\; \sup_N \left\| \sum_{n=1}^{N} c_n x_n \right\|.$$

Then the following statements hold.

(a) *Y is a Banach space.*

(b) *If $\{x_n\}$ is a weak basis for X, then Y is topologically isomorphic to X via the mapping $(c_n) \mapsto \sum c_n x_n$.* ♢

An immediate consequence is that the partial sum operators for a weak basis are *strongly* continuous.

Corollary 4.29. *Let $\{x_n\}$ be a weak basis for a Banach space X, with associated coefficient functionals $\{a_n\}$. Then the following statements hold.*

(a) $\sup_N \|S_N x\| < \infty$ *for each* $x \in X$.

(b) *Each* S_N *is strongly continuous, and* $\mathcal{C} = \sup_N \|S_N\| < \infty$.

(c) $|||x||| = \sup_N \|S_N x\|$ *forms a norm on* X *that is equivalent to the initial norm* $\|\cdot\|$, *and we have* $\|\cdot\| \le |||\cdot||| \le \mathcal{C}\,\|\cdot\|$.

(d) *Each coefficient functional* a_n *is strongly continuous, and*

$$1 \;\le\; \|a_n\|\,\|x_n\| \;\le\; 2\mathcal{C}, \qquad n \in \mathbf{N}. \tag{4.7}$$

(e) $\{x_n\}$ *is a weak Schauder basis for* X.

Proof. (a), (b). Let Y be as in Theorem 4.28. Then $T\colon X \to Y$ defined by $T(c_n) = \sum c_n x_n$ (converging weakly) is a topological isomorphism of X onto Y. Suppose that $x \in X$. Then we have by definition that the series $x = \sum a_n(x)\,x_n$ converges weakly and the scalars $a_n(x)$ are unique, so $T^{-1}x = (a_n(x))$. Hence

$$\begin{aligned}\sup_N \|S_N x\| \;&=\; \sup_N \left\|\sum_{n=1}^N a_n(x)\,x_n\right\| \\ &=\; \big\|(a_n(x))\big\|_Y \;=\; \|T^{-1}x\|_Y \;\le\; \|T^{-1}\|\,\|x\| \;<\; \infty.\end{aligned}$$

(c) It is easy to see that $|||\cdot|||$ has the properties of at least a seminorm. Given $x \in X$ we have

$$|||x||| \;=\; \sup_N \|S_N x\| \;\le\; \sup_N \|S_N\|\,\|x\| \;=\; \mathcal{C}\,\|x\|$$

and

$$\|x\| \;=\; \lim_{N\to\infty} \|S_N x\| \;\le\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows that $|||\cdot|||$ is a norm that is equivalent to the initial norm $\|\cdot\|$.

(d) Since each S_N is continuous and $a_N(x)\,x_N = S_N x - S_{N-1}x$ for $N \ge 2$, each a_N is continuous. The proof of equation (4.7) then follows just as it does in the proof of Theorem 4.13.

(e) This now follows from the definition of weak Schauder basis. □

Theorem 4.30 (Weak Basis Theorem). *Every weak basis for a Banach space* X *is a strong basis for* X, *and conversely.*

Proof. By Theorem 4.27, all strong bases are weak bases.

For the converse, assume that $\{x_n\}$ is a weak basis for X, and let $\{a_n\}$ be the associated sequence of coefficient functionals. By Corollary 4.29, $a_n \in X^*$ for each n. Moreover, by the uniqueness of the representations in equation (4.6), we must have $\langle x_m, a_n\rangle = \delta_{mn}$, so $\{a_n\}$ is biorthogonal to $\{x_n\}$. Further,

$\sup_N \|S_N\| < \infty$ by Corollary 4.29. Therefore, by Theorem 5.12, it suffices to show that $\{x_n\}$ is complete.

Assume that $x^* \in X^*$ satisfies $\langle x_n, x^* \rangle = 0$ for every n. Then for each $x \in X$, we have by equation (4.6) that

$$\langle x, x^* \rangle = \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} \langle x, a_n \rangle\, x_n,\ x^* \right\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, a_n \rangle \langle x_n, x^* \rangle = 0.$$

Hence $x^* = 0$, so $\{x_n\}$ is complete. □

Now we turn our attention to weak* bases. Although a nonseparable space cannot possess any strong bases (and therefore by Theorem 4.30 cannot possess any weak bases either), our first example shows that a nonseparable Banach space can possess a weak* basis.

Example 4.31. Let $X = \ell^1$, and note that $X^* = \ell^\infty$ is not separable. The sequence of standard basis vectors $\{\delta_n\}$ is contained in ℓ^∞, although it does not form a strong basis for this space. We will show that $\{\delta_n\}$ is a weak* basis for ℓ^∞. Choose any $y = (y_n) \in \ell^\infty$. Then for any $x = (x_n) \in \ell^1$, we have

$$\lim_{N\to\infty} \left\langle x,\ \sum_{n=1}^{N} y_n \delta_n \right\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, \delta_n \rangle\, y_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n y_n = \langle x, y \rangle.$$

Hence $y = \sum y_n \delta_n$ in the weak* topology (even though this series need not converge strongly), and by Exercise 4.27 this representation is unique. Therefore $\{\delta_n\}$ is a weak* basis for ℓ^∞. ♢

Although every strong or weak basis is a strong Schauder basis, the following example shows that a weak* basis need not be a weak* Schauder basis. We assign the proof of the statements made in this example as Exercise 4.28.

Example 4.32. Let $X = c_0$, so $X^* = \ell^1$. Let $\{\delta_n\}$ be the standard basis for ℓ^1, and define

$$\begin{aligned}
x_1 &= \delta_1, \\
x_n &= (-1)^n \delta_1 + \delta_n = ((-1)^n, 0, \dots, 0, 1, 0, \dots) \quad \text{for } n > 1, \\
y_1 &= (1, -1, 1, -1, 1, -1, \dots), \\
y_n &= \delta_n \quad \text{for } n > 1.
\end{aligned}$$

Then $\{x_n\} \subseteq \ell^1$ and $\{y_n\} \subseteq \ell^\infty$, and $\{x_n\}$ is a strong Schauder basis for ℓ^1 whose biorthogonal system is $\{y_n\}$ (Exercise 4.28).

Since strong convergence in X^* implies weak and weak* convergence in X^*, it follows that every $x \in \ell^1$ can be written $x = \sum \langle x, y_n \rangle\, x_n$ with strong, weak, and weak* convergence of this series. Since $\{x_n\}$ is a strong and hence weak basis, this is the unique way to represent x as $x = \sum c_n x_n$ with strong or weak convergence of the series. A separate calculation is required to determine if this is the unique representation of x with respect to weak* convergence of the series, and indeed this is the case. Hence $\{x_n\}$ is also a weak* basis for ℓ^1.

However, the sequence y_1 does not belong to c_0, and the functional determined by y_1 is not weak* continuous on ℓ^1. Hence $\{x_n\}$ is not a Schauder weak* basis for ℓ^1. ♢

Our final example shows that a strong basis for X^* need not be a weak* basis for X^* (see Exercise 4.29).

Example 4.33. Let $X = c_0$, so that $X^* = \ell^1$. Let $\{\delta_n\}$ be the standard basis for ℓ^1, and define

$$\begin{aligned} x_1 &= \delta_1, \\ x_n &= \delta_n - \delta_{n-1} = (0,\dots,0,-1,1,0,\dots) \quad \text{for } n > 1, \\ y_n &= (0,\dots,0,0,1,1,1,\dots) \quad \text{for } n \in \mathbf{N}. \end{aligned}$$

Then $\{x_n\}$ is a strong Schauder basis for ℓ^1 whose biorthogonal system is $\{y_n\}$ (Exercise 4.29). As observed in Example 4.32, this implies that $x \in \ell^1$ can be written $x = \sum \langle x, y_n\rangle\, x_n$ with strong, weak, and weak* convergence of this series. However, with respect to weak* convergence of the series we have $\sum x_n = 0$ so $\{x_n\}$ is not "weak* ω-independent." Therefore, $x = \sum \langle x, y_n\rangle\, x_n$ is not a unique representation of x in the form $x = \sum c_n x_n$ with respect to weak* convergence. Hence $\{x_n\}$ is not a weak* basis for ℓ^1, even though it is a strong and weak Schauder basis for ℓ^1. ♢

Exercises

4.25. Prove Theorem 4.27.

4.26. Prove Theorem 4.28.

4.27. Finish the details of Example 4.31.

4.28. Prove the statements made in Example 4.32

4.29. Prove the statements made in Example 4.33.

5

Biorthogonality, Minimality, and More About Bases

In this chapter we explore some of the many "shades of grey" in the meaning of independence in infinite-dimensional Banach spaces, and we apply this knowledge to derive additional results about Schauder bases.

5.1 The Connection between Minimality and Biorthogonality

If $\{x_n\}$ is a Schauder basis with associated coefficient functionals $\{a_n\}$, then we have the biorthogonality condition $\langle x_m, a_n \rangle = \delta_{mn}$. Unfortunately, the next example shows that the existence of a biorthogonal sequence is not sufficient by itself to guarantee that we have a Schauder basis, even when combined with completeness.

Example 5.1. Complete with biorthogonal sequence $\not\Longrightarrow$ *basis.*

Recall that the space $C(\mathbf{T})$ consisting of the complex-valued, continuous, 1-periodic functions on $\mathbf{R}$ is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$ (see Exercise 4.24). Define $e_n(t) = e^{2\pi i n t}$ for $n \in \mathbf{Z}$. Not only are these functions elements of $C(\mathbf{T})$, but they define continuous linear functionals on $C(\mathbf{T})$ via the rule

$$\langle f, e_n \rangle \;=\; \int_0^1 f(t)\, e^{-2\pi i n t}\, dt, \qquad f \in C(\mathbf{T}), \tag{5.1}$$

because

$$|\langle f, e_n \rangle| \;\le\; \int_0^1 |f(t)|\, dt \;\le\; \|f\|_\infty.$$

Thus, we can consider e_n to be an element of $C(\mathbf{T})^*$ in the sense that we identify the function e_n with the functional on $C(\mathbf{T})$ that it determines. Further, $\{e_n\}_{n\in\mathbf{Z}}$ is its own biorthogonal system since $\langle e_m, e_n \rangle = \delta_{mn}$. The Weierstrass Approximation Theorem for trigonometric polynomials, which we will prove

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in Chapter 13, states that if $f \in C(\mathbf{T})$ then there exists some integer N and scalars c_n such that $\left\|f - \sum_{n=-N}^{N} c_n e_n\right\|_\infty < \varepsilon$. Equivalently, this says that $\operatorname{span}\{e_n\}_{n\in\mathbf{Z}}$ is dense in $C(\mathbf{T})$, so $\{e_n\}_{n\in\mathbf{Z}}$ is complete in $C(\mathbf{T})$.

Thus, $\{e_n\}_{n\in\mathbf{Z}}$ is complete in $C(\mathbf{T})$ and has a biorthogonal sequence in $C(\mathbf{T})^*$. However, we shall see that it is not a basis for $C(\mathbf{T})$. Since this sequence is indexed by $\mathbf{Z}$ rather than $\mathbf{N}$, in order to discuss convergence of partial sums we must fix an ordering of this index set. We choose the "natural" ordering $\{0, -1, 1, -2, 2, \dots\}$, which means that we are considering the sequence of partial sums $\sum_{n=0}^{0}$, $\sum_{n=-1}^{0}$, $\sum_{n=-1}^{1}$, $\sum_{n=-2}^{1}$, $\sum_{n=-2}^{2}$, etc. Suppose that $\{e_n\}_{n\in\mathbf{Z}}$ was a basis for $C(\mathbf{T})$ with respect to this ordering of $\mathbf{Z}$. Then given $f \in C(\mathbf{T})$, the partial sums of its basis expansion converge to f. Considering the "symmetric" partial sums $\sum_{n=-N}^{N}$ in particular, this implies that

$$f \;=\; \lim_{N\to\infty} \sum_{n=-N}^{N} \langle f, e_n\rangle\, e_n, \tag{5.2}$$

where the limit is in the norm $\|\cdot\|_\infty$. When the limit in equation (5.2) exists, it is customary to write $f = \sum_{n\in\mathbf{Z}} \langle f, e_n\rangle\, e_n$ and call this the Fourier series representation of f. However, it is known that there exist continuous functions $f \in C(\mathbf{T})$ whose Fourier series representation does not converge uniformly. Hence $\{e_n\}_{n\in\mathbf{Z}}$ is not a basis for $C(\mathbf{T})$, at least with respect to the natural ordering of $\mathbf{Z}$ given above. We will discuss this in more detail in Chapter 14, or see [Gra04], [Kat04], [Heil]. ◊

Remark 5.2. (a) In contrast, we know that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, and hence the Fourier series of every function $f \in L^2(\mathbf{T})$ converges unconditionally in L^2-norm to f (see Example 1.52). However, we should not be misled into thinking that the difference between this fact and Example 5.1 is that $L^2(\mathbf{T})$ is a Hilbert space while $C(\mathbf{T})$ is not. We will give more examples in the coming sections that illustrate this point.

(b) Although we will not need this information, the dual space of $C(\mathbf{T})$ can be characterized as the space of all complex Radon measures on the torus. More generally, one of the *Riesz Representation Theorems* states that if X is a locally convex Hausdorff topological space then $C_0(X)^*$ is isomorphic to the space of complex Radon measures on X, see [Fol99]. ◊

While having a biorthogonal system is not equivalent to being a basis, it is equivalent to something else. We explore this next.

Definition 5.3 (Minimal Sequences). We say that a sequence $\{x_n\}$ in a Banach space X is *minimal* if no vector x_m lies in the closed span of the other vectors x_n, i.e.,

$$\forall\, m \in \mathbf{N}, \quad x_m \notin \overline{\operatorname{span}}\{x_n\}_{n\neq m}.$$

A sequence that is both minimal and complete is said to be *exact.* ◊

We will show that minimality and the existence of a biorthogonal sequence are equivalent. This is easy to prove in Hilbert spaces by making use of orthogonal complements (see Exercise 1.47), but to prove this for generic Banach spaces we will need to employ the Hahn–Banach Theorem.

Lemma 5.4. *Let $\{x_n\}$ be a sequence in a Banach space X.*

(a) $\exists\,\{a_n\} \subseteq X^*$ *biorthogonal to* $\{x_n\} \iff \{x_n\}$ *is minimal.*

(b) $\exists$ *unique* $\{a_n\} \subseteq X^*$ *biorthogonal to* $\{x_n\} \iff \{x_n\}$ *is exact.*

Proof. (a) $\Rightarrow$. Suppose that $\{a_n\} \subseteq X^*$ is biorthogonal to $\{x_n\}$. Fix any $m \in \mathbf{N}$, and choose $z \in \operatorname{span}\{x_n\}_{n \neq m}$, say $z = \sum_{j=1}^{N} c_{n_j} x_{n_j}$. Then

$$\langle z, a_m \rangle \;=\; \sum_{j=1}^{N} c_{n_j} \,\langle x_{n_j}, a_m \rangle \;=\; 0,$$

since $x_{n_j} \neq x_m$ for each j. Thus $a_m = 0$ on $\operatorname{span}\{x_n\}_{x \neq m}$, and since a_m is continuous we therefore have $\langle z, a_m \rangle = 0$ for all $z \in \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. However $\langle x_m, a_m \rangle = 1$, so we must have $x_m \notin \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. Therefore $\{x_n\}$ is minimal.

$\Leftarrow$. Suppose that $\{x_n\}$ is minimal. Fix m, and define $E = \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. This is a closed subspace of X that does not contain x_m. Therefore, by the Hahn–Banach Theorem (Corollary 2.4) there is a functional $a_m \in X^*$ such that

$$\langle x_m, a_m \rangle \;=\; 1 \qquad \text{and} \qquad \langle x, a_m \rangle \;=\; 0 \text{ for } x \in E.$$

Repeating this for each $m \in N$ we obtain a sequence $\{a_n\}$ that is biorthogonal to $\{x_n\}$.

(b) We assign this part as Exercise 5.1. □

Example 5.5. By the Weierstrass Approximation Theorem, the sequence of monomials $\{x^k\}_{k \geq 0}$ is complete in $C[0,1]$. However, by Exercise 2.4 the proper subsequence $\{x^{2k}\}_{k \geq 0}$ is also complete in $C[0,1]$. Therefore $x \in \overline{\operatorname{span}}\{x^{2k}\}_{k \geq 0}$, so $\{x^k\}_{k \geq 0}$ is not minimal and consequently does not possess a biorthogonal system. ◊

The following remarkable result characterizes the sequences of monomials that are complete in $C[0,1]$ or $C[a,b]$ (see [DM72] for proof). In the statement of this result, we implicitly omit any terms of the form $1/0$ from the given series.

Theorem 5.6 (Müntz–Szász Theorem). *Let $0 \leq n_1 \leq n_2 \leq \cdots$ be an increasing sequence of nonnegative integers.*

(a) $\{x^{n_k}\}_{k \in \mathbf{N}}$ *is complete in* $C[0,1]$ *if and only if* $n_1 = 0$ *and* $\sum 1/n_k = \infty$.

(b) *If* $0 < a < b < \infty$ *then* $\{x^{n_k}\}_{k \in \mathbf{N}}$ *is complete in* $C[a,b]$ *if and only if* $\sum 1/n_k = \infty$. ◊

Exercises

5.1. Prove part (b) of Lemma 5.4.

5.2. Fix $1 \le p < \infty$. Show that $\{x^k\}_{k\ge 0}$ is complete but not minimal in $L^p[0,1]$, and is not complete in $L^\infty[0,1]$.

5.2 Shades of Grey: Independence

In finite dimensions, a sequence $\{x_1, \dots, x_n\}$ is minimal if and only if it is linearly independent, and in this case it is a basis for its span. These simple facts do not extend to infinite sequences in infinite-dimensional spaces. In this section we will consider some of the "shades of grey" in the meaning of independence in infinite dimensions. Some of the terms given in the next definition were introduced earlier, but we restate them here for convenience.

Definition 5.7. A sequence $\{x_n\}$ in a Banach space X is:

(a) *finitely linearly independent* (or *finitely independent* or simply *independent* for short) if $\sum_{n=1}^{N} c_n x_n = 0$ implies $c_1 = \dots = c_N = 0$,

(b) *ω-independent* if $\sum_{n=1}^{\infty} c_n x_n$ converges and equals 0 only when $c_n = 0$ for every n,

(c) *minimal* if $x_m \notin \overline{\operatorname{span}}\{x_n\}_{n\ne m}$ for every m,

(d) a *basic sequence* if it is a Schauder basis for $\overline{\operatorname{span}}\{x_n\}$. ♢

In particular, a Schauder basis for X is a basic sequence that is complete. Completeness is essentially a spanning-type property, whereas we are most interested at the moment in independence-type properties. Hence we focus in this section on basic sequences rather than bases.

We have the following implications among the independence properties introduced in Definition 5.7.

Theorem 5.8. *Let $\{x_n\}$ be a sequence in a Banach space X. Then the following statements hold.*

(a) *$\{x_n\}$ is a basic sequence $\implies$ $\{x_n\}$ is minimal.*

(b) *$\{x_n\}$ is minimal $\implies$ $\{x_n\}$ is ω-independent.*

(c) *$\{x_n\}$ is ω-independent $\implies$ $\{x_n\}$ is finitely independent.*

Proof. (a) Assume that $\{x_n\}$ is a basic sequence in X. Then $\{x_n\}$ is a basis for $M = \overline{\operatorname{span}}\{x_n\}$, so there exists a sequence $\{a_n\} \subseteq M^*$ that is biorthogonal to $\{x_n\}$.

Fix $m \in \mathbf{N}$, and define $E_m = \operatorname{span}\{x_n\}_{n\ne m}$. Then, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have $\langle x, a_m\rangle = 0$ for every $x \in E_m$. Since a_m is continuous on M, this implies $\langle x, a_m \rangle = 0$ for every $x \in \overline{E_m} = \overline{\text{span}}\{x_n\}_{n \neq m}$. However, we know that $\langle x_m, a_m \rangle = 1$, so we conclude that $x_m \notin \overline{E_m}$. Hence $\{x_n\}$ is minimal.

(b) Suppose that $\{x_n\}$ is minimal and $\sum c_n x_n$ converges and equals 0. Suppose that there exists some m such that $c_m \neq 0$. Then

$$x_m = -\frac{1}{c_m} \sum_{n \neq m} c_n x_n \in \overline{\text{span}}\{x_n\}_{n \neq m},$$

which contradicts the definition of minimality. Therefore the sequence $\{x_n\}$ is ω-independent.

(c) This is immediate. □

The following examples show that none of the converse implications in Theorem 5.8 hold in general, even in Hilbert spaces and even if we assume completeness.

Example 5.9. Minimal and complete $\not\Rightarrow$ *basis.*

(a) Example 5.1 shows that the sequence $\{e^{2\pi int}\}_{n \in \mathbf{Z}}$ is minimal and complete in $C(\mathbf{T})$ but is not a basis for $C(\mathbf{T})$.

(b) We will give a Hilbert space example from [KS35] of a sequence that is minimal and complete but not a basis. Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space H, and let $x_n = e_n + e_1$ for $n \geq 2$. Consider the sequence $\{x_n\}_{n \geq 2}$. Given m, $n \geq 2$, we have

$$\langle x_m, e_n \rangle = \langle e_m, e_n \rangle + \langle e_1, e_n \rangle = \delta_{mn} + 0.$$

Hence the sequence $\{e_n\}_{n \geq 2}$ is biorthogonal to $\{x_n\}_{n \geq 2}$, so $\{x_n\}_{n \geq 2}$ is minimal.

If $x \in H$ satisfies $\langle x, x_n \rangle = 0$ for every $n \geq 2$, then $\langle x, e_n \rangle = -\langle x, e_1 \rangle$ for all $n \geq 2$. Therefore, by the Plancherel Equality,

$$\sum_{n=1}^{\infty} |\langle x, e_1 \rangle|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2 < \infty,$$

so we must have $\langle x, e_1 \rangle = 0$. But then $\langle x, e_n \rangle = 0$ for every n, so $x = 0$ and $\{x_n\}_{n \geq 2}$ is complete. Since we have already shown that $\{x_n\}_{n \geq 2}$ is minimal, we conclude that it is exact.

However, we will show that $\{x_n\}_{n \geq 2}$ is not a basis for H. Since the sequence $(1/n)_{n \in \mathbf{N}}$ belongs to ℓ^2, the series $x = \sum_{n=1}^{\infty} e_n/n$ converges in H. Suppose that we could write $x = \sum_{n=2}^{\infty} c_n x_n$, with convergence of the series in the norm of H. Then for each $m \geq 2$ we would have

$$\frac{1}{m} = \langle x, e_m \rangle = \left\langle \sum_{n=2}^{\infty} c_n x_n,\ e_m \right\rangle = \sum_{n=2}^{\infty} c_n \langle e_n + e_1, e_m \rangle = c_m.$$

But then

$$\sum_{n=2}^{\infty} c_n x_n \;=\; \sum_{n=2}^{\infty} \frac{1}{n} x_n \;=\; \sum_{n=2}^{\infty} \frac{1}{n}\,(e_n + e_1),$$

which is a contradiction because this series does not converge (consider the norms of the partial sums). Hence we cannot represent the vector x in the form $x = \sum_{n=2}^{\infty} c_n x_n$, so $\{x_n\}_{n\geq 2}$ is not a basis.

An interesting fact about this example is that while $\{x_n\}_{n\geq 2}$ is exact (both minimal and complete), its biorthogonal sequence $\{e_n\}_{n\geq 2}$ is not complete! In contrast, we will see in Corollary 5.22 that the biorthogonal sequence to a Schauder basis for a Hilbert space (or a reflexive Banach space) must be complete.

(c) We give another, even more interesting, example of a sequence in a Hilbert space that is exact but not a basis. Consider the trigonometric system $\{e_n\}_{n\in\mathbf{Z}}$ where $e_n(t) = e^{2\pi i n t}$, which forms an orthonormal basis for $L^2(\mathbf{T})$. Define functions $f_n \in L^2(\mathbf{T})$ by

$$f_n(t) \;=\; t e_n(t) \;=\; t\, e^{2\pi i n t}, \qquad n \neq 0.$$

More precisely, we define $f_n(t) = t e_n(t)$ for $t \in [0,1)$, and then extend f_n 1-periodically to $\mathbf{R}$. Also define

$$g_n(t) \;=\; \frac{e_n(t) - 1}{t} \;=\; \frac{e_n(t) - e_0(t)}{t} \qquad n \neq 0.$$

A direct calculation shows that

$$\|g_n\|_{L^2}^2 \;=\; \int_0^1 |g_n(t)|^2\, dt \;=\; 4\pi n \int_0^{\pi n} \frac{\sin^2 u}{u^2}\, du \;<\; \infty, \tag{5.3}$$

so $g_n \in L^2(\mathbf{T})$. Further, for integers $m, n \neq 0$ we have

$$\langle f_m, g_n \rangle \;=\; \int_0^1 t e_m(t)\, \overline{\frac{e_n(t) - e_0(t)}{t}}\, dt \;=\; \langle e_m, e_n \rangle - \langle e_m, e_0 \rangle \;=\; \delta_{mn} - 0.$$

Therefore $\{g_n\}_{n\neq 0}$ is biorthogonal to $\{f_n\}_{n\neq 0}$, so each of these sequences is minimal in $L^2(\mathbf{T})$.

Now suppose that $f \in L^2(\mathbf{T})$ is such that $\langle f, f_n \rangle = 0$ for all $n \neq 0$. The function $g(t) = t f(t)$ belongs to $L^2(\mathbf{T})$, and for each $n \neq 0$ we have

$$\langle g, e_n \rangle \;=\; \int_0^1 t f(t)\, e^{-2\pi i n t}\, dt \;=\; \int_0^1 f(t)\, \overline{f_n(t)}\, dt \;=\; \langle f, f_n \rangle \;=\; 0.$$

Since $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, this implies that

$$g \;=\; \sum_{n\in\mathbf{Z}} \langle g, e_n \rangle\, e_n \;=\; \langle g, e_0 \rangle\, e_0. \tag{5.4}$$

As e_0 is the constant function 1, we therefore have

$$f(t) \;=\; \frac{g(t)}{t} \;=\; \frac{\langle g, e_0\rangle\, e_0(t)}{t} \;=\; \frac{c}{t} \;\text{ a.e.},$$

where c is the constant $\langle g, e_0\rangle$. If $c \neq 0$ then $f(t) = c/t \notin L^2(\mathbf{T})$, which is a contradiction. Therefore $c = 0$, so $f = 0$ a.e. Hence $\{f_n\}_{n\neq 0}$ is complete in $L^2(\mathbf{T})$.

Since $\{f_n\}_{n\neq 0}$ is both minimal and complete, it is exact in $L^2(\mathbf{T})$. Suppose that it was a basis for $L^2(\mathbf{T})$. Then it would have a finite basis constant $\mathcal{C}$. Since $\{g_n\}_{n\neq 0}$ is the biorthogonal sequence, Theorem 4.13(d) implies that

$$1 \;\leq\; \|f_n\|_{L^2}\, \|g_n\|_{L^2} \;\leq\; 2\mathcal{C}, \qquad n \neq 0.$$

As the functions f_n all have identical norms, this implies that $\sup \|g_n\|_{L^2} < \infty$. However, since

$$\lim_{n\to\infty} \int_0^{\pi n} \frac{\sin^2 u}{u^2}\, du \;=\; \frac{\pi}{2},$$

it follows from equation (5.3) that $\lim_{n\to\infty} \|g_n\|_{L^2} = \infty$. This is a contradiction, so $\{f_n\}_{n\neq 0}$ cannot be a basis for $L^2(\mathbf{T})$, no matter what ordering of $\mathbf{Z}\backslash\{0\}$ we choose.

In contrast to the example from part (a), we will show that this biorthogonal sequence $\{g_n\}_{n\neq 0}$ is complete in $L^2(\mathbf{T})$. Suppose $h \in L^2(\mathbf{T})$ satisfies $\langle h, g_n\rangle = 0$ for $n \neq 0$. If we define $g_0(t) = \frac{e^{-2\pi i 0\cdot t}-1}{t} = 0$, then $\langle h, g_n\rangle = 0$ for all $n \in \mathbf{Z}$. Set $g(t) = h(t)\, \frac{e^{2\pi i t}-1}{t} \in L^2(\mathbf{T})$. Then for every $m \in \mathbf{Z}$ we have

$$\begin{aligned}
\langle g, e_m\rangle \;&=\; \int_0^1 g(t)\, e^{-2\pi i m t}\, dt \\
&=\; \int_0^1 h(t)\, \frac{e^{-2\pi i (m-1)t} - 1 + 1 - e^{-2\pi i m t}}{t}\, dt \\
&=\; \langle h, g_{m-1}\rangle - \langle h, g_m\rangle \;=\; 0.
\end{aligned}$$

Therefore $g = 0$ a.e., which implies $h = 0$ a.e., so we conclude that $\{g_n\}_{n\neq 0}$ is complete. We refer to [HY10] for other results inspired by this example. $\diamondsuit$

Example 5.10. ω-independent and complete $\not\Longrightarrow$ minimal.

(a) Let X be a Banach space that has a sequence that is exact but is not a basis for X (e.g., Example 5.9 shows that such sequences exist in any separable Hilbert space). Then by Theorem 5.12, which we will prove shortly, there exists some $y \in X$ such that the series $\sum_{n=1}^{\infty} \langle y, a_n\rangle\, x_n$ does not converge, where $\{a_n\}$ is the biorthogonal sequence to $\{x_n\}$. Consider the new sequence $\{y\} \cup \{x_n\}$. This sequence is complete, and since $y \in X = \overline{\mathrm{span}}\{x_n\}$ it cannot be minimal. However, we will show that $\{y\} \cup \{x_n\}$ is ω-independent.

Suppose $c_0 y + \sum_{n=1}^{\infty} c_n x_n = 0$, i.e., the series converges and equals zero. If $c_0 \neq 0$ then $y = -\frac{1}{c_0}\sum_{n=1}^{\infty} c_n x_n$. In this case, the biorthogonality of $\{x_n\}$ and $\{a_n\}$ implies that $\langle y, a_n\rangle = -c_n/c_0$. But then $\sum_{n=1}^{\infty} \langle y, a_n\rangle\, x_n$ converges, which is a contradiction. Therefore, we must have $c_0 = 0$, and hence $\sum_{n=1}^{\infty} c_n x_n = 0$. However, $\{x_n\}$ is minimal and therefore ω-independent, so this implies that every c_n is zero. Consequently $\{y\} \cup \{x_n\}$ is ω-independent and complete, but it is not minimal.

(b) Here is another example of a complete ω-independent sequence that is not minimal. By the Weierstrass Approximation Theorem, the set of monomials $\{x^k\}_{k\geq 0}$ is complete in the space $C[0,1]$ of continuous functions on $[0,1]$. However, Exercise 2.4 shows that $\{x^{2k}\}_{k\geq 0}$ is also complete in $C[0,1]$, so $\{x^k\}_{k\geq 0}$ cannot be minimal. Alternatively, this also follows from the Müntz–Szász Theorem (Theorem 5.6). Still, we will show that $\{x^k\}_{k\geq 0}$ is ω-independent.

Suppose that $\sum_{k=0}^{\infty} c_k x^k = 0$, where the series converges uniformly on $[0,1]$. Then by Exercise 1.29, the function $f(x) = \sum_{k=0}^{\infty} c_k x^k$ is well defined and infinitely differentiable on $(-1,1)$, and by hypothesis we have $f = 0$ on $[0,1)$. Taking limits from the right, we see that $f^{(n)}(0) = 0$ for every $n \geq 0$. Considering $n = 0$, we see that $c_0 = f(0) = 0$. Since power series can be differentiated term by term, we have $f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$, and therefore $c_1 = f'(0) = 0$. Continuing in this way we obtain $c_k = 0$ for every k. Hence $\{x^k\}_{k\geq 0}$ is ω-independent.

It is interesting to note that the sequence $\{x^k\}_{k\geq 0}$ contains *no* minimal subsequences. Indeed, by the Müntz–Szász Theorem, if $\{x^{n_k}\}_{k\in\mathbf{N}}$ is complete in $C[0,1]$, then we can remove any particular monomial x^{n_j} (except for the constant polynomial $x^0 = 1$) and still have a complete sequence. ◇

Example 5.11. Finitely independent and complete $\not\Longrightarrow$ *ω-independent.*

(a) Let $\alpha, \beta \in \mathbf{C}$ be fixed nonzero scalars such that $|\alpha/\beta| > 1$. Let $\{\delta_n\}_{n\in\mathbf{N}}$ be the standard basis for ℓ^2, and define $x_0 = \delta_1$ and $x_n = \alpha\delta_n + \beta\delta_{n+1}$ for $n \in \mathbf{N}$. By Exercise 1.46, $\{x_n\}_{n\geq 0}$ is complete and finitely independent in ℓ^2, but is not ω-independent.

(b) We give another example of a complete, finitely independent sequence that is not ω-independent. Let X be a Banach space that has a basis, and let $\{x_n\}$ be a basis for X. Let $\{a_n\} \subseteq X^*$ be its biorthogonal sequence, and let $x \in X$ be any element such that $\langle x, a_n\rangle \neq 0$ for every n, such as

$$x = \sum_n \frac{x_n}{2^n\,\|x_n\|}.$$

Note that x cannot equal any x_n because $\langle x_n, a_m\rangle = 0$ when $m \neq n$. Consider the new sequence $\{x\} \cup \{x_n\}_{n\in\mathbf{N}}$. This is certainly complete, and we have $-x + \sum \langle x, a_n\rangle\, x_n = 0$, so it is not ω-independent. However, we claim that it is finitely independent. Suppose that $c_0 x + \sum_{n=1}^{N} c_n x_n = 0$. Substituting $x = \sum \langle x, a_n\rangle\, x_n$, it follows that

$$\sum_{n=1}^{N} \bigl(c_0 \langle x, a_n\rangle + c_n\bigr) x_n \;+\; \sum_{n=N+1}^{\infty} c_0 \langle x, a_n\rangle x_n \;=\; 0.$$

However, $\{x_n\}$ is a basis, so this is only possible if $c_0 \langle x, a_n\rangle + c_n = 0$ for $n = 1, \dots, N$ and $c_0 \langle x, a_n\rangle = 0$ for $n > N$. Since no $\langle x, a_n\rangle$ is zero we therefore must have $c_0 = 0$. But then $c_1 = \cdots = c_N = 0$, so $\{x\} \cup \{x_n\}$ is finitely independent. $\diamondsuit$

Exercises

5.3. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Show that each of the following two sequences are ω-independent and complete in H, but are not minimal.

(a) $\{e_1\} \cup \{e_n + e_{n+1}\}_{n\in\mathbf{N}}$.

(b) $\{e_1\} \cup \{e_1 + \frac{1}{n} e_n\}_{n\geq 2}$.

5.4. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H, and for each $n \in \mathbf{N}$ define $x_n = \sum_{k=1}^{n} e_k/k$. Show that $\{x_n\}$ is minimal and complete in H, but is not a Schauder basis for H.

5.5. Let $\{x_n\}$ be a sequence in a Banach space X. Show that if $\{x_n\}$ is ω-independent and for every $x \in X$ there exist some scalars (c_n) such that $x = \sum c_n x_n$, then $\{x_n\}$ is a basis for X.

5.6. Let X be a Banach space. Let us say that a sequence $\{P_N\} \subseteq \mathcal{B}(X)$ is a *family of partial sum projections* if: (i) $P_N^2 = P_N$, (ii) $P_N x \to x$ as $N \to \infty$ for each $x \in X$, (iii) $\dim(\mathrm{range}(P_N)) = N$, and (iv) $P_N P_M = P_{\min\{M,N\}}$.

(a) Show that if $\{x_n\}$ is a basis for X, then $\{S_N\}$ is a family of partial sum projections.

(b) Suppose that $\{P_N\}$ is a family of partial sum projections. Show that if there exist nonzero vectors $x_1 \in \mathrm{range}(P_1)$ and $x_n \in \mathrm{range}(P_n) \cap \ker(P_{n-1})$ for $n > 1$, then $\{x_n\}$ is a basis for X.

5.3 A Characterization of Schauder Bases

If $\{x_n\}$ is a minimal sequence in a Banach space X, then it has a biorthogonal sequence $\{a_n\} \subseteq X^*$. Therefore, even though we do not know whether $\{x_n\}$ is a basis, we can define *partial sum operators*

$$S_N x \;=\; \sum_{k=1}^{N} \langle x, a_n\rangle x_n, \qquad x \in X. \tag{5.5}$$

Each S_N is a bounded operator on X since each a_n is continuous by hypothesis. The sequence $\{x_n\}$ is a basis if and only if $S_N x \to x$ for each $x \in X$. The next theorem gives some equivalent characterizations of when this happens. In particular, we see that a Schauder basis is precisely an exact sequence whose basis constant $\mathcal{C} = \sup_N \|S_N\|$ is finite.

Theorem 5.12. *Given a sequence $\{x_n\}$ in a Banach space X, the following statements are equivalent.*

(a) *$\{x_n\}$ is a basis for X.*

(b) *There exists a biorthogonal sequence $\{a_n\} \subseteq X^*$ such that*

$$\forall\, x \in X, \quad x \;=\; \lim_{N\to\infty} S_N x \;=\; \sum_{n=1}^{\infty} \langle x, a_n\rangle\, x_n.$$

(c) *$\{x_n\}$ is complete and there exists a biorthogonal sequence $\{a_n\} \subseteq X^*$ such that the series $\sum \langle x, a_n\rangle\, x_n$ converges for each $x \in X$.*

(d) *$\{x_n\}$ is exact and* $\sup_N \|S_N x\| < \infty$ *for all $x \in X$.*

(e) *$\{x_n\}$ is exact and* $\sup_N \|S_N\| < \infty$.

Proof. (e) $\Rightarrow$ (b). Assume that statement (e) holds, and choose any $x \in \operatorname{span}\{x_n\}$, say $x = \sum_{n=1}^{M} c_n x_n$. Then, since S_N is linear and $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have for each $N \geq M$ that

$$S_N x \;=\; S_N\Biggl(\sum_{m=1}^{M} c_m x_m\Biggr) \;=\; \sum_{m=1}^{M} c_m\, S_N x_m \;=\; \sum_{m=1}^{M} c_m x_m \;=\; x.$$

Therefore, we trivially have $x = \lim_{N\to\infty} S_N x = \sum \langle x, a_n\rangle\, x_n$ whenever x lies in the dense subspace $\operatorname{span}\{x_n\}$.

At this point we could simply appeal to Exercise 2.31 to draw the conclusion that $x = \lim_{N\to\infty} S_N x$ for arbitrary $x \in X$, but we will write out the argument in detail. Let $\mathcal{C} = \sup_N \|S_N\|$, and let x be any element of X. Since $\operatorname{span}\{x_n\}$ is dense in X, given $\varepsilon > 0$ we can find an element $y \in \operatorname{span}\{x_n\}$ such that $\|x - y\| < \varepsilon/(1+\mathcal{C})$, say $y = \sum_{m=1}^{M} c_m x_m$. Then for $N \geq M$ we have

$$\begin{aligned}
\|x - S_N x\| \;&\leq\; \|x-y\| \;+\; \|y - S_N y\| \;+\; \|S_N y - S_N x\| \\
&\leq\; \|x-y\| \;+\; 0 \;+\; \|S_N\|\, \|x-y\| \\
&\leq\; (1+\mathcal{C})\, \|x-y\| \\
&<\; \varepsilon.
\end{aligned}$$

Thus $x = \lim_{N\to\infty} S_N x = \sum \langle x, a_n\rangle\, x_n$.

We assign the proof of the remaining implications as Exercise 5.7. □

Unfortunately, given an exact sequence $\{x_n\}$, it can be very difficult to determine whether any of the hypotheses of Theorem 5.12 hold.

Example 5.13. Fix $0 < \alpha < 1/2$, and set $\varphi(t) = |t - \frac{1}{2}|^\alpha$ for $t \in [0,1]$. Observe that both φ and its pointwise reciprocal $\widetilde{\varphi}(t) = 1/\varphi(t) = |t - \frac{1}{2}|^{-\alpha}$ belong to $L^2(\mathbf{T})$. Set $e_n(t) = e^{2\pi int}$, and recall that $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$. Now consider the sequence

$$\{e_n\varphi\}_{n\in\mathbf{Z}} \;=\; \{e^{2\pi int}\,\varphi(t)\}_{n\in\mathbf{Z}},$$

which we call a sequence of *weighted exponentials* (we will study such sequences in detail in Section 10.3). Given $m, n \in \mathbf{Z}$ we have

$$\begin{aligned}\langle e_m\varphi, e_n\widetilde{\varphi}\,\rangle \;&=\; \int_0^1 e_m(t)\,\varphi(t)\,\overline{e_n(t)\,\widetilde{\varphi}(t)}\,dt\\ &=\; \int_0^1 e_m(t)\,\overline{e_n(t)}\,dt \;=\; \langle e_m, e_n\rangle \;=\; \delta_{mn}.\end{aligned}$$

Hence $\{e_n\widetilde{\varphi}\}_{n\in\mathbf{Z}}$ is biorthogonal to $\{e_n\varphi\}_{n\in\mathbf{Z}}$, so each of these sequences is minimal in $L^2(\mathbf{T})$. By Exercise 5.9, they are also complete, and hence are exact. It is a much more subtle fact, due to Babenko [Bab48], that $\{e_n\varphi\}_{n\in\mathbf{Z}}$ is a conditional basis for $L^2(\mathbf{T})$. Babenko's paper is in Russian, but his proof is discussed in the text by Singer, see [Sin70, Example 11.2, pp. 351–354]. $\diamondsuit$

The difficulty in Example 5.13 is showing that the series

$$f \;=\; \sum_{n\in\mathbf{Z}} \langle f, e_n\widetilde{\varphi}\rangle\, e_n\varphi$$

converges for each function $f \in L^2(\mathbf{T})$. The convergence is conditional, and is with respect to the "natural" ordering $\mathbf{Z} = \{0, -1, 1, -2, 2, \dots\}$. This was proved directly by Babenko, but it was later shown by by Hunt, Muckenhoupt, and Wheeden that the specific function φ used in Example 5.13 can be replaced by any function φ such that $|\varphi|^2$ belongs to the class of $\mathcal{A}_2$ *weights* on $\mathbf{T}$, which are defined as follows.

Definition 5.14 ($\mathcal{A}_2$ weight). A nonnegative function $w \in L^1(\mathbf{T})$ is an $\mathcal{A}_2$ *weight* if

$$\sup_I \left(\frac{1}{|I|}\int_I w(t)\,dt\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(t)}\,dt\right) \;<\; \infty,$$

where the supremum is taken over all intervals $I \subseteq \mathbf{R}$ (recall that functions in $L^2(\mathbf{T})$ are 1-periodic on $\mathbf{R}$). The class of $\mathcal{A}_2$ weights on $\mathbf{T}$ is denoted by $\mathcal{A}_2(\mathbf{T})$. $\diamondsuit$

Thus, a necessary condition for w to be an $\mathcal{A}_2$ weight is that $1/w$ be integrable, but a little more is required to actually be an $\mathcal{A}_2$ weight—the averages of w and $1/w$ on intervals I must be "complementary." The following result is an equivalent formulation of the theorem of Hunt, Muckenhoupt, and Wheeden [HMW73]. There are also extensions of this result to $p \neq 2$, see [HMW73] or [Gra04].

Theorem 5.15. *Given* $\varphi \in L^2(\mathbf{T})$, *the following statements are equivalent.*

(a) $\{e^{2\pi int}\varphi(t)\}_{n\in\mathbf{Z}}$ *is a Schauder basis for* $L^2(\mathbf{T})$ *with respect to the ordering* $\mathbf{Z} = \{0, -1, 1, -2, 2, \dots\}$.

(b) $|\varphi|^2 \in \mathcal{A}_2(\mathbf{T})$. $\diamondsuit$

If we set $\varphi(t) = |t - \frac{1}{2}|^\alpha$ with $0 < \alpha < 1/2$ then $|\varphi|^2$ is an example of an $\mathcal{A}_2$ weight (Exercise 5.9). In Section 10.3 we will characterize those functions φ such that the system of weighted exponentials $\{e^{2\pi int}\varphi(t)\}_{n\in\mathbf{Z}}$ is complete, exact, an unconditional basis, or an orthonormal basis for $L^2(\mathbf{T})$. In particular, we will see that $\{e^{2\pi int}\varphi(t)\}_{n\in\mathbf{Z}}$ is an unconditional basis if and only if there exist constants A, $B > 0$ such that $A \le |\varphi(t)|^2 \le B$ a.e.

Exercises

5.7. Prove the remaining implications in Theorem 5.12.

5.8. Let $\{x_n\}$ be a basis for a Hilbert space H and $\{y_n\}$ be a basis for a Hilbert space K. This exercise will show that the tensor product sequence $\{x_m \otimes y_n\}_{m,n\in\mathbf{N}}$ is a basis for the tensor product space $H \otimes K = \mathcal{B}_2(H,K)$ (see Appendix B for definitions).

Let $\{a_n\}$, $\{b_n\}$ and $\mathcal{C}_X$, $\mathcal{C}_Y$ denote the biorthogonal systems and basis constants for $\{x_n\}$, $\{y_n\}$, respectively. Order $\mathbf{N} \times \mathbf{N}$ as follows:

$$\begin{aligned}\mathbf{N} \times \mathbf{N} \;=\; \bigl\{&(1,1),\\ &(2,1),\,(2,2),\,(1,2),\\ &(3,1),\,(3,2),\,(3,3),\,(2,3),\,(1,3),\\ &(4,1),\,(4,2),\,(4,3),\,(4,4),\,(3,4),\,(2,4),\,(1,4),\\ &\quad\dots\bigr\},\end{aligned}$$

and let $\{z_k\}_{k\in\mathbf{N}}$ denote $\{x_m \otimes y_n\}_{m,n\in\mathbf{N}}$ arranged according to the above ordering of $\mathbf{N} \times \mathbf{N}$. Let S_N^X, S_N^Y, and S_N^Z denote the partial sum operators associated with $\{x_m\}$, $\{y_n\}$, and $\{z_k\}$, respectively.

(a) Show that $\{x_m \otimes y_n\}_{m,n\in\mathbf{N}}$ is exact in $H \otimes K$, and its biorthogonal system is $\{a_m \otimes b_n\}_{m,n\in\mathbf{N}}$.

(b) Prove the following relationships among the partial sum operators (where we let S_0^X and S_0^Y be the zero operators):

$$\begin{aligned}
S^Z_{N^2} &= S^X_N \otimes S^Y_N, \\
S^Z_{N^2+\ell} &= S^X_N \otimes S^Y_N + (S^X_{N+1} - S^X_N) \otimes S^Y_\ell, \quad \ell = 1, \dots, N+1, \\
S^Z_{N^2+N+1+\ell} &= S^X_{N+1} \otimes S^Y_{N+1} - S^X_\ell \otimes (S^Y_{N+1} - S^Y_N), \quad \ell = 1, \dots, N.
\end{aligned}$$

(c) Show that $\|S^Z_N\| \le 3\,\mathcal{C}_X\,\mathcal{C}_Y$ for each $N \in \mathbf{N}$, and conclude that $\{z_k\}$ is a basis for $H \otimes K$.

5.9. Fix $0 < \alpha < 1/2$, and let $\varphi(t) = |t - \frac{1}{2}|^\alpha$ be as in Example 5.13.

(a) Prove that $\{e^{2\pi int}\varphi(t)\}_{n\in\mathbf{Z}}$ is exact in $L^2(\mathbf{T})$, and its biorthogonal system $\{e^{2\pi int}\widetilde{\varphi}(t)\}_{n\in\mathbf{Z}}$ is also exact (compare Exercise 10.9).

(b) Prove that $|\varphi|^2 \in \mathcal{A}_2(\mathbf{T})$.

5.4 A Characterization of Minimal Sequences and Schauder Bases

As Theorem 5.12 illustrates, even though a minimal sequence $\{x_n\}$ need not be a basis, it is often still useful to consider the corresponding partial sum operators S_N defined in equation (5.5). To motivate the next result, suppose that $\{x_n\}$ is a basis with basis constant $\mathcal{C}$. Then given any $N \ge M$ and scalars $c_1, \dots, c_N$, we have

$$\begin{aligned}
\biggl\|\sum_{n=1}^{M} c_n x_n\biggr\| &= \biggl\|S_M\Bigl(\sum_{n=1}^{N} c_n x_n\Bigr)\biggr\| \\
&\le \|S_M\| \, \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\| \\
&\le \mathcal{C}\biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|. \qquad (5.6)
\end{aligned}$$

We will prove a characterization of minimal sequences that involves a similar estimate. However, in contrast to equation (5.6), the characterization of minimal sequences allows constants C_M that depend on M.

Theorem 5.16. *Let $\{x_n\}$ be a sequence in a Banach space X with all vectors $x_n \ne 0$. Then the following two statements are equivalent.*

(a) $\{x_n\}$ *is minimal.*

(b) $\forall\, M,\ \exists\, C_M \ge 1$ *such that*

$$\forall\, N \ge M, \quad \forall\, c_0, \dots, c_N, \quad \biggl\|\sum_{n=1}^{M} c_n x_n\biggr\| \le C_M \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|.$$

Proof. (a) ⇒ (b). Assume that $\{x_n\}$ is minimal. By Lemma 5.4, there exists a sequence $\{a_n\} \subseteq X^*$ biorthogonal to $\{x_n\}$. Therefore, given $N \geq M$ and scalars $c_0, \dots, c_N$ we have

$$\left\|\sum_{n=1}^{M} c_n x_n\right\| \;=\; \left\|S_M\left(\sum_{n=1}^{N} c_n x_n\right)\right\| \;\leq\; \|S_M\| \left\|\sum_{n=1}^{N} c_n x_n\right\|.$$

Hence statement (b) follows with $C_M = \|S_M\|$.

(b) ⇒ (a). Assume that statement (b) holds, and let $E = \operatorname{span}\{x_n\}$. Set $C_0 = 0$. Then given $x = \sum_{n=1}^{N} c_n x_n \in E$ and $1 \leq M \leq N$, we have

$$\begin{aligned} |c_M|\, \|x_M\| \;=\; \|c_M x_M\| \;&\leq\; \left\|\sum_{n=1}^{M} c_n x_n\right\| \;+\; \left\|\sum_{n=1}^{M-1} c_n x_n\right\| \\ &\leq\; C_M \left\|\sum_{n=1}^{N} c_n x_n\right\| \;+\; C_{M-1} \left\|\sum_{n=1}^{N} c_n x_n\right\| \\ &=\; (C_M + C_{M-1})\, \|x\|. \end{aligned}$$

As $x_M \neq 0$, we therefore have

$$|c_M| \;\leq\; \frac{C_M + C_{M-1}}{\|x_M\|}\, \|x\|, \qquad 1 \leq M \leq N. \tag{5.7}$$

In particular, if $x = 0$ then $c_1 = \cdots = c_N = 0$, so $\{x_n\}$ is finitely linearly independent. Since E is the finite linear span of $\{x_n\}$, this implies that $\{x_n\}$ is a Hamel basis for E. That is, every element $x \in E$ has a *unique* representation of the form $x = \sum_{n=1}^{\infty} a_n(x)\, x_n$ where only finitely many of the scalars $a_n(x)$ are nonzero. By equation (5.7),

$$|a_n(x)| \;\leq\; \frac{C_n + C_{n-1}}{\|x_n\|}\, \|x\|, \qquad x \in E,$$

so a_n is continuous on the subspace E. By the Hahn–Banach Theorem (Corollary 2.2), there is a continuous extension of a_n to all of X, which we also refer to as a_n. Consequently $\{x_n\}$ is minimal since $\{a_n\} \subseteq X^*$ is biorthogonal to $\{x_n\} \subseteq X$. □

Given an exact sequence $\{x_n\}$, the next result states that the constants C_M appearing in Theorem 5.16 are uniformly bounded in M if and only if $\{x_n\}$ is a basis for X.

Theorem 5.17. *If $\{x_n\}$ is a sequence in a Banach space X, then the following statements are equivalent.*

(a) *$\{x_n\}$ is a basis for X.*

(b) *$\{x_n\}$ is complete, $x_n \neq 0$ for all n, and there exists $C \geq 1$ such that*

$$\forall\, N \geq M, \quad \forall\, c_1, \dots, c_N, \quad \left\| \sum_{n=1}^{M} c_n x_n \right\| \;\leq\; C \left\| \sum_{n=1}^{N} c_n x_n \right\|. \tag{5.8}$$

Further, in case these hold, the best constant C in equation (5.8) *is the basis constant $C = \mathcal{C} = \sup_N \|S_N\|$.*

Proof. (a) ⇒ (b). This follows as in equation (5.6).

(b) ⇒ (a). Suppose that statement (b) holds. Theorem 5.16 implies that $\{x_n\}$ is minimal, so there exists a biorthogonal system $\{a_n\} \subseteq X^*$. Since $\{x_n\}$ is complete, it suffices by Theorem 5.12 to show that $\sup_N \|S_N\| < \infty$.

Suppose that $x = \sum_{n=1}^{M} c_n x_n \in \operatorname{span}\{x_n\}$. Then:

$$N \leq M \quad \Longrightarrow \quad \|S_N x\| \;=\; \left\| \sum_{n=1}^{N} c_n x_n \right\| \;\leq\; C \left\| \sum_{n=1}^{M} c_n x_n \right\| \;=\; C\,\|x\|,$$

$$N > M \quad \Longrightarrow \quad \|S_N x\| \;=\; \left\| \sum_{n=1}^{M} c_n x_n \right\| \;=\; \|x\|.$$

As $C \geq 1$ we therefore have

$$\forall\, x \in \operatorname{span}\{x_n\}, \quad \forall\, N \in \mathbf{N}, \quad \|S_N x\| \;\leq\; C\,\|x\|.$$

However, each S_N is continuous and $\operatorname{span}\{x_n\}$ is dense in X, so we have $\|S_N x\| \leq C\,\|x\|$ for all $x \in X$ and $N \in \mathbf{N}$. Thus $\sup_N \|S_N\| \leq C < \infty$, and this argument also shows that the smallest possible value for C is $C = \mathcal{C} = \sup_N \|S_N\|$. □

Exercises

5.10. Let X be a complex Banach space. By Exercise 1.7, the vector space $X_{\mathbf{R}} = X$ over the real field is a real Banach space. Let $\{x_n\}$ be a fixed sequence in X. Show that

$$\{x_n\} \text{ is a basis for } X \quad \Longleftrightarrow \quad \{x_1, ix_1, x_2, ix_2, \dots\} \text{ is a basis for } X_{\mathbf{R}}.$$

5.11. Let X be a Banach space. Show that $\{x_n\} \subseteq X$ is a monotone basis for X if and only if $\{x_n\}$ is complete, $x_n \neq 0$ for every n, and

$$\forall\, N \in \mathbf{N}, \quad \forall\, c_1, \dots, c_N, c_{N+1} \in \mathbf{F}, \quad \left\| \sum_{n=1}^{N} c_n x_n \right\| \;\leq\; \left\| \sum_{n=1}^{N+1} c_n x_n \right\|.$$

5.12. Prove that the Schauder system is a monotone basis for $C[0,1]$.

5.13. Let X be a Banach space. Show that X has a monotone basis if and only if there exists a sequence of operators $\{P_N\} \subseteq \mathcal{B}(X)$ such that for each $N \in \mathbf{N}$ we have

(i) $\|P_N\| = 1$,

(ii) $P_N^2 = P_N$,

(iii) $\dim(\mathrm{range}(P_N)) = N$,

(iv) $\mathrm{range}(P_N) \subseteq \mathrm{range}(P_{N+1})$, and

(v) $\bigcup \mathrm{range}(P_N)$ is dense in X.

5.5 The Haar System in $L^p[0,1]$

Set $\chi = \chi_{[0,1]}$, and let $\psi_{n,k}$ be as defined in Example 1.54. Then by Exercise 1.50, the *Haar system*

$$\mathcal{H} \;=\; \{\chi\} \,\cup\, \{\psi_{n,k}\}_{n\ge 0,\, k=0,\dots,2^n-1},$$

forms an orthonormal basis for $L^2[0,1]$. Let us consider what happens if we take $p \neq 2$.

Since $L^\infty[0,1]$ is not separable, it cannot have a basis (but even so, see Exercise 5.15 for more on the Haar system in $L^\infty[0,1]$). Therefore we focus on $1 \le p < \infty$. Note that $\chi + \psi_{0,0} = 2\chi_{[0,1/2)}$, and by continuing to form finite sums we see that $\chi_{[\frac{k}{2^n},\frac{k+1}{2^n})}$, the characteristic function of the dyadic interval $[\frac{k}{2^n},\frac{k+1}{2^n})$, belongs to $\mathrm{span}(\mathcal{H})$ for each $n \ge 0$ and $k = 0,\dots,2^n-1$. Consequently,

$$\mathrm{span}(\mathcal{H}) \;=\; \mathrm{span}\Big\{\chi_{[\frac{k}{2^n},\frac{k+1}{2^n})} : n \ge 0,\, k = 0,\dots,2^n-1\Big\}.$$

The right-hand set on the line above is dense in $L^p[0,1]$, so the Haar system is complete in $L^p[0,1]$ when p is finite. We will use Exercise 5.11 to show that $\mathcal{H}$ is a a basis for $L^p[0,1]$.

Enumerate the Haar system as

$$\{h_1, h_2, \dots\} \;=\; \{\chi,\, \psi_{0,0},\, \psi_{1,0},\, \psi_{1,1},\, \psi_{2,0},\, \psi_{2,1},\, \psi_{2,2},\, \dots\}. \tag{5.9}$$

Fix $N > 1$ and scalars $c_1,\dots,c_N$, and consider the functions

$$g_{N-1} \;=\; \sum_{n=1}^{N-1} c_n h_n \qquad \text{and} \qquad g_N \;=\; \sum_{n=1}^{N} c_n h_n.$$

Note that g_{N-1} and g_N agree except possibly on the dyadic interval I where h_N is nonzero. Let I_1, I_2 denote the left and right halves of I. Then g_{N-1} takes a constant value c on I, and there is a constant d_N such that $g_N = c + d_N$

on I_1 and $g_N = c - d_N$ on I_2. Let m be the integer such that the length of I is 2^{-m}. Then

$$\begin{aligned}
\int_0^1 \biggl|\sum_{n=1}^{N} c_n h_n(t)\biggr|^p dt \;-\; \int_0^1 \biggl|\sum_{n=1}^{N-1} c_n h_n(t)\biggr|^p dt
&= \int_0^1 |g_N(t)|^p\,dt - \int_0^1 |g_{N-1}(t)|^p\,dt \\
&= \int_{I_1} |c+d_N|^p\,dt + \int_{I_2} |c-d_N|^p\,dt - \int_I |c|^p\,dt \\
&= \frac{|c+d_N|^p}{2^{m+1}} + \frac{|c-d_N|^p}{2^{m+1}} - \frac{|c|^p}{2^m} \\
&= 2^{-m-1}\bigl(|c+d_N|^p + |c-d_N|^p - 2\,|c|^p\bigr). \qquad (5.10)
\end{aligned}$$

The quantity in equation (5.10) is nonnegative by Exercise 5.14. Therefore

$$\begin{aligned}
\biggl\|\sum_{n=1}^{N-1} c_n h_n\biggr\|_p &= \biggl(\int_0^1 \biggl|\sum_{n=1}^{N-1} c_n h_n(t)\biggr|^p dt\biggr)^{1/p} \\
&\le \biggl(\int_0^1 \biggl|\sum_{n=1}^{N} c_n h_n(t)\biggr|^p dt\biggr)^{1/p} = \biggl\|\sum_{n=1}^{N} c_n h_n\biggr\|_p,
\end{aligned}$$

so it follows from Exercise 5.11 that $\mathcal{H} = \{h_n\}$ is a monotone basis for $L^p[0,1]$. Thus, we have proved the following result.

Theorem 5.18. *Given $1 \le p < \infty$, the Haar system is a monotone basis for $L^p[0,1]$ with respect to the ordering given in equation* (5.9). ♢

It can be shown that the Haar system is an unconditional basis for $L^p[0,1]$ when $1 < p < \infty$. This was proved by Paley [Pal32], and a short and seemingly "magical" proof has been given by Burkholder [Bur88]. We will prove in Section 6.4 that the Haar system is a conditional basis for $L^1[0,1]$.

Exercises

5.14. Fix $1 \le p < \infty$, and prove the following statements.

(a) $(1+t)^p \le 2^{p-1}\,(1+t^p)$ for all $t \ge 1$.

(b) If $a,\, b \in \mathbf{F}$ then $|a+b|^p \le 2^{p-1}\,(|a|^p + |b|^p)$.

(c) The quantity appearing in equation (5.10) is nonnegative.

5.15. Let $\{h_n\}$ be the Haar system, enumerated as in equation (5.9).

(a) What is the biorthogonal system for $\{h_n\}$ as a basis for $L^p[0,1]$?

(b) Show that $\{h_n\}$ is a basic sequence in $L^\infty[0,1]$.

(c) Show that $C[0,1] \subseteq \overline{\text{span}}\{h_n\}$ (closure in L^∞-norm). In fact, show that if $f \in C[0,1]$ then the series $\sum \langle f, h_n \rangle\, h_n$ converges uniformly to f.

5.16. (a) Suppose that $f \in C(\mathbf{R})$ is Hölder continuous with exponent α, i.e., there exists a constant $K > 0$ such that $|f(x) - f(y)| \le K\,|x-y|^\alpha$ for all x, $y \in [0,1]$ (compare Exercise 1.23). Show that there exists a constant $C > 0$ such that

$$|\langle f, \psi_{n,k}\rangle| \;\le\; C\,2^{-n(\alpha+1/2)}, \qquad n \ge 0,\, k \in \mathbf{Z}.$$

(b) How does the conclusion change if we only assume that f is Hölder continuous at a point x, i.e., there exist K, $\delta > 0$ such that $|f(x) - f(y)| \le K\,|x-y|^\alpha$ for all y with $|x-y| < \delta$?

5.6 Duality for Bases

Let π denote the canonical embedding of a Banach space X into its double-dual X^{**}. That is, if $x \in X$ then $\pi(x) \in X^{**}$ is the continuous linear functional on X^* defined by $\langle x^*, \pi(x)\rangle = \langle x, x^*\rangle$ for $x^* \in X^*$ (see Definition 2.7).

Suppose that $\{x_n\}$ is a minimal sequence in X. Then by Lemma 5.4, it has a biorthogonal system $\{a_n\} \subseteq X^*$. Consider the sequence $\{\pi(x_n)\} \subseteq X^{**}$. For m, $n \in \mathbf{N}$ we have

$$\langle a_m, \pi(x_n)\rangle \;=\; \langle x_n, a_m\rangle \;=\; \delta_{mn}.$$

Therefore $\{\pi(x_n)\}$ is a sequence in X^{**} that is biorthogonal to $\{a_n\}$ in X^*. Hence $\{a_n\}$ is a minimal sequence in X^*. This proves the following result.

Lemma 5.19. *If $\{x_n\}$ is a minimal sequence in a Banach space X and $\{a_n\} \subseteq X^*$ is biorthogonal to $\{x_n\}$, then $\{a_n\}$ is minimal in X^* and $\{\pi(x_n)\}$ is a biorthogonal sequence in X^{**}.* $\diamondsuit$

However, in general we cannot replace the word "minimal" in Lemma 5.19 by "exact." We have trivial counterexamples whenever X^* is nonseparable, for in this case the countable sequence $\{a_n\}$ cannot possibly be complete in X^*. For example, the standard basis $\{\delta_n\}$ is a basis for $X = \ell^1$, and hence is exact in ℓ^1, but its biorthogonal system (which is also $\{\delta_n\}$) is not complete in $X^* = \ell^\infty$.

Less trivially, even if X^* is separable, it is not true that the dual of an exact sequence need be exact. In fact, this can fail even in Hilbert spaces.

Example 5.20. Given an orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$ for a separable Hilbert space H, Example 5.9(b) constructs a sequence $\{x_n\}_{n\ge 2}$ that is exact but whose biorthogonal sequence is $\{e_n\}_{n\ge 2}$, which is not complete. $\diamondsuit$

However, instead of just being exact, suppose that $\{x_n\}$ is a basis for X. Again, if X^* is not separable then the biorthogonal sequence cannot possibly be a basis for X^*. On the other hand, $\{a_n\}$ is minimal and therefore is exact as a subset of its closed span. Will $\{a_n\}$ be a basis for $\overline{\text{span}}\{a_n\}$? The next theorem shows that this much does always hold.

Theorem 5.21. *Let X be a Banach space. If $\{x_n\}$ is a basis for X, then its biorthogonal system $\{a_n\}$ is a basis for $\overline{\text{span}}\{a_n\}$ in X^*.*

Proof. By Lemma 5.19 $\{a_n\}$ is exact in $\overline{\text{span}}\{a_n\}$, and $\{\pi(x_n)\}$ is a biorthogonal system in X^{**}. Therefore, by Theorem 5.12, we need only show that the partial sum operators T_N associated with the sequence $\{a_n\}$ are uniformly bounded in operator norm. These partial sum operators have the form

$$T_N(x^*) = \sum_{n=1}^{N} \langle x^*, \pi(x_n)\rangle\, a_n = \sum_{n=1}^{N} \langle x_n, x^*\rangle\, a_n, \qquad x^* \in \overline{\text{span}}\{a_n\}.$$

As usual, let S_N denote the partial sum operators associated with the basis $\{x_n\}$. Since S_N is a continuous linear mapping of X into itself, it has an adjoint $S_N^*\colon X^* \to X^*$. In fact, if $x \in X$ and $x^* \in X^*$ then, by definition of the adjoint,

$$\begin{aligned}
\big\langle x,\, S_N^*(x^*)\big\rangle = \big\langle S_N x,\, x^*\big\rangle &= \Big\langle \sum_{n=1}^{N} \langle x, a_n\rangle\, x_n,\; x^* \Big\rangle \\
&= \sum_{n=1}^{N} \langle x, a_n\rangle\, \langle x_n, x^*\rangle \\
&= \Big\langle x,\, \sum_{n=1}^{N} \langle x_n, x^*\rangle\, a_n \Big\rangle = \big\langle x,\, T_N(x^*)\big\rangle.
\end{aligned}$$

Therefore $T_N = S_N^*$, and hence $\|T_N\| = \|S_N^*\| = \|S_N\|$. Consequently, $\sup_N \|T_N\| = \sup_N \|S_N\| < \infty$. $\square$

By Example 5.20, the dual of an exact system need not itself be exact, even in a Hilbert space. In striking contrast we show next that the dual of a basis for a reflexive Banach space X is a basis for X^*.

Corollary 5.22. *If $\{x_n\}$ is a basis for a reflexive Banach space X, then its biorthogonal system $\{a_n\}$ is a basis for X^*.*

Proof. Theorem 5.21 implies that $\{a_n\}$ is a basis for for $\overline{\text{span}}\{a_n\}$ in X^*, so we need only show that $\{a_n\}$ is complete in X^*. Suppose that $x^{**} \in X^{**}$ satisfies $\langle a_n, x^{**}\rangle = 0$ for every n. Since X is reflexive, $X^{**} = \pi(X)$, and therefore $x^{**} = \pi(x)$ for some $x \in X$. But then $\langle x, a_n\rangle = \langle a_n, \pi(x)\rangle = \langle a_n, x^{**}\rangle = 0$ for every n. Hence $x = \sum \langle x, a_n\rangle\, x_n = 0$, which implies that $x^{**} = \pi(x) = 0$. Consequently $\{a_n\}$ is complete in X^* by Corollary 2.5. $\square$

If $\{x_n\}$ is a basis for a Banach space X and its biorthogonal system $\{a_n\}$ is a basis for X^*, then we say that $\{x_n\}$ is a *shrinking basis* for X. In particular, every basis for a reflexive Banach space is shrinking.

We use Corollary 5.22 to show that the dual systems of equivalent bases in Hilbert spaces are themselves equivalent.

Corollary 5.23. *Let H be a Hilbert space. Let $\{x_n\}$ be a basis for H with biorthogonal system $\{a_n\}$, and let $\{y_n\}$ be a basis for H with biorthogonal system $\{b_n\}$. If $\{x_n\} \sim \{y_n\}$, then $\{a_n\} \sim \{b_n\}$.*

Proof. Since Hilbert spaces are self-dual, Corollary 5.22 implies that $\{a_n\}$ is a basis for H with biorthogonal system $\{x_n\}$, and $\{b_n\}$ is a basis for H with biorthogonal system $\{y_n\}$. If $\{x_n\} \sim \{y_n\}$, then there exists a topological isomorphism $T\colon H \to H$ such that $Tx_n = y_n$ for every n. The adjoint mapping T^* is also a topological isomorphism of H onto itself, and we have for each $m, n \in \mathbf{N}$ that

$$\langle x_m, T^*b_n\rangle \;=\; \langle Tx_m, b_n\rangle \;=\; \langle y_m, b_n\rangle \;=\; \delta_{mn} \;=\; \langle x_m, a_n\rangle.$$

Since $\{x_n\}$ is complete, this implies that $T^*b_n = a_n$ for every n, and therefore $\{a_n\} \sim \{b_n\}$. □

Exercises

5.17. Let $\{x_n\}$ be a basis for a Banach space X, with biorthogonal system $\{a_n\} \subseteq X^*$.

(a) Show that if $\{x_n\}$ is a bounded basis for X, then $\{a_n\}$ is a bounded basis for $\overline{\text{span}}\{a_n\}$ in X^*.

(b) If $\{x_n\}$ is a normalized basis for X, must $\{a_n\}$ be a normalized basis for $\overline{\text{span}}\{a_n\}$?

5.18. Suppose that $\{x_n\}$ is a minimal system in a Banach space X, and it has a biorthogonal system $\{a_n\}$ that is a basis for X^*. Prove that $\{x_n\}$ is a basis for X.

5.7 Perturbations of Bases

Given a basis $\{x_n\}$ for a Banach space X, it is often useful to have some idea of how much the elements x_n can be perturbed so that the resulting new sequence remains a basis for X, or at least a basis for its closed linear span. The first result of this type was proved by Paley and Wiener [PW34] in the context of Hilbert spaces, although the same proof extends to Banach spaces.

Theorem 5.24 (Paley–Wiener). *Let* $\{x_n\}$ *be a basis for a Banach space* X. *If* $\{y_n\} \subseteq X$ *and there exists a constant* $0 \le \lambda < 1$ *such that*

$$\left\|\sum_{n=1}^{N} c_n\,(x_n - y_n)\right\| \;\le\; \lambda\left\|\sum_{n=1}^{N} c_n x_n\right\|, \qquad N \in \mathbf{N},\, c_1,\dots,c_N \in \mathbf{F},$$

then $\{y_n\}$ *is a basis for* X, *and* $\{y_n\}$ *is equivalent to* $\{x_n\}$.

Proof. Let $\{a_n\}$ be the sequence of coefficient functionals associated with $\{x_n\}$. Given $x \in X$, the series $x = \sum \langle x, a_n\rangle\, x_n$ converges, and we have by hypothesis that

$$\left\|\sum_{n=M+1}^{N} \langle x, a_n\rangle\,(x_n - y_n)\right\| \;\le\; \lambda\left\|\sum_{n=M+1}^{N} \langle x, a_n\rangle\, x_n\right\| \tag{5.11}$$

whenever $M < N$. Since $\sum \langle x, a_n\rangle\, x_n$ is a Cauchy series, we conclude that $\sum \langle x, a_n\rangle\,(x_n - y_n)$ is Cauchy as well and hence converges. Define $Tx = \sum \langle x, a_n\rangle\,(x_n - y_n)$. Then T is linear, and by taking $M = 0$ and letting $N \to \infty$ in equation (5.11) we see that

$$\|Tx\| \;=\; \left\|\sum_{n=1}^{\infty} \langle x, a_n\rangle\,(x_n - y_n)\right\| \;\le\; \lambda\left\|\sum_{n=1}^{\infty} \langle x, a_n\rangle\, x_n\right\| \;=\; \lambda\,\|x\|.$$

Thus T is bounded and $\|T\| \le \lambda < 1$. Exercise 2.40 therefore implies that $I - T$ is a topological isomorphism of X onto itself. Given $m \in \mathbf{N}$, we have

$$(I-T)x_m \;=\; x_m - \sum_n \langle x_m, a_n\rangle\,(x_n - y_n) \;=\; x_m - (x_m - y_m) \;=\; y_m.$$

By Lemma 4.18 and the definition of equivalent bases, we conclude that $\{y_n\}$ is a basis for X that is equivalent to $\{x_n\}$. □

Using the gross estimate $|\langle x, a_n\rangle| \le \|x\|\,\|a_n\|$, we obtain the following corollary (see Exercise 5.19).

Corollary 5.25. *Let* $\{x_n\}$ *be a basis for a Banach space* X, *with associated coefficient functionals* $\{a_n\} \subseteq X^*$. *If* $\{y_n\} \subseteq X$ *and*

$$\lambda \;=\; \sum_n \|a_n\|\,\|x_n - y_n\| \;<\; 1,$$

then $\{y_n\}$ *is a basis for* X *that is equivalent to* $\{x_n\}$. ◇

Corollary 5.25 does not remain valid if we assume only that $\lambda \le 1$ (see Exercise 5.19). However, the next result shows that we can allow λ to be any finite positive number if we impose the extra requirement that $\{y_n\}$ be complete. This proof has a different flavor than the preceding ones, as it makes use of facts about compact operators, which are briefly reviewed in Appendix B.

Theorem 5.26. *Let $\{x_n\}$ be a basis for a Banach space X, with associated coefficient functionals $\{a_n\} \subseteq X^*$. If $\{y_n\}$ is a complete sequence in X and*

$$\lambda = \sum_n \|a_n\| \, \|x_n - y_n\| < \infty,$$

then $\{y_n\}$ is a basis for X that is equivalent to $\{x_n\}$.

Proof. If $x \in X$, then the series $Tx = \sum \langle x, a_n\rangle (x_n - y_n)$ converges absolutely in X since

$$\sum_n |\langle x, a_n\rangle| \, \|x_n - y_n\| \le \sum_n \|x\| \, \|a_n\| \, \|x_n - y_n\| \le \lambda \, \|x\|. \tag{5.12}$$

This also shows that T is a bounded operator and $\|T\| \le \lambda$. For each $N \in \mathbf{N}$, define the bounded, finite-rank operator $T_N \colon X \to X$ by $T_N x = \sum_{n=1}^N \langle x, a_n\rangle (x_n - y_n)$. A computation similar to the one in equation (5.12) shows that

$$\|T - T_N\| \le \sum_{n=N+1}^{\infty} \|a_n\| \, \|x_n - y_n\| \to 0 \quad \text{as } N \to \infty.$$

Each T_N is compact by Theorem B.5(b), and operator norm limits of compact operators are compact by Theorem B.5(c), so we conclude that T is a compact operator on X.

Now we will show that $\{y_n\}$ is an ω-independent sequence. Suppose that $\sum c_n y_n = 0$ for some choice of scalars $c_n \in \mathbf{F}$. Let N be large enough that $\sum_{n=N+1}^{\infty} \|a_n\| \, \|x_n - y_n\| < 1$. Corollary 5.25 then implies that $\{x_1, \dots, x_N, y_{N+1}, y_{N+2}, \dots\}$ is a basis for X. Set $X_N = \operatorname{span}\{x_1, \dots, x_N\}$ and $Y_N = \overline{\operatorname{span}}\{y_{N+1}, y_{N+2}, \dots\}$. Then $X_N \cap Y_N = \{0\}$ and $X_N + Y_N = X$, so by Exercise 5.21 the *codimension* of Y_N in X is $\operatorname{codim}(Y_N) = \dim(X_N) = N$. Suppose that $c_k \ne 0$ for some $1 \le k \le N$. Then

$$y_k = -\frac{1}{c_k} \sum_{n \ne k} c_n y_n \in \overline{\operatorname{span}}\{y_n\}_{n \ne k}. \tag{5.13}$$

If we set $Z_N = \operatorname{span}\{y_n : 1 \le n \le N,\, n \ne k\}$, then by combining equation (5.13) with the fact that $\{y_n\}$ is complete, we see that $Z_N + Y_N = X$. But then $\operatorname{codim}(Y_N) \le \dim(Z_N) \le N - 1$, which contradicts Exercise 5.21. Hence we must have $c_1 = \cdots = c_N = 0$. Then

$$\sum_{n=1}^{N} c_n x_n - \sum_{n=N+1}^{\infty} c_n y_n = \sum_{n=1}^{\infty} c_n y_n = 0,$$

so $c_n = 0$ for all n since $\{x_1, \dots, x_N, y_{N+1}, y_{N+2}, \dots\}$ is a basis. Hence $\{y_n\}$ is ω-independent.

Suppose now that $(I-T)x = 0$ for some $x \in X$. Then

$$0 \;=\; x - Tx \;=\; \sum_n \langle x, a_n\rangle\, x_n + \sum_n \langle x, a_n\rangle\, (x_n - y_n) \;=\; \sum_n \langle x, a_n\rangle\, y_n,$$

so $\langle x, a_n\rangle = 0$ for every n since $\{y_n\}$ is ω-independent. Therefore $x = 0$, so $\ker(I-T) = \{0\}$. Since T is compact, the Fredholm Alternative (Theorem B.6) implies that $I-T$ is a topological isomorphism of X onto itself. Since we have $(I-T)x_m = y_m$ for every m, it follows from Lemma 4.18 that $\{y_n\}$ is a basis equivalent to $\{x_n\}$. □

The perturbation theorems we have presented are typical examples that apply to general bases. More refined versions of these results are known, and often it is possible to derive sharper results for given classes of Banach spaces. One survey of basis perturbations appears in [RH71].

Exercises

5.19. Prove Corollary 5.25. Does Theorem 5.24 or Corollary 5.25 remain valid if $\lambda = 1$?

5.20. (a) Prove the *Krein–Milman–Rutman Theorem*: If $\{x_n\}$ is a basis for a Banach space X, then there exist constants ε_n such that if $\{y_n\}$ is a sequence in X satisfying $\|x_n - y_n\| < \varepsilon_n$ for all n, then $\{y_n\}$ is a basis for X that is equivalent to $\{x_n\}$.

(b) Suppose that X is a Banach space that has a basis. Show that any dense subset E of X contains a basis for X.

(c) Show that there exists a basis $\{p_n\}$ for $C[0,1]$ such that each p_n is a polynomial. Contrast this with the fact that $\{x^k\}_{k\ge 0}$ is not a basis for $C[0,1]$, and the Schauder system is a basis for $C[0,1]$ that does not consist of polynomials.

5.21. This exercise defines and presents some facts about quotient spaces for vector spaces analogous to those for quotient groups in abstract algebra.

Let M be a subspace of a vector space V.

(a) Define $x \sim y$ if $x - y \in M$. Show that $\sim$ is an equivalence relation on V, and the equivalence class of x is $[x] = M + x = \{m + x : m \in M\}$.

(b) The *quotient space* is $V/M = \{M + x : x \in V\}$. Show that

$$(x+M) + (y+M) = (x+y) + M \qquad \text{and} \qquad c(x+M) = (cx) + M$$

are well-defined operations on V/M, and V/M is a vector space with respect to these operations.

(c) Prove the *Isomorphism Theorem* for vector spaces: If V, X are vector spaces and $T\colon V \to W$ is linear and surjective with kernel $M = \ker(T)$, then $\psi(x+M) = Tx$ is a well-defined linear bijection of V/M onto W.

(d) If there exists a subspace $N \subseteq V$ such that $M \cap N = \{0\}$ and $M + N = \{m + n : m \in M, n \in N\} = V$, then we define the *codimension* of M to be $\operatorname{codim}(M) = \dim(N)$. Show that the codimension is independent of the choice of subspace N, and $\operatorname{codim}(M) = \dim(V/M)$.

6

Unconditional Bases in Banach Spaces

A Schauder basis provides unique series representations $x = \sum \langle x, a_n \rangle x_n$ of each vector in a Banach space. However, conditionally convergent series are delicate in many respects. For example, if $x = \sum \langle x, a_n \rangle x_n$ converges conditionally and (λ_n) is a bounded sequence of scalars, then the series $\sum \lambda_n \langle x, a_n \rangle x_n$ may not converge. Unconditionality is an important property, and in many applications we greatly prefer a basis that is unconditional over one that is conditional. Therefore we study unconditional bases in more detail in this chapter.

6.1 Basic Properties and the Unconditional Basis Constant

We can reformulate unconditionality of a basis as follows (see Exercise 6.1).

Lemma 6.1. *Given a sequence $\{x_n\}$ in a Banach space X, the following two statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for X.*

(b) *$\{x_{\sigma(n)}\}$ is a basis for X for every permutation σ of $\mathbf{N}$.*

In this case, if $\{a_n\}$ is the sequence of coefficient functionals for $\{x_n\}$, then $\{a_{\sigma(n)}\}$ is the sequence of coefficient functionals for $\{x_{\sigma(n)}\}$. $\diamondsuit$

By Lemma 4.18, topological isomorphisms preserve the property of being a basis. The same is true of unconditional bases (see Exercise 6.2).

Lemma 6.2. (a) *Unconditional bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is an unconditional basis for a Banach space X and $T\colon X \to Y$ is a topological isomorphism, then $\{Tx_n\}$ is an unconditional basis for Y.*

(b) *Bounded unconditional bases are likewise preserved by topological isomorphisms.* $\diamondsuit$

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Recall from Definition 4.19 that two bases $\{x_n\}$ and $\{y_n\}$ are *equivalent* if there exists a topological isomorphism T such that $Tx_n = y_n$ for every n. We will see in Section 7.2 that all bounded unconditional bases for a Hilbert space are equivalent, and in fact they are equivalent to orthonormal bases. Up to isomorphisms, the only other infinite-dimensional Banach spaces that have a basis and in which all bounded unconditional bases are equivalent are the sequence spaces c_0 and ℓ^1 [LP68], [LZ69].

Notation 6.3. We will associate three types of partial sum operators with a given unconditional basis $\{x_n\}$ for a Banach space X. Let $\{a_n\}$ be the biorthogonal system to $\{x_n\}$. First, to each finite set $F \subseteq \mathbf{N}$ we associate the partial sum operator $S_F\colon X \to X$ defined by

$$S_F(x) \;=\; \sum_{n\in F} \langle x, a_n\rangle\, x_n, \qquad x \in X.$$

Second, to each finite set $F \subseteq \mathbf{N}$ and each set of scalars $\mathcal{E} = \{\varepsilon_n\}_{n\in F}$ satisfying $\varepsilon_n = \pm 1$ for all n, we associate the operator $S_{F,\mathcal{E}}\colon X \to X$ defined by

$$S_{F,\mathcal{E}}(x) \;=\; \sum_{n\in F} \varepsilon_n\, \langle x, a_n\rangle\, x_n, \qquad x \in X.$$

Finally, to each finite set $F \subseteq \mathbf{N}$ and each collection of scalars $\Lambda = \{\lambda_n\}_{n\in F}$ satisfying $|\lambda| \le 1$ for all n, we associate the operator $S_{F,\Lambda}\colon X \to X$ defined by

$$S_{F,\Lambda}(x) \;=\; \sum_{n\in F} \lambda_n\, \langle x, a_n\rangle\, x_n, \qquad x \in X.$$

Note that while the operators S_F are projections in the sense that $S_F^2 = S_F$, the operators $S_{F,\mathcal{E}}$ and $S_{F,\Lambda}$ need not be projections in this sense. ♢

Applying Theorem 3.10, we obtain the following facts about unconditional bases, where the suprema are implicitly taken over all F, $\mathcal{E}$, Λ described in Notation 6.3. The proof of this result is Exercise 6.3.

Theorem 6.4. *If $\{x_n\}$ is an unconditional basis for a Banach space X, then the following statements hold.*

(a) *The following three quantities are finite for each $x \in X$:*

$$|||x||| \;=\; \sup_F \|S_F(x)\|,$$

$$|||x|||_{\mathcal{E}} \;=\; \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}(x)\|,$$

$$|||x|||_{\Lambda} \;=\; \sup_{F,\Lambda} \|S_{F,\Lambda}(x)\|.$$

(b) *The following three numbers are finite:*

$$\mathcal{K} \;=\; \sup_F \|S_F\|, \qquad \mathcal{K}_{\mathcal{E}} \;=\; \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}\|, \qquad \mathcal{K}_{\Lambda} \;=\; \sup_{F,\Lambda} \|S_{F,\Lambda}\|.$$

(c) $|||\cdot||| \le |||\cdot|||_{\mathcal{E}} \le 2\,|||\cdot|||$ *and* $\mathcal{K} \le \mathcal{K}_{\mathcal{E}} \le 2\mathcal{K}$.

(d) *If* $\mathbf{F} = \mathbf{R}$ *then* $|||\cdot|||_{\mathcal{E}} = |||\cdot|||_{\Lambda}$ *and* $\mathcal{K}_{\mathcal{E}} = \mathcal{K}_{\Lambda}$.

(e) *If* $\mathbf{F} = \mathbf{C}$ *then* $|||\cdot|||_{\mathcal{E}} \le |||\cdot|||_{\Lambda} \le 2\,|||\cdot|||_{\mathcal{E}}$ *and* $\mathcal{K}_{\mathcal{E}} \le \mathcal{K}_{\Lambda} \le 2\mathcal{K}_{\mathcal{E}}$.

(f) $|||\cdot|||$, $|||\cdot|||_{\mathcal{E}}$, *and* $|||\cdot|||_{\Lambda}$ *form norms on* X, *each equivalent to the initial norm* $\|\cdot\|$, *with*

$$\|\cdot\| \;\le\; |||\cdot||| \;\le\; \mathcal{K}\,\|\cdot\|,$$
$$\|\cdot\| \;\le\; |||\cdot|||_{\mathcal{E}} \;\le\; \mathcal{K}_{\mathcal{E}}\,\|\cdot\|,$$
$$\|\cdot\| \;\le\; |||\cdot|||_{\Lambda} \;\le\; \mathcal{K}_{\Lambda}\,\|\cdot\|. \qquad \diamondsuit$$

Notation 6.5. Given an unconditional basis $\{x_n\}$ for a Banach space X, we will let the constants $\mathcal{K}$, $\mathcal{K}_{\mathcal{E}}$, and $\mathcal{K}_{\Lambda}$ and the norms $|||\cdot|||$, $|||\cdot|||_{\mathcal{E}}$, and $|||\cdot|||_{\Lambda}$ be as described in Theorem 6.4. $\diamondsuit$

Definition 6.6 (Unconditional Basis Constant). If $\{x_n\}$ is an unconditional basis for a Banach space X, then the number $\mathcal{K}_{\mathcal{E}}$ is called the *unconditional basis constant* for $\{x_n\}$. $\diamondsuit$

Comparing the number $\mathcal{K}$ to the basis constant $\mathcal{C}$ from Definition 4.14, we see that $\mathcal{C} \le \mathcal{K}$. In fact, if we let $\mathcal{C}_\sigma$ be the basis constant for the permuted basis $\{x_{\sigma(n)}\}$, then $\mathcal{K} = \sup \mathcal{C}_\sigma$, where we take the supremum over all permutations σ of $\mathbf{N}$.

The unconditional basis constant $\mathcal{K}_{\mathcal{E}}$ implicitly depends on the norm for X, and changing the norm to some other equivalent norm may change the value of the basis constant. For example, the unconditional basis constant for $\{x_n\}$ with respect to the equivalent norm $|||\cdot|||_{\mathcal{E}}$ is precisely 1 (compare Theorem 4.15 for the analogous statement for the basis constant).

Exercises

6.1. Prove Lemma 6.1.

6.2. Prove Lemma 6.2.

6.3. Prove Theorem 6.4.

6.4. Let $\{x_n\}$ be an unconditional basis for a Banach space X, with associated coefficient functionals $\{a_n\}$.

(a) Prove that $\{a_n\}$ is an unconditional basic sequence in X^*.

(b) Show that if X is reflexive, then $\{a_n\}$ is an unconditional basis for X^*.

6.5. Use Orlicz's Theorem to prove that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ cannot be an unconditional basis for $L^p(\mathbf{T})$ when $1 \le p < 2$. Argue by duality to show that it also cannot be an unconditional basis when $2 < p < \infty$. (See Chapter 14 for proof that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a basis for $L^p(\mathbf{T})$ when $1 < p < \infty$, but is not a basis for $L^1(\mathbf{T})$ or $C(\mathbf{T})$.)

6.2 Characterizations of Unconditional Bases

The next result gives several equivalent formulations of unconditional bases. We include the proofs of more implications than are strictly needed, in order to illustrate some different approaches to the proof.

Theorem 6.7. *Let $\{x_n\}$ be a complete sequence in a Banach space X such that $x_n \neq 0$ for every n. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for X.*

(b) $\exists\, C_1 \geq 1, \quad \forall\, c_1, \dots, c_N, \quad \forall\, \varepsilon_1, \dots, \varepsilon_N = \pm 1,$

$$\Biggl\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \Biggr\| \;\leq\; C_1 \Biggl\| \sum_{n=1}^{N} c_n x_n \Biggr\|. \tag{6.1}$$

(c) $\exists\, C_2 \geq 1, \quad \forall\, b_1, \dots, b_N, \quad \forall\, c_1, \dots, c_N,$

$$|b_1| \leq |c_1|, \dots, |b_N| \leq |c_N| \quad \Longrightarrow \quad \Biggl\| \sum_{n=1}^{N} b_n x_n \Biggr\| \;\leq\; C_2 \Biggl\| \sum_{n=1}^{N} c_n x_n \Biggr\|.$$

(d) $\exists\, 0 < C_3 \leq 1 \leq C_4 < \infty, \quad \forall\, c_1, \dots, c_N,$

$$C_3 \Biggl\| \sum_{n=1}^{N} |c_n|\, x_n \Biggr\| \;\leq\; \Biggl\| \sum_{n=1}^{N} c_n x_n \Biggr\| \;\leq\; C_4 \Biggl\| \sum_{n=1}^{N} |c_n|\, x_n \Biggr\|.$$

(e) *$\{x_n\}$ is a basis, and for each bounded sequence of scalars $\Lambda = (\lambda_n)$ there exists a continuous linear operator $T_\Lambda \colon X \to X$ such that $T_\Lambda(x_n) = \lambda_n x_n$ for all $n \in \mathbf{N}$.*

Further, in case these hold, the best constant C_1 in equation (6.1) *is the unconditional basis constant $C_1 = \mathcal{K}_{\mathcal{E}} = \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}\|$.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\{x_n\}$ is an unconditional basis for X, with coefficient functionals $\{a_n\}$. Choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$, and set $x = \sum_{n=1}^{N} c_n x_n$. Then $\langle x, a_n \rangle = c_n$ if $n \leq N$, while $\langle x, a_n \rangle = 0$ if $n > N$. Therefore

$$\sum_{n=1}^{N} \varepsilon_n c_n x_n \;=\; \sum_{n \in F} \varepsilon_n \langle x, a_n \rangle\, x_n \;=\; S_{F,\mathcal{E}}(x),$$

where $F = \{1, \dots, N\}$ and $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_N\}$. By definition of $|\!|\!| \cdot |\!|\!|_{\mathcal{E}}$ and by Theorem 6.4(f), we therefore have

$$\Biggl\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \Biggr\| \;=\; \|S_{F,\mathcal{E}}(x)\| \;\leq\; |\!|\!| x |\!|\!|_{\mathcal{E}} \;\leq\; \mathcal{K}_{\mathcal{E}}\, \|x\| \;=\; \mathcal{K}_{\mathcal{E}} \Biggl\| \sum_{n=1}^{N} c_n x_n \Biggr\|.$$

Thus statement (b) holds with $C_1 = \mathcal{K}_{\mathcal{E}}$.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds, and let σ be any permutation of $\mathbf{N}$. We must show that $\{x_{\sigma(n)}\}$ is a basis for X. By hypothesis, $\{x_{\sigma(n)}\}$ is complete with every element nonzero. Therefore, by Theorem 5.17 it suffices to show that there is a constant C_σ such that

$$\forall\, N \geq M, \quad \forall\, c_{\sigma(1)}, \dots, c_{\sigma(N)}, \quad \left\|\sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)}\right\| \;\leq\; C_\sigma \left\|\sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)}\right\|.$$

To this end, fix any $N \geq M$ and choose any scalars $c_{\sigma(1)}, \dots, c_{\sigma(N)}$. Define $c_n = 0$ for $n \notin \{\sigma(1), \dots, \sigma(N)\}$. Let $L = \max\{\sigma(1), \dots, \sigma(N)\}$, and define

$$\varepsilon_n = 1 \qquad \text{and} \qquad \gamma_n \;=\; \begin{cases} 1, & \text{if } n \in \{\sigma(1), \dots, \sigma(M)\}, \\ -1, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}
\left\|\sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)}\right\| \;&=\; \left\|\sum_{n=1}^{L} \left(\frac{\varepsilon_n + \gamma_n}{2}\right) c_n x_n\right\| \\
&\leq\; \frac{1}{2}\left\|\sum_{n=1}^{L} \varepsilon_n c_n x_n\right\| \;+\; \frac{1}{2}\left\|\sum_{n=1}^{L} \gamma_n c_n x_n\right\| \\
&\leq\; \frac{C_1}{2}\left\|\sum_{n=1}^{L} c_n x_n\right\| \;+\; \frac{C_1}{2}\left\|\sum_{n=1}^{L} c_n x_n\right\| \\
&=\; C_1 \left\|\sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)}\right\|.
\end{aligned}$$

This is the desired result, with $C_\sigma = C_1$.

(a) $\Rightarrow$ (c). Suppose that $\{x_n\}$ is an unconditional basis for X, with coefficient functionals $\{a_n\}$. Choose any scalars $c_1, \dots, c_N$ and $b_1, \dots, b_N$ such that $|b_n| \leq |c_n|$ for every n. Define $x = \sum_{n=1}^{N} c_n x_n$, and note that $c_n = \langle x, a_n\rangle$. Let λ_n be such that $b_n = \lambda_n c_n$. Since $|b_n| \leq |c_n|$ we can take $|\lambda_n| \leq 1$ for every n. Therefore, if we define $F = \{1, \dots, N\}$ and $\Lambda = \{\lambda_1, \dots, \lambda_N\}$, then

$$\sum_{n=1}^{N} b_n x_n \;=\; \sum_{n\in F} \lambda_n c_n x_n \;=\; \sum_{n \in F} \lambda_n \langle x, a_n\rangle\, x_n \;=\; S_{F,\Lambda}(x).$$

Hence

$$\left\|\sum_{n=1}^{N} b_n x_n\right\| \;=\; \|S_{F,\Lambda}(x)\| \;=\; |\!|\!|x|\!|\!|_{\Lambda} \;\leq\; \mathcal{K}_\Lambda \,\|x\| \;=\; \mathcal{K}_\Lambda \left\|\sum_{n=1}^{N} c_n x_n\right\|.$$

Thus statement (c) holds with $C_2 = \mathcal{K}_\Lambda$.

(b) ⇒ (c). Suppose that statement (b) holds. Choose any $N > 0$, and any scalars b_n, c_n such that $|b_n| \le |c_n|$ for each $n = 1, \dots, N$. Let $|\lambda_n| \le 1$ be such that $b_n = \lambda_n c_n$. Let $\alpha_n = \mathrm{Re}(\lambda_n)$ and $\beta_n = \mathrm{Im}(\lambda_n)$. Since the α_n are real and satisfy $|\alpha_n| \le 1$, Carathéodory's Theorem (Theorem 3.13) implies that we can find scalars $t_m \ge 0$ and signs $\varepsilon_m^n = \pm 1$, for $m = 1, \dots, N+1$ and $n = 1, \dots, N$, such that

$$\sum_{m=1}^{N+1} t_m \;=\; 1 \qquad \text{and} \qquad \sum_{m=1}^{N+1} \varepsilon_m^n\, t_m \;=\; \alpha_n \quad \text{for } n = 1, \dots, N.$$

Hence,

$$\begin{aligned}
\Bigl\|\sum_{n=1}^{N} \alpha_n c_n x_n\Bigr\| &= \Bigl\|\sum_{n=1}^{N} \sum_{m=1}^{N+1} \varepsilon_m^n t_m c_n x_n\Bigr\| \\
&= \Bigl\|\sum_{m=1}^{N+1} t_m \sum_{n=1}^{N} \varepsilon_m^n c_n x_n\Bigr\| \\
&\le \sum_{m=1}^{N+1} t_m \Bigl\|\sum_{n=1}^{N} \varepsilon_m^n c_n x_n\Bigr\| \\
&\le \sum_{m=1}^{N+1} t_m\, C_1 \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\| \\
&= C_1 \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|.
\end{aligned}$$

A similar formula holds for the imaginary parts β_n (which are zero if $\mathbf{F} = \mathbf{R}$), so

$$\begin{aligned}
\Bigl\|\sum_{n=1}^{N} b_n x_n\Bigr\| &= \Bigl\|\sum_{n=1}^{N} \lambda_n c_n x_n\Bigr\| \\
&\le \Bigl\|\sum_{n=1}^{N} \alpha_n c_n x_n\Bigr\| + \Bigl\|\sum_{n=1}^{N} \beta_n c_n x_n\Bigr\| \\
&\le 2C_1 \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|.
\end{aligned}$$

Therefore statement (c) holds with $C_2 = 2C_1$.

(c) ⇒ (a). Suppose that statement (c) holds, and let σ be any permutation of $\mathbf{N}$. We must show that $\{x_{\sigma(n)}\}$ is a basis for X. By hypothesis, $\{x_{\sigma(n)}\}$ is complete in X and every element $x_{\sigma(n)}$ is nonzero. Therefore, by Theorem 5.17 it suffices to show that there is a constant C_σ such that

$$\forall\, N \ge M, \quad \forall\, c_{\sigma(1)}, \dots, c_{\sigma(N)}, \quad \left\| \sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)} \right\| \;\le\; C_\sigma \left\| \sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)} \right\|.$$

To this end, fix any $N \ge M$ and choose any scalars $c_{\sigma(1)}, \dots, c_{\sigma(N)}$. Define $c_n = 0$ for $n \notin \{\sigma(1), \dots, \sigma(N)\}$. Let $L = \max\{\sigma(1), \dots, \sigma(N)\}$ and define

$$\lambda_n \;=\; \begin{cases} 1, & \text{if } n \in \{\sigma(1), \dots, \sigma(M)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \left\| \sum_{n=1}^{M} c_{\sigma(n)} x_{\sigma(n)} \right\| \;&=\; \left\| \sum_{n=1}^{L} \lambda_n c_n x_n \right\| \\ &\le\; C_2 \left\| \sum_{n=1}^{L} c_n x_n \right\| \\ &=\; C_2 \left\| \sum_{n=1}^{N} c_{\sigma(n)} x_{\sigma(n)} \right\|. \end{aligned}$$

This is the desired result, with $C_\sigma = C_2$.

(c) $\Rightarrow$ (d). Assume that statement (c) holds, and choose any scalars $c_1, \dots, c_N$. Let $b_n = |c_n|$. Then we have both $|b_n| \le |c_n|$ and $|c_n| \le |b_n|$, so statement (c) implies

$$\left\| \sum_{n=1}^{N} b_n x_n \right\| \;\le\; C_2 \left\| \sum_{n=1}^{N} c_n x_n \right\| \qquad \text{and} \qquad \left\| \sum_{n=1}^{N} c_n x_n \right\| \;\le\; C_2 \left\| \sum_{n=1}^{N} b_n x_n \right\|.$$

Therefore statement (d) holds with $C_3 = 1/C_2$ and $C_4 = C_2$.

(d) $\Rightarrow$ (c). Assume that statement (d) holds. Choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$. Then, by statement (d),

$$\left\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \right\| \;\le\; C_4 \left\| \sum_{n=1}^{N} |\varepsilon_n c_n|\, x_n \right\| \;=\; C_4 \left\| \sum_{n=1}^{N} |c_n|\, x_n \right\| \;\le\; \frac{C_4}{C_3} \left\| \sum_{n=1}^{N} c_n x_n \right\|.$$

Hence statement (c) holds with $C_2 = C_4/C_3$.

(a) $\Rightarrow$ (e). Let $\{x_n\}$ be an unconditional basis for X, with coefficient functionals $\{a_n\}$. Let (λ_n) be any bounded sequence of scalars, and let $M = \sup |\lambda_n|$. Fix any $x \in X$. Then the series $x = \sum \langle x, a_n \rangle\, x_n$ converges unconditionally. Hence, by Theorem 3.10(f), the series $T_\Lambda(x) = \sum \lambda_n \langle x, a_n \rangle\, x_n$ converges. Clearly $T_\Lambda \colon X \to X$ defined in this way is linear, and we have

$$\|T_\Lambda(x)\| \;=\; M\left\|\sum_n \frac{\lambda_n}{M}\,\langle x, a_n\rangle\, x_n\right\| \;\le\; M\mathcal{K}_\Lambda \left\|\sum_n \langle x, a_n\rangle\, x_n\right\| \;=\; M\mathcal{K}_\Lambda\,\|x\|.$$

Therefore T_Λ is continuous. Finally, the biorthogonality of $\{x_n\}$ and $\{a_n\}$ ensures that $T_\Lambda(x_n) = \lambda_n x_n$ for every n.

(e) ⇒ (a). Suppose that statement (e) holds. Since $\{x_n\}$ is a basis, there exists a biorthogonal sequence $\{a_n\} \subseteq X^*$ such that the series $x = \sum \langle x, a_n\rangle\, x_n$ converges and is the unique expansion of x in terms of the vectors x_n. We must show that this series converges unconditionally. Let $\Lambda = (\lambda_n)$ be any sequence of scalars such that $|\lambda_n| \le 1$ for every n. Then, by hypothesis, there exists a continuous mapping $T_\Lambda \colon X \to X$ such that $T_\Lambda(x_n) = \lambda_n x_n$ for every n. The continuity of T_Λ implies that

$$T_\Lambda(x) \;=\; T_\Lambda\!\left(\sum_n \langle x, a_n\rangle\, x_n\right) \;=\; \sum_n \langle x, a_n\rangle\, T_\Lambda(x_n) \;=\; \sum_n \lambda_n\, \langle x, a_n\rangle\, x_n.$$

That is, the rightmost series on the line above converges for every choice of bounded scalars, so Theorem 3.10(f) tells us that the series $x = \sum \langle x, a_n\rangle\, x_n$ converges unconditionally. □

Exercises

6.6. Let X be a real Banach space, and suppose that $\{x_n\}$ is an unconditional basis for X with unconditional basis constant $\mathcal{K}_\mathcal{E} = 1$. Given $x = \sum a_n x_n$ and $y = \sum b_n y_n$ in X, declare that $x \le y$ if $a_n \le b_n$ for every n. Show that $\le$ is a partial order on X, and X is a Banach lattice in the sense of Definition 3.35. Using the notation of that definition, show that $x \vee y = \sum \max\{a_n, b_n\}\, x_n$, $x \wedge y = \sum \min\{a_n, b_n\}\, x_n$, and $|x| = \sum |a_n|\, x_n$.

6.7. Set $F = \mathbf{R}$. The Haar system is an orthonormal basis for $L^2[0,1]$, so by Exercise 6.6 there is a partial ordering $\le$ on $L^2[0,1]$ induced by this unconditional basis. There is also the ordinary partial ordering $\le$ on $L^2[0,1]$ defined by $f \le g$ if $f(t) \le g(t)$ for a.e. t. Do these two orderings coincide?

6.3 Conditionality of the Schauder System in $C[0,1]$

We saw in Section 4.5 that the Schauder system is a basis for $C[0,1]$. Now we will show that this basis is conditional. We do this indirectly—we will not explicitly construct an element of $C[0,1]$ whose basis representation converges conditionally, but rather will use Theorem 6.7 to demonstrate that the unconditional basis constant for the Schauder system must be infinite.

Using the notation of Section 4.3, the elements of the Schauder system are the box function $\chi = \chi_{[0,1]}$, the function $\ell(t) = t$, and the dilated and

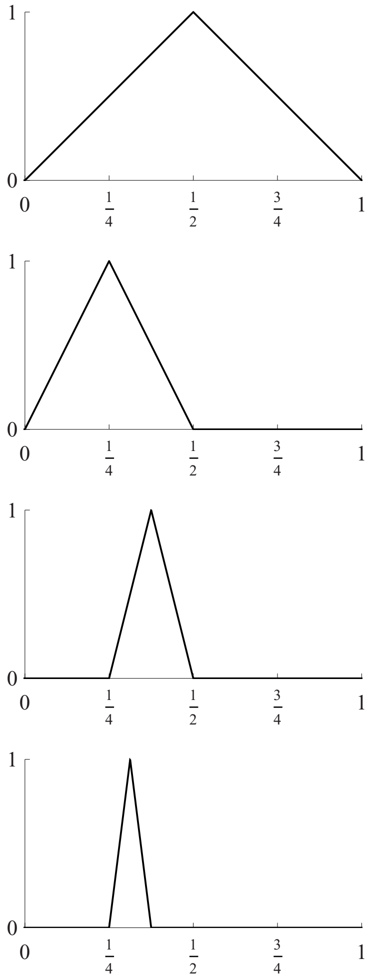

Fig. 6.1. From top to bottom: The functions t_1, t_2, t_3, and t_4.

translated hat functions $s_{n,k}(t) = W(2^n t - k)$, where W is the hat function of height 1 supported on $[0,1]$. We select a subsequence of the Schauder system by defining:

$$\begin{aligned}
t_1 &= s_{0,0} &&\big(\text{hat function on } I_1 = \big[0,1\big]\big),\\
t_2 &= s_{1,0} &&\big(\text{hat function on } I_2 = \big[0,\tfrac{1}{2}\big]\big),\\
t_3 &= s_{2,1} &&\big(\text{hat function on } I_3 = \big[\tfrac{1}{4},\tfrac{1}{2}\big]\big),\\
t_4 &= s_{3,2} &&\big(\text{hat function on } I_4 = \big[\tfrac{1}{4},\tfrac{3}{8}\big]\big),
\end{aligned}$$

$$t_5 = s_{4,5} \qquad \left(\text{hat function on } I_5 = \left[\tfrac{5}{16}, \tfrac{3}{8}\right]\right),$$
$$t_6 = s_{5,10} \qquad \left(\text{hat function on } I_6 = \left[\tfrac{5}{16}, \tfrac{11}{32}\right]\right),$$

etc., where we alternate choosing the left or right half of I_{N-1} as the interval I_N on which the hat function t_N is supported (see Figure 6.1).

Now consider the function $g_N = \sum_{n=1}^{N} t_n$. Our goal is not to show that g_N converges uniformly (in fact, it does not), but rather to compute its norm and to compare this to the norm of $h_N = \sum_{n=1}^{N} (-1)^{n+1} t_n$ (see the illustration in Figure 6.2).

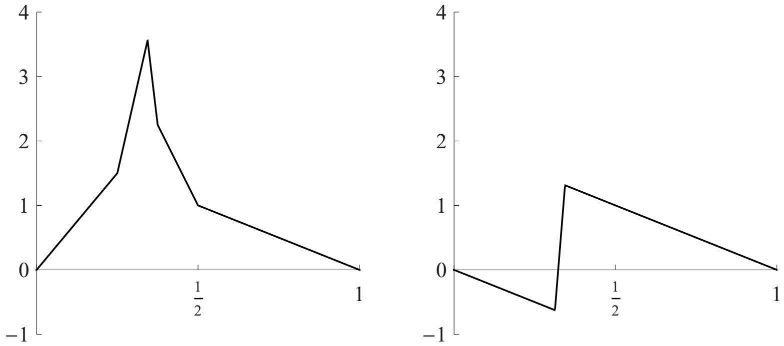

Fig. 6.2. The functions g_5 (left) and h_5 (right).

The functions g_{N-1} and g_N agree everywhere except on the interval I_N. Let μ_N be the midpoint of I_N. The function g_{N-1} is linear on the interval I_N, and g_N achieves its global maximum at the midpoint μ_N. By construction, for $N \geq 3$ one endpoint of I_N is μ_{N-2} and the other is μ_{N-1}. Letting $a_N = g_N(\mu_N)$ be the global maximum of g_N, we have

$$a_N = 1 + \frac{a_{N-1} + a_{N-2}}{2}.$$

By Exercise 6.8, a_N increases without bound as $N \to \infty$.

On the other hand, a similar analysis of $h_N = \sum_{n=1}^{N} (-1)^{n+1} t_n$ shows that we always have $|h_N(t)| \leq 2$ (Exercise 6.8), so $b_N = \|h_N\|_\infty \leq 2$. Consequently there can be no finite constant C such that

$$\left\| \sum_{n=1}^{N} t_n \right\|_\infty = a_N \leq C b_N = C \left\| \sum_{n=1}^{N} (-1)^{n+1} t_n \right\|_\infty, \qquad N \in \mathbf{N}.$$

Considering hypothesis (c) of Theorem 6.7, we conclude that the Schauder system cannot be unconditional.

Exercises

6.8. Show that $a_N \to \infty$ and $0 \le b_N \le 2$ for each N.

6.4 Conditionality of the Haar System in $L^1[0,1]$

By Theorem 5.18, the Haar system is a basis for $L^p[0,1]$ for each $1 \le p < \infty$, at least with respect to the ordering given in equation (5.9). We will show that this basis is conditional when $p = 1$ by taking an indirect approach similar to the one we used to prove that the Schauder system is conditional.

Set $\chi = \chi_{[0,1]}$, and let $\psi_{n,k}$ be as defined in Example 1.54. For this proof, we only need to deal with the elements of the Haar system that are nonzero at the origin. Normalizing so that each function has unit L^1-norm, these are the functions χ and

$$k_n \;=\; 2^{n/2}\psi_{n,0} \;=\; 2^n\left(\chi_{[0,2^{-n-1})} - \chi_{[2^{-n-1},2^{-n})}\right), \qquad n \ge 0.$$

Fix $N > 0$ and define

$$f_N \;=\; \chi \,+\, \sum_{n=0}^{2N} k_n.$$

Examining the graphs of the functions k_n, we see that there is a great deal of cancellation in this sum, leaving us with

$$f_N \;=\; 2^{2N+1}\,\chi_{[0,2^{-2N-1})}.$$

In particular, f_N is a unit vector in $L^1[0,1]$.

Now we form a "subseries" of the series defining f_N. Specifically, we take

$$g_N \;=\; \sum_{\substack{n=0\\ n \text{ even}}}^{2N} k_n.$$

Looking at the graphs in Figure 6.3, we see that $g_0 = -1$ on $[\frac{1}{2},1)$, $g_1 = 4-1 = -3$ on $[\frac{1}{8},\frac{1}{4})$, and $g_2 = 1+4-16 = -11$ on $[\frac{1}{32},\frac{1}{16})$. In general, since k_n is -1 only on an interval where each of $k_0,\dots,k_{n-1}$ are identically 1, we see that

$$g_N(x) \;=\; \left(\sum_{n=0}^{N-1} 4^n\right) - 4^N \;=\; -\frac{2}{3}\,4^N - \frac{1}{3}, \qquad \tfrac{1}{2}\,4^{-N} \le x < 4^{-N}.$$

Therefore the L^1-norm of g_N on this particular interval is

$$\int_{\frac{1}{2}4^{-N}}^{4^{-N}} |g_N(t)|\,dt \;=\; \left(\frac{2}{3}\,4^N + \frac{1}{3}\right)\frac{1}{2}\,4^{-N} \;\ge\; \frac{1}{3}.$$

However, $g_N = g_{N-1}$ on the interval $[4^{-N}, 1]$, so the total L^1-norm of g_N is at least

$$\|g_N\|_{L^1} \;\ge\; \sum_{n=0}^{N} \int_{\frac{1}{2}\,4^{-n}}^{4^{-n}} |g_n(t)|\,dt \;\ge\; \frac{N+1}{3}.$$

Since $\|f_N\|_{L^1} = 1$ for every N, criterion (c) of Theorem 6.7 implies that the Haar system cannot be an unconditional basis for $L^1[0,1]$.

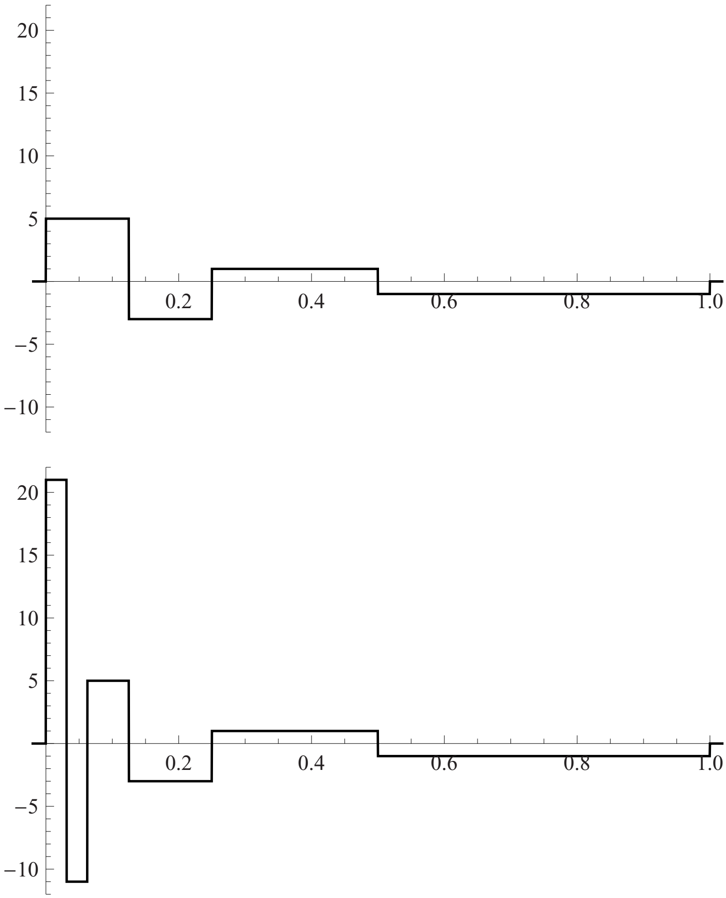

Fig. 6.3. The functions g_1 (top) and g_2 (bottom).

The facts that the Schauder system is conditional in $C[0,1]$ and the Haar system is conditional in $L^1[0,1]$ are special cases of the deeper fact that these two spaces contain no unconditional bases whatsoever! For proof, we refer to [LT77], [Sin70].

7

Bessel Sequences and Bases in Hilbert Spaces

In this chapter and the next we focus on bases and basis-like systems in Hilbert spaces. Our goal in this chapter is to understand bounded unconditional bases in Hilbert spaces, but in order to do this, we first need to study sequences that need not be bases but which do have a property that is reminiscent of Bessel's Inequality for orthonormal bases. These *Bessel sequences* will also be very useful to us in Chapter 8 when we consider *frames* in Hilbert spaces.

7.1 Bessel Sequences in Hilbert Spaces

Bessel sequences are defined as follows.

Definition 7.1 (Bessel Sequence). A sequence $\{x_n\}$ in a Hilbert space H is a *Bessel sequence* if

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n \rangle|^2 \;<\; \infty. \qquad \diamondsuit$$

Thus, if $\{x_n\}$ is a Bessel sequence, then the *analysis operator* C that takes an element x to the sequence of coefficients $Cx = \bigl(\langle x, x_n\rangle\bigr)$ maps H into ℓ^2. By applying either the Uniform Boundedness Principle or the Closed Graph Theorem, this mapping must be bounded. The next theorem, whose proof is Exercise 7.2, states several additional properties possessed by Bessel sequences (parts (a)–(c) of this exercise can also be derived by applying Exercise 3.8 with $X = H$ and $p = 2$).

Theorem 7.2. *Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H. If we define $Cx = \bigl(\langle x, x_n\rangle\bigr)$ for $x \in H$, then the following statements hold.*

(a) *C is a bounded mapping of H into ℓ^2, and therefore there exists a constant $B > 0$ such that*

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n \rangle|^2 \;\le\; B\, \|x\|^2. \tag{7.1}$$

C. Heil, *A Basis Theory Primer: Expanded Edition*, Applied and Numerical Harmonic Analysis, 189
DOI 10.1007/978-0-8176-4687-5_7, © Springer Science+Business Media, LLC 2011

(b) *If* $(c_n) \in \ell^2$, *then the series* $\sum c_n x_n$ *converges unconditionally in* H, *and* $Rc = \sum c_n x_n$ *defines a bounded map of* ℓ^2 *into* H.

(c) $R = C^*$ *and* $\|R\| = \|C\| \le B^{1/2}$. *Consequently,*

$$\forall\, (c_n) \in \ell^2, \qquad \Big\|\sum_n c_n x_n\Big\|^2 \;\le\; B \sum_n |c_n|^2.$$

(d) *If* $\{x_n\}$ *is complete, then* C *is injective and* range(R) *is dense in* H. ◊

Comparing equation (7.1) to Bessel's Inequality for orthonormal bases (Theorem 1.49), we see the motivation for the name "Bessel sequence." However, a Bessel sequence need not be orthonormal and need not be a basis (Exercise 7.1).

Definition 7.3. Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H.

(a) A constant B such that equation (7.1) holds is called a *Bessel bound* or an *upper frame bound* for $\{x_n\}$ (compare Definition 8.2). The smallest such constant B is called the *optimal Bessel bound*.

(b) The operator $C\colon H \to \ell^2$ defined in Theorem 7.2 is called the *analysis operator* or the *coefficient mapping* for $\{x_n\}$, and its adjoint $R\colon \ell^2 \to H$ is the *synthesis operator* or the *reconstruction operator* for $\{x_n\}$.

(c) The *frame operator* for $\{x_n\}$ is $S = RC\colon H \to H$.

(d) The *Gram operator* or *Gram matrix* for $\{x_n\}$ is $G = CR\colon \ell^2 \to \ell^2$. ◊

Note that the optimal Bessel bound is precisely $\|C\|^2$.

We will study *frames* in detail in Chapter 8. These are Bessel sequences which also possess a "lower frame bound" in the sense that there is a constant $A > 0$ such that $A\,\|x\|^2 \le \sum |\langle x, x_n\rangle|^2$ for $x \in H$. The synthesis operator for a frame is sometimes called the *pre-frame operator* (and this terminology is sometimes applied to generic Bessel sequences as well).

Since the analysis and synthesis operators associated to a Bessel sequence $\{x_n\}$ are bounded, the frame and Gram operators are bounded as well. Moreover, $S = C^*C = RR^*$ and $G = CC^* = R^*R$ are self-adjoint and positive in the sense of Definition 2.14. By definition,

$$Sx \;=\; RCx \;=\; \sum_n \langle x, x_n\rangle\, x_n, \qquad x \in H,$$

and therefore

$$\langle Sx, x\rangle \;=\; \sum_n |\langle x, x_n\rangle|^2. \tag{7.2}$$

In particular, an orthonormal basis is a Bessel sequence, and the frame operator for an orthonormal basis is $S = I$. However, there exist Bessel sequences whose frame operator is $S = I$ but which are neither orthonormal nor bases (see Exercise 7.1).

We have the following equivalent characterizations of Bessel sequences (see Exercise 7.3).

Theorem 7.4. *Let $\{x_n\}$ be a sequence in a Hilbert space H, and let $\{\delta_n\}$ be the standard basis for ℓ^2. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a Bessel sequence in H.*

(b) *There exists a constant $B > 0$ and a dense set $E \subseteq H$ such that*

$$\forall\, x \in E, \quad \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2.$$

(c) *There exists a constant $B > 0$ such that*

$$\forall\, N \in \mathbf{N}, \quad \forall\, c_1, \dots, c_N \in \mathbf{F}, \quad \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2. \tag{7.3}$$

(d) *The series $\sum c_n x_n$ converges for each sequence $(c_n) \in \ell^2$.*

(e) *There exists a bounded operator $R\colon \ell^2 \to H$ such that $R\delta_n = x_n$ for each $n \in \mathbf{N}$.*

(f) *There exists an orthonormal sequence $\{e_n\}$ in H and a bounded operator $T \in \mathcal{B}(H)$ such that $Te_n = x_n$ for each $n \in \mathbf{N}$.*

Further, when these hold, the operator R appearing in part (e) *is the synthesis operator for $\{x_n\}$, and $\overline{\operatorname{span}}\{x_n\} = \overline{\operatorname{range}(R)}$.* $\diamondsuit$

Now we consider the Gram operator G associated with a Bessel sequence. Since G is a bounded mapping of ℓ^2 into itself, it can be represented as multiplication by an infinite matrix. We identify the Gram operator and the matrix that represents it. The form of this matrix is given in the next result, whose proof is Exercise 7.4.

Theorem 7.5. *Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H. Then the matrix for the Gram operator G is*

$$G \;=\; \bigl[\,\langle x_n, x_m\rangle\,\bigr]_{m,n\in\mathbf{N}}. \qquad \diamondsuit$$

That is, if we think of $c = (c_n) \in \ell^2$ as a column vector, then Gc is the product of the infinite matrix $\bigl[\,\langle x_n, x_m\rangle\,\bigr]_{m,n\in\mathbf{N}}$ with the vector $c = (c_n)$. The mth entry of Gc is $(Gc)_m = \sum_n c_n \langle x_n, x_m\rangle$.

We can extend the notion of a Gram matrix to sequences that are not Bessel. Given any sequence $\{x_n\}$ in a Hilbert space H, we call $G = \bigl[\,\langle x_n, x_m\rangle\,\bigr]_{m,n\in\mathbf{N}}$ the *Gram matrix* or the *Gramian* for $\{x_n\}$. However, it is important to note that this matrix need not define a bounded mapping on ℓ^2. In fact, the following converse to Theorem 7.5 shows that this happens exactly for Bessel sequences.

Theorem 7.6. *Let $\{x_n\}$ be a sequence in a Hilbert space H, and let G be its Gram matrix. If either:*

(a) *G is a bounded map of $\big(c_{00}, \|\cdot\|_{\ell^2}\big)$ into ℓ^2, i.e., there exists a constant $B > 0$ such that $\|Gc\|_{\ell^2} \le B\,\|c\|_{\ell^2}$ for all finite sequences c, or*

(b) *multiplication by G is a well-defined mapping of ℓ^2 into itself, i.e., for each $c = (c_n) \in \ell^2$ the series $(Gc)_m = \sum_n c_n \langle x_n, x_m\rangle$ converges for each $m \in \mathbf{N}$ and the sequence $Gc = \big((Gc)_m\big)_{m\in\mathbf{N}}$ belongs to ℓ^2,*

then $\{x_n\}$ is a Bessel sequence.

Proof. (a) Choose any finite sequence $c = (c_1, \dots, c_N, 0, 0, \dots) \in c_{00}$. Then

$$\begin{aligned}\langle Gc, c\rangle &= \sum_{m=1}^{N} (Gc)_m\, \overline{c_m} \\ &= \sum_{m=1}^{N} \left(\sum_{n=1}^{N} \langle x_n, x_m\rangle\, c_n\right) \overline{c_m} \\ &= \sum_{m=1}^{N}\sum_{n=1}^{N} c_n\, \langle x_n, x_m\rangle\, \overline{c_m} \\ &= \left\langle \sum_{n=1}^{N} c_n x_n,\ \sum_{m=1}^{N} c_m x_m \right\rangle \\ &= \left\| \sum_{n=1}^{N} c_n x_n \right\|^2.\end{aligned}$$

On the other hand,

$$\langle Gc, c\rangle \;\le\; \|Gc\|_{\ell^2}\, \|c\|_{\ell^2} \;\le\; B\, \|c\|_{\ell^2}^2 \;=\; B \sum_{n=1}^{N} |c_n|^2.$$

Combining these two estimates, we see that equation (7.3) holds, and therefore Theorem 7.4 implies that $\{x_n\}$ is a Bessel sequence.

(b) The well-defined hypothesis of this part precisely fulfills the hypotheses of Exercise 2.34. That exercise therefore implies that $c \mapsto Gc$ is a bounded mapping on ℓ^2, so we conclude from part (a) that $\{x_n\}$ is a Bessel sequence. □

If $\{x_n\}$ is a Bessel sequence, then it follows from the proof of Theorem 7.6, or directly from the fact that $G = R^*R$, that we have the useful equality

$$\forall\, c = (c_n) \in \ell^2, \quad \langle Gc, c\rangle \;=\; \|Rc\|^2 \;=\; \left\| \sum_n c_n x_n \right\|^2.$$

Example 7.7. Consider the sequence of monomials $\{x^k\}_{k\ge 0}$. By Example 1.29 or Theorem 5.6, the monomials are complete but are not a basis for $C[0,1]$, and by Exercise 5.2, the same is true in the space $L^2[0,1]$. The Gram matrix for the monomials is

$$G = \left[\langle x^n, x^m\rangle\right]_{m,n\ge 0} = \left[\tfrac{1}{m+n+1}\right]_{m,n\ge 0} = \mathcal{H},$$

which is the famous *Hilbert matrix*. It is not obvious, but the Hilbert matrix determines a bounded mapping on $\ell^2(\mathbf{N}\cup\{0\})$. Exercise 7.12 shows that $\|\mathcal{H}\| \le 4$, and in fact it is known that the operator norm of the Hilbert matrix is precisely $\|\mathcal{H}\| = \pi$ [Cho83]. Theorem 7.6 therefore implies that $\{x^k\}_{k\ge 0}$ is a Bessel sequence in $L^2[0,1]$. ♢

All Bessel sequences must be bounded above in norm (Exercise 7.5), but not all norm-bounded sequences are Bessel sequences (see Exercise 7.1). On the other hand, we end this section by making use of Orlicz's Theorem to prove that all unconditional bases that are norm-bounded above are examples of Bessel sequences. Various examples of other systems that are or are not Bessel sequences are considered in the Exercises.

Theorem 7.8. *Let H be a Hilbert space. Every unconditional basis for H that is norm-bounded above is a Bessel sequence in H.*

Proof. Let $\{x_n\}$ be an unconditional basis for H such that $\sup\|x_n\| < \infty$, and let $\{y_n\}$ be its biorthogonal system in H. By Theorem 4.13 we have for each n that $1 \le \|x_n\|\,\|y_n\| \le 2\mathcal{C}$ where $\mathcal{C}$ is the basis constant. Hence $\inf\|y_n\| > 0$.

By Exercise 6.4, $\{y_n\}$ is an unconditional basis for H and $\{x_n\}$ is its biorthogonal sequence. Therefore, given $x \in H$, the series $x = \sum \langle x, x_n\rangle\, y_n$ converges unconditionally. By Orlicz's Theorem (Theorem 3.16), it follows that

$$\sum_n |\langle x, x_n\rangle|^2\, \|y_n\|^2 = \sum_n \big\|\langle x, x_n\rangle\, y_n\big\|^2 < \infty.$$

Consequently, since $\{y_n\}$ is norm-bounded below, $\sum|\langle x, x_n\rangle|^2 < \infty$ for each $x \in H$. Therefore $\{x_n\}$ is a Bessel sequence. □

Exercises

7.1. Let H be a separable Hilbert space. For each of the following, construct a sequence $\{x_n\}$ that has the specified property.

(a) A bounded sequence that is not a Bessel sequence.

(b) A Bessel sequence that is a nonorthogonal basis for H.

(c) A Bessel sequence that is not a basis for H but has frame operator $S = I$.

(d) A Bessel sequence such that $\{x_n\}_{n\in\mathbf{N}\setminus F}$ is complete for every finite $F \subseteq \mathbf{N}$.

(e) An unconditional basis that is not a Bessel sequence.

(f) A normalized conditional basis that is a Bessel sequence.

(g) A normalized conditional basis that is not a Bessel sequence.

(h) A basis that is Bessel but whose biorthogonal sequence is not Bessel.

7.2. Give a direct proof of Theorem 7.2.

7.3. Prove Theorem 7.4.

7.4. Prove Theorem 7.5.

7.5. Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H and let B be a Bessel bound.

(a) Show that $\|x_n\|^2 \le B$ for every $n \in \mathbf{N}$. Thus Bessel sequences are bounded above in norm.

(b) Show that if $\|x_m\|^2 = B$ for any particular m, then $x_n \perp x_m$ for all $n \neq m$.

7.6. Let H, K be Hilbert spaces. Show that if $\{x_n\}$ is a Bessel sequence in H and $L \in \mathcal{B}(H,K)$, then $\{Lx_n\}$ is a Bessel sequence in K.

7.7. Suppose that H is a Hilbert space contained in another Hilbert space K. Given a sequence $\{x_n\}$ in H, show that $\{x_n\}$ is a Bessel sequence in H if and only if it is a Bessel sequence in K.

7.8. Let $\{x_n\}$ be a sequence in a Hilbert space H.

(a) If $\sum |\langle x, x_n\rangle|^2 < \infty$ for all x in a dense set $E \subseteq H$, must $\{x_n\}$ be a Bessel sequence?

(b) If there exists a constant $B > 0$ such that $\sum |\langle x, x_n\rangle|^2 \le B\,\|x\|^2$ for all x in a complete set $E \subseteq H$, must $\{x_n\}$ be a Bessel sequence?

7.9. Show that a sequence $\{x_n\}$ in a Hilbert space H is a Bessel sequence if either of the following two conditions holds:

(a) $\sum_m \sum_n |\langle x_m, x_n\rangle|^2 < \infty$, or

(b) $\sup_m \sum_n |\langle x_m, x_n\rangle| < \infty$.

Observe that hypothesis (a) is quite restrictive, e.g., it is not satisfied by any infinite orthonormal sequence.

7.10. Suppose that $\{x_n\}$ is a Bessel sequence that is a basis for a Hilbert space H. Let $\{y_n\}$ be the biorthogonal sequence, and let B be a Bessel bound.

(a) Show that

$$\forall\, x \in H, \quad \frac{1}{B}\,\|x\|^2 \;\le\; \sum_n |\langle x, y_n\rangle|^2.$$

We say that $\{y_n\}$ has a *lower frame bound* of B^{-1}; compare Definition 8.2. Note that $\{y_n\}$ need not be a Bessel sequence; see Exercise 7.1(h).

(b) Show that for all $N \in \mathbf{N}$ and $c_1, \dots, c_N \in \mathbf{F}$ we have

$$\frac{1}{B} \sum_{n=1}^{N} |c_n|^2 \;\le\; \biggl\| \sum_{n=1}^{N} c_n y_n \biggr\|^2.$$

7.11. Let $\{x_n\}$, $\{y_n\}$ be Bessel sequences in separable Hilbert spaces H, K, respectively. Show that the tensor product sequence $\{x_m \otimes y_n\}_{m,n \in \mathbf{N}}$ is a Bessel sequence in $H \otimes K = \mathcal{B}_2(H,K)$ (see Appendix B for definitions).

7.12. The Hilbert matrix is

$$\mathcal{H} \;=\; \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & 1/5 & \\ 1/3 & 1/4 & 1/5 & 1/6 & \\ 1/4 & 1/5 & 1/6 & 1/7 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Define

$$C \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \\ 1/3 & 1/3 & 1/3 & 0 & \\ 1/4 & 1/4 & 1/4 & 1/4 & \\ \vdots & & & & \ddots \end{bmatrix} \quad \text{and} \quad L \;=\; \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/2 & 1/3 & 1/4 & \\ 1/3 & 1/3 & 1/3 & 1/4 & \\ 1/4 & 1/4 & 1/4 & 1/4 & \\ \vdots & & & & \ddots \end{bmatrix},$$

and prove the following statements.

(a) $L = CC^*$, so $L \ge 0$ (i.e., L is a positive operator).

(b) $I - (I-C)(I-C)^* = \operatorname{diag}(1, 1/2, 1/3, 1/4, \dots)$, the diagonal matrix with entries 1, 1/2, ... on the diagonal.

(c) $\|(I-C)\|^2 = \|(I-C)(I-C)^*\| \le 1$.

(d) $\|C\| \le 2$ and $\|L\| \le 4$.

Remark: It is a fact (though not so easy to prove) that if A, B are symmetric matrices and $a_{ij} \le b_{ij}$ for all $i, j \in \mathbf{N}$, then $\|A\| \le \|B\|$. Consequently, $\|\mathcal{H}\| \le \|L\| \le 4$.

7.2 Unconditional Bases and Riesz Bases in Hilbert Spaces

Let H be a separable Hilbert space. We saw in Example 4.21 that all orthonormal bases in H are equivalent. We will show in this section that the class of bases that are equivalent to orthonormal bases coincides with the class of bounded unconditional bases for H, and we will discuss some of the properties of such bases.

Definition 7.9 (Riesz Basis). Let $\{x_n\}$ be a sequence in a Hilbert space H.

(a) $\{x_n\}$ is a *Riesz basis* if it is equivalent to some (and therefore every) orthonormal basis for H.

(b) $\{x_n\}$ is a *Riesz sequence* if it is a Riesz basis for its closed span in H. ◇

Note that all Riesz bases are equivalent since all orthonormal bases are equivalent. Also, since all orthonormal bases are Bessel sequences, any Riesz basis $\{x_n\}$ must be a Bessel sequence (see Exercise 7.6). Hence we have at hand the tools discussed in Section 7.1. In particular, if $\{x_n\}$ is a Riesz basis, then we know that the analysis operator $Cx = \big(\langle x, x_n\rangle\big)$ is a bounded mapping of H into ℓ^2, and its adjoint is the synthesis operator $Rc = \sum c_n x_n$ for $c = (c_n) \in \ell^2$, where this series converges unconditionally in H.

As with bases or unconditional bases, the image of a Riesz basis under a topological isomorphism is a Riesz basis.

Lemma 7.10. *Riesz bases are preserved by topological isomorphisms. Specifically, if $\{x_n\}$ is a Riesz basis for a Hilbert space H and $T\colon H \to K$ is a topological isomorphism, then $\{Tx_n\}$ is a Riesz basis for K.*

Proof. Since H possesses a basis, it is separable. Therefore K, being topologically isomorphic to H, is separable as well. By Exercise 1.71, all separable Hilbert spaces are isometrically isomorphic, so there exists an isometry Z that maps H onto K. Further, by the definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $U\colon H \to H$ such that $Ue_n = x_n$. Since Z is an isometric isomorphism, the sequence $\{Ze_n\}$ is an orthonormal basis for K. Hence, TUZ^{-1} is a topological isomorphism of K onto itself which has the property that $TUZ^{-1}(Ze_n) = TUe_n = Tx_n$. Hence $\{Tx_n\}$ is equivalent to an orthonormal basis for K, so we conclude that $\{Tx_n\}$ is a Riesz basis for K. □

This yields one half of our characterization of Riesz bases.

Theorem 7.11. *Every Riesz basis for a Hilbert space H is a bounded unconditional basis for H.*

Proof. Let $\{x_n\}$ be a Riesz basis for a Hilbert space H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$ for every n. However, $\{e_n\}$ is a bounded unconditional basis, and bounded unconditional bases are preserved by topological isomorphisms by Lemma 6.2(b), so $\{x_n\}$ must be a bounded unconditional basis for H. □

Before presenting the converse to this result, we prove that Riesz bases are interchangeable with their dual systems in the following sense.

Lemma 7.12. *Let $\{x_n\}$ be a basis for a Hilbert space H, with biorthogonal system $\{y_n\}$. Then the following statements are equivalent.*

(a) $\{x_n\}$ *is a Riesz basis for* H.

(b) $\{y_n\}$ *is a Riesz basis for* H.

(c) $\{x_n\} \sim \{y_n\}$.

Proof. (a) $\Rightarrow$ (b), (c). If $\{x_n\}$ is a Riesz basis for H, then $\{x_n\} \sim \{e_n\}$ for some orthonormal basis $\{e_n\}$ of H. By Corollary 5.23, $\{x_n\}$ and $\{e_n\}$ have equivalent biorthogonal systems. However, $\{e_n\}$ is biorthogonal to itself, so this implies $\{y_n\} \sim \{e_n\} \sim \{x_n\}$. Hence $\{y_n\}$ is equivalent to $\{x_n\}$, and $\{y_n\}$ is a Riesz basis for H.

(b) $\Rightarrow$ (a), (c). By Corollary 5.22, $\{y_n\}$ is a basis for H with biorthogonal system $\{x_n\}$. Therefore, this argument follows symmetrically.

(c) $\Rightarrow$ (a), (b). Assume that $\{x_n\} \sim \{y_n\}$. Then there exists a topological isomorphism $T\colon H \to H$ such that $Tx_n = y_n$ for every n. Given $x \in H$, we therefore have

$$x = \sum_n \langle x, y_n\rangle\, x_n = \sum_n \langle x, Tx_n\rangle\, x_n,$$

so

$$\langle Tx, x\rangle = \Bigl\langle \sum_n \langle x, Tx_n\rangle\, Tx_n,\ x \Bigr\rangle = \sum_n |\langle x, Tx_n\rangle|^2 \geq 0.$$

Thus T is a continuous and positive linear operator on H, and therefore has a continuous and positive square root $T^{1/2}$ by Theorem 2.18. Similarly, T^{-1} is positive and has a positive square root. Consequently, $T^{1/2}$ is a topological isomorphism. Further, $T^{1/2}$ is self-adjoint, so

$$\langle T^{1/2}x_m, T^{1/2}x_n\rangle = \langle x_m, T^{1/2}T^{1/2}x_n\rangle = \langle x_m, Tx_n\rangle = \langle x_m, y_n\rangle = \delta_{mn}.$$

Hence $\{T^{1/2}x_n\}$ is an orthonormal sequence in H, and it is complete since $\{x_n\}$ is complete and $T^{1/2}$ is a topological isomorphism. Therefore $\{x_n\}$ is the image of the orthonormal basis $\{T^{1/2}x_n\}$ under the topological isomorphism $T^{-1/2}$, so $\{x_n\}$ is a Riesz basis. By symmetry, $\{y_n\}$ is a Riesz basis as well. □

Now we can prove that Riesz bases and bounded unconditional bases are equivalent, and we also give several other equivalent formulations of Riesz bases. We include the proofs of more implications than are strictly necessary. Additional characterizations of Riesz bases will be given in Theorem 8.32.

Theorem 7.13. *Let $\{x_n\}$ be a sequence in a Hilbert space H. Then the following statements are equivalent.*

(a) $\{x_n\}$ *is a Riesz basis for* H.

(b) $\{x_n\}$ *is a bounded unconditional basis for* H.

(c) $\{x_n\}$ *is a basis for* H, *and*

$$\sum_n c_n x_n \text{ converges} \quad \Longleftrightarrow \quad \sum_n |c_n|^2 < \infty.$$

(d) $\{x_n\}$ *is complete in* H *and there exist constants* A, $B > 0$ *such that*

$$\forall\, c_1, \dots, c_N, \quad A \sum_{n=1}^{N} |c_n|^2 \;\le\; \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2. \tag{7.4}$$

(e) *There is an equivalent inner product* $(\cdot,\cdot)$ *for* H *such that* $\{x_n\}$ *is an orthonormal basis for* H *with respect to* $(\cdot,\cdot)$.

(f) $\{x_n\}$ *is a complete Bessel sequence and possesses a biorthogonal system* $\{y_n\}$ *that is also a complete Bessel sequence.*

(g) $\{x_n\}$ *is complete, and multiplication of vectors in* ℓ^2 *by the Gram matrix* $G = \bigl[\langle x_n, x_m\rangle\bigr]_{m,n\in\mathbf{N}}$ *defines a topological isomorphism of* ℓ^2 *onto itself.*

Proof. (a) ⇒ (b). This is Theorem 7.11.

(a) ⇒ (e). If $\{x_n\}$ is a Riesz basis for H, then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Tx_n = e_n$ for every n. Define

$$(x,y) \;=\; \langle Tx, Ty\rangle \qquad \text{and} \qquad |||x|||^2 \;=\; (x,x) \;=\; \langle Tx, Tx\rangle \;=\; \|Tx\|^2.$$

It is easy to see that $(\cdot,\cdot)$ is an inner product for H, and by applying Exercise 2.37 we obtain $\|T^{-1}\|^{-1}\, \|x\| \;\le\; |||x||| \;\le\; \|T\|\, \|x\|$. Hence $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms for H, and so $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are equivalent inner products. Since

$$(x_m, x_n) \;=\; \langle Tx_m, Tx_n\rangle \;=\; \langle e_m, e_n\rangle \;=\; \delta_{mn},$$

the sequence $\{x_n\}$ is orthonormal with respect to $(\cdot,\cdot)$. Suppose $x \in H$ satisfies $(x, x_n) = 0$ for every n. Then $0 = (x, x_n) = \langle Tx, Tx_n\rangle = \langle Tx, e_n\rangle$ for every n, so $Tx = 0$ since $\{e_n\}$ is complete with respect to $\langle\cdot,\cdot\rangle$. Since T is a topological isomorphism, we therefore have $x = 0$, so $\{x_n\}$ is complete with respect to $(\cdot,\cdot)$. A complete orthonormal sequence is an orthonormal basis, so statement (e) holds.

(a) ⇒ (g). Suppose that $\{x_n\}$ is a Riesz basis for H. Since all Riesz bases and orthonormal bases are equivalent, there exists a topological isomorphism $T\colon \ell^2 \to H$ such that $T\delta_n = x_n$, where $\{\delta_n\}$ is the standard basis for ℓ^2. Note that since $\{x_n\}$ is a Bessel sequence, T is precisely the synthesis operator R for $\{x_n\}$. Hence $G = R^*R = T^*T$ is also a topological isomorphism.

(b) ⇒ (f). Suppose that $\{x_n\}$ is a bounded unconditional basis for H, and let $\{y_n\}$ be its biorthogonal system. Since H is reflexive, Exercise 6.4 implies that $\{y_n\}$ is also an unconditional basis for H. Also, by Theorem 4.13,

$1 \le \|x_n\|\,\|y_n\| \le 2\mathcal{C}$ where $\mathcal{C}$ is the basis constant for $\{x_n\}$. Hence $\{y_n\}$ is a bounded unconditional basis for H. All bounded unconditional bases are Bessel sequences by Theorem 7.8, so statement (f) follows.

(c) $\Rightarrow$ (a). Let $\{e_n\}$ be an orthonormal basis for H. Then, by Theorem 4.20, statement (c) implies that $\{x_n\} \sim \{e_n\}$, so $\{x_n\}$ is a Riesz basis for H.

(d) $\Rightarrow$ (c). Suppose that statement (d) holds. Taking $c_m = 1$ and $c_n = 0$ for $n \ne m$, we see from equation (7.4) that $\|x_m\|^2 \ge B^{-1}$. Hence each x_m is nonzero. Choose any $M < N$, and scalars $c_1, \dots, c_N$. Then, by equation (7.4),

$$\left\|\sum_{n=1}^{M} c_n x_n\right\|^2 \;\le\; B\sum_{n=1}^{M} |c_n|^2 \;\le\; B\sum_{n=1}^{N} |c_n|^2 \;\le\; \frac{B}{A}\left\|\sum_{n=1}^{N} c_n x_n\right\|^2.$$

Since $\{x_n\}$ is complete and every x_n is nonzero, Theorem 5.17 implies that $\{x_n\}$ is a basis for H.

It remains to show that $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$. Given a sequence of scalars (c_n) and $M < N$, we have by equation (7.4) that

$$A\sum_{n=M+1}^{N} |c_n|^2 \;\le\; \left\|\sum_{n=M+1}^{N} c_n x_n\right\|^2 \;\le\; B\sum_{n=M+1}^{N} |c_n|^2.$$

Therefore, $\sum c_n x_n$ is a Cauchy series in H if and only if $\sum |c_n|^2$ is a Cauchy series of real numbers. Hence one series converges if and only if the other series converges.

(e) $\Rightarrow$ (d). Suppose that $(\cdot,\cdot)$ is an equivalent inner product for H such that $\{x_n\}$ is an orthonormal basis with respect to $(\cdot,\cdot)$. Let $|||\cdot|||$ denote the norm induced by $(\cdot,\cdot)$. Then there exist constants $A,\, B > 0$ such that

$$\forall\, x \in H, \quad A\,|||x|||^2 \;\le\; \|x\|^2 \;\le\; B\,|||x|||^2. \tag{7.5}$$

Given $x \in H$, we have the orthonormal basis expansion $x = \sum (x, x_n)\, x_n$, where the series converges with respect to $|||\cdot|||$. Since $\|\cdot\|$ is equivalent to $|||\cdot|||$, this series also converges with respect to $\|\cdot\|$. Hence $\operatorname{span}\{x_n\}$ is dense and therefore $\{x_n\}$ is complete, with respect to both norms.

Now choose any scalars $c_1, \dots, c_N$. Then by the Plancherel Equality (Theorem 1.50), $\left|\left|\left| \sum_{n=1}^{N} c_n x_n \right|\right|\right|^2 = \sum_{n=1}^{N} |c_n|^2$. Combined with equation (7.5), this implies that

$$A\sum_{n=1}^{N} |c_n|^2 \;\le\; \left\|\sum_{n=1}^{N} c_n x_n\right\|^2 \;\le\; B\sum_{n=1}^{N} |c_n|^2,$$

so statement (d) holds.

(f) $\Rightarrow$ (b). Suppose that $\{x_n\}$ and $\{y_n\}$ are biorthogonal Bessel systems that are each complete in H. Given $x \in H$, we have $\big(\langle x, y_n\rangle\big) \in \ell^2$ since $\{y_n\}$

is Bessel. Hence $z = \sum \langle x, y_n \rangle x_n$ converges unconditionally by Theorem 7.2. By biorthogonality, $\langle z, y_n \rangle = \langle x, y_n \rangle$ for every n, and so $z = x$ since $\{y_n\}$ is complete. Thus $x = \sum \langle x, y_n \rangle x_n$ with unconditional convergence. Biorthogonality ensures that this representation is unique, so $\{x_n\}$ is an unconditional basis for H. Both $\{x_n\}$ and $\{y_n\}$ are bounded above in norm since they are Bessel sequences. Also, $1 \le \|x_n\| \, \|y_n\| \le 2\mathcal{C}$, where $\mathcal{C}$ is the basis constant for $\{x_n\}$, so $\{x_n\}$ and $\{y_n\}$ are bounded below in norm. Therefore $\{x_n\}$ is a bounded unconditional basis for H.

(f) $\Rightarrow$ (g). Suppose that $\{x_n\}$, $\{y_n\}$ are biorthogonal sequences that are each complete Bessel sequences. Let C, R be the analysis and synthesis operators for $\{x_n\}$, and let D, V be the analysis and synthesis operators for $\{y_n\}$. These are all bounded since $\{x_n\}$ and $\{y_n\}$ are Bessel. By biorthogonality, if $c \in \ell^2$, then

$$CVc \;=\; \biggl(\Bigl\langle \sum_n c_n y_n,\; x_m \Bigr\rangle\biggr)_{m\in\mathbf{N}} \;=\; (c_m)_{m\in\mathbf{N}} \;=\; c.$$

Further, if $x \in H$, then $RDx = \sum \langle x, y_n \rangle x_n$, and biorthogonality implies that $\langle RDx, y_n \rangle = \langle x, y_n \rangle$ for each n. Since $\{y_n\}$ is complete, this implies that $RDx = x$. Symmetric arguments show that VC and DR are identity operators as well. Finally, $G = CR$, so $L = DV$ is a bounded operator that satisfies

$$GL \;=\; CRDV \;=\; CV \;=\; I \qquad \text{and} \qquad LG \;=\; DVCR \;=\; DR \;=\; I.$$

Hence G has a bounded two-sided inverse, and therefore is a topological isomorphism.

(g) $\Rightarrow$ (d). Assume that $\{x_n\}$ is complete and the Gram matrix G defines a topological isomorphism of ℓ^2 onto itself. Then G is bounded, so we have by Theorem 7.6 that $\{x_n\}$ is a Bessel sequence, and therefore $\langle Gc, c \rangle = \bigl\|\sum c_n x_n\bigr\|^2 \ge 0$ for all $c = (c_n) \in \ell^2$. Hence G is a positive operator on ℓ^2, and in fact it is positive definite since it is a topological isomorphism. Exercise 2.45 therefore implies that $|||c||| = \langle Gc, c \rangle$ is an equivalent norm on ℓ^2. Hence there exist constants A, $B > 0$ such that $A\,|||c|||^2 \le \|c\|_{\ell^2}^2 \le B\,|||c|||^2$ for all $c \in \ell^2$, and this implies that statement (d) holds. $\square$

Exercises

7.13. Given a Riesz basis $\{x_n\}$ in a Hilbert space H, prove that the following statements are equivalent.

(a) $\sum c_n x_n$ converges.

(b) $\sum c_n x_n$ converges unconditionally.

(c) $\sum |c_n|^2 < \infty$.

7.14. Show that every basis for a finite-dimensional vector space V is a Riesz basis for V (with respect to any inner product on V).

7.15. Exhibit an unconditional basis for a Hilbert space H that is not a Riesz basis for H.

7.16. Show that if $\{x_n\}$ is a complete sequence in a Hilbert space H that satisfies $\left\|\sum_{n=1}^{N} c_n x_n\right\|^2 = \sum_{n=1}^{N} |c_n|^2$ for any $N \in \mathbf{N}$ and $c_1, \dots, c_N \in \mathbf{F}$, then $\{x_n\}$ is an orthonormal basis for H.

7.17. Let $\{x_n\}$, $\{y_n\}$ be Riesz bases for Hilbert spaces H, K, respectively. Show that the tensor product sequence $\{x_m \otimes y_n\}_{m,n\in\mathbf{N}}$ is a Riesz basis for $H \otimes K = \mathcal{B}_2(H, K)$ (see Appendix B for definitions).

7.18. Let $\{x_n\}$ be an orthonormal basis for a Hilbert space H. Suppose $\{y_n\}$ is a sequence in H and there exists $0 < \lambda < 1$ such that

$$\left\|\sum_{n=1}^{N} c_n\,(x_n - y_n)\right\|^2 \;\le\; \lambda \sum_{n=1}^{N} |c_n|^2, \qquad N \in \mathbf{N},\ c_1, \dots, c_N \in \mathbf{F}.$$

Show that $\{y_n\}$ is a Riesz basis for H.

7.19. Let $\{x_n\}$ be an orthonormal basis for a Hilbert space H. Let $T_k \in \mathcal{B}(H)$ and $a_{nk} \in \mathbf{F}$ be such that

$$\lambda \;=\; \sum_{k=1}^{\infty} \|T_k\| \left(\sup_n |a_{nk}|\right) \;<\; 1.$$

Assume that the series

$$y_n \;=\; x_n \;+\; \sum_{k=1}^{\infty} a_{nk}\, T_k e_n$$

converges for each $n \in \mathbf{N}$. Show that $\{y_n\}$ is a Riesz basis for H.

7.20. In this exercise we will use the abbreviation $e_b(x) = e^{2\pi i b x}$, where $b \in \mathbf{R}$. Also, we identify the Hilbert space $L^2(\mathbf{T})$ with $L^2[-\frac{1}{2}, \frac{1}{2}]$.

Fix $\lambda_n \in \mathbf{C}$ and assume that

$$\delta \;=\; \sup_{n\in\mathbf{Z}} |n - \lambda_n| \;<\; \infty.$$

(a) Define bounded linear operators T_k on $L^2[-\frac{1}{2}, \frac{1}{2}]$ by

$$T_k f(x) \;=\; x^k f(x).$$

Show that the operator norm of T_k is $\|T_k\| = 2^{-k}$.

(b) Define

$$a_{nk} = -\frac{\big(2\pi i(\lambda_n - n)\big)^k}{k!}.$$

Show that

$$e_n - e_{\lambda_n} = \sum_{k=1}^{\infty} a_{nk}\, T_k e_n, \qquad n \in \mathbf{Z},$$

where the series converge absolutely in $L^2[-\frac{1}{2}, \frac{1}{2}]$.

(c) Show that if

$$\delta < (\ln 2)/\pi \approx 0.22\ldots,$$

then $\{e^{2\pi i \lambda_n x}\}_{n\in\mathbf{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$.

Remark: This result is due to Duffin and Eachus [DE42], but it is not quite the best possible. *Kadec's* $\frac{1}{4}$*-Theorem* [Kad64] states that if $\delta < \frac{1}{4}$ then $\{e^{2\pi i \lambda_n x}\}_{n\in\mathbf{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$, and it is known that $\frac{1}{4}$ is the optimal value. For a more detailed discussion, we refer to [You01].

8

Frames in Hilbert Spaces

So far in this volume we have mostly focused on *bases*, which provide unique series representations of vectors in Banach spaces in terms of the basis vectors. Intuitively, uniqueness seems to be important, if not essential, to any practical use of series representations. Yet in many applications uniqueness turns out to be more of a hindrance than a help. For example, suppose that $\{x_n\}$ is a basis for a Banach space. Then $x = \sum \langle x, a_n \rangle \, x_n$ uniquely, so x is characterized by the information appearing in the sequence of basis coefficients $\big(\langle x, a_n \rangle\big)$. Considering a data transmission application, if even one single coefficient $\langle x, a_n \rangle$ is lost from this sequence during transmission, then the receiver has no hope of reconstructing x from the received coefficients. If somehow there was some redundancy built into the coefficients, then we might still be able to reconstruct x from the remaining coefficients.

In this chapter we will study *frames*, which provide basis-like but usually redundant series representations of vectors in a Hilbert space. As suggested by the preceding discussion, frames have found many applications in engineering, but are also important tools in pure mathematics. For example, frames play key roles in wavelet theory, time-frequency analysis, the theory of shift-invariant spaces, sampling theory, and many other areas.

Some of the concrete frames used in those applications will be presented in Part III. In this chapter we concentrate on the abstract theory of frames. Frames were first introduced by Duffin and Schaeffer [DS52] during their study of nonharmonic Fourier series, and that paper is still an elegant introduction to frames. Many of the proofs that we present are directly inspired by Duffin and Schaeffer's paper. Other important sources include the classic text by Young [You01] and the papers and texts by Daubechies [Dau92] and Christensen [Chr03]. Other references will be noted as we progress through the chapter.

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8.1 Definition and Motivation

Each orthonormal basis $\{e_n\}$ for a Hilbert space H satisfies the Plancherel Equality, which states that $\sum |\langle x, e_n\rangle|^2 = \|x\|^2$ for all $x \in H$. However, a sequence can satisfy the Plancherel Equality without being orthonormal or a basis. Here is an (elementary) finite-dimensional example.

Example 8.1. Let $H = \mathbf{R}^2$, and set

$$x_1 = (1,0), \quad x_2 = (0,1), \quad x_3 = \big(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\big), \quad x_4 = \big(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\big).$$

Each of $\{x_1, x_2\}$ and $\{x_3, x_4\}$ is an orthonormal basis for $\mathbf{R}^2$, so

$$\sum_{n=1}^{4} |\langle x, x_n\rangle|^2 \;=\; 2\,\|x\|^2, \qquad x \in \mathbf{R}^2.$$

Therefore the family $\{2^{-1/2}x_n\}_{n=1}^4$ satisfies the Plancherel Equality, but it is not orthogonal and is not a basis for $\mathbf{R}^2$. ◇

We will call a sequence that satisfies the Plancherel Equality a *Parseval frame*. Exercise 8.1 gives another, less trivial, example of a Parseval frame. Specifically, if we take

$$x_1 = (0,1), \quad x_2 = \big(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\big), \quad x_3 = \big(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\big),$$

then $\sum_{n=1}^{3} |\langle x, x_n\rangle|^2 = \frac{3}{2}\,\|x\|^2$ for all $x \in \mathbf{R}^2$. Therefore, if we set $c = (2/3)^{1/2}$, then $\{cx_1, cx_2, cx_3\}$ is a Parseval frame, but it is not a union of orthonormal bases. This system is affectionately referred to as the "Mercedes frame" (see Figure 8.1). Using a little three-dimensional visualization, we realize that the Mercedes frame is the orthogonal projection of a certain orthonormal basis for $\mathbf{R}^3$ onto a two-dimensional plane. We will see in Corollary 8.34 that all frames can be realized in a similar manner.

While a Parseval frame is required to precisely satisfy the Plancherel Equality, the definition of a generic frame imposes a less stringent requirement.

Definition 8.2 (Frame). A sequence $\{x_n\}$ in a Hilbert space H is a *frame* for H if there exist constants A, $B > 0$ such that the following *pseudo-Plancherel formula* holds:

$$\forall\, x \in H, \quad A\,\|x\|^2 \;\le\; \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2. \tag{8.1}$$

The constants A, B are called *frame bounds*. We refer to A as a *lower frame bound*, and to B as an *upper frame bound*. The largest possible lower frame bound is called the *optimal lower frame bound*, and the smallest possible upper frame bound is the *optimal upper frame bound*. ◇

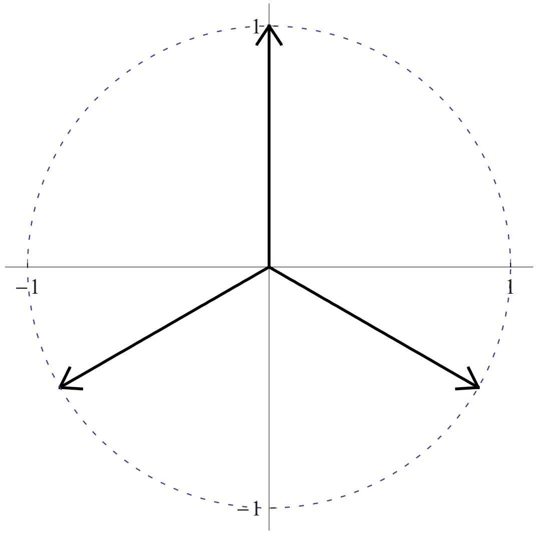

Fig. 8.1. The three vectors of the Mercedes frame. A dashed unit circle is included for comparison, and also to motivate its alternative name (the *peace frame*).

Thus, $\{x_n\}$ is a frame if $|||x||| = \|(\langle x, x_n\rangle)\|_{\ell^2}$ is an equivalent norm for H, and if it is possible to take $A = B = 1$ then we actually have $\|x\| = \|(\langle x, x_n\rangle)\|_{\ell^2}$ (and in this case we call $\{x_n\}$ a Parseval frame, see Definition 8.3). Although the frame definition says nothing explicitly about basis or basis-like properties of $\{x_n\}$, we will see that the norm equivalence requirement alone implies unconditionally convergent, basis-like representations of vectors in H.

Frames were introduced by Duffin and Schaeffer in their 1952 paper on nonharmonic Fourier series [DS52], and much of the abstract theory of frames was elegantly laid out in that paper. Young's text [You01], whose first edition appeared in 1980, contains a beautiful development of abstract frames and their applications to nonharmonic Fourier series. Frames for $L^2(\mathbf{R})$ based on time-frequency or time-scale translates of functions were constructed by Daubechies, Grossmann, and Meyer in [DGM86], and the paper [Dau90] by Daubechies extensively analyzed frames in these settings. These papers and others spurred a dramatic development of wavelet theory and frame theory in the following years; see the texts [Dau92], [Grö01], [Chr03], or the research survey [HW89].

The following special types of frames will be important.

Definition 8.3. Let $\{x_n\}$ be a frame for a Hilbert space H.

(a) We say that $\{x_n\}$ is a *tight frame* if we can choose $A = B$ as frame bounds. In this case, we usually refer to A as "a frame bound" for $\{x_n\}$, or say that $\{x_n\}$ is an A-*tight frame*.

(b) We say that $\{x_n\}$ is a *Parseval frame* if $A = B = 1$ are frame bounds. Thus a Parseval frame is a 1-tight frame.

(c) We say that $\{x_n\}$ is an *exact frame* if it ceases to be a frame whenever any single element is deleted from the sequence. ♢

On occasion we need to deal with sequences that do not satisfy the frame condition on all of H but are frames for their closed spans.

Definition 8.4. A sequence $\{x_n\}$ in a Hilbert space H is called a *frame sequence* if it is a frame for $\overline{\text{span}}\{x_n\}$. ♢

We make some basic observations about frames.

Remark 8.5. (a) Every orthonormal basis is an exact Parseval frame, and conversely every exact Parseval frame is an orthonormal basis (Exercise 8.7). On the other hand, we have already seen examples of Parseval frames that are not exact, not orthonormal, and not bases.

(b) A frame is a sequence, not a set, and hence repetitions of elements are allowed. Also, the zero vector is allowed to be an element of a frame. This gives us more trivial examples of frames that are not bases, e.g., if $\{e_n\}$ is an orthonormal basis for H then $\{0, e_1, e_2, \dots\}$ is a Parseval frame that is not a basis. We must beware of the zero vector when dealing with frames, but we should not be misled into thinking that the only differences between frames and bases arise from trivialities such as repeating elements or including the zero vector. There are many nontrivial and interesting examples of frames, even in finite dimensions.

(c) If $\{x_n\}$ is a frame then $\sum |\langle x, x_n \rangle|^2$ is an absolutely convergent series of nonnegative scalars $|\langle x, x_n \rangle|^2$, and therefore it converges unconditionally by Lemma 3.3. In particular, $\sum |\langle x, x_{\sigma(n)} \rangle|^2 = \sum |\langle x, x_n \rangle|^2$ for any permutation σ of $\mathbf{N}$. Hence every rearrangement $\{x_{\sigma(n)}\}$ of a frame is a frame, and therefore it usually does not matter what countable set we use to index a frame. Many series involving frames will converge unconditionally—but not all. We will consider this issue in more detail in Section 8.6.

(d) Every frame is a Bessel sequence, and hence all of the results derived in Section 7.1 apply to frames. However, not every Bessel sequence is a frame (consider part (c) of Example 8.6). When trying to show that a given sequence is a frame it is often comparatively easy to show that it is a Bessel sequence. The difficulty usually lies in establishing the existence of a lower frame bound.

(e) A Bessel sequence need not be complete, but it follows from Definition 8.2 that frames must be complete. For, if we have $\langle x, x_n \rangle = 0$ for every n, then $A\,\|x\|^2 \le \sum |\langle x, x_n \rangle|^2 = 0$ and therefore $x = 0$. Since any space that contains a countable complete subset is separable (Theorem 1.27), we conclude that a Hilbert space that possesses a frame is separable. Conversely, every separable Hilbert space has an orthonormal basis and hence has frames.

(f) Some authors use the term "normalized frame" for what we call a Parseval frame, and other authors define a "normalized frame" to be a frame $\{x_n\}$ such that $\|x_n\| = 1$ for all n. The latter terminology is more in line with the terminology for *normalized bases* introduced in Definition 4.5, but because of this ambiguity in meaning, we will avoid using the term *normalized* in connection with frames.[1] Currently, the preferred terminology for a frame that satisfies $\|x_n\| = 1$ for all n is *uniform norm frame* or *equal norm frame*.

(g) At first glance, "exact frame" may seem to be another unfortunate choice of terminology since we already have defined an *exact sequence* to be a sequence that is both minimal and complete. However, we will see that a frame is an exact sequence if and only if it is an exact frame, so there is no ambiguity with this terminology in the end. We will also see that a frame is an exact frame if and only if it is a basis for H, in which case it is actually a Riesz basis for H. An inexact frame is redundant or overcomplete in the sense that a proper subset of the frame is still complete (in fact, still a frame). Hence, we often use the terms inexact, overcomplete, or redundant to describe a frame that is not a basis.

(h) Our focus in this volume is on infinite sequences $\{x_n\}_{n\in\mathbf{N}}$ that are frames for infinite-dimensional Hilbert spaces. However, we can certainly consider frames for finite-dimensional spaces. Exercise 8.11 will show that a sequence $\{v_1, \dots, v_n\}$ is a frame for a d-dimensional Hilbert space H if and only if $\{v_1, \dots, v_n\}$ is a spanning set for H, and it is an exact frame if and only if it is a Hamel basis for H. While this seems to suggest that frames for finite-dimensional spaces (often referred to as *finite frames*) are trivial, quite the opposite is true. The elegant characterization of finite uniform norm tight frames (FUNTFs) by Benedetto and Fickus [BF03] has inspired a great deal of research, and finite frames play important roles in modern signal-processing applications such as Σ-Δ quantization schemes [BPY06]. We refer to the text [HKLW07] and the survey paper [CFKLT06] for more on finite frames. $\diamondsuit$

Looking ahead, the "frame miracle" is that even though the definition of a frame is a statement about inner products that manifestly does not imply that the sequence is a basis, a frame $\{x_n\}$ nonetheless yields basis-like expansions of the form $x = \sum a_n(x)\, x_n$ for $x \in H$. We can even choose functionals a_n that are continuous, and so we have $x = \sum \langle x, y_n\rangle\, x_n$ for some $y_n \in H$. Further, there is a canonical choice for the dual system $\{y_n\}$, and using that choice these "frame expansions" converge *unconditionally* for every $x \in H$. We have almost every advantage of an unconditional basis—except that the scalars in this expansion need not be unique in general. We will prove these facts for general frames in Section 8.2. Many of these facts have elegant direct proofs for the case of tight frames, and are assigned as exercises at the end of this section. In particular, Exercise 8.5 shows that if $\{x_n\}$ is an A-tight frame then $x = \frac{1}{A} \sum \langle x, x_n\rangle\, x_n$ for all $x \in H$.

[1] Thanks to Larry Baggett for introducing the term *Parseval frame*.

In the remainder of this section, we give some examples that illustrate various features of frames. We start with some simple examples that show that tightness and exactness are distinct properties for frames.

Example 8.6. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H.

(a) $\{e_n\}$ is a tight exact frame for H with frame bounds $A = B = 1$ (hence is a Parseval frame).

(b) $\{e_1, e_1, e_2, e_2, e_3, e_3, \dots\}$ is a tight inexact frame with bounds $A = B = 2$, but it is not orthogonal and it is not a basis, although it does contain an orthonormal basis. Similarly, if $\{f_n\}$ is another orthonormal basis for H then $\{e_n\} \cup \{f_n\}$ is a tight inexact frame for H.

(c) $\{e_1,\, e_2/2,\, e_3/3,\, \dots\}$ is a complete orthogonal sequence and it is a basis for H, but it does not possess a lower frame bound and hence is not a frame.

(d) $\{e_1,\, e_2/\sqrt{2},\, e_2/\sqrt{2},\, e_3/\sqrt{3},\, e_3/\sqrt{3},\, e_3/\sqrt{3},\, \dots\}$ is an inexact Parseval frame, and no nonredundant subsequence is a frame. Further, while this sequence does contain an orthogonal basis, that basis is not a Riesz basis because it is not norm-bounded below.

(e) $\{2e_1, e_2, e_3, \dots\}$ is a nontight exact frame with frame bounds $A = 1$, $B = 2$. ♢

The preceding example suggests the question: Does every frame contain a basis as a subset? The example in part (d) shows that if we allow $\inf \|x_n\| = 0$, then there exist frames that do not contain a Riesz basis. The first example of a frame that is norm-bounded below but contains no Riesz bases as subsets was given by Seip [Sei95]. In that article, Seip obtained a variety of deep results related to the question of when a system of nonharmonic complex exponentials $\{e^{2\pi i \lambda_n t}\}_{n\in\mathbf{N}}$ that forms a frame for $L^2[0,1]$ will contain a Riesz basis, or when a Riesz sequence of exponentials can be extended to form a frame for $L^2[0,1]$. Casazza and Christensen also constructed a frame that is norm-bounded below and which does not contain a Riesz basis [CC98a], and they further showed in [CC98b] that this frame contains no subsets that are Schauder bases (see Example 8.45).

Systems of exponentials are very interesting and have applications in many areas, so let us take a closer look at them. We will focus on "harmonic" or "lattice" sequences whose frequencies form a subgroup of $\mathbf{R}$. Nonharmonic systems are considerably more difficult to understand, e.g., see the text by Young [You01].

Example 8.7 (Trigonometric Systems Revisited). We will consider the trigonometric system $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ in $L^2(\mathbf{T})$, where b is a fixed positive real number. Since functions in $L^2(\mathbf{T})$ are 1-periodic, we are implicitly considering $e^{2\pi ibnt}$ to be defined on the interval $[0,1)$ and then extended 1-periodically to $\mathbf{R}$.

(a) When $b = 1$ we know that $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ forms an orthonormal basis for $L^2(\mathbf{T})$ (see the discussion in Example 1.52 and Section 4.6).

(b) Suppose that $b > 1$. Considered on all of $\mathbf{R}$, the function $e^{2\pi ibnt}$ is $1/b$-periodic. However, we are restricting our attention to $[0,1)$. Since $1/b < 1$, the interval $[0,1/b]$ is properly contained in $[0,1)$. Therefore, for those t such that t and $t+1/b$ both belong to $[0,1)$ we have $e^{2\pi ibnt} = e^{2\pi ibn(t+1/b)}$. Taking finite linear combinations and limits, every function in $\overline{\text{span}}\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ must exhibit a similar behavior. However, not every function in $L^2(\mathbf{T})$ satisfies $f(t) = f(t+1/b)$, so there are functions in $L^2(\mathbf{T})$ that are not in the closed span of $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$. For example, the function $f(t) = t$, $t \in [0,1)$, is one of these. Hence $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is incomplete and therefore cannot be a frame for $L^2(\mathbf{T})$ (see Exercise 8.9).

(c) Now consider $0 < b < 1$. It is still true that $e^{2\pi ibnt}$ is $1/b$-periodic when we consider all $t \in \mathbf{R}$, but now we have $1/b > 1$. Hence the argument used in part (b) does not apply to this case. Moreover, for special choices of b we can easily see that $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{T})$. For example, if $b = 1/2$ then we can write $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ as a union of two orthonormal bases for $L^2(\mathbf{T})$:

$$\begin{aligned}\left\{e^{2\pi int/2}\right\}_{n\in\mathbf{Z}} &= \left\{e^{2\pi int}\right\}_{n\in\mathbf{Z}} \cup \left\{e^{2\pi i(n+1/2)t}\right\}_{n\in\mathbf{Z}} \\ &= \left\{e^{2\pi int}\right\}_{n\in\mathbf{Z}} \cup \left\{e^{\pi it}\, e^{2\pi int}\right\}_{n\in\mathbf{Z}}.\end{aligned}$$

Hence this system is a tight frame with frame bound 2. Similarly, if $1/b = M \in \mathbf{N}$ then $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is a union of M orthonormal bases and therefore is an M-tight frame. We often say that a sequence that is a union of M orthonormal bases or Riesz bases is "M-times overcomplete," or that it has "redundancy M."

What if $1/b$ is not an integer? For example, if $1/b = \sqrt{2}$ then there is certainly no way to divide $\{e^{2\pi int/\sqrt{2}}\}_{n\in\mathbf{Z}}$ into "$\sqrt{2}$-many orthonormal bases." Yet Exercise 8.9 shows that $\{e^{2\pi int/\sqrt{2}}\}_{n\in\mathbf{Z}}$ is a tight frame, and the frame bound is exactly $\sqrt{2}$. In some sense, this system has redundancy $\sqrt{2}$, even though we cannot interpret redundancy as meaning that a sequence is a union of bases. *Quantifying* what redundancy means is surprisingly difficult, and we refer to the papers [BCHL06a], [BCHL06b] for results in this direction (see also the survey paper [Hei07]).

The system $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ illustrates another subtlety about the meaning of redundancy. Exercise 8.9 shows that $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is a tight frame but is not a basis for $L^2(\mathbf{T})$ when $0 < b < 1$. The frames discussed in Example 8.6 that fail to be bases do so because they contain linearly dependent subsets, but we will show that this is not the case for $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$.

Suppose that $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ contained a finite dependent subset. Then we would have

$$\sum_{n=-N}^{N} c_n e^{2\pi ibnt} = 0$$

for some $N \in \mathbf{N}$ and scalars c_n not all zero. Technically, this is an equality of functions in $L^2(\mathbf{T})$, which means that it holds for almost every t. However, since both sides are continuous, we can assume without loss of generality that equality holds for all t. Let p be the polynomial $p(z) = \sum_{n=-N}^{N} c_n z^{n+N}$. Then for $z = e^{2\pi i b t}$ with $t \in \mathbf{R}$ we have

$$p(z) = \sum_{n=-N}^{N} c_n e^{2\pi i b(n+N)t} = e^{2\pi i b N t} \sum_{n=-N}^{N} c_n e^{2\pi i b n t} = 0.$$

Hence p has uncountably many roots in the complex plane. This contradicts the Fundamental Theorem of Algebra, which states that a nontrivial polynomial has at most finitely many roots. Therefore, even though $\{e^{2\pi i b n t}\}_{n\in\mathbf{Z}}$ is a redundant frame, it is finitely linearly independent.

In summary, if $0 < b < 1$ then $\{e^{2\pi i b n t}\}_{n\in\mathbf{Z}}$ is a tight frame with "redundancy $1/b$" that is not a basis yet provides basis-like representations of vectors in $L^2(\mathbf{T})$. This frame is redundant, even though it is finitely linearly independent. Exercise 8.9 shows that this frame is not ω-independent, and therefore is not minimal. In particular, the constant function 1 (which is $e^{2\pi i b 0 t}$) belongs to $\overline{\operatorname{span}}\{e^{2\pi i b n t}\}_{n\neq 0}$, so this smaller set is still complete, and Exercise 8.6 shows that this smaller set is still a frame for $L^2(\mathbf{T})$. $\diamondsuit$

Remark 8.8. Although it takes some of the "magic" out of it, we can get an important insight into Example 8.7 by recasting it in terms of orthogonal projections. Since the trigonometric system $\{e^{2\pi i n t}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,1]$, a simple change of variables tells us that $\{b^{-1/2} e^{2\pi i b n t}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0, b^{-1}]$. If $0 < b < 1$ then we can think of $L^2[0,1]$ as being a closed subspace of $L^2[0, b^{-1}]$ by extending a function $f \in L^2[0,1]$ by zero on $(1, b^{-1}]$. With this understanding, the mapping $P\colon L^2[0, b^{-1}] \to L^2[0,1]$ defined by $Pf = f \cdot \chi_{[0,1]}$ is an orthogonal projection. By Exercise 8.8, the orthogonal projection of an orthonormal basis onto a closed subspace is a Parseval frame for that subspace. Hence the sequence $\{b^{-1/2} e^{2\pi i b n t}\}_{n\in\mathbf{Z}}$ in $L^2[0,1]$ is simply the image of an orthonormal basis under an orthogonal projection, and therefore is a Parseval frame. On the other hand, if $b > 1$ then $L^2[0, b^{-1}]$ is a proper subspace of $L^2[0,1]$, which yields (why?) the incompleteness statement of Example 8.7(b).

Later we will see that the fact that an orthogonal projection of an orthonormal basis is a Parseval frame has a converse, see Corollary 8.34. $\diamondsuit$

Example 8.7 shows that some redundant frames contain no finitely dependent subsets, and also shows that some redundant frames are unions of orthonormal bases while others are not. We have also seen that some frames contain no bases as subsets. Feichtinger asked a very natural related question, and made the following conjecture (recall that a Riesz sequence is a sequence $\{x_n\}$ that is a Riesz basis for its closed span within H).

Conjecture 8.9 (Feichtinger Conjecture). *If $\{x_n\}$ is a frame that is norm-bounded below, then $\{x_n\}$ can be written as the union of finitely many Riesz sequences.* ♢

The reason that we include norm-boundedness in this conjecture is that there exist trivial counterexamples if we allow the norms of the frame elements to converge to zero, e.g., see Example 8.6(d).

At the time of writing, there are many classes of frames for which the Feichtinger Conjecture is known to be true (e.g., see [Grö03], [BCHL06a], [BoS06]), but there are no known counterexamples. More surprisingly, Casazza and Tremain [CT06] have shown that the Feichtinger Conjecture is equivalent to an entire suite of other open conjectures from different areas of mathematics and engineering, including the 1959 *Kadison–Singer Conjecture*, which is one of the deepest open problems in operator theory today. For surveys and references, see [CE07], [CMTW06].

The frames of complex exponentials discussed in Example 8.7 are related to many other types of frames that arise naturally in theory and applications. Some of these are listed in the next example, and there are many variations on these types of frames.

Example 8.10. (a) Given $\varphi \in L^2(\mathbf{T})$, a *frame of weighted exponentials* is a frame for $L^2(\mathbf{T})$ of the form $\{e^{2\pi i b_n t}\varphi(t)\}_{n\in\mathbf{N}}$, where $b_n \in \mathbf{R}$. Often we require that the set $\{b_n\}$ have some special structure, e.g., $\{b_n\} = b\mathbf{Z}$ for some fixed $b > 0$. A specific system of weighted exponentials of the form $\{e^{2\pi i n t}\varphi(t)\}_{n\in\mathbf{N}}$ was considered in Example 5.13, and the unweighted system $\{e^{2\pi i b n t}\}_{n\in\mathbf{N}}$ was studied above in Example 8.7. We will consider the weighted systems $\{e^{2\pi i n t}\varphi(t)\}_{k\in\mathbf{Z}}$ in detail in Section 10.3.

(b) Given $g \in L^2(\mathbf{R})$, a *frame of translates* is a frame sequence in $L^2(\mathbf{R})$ of the form $\{g(t - a_k)\}_{k\in\mathbf{N}}$, where $a_k \in \mathbf{R}$. We will consider *lattice frames of translates* of the form $\{g(t - ak)\}_{k\in\mathbf{Z}}$ in Section 10.4, and they also play an important role in Chapter 12.

(c) Given $g \in L^2(\mathbf{R})$, a *Gabor frame* is a frame for $L^2(\mathbf{R})$ of the form

$$\{e^{2\pi i b_n t} g(t - a_n)\}_{n\in\mathbf{N}},$$

where $a_n,\, b_n \in \mathbf{R}$. If $a,\, b > 0$ are fixed and $\{e^{2\pi i b n t} g(t - ak)\}_{k,n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{R})$ then we call it a *lattice Gabor frame*. We will study lattice Gabor frames in Chapter 11.

(d) Given $\psi \in L^2(\mathbf{R})$, a *wavelet frame* is a frame for $L^2(\mathbf{R})$ of the form

$$\{a_n^{1/2}\psi(a_n t - b_n)\}_{n\in\mathbf{N}},$$

where $a_n > 1$ and $b_n \in \mathbf{R}$. Some of the most common wavelet frames are *dyadic wavelet frames* based on dilation by a factor of 2, e.g., a frame of the form $\{2^{n/2}\psi(2^n t - k)\}_{k,n\in\mathbf{Z}}$. We will study dyadic wavelet frames in Chapter 12. ♢

In the examples above, we cannot choose φ, g, ψ, a_n, or b_n at random and hope to obtain a frame. Often we can find some specific functions and parameters that will yield a frame, but constructing frames that have extra desirable properties is usually more difficult. Moreover, it is extremely difficult or impossible to explicitly characterize *all* possible choices of φ, g, ψ, a_n, b_n that yield frames. We will address some of these issues in Chapters 10–12.

Remark 8.11. We call $e^{2\pi ibt}g(t-a)$ a *time-frequency translate* of g and $a^{1/2}\psi(at-b)$ a *time-scale translate* of ψ. Thus a Gabor frame is a frame constructed from time-frequency translates of a function g, while a wavelet frame is a frame constructed from time-scale translates of ψ. ◇

As we have mentioned, Gabor frames play an important role in time-frequency analysis, which is a type of local harmonic analysis. To give one illustration of why we are interested in time-frequency frames rather than time-frequency Riesz bases, we quote two versions of the *Balian–Low Theorems*. Essentially, these theorems tell us that we simply cannot construct useful Gabor systems that are Riesz bases for $L^2(\mathbf{R})$, and so we absolutely need the added flexibility of frames in order to construct "useful" Gabor systems.

Theorem 8.12. *Fix $g \in L^2(\mathbf{R})$ and $a,\, b > 0$.*

(a) (Classical Balian–Low Theorem) *If $\{e^{2\pi ibmt}g(t-an)\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$, then*

$$\biggl(\int_{-\infty}^{\infty} |tg(t)|^2\,dt\biggr)\biggl(\int_{-\infty}^{\infty} |\xi\widehat{g}(\xi)|^2\,d\xi\biggr) = \infty,$$

where $\widehat{g}$ denotes the Fourier transform of g (see Definition 9.7).

(b) (Amalgam Balian–Low Theorem) *If $\{e^{2\pi ibmt}g(t-an)\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$, then either g is not continuous or*

$$\sum_{k\in\mathbf{Z}} \|g\,\chi_{[k,k+1]}\|_{L^\infty} \;=\; \infty. \qquad \diamondsuit$$

We will discuss the Balian–Low Theorems in Section 11.8, and we refer to the survey paper [BHW95] for historical discussion and references. Qualitatively, both versions of the Balian–Low Theorem (which is known familiarly as the *BLT*)[2] say that if a lattice Gabor frame is not redundant, then the generating function g has such poor time-frequency localization that it is essentially useless for nontrivial applications. For example, the easiest example of a Gabor orthonormal basis is obtained by taking $g = \chi_{[0,1]}$ and $a = b = 1$ (see Exercise 11.5). This function g is discontinuous. Further, while g has excellent decay in time (in fact, it is zero outside of $[0,1]$), its Fourier transform is $\widehat{g}(\xi) = e^{-\pi i\xi}\,(\sin \pi\xi)/(\pi\xi)$, which decays only on the order of $1/|\xi|$ and is not even integrable. On the other hand, it is possible to create very nice functions g that generate redundant Gabor frames for $L^2(\mathbf{R})$ (see Section 11.2). These are the frames that are used in practice in time-frequency analysis.

[2]In the United States, a *BLT* is a Bacon, Lettuce, and Tomato sandwich.

Exercises

8.1. Prove that the Mercedes frame is a tight frame for $\mathbf{R}^2$ with frame bound $A = 3/2$.

8.2. Show that if $\{x_n\}$ is a frame for a Hilbert space H and $\{y_n\}$ is a Bessel sequence in H, then $\{x_n\} \cup \{y_n\}$ is a frame for H.

8.3. Show that if $\{x_n\}_{n\in\mathbf{N}}$ is a Riesz basis and $J \subseteq \mathbf{N}$, then $\{x_n\}_{n\in J}$ is a Riesz basis for $\overline{\operatorname{span}}\{x_n\}_{n\in J}$. Does the analogous statement hold for frames?

8.4. Let $\{x_n\}$ be a sequence in a Hilbert space H, and let E be a dense subset of H. Show that if there exist some $A, B > 0$ such that $A\,\|x\|^2 \leq \sum |\langle x, x_n\rangle|^2 \leq B\,\|x\|^2$ for $x \in E$, then $\{x_n\}$ is a frame for H. Thus it suffices to establish the frame condition on some dense (and hopefully "nice") subset of H (compare hypothesis (b) of Theorem 7.4).

8.5. Let $\{x_n\}$ be an A-tight frame for a Hilbert space H. Show that the frame operator for $\{x_n\}$ (see Definition 7.3) is $S = AI$, and use this to show that $x = A^{-1} \sum \langle x, x_n\rangle\, x_n$ for $x \in H$.

8.6. Let $\{x_n\}$ be an A-tight frame in a Hilbert space H.

(a) Show that $\|x_n\|^2 \leq A$ for every $n \in \mathbf{N}$.

(b) Show that if $\|x_m\|^2 < A$ for some m, then $\{x_n\}_{n\neq m}$ is a frame for H, and the optimal lower frame bound for $\{x_n\}_{n\neq m}$ is $A - \|x_m\|^2$ (note that this new frame might not be tight).

(c) Show that if $\|x_m\|^2 = A$ for some m, then $x_m \perp x_n$ for all $n \neq m$.

8.7. Let $\{x_n\}$ be an A-tight frame for a Hilbert space H. Show directly that the following statements are equivalent.

(a) $\|x_n\|^2 = A$ for every n.

(b) $\{x_n\}$ is an orthogonal (but not necessarily orthonormal) sequence with no zero elements.

(c) $\{x_n\}$ is a basis for H.

(d) $\{x_n\}$ is ω-independent.

(e) $\{x_n\}$ is an exact frame.

Use these equivalences to show that $\{x_n\}$ is an exact Parseval frame for H if and only if it is an orthonormal basis for H.

8.8. Let P be the orthogonal projection of a Hilbert space H onto a closed subspace M. Show that if $\{x_n\}$ is an orthonormal basis for H, then $\{Px_n\}$ is a Parseval frame for M.

8.9. Given $\lambda \in \mathbf{R}$, let $e_\lambda(t) = e^{2\pi i \lambda t}$ for $t \in [0,1)$.

(a) Show that if $b > 1$, then $\overline{\text{span}}\{e_{bn}\}_{n\in\mathbf{Z}}$ is a proper subspace of $L^2(\mathbf{T})$. In particular, find a nonzero function in $L^2(\mathbf{T})$ that is orthogonal to e_{bn} for every $n \in \mathbf{Z}$.

(b) Show that if $0 < b < 1$ then $\{e_{bn}\}_{n\in\mathbf{Z}}$ is a tight frame for $L^2(\mathbf{T})$, and find the frame bound. Show directly that this sequence is not orthogonal, and demonstrate that it is not a basis by finding two distinct ways to write the constant function as $1 = \sum c_n e_{bn}$, where the series converges in L^2-norm. Show that $\{e_{bn}\}_{n\neq 0}$ is a frame for $L^2(\mathbf{T})$, and therefore $\{e_{bn}\}_{n\in\mathbf{Z}}$ is inexact.

8.10. (a) Prove the following perturbation result for frames. Suppose that $\{x_n\}$ is a frame for a Hilbert space H with frame bounds A, B, and $\{y_n\} \subseteq H$ is such that $\{f_n - g_n\}$ is a Bessel sequence with Bessel bound K. Show that $\{g_n\}$ is a frame if $K < A$.

(b) Show that if $\{h_n\}$ is a sequence in H that satisfies $K = \sum \|h_n\|^2 < \infty$, then $\{h_n\}$ is a Bessel sequence with Bessel bound K.

(c) Exercise 8.9 showed that $\{e^{2\pi i b n t}\}_{n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{T})$ when $0 < b \le 1$. Combine this with parts (a) and (b) of this exercise to formulate and prove a theorem establishing a sufficient condition on numbers $\lambda_n \in \mathbf{R}$ so that $\{e^{2\pi i \lambda_n t}\}_{n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{T})$. Do you think your result is optimal?

8.2 Frame Expansions and the Frame Operator

In this section we will show that a frame yields unconditionally convergent, basis-like representations of vectors in a Hilbert space. Note that frames are Bessel sequences, so all of the facts in Theorems 7.2 and 7.4 regarding Bessel sequences also apply to frames.

In the statement of the next result, we use the operator notation $U \le V$ introduced in Definition 2.14. Specifically, $U \le V$ if and only if $\langle Ux, x\rangle \le \langle Vx, x\rangle$ for every $x \in H$.

Theorem 8.13. *Let $\{x_n\}$ be a frame for a Hilbert space H with frame bounds A, B. Then the following statements hold.*

(a) *The frame operator S is a topological isomorphism of H onto itself, and $AI \le S \le BI$.*

(b) *S^{-1} is a topological isomorphism, and $B^{-1}I \le S^{-1} \le A^{-1}I$.*

(c) *$\{S^{-1}x_n\}$ is a frame for H with frame bounds B^{-1}, A^{-1}.*

(d) *For each $x \in H$,*

$$x = \sum_n \langle x, S^{-1}x_n\rangle\, x_n = \sum_n \langle x, x_n\rangle\, S^{-1}x_n, \tag{8.2}$$

and these series converge unconditionally in the norm of H.

(e) *If the frame is A-tight, then* $S = AI$, $S^{-1} = A^{-1}I$, *and*

$$\forall\, x \in H, \quad x \;=\; A^{-1} \sum \langle x, x_n \rangle\, x_n.$$

Proof. (a) Since $\{x_n\}$ is a Bessel sequence, the frame operator S is a continuous positive operator on H. Further, equation (7.2) tells us that $\langle Sx, x\rangle = \sum |\langle x, x_n\rangle|^2$. Since $\langle AIx, x\rangle = A\,\|x\|^2$, the frame definition can be rewritten as

$$\langle AIx, x\rangle \;\le\; \langle Sx, x\rangle \;\le\; \langle BIx, x\rangle, \qquad x \in H.$$

In operator notation, this says that $AI \le S \le BI$.

Applying the Cauchy–Bunyakovski–Schwarz Inequality to $\langle Sx, x\rangle$, we have

$$A\,\|x\|^2 \;=\; \langle AIx, x\rangle \;\le\; \langle Sx, x\rangle \;\le\; \|Sx\|\,\|x\|.$$

Hence $A\,\|x\| \le \|Sx\|$ for all $x \in H$, so it follows from Exercise 2.38 that S has closed range and $S\colon H \to \mathrm{range}(S)$ is a topological isomorphism. It therefore only remains to show that $\mathrm{range}(S) = H$.

Suppose that $y \in H$ was orthogonal to $\mathrm{range}(S)$, i.e., $\langle Sx, y\rangle = 0$ for every $x \in H$. Then $A\,\|y\|^2 = \langle AIy, y\rangle \le \langle Sy, y\rangle = 0$, so $y = 0$. Thus $\mathrm{range}(S)^\perp = \{0\}$, and since $\mathrm{range}(S)$ is a closed subspace we therefore have $\mathrm{range}(S) = \mathrm{range}(S)^{\perp\perp} = \{0\}^\perp = H$. Thus S is surjective.

(b) Since S is a positive topological isomorphism, the same is true of S^{-1}. Also, since $AI \le S$, we have $\langle AIy, y\rangle \le \langle Sy, y\rangle$ for every vector $y \in H$. Applying this with $y = S^{-1}x$, we see that

$$\begin{aligned} 0 \;\le\; A\,\|S^{-1}x\|^2 \;&=\; \big\langle AI(S^{-1}x),\, S^{-1}x\big\rangle \\ &\le\; \big\langle S(S^{-1}x),\, S^{-1}x\big\rangle \\ &=\; \big\langle x,\, S^{-1}x\big\rangle \\ &\le\; \|x\|\,\|S^{-1}x\|. \end{aligned}$$

Consequently $\|S^{-1}x\| \le A^{-1}\,\|x\|$, and therefore

$$\big\langle S^{-1}x,\, x\big\rangle \;\le\; \|S^{-1}x\|\,\|x\| \;\le\; A^{-1}\,\|x\|^2 \;=\; \big\langle A^{-1}Ix,\, x\big\rangle.$$

Hence $S^{-1} \le A^{-1}I$.

To prove the inequality $S^{-1} \ge B^{-1}I$, we will use Lemma 2.17. That lemma, which is the Cauchy–Bunyakovski–Schwarz Inequality applied to the inner product $(x, y) = \langle S^{-1}x, y\rangle$, tells us that

$$\langle S^{-1}u, v\rangle^2 \;\le\; \langle S^{-1}u, u\rangle\,\langle S^{-1}v, v\rangle, \qquad u, v \in H.$$

Taking $u = Sx$ and $v = x$, we therefore have

$$\begin{aligned} \|x\|^4 \;=\; \langle x, x\rangle^2 \;&=\; \big\langle S^{-1}(Sx),\, x\big\rangle^2 \\ &\le\; \big\langle S^{-1}(Sx),\, Sx\big\rangle\,\big\langle S^{-1}x,\, x\big\rangle \end{aligned}$$

$$\begin{aligned} &= \langle x, Sx\rangle \left\langle S^{-1}x,\, x\right\rangle \\ &\leq B\,\|x\|^2 \left\langle S^{-1}x,\, x\right\rangle. \end{aligned}$$

Hence $\langle S^{-1}x, x\rangle \geq B^{-1}\,\|x\|^2 = \langle B^{-1}Ix, x\rangle$, so $S^{-1} \geq B^{-1}I$.

(c) The fact that S is self-adjoint implies that S^{-1} is self-adjoint as well. Therefore,

$$\begin{aligned} \sum_n \langle x, S^{-1}x_n\rangle\, S^{-1}x_n &= \sum_n \langle S^{-1}x, x_n\rangle\, S^{-1}x_n \\ &= S^{-1}\Bigl(\sum_n \langle S^{-1}x, x_n\rangle\, x_n\Bigr) \\ &= S^{-1}S(S^{-1}x) \;=\; S^{-1}x. \end{aligned}$$

Consequently,

$$\sum_n |\langle x,\, S^{-1}x_n\rangle|^2 \;=\; \sum_n \left\langle x,\, S^{-1}x_n\right\rangle \left\langle S^{-1}x_n,\, x\right\rangle \;=\; \left\langle S^{-1}x,\, x\right\rangle.$$

Applying the fact that $B^{-1}I \leq S^{-1} \leq A^{-1}I$, we conclude that

$$\frac{1}{B}\,\|x\|^2 \;\leq\; \sum_n |\langle x,\, S^{-1}x_n\rangle|^2 \;\leq\; \frac{1}{A}\,\|x\|^2, \qquad x \in H.$$

(d) Since $\{x_n\}$ and $\{S^{-1}x_n\}$ are Bessel sequences, $\sum c_n x_n$ and $\sum c_n S^{-1}x_n$ converge unconditionally for each $(c_n) \in \ell^2$. Since $\bigl(\langle x, x_n\rangle\bigr)$ and $\bigl(\langle x, S^{-1}x_n\rangle\bigr)$ belong to ℓ^2 and S, S^{-1} are continuous, we have

$$x \;=\; S(S^{-1}x) \;=\; \sum_n \langle S^{-1}x, x_n\rangle\, x_n \;=\; \sum_n \langle x, S^{-1}x_n\rangle\, x_n$$

and

$$x \;=\; S^{-1}(Sx) \;=\; S^{-1}\Bigl(\sum_n \langle x, x_n\rangle\, x_n\Bigr) \;=\; \sum_n \langle x, x_n\rangle\, S^{-1}x_n,$$

with unconditional convergence of the series.

(e) This follows from the preceding statements, and was also established directly in Exercise 8.5. □

Thus each frame $\{x_n\}$ has a dual system $\{S^{-1}x_n\}$ associated to it that is also a frame. Unlike bases, this dual system need not be biorthogonal to $\{x_n\}$, and it need not be unique. We will explore these issues in more detail in the following sections.

Definition 8.14 (Canonical Dual Frame). Let $\{x_n\}$ be a frame with frame operator S. The frame $\{S^{-1}x_n\}$ is called the *canonical dual frame* or the *standard dual frame* for $\{x_n\}$. ◇

Notation 8.15. Given a frame $\{x_n\}$, we will write $\widetilde{x}_n = S^{-1}x_n$, so the canonical dual frame is $\{\widetilde{x}_n\}$. Thus if $\{x_n\}$ is a frame then

$$x = \sum_n \langle x, \widetilde{x}_n\rangle\, x_n = \sum_n \langle x, x_n\rangle\, \widetilde{x}_n, \qquad x \in H, \tag{8.3}$$

with unconditional convergence of these series. If we let C, R be the analysis and synthesis operators for $\{x_n\}$ and $\widetilde{C}$, $\widetilde{R}$ the analysis and synthesis operators for $\{\widetilde{x}_n\}$, then equation (8.3) says that

$$x = R\widetilde{C}x = \widetilde{R}Cx, \qquad x \in H.$$

That is, analysis followed by the appropriate synthesis is the identity (though synthesis followed by analysis will not be the identity in general).

Note that

$$\widetilde{R}\widetilde{C}x = \sum_n \langle x, \widetilde{x}_n\rangle\, \widetilde{x}_n = S^{-1}\Bigl(\sum_n \langle x, \widetilde{x}_n\rangle\, x_n\Bigr) = S^{-1}x,$$

so the frame operator for $\{\widetilde{x}_n\}$ is S^{-1}. Consequently, the canonical dual of $\{\widetilde{x}_n\}$ is $\{x_n\}$, because

$$\widetilde{\widetilde{x}}_n = (S^{-1})^{-1}\,\widetilde{x}_n = SS^{-1}x_n = x_n. \qquad \diamondsuit$$

Sometimes the canonical dual frame is simply referred to as "the dual frame," although this can be misleading because there are usually many dual frames in the following sense.

Definition 8.16 (Alternative Duals). Let $\{x_n\}$ be a frame for a Hilbert space H. A sequence $\{y_n\} \subseteq H$ such that

$$x = \sum_n \langle x, y_n\rangle\, x_n, \qquad x \in H,$$

is called an *alternative dual* of $\{x_n\}$. If $\{y_n\}$ is a frame (which need not be the case in general), then it is an *alternative dual frame*, or simply a *dual frame* for short, of $\{x_n\}$. $\diamondsuit$

For example, if $\{e_n\}$ is an orthonormal basis for H then the frame $\{e_1, e_1, e_2, e_2, \dots\}$ has infinitely many duals. The canonical dual frame is $\{e_1/2, e_1/2, e_2/2, e_2/2, \dots\}$, but $\{e_1, 0, e_2, 0, \dots\}$ and $\{0, e_1, 0, e_2, \dots\}$ are examples of alternative dual frames.

The canonical dual frame is certainly the dual that we encounter most often. However, in applications the use of an alternative dual can sometimes lead to frame expansions that are "better behaved" than those obtained using the canonical dual [LO04], [BLPY10]. To illustrate this, consider the frame $\{e_1, e_1, e_2, e_2, \dots\}$. If we use the alternative dual $\{e_1, 0, e_2, 0, \dots\}$ then at least "half" of the coefficients in the frame expansion will be zero, while if we use the canonical dual frame $\{e_1/2, e_1/2, e_2/2, e_2/2, \dots\}$ then typically every coefficient in the frame expansion will be nonzero. While this example is trivial, the need for "well-concentrated" frame coefficients is important in many applications. For a survey of *sparse representations* with extensive references, we refer to [BDE09].

In the remainder of this section we will derive some abstract properties of alternative duals. First we show that every inexact frame has multiple dual frames.

Lemma 8.17. *If $\{x_n\}$ is a frame for a Hilbert space H, then $\{x_n\}$ has a unique dual frame if and only if it is exact (and in this case the unique dual is the canonical dual).*

Proof. $\Rightarrow$. The easiest proof of this fact uses some machinery from Section 8.3, and so we assign it as Exercise 8.23.

$\Leftarrow$. Suppose that $\{x_n\}$ is an inexact frame, and let $\{\widetilde{x}_n\}$ denote its canonical dual frame. If $x_m = 0$ for some m then $\widetilde{x}_m = S^{-1}x_m = 0$, and in this case we can obtain a new dual frame by replacing $\widetilde{x}_m$ by any nonzero vector in H that we like.

So, consider the more interesting case where $x_n \neq 0$ for every n. By definition of exact frame, there exists some frame element x_m that can be removed from $\{x_n\}_{n\in\mathbf{N}}$ yet still leave a frame. That is, $\{x_n\}_{n\neq m}$ is a frame for H for some index m. This frame has a canonical dual frame, say $\{y_n\}_{n\neq m}$. Set $y_m = 0$. Then $\{y_n\}_{n\in\mathbf{N}}$ is a frame, and given $x \in H$ we have

$$\sum_{n=1}^{\infty} \langle x, y_n\rangle\, x_n \;=\; \langle x, y_m\rangle\, x_m \;+\; \sum_{n\neq m} \langle x, y_n\rangle\, x_n \;=\; 0 + x \;=\; x.$$

Thus $\{y_n\}_{n\in\mathbf{N}}$ is a dual frame for $\{x_n\}_{n\in\mathbf{N}}$, but it is not the canonical dual $\{\widetilde{x}_n\}_{n\in\mathbf{N}}$ since $y_m = 0$ while $\widetilde{x}_m = S^{-1}x_m \neq 0$. □

The proof of the following theorem is Exercise 8.16.

Theorem 8.18. *Let $\{x_n\}$ be a frame for a Hilbert space H.*

(a) *If $\{y_n\}$ is a Bessel sequence in H that is an alternative dual for $\{x_n\}$, then $\{y_n\}$ is a frame and $\{x_n\}$ is an alternative dual frame for $\{y_n\}$.*

(b) *A Bessel sequence $\{y_n\}$ is the canonical dual frame for $\{x_n\}$ if and only if it is an alternative dual and* $\mathrm{range}(C) = \mathrm{range}(\widetilde{C})$, *where C is the analysis operator for $\{x_n\}$ and $\widetilde{C}$ is the analysis operator for $\{y_n\}$.* ◇

The next result deals with the orthogonal projection of frames onto closed subspaces.

Theorem 8.19. *Let $\{x_n\}$ be a frame for a Hilbert space H, with canonical dual frame $\{\widetilde{x}_n\}$. Let P be the orthogonal projection of H onto a closed subspace M.*

(a) *$\{Px_n\}$ is a frame for M with the same frame bounds as $\{x_n\}$, and $\{P\widetilde{x}_n\}$ is an alternative dual frame for $\{Px_n\}$.*

(b) *$\{P\widetilde{x}_n\}$ is the canonical dual frame for $\{Px_n\}$ if and only if $PS = SP$, where S is the frame operator for $\{x_n\}$.*

Proof. (a) The proof that $\{Px_n\}$ is a frame relies on the fact that $Px = x$ for $x \in M$ and the self-adjointness of P, and is assigned as Exercise 8.17. By the same argument $\{P\widetilde{x}_n\}$ is a frame for M, and given $x \in M$ we have

$$\sum_n \langle x, P\widetilde{x}_n\rangle\, Px_n \;=\; P\biggl(\sum_n \langle Px, \widetilde{x}_n\rangle\, x_n\biggr) \;=\; P(Px) \;=\; x,$$

so $\{P\widetilde{x}_n\}$ is an alternative dual frame for $\{Px_n\}$.

(b) Suppose that $\{P\widetilde{x}_n\}$ is the canonical dual frame for $\{Px_n\}$. Let T be the frame operator for $\{Px_n\}$ as a frame for M. Then we have

$$T^{-1}(Px_n) \;=\; P\widetilde{x}_n \;=\; PS^{-1}x_n, \qquad n \in \mathbf{N}.$$

Since $\{x_n\}$ is complete and S^{-1}, T^{-1}, and P are continuous, it follows that $T^{-1}Px = PS^{-1}x$ for every $x \in H$. Since S is a topological isomorphism on H and T is a topological isomorphism on M, we therefore have

$$PS \;=\; TT^{-1}PS \;=\; TPS^{-1}S \;=\; TP.$$

Since P, S, and T are self-adjoint, by taking adjoints we obtain

$$SP \;=\; (PS)^* \;=\; (TP)^* \;=\; PT.$$

Hence $SP = SPP = PTP = PPS = PS$.

The converse implication is Exercise 8.17. □

If M is a closed subspace of a Hilbert space H, then we know how to use an orthonormal basis for M to represent the orthogonal projection of H onto M (see Theorem 1.49). The next result shows that we can use an arbitrary frame for M in place of an orthonormal basis to represent an orthogonal projection.

Theorem 8.20. *Let $\{x_n\}$ be a frame sequence in a Hilbert space H, and let $M = \overline{\operatorname{span}}\{x_n\}$. Let $\{\widetilde{x}_n\}$ be the canonical dual frame of $\{x_n\}$ as a frame for M. Then*

$$Px \;=\; \sum_n \langle x, \widetilde{x}_n\rangle\, x_n, \qquad x \in H, \tag{8.4}$$

is the orthogonal projection of H onto M.

Proof. Let P be the orthogonal projection of H onto M. To show that P has the form given in equation (8.4), observe that $P\widetilde{x}_n = \widetilde{x}_n$ since $\widetilde{x}_n \in M$. Given $x \in H$ we have $Px \in M$, so using frame expansions in M and the fact that P is self-adjoint it follows that

$$Px \;=\; \sum_n \langle Px, \widetilde{x}_n\rangle\, x_n \;=\; \sum_n \langle x, P\widetilde{x}_n\rangle\, x_n \;=\; \sum_n \langle x, \widetilde{x}_n\rangle\, x_n. \quad \square$$

Exercises

8.11. Let $v_1, \dots, v_n$ be vectors in $\mathbf{F}^d$. Give direct proofs of the following statements.

(a) $\{v_1, \dots, v_n\}$ is a Bessel sequence in $\mathbf{F}^d$, its synthesis operator corresponds to multiplication by the matrix R that has $v_1, \dots, v_n$ as columns, and its analysis operator is $C = R^*$, the Hermitian of the matrix R.

(b) The following are equivalent: (i) $\{v_1, \dots, v_n\}$ spans $\mathbf{F}^d$, (ii) $S = C^*C$ is positive definite, (iii) $\{v_1, \dots, v_n\}$ is a frame for $\mathbf{F}^d$. Further, in case these hold, the optimal frame bounds for $\{v_1, \dots, v_n\}$ are λ_1, λ_d, where λ_1 is the smallest eigenvalue of S and λ_d is the largest eigenvalue.

(c) The following are equivalent: (i) $\{v_1, \dots, v_n\}$ is linearly independent, (ii) $G = CC^*$ is positive definite, (iii) $\{v_1, \dots, v_n\}$ is a Riesz sequence in $\mathbf{F}^d$.

8.12. Let $\{v_1, \dots, v_N\}$ be a finitely many vectors in a Hilbert space H.

(a) Show that $\{v_1, \dots, v_N\}$ is a frame sequence, i.e., it is a frame for $\operatorname{span}\{v_1, \dots, v_N\}$.

(b) Assume $\{v_1, \dots, v_N\}$ is linearly independent, and let A, B be frame bounds for $\{v_1, \dots, v_N\}$ as a frame for its span. Show that

$$\forall\, c_1, \dots, c_N, \quad A \sum_{k=1}^{N} |c_k|^2 \le \left\| \sum_{k=1}^{N} c_k v_k \right\|^2 \le B \sum_{k=1}^{N} |c_k|^2.$$

8.13. A sequence $\{x_n\}$ in a Banach space X is called a *quasibasis* if there exist $a_n \in X^*$ such that $x = \sum \langle x, a_n \rangle \, x_n$ for all $x \in X$. All Schauder bases and frames are therefore quasibases. Exhibit a quasibasis for a Hilbert space that is neither a basis nor a frame.

8.14. Given a sequence $\{x_n\}$ in a Hilbert space H, show that $\{x_n\}$ is a frame with frame bounds A, B if and only if $Sx = \sum \langle x, x_n \rangle \, x_n$ is a well-defined positive linear mapping of H into H that satisfies $AI \le S \le BI$.

8.15. This exercise will give an alternative proof that the frame operator S is a topological isomorphism.

(a) Show that if $U, V \in \mathcal{B}(H)$ are positive and $U \le V$ then $\|U\| \le \|V\|$.

(b) Let S be the frame operator for a frame $\{x_n\}$ that has frame bounds A, B. Prove the operator inequalities

$$0 \;\le\; I - \frac{1}{B} S \;\le\; \frac{B-A}{B} I$$

and

$$0 \;\le\; I - \frac{2}{A+B} S \;\le\; \frac{B-A}{B+A} I,$$

and use either one of these to show that S is a topological isomorphism.

8.16. Prove Theorem 8.18.

8.17. Complete the proof of Theorem 8.19.

8.18. Let $\{x_n\}$ be a frame for a Hilbert space H. Let C, R, S be the analysis, synthesis, and frame operators for $\{x_n\}$, and let $\widetilde{C}$, $\widetilde{R}$ be the analysis and synthesis operators for the canonical dual frame $\{\widetilde{x}_n\}$.

(a) Show that $\widetilde{C} = CS^{-1}$ and $\widetilde{R} = S^{-1}R$.

(b) Show that the orthogonal projection P of ℓ^2 onto range(C) is

$$Pc \;=\; C\widetilde{R}c \;=\; CS^{-1}Rc = \biggl\{\biggl\langle \sum_n c_n \widetilde{x}_n,\; x_k \biggr\rangle\biggr\}_{k\in\mathbf{N}}, \qquad c = (c_n) \in \ell^2.$$

8.19. Let $\{x_n\}$ be a frame for a Hilbert space H and let $C\colon H \to \ell^2$ be its analysis operator. Let $\{\delta_n\}$ be the standard basis for ℓ^2, and prove that the following two statements are equivalent.

(a) $\{y_n\}$ is an alternative dual frame for $\{x_n\}$.

(b) There is a bounded left-inverse V of C such that $y_n = V\delta_n$ for all $n \in \mathbf{N}$ (that is, $V\colon \ell^2 \to H$ is bounded, $VC = I$, and $y_n = V\delta_n$).

8.20. Let $T \in \mathcal{B}(H)$ be a Hilbert–Schmidt operator on a Hilbert space H (see Definition B.7). Show that if $\{x_n\}$ is a frame for H with canonical dual frame $\{\widetilde{x}_n\}$, then the Hilbert–Schmidt norm of T is given by $\|T\|_{\mathrm{HS}}^2 = \sum \langle Tx_n, T\widetilde{x}_n\rangle$.

8.3 Overcompleteness

Now we will prove some results related to the uniqueness of frame expansions. Our first result shows that among all choices of scalars (c_n) for which $x = \sum c_n x_n$, the scalars $c_n = \langle x, \widetilde{x}_n\rangle$ associated with the canonical dual frame $\{\widetilde{x}_n\}$ have the minimal ℓ^2-norm.

Theorem 8.21. *Let $\{x_n\}$ be a frame for a Hilbert space H, and fix $x \in H$. If $x = \sum c_n x_n$ for some scalars (c_n), then*

$$\sum_n |c_n|^2 \;=\; \sum_n \bigl|\langle x, \widetilde{x}_n\rangle\bigr|^2 \;+\; \sum_n \bigl|\langle x, \widetilde{x}_n\rangle - c_n\bigr|^2.$$

In particular, the sequence $\bigl(\langle x, \widetilde{x}_n\rangle\bigr)$ has the minimal ℓ^2-norm among all such sequences (c_n).

Proof. By equation (8.2) we have $x = \sum a_n x_n$ where $a_n = \langle x, \widetilde{x}_n\rangle$. Let (c_n) be any sequence of scalars such that $x = \sum c_n x_n$. Since $\sum |a_n|^2 < \infty$, there is nothing to prove if $\sum |c_n|^2 = \infty$, so we may assume that $(c_n) \in \ell^2$. Using the self-adjointness of S^{-1}, we compute that

$$\begin{aligned}
\langle x, S^{-1}x\rangle &= \left\langle \sum_n a_n x_n,\ S^{-1}x \right\rangle \\
&= \sum_n a_n \langle \widetilde{x}_n, x\rangle \\
&= \sum_n a_n \overline{a_n} \ = \ \big\langle (a_n),\, (a_n) \big\rangle_{\ell^2}
\end{aligned}$$

and

$$\begin{aligned}
\langle x, S^{-1}x\rangle &= \left\langle \sum_n c_n x_n,\ S^{-1}x \right\rangle \\
&= \sum_n c_n \langle \widetilde{x}_n, x\rangle \\
&= \sum_n c_n \overline{a_n} \ = \ \big\langle (c_n),\, (a_n) \big\rangle_{\ell^2}.
\end{aligned}$$

Therefore $(c_n - a_n)$ is orthogonal to (a_n) in ℓ^2, so by the Pythagorean Theorem we have

$$\|(c_n)\|_{\ell^2}^2 \ = \ \|(c_n - a_n) + (a_n)\|_{\ell^2}^2 \ = \ \|(c_n - a_n)\|_{\ell^2}^2 \ + \ \|(a_n)\|_{\ell^2}^2. \quad \square$$

As a consequence, we obtain a remarkable formula for $\sum_{n \neq m} |\langle x_m, \widetilde{x}_n\rangle|^2$. This formula and its consequences will play an important role in characterizing the class of exact frames in Section 8.4. Compare parts (b) and (c) of the next result to Exercise 8.6, which obtains similar results for tight frames.

Theorem 8.22. *Let $\{x_n\}$ be a frame for a Hilbert space H.*

(a) *For each $m \in \mathbf{N}$,*

$$\sum_{n \neq m} |\langle x_m, \widetilde{x}_n\rangle|^2 \ = \ \frac{1 \ - \ |\langle x_m, \widetilde{x}_m\rangle|^2 \ - \ |1 - \langle x_m, \widetilde{x}_m\rangle|^2}{2}. \tag{8.5}$$

(b) *If $\langle x_m, \widetilde{x}_m\rangle = 1$, then $\langle x_m, \widetilde{x}_n\rangle = 0$ for $n \neq m$.*

(c) *The removal of a vector from a frame leaves either a frame or an incomplete set. Specifically,*

$$\begin{aligned}
\langle x_m, \widetilde{x}_m\rangle \neq 1 \quad &\Longrightarrow \quad \{x_n\}_{n \neq m} \text{ is a frame}, \\
\langle x_m, \widetilde{x}_m\rangle = 1 \quad &\Longrightarrow \quad \{x_n\}_{n \neq m} \text{ is incomplete.}
\end{aligned}$$

Proof. (a) Fix any m, and let $a_n = \langle x_m, \widetilde{x}_n\rangle$. Then $x_m = \sum a_n x_n$ by equation (8.2). However, we also have $x_m = \sum \delta_{mn} x_n$, so Theorem 8.21 implies that

$$1 = \sum_n |\delta_{mn}|^2 = \sum_n |a_n|^2 + \sum_n |a_n - \delta_{mn}|^2$$
$$= |a_m|^2 + \sum_{n \neq m} |a_n|^2 + |a_m - 1|^2 + \sum_{n \neq m} |a_n|^2.$$

Rearranging, we find that

$$\sum_{n \neq m} |a_n|^2 = \frac{1 - |a_m|^2 - |a_m - 1|^2}{2}.$$

(b) Suppose that $\langle x_m, \widetilde{x}_m \rangle = 1$. Then $\sum_{n \neq m} |\langle x_m, \widetilde{x}_n \rangle|^2 = 0$ by equation (8.5). Hence $\langle \widetilde{x}_m, x_n \rangle = 0$ for $n \neq m$.

(c) Suppose that $\langle x_m, \widetilde{x}_m \rangle = 1$. Then by part (b), $\widetilde{x}_m$ is orthogonal to x_n for every $n \neq m$. However, $\widetilde{x}_m \neq 0$ since $\langle \widetilde{x}_m, x_m \rangle = 1 \neq 0$. Therefore $\{x_n\}_{n \neq m}$ is incomplete in this case.

On the other hand, suppose that $\langle x_m, \widetilde{x}_m \rangle \neq 1$, and set $a_n = \langle x_m, \widetilde{x}_n \rangle$. We have $x_m = \sum a_n x_n$ by equation (8.2). Since $a_m \neq 1$, we therefore have $x_m = \frac{1}{1-a_m} \sum_{n \neq m} a_n x_n$. Hence, for each $x \in H$,

$$|\langle x, x_m \rangle|^2 = \left| \frac{1}{1 - a_m} \sum_{n \neq m} a_n \langle x, x_n \rangle \right|^2 \leq C \sum_{n \neq m} |\langle x, x_n \rangle|^2,$$

where $C = |1 - a_m|^{-2} \sum_{n \neq m} |a_n|^2 > 0$. Therefore,

$$\sum_n |\langle x, x_n \rangle|^2 = |\langle x, x_m \rangle|^2 + \sum_{n \neq m} |\langle x, x_n \rangle|^2 \leq (1 + C) \sum_{n \neq m} |\langle x, x_n \rangle|^2.$$

Hence, if we let A, B be frame bounds for $\{x_n\}$ then

$$\frac{A}{1+C} \|x\|^2 \leq \frac{1}{1+C} \sum_n |\langle x, x_n \rangle|^2 \leq \sum_{n \neq m} |\langle x, x_n \rangle|^2 \leq B \|x\|^2,$$

so $\{x_n\}_{n \neq m}$ is a frame with frame bounds $A/(1 + C)$, B. □

As a corollary, we find that a frame is exact if and only if it is biorthogonal to its dual frame. We assign the proof as Exercise 8.21 (compare Exercise 8.7 for the case of tight frames).

Corollary 8.23. *If $\{x_n\}$ is a frame for a Hilbert space H, then the following statements are equivalent.*

(a) *$\{x_n\}$ is an exact frame.*

(b) *$\{x_n\}$ and $\{\widetilde{x}_n\}$ are biorthogonal.*

(c) *$\langle x_n, \widetilde{x}_n \rangle = 1$ for all n.*

Consequently, if $\{x_n\}$ is an A-tight frame, then the following statements are equivalent.

(a') $\{x_n\}$ *is an exact frame.*

(b') $\{x_n\}$ *is an orthogonal basis for H.*

(c') $\|x_n\|^2 = A$ *for all n.* ◊

In Example 8.6(d), we constructed a frame that is not norm-bounded below. The following result shows that all frames are norm-bounded above, and all exact frames are norm-bounded below (the proof is Exercise 8.22).

Corollary 8.24. *If $\{x_n\}$ is a frame for a Hilbert space H with frame bounds A, B, then the following statements hold.*

(a) $\{x_n\}$ *is norm-bounded above, with $\|x_n\|^2 \le B$ for every $n \in \mathbf{N}$.*

(b) *If $\{x_n\}$ is exact then it is norm-bounded below, with $A \le \|x_n\|^2$ for every $n \in \mathbf{N}$.* ◊

Exercises

8.21. Prove Corollary 8.23.

8.22. Prove Corollary 8.24.

8.23. Show that if $\{x_n\}$ is an exact frame for a Hilbert space H then it has a unique dual frame (the canonical dual).

8.4 Frames and Bases

Now we will determine the exact relationship that holds between frames and bases. Our approach in this section follows [HW89]; see [You01] for an alternative approach.

Theorem 8.25. *An inexact frame is not a basis.*

Proof. Assume that $\{x_n\}$ is an inexact frame. Then by definition, $\{x_n\}_{n \ne m}$ is a frame for some m, and therefore is complete. However, no proper subset of a basis can be complete, so $\{x_n\}$ cannot be a basis. Alternatively, we can see this directly by writing $x_m = \sum \langle x_m, \widetilde{x}_n \rangle \, x_n$ and $x_m = \sum \delta_{mn} x_n$. By Theorem 8.22, the fact that $\{x_n\}_{n \ne m}$ is a frame implies that $\langle x_m, \widetilde{x}_m \rangle \ne 1$. Hence we have two distinct representations of x_m in terms of the frame elements, so $\{x_n\}$ cannot be a basis. □

We observe next that frames are preserved by topological isomorphisms (see Exercise 8.24).

Lemma 8.26. *Frames are preserved by topological isomorphisms. Specifically, if $\{x_n\}$ is a frame for a Hilbert space H and $T\colon H \to K$ is a topological isomorphism, then $\{Tx_n\}$ is a frame for K and the following additional statements hold as well.*

(a) *If A, B are frame bounds for $\{x_n\}$, then $\{Tx_n\}$ has frame bounds $A\,\|T^{-1}\|^{-2}$, $B\,\|T\|^2$.*

(b) *If S is the frame operator for $\{x_n\}$, then $\{Tx_n\}$ has frame operator TST^*.*

(c) *$\{x_n\}$ is exact if and only if $\{Tx_n\}$ is exact.* $\diamondsuit$

We can now show that the class of exact frames for H coincides with the class of bounded unconditional bases for H. By Theorem 7.13, this further coincides with the class of Riesz bases for H.

Theorem 8.27. *Let $\{x_n\}$ be a sequence in a Hilbert space H. Then $\{x_n\}$ is an exact frame for H if and only if it is a bounded unconditional basis for H.*

Proof. $\Rightarrow$. Assume that $\{x_n\}$ is an exact frame for H. Then $\{x_n\}$ is norm-bounded both above and below by Corollary 8.24. We have from equation (8.2) that $x = \sum \langle x, \widetilde{x}_n\rangle\, x_n$ for all x, with unconditional convergence of this series. Corollary 8.23 implies that $\{x_n\}$ and $\{\widetilde{x}_n\}$ are biorthogonal, so this representation is unique, and therefore $\{x_n\}$ is a bounded unconditional basis for H.

$\Leftarrow$. Assume that $\{x_n\}$ is a bounded unconditional basis for H. Then $\{x_n\}$ is a Riesz basis by Theorem 7.13. Therefore, by the definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$ for all n. However, $\{e_n\}$ is an exact frame and exact frames are preserved by topological isomorphisms (Lemma 8.26), so $\{x_n\}$ must be an exact frame for H. $\square$

We can explicitly exhibit the topological isomorphism T that appears in the proof of Theorem 8.27. Since S is a positive operator that is a topological isomorphism of H onto itself, it has a square root $S^{1/2}$ that is a positive topological isomorphism (see Theorem 2.18). Similarly, S^{-1} has a square root $S^{-1/2}$, and it is easy to verify that $(S^{1/2})^{-1} = S^{-1/2}$. Since $\{x_n\}$ is exact, $\{x_n\}$ and $\{\widetilde{x}_n\}$ are biorthogonal by Corollary 8.23. Therefore,

$$\langle S^{-1/2}x_m, S^{-1/2}x_n\rangle \;=\; \langle x_m, S^{-1/2}S^{-1/2}x_n\rangle \;=\; \langle x_m, \widetilde{x}_n\rangle \;=\; \delta_{mn}.$$

Thus $\{S^{-1/2}x_n\}$ is an orthonormal sequence. Moreover, it is complete since topological isomorphisms preserve complete sequences. Therefore $\{S^{-1/2}x_n\}$ is an orthonormal basis for H, and the topological isomorphism $T = S^{1/2}$ maps this orthonormal basis onto the frame $\{x_n\}$.

We can consider the sequence $\{S^{-1/2}x_n\}$ for any frame, not just exact frames. If $\{x_n\}$ is inexact then $\{S^{-1/2}x_n\}$ will not be an orthonormal basis for H, but we show next that it will be a Parseval frame for H.

Corollary 8.28. *Let $\{x_n\}$ be a frame for a Hilbert space H.*

(a) *$S^{-1/2}$ is a topological isomorphism of H onto itself, and $\{S^{-1/2}x_n\}$ is a Parseval frame for H.*

(b) *$\langle x_n, \widetilde{x}_n\rangle = \|S^{-1/2}x_n\|^2$, and $0 \le \langle x_n, \widetilde{x}_n\rangle \le 1$ for every n.*

(c) *$\{x_n\}$ is an exact frame if and only if $\{S^{-1/2}x_n\}$ is an orthonormal basis for H.*

Proof. (a) $S^{-1/2}$ is a topological isomorphism because S is. Since frames are preserved by topological isomorphisms, we therefore have that $\{S^{-1/2}x_n\}$ is a frame for H. For each $x \in H$,

$$\sum_n \langle x, S^{-1/2}x_n\rangle\, S^{-1/2}x_n \;=\; S^{-1/2}SS^{-1/2}x \;=\; x \;=\; Ix,$$

and it follows from this that $\{S^{-1/2}x_n\}$ is a Parseval frame.

(b) Since $S^{-1/2}$ is self-adjoint, we have

$$\langle x_n, \widetilde{x}_n\rangle \;=\; \big\langle x_n,\, S^{-1}x_n\big\rangle \;=\; \big\langle S^{-1/2}x_n,\, S^{-1/2}x_n\big\rangle \;=\; \|S^{-1/2}x_n\|^2.$$

Since $\{S^{-1/2}x_n\}$ is 1-tight, it follows from Corollary 8.24 that $\|S^{-1/2}x_n\|^2 \le 1$ for every n.

(c) Since $S^{-1/2}$ is a topological isomorphism, $\{x_n\}$ is exact if and only if $\{S^{-1/2}x_n\}$ is exact. Statement (c) follows by combining this with the fact that a Parseval frame is exact if and only if it is an orthonormal basis (see Exercise 8.7). □

We call $\{S^{-1/2}x_n\}$ the *canonical Parseval frame* associated with $\{x_n\}$.

Exercises

8.24. Prove Lemma 8.26.

8.25. Let $\{x_n\}$ be a frame for a Hilbert space H. Show that if $\{x_n\}$ is a Riesz basis for H and M is a proper closed subspace of H, then $\{Px_n\}$ is a frame but is not a Riesz basis for M, where P denotes the orthogonal projection of H onto M.

8.5 Characterizations of Frames

In this section we will prove some equivalent formulations of frames and Riesz bases in terms of the analysis and synthesis operators, and as images of orthonormal bases for H.

Theorem 8.29. *Given a sequence* $\{x_n\}$ *in a Hilbert space* H, *the following statements are equivalent.*

(a) $\{x_n\}$ *is a frame for* H.

(b) *There exist* $A,\, B > 0$ *such that* $A\,\|x\|^2 \le \|Cx\|_{\ell^2}^2 \le B\,\|x\|^2$ *for every* $x \in H$.

(c) *The analysis operator* $Cx = \big(\langle x, x_n\rangle\big)$ *maps* H *bijectively onto a closed subspace of* ℓ^2.

(d) *The synthesis operator* $Rc = \sum c_n x_n$ *is well defined for each* $c = (c_n) \in \ell^2$, *and maps* ℓ^2 *onto* H.

(e) $\{x_n\}$ *is a complete Bessel sequence, and there exist* $A,\, B > 0$ *such that* $A\,\|c\|_{\ell^2}^2 \le \|Rc\|^2 \le B\,\|c\|_{\ell^2}^2$ *for* $c \in \ker(R)^\perp$, *i.e.,*

$$A \sum_n |c_n|^2 \le \Big\| \sum_n c_n x_n \Big\|^2 \le B \sum_n |c_n|^2$$

for those $c = (c_n)$ *that are contained in* $\ker(R)^\perp$.

In case these hold, the constants A, B *appearing in statements* (b) *and* (e) *are frame bounds for* $\{x_n\}$. *Further,* $A = \|R^\dagger\|^{-2}$ *is a lower frame bound for* $\{x_n\}$, *where* $R^\dagger$ *is the pseudoinverse of the synthesis operator* R.

Proof. (a) $\Leftrightarrow$ (b). Statement (b) is simply a restatement of the definition of a frame.

(b) $\Rightarrow$ (c). Suppose that there exist constants $A,\, B > 0$ such that $A\,\|x\|^2 \le \|Cx\|_{\ell^2}^2 \le B\,\|x\|^2$ for every $x \in H$. Then $Cx \in \ell^2$, C is a bounded injective mapping of H into ℓ^2, and C has closed range by Exercise 2.38.

(c) $\Rightarrow$ (b). Suppose that C maps H injectively into ℓ^2 and has closed range. Then $\sum |\langle x, x_n\rangle|^2 = \|Cx\|_{\ell^2}^2 < \infty$ for every x, so $\{x_n\}$ is a Bessel sequence. Theorem 7.2 therefore implies that C is a bounded mapping of H into ℓ^2. As C is injective and has closed range, it is a bounded bijection of H onto the Hilbert space range(C). Consequently, by the Inverse Mapping Theorem, $C\colon H \to \mathrm{range}(C)$ is a topological isomorphism. By Exercise 2.38, there exists a constant $A > 0$ such that $A\,\|x\|^2 \le \|Cx\|_{\ell^2}^2$ for all $x \in H$. Since C is bounded, we also have $\|Cx\|_{\ell^2}^2 \le \|C\|^2\,\|x\|^2$, so statement (b) holds.

(b) $\Rightarrow$ (e). If statement (b) holds, then $\{x_n\}$ is a Bessel sequence with Bessel bound B, and the analysis operator C has closed range. Therefore $\ker(R)^\perp = \mathrm{range}(R^*) = \mathrm{range}(C)$, and Theorem 7.2 tells us that the synthesis operator R is well defined on ℓ^2 with $\|R\|^2 \le B$. Therefore $\|Rc\|^2 \le B\,\|c\|_{\ell^2}^2$ for all $c \in \ell^2$, so the upper inequality in statement (e) holds not only on $\ker(R)^\perp$, but on all of ℓ^2.

For the lower inequality, fix $c \in \ker(R)^\perp = \mathrm{range}(C)$. Then $c = Cx$ for some $x \in H$, so by using the fact $R = C^*$ and the inequality $A\,\|x\|^2 \le \|Cx\|_{\ell^2}^2$ we obtain

$$\begin{aligned} A\,\|c\|_{\ell^2}^4 \;=\; A\,|\langle c, c\rangle|^2 \;&=\; A\,|\langle c, Cx\rangle|^2 \\ &=\; A\,|\langle Rc, x\rangle|^2 \\ &\le\; A\,\|Rc\|^2\,\|x\|^2 \\ &\le\; \|Rc\|^2\,\|Cx\|_{\ell^2}^2 \\ &=\; \|Rc\|^2\,\|c\|_{\ell^2}^2. \end{aligned}$$

This gives the lower inequality $A\,\|c\|_{\ell^2}^2 \le \|Rc\|^2$ for $c \in \ker(R)^\perp$.

(e) $\Rightarrow$ (d). Suppose that statement (e) holds. Then R maps ℓ^2 into H since $\{x_n\}$ is Bessel, and R has dense range since $\{x_n\}$ is complete (see Theorem 7.2). Therefore, to show that statement (d) holds, we need to show that R is surjective.

Choose any $y \in H$. Then since range(R) is dense in H, we can find vectors $y_n \in \text{range}(R)$ such that $y_n \to y$. For each vector y_n, we can choose a sequence $c_n \in \ker(R)^\perp$ such that $Rc_n = y_n$. By hypothesis,

$$A\,\|c_m - c_n\|_{\ell^2}^2 \;\le\; \|Rc_m - Rc_n\|^2 \;=\; \|y_m - y_n\|^2,$$

so $\{c_n\}$ is a Cauchy sequence in ℓ^2. Hence there exists some sequence $c \in \ell^2$ such that $\|c - c_n\|_{\ell^2} \to 0$, and therefore $Rc_n \to Rc$ in H since R is continuous. However, $Rc_n = y_n \to y$, so $y = Rc \in \text{range}(R)$. Thus R is surjective.

(d) $\Rightarrow$ (a). Suppose that R is well defined and maps ℓ^2 onto H. Then $\{x_n\}$ is a Bessel sequence by Theorem 7.4, so we only need to establish the existence of a lower frame bound. Since R has closed range, Theorem 2.33 implies that it has a bounded pseudoinverse $R^\dagger\colon H \to \ell^2$ that satisfies $RR^\dagger x = x$ for every $x \in \text{range}(R) = H$. Write the components of the sequence $R^\dagger x$ as $R^\dagger x = \big((R^\dagger x)_n\big)$. Then we have

$$x \;=\; RR^\dagger x \;=\; \sum_n (R^\dagger x)_n\, x_n, \qquad x \in H,$$

so

$$\begin{aligned} \|x\|^4 \;=\; |\langle x, x\rangle|^2 \;&=\; \left|\sum_n (R^\dagger x)_n\, \langle x, x_n\rangle\right|^2 \\ &\le\; \left(\sum_n |(R^\dagger x)_n|^2\right)\left(\sum_n |\langle x, x_n\rangle|^2\right) \\ &=\; \big\|R^\dagger x\big\|_{\ell^2}^2 \sum_n |\langle x, x_n\rangle|^2 \\ &\le\; \big\|R^\dagger\big\|^2\,\|x\|^2 \sum_n |\langle x, x_n\rangle|^2. \end{aligned}$$

Rearranging, we find that $\{x_n\}$ has a lower frame bound of $\|R^\dagger\|^{-2}$.

Proof of the additional statements. Assume that statements (a)–(e) hold. The numbers A, B in statement (b) are clearly frame bounds for $\{x_n\}$, and the proof of (d) $\Rightarrow$ (a) showed that $\|R^\dagger\|^{-2}$ is a lower frame bound. Hence the remaining issue is to show that the constants A, B appearing in statement (e) are frame bounds for $\{x_n\}$.

Let S be the frame operator and $\{\widetilde{x}_n\}$ the canonical dual frame. Given $x \in H$ we have

$$\big(\langle x, \widetilde{x}_n\rangle\big) \;=\; \big(\langle S^{-1}x, x_n\rangle\big) \;=\; C(S^{-1}x) \;\in\; \mathrm{range}(C) \;=\; \ker(R)^\perp,$$

where we have used the fact that range(C) is closed. Therefore, by statement (e),

$$A \sum_n |\langle x, \widetilde{x}_n\rangle|^2 \;\le\; \Big\|\sum_n \langle x, \widetilde{x}_n\rangle\, x_n\Big\|^2 \;=\; \|x\|^2.$$

Hence $\{\widetilde{x}_n\}$ has an upper frame bound of A^{-1}, and a similar calculation shows it has a lower frame bound of B^{-1}. Since $\{x_n\}$ is the canonical dual of $\{\widetilde{x}_n\}$, Theorem 8.13 implies that A, B are frame bounds for $\{x_n\}$. □

We can also characterize frames as the image of orthonormal bases under surjective maps. We state the next result for infinite-dimensional spaces, and discuss the modification needed for finite-dimensional spaces after the proof.

Corollary 8.30. *Let H be a separable, infinite-dimensional Hilbert space. Then a sequence $\{x_n\}$ is a frame for H if and only if there exists an orthonormal basis $\{e_n\}$ for H and a surjective operator $T \in \mathcal{B}(H)$ such that $Te_n = x_n$ for every $n \in \mathbf{N}$.*

Proof. $\Rightarrow$. Suppose $\{x_n\}$ is a frame for H. Let $\{\delta_n\}$ be the standard basis for ℓ^2, and let $\{e_n\}$ be any orthonormal basis for H. Since H and ℓ^2 are each separable, infinite-dimensional Hilbert spaces, there exists an isometric isomorphism $U\colon H \to \ell^2$ such that $Ue_n = \delta_n$ for each n. By Theorem 8.29, the synthesis operator R is a bounded map of ℓ^2 onto H, and we have $R\delta_n = x_n$ for every n. Therefore $UR\colon H \to H$ is surjective and satisfies $URe_n = x_n$.

$\Leftarrow$. If $\{x_n\}$ is the image of an orthonormal basis $\{e_n\}$ under a bounded surjective map T, then $\{x_n\}$ is a Bessel sequence by Exercise 7.6, and therefore the synthesis operator R maps ℓ^2 boundedly into H. Choose any $y \in H$. Then $y = Tx$ for some x since T is surjective. Since $\{e_n\}$ is an orthonormal basis for H, we have $x = \sum \langle x, e_n\rangle\, e_n$. Since T is bounded, it follows that

$$y \;=\; Tx \;=\; \sum_n \langle x, e_n\rangle\, Te_n \;=\; \sum_n \langle x, e_n\rangle\, x_n \;=\; Rc,$$

where $c = \big(\langle x, e_n\rangle\big) \in \ell^2$. Hence R is surjective, and therefore $\{x_n\}$ is a frame by Theorem 8.29. □

Suppose that $\{x_1, \dots, x_m\}$ is a frame for a d-dimensional Hilbert space H. By Exercise 8.11, this happens if $\{x_1, \dots, x_m\}$ is a spanning set, so we must have $m \geq n$. If $m > n$ then $\{x_1, \dots, x_m\}$ is a redundant frame, and in this case H and $\ell^2(\{1, \dots, m\})$ are not isomorphic since they have different dimensions. Hence the proof of Corollary 8.30 does not quite carry over to the finite-dimensional setting. On the other hand, we simply have to modify the statement of the corollary as follows.

Corollary 8.31. *Let H be a d-dimensional Hilbert space, and let K be an m-dimensional Hilbert space. Then a sequence $\{x_1, \dots, x_m\}$ is a frame for H if and only if there exists an orthonormal basis $\{e_1, \dots, e_m\}$ for K and a surjective linear operator $T\colon K \to H$ such that $Te_n = x_n$ for every $n = 1, \dots, m$.* ◇

Since a Riesz basis is a frame with extra "independence" properties, we expect that we can give a characterization of Riesz bases that is similar in flavor to Theorem 8.29. This is given in the next result, which complements the characterizations of Riesz bases given in Theorem 7.13. We assign the proof of Theorem 8.32 as Exercise 8.27.

Theorem 8.32. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following statements are equivalent.*

(a) *$\{x_n\}$ is a Riesz basis for H.*

(b) *The analysis operator $Cx = \big(\langle x, x_n\rangle\big)$ maps H bijectively onto ℓ^2.*

(c) *The synthesis operator $Rc = \sum c_n x_n$ is well defined for each $c = (c_n) \in \ell^2$, and maps ℓ^2 bijectively onto H.*

(d) *There exists an orthonormal basis $\{e_n\}$ for H and a bounded bijection $T \in \mathcal{B}(H)$ such that $Te_n = x_n$ for every $n \in \mathbf{N}$.*

(e) *$\{x_n\}$ is an ω-independent frame.*

(f) *$\{x_n\}$ is an ℓ^2-*independent frame*, i.e., it is a frame and if $\sum c_n x_n = 0$ for some sequence $(c_n) \in \ell^2$ then $c_n = 0$ for every n.* ◇

Note the subtle difference between statements (e) and (f) in Theorem 8.32. Whereas ω-independence requires that $c_n = 0$ for all n whenever (c_n) is *any* sequence such that $\sum c_n x_n$ converges and equals zero, ℓ^2-independence only requires that this conclusion hold under the additional assumption that (c_n) belongs to ℓ^2. In general, ω-independence is not equivalent to ℓ^2-independence, even for Bessel sequences (see Exercise 8.28).

Using Theorem 8.29, we will show that every frame is equivalent (in the sense of topological isomorphism) to a particular kind of frame sequence in ℓ^2. This result is due to Holub [Hol94] (see also Aldroubi [Ald95]).

Corollary 8.33. *Let $\{x_n\}$ be a sequence in a Hilbert space H, and let $\{\delta_n\}$ be the standard basis for ℓ^2.*

(a) $\{x_n\}$ *is a frame for* H *if and only if there exists a closed subspace* M *of* ℓ^2 *and a topological isomorphism* $T\colon M \to H$ *such that* $x_n = TP_M\delta_n$ *for every* n, *where* P_M *is the orthogonal projection of* ℓ^2 *onto* M.

(b) $\{x_n\}$ *is a Parseval frame for* H *if and only if we can take the operator* T *in part* (a) *to be an isometric isomorphism.*

Further, in case either of these statements holds we can take the subspace M *to be* $M = \mathrm{range}(C)$, *where* C *is the analysis operator for* $\{x_n\}$.

Proof. Suppose that $\{x_n\}$ is a frame for H. Let $M = \ker(R)^\perp = \mathrm{range}(C)$, and define $T\colon M \to H$ by $Tc = Rc$ for $c \in M$. Then T is a bounded bijection, and hence is a topological isomorphism. Further, $\ker(P_M) = \ker(R) = M^\perp$, so it follows that $RP_M\delta_n = R\delta_n$, and therefore $TP_M\delta_n = RP_M\delta_n = R\delta_n = x_n$.

We assign the remainder of the proof as Exercise 8.29. □

Viewing Corollary 8.33 from a different angle, we obtain the *Naimark Duality Theorem* for frames, which states that all frames can be obtained as orthogonal projections of Riesz bases (compare this result to Corollaries 8.30 and 8.31). The proof of the next result is Exercise 8.30.

Corollary 8.34 (Naimark Duality). *Let* $\{x_n\}$ *be a sequence in a Hilbert space* H.

(a) $\{x_n\}$ *is a frame for* H *if and only if there exists a Hilbert space* $K \supseteq H$ *and a Riesz basis* $\{e_n\}$ *for* K *such that* $P_H e_n = x_n$ *for each* n, *where* P_H *is the orthogonal projection of* K *onto* H.

(b) $\{x_n\}$ *is a Parseval frame for* H *if and only if there exists a Hilbert space* $K \supseteq H$ *and an orthonormal basis* $\{e_n\}$ *for* K *such that* $P_H e_n = x_n$ *for each* n. ♢

Corollary 8.33 is so named because it can be derived from *Naimark's Dilation Theorem*, see the paper [Cza08] by Czaja for references. It has been independently discovered several times, and appears to have been first stated explicitly by Han and Larson [HL00].

Exercises

8.26. Let $\{x_n\}$ be a frame for a Hilbert space H. Let C, R be the analysis and synthesis operators for $\{x_n\}$, and let $\widetilde{C}$, $\widetilde{R}$ be the analysis and synthesis operators for the canonical dual frame $\{\widetilde{x}_n\}$.

(a) Show that $\mathrm{range}(C) = \mathrm{range}(\widetilde{C})$.

(b) Show that the pseudoinverse $R^\dagger$ of the synthesis operator R is $R^\dagger = \widetilde{C}$, so

$$R^\dagger x \;=\; \widetilde{C}x \;=\; \bigl\{\langle x, \widetilde{x}_n\rangle\bigr\} \;=\; \bigl\{\langle x, S^{-1}x_n\rangle\bigr\}, \qquad x \in H.$$

(c) Show that the optimal frame bounds for $\{x_n\}$ are

$$A = \|\widetilde{C}\|^{-2} = \|S^{-1}\|^{-1} \qquad \text{and} \qquad B = \|C\|^2 = \|S\|.$$

8.27. Prove Theorem 8.32.

8.28. Let $\{x_n\}$ be a Bessel sequence in a Hilbert space H, and let $R\colon \ell^2 \to H$ be its synthesis operator.

(a) Show that $\{x_n\}$ is complete if and only if R has dense range.

(b) Show that $\{x_n\}$ is ℓ^2-independent if and only if R is injective.

(c) Now assume that $\{x_n\}$ is a Bessel sequence that is a conditional Schauder basis for H (see Exercise 7.1 for an example). Show that there exists some $x \in H$ such that $\big(\langle x, \widetilde{x}_n\rangle\big) \notin \ell^2$, where $(\widetilde{x}_n)$ is the dual basis to $\{x_n\}$. Show that the sequence $\{x_n\} \cup \{x\}$ is ℓ^2-independent but not ω-independent.

8.29. Prove the remaining statements in Corollary 8.33.

8.30. Prove Corollary 8.34.

8.31. (a) Exhibit a frame for $\mathbf{F}^d$ that contains infinitely many nonzero vectors.

(b) Show that if $\{x_n\}_{n\in I}$ is a frame for $\mathbf{F}^d$ and $\inf_{n\in I} \|x_n\| > 0$, then I is finite.

8.32. Let $\{x_n\}$ be a frame for a Hilbert space H and let $\{y_n\}$ be a frame for a Hilbert space K. Inspired by Definition 4.19, declare $\{x_n\}$ and $\{y_n\}$ to be *equivalent* if there exists a topological isomorphism $T\colon H \to K$ such that $Tx_n = y_n$ for every n.

(a) Show that $\{x_n\}$ and $\{y_n\}$ are equivalent if and only if $\mathrm{range}(C_X) = \mathrm{range}(C_Y)$, where C_X, C_Y are the analysis operators for $\{x_n\}$ and $\{y_n\}$.

(b) Show that part (a) is equivalent to:

$$\forall\, (c_n) \in \ell^2, \quad \sum c_n x_n = 0 \iff \sum c_n y_n = 0.$$

(c) Exhibit a frame $\{x_n\}$ and a permutation σ of $\mathbf{N}$ that fixes all but finitely many elements of $\mathbf{N}$ such that $\{x_n\}$ is not equivalent to $\{x_{\sigma(n)}\}$. Show that this cannot happen if $\{x_n\}$ is an exact frame.

(d) Exhibit a frame $\{x_n\}$ and a sequence of signs $\varepsilon_n = \pm 1$ such that $\{x_n\}$ is not equivalent to $\{\varepsilon_n x_n\}$. Show that this cannot happen if $\{x_n\}$ is an exact frame.

Remark: Parts (c) and (d) illustrate some of the weaknesses of this notion of equivalence for frames. A more robust notion of frame equivalence was introduced by Balan and Landau in [BL07].

8.33. Show that $\{x_n\}$ is a Parseval frame for a Hilbert space H if and only if there exists a Hilbert space K and a Parseval frame $\{y_n\}$ for K such that $\{(x_n, y_n)\}$ is an orthonormal basis for $H \times K$.

8.34. Let H be an infinite-dimensional separable Hilbert space. Show that if M is an infinite-dimensional closed subspace of ℓ^2, then there exists a frame $\{x_n\}$ for H such that $\mathrm{range}(C) = M$, where C is the analysis operator.

8.35. This exercise is about *superframes* as introduced by Balan [Bal98] and Han and Larson [HL00]. Let $\{x_n\}$ be a frame for a Hilbert space H and let $\{y_n\}$ be a frame for a Hilbert space K. Let C_X be the analysis operator for $\{x_n\}$ and C_Y the analysis operator for $\{y_n\}$.

(a) Show that if $\mathrm{range}(C_X) \perp \mathrm{range}(C_Y)$ then $\{(x_n, y_n)\}$ is a frame for $H \times K$, and it is a Parseval frame if and only if both $\{x_n\}$ and $\{y_n\}$ are Parseval. (When $\mathrm{range}(C_X) \perp \mathrm{range}(C_Y)$, we say that $\{x_n\}$ and $\{y_n\}$ are *orthogonal frames.*)

(b) Suppose that, in addition, we have $\mathrm{range}(C_X) \oplus \mathrm{range}(C_Y) = \ell^2$. Show that $\{(x_n, y_n)\}$ is a Riesz basis for $H \times K$, and it is an orthonormal basis if and only if $\{x_n\}$ and $\{y_n\}$ are each Parseval frames (in this case, we call $\{(x_n, y_n)\}$ a *superframe*).

(c) Give examples of frames $\{x_n\}$ and $\{y_n\}$ that are not Riesz bases but are such that $\{(x_n, y_n)\}$ is a Riesz basis for $H \times H$.

(d) Given a frame $\{x_n\}$ for H, show there exists a frame $\{y_n\}$ for H such that $\{(x_n, y_n)\}$ is a Riesz basis for $H \times H$.

8.36. This exercise gives another way to construct frames for "larger" spaces from "smaller" ones.

(a) Let $\{x_n\}$, $\{y_n\}$ be frames for Hilbert spaces H, K, respectively. Show that the tensor product sequence $\{x_m \otimes y_n\}_{m,n \in \mathbf{N}}$ is a frame for $H \otimes K = \mathcal{B}_2(H, K)$ (see Appendix B for definitions).

(b) Let E, F be measurable subsets of $\mathbf{R}$. Show that if $\{f_n\}$ is a frame for $L^2(E)$ and $\{g_n\}$ is a frame for $L^2(F)$, then $\{\overline{f_m(x)} g_n(y)\}_{m,n \in \mathbf{N}}$ is a frame for $L^2(E \times F)$.

8.6 Convergence of Frame Series

In this section we examine the convergence of $\sum c_n x_n$ for arbitrary sequences of scalars when $\{x_n\}$ is a frame. By Theorem 7.2, one of the important facts about such series are that if $\{x_n\}$ is a frame and $\sum |c_n|^2 < \infty$, then $\sum c_n x_n$ converges (unconditionally) in H. The following example shows that the converse does not hold in general.

Example 8.35. Let $\{x_n\}$ be any frame that includes infinitely many zero elements. Let $c_n = 1$ whenever $x_n = 0$, and let $c_n = 0$ when $x_n \neq 0$. Then $\sum c_n x_n = 0$, even though $\sum |c_n|^2 = \infty$.

Less trivially, let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Define $x_n = n^{-1} e_n$ and $y_n = (1 - n^{-2})^{1/2} e_n$. Then $\{x_n\} \cup \{y_n\}$ is a Parseval frame. Let $x = \sum n^{-1} e_n$. This is an element of H since $\sum n^{-2} < \infty$. With respect to the frame $\{x_n\} \cup \{y_n\}$ we can write $x = \sum (1 \cdot x_n + 0 \cdot y_n)$, and $\sum (1^2 + 0^2) = \infty$. ♢

The frame representations in Example 8.35 are not the ones corresponding to the canonical dual frame representations given in equation (8.3). If we restrict our attention to just the expansions $x = \sum \langle x, \widetilde{x}_n \rangle x_n$ then we always have $\sum |\langle x, \widetilde{x}_n \rangle|^2 < \infty$ since the canonical dual frame $\{\widetilde{x}_n\}$ is a frame. However, nonuniqueness is one of the major reasons that we are interested in frames, so it is important to consider alternative representations of elements with respect to a frame $\{x_n\}$.

Most "practical" examples of frames are norm-bounded below. For these frames, we can completely characterize when $\sum c_n x_n$ will converge unconditionally. The next result and the various examples given in this section are from [Hei90].

Theorem 8.36. *If $\{x_n\}$ is a frame that is norm-bounded below, then*

$$\sum_n |c_n|^2 < \infty \quad \Longleftrightarrow \quad \sum_n c_n x_n \text{ converges unconditionally.}$$

Proof. $\Leftarrow$. Assume that $\sum c_n x_n$ converges unconditionally. Then Orlicz's Theorem (Theorem 3.16) implies that $\sum |c_n|^2 \|x_n\|^2 = \sum \|c_n x_n\|^2 < \infty$, and since $\{x_n\}$ is norm-bounded below, it follows that $\sum |c_n|^2 < \infty$. □

By Exercise 7.13, if $\{x_n\}$ is an exact frame then $\sum c_n x_n$ converges if and only if it converges unconditionally, and this happens precisely when $(c_n) \in \ell^2$. The next example shows that, for an inexact frame, $\sum c_n x_n$ may converge conditionally, even if the frame is norm-bounded below.

Example 8.37. Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space H. Then $\{e_1, e_1, e_2, e_2, \dots\}$ is a frame that is norm-bounded below. The series

$$e_1 - e_1 + \frac{e_2}{\sqrt{2}} - \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{3}} - \frac{e_3}{\sqrt{3}} + \cdots \tag{8.6}$$

converges strongly in H to 0, but the series

$$e_1 + e_1 + \frac{e_2}{\sqrt{2}} + \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{3}} + \frac{e_3}{\sqrt{3}} + \cdots$$

does not converge. Therefore, the series in equation (8.6) converges conditionally (see Theorem 3.10). Since $(n^{-1/2}) \notin \ell^2$, the conditionality of the convergence also follows from Theorem 8.36. ♢

By Exercise 7.13, if $\{x_n\}$ is an exact frame then the three statements, (i) $\sum |c_n|^2 < \infty$, (ii) $\sum c_n x_n$ converges, and (iii) $\sum c_n x_n$ converges unconditionally, are equivalent. By Example 8.37, these equivalences may fail if the frame is not exact. However, we can certainly construct inexact frames for which these equivalences remain valid. For example, if $\{e_n\}$ is an orthonormal basis for a Hilbert space H, then these equivalences hold for the frame $\{x_n\} = \{e_1, e_1, e_2, e_3, \dots\}$. Our next theorem will show that it is precisely the frames that are Riesz bases plus finitely many elements that have this property.

Definition 8.38. Let $\{x_n\}$ be a frame for a Hilbert space H.

(a) We call $\{x_n\}$ a *Besselian frame* if $\sum c_n x_n$ converges only for $(c_n) \in \ell^2$.

(b) $\{x_n\}$ is an *unconditional frame* if $\sum c_n x_n$ converges if and only if it converges unconditionally.

(c) $\{x_n\}$ is a *near-Riesz basis* if there is a finite set $F \subseteq \mathbf{N}$ such that $\{x_n\}_{n \notin F}$ is a Riesz basis for H. ◇

In particular, the frame in Example 8.37 is norm-bounded below but is not unconditional, not Besselian, and not a near-Riesz basis.

The following theorem, and the terminology in Definition 8.38, is due to Holub [Hol94].

Theorem 8.39. *Given a frame $\{x_n\}$ for a Hilbert space H, the following statements are equivalent.*

(a) $\{x_n\}$ *is a near-Riesz basis.*

(b) $\{x_n\}$ *is Besselian.*

(c) $\mathrm{range}(C)^\perp = \ker(R)$ *is finite dimensional.*

Moreover, if $\{x_n\}$ is norm-bounded below, then statements (a)–(c) *are also equivalent to the following statement.*

(d) $\{x_n\}$ *is an unconditional frame.*

Proof. We will prove some portions of the theorem, and assign the remaining details and implications as Exercise 8.37.

Note that by applying Corollary 8.33, it suffices to prove the theorem for a frame of the form $\{P\delta_n\}$, where P is the orthogonal projection of ℓ^2 onto a closed subspace M and $\{\delta_n\}$ is the standard basis for ℓ^2. Such a frame is Parseval, and we have $\mathrm{range}(C) = M$.

(a) $\Rightarrow$ (b). Exercise.

(b) $\Rightarrow$ (c). Suppose that $\mathrm{range}(C)^\perp$ is infinite dimensional, and let $\{\phi_n\}_{n\in\mathbf{N}}$ be an orthonormal basis for $\mathrm{range}(C)^\perp = M^\perp = \ker(P)$. We will show that $\{x_n\}$ is not a Besselian frame.

Let $m_1 = 1$. Since $\phi_1 = \sum \langle \phi_1, \delta_n \rangle\, \delta_n$ is a unit vector and

$$\sum_n \langle \phi_1, \delta_n \rangle P\delta_n \;=\; P\biggl(\sum_n \langle \phi_1, \delta_n \rangle \delta_n\biggr) \;=\; P\phi_1 \;=\; 0,$$

we can find an integer N_1 such that

$$\biggl\|\sum_{n=1}^{N_1} \langle \phi_1, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \;\geq\; \frac{1}{2} \qquad \text{and} \qquad \biggl\|\sum_{n=1}^{N_1} \langle \phi_1, \delta_n \rangle P\delta_n\biggr\|_{\ell^2} \;\leq\; 1.$$

Since $\{\phi_n\}$ is orthonormal, it follows from Bessel's Inequality that $\phi_n \xrightarrow{\text{w}} 0$ (see Exercise 2.52). That is,

$$\forall\, x \in \ell^2, \quad \lim_{k\to\infty} \langle \phi_k, x \rangle \;=\; 0.$$

Hence we can find an $m_2 > m_1$ such that

$$\biggl\|\sum_{n=1}^{N_1} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \;\leq\; \sum_{n=1}^{N_1} |\langle \phi_{m_2}, \delta_n \rangle| \;\leq\; \frac{1}{4}.$$

Since

$$\begin{aligned}
\biggl\|\sum_{n=N_1+1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \;&\geq\; \biggl\|\sum_{n=1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \;-\; \biggl\|\sum_{n=1}^{N_1} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \\
&\geq\; \biggl\|\sum_{n=1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \;-\; \frac{1}{4} \\
&\to\; 1 \;-\; \frac{1}{4} \;=\; \frac{3}{4} \quad \text{as } N_2 \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\biggl\|\sum_{n=N_1+1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle P\delta_n\biggr\|_{\ell^2} \\
&\qquad\leq\; \biggl\|\sum_{n=1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle P\delta_n\biggr\|_{\ell^2} \;+\; \biggl\|P\biggl(\sum_{n=1}^{N_1} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr)\biggr\|_{\ell^2} \\
&\qquad\leq\; \biggl\|\sum_{n=1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle P\delta_n\biggr\|_{\ell^2} \;+\; \|P\|\,\frac{1}{4} \\
&\qquad\to\; 0 \;+\; \frac{1}{4} \;=\; \frac{1}{4} \quad \text{as } N_2 \to \infty,
\end{aligned}$$

we can choose N_2 large enough that we have both

$$\biggl\|\sum_{n=N_1+1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle \delta_n\biggr\|_{\ell^2} \;\geq\; \frac{1}{2} \qquad \text{and} \qquad \biggl\|\sum_{n=N_1+1}^{N_2} \langle \phi_{m_2}, \delta_n \rangle P\delta_n\biggr\|_{\ell^2} \;\leq\; \frac{1}{2}.$$

Continuing in this way, we can find $1 = m_1 < m_2 < \cdots$ and $N_1 < N_2 < \cdots$ such that for each k we have

$$\Bigg\| \sum_{n=N_k+1}^{N_{k+1}} \langle \phi_{m_k}, \delta_n \rangle \, \delta_n \Bigg\|_{\ell^2} \geq \frac{1}{2} \qquad \text{and} \qquad \Bigg\| \sum_{n=N_k+1}^{N_{k+1}} \langle \phi_{m_k}, \delta_n \rangle \, P\delta_n \Bigg\|_{\ell^2} \leq \frac{1}{2^k}.$$

For each $k \in \mathbf{N}$, define

$$c_n = k^{-1/2} \langle \phi_{m_k}, \delta_n \rangle, \qquad N_k + 1 \leq n \leq N_{k+1}.$$

Exercise: The series

$$\sum_{n=N_1+1}^{\infty} c_n P\delta_n = \sum_{k=1}^{\infty} \sum_{n=N_k+1}^{N_{k+1}} k^{-1/2} \langle \phi_{m_k}, \delta_n \rangle \, P\delta_n$$

is Cauchy and therefore converges. However, $\sum |c_n|^2 = \infty$, so we conclude that $\{P\delta_n\}$ is not a Besselian frame for M.

(c) $\Rightarrow$ (a). Suppose that $\mathrm{range}(C)^\perp$ is finite dimensional. Since P is the orthogonal projection onto $M = \mathrm{range}(C)$, the operator $I - P$ is the orthogonal projection onto $\mathrm{range}(C)^\perp$. By Exercise 1.53,

$$\sum_n \|(I-P)\delta_n\|_{\ell^2}^2 = \dim\bigl(\mathrm{range}(C)^\perp\bigr) < \infty,$$

so we can find some $N \in \mathbf{N}$ such that

$$\sum_{n=N+1}^{\infty} \|\delta_n - P\delta_n\|_{\ell^2}^2 < \infty.$$

Since $\|\delta_n\|_{\ell^2} = 1$, Theorem 5.26 implies that $\{\delta_1, \dots, \delta_N, P\delta_{N+1}, P\delta_{N+2}, \dots\}$ is a basis for ℓ^2 that is equivalent to $\{\delta_n\}$. Therefore this sequence is a Riesz basis for ℓ^2, and hence $\{P\delta_n\}_{n=N+1}^\infty$ is a Riesz basis for its closed span K, which is contained within M. Since $\overline{\mathrm{span}}\{P\delta_n\}_{n=N+1}^\infty = K$ and $\overline{\mathrm{span}}\{P\delta_n\}_{n\in\mathbf{N}} = M$, there exist sets $F \subseteq \{1, \dots, N\}$ such that the closed span of $\{P\delta_n\}_{n\in F} \cup \{P\delta_n\}_{n=N+1}^\infty$ is M. Exercise: If we let F be a minimal such set, then this sequence is a Riesz basis for M, and hence $\{P\delta_n\}_{n\in\mathbf{N}}$ is a near-Riesz basis for M.

(b) $\Leftrightarrow$ (d). Exercise. $\square$

Exercises

8.37. Complete the proof of Theorem 8.39.

8.7 Excess

Now we will take a closer look at the overcompleteness and undercompleteness of sequences, especially for frames and frame sequences. The results of this section are mostly taken from [BCHL03].

Notation 8.40. It will be convenient in this section to consider cardinalities to be either finite or infinite, with no distinction between infinite sets of different sizes. Therefore, given a set E we will let $|E|$ denote its cardinality if E is finite; otherwise we set $|E| = \infty$. Likewise, given a subspace S of a vector space, if S is finite-dimensional then $\dim(S)$ will denote its dimension; otherwise we set $\dim(S) = \infty$. $\diamondsuit$

Using this convention, we define the excess and deficit of a sequence as follows.

Definition 8.41. Let $\mathcal{F} = \{x_n\}$ be a sequence in a separable Hilbert space H.

(a) The *deficit* of $\mathcal{F}$ is

$$d(\mathcal{F}) \;=\; \inf\bigl\{|\mathcal{G}| : \mathcal{G} \subseteq H \text{ and } \overline{\operatorname{span}}(\mathcal{F} \cup \mathcal{G}) = H\bigr\}.$$

(b) The *excess* of $\mathcal{F}$ is

$$e(\mathcal{F}) \;=\; \sup\bigl\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F} \text{ and } \overline{\operatorname{span}}(\mathcal{F}\backslash\mathcal{G}) = \overline{\operatorname{span}}(\mathcal{F})\bigr\}. \qquad \diamondsuit$$

The infimum and supremum in Definition 8.41 are actually achieved, so the deficit is the cardinality of the smallest set that we need to add to $\mathcal{F}$ so that it becomes complete in H, and the excess is the cardinality of the largest set that we can remove from $\mathcal{F}$ without changing its closed span.

Note that a frame for a Hilbert space H has zero deficit, as does any complete sequence. On the other hand, a Riesz sequence in H has zero excess, and it follows from Theorem 8.22 that if a frame has zero excess then it is an exact frame and hence is a Riesz basis for H.

The following result, proved in [BCHL03], relates the deficit and excess to the analysis and synthesis operators.

Lemma 8.42. *Let $\mathcal{F} = \{x_n\}$ be a Bessel sequence in a separable Hilbert space H, with analysis operator C and synthesis operator R.*

(a) $d(\mathcal{F}) = \dim\bigl(\ker(C)\bigr) = \dim\bigl(\operatorname{range}(R)^\perp\bigr)$.

(b) $e(\mathcal{F}) \ge \dim\bigl(\operatorname{range}(C)^\perp\bigr) = \dim\bigl(\ker(R)\bigr)$.

(c) *$\{x_n\}$ is a near-Riesz basis if and only if it is a frame with finite excess.*

(d) *If $\mathcal{F}$ is a frame and $\{\widetilde{x}_n\}$ is its canonical dual frame, then*

$$e(\mathcal{F}) \;=\; \dim\bigl(\operatorname{range}(C)^\perp\bigr) \;=\; \dim\bigl(\ker(R)\bigr) \;=\; \sum_n \bigl(1 - \langle x_n, \widetilde{x}_n\rangle\bigr). \quad (8.7)$$

Proof. (a) If we add an orthonormal basis $\mathcal{E}$ for $\overline{\text{span}}(\mathcal{F})^{\perp}$ to $\mathcal{F}$, then the new sequence $\mathcal{F} \cup \mathcal{E}$ will be complete, and this is the smallest size set that we can add to make $\mathcal{F}$ into a complete sequence. Hence the deficit of $\mathcal{F}$ is the dimension of $\overline{\text{span}}(\mathcal{F})^{\perp}$. Theorem 7.4 tells us that $\overline{\text{span}}(\mathcal{F}) = \overline{\text{range}(R)} = \ker(C)^{\perp}$. Therefore, since $R = C^*$, Theorem 2.13 implies that $\overline{\text{span}}(\mathcal{F})^{\perp} = \ker(C)$.

(b) Let $\{a_1, \dots, a_N\}$ be linearly independent sequences in $\ker(R)$, where we take $N = \dim(\ker(R))$ if $\ker(R)$ is finite dimensional and N arbitrary but finite otherwise. Denote the components of a_n by $a_n = (a_{n,k})_{k \in \mathbf{N}}$. Then

$$Ra_n \;=\; \sum_{k=1}^{\infty} a_{n,k} x_k \;=\; 0, \qquad n = 1, \dots, N, \tag{8.8}$$

or, in terms of an infinite matrix equation,

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots \\ \vdots & \vdots & \cdots \\ a_{N,1} & a_{N,2} & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{8.9}$$

The matrix on the left of equation (8.9) has N linearly independent rows, i.e., it has row rank N. The same Gaussian elimination argument used for finite matrices implies that this matrix has column rank N as well (Exercise 8.38). Let $F = \{k_1, \dots, k_N\}$ denote the indices of a set of N linearly independent columns of this matrix. We will show that $\{x_n\}_{n \notin F}$ is complete in $\overline{\text{span}}(\mathcal{F})$.

Suppose that $x \in \overline{\text{span}}(\mathcal{F})$ satisfies $\langle x_n, x \rangle = 0$ for $n \notin F$. Then from equation (8.8) we have

$$0 \;=\; \langle Ra_n, x \rangle \;=\; \sum_{k=1}^{\infty} a_{n,k} \, \langle x_k, x \rangle \;=\; \sum_{i=1}^{m} a_{n,k_i} \, \langle x_{k_i}, x \rangle, \qquad n = 1, \dots, N.$$

That is,

$$\begin{bmatrix} a_{1,k_1} & \cdots & a_{1,k_m} \\ \vdots & \ddots & \vdots \\ a_{m,k_1} & \cdots & a_{m,k_m} \end{bmatrix} \begin{bmatrix} \langle x_{k_1}, x \rangle \\ \vdots \\ \langle x_{k_N}, x \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

However, the matrix on the left-hand side of this equation is invertible, so this implies that $\langle x_{k_i}, x \rangle = 0$ for $i = 1, \dots, N$. Hence $\langle x_n, x \rangle = 0$ for all $n \in \mathbf{N}$, so $x = 0$. Therefore $\{x_n\}_{n \notin F}$ is complete, so $e(\mathcal{F}) \geq N$.

(c) If $\mathcal{F}$ is a near-Riesz basis, then there is a finite set $F \subseteq \mathbf{N}$ such that $\{x_n\}_{n \notin F}$ is a Riesz basis. Let $k = |F|$. If we remove $2k+1$ elements from the original sequence $\mathcal{F} = \{x_n\}_{n \in \mathbf{N}}$, then at least $k+1$ of these indices must be coming from $\mathbf{N} \backslash F$. Yet $\text{span}\{x_n\}_{n \in \mathbf{F}}$ is at most k-dimensional, so removing $2k+1$ elements from $\mathcal{F}$ must leave us with an incomplete sequence. Hence the excess of $\mathcal{F}$ is finite.

Conversely, suppose that $\mathcal{F}$ is a frame with finite excess, i.e., there is some finite set $F \subseteq \mathbf{N}$ such that $\{x_n\}_{n\notin F}$ is complete, but any subset of this is incomplete. By Theorem 8.22, the removal of a single vector from a frame leaves either a frame or an incomplete set, and by induction this extends to the removal of finitely many elements from a frame. Therefore $\{x_n\}_{n\notin F}$ must be a frame. Yet if any additional element is removed from $\{x_n\}_{n\notin F}$ then it becomes incomplete, so $\{x_n\}_{n\notin F}$ is an exact frame and therefore is a Riesz basis for H.

(d) We prove the final equality in equation (8.7) first. Recall from Exercise 8.18 that the orthogonal projection of ℓ^2 onto $\ker(R) = \mathrm{range}(C)^\perp$ is given by $P = I - CS^{-1}R$. Letting $\{\delta_n\}$ be the standard basis for ℓ^2, and using Exercise 1.53, we therefore have

$$\begin{aligned} e(\mathcal{F}) \;=\; \dim\bigl(\ker(R)\bigr) \;&=\; \sum_n \|P\delta_n\|_{\ell^2}^2 \\ &=\; \sum_n \langle \delta_n, P\delta_n\rangle \\ &=\; \sum_n \Bigl(1 - \bigl\langle R\delta_n,\, S^{-1}R\delta_n\bigr\rangle\Bigr) \\ &=\; \sum_n \bigl(1 - \langle x_n, \widetilde{x}_n\rangle\bigr). \end{aligned}$$

It remains to show that $e(\mathcal{F}) = \dim\bigl(\ker(R)\bigr)$. If $\dim\bigl(\ker(R)\bigr) = \infty$, then the proof of part (b) shows that $e(\mathcal{F}) \ge \dim\bigl(\ker(R)\bigr) = \infty$. Therefore, suppose that $\dim\bigl(\ker(R)\bigr) < \infty$. Then $\mathcal{F}$ is a near-Riesz basis by Theorem 8.39, so there is a finite set F such that $\{x_n\}_{n\notin F}$ is a Riesz basis. By applying Corollary 8.33, it suffices to assume that $\{x_n\} = \{P\delta_n\}$ where P is the orthogonal projection of ℓ^2 onto the closed subspace $M = \mathrm{range}(C)$. In this setting, our assumption is that $\{P\delta_n\}_{n\notin F}$ is a Riesz basis for M. Note that $\ker(R) = \mathrm{range}(C)^\perp = M^\perp = \ker(P)$.

Let $K = \overline{\mathrm{span}}\{\delta_n\}_{n\notin F}$, and suppose that $a = (a_n) \in \ell^2$. Then $Pa \in M$, so since $\{P\delta_n\}_{n\notin F}$ is a Riesz basis for M we can write $Pa = \sum_{n\notin F} c_n P\delta_n$ for a unique choice of scalars c_n. Set $y = \sum_{n\notin F} c_n\delta_n \in K$ and observe that $Py = Pa$. Therefore $z = a - y \in \ker(P)$, so $a = y + z \in K + \ker(P)$, and therefore $\ell^2 = K + \ker(P)$. On the other hand, if $a \in K \cap \ker(P)$, then $a = \sum_{n\notin F} a_n\delta_n$ and $0 = Pa = \sum_{n\notin F} a_n P\delta_n$. By uniqueness, we conclude that $a_n = 0$ for all $n \notin F$, so $a = 0$. Thus we have both $\ell^2 = K + \ker(P)$ and $K \cap \ker(P) = \{0\}$. In the language of Exercise 5.21, this tells us that the *codimension* of K is $\mathrm{codim}(K) = \dim\bigl(\ker(P)\bigr) = \dim\bigl(\ker(R)\bigr)$. However, we also have $\ell^2 = K + K^\perp$ and $K \cap K^\perp = \{0\}$. Exercise 5.21 shows that the codimension of a space is independent of the choice of complementary subspace, so we conclude that $\mathrm{codim}(K) = \dim(K^\perp) = |F|$. Therefore, $e(\mathcal{F}) = |F| = \dim\bigl(\ker(R)\bigr)$. □

Example 8.43. (a) If $\mathcal{F}$ is a Bessel sequence that is not a frame, then it is possible that $e(\mathcal{F})$ can strictly exceed $\dim\bigl(\ker(R)\bigr)$. For example, let $\{e_n\}$ be an orthonormal basis for a Hilbert space H, and set $f = \sum e_n/n$. Then $\mathcal{F} = \{e_n/n\}_{n\in\mathbf{N}} \cup \{f\}$ is a Bessel sequence but it is not a frame, and we have $e(\mathcal{F}) = 1$ while $\dim\bigl(\ker(R)\bigr) = 0$. It is similarly possible to construct Bessel sequences such that $e(\mathcal{F})$ is any specified finite value or infinity yet $\dim\bigl(\ker(R)\bigr) = 0$.

(b) Another interesting example is considered in Example 11.34. As discussed there, the Gabor system $\mathcal{F} = \{e^{2\pi inx}\phi(x-n)\}_{m,n\in\mathbf{Z}}$ generated by the Gaussian function $\phi(x) = e^{-\pi x^2}$ is a Bessel sequence with excess $e(\mathcal{F}) = 1$. However, by Exercise 11.26 this system has $\dim\bigl(\ker(R)\bigr) = 0$. In contrast to the example given in part (a), the elements of this Bessel sequence all have identical norms. ♢

If $\mathcal{F}$ is a frame with finite excess, then some finite subset can be removed and leave a Riesz basis. If $\mathcal{F}$ is a frame with infinite excess, then by definition it is possible to remove some infinite subset yet still leave a complete set. We will see that it is not always possible to find an infinite subset that can be removed and leave a frame. The next theorem, proved in [BCHL03], characterizes those Parseval frames for which this can be done (and a general result for arbitrary frames is also proved in that paper). Surprisingly, the ability to remove infinitely many elements and leave a frame is determined by what happens when single elements are removed from the frame.

Theorem 8.44. *Let $\mathcal{F} = \{x_n\}_{n\in\mathbf{N}}$ be a Parseval frame for a Hilbert space H, and let $\mathcal{G} = \{x_{n_k}\}_{k\in\mathbf{N}}$ be a subsequence of $\mathcal{F}$. Then the following statements are equivalent.*

(a) *$\mathcal{F}\backslash\{x_{n_k}\}$ is complete (and hence a frame) for each $k \in \mathbf{N}$, and there exists a single constant $L > 0$ that is a lower frame bound for each frame $\mathcal{F}\backslash\{x_{n_k}\}$.*

(b) $\sup_{k\in\mathbf{N}} \|x_{n_k}\| < 1$.

In case these hold, for each $0 < \varepsilon < L$ there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $\mathcal{F}\backslash\mathcal{G}_\varepsilon$ is a frame for H with frame bounds $L-\varepsilon$, 1.

Proof. For simplicity of notation, let $y_k = x_{n_k}$ for $k \in \mathbf{N}$.

By Exercise 8.6, the optimal lower frame bound for $\mathcal{F}\backslash\{y_k\}$ is $1 - \|y_k\|^2$, so it follows from this that statements (a) and (b) are equivalent. Therefore, our task is show that if $0 < \varepsilon < L$ is given then we can find the desired subsequence $\mathcal{G}_\varepsilon$.

By hypothesis, the frame operator S for $\mathcal{F}$ is the identity:

$$\forall\, f \in H, \quad f \;=\; Sf \;=\; \sum_n \langle f, x_n\rangle\, x_n.$$

We are given that $\mathcal{F}\backslash\{y_k\}$ is a frame with lower frame bound L, so since the optimal lower frame bound for $\mathcal{F}\backslash\{y_k\}$ is $1 - \|y_k\|^2$, we must have

$$L \le 1 - \|y_k\|^2, \qquad k \in \mathbf{N}.$$

Since $\{y_k\}_{k\in\mathbf{N}}$ is a subset of the Parseval frame $\mathcal{F}$, we have for each $m \in \mathbf{N}$ that $\sum_k |\langle y_m, y_k\rangle|^2 \le \|y_m\|^2 < \infty$. Therefore,

$$\forall\, m \in \mathbf{N}, \quad \lim_{k\to\infty} \langle y_m, y_k\rangle = 0.$$

Appealing to Exercise 8.39, we can extract a subsequence $\mathcal{G}_\varepsilon = \{y_{m_k}\}_{k\in\mathbf{N}}$ with the property that

$$\sum_{\substack{j,k\in\mathbf{N},\\ k\ne j}} |\langle y_{m_k}, y_{m_j}\rangle| < \varepsilon. \tag{8.10}$$

We claim that $\mathcal{F}\backslash\mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L-\varepsilon$. To see this, consider the operator

$$Tf = \sum_{k=1}^\infty \langle f, y_{m_k}\rangle\, y_{m_k}.$$

This is a bounded operator since $\mathcal{G}_\varepsilon$ is a Bessel sequence. We have

$$\begin{aligned}
\|Tf\|^2 &= \left\langle \sum_{k=1}^\infty \langle f, y_{m_k}\rangle\, y_{m_k},\ \sum_{j=1}^\infty \langle f, y_{m_j}\rangle\, y_{m_j} \right\rangle \\
&= \sum_{k=1}^\infty |\langle f, y_{m_k}\rangle|^2\, \|y_{m_k}\|^2 + \sum_{\substack{j,k\in\mathbf{N},\\ k\ne j}} \langle f, y_{m_k}\rangle\, \langle y_{m_j}, f\rangle\, \langle y_{m_k}, y_{m_j}\rangle \\
&\le \Bigl(\sup_k \|y_{m_k}\|^2\Bigr) \langle Tf, f\rangle + \\
&\qquad \|f\|^2 \Bigl(\sup_k \|y_{m_k}\|^2\Bigr) \Biggl(\sum_{\substack{j,k\in\mathbf{N},\\ k\ne j}} |\langle y_{m_k}, y_{m_j}\rangle|\Biggr) \\
&\le (1-L)\,\|Tf\|\,\|f\| + \|f\|^2\,(1-L)\,\varepsilon.
\end{aligned}$$

Taking the suprema over all unit vectors f yields

$$\|T\|^2 \le (1-L)\,\|T\| + (1-L)\,\varepsilon,$$

and after some algebra we obtain the estimate $\|T\| \le 1-L+\varepsilon$. Consequently,

$$\sum_n |\langle f, x_n\rangle|^2 - \sum_{k=1}^\infty |\langle f, y_{m_k}\rangle|^2 = \|f\|^2 - \langle Tf, f\rangle \ge (L-\varepsilon)\,\|f\|^2.$$

Thus $\mathcal{F}\backslash\mathcal{G}_\varepsilon$ is a frame with lower frame bound $L-\varepsilon$. □

As we mentioned earlier, Casazza and Christensen [CC98a] constructed a Parseval frame $\mathcal{F}$ that is norm-bounded below but contains no subsets that are Riesz bases or even Schauder bases for H. Following [BCHL03], we now show that their frame has infinite excess, yet there is no way to choose an infinite subset $\mathcal{G}$ so that $\mathcal{F}\backslash\mathcal{G}$ is still a frame.

Example 8.45. Let H be a separable Hilbert space. Index an orthonormal basis for H as $\{e_j^n\}_{n\in\mathbf{N},\, j=1,\dots,n}$. Set $H_n = \operatorname{span}\{e_1^n, \dots, e_n^n\}$, and define

$$f_j^n \;=\; e_j^n - \frac{1}{n}\sum_{i=1}^{n} e_i^n, \qquad j = 1, \dots, n,$$

$$f_{n+1}^n \;=\; \frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_i^n.$$

Then $\mathcal{F}_n = \{f_1^n, \dots, f_{n+1}^n\}$ is a Parseval frame for H_n [CC98a, Lem. 2.5]. Since H_n is n-dimensional, at most one element can be removed from $\mathcal{F}_n$ if the remaining elements are to span H_n. Moreover f_{n+1}^n is orthogonal to $f_1^n, \dots, f_n^n$, so the element f_{n+1}^n cannot be removed. If one of the other elements is removed, say f_1^n, then since

$$\sum_{j=2}^{n+1} |\langle e_1^n, f_j^n\rangle|^2 \;=\; \left(\sum_{j=2}^{n}\frac{1}{n^2}\right) + \frac{1}{\sqrt{n}^2} \;=\; \frac{2}{n} - \frac{1}{n^2},$$

the lower frame bound for $\mathcal{F}_n\backslash\{f_1^n\}$ as a frame for H_n is at most $2/n - 1/n^2$.

The sequence $\mathcal{F} = \{f_j^n\}_{n\in\mathbf{N},\, j=1,\dots,n+1}$ is a Parseval frame for H with infinite excess. Suppose that $\mathcal{G}$ is any infinite subset of $\mathcal{F}$ such that $\mathcal{F}\backslash\mathcal{G}$ is complete. Then $\mathcal{G}$ cannot contain any elements of the form f_{n+1}^n. Hence $\mathcal{G} = \{f_{j_k}^{n_k}\}_{k\in\mathbf{N}}$ with $n_1 < n_2 < \cdots$ and $j_k \le n_k$ for every k. But then the lower frame bound for $\mathcal{F}\backslash\mathcal{G}$ can be at most $2/n_k - 1/n_k^2$ for every k, which implies that $\mathcal{F}\backslash\mathcal{G}$ cannot have a positive lower frame bound and therefore is not a frame. $\diamondsuit$

The proof that the frame discussed in Example 8.45 contains no subsets that are Riesz bases or Schauder bases is obtained by calculating the basis constants for subsequences that are bases for H_n. These basis constants tend to infinity with n, and so the basis constant of any ω-independent subsequence must be infinite. For a detailed proof, see [CC98a] or [Chr03].

For arbitrary frames, it is difficult to say much more about excess than what appears in Theorem 8.44. However, many frames that are used in practice, such as Gabor frames, have an internal structure that can be used to derive stronger and more sophisticated statements about excess. For example, we will give a test from [BCHL03] that determines the deficit or excess of some sequences.

Definition 8.46. Let C be the analysis operator for a Bessel sequence $\{x_n\}$ in a separable Hilbert space H. If there exist bounded operators $U\colon H \to H$ and $V\colon \ell^2 \to \ell^2$ such that $VC = CU$, then we say that (U, V) is a pair of *intertwining operators* for $\{x_n\}$. $\diamondsuit$

The existence of intertwining operators implies the following facts (see Exercise 8.41).

Lemma 8.47. *If (U, V) is a pair of intertwining operators for a Bessel sequence $\{x_n\}$, then the following statements hold:*

(a) $U\bigl(\ker(C)\bigr) \subseteq \ker(C)$,

(b) $V^*\bigl(\ker(R)\bigr) \subseteq \ker(R)$,

(c) $V\bigl(\overline{\mathrm{range}(C)}\bigr) \subseteq \overline{\mathrm{range}(C)}$,

(d) $U^*\bigl(\overline{\mathrm{range}(R)}\bigr) \subseteq \overline{\mathrm{range}(R)}$. $\diamondsuit$

In another terminology, statement (a) says that $\ker(C)$ is invariant under U, and so forth.

Using this, we can show that many sequences that have an intertwining pair of operators can have only zero or infinite deficit or excess.

Theorem 8.48. *Suppose that $\mathcal{F} = \{x_n\}$ is a Bessel sequence in a separable Hilbert space that has a pair of intertwining operators (U, V).*

(a) *If U^* has no eigenvalues, then either $\overline{\mathrm{span}}\{x_n\}$ is infinite dimensional or $x_n = 0$ for every n.*

(b) *If U has no eigenvalues, then either $d(\mathcal{F}) = 0$ or $d(\mathcal{F}) = \infty$.*

(c) *If $\mathcal{F}$ is a frame and V^* has no eigenvalues, then either $e(\mathcal{F}) = 0$ or $e(\mathcal{F}) = \infty$.*

Proof. First we make a generic observation about bounded operators on H that have no eigenvalues. Suppose that T is such an operator, and M is a closed subspace of H that is invariant under T. If M has finite dimension, then $T|_M$ maps the finite-dimensional vector space M into itself. Consequently, if $M \neq \{0\}$ then $T|_M$ must have an eigenvalue λ, and this must also be an eigenvalue of T, which is a contradiction. Therefore any nontrivial closed subspace that is invariant under T must be infinite dimensional.

(a) This follows from part (d) of Lemma 8.47 and the fact that $\overline{\mathrm{span}}(\mathcal{F}) = \overline{\mathrm{range}(R)}$.

(b) Since $d(\mathcal{F}) = \dim\bigl(\ker(C)\bigr)$, this follows from Lemma 8.47(a).

(c) If $\mathcal{F}$ is a frame then $e(\mathcal{F}) = \dim\bigl(\ker(R)\bigr)$, so this follows from Lemma 8.47(b). $\square$

Example 8.49. Consider the lattice trigonometric system $\mathcal{E}_b = \{e^{2\pi ibnt}\}_{n\in\mathbf{Z}} = \{e_{bn}\}_{n\in\mathbf{Z}}$ presented in Example 8.7. This system is incomplete in $L^2(\mathbf{T})$ when $b > 1$, an orthonormal basis when $b = 1$, and an overcomplete frame when $0 < b < 1$. Let $U\colon L^2(\mathbf{T}) \to L^2(\mathbf{T})$ be the multiplication operator

$$Uf(x) \;=\; e^{2\pi ibt} f(t), \qquad f \in L^2(\mathbf{T}),$$

and let $V\colon \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ be the right-shift operator

$$V(c_n)_{n\in\mathbf{Z}} \;=\; (c_{n-1})_{n\in\mathbf{Z}}.$$

Note that

$$\langle Uf, e_{bn}\rangle \;=\; \int_0^1 e^{2\pi ibt}\, f(t)\, e^{-2\pi ibnt}\, dt \;=\; \langle f, e_{b(n-1)}\rangle.$$

Letting C be the analysis operator for $\mathcal{E}_b$, we therefore have that

$$CUf \;=\; \bigl\{\langle Uf, e_{bn}\rangle\bigr\}_{n\in\mathbf{Z}} \;=\; \bigl\{\langle f, e_{b(n-1)}\rangle\bigr\}_{n\in\mathbf{Z}} \;=\; V\bigl\{\langle f, e_{bn}\rangle\bigr\}_{n\in\mathbf{Z}} \;=\; VCf.$$

Therefore (U,V) is an intertwining pair for $\mathcal{E}_b$. Further, by Exercise 8.42, none of U, V, U^*, or V^* has an eigenvalue, so we conclude that the deficit and the excess of $\mathcal{E}_b$ can only be zero or infinity. For $b = 1$ we know that $\mathcal{E}_1 = \{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, so this simply confirms what we already knew: $d(\mathcal{E}_b) = 0 = e(\mathcal{E}_b)$. On the other hand, for $b \neq 1$ it does give us new information.

If $b > 1$, then Example 8.7 showed us that $\mathcal{E}_b$ is incomplete, but it did not tell us how large the deficit is. Incompleteness means that the deficit is nonzero, but now we know that $\mathrm{span}(\mathcal{E}_b)^\perp$ is actually an infinite-dimensional space when $b > 1$.

Similarly, for $0 < b < 1$ we knew from Example 8.7 that $\mathcal{E}_b$ is overcomplete, but we did not know how large the excess was. Since overcompleness means that the excess is nonzero, it follows from the discussion above that the excess must actually be infinite. Moreover, by Exercise 8.42 it is possible to remove an infinite subset from $\mathcal{E}_b$ yet still leave a frame (and not just a complete set). Thus $\mathcal{E}_b$ is "infinitely redundant" when $0 < b < 1$, even though it is finitely linearly independent. $\diamondsuit$

For more sophisticated results on excess and properties of *localized frames*, we refer to [BCHL06a], [BCHL06b], [BCL10].

Exercises

8.38. Prove that the matrix appearing on the left-hand side of equation (8.9) has column rank N, i.e., its columns span an N-dimensional subspace of $\mathbf{F}^N$.

8.39. Verify the claim in the proof of Theorem 8.44 that there exists a subsequence such that equation (8.10) is satisfied.

8.40. Let $\mathcal{F} = \{x_n\}$ be a frame for a Hilbert space H with frame bounds A, B and canonical dual frame $\{\widetilde{x}_n\}$. Let $\mathcal{G} = \{x_{n_k}\}$ be a subsequence of $\{x_n\}$, and show that the following statements are equivalent.

(a) $\{x_n\}_{n \neq n_k}$ is complete (and hence a frame) for each $k \in \mathbf{N}$, and there exists a single constant $L > 0$ that is a lower frame bound for each frame $\{x_n\}_{n \neq n_k}$.

(b) $\sup_k \langle x_{n_k}, \widetilde{x}_{n_k} \rangle < 1$.

Show further that if these statements hold, then there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $\mathcal{F} \backslash \mathcal{G}_\varepsilon$ is a frame for H with frame bounds $L(A/B) - \varepsilon$, 1.

Remark: It is shown in [BCHL03] that $\mathcal{G}_\varepsilon$ can be chosen so that $\mathcal{F} \backslash \mathcal{G}_\varepsilon$ has frame bounds $L - \varepsilon$, 1.

8.41. Prove Lemma 8.47.

8.42. (a) Prove that the operators U, V, U^*, and V^* appearing in Example 8.49 have no eigenvalues.

(b) Show that if $0 < b < 1$, then there is an infinite set $J \subseteq \mathbf{Z}$ such that $\{e^{2\pi ibnt}\}_{n \notin J}$ is a frame for $L^2(\mathbf{T})$.

Part III

Bases and Frames in Applied Harmonic Analysis

9

The Fourier Transform on the Real Line

In Part II we developed the abstract theory of bases and frames. Now in Part III we turn to more concrete settings, and examine some frames and bases that have a specific structure. In Chapter 10 we will consider systems of weighted exponentials and systems of translates, while in Chapter 11 we turn to Gabor systems and then to wavelets in Chapter 12. Part IV, consisting of Chapters 13 and 14, is devoted to Fourier series.

The Fourier transform is a fundamental mathematical tool that will be very useful to us throughout Part III. In this chapter we present a short review of some of the most important properties of the Fourier transform. We will sketch the main ideas, and refer to texts such as [Ben97], [DM72], [Gra04], [Kat04], [Heil] for complete details and proofs. The flavor of the proofs of the theorems in this section is quite similar to those of the analogous results for Fourier series that we will develop in detail in Chapter 13. Hence the interested reader can use that chapter as a stepping stone to a broader study of Fourier analysis. Indeed, from an abstract point of view, Fourier series correspond to the Fourier transform of functions on the torus $\mathbf{T}$, while in this chapter we are interested in the Fourier transform of functions on the real line $\mathbf{R}$ (for details on the abstract Fourier transform we refer to [Rud62] or [Fol95]).

One notational change from the preceding chapters is that throughout Parts III and IV the scalar field will always be complex.

Notation 9.1 (Scalars Are Complex). In the abstract development in Parts I and II it was convenient to allow $\mathbf{F}$ to denote a generic choice of either the real or complex scalar field. Because we are now dealing with concrete systems, it will be better to fix the scalar field. Therefore, throughout all of Parts III and IV, the scalar field will be $\mathbf{F} = \mathbf{C}$. In particular, functions and sequences will generally be complex valued. $\diamondsuit$

Also, as we will be dealing with complex exponential functions very often, we introduce the following notation.

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Notation 9.2 (Complex Exponentials). Given $\lambda \in \mathbf{R}$ we will use the abbreviation

$$e_\lambda(x) = e^{2\pi i\lambda x}, \qquad x \in \mathbf{R}. \qquad \diamondsuit \tag{9.1}$$

In particular, the trigonometric system is $\{e_n\}_{n\in\mathbf{Z}}$, although sometimes for emphasis we write out $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$.

9.1 Summary: Main Properties of the Fourier Transform on the Real Line

For the convenience of the reader, we summarize in this section the main properties of the Fourier transform that we will need in the coming chapters.

The Fourier transform can be defined on many spaces of functions or distributions, but mostly we will use it on the Banach spaces $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$. As an operator on these spaces we usually denote the Fourier transform by $\mathcal{F}$, and we denote the action of the Fourier transform of a function f by

$$\widehat{f} = \mathcal{F}(f).$$

The precise definition of the Fourier transform will be given later. However, what is most important to us is not the actual definition, but rather the properties of the Fourier transform that appear in Theorems 9.3 and 9.5 below.

The first important fact is that the Fourier transform is unitary on $L^2(\mathbf{R})$ and bounded on $L^1(\mathbf{R})$.

Theorem 9.3 (Properties of the Fourier Transform).

(a) *The Fourier transform $\mathcal{F}$ is a bounded, injective linear mapping of $L^1(\mathbf{R})$ into $C_0(\mathbf{R})$.*

(b) *The Fourier transform $\mathcal{F}$ is a unitary mapping of $L^2(\mathbf{R})$ onto itself.* $\diamondsuit$

The second important fact is the way in which the Fourier transform interacts with the translation, modulation, and dilation operators.

Notation 9.4 (Translation, Modulation, Dilation). We define the following operations on functions $f\colon \mathbf{R} \to \mathbf{C}$.

$$\begin{aligned} \text{Translation:} \quad & (T_a f)(x) = f(x-a), && a \in \mathbf{R}. \\ \text{Modulation:} \quad & (M_b f)(x) = e^{2\pi ibx} f(x), && b \in \mathbf{R}. \\ \text{Dilation:} \quad & (D_r f)(x) = r^{1/2} f(rx), && r > 0. \qquad \diamondsuit \end{aligned}$$

The translation and modulation operators T_a and M_b are isometries on $L^p(\mathbf{R})$ for each $1 \le p \le \infty$. The dilation operator D_r is isometric on $L^2(\mathbf{R})$, and is a multiple of an isometry on $L^p(\mathbf{R})$ for all other p.

Theorem 9.5. (a) *The Fourier transform interchanges translation with modulation: For all* $f \in L^1(\mathbf{R})$ *or* $f \in L^2(\mathbf{R})$ *we have*

$$(T_a f)^\wedge(\xi) \;=\; M_{-a}\widehat{f}(\xi) \;=\; e^{-2\pi i a\xi}\,\widehat{f}(\xi). \tag{9.2}$$

(b) *The Fourier transform interchanges modulation with translation: For all* $f \in L^1(\mathbf{R})$ *or* $f \in L^2(\mathbf{R})$ *we have*

$$(M_b f)^\wedge(\xi) \;=\; T_b\widehat{f}(\xi) \;=\; \widehat{f}(\xi - b). \tag{9.3}$$

(c) *The Fourier transform interchanges dilation with a reciprocal dilation: For all* $f \in L^1(\mathbf{R})$ *or* $f \in L^2(\mathbf{R})$ *we have*

$$(D_r f)^\wedge(\xi) \;=\; D_{1/r}\widehat{f}(\xi) \;=\; \frac{1}{r}\,\widehat{f}(\xi/r). \tag{9.4}$$

The equalities in equations (9.2), (9.3), *and* (9.4) *hold pointwise everywhere if* $f \in L^1(\mathbf{R})$, *and pointwise almost everywhere if* $f \in L^2(\mathbf{R})$. $\diamondsuit$

Properties such as being a basis or a frame are preserved by unitary maps, so by applying the Fourier transform to a sequence $\{f_n\}$ we obtain a new sequence $\{\widehat{f_n}\}$ that has a different structure but still has the same basis or frame properties as $\{f_n\}$. Because of the way the Fourier transform interacts with translation, modulation, and dilation, it is often easier to work with $\{\widehat{f_n}\}$ than $\{f_n\}$.

Corollary 9.6. *Let* $\{f_n\}$ *be a sequence in* $L^2(\mathbf{R})$. *Then* $\{f_n\}$ *is a Schauder basis, Riesz basis, Bessel sequence, or frame for* $L^2(\mathbf{R})$ *if and only if the same is true of* $\{\widehat{f_n}\}$. $\diamondsuit$

In the remainder of this chapter we will motivate the Fourier transform and prove Theorems 9.3 and 9.5 and some related results.

9.2 Motivation: The Trigonometric System

Every *locally compact abelian group* (LCA group or LCAG) has a Fourier transform associated to it. In particular, the real line $\mathbf{R}$ and the torus $\mathbf{T}$ are LCA groups, and each has an associated Fourier transform. Although we did not discuss it in these terms, whenever we worked with the trigonometric system $\{e^{2\pi i n x}\}_{n\in\mathbf{Z}}$ in earlier chapters, we were dealing with the Fourier transform for the torus. In the coming sections we will introduce the Fourier transform for the real line, but in order to motivate it we first take another look at the trigonometric system and the Fourier transform on $\mathbf{T}$.

Recall from Theorem 4.25 that the trigonometric system $\{e_n\}_{n\in\mathbf{Z}}$ forms an orthonormal basis for $L^2(\mathbf{T})$. Therefore every function $f \in L^2(\mathbf{T})$ can be written as

$$f = \sum_{n\in\mathbf{Z}} \langle f, e_n\rangle\, e_n. \tag{9.5}$$

Thinking of the operators associated with a frame motivates us to view the representation in equation (9.5) as consisting of two parts. First, the function f has *Fourier coefficients*

$$\widehat{f}(n) = \langle f, e_n\rangle = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx, \qquad n \in \mathbf{Z}.$$

The sequence

$$\widehat{f} = \big(\widehat{f}(n)\big)_{n\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$$

is the *Fourier transform of* f, and the mapping $\mathcal{F}\colon f \mapsto \widehat{f}$ is the *Fourier transform* for the torus. Using the frame terminology from Chapter 8, the Fourier transform is precisely the analysis operator C for the Parseval frame $\{e_n\}_{n\in\mathbf{Z}}$, so $\widehat{f} = \mathcal{F}f = Cf$. The Fourier transform is an analysis operation—we take a function $f \in L^2(\mathbf{T})$ and analyze it, converting it from a function on $\mathbf{T}$ into a sequence $\widehat{f}$ indexed by $\mathbf{Z}$.

The second part of equation (9.5) is the recovery of f from its Fourier transform $\widehat{f} = \big(\widehat{f}(n)\big)_{n\in\mathbf{Z}}$. We can view this recovery both as being the inverse $\mathcal{F}^{-1}$ of the Fourier transform operator and as the synthesis operator R for the Parseval frame $\{e_n\}_{n\in\mathbf{Z}}$. The synthesis operator $\mathcal{F}^{-1} = R$ maps $\ell^2(\mathbf{Z})$ back to $L^2(\mathbf{R})$. We usually denote its action on $c \in \ell^2(\mathbf{Z})$ by $\check{c}$. Writing $c = (c_n)_{n\in\mathbf{Z}}$, the 1-periodic function $\check{c}$ is

$$\check{c}(x) = (\mathcal{F}^{-1}c)(x) = (Rc)(x) = \sum_{n\in\mathbf{Z}} c_n e_n(x) = \sum_{n\in\mathbf{Z}} c_n e^{2\pi i n x},$$

where the series converges unconditionally in $L^2(\mathbf{R})$. Equation (9.5) is simply the frame representation $f = RCf$:

$$f(x) = RCf(x) = \sum_{n\in\mathbf{Z}} \langle f, e_n\rangle\, e_n(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x}, \tag{9.6}$$

which we can also write as

$$f = \mathcal{F}^{-1}(\mathcal{F}f) = \big(\widehat{f}\,\big)^{\vee}, \qquad f \in L^2(\mathbf{T}). \tag{9.7}$$

Analysis followed by synthesis, or the Fourier transform followed by the inverse Fourier transform, is the identity operator, and this is expressed in equations (9.5), (9.6), and (9.7), which are identical equalities written using different notation. Since the trigonometric system is an orthonormal basis for $L^2(\mathbf{T})$, it is also true that synthesis followed by analysis is the identity:

$$c = \mathcal{F}(\mathcal{F}^{-1}c) = \big(\check{c}\big)^{\wedge}, \qquad c \in \ell^2(\mathbf{Z}).$$

By focusing solely on $L^2(\mathbf{T})$ we are glossing over many interesting aspects of the Fourier transform, some of which will be discussed in greater detail in Chapters 13 and 14. However, the main point for us here is that the Fourier transform and its inverse are analysis and synthesis operators associated with the trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$.

9.3 The Fourier Transform on $L^1(\mathbf{R})$

The Fourier transform for the real line is also related to analysis and synthesis operators based on the complex exponentials, but since the functions $e^{2\pi inx}$ are not square integrable on $\mathbf{R}$, it is not immediately obvious how we can use them to analyze functions in $L^2(\mathbf{R})$. The sequence $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is not even contained in $L^2(\mathbf{R})$, and it certainly does not form an orthonormal basis for $L^2(\mathbf{R})$. Even so, there are very strong analogies between the Fourier transform on the real line and the version on the torus that we discussed in Section 9.2 (and these analogies are reflections of the definition of the Fourier transform on abstract LCA groups).

Although $e^{2\pi inx}$ is not square integrable, it is a bounded function, and so if we take $f \in L^1(\mathbf{R})$ rather than $L^2(\mathbf{R})$ then the "inner product" of f with $e^{2\pi inx}$ is well defined. Since we are no longer dealing with periodic functions it will be important to consider all real frequencies of the exponentials, not just integer frequencies n. We will no longer be relying on the existence of a *basis* of complex exponential functions, and we will need to consider $e_\xi(x) = e^{2\pi i\xi x}$ for all real values of ξ. We no longer have a countable sequence that forms a frame, but we will see that we still have analysis and synthesis.

Our *analysis operator* on $L^1(\mathbf{R})$ is defined as follows.

Definition 9.7 (Fourier Transform on $L^1(\mathbf{R})$). The *Fourier transform* of $f \in L^1(\mathbf{R})$ is the function $\widehat{f}\colon \mathbf{R} \to \mathbf{C}$ defined by

$$\widehat{f}(\xi) \;=\; \langle f, e_\xi\rangle \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx, \qquad \xi \in \mathbf{R}. \tag{9.8}$$

As an operator, the *Fourier transform* is the mapping $\mathcal{F}\colon f \mapsto \widehat{f}$. $\diamondsuit$

For notational clarity, we sometimes write $f^\wedge$ or $(f)^\wedge$ instead of $\widehat{f}$.

Even though a function $f \in L^1(\mathbf{R})$ is only defined almost everywhere, its Fourier transform $\widehat{f}$ is defined for every $\xi \in \mathbf{R}$ since the integral in equation (9.8) "converges absolutely":

$$\int_{-\infty}^{\infty} |f(x)\, e^{-2\pi i\xi x}|\, dx \;=\; \int_{-\infty}^{\infty} |f(x)|\, dx \;=\; \|f\|_{L^1} \;<\; \infty. \tag{9.9}$$

Both $f\colon \mathbf{R} \to \mathbf{C}$ and $\widehat{f}\colon \mathbf{R} \to \mathbf{C}$ are complex-valued functions on $\mathbf{R}$, though $\widehat{f}$ is defined everywhere while f is only defined almost everywhere. In fact, we show next that $\widehat{f}$ is a continuous function on $\mathbf{R}$.

Lemma 9.8. *If $f \in L^1(\mathbf{R})$ then $\widehat{f}$ is bounded and uniformly continuous on $\mathbf{R}$, and*

$$\|\widehat{f}\,\|_\infty \;\le\; \|f\|_{L^1}. \tag{9.10}$$

Proof. Given ξ, $\eta \in \mathbf{R}$ we have

$$\begin{aligned}
\bigl|\widehat{f}(\xi+\eta) - \widehat{f}(\xi)\bigr| &= \left|\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i(\xi+\eta)x}\, dx \;-\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx\right| \\
&\le \int_{-\infty}^{\infty} |f(x)|\, |e^{-2\pi i\xi x}|\, |e^{-2\pi i\eta x} - 1|\, dx \\
&= \int_{-\infty}^{\infty} |f(x)|\, |e^{-2\pi i\eta x} - 1|\, dx.
\end{aligned}$$

Note that the final quantity above is independent of ξ. For almost every x (those where $f(x)$ is defined), we have

$$\lim_{\eta\to 0} |f(x)|\, |e^{-2\pi i\eta x} - 1| \;=\; 0.$$

Also,

$$|f(x)|\, |e^{-2\pi i\eta x} - 1| \;\le\; 2|f(x)| \;\in\; L^1(\mathbf{R}),$$

so the Lebesgue Dominated Convergence Theorem (Theorem A.24) implies that

$$\sup_{\xi\in\mathbf{R}} \bigl|\widehat{f}(\xi+\eta) - \widehat{f}(\xi)\bigr| \;\le\; \int_{-\infty}^{\infty} |f(x)|\, |e^{-2\pi i\eta x} - 1|\, dx \;\to\; 0 \qquad \text{as } \eta\to 0.$$

Hence $\widehat{f}$ is uniformly continuous on $\mathbf{R}$.

Boundedness follows from equation (9.9), because

$$|\widehat{f}(\xi)| \;=\; \left|\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx\right| \;\le\; \int_{-\infty}^{\infty} |f(x)\, e^{-2\pi i\xi x}|\, dx \;=\; \|f\|_{L^1}.$$

Taking the supremum over $\xi \in \mathbf{R}$, we see that $\|\widehat{f}\,\|_\infty \le \|f\|_{L^1}$. □

When we dealt with the Fourier transform on the torus, the Fourier transform of a function $f \in L^2(\mathbf{T})$ was a *sequence* $\widehat{f} = \bigl(\widehat{f}(n)\bigr)_{n\in Z} \in \ell^2(\mathbf{Z})$. Now the Fourier transform of $f \in L^1(\mathbf{R})$ is a continuous function $\widehat{f} \in C_b(\mathbf{R})$. We will see how to define $\widehat{f}$ when $f \in L^2(\mathbf{R})$ in Section 9.4.

Example 9.9. The characteristic function $\chi_{[-T,T]}$ belongs to $L^1(\mathbf{R})$, and its Fourier transform is

$$(\chi_{[-T,T]})^\wedge(\xi) \;=\; \int_{-T}^{T} e^{-2\pi i\xi x}\, dx \;=\; \begin{cases} \frac{\sin 2\pi T\xi}{\pi\xi}, & \xi \ne 0, \\ 2T, & \xi = 0. \end{cases} \tag{9.11}$$

This is a continuous function on $\mathbf{R}$. We usually just write $(\chi_{[-T,T]})^\wedge(\xi) = \frac{\sin 2\pi T\xi}{\pi\xi}$, and assume it is defined appropriately at the origin.

An important special case is the *sinc function* $d_\pi = (\chi_{[-\frac{1}{2},\frac{1}{2}]})^\wedge$, which is given explicitly as

$$d_\pi(\xi) \;=\; (\chi_{[-\frac{1}{2},\frac{1}{2}]})^\wedge(\xi) \;=\; \frac{\sin \pi\xi}{\pi\xi}. \tag{9.12}$$

While the sinc function is continuous, it is not integrable on $\mathbf{R}$ (Exercise 9.1). On the other hand, d_π is continuous and $d_\pi(\xi) \to 0$ as $|\xi| \to \infty$, so $d_\pi \in C_0(\mathbf{R})$. Recalling that $C_0(\mathbf{R})$ is a Banach space with respect to the uniform norm, we observe that

$$\big\|(\chi_{[-\frac{1}{2},\frac{1}{2}]})^\wedge\big\|_\infty \;=\; \|d_\pi\|_\infty \;=\; 1,$$

so d_π is a unit vector in $C_0(\mathbf{R})$. $\diamondsuit$

Example 9.9 shows us that $f \in L^1(\mathbf{R})$ does not imply $\widehat{f} \in L^1(\mathbf{R})$ in general, so the Fourier transform does not map $L^1(\mathbf{R})$ into itself. Instead, we will show that $\mathcal{F}$ maps $L^1(\mathbf{R})$ into $C_0(\mathbf{R})$.

Theorem 9.10 (Riemann–Lebesgue Lemma). *The Fourier transform maps $L^1(\mathbf{R})$ into $C_0(\mathbf{R})$:*

$$f \in L^1(\mathbf{R}) \implies \widehat{f} \in C_0(\mathbf{R}).$$

Proof. We saw in Lemma 9.8 that $\widehat{f}$ is continuous, so our task is to show that $\widehat{f}$ decays to zero at $\pm\infty$. Since $e^{-\pi i} = -1$, for $\xi \neq 0$ we have

$$\begin{aligned}
\widehat{f}(\xi) \;&=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx \qquad (9.13)\\
&=\; -\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, e^{-2\pi i\xi(\frac{1}{2\xi})}\, dx\\
&=\; -\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi(x+\frac{1}{2\xi})}\, dx\\
&=\; -\int_{-\infty}^{\infty} f\Big(x - \frac{1}{2\xi}\Big)\, e^{-2\pi i\xi x}\, dx. \qquad (9.14)
\end{aligned}$$

Averaging equalities (9.13) and (9.14) yields

$$\widehat{f}(\xi) \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \Big(f(x) - f\Big(x - \frac{1}{2\xi}\Big)\Big)\, e^{-2\pi i\xi x}\, dx.$$

Using the strong continuity of translation proved in Exercise 9.2, it follows that

$$|\widehat{f}(\xi)| \;\le\; \frac{1}{2}\int_{-\infty}^{\infty} \Big|f(x) - f\Big(x - \frac{1}{2\xi}\Big)\Big|\, dx \;=\; \frac{1}{2}\,\|f - T_{\frac{1}{2\xi}} f\|_{L^1} \;\to\; 0$$

as $|\xi| \to \infty$. Therefore $\widehat{f} \in C_0(\mathbf{R})$. $\square$

Since equation (9.10) tells us that $\|\widehat{f}\|_\infty \le \|f\|_{L^1}$, the Fourier transform is a bounded mapping of $L^1(\mathbf{R})$ into $C_0(\mathbf{R})$, and its operator norm is at most 1. In fact, since $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ is a unit vector in $L^1(\mathbf{R})$, and its Fourier transform, the sinc function d_π, is a unit vector in $C_0(\mathbf{R})$, the operator norm of $\mathcal{F}$ is precisely

$$\|\mathcal{F}\|_{L^1 \to C_0} \;=\; 1. \tag{9.15}$$

However, $\mathcal{F}\colon L^1(\mathbf{R}) \to C_0(\mathbf{R})$ is not an isometry, and it can be shown that its range is a dense but proper subspace of $C_0(\mathbf{R})$ (compare Exercise 13.26 for the analogous statement for the Fourier transform on the torus).

Now we turn to synthesis for the Fourier transform. While synthesis for the Fourier transform on the torus converted a sequence c into a periodic function $\check{c}$, for the real line it transforms a function f into another function $\check{f}$.

Definition 9.11. The *inverse Fourier transform* of $f \in L^1(\mathbf{R})$ is

$$\check{f}(\xi) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{2\pi i \xi x}\, dx, \qquad \xi \in \mathbf{R}.$$

We also write $\mathcal{F}^{-1} f = \check{f}$ for the inverse Fourier transform of $f \in L^1(\mathbf{R})$. ◇

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. Therefore, $\check{f}$ and $\widehat{f}$ have many similarities, and analysis and synthesis are more “symmetric” for the Fourier transform on the real line than on the torus.

In particular, the Fourier transform and the inverse Fourier transform both map $L^1(\mathbf{R})$ into $C_0(\mathbf{R})$. However, if $f \in L^1(\mathbf{R})$ then $\widehat{f}$ need not belong to $L^1(\mathbf{R})$, and so we cannot compose the Fourier transform with the inverse Fourier transform in general. Thus our terminology is a little misleading, as the operator $\mathcal{F}^{-1}$ given in Definition 9.11 is not literally the inverse of the Fourier transform $\mathcal{F}\colon L^1(\mathbf{R}) \to C_0(\mathbf{R})$. However, we will show that the operator $\mathcal{F}^{-1}$ does play the role of synthesis and an inverse to $\mathcal{F}$ if we impose appropriate restrictions. Specifically, if f and $\widehat{f}$ are both integrable then f can be recovered from $\widehat{f}$ via the synthesis operation $\mathcal{F}^{-1}$.

Theorem 9.12 (Inversion Formula). *If $f,\ \widehat{f} \in L^1(\mathbf{R})$, then f and $\widehat{f}$ are continuous, and*

$$f(x) \;=\; \big(\widehat{f}\,\big)^{\vee}(x) \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi \tag{9.16}$$

with equality holding pointwise everywhere. Similarly,

$$f(x) \;=\; \big(\check{f}\,\big)^{\wedge}(x) \;=\; \int_{-\infty}^{\infty} \check{f}(\xi)\, e^{-2\pi i \xi x}\, d\xi$$

for every x. ◇

Theorem 9.12 should be compared to Theorem 13.25 for Fourier series, which states that if $f \in L^1(\mathbf{T})$ is such that $\widehat{f} \in \ell^1(\mathbf{T})$, then f is continuous and $f(x) = \big(\widehat{f}\big)^{\vee}(x)$ for all $x \in \mathbf{T}$. The proof of Theorem 9.12 is quite similar to the proof of Theorem 13.25 that we present in Chapter 13, and therefore the proof of Theorem 9.12 will be omitted.

Remark 9.13. We are abusing terminology in Theorem 9.12 when we say that "f is continuous." An element of $L^1(\mathbf{R})$ is an equivalence class of functions that are equal almost everywhere, so it does not make literal sense to say that $f \in L^1(\mathbf{R})$ is continuous. What we really mean is that there is a representative of f that is a continuous function, or, in other words, there is some continuous function g such that f is the equivalence class of functions that are equal to g almost everywhere. □

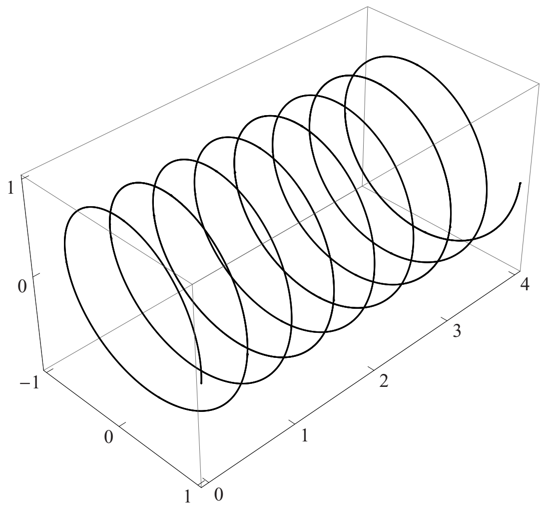

Fig. 9.1. Graph of $e_\xi(x) = e^{2\pi i \xi x}$ for $\xi = 2$ and $0 \le x \le 4$.

We expand on the meaning of equation (9.16). Picture the complex exponential

$$e_\xi(x) \;=\; e^{2\pi i \xi x} \;=\; \cos(2\pi\xi x) + i\sin(2\pi\xi x)$$

as a function of x. While x lies in $\mathbf{R}$, the function values $e_\xi(t)$ are complex numbers that lie on the unit circle S^1 in $\mathbf{C}$. As x ranges through the real line, the values $e_\xi(x) = e^{2\pi i \xi x}$ move around the unit circle S^1. If $\xi > 0$, then as x increases through an interval of length $1/\xi$, the values $e_\xi(x) = e^{2\pi i \xi x}$ move once around S^1 in the counter-clockwise direction. If ξ is negative, the same is true except that the values $e_\xi(x) = e^{2\pi i \xi x}$ circle around S^1 in the opposite direction. The function e_ξ is periodic with period $1/\xi$, and we say that e_ξ has *frequency* ξ. The graph of e_ξ is

$$\Gamma_\xi \;=\; \big\{(x, e^{2\pi i \xi x}) : x \in \mathbf{R}\big\} \;\subseteq\; \mathbf{R} \times \mathbf{C}.$$

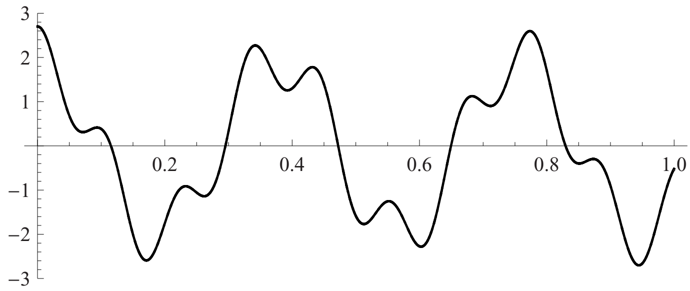

Fig. 9.2. Graph of $\varphi(x) = 2\cos(2\pi\sqrt{7}x) + 0.7\cos(2\pi 9x)$.

Identifying $\mathbf{R} \times \mathbf{C}$ with $\mathbf{R} \times \mathbf{R}^2 = \mathbf{R}^3$, the graph Γ_ξ is a helix in $\mathbf{R}^3$ coiling around the x-axis, which runs down the center of the helix (see Figure 9.1). The function e_ξ is a "pure tone" in some sense.

For a given fixed ξ, the function $\widehat{f}(\xi)\, e^{2\pi i\xi x}$ is a pure tone whose *amplitude* is the scalar $\widehat{f}(\xi)$. Given two frequencies η, ξ and amplitudes $\widehat{f}(\eta)$, $\widehat{f}(\xi)$, a function φ of the form

$$\varphi(x) \;=\; \widehat{f}(\eta)\, e^{2\pi i\eta x} + \widehat{f}(\xi)\, e^{2\pi i\xi x}$$

is a superposition of two pure tones (see the illustration of the real part of such a superposition in Figure 9.2). The real part of a superposition of 75 pure tones with randomly chosen frequencies and amplitudes is shown in Figure 9.3.

The Inversion Formula is an extreme version of such a superposition. It says that any function f (so long as f and $\widehat{f}$ are integrable) can be represented as an integral (in effect, a continuous sum) of pure tones $\widehat{f}(\xi)\, e^{2\pi i\xi x}$ over all possible frequencies $\xi \in \mathbf{R}$. By superimposing all the pure tones with the correct amplitudes, we create any function that we like. Of course, the "superposition" is an integral, not a finite or even countable sum, but still we are combining our very simple building blocks e_ξ to create very complicated functions f via the Inversion Formula. Equation (9.6) is an analogue of this for functions $f \in L^2(\mathbf{T})$. On the torus, we superimpose pure tones $e_n(x) = e^{2\pi i n x}$, all of which are 1-periodic, in order to synthesize arbitrary square integrable 1-periodic functions. On the real line, our pure tones $e_\xi(x) = e^{2\pi i\xi x}$ share no common period when we consider all $\xi \in \mathbf{R}$, and we synthesize arbitrary functions on $\mathbf{R}$ (as long as f, $\widehat{f} \in L^1(\mathbf{R})$).

A consequence of Theorem 9.12 is that functions in $L^1(\mathbf{R})$ are completely determined by their Fourier transforms.

Corollary 9.14 (Uniqueness Theorem). *If* $f \in L^1(\mathbf{R})$ *then*

$$f = 0 \text{ a.e.} \quad \Longleftrightarrow \quad \widehat{f} = 0 \text{ a.e.}$$

Consequently, the Fourier transform $\mathcal{F}\colon L^1(\mathbf{R}) \to C_0(\mathbf{R})$ *is injective.* $\diamondsuit$

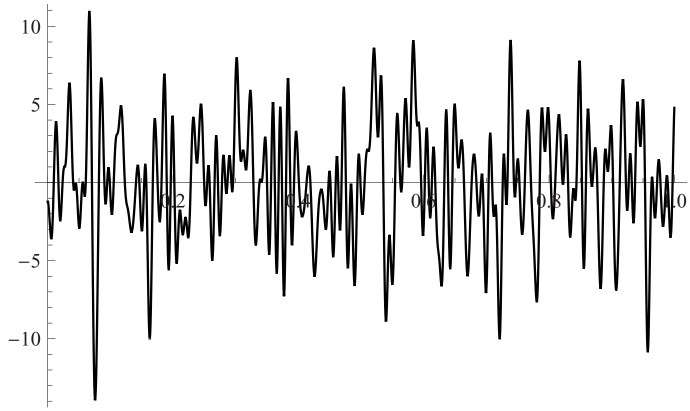

Fig. 9.3. Graph of 75 superimposed pure tones: $\varphi(x) = \sum_{k=1}^{75} \widehat{f}(\xi_k)\, \cos(2\pi\xi_k x)$.

One important property of the Fourier transform is that it interchanges translation with modulation, modulation with translation, and dilation with a reciprocal dilation. This was stated explicitly in Theorem 9.5. The proof of that theorem for $f \in L^1(\mathbf{R})$ follows by making simple changes of variable. For example, if we fix $f \in L^1(\mathbf{R})$ and $a \in \mathbf{R}$ then we have

$$\begin{aligned} (T_a f)^\wedge(\xi) &= \int f(x-a)\, e^{-2\pi i \xi x}\, dx \\ &= \int f(x)\, e^{-2\pi i \xi (x+a)}\, dx \\ &= e^{-2\pi i a \xi} \int f(x)\, e^{-2\pi i \xi x}\, dx \\ &= e^{-2\pi i a \xi}\, \widehat{f}(\xi) \;=\; M_{-a}\widehat{f}(\xi). \end{aligned} \tag{9.17}$$

This proves equation (9.2) in Theorem 9.5 when $f \in L^1(\mathbf{R})$. We assign the proof of the equalities in equations (9.3) and (9.4) for $f \in L^1(\mathbf{R})$ as Exercise 9.3.

Another important property of the Fourier transform is that it interchanges decay of a function f with smoothness of its Fourier transform $\widehat{f}$.

Theorem 9.15. *Given $f \in L^1(\mathbf{R})$, if $x^m f(x) \in L^1(\mathbf{R})$ for some $m \in \mathbf{N}$ then*

$$\widehat{f} \in C_0^m(\mathbf{R}),$$

i.e., $\widehat{f}$ is m-times differentiable and $\widehat{f}, \widehat{f}\,', \dots, \widehat{f}^{(m)} \in C_0(\mathbf{R})$. Furthermore, we have in this case that $x^k f(x) \in L^1(\mathbf{R})$ for $k = 0, \dots, m$, and the kth derivative of $\widehat{f}$ is the Fourier transform of $(-2\pi i x)^k f(x)$:

$$\widehat{f}^{(k)} \;=\; \frac{d^k}{d\xi^k}\widehat{f} \;=\; \bigl((-2\pi i x)^k f(x)\bigr)^\wedge, \qquad k = 0, \dots, m. \qquad \diamondsuit \tag{9.18}$$

We can guess that equation (9.18) should hold by formally exchanging a derivative and an integral:

$$\begin{aligned}\frac{d}{d\xi}\widehat{f}(\xi) &= \frac{d}{d\xi}\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\, \frac{d}{d\xi} e^{-2\pi i\xi x}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\, (-2\pi i x)\, e^{-2\pi i\xi x}\, dx \\ &= (-2\pi i x f(x))^{\wedge}(\xi).\end{aligned}$$

Essentially, the proof of Theorem 9.15 is the justification of this interchange, and this is done by applying the Lebesgue Dominated Convergence Theorem (Exercise 9.4).

Similarly, smoothness is interchanged with decay under the Fourier transform.

Theorem 9.16. *Let $f \in L^1(\mathbf{R})$ and $m \in \mathbf{N}$ be given. If f is everywhere m-times differentiable and $f, f', \dots, f^{(m)} \in L^1(\mathbf{R})$, then*

$$(f^{(k)})^{\wedge}(\xi) = (2\pi i\xi)^k\, \widehat{f}(\xi), \qquad k = 0, \dots, m.$$

Consequently,

$$|\widehat{f}(\xi)| \le \frac{\|f^{(m)}\|_{L^1}}{|2\pi\xi|^m}, \qquad \xi \ne 0. \qquad \diamondsuit \tag{9.19}$$

The proof of Theorem 9.16 is somewhat more subtle than that of Theorem 9.15 and will be omitted. In any case, the point is that the smoother an integrable function f is, the faster its Fourier transform $\widehat{f}$ will decay at infinity. In particular, if f is smooth enough then $\widehat{f}$ will be integrable.

Corollary 9.17. *If $f \in L^1(\mathbf{R})$ is twice differentiable and $f'' \in L^1(\mathbf{R})$, then $\widehat{f}$ decays like $C/|\xi|^2$ and therefore $\widehat{f} \in L^1(\mathbf{R})$. In particular,*

$$f \in C_c^2(\mathbf{R}) \quad \Longrightarrow \quad \widehat{f} \in L^1(\mathbf{R}).$$

Proof. Since $\widehat{f}$ is continuous it is bounded near the origin. Also, since f'' is integrable, Theorem 9.16 tells us that $|\widehat{f}(\xi)| \le C/|\xi|^2$ away from the origin. The combination of these facts implies that $\widehat{f}$ is integrable. □

Note that if a function f has both smoothness and decay, then these are interchanged by the Fourier transform, so $\widehat{f}$ has both decay and smoothness. We collect those functions that have an extreme amount of simultaneous smoothness and decay to form the Schwartz class.

Definition 9.18 (Schwartz Class). The *Schwartz class* $\mathcal{S}(\mathbf{R})$ consists of all infinitely differentiable functions $f\colon \mathbf{R} \to \mathbf{C}$ that satisfy

$$\|x^m f^{(n)}(x)\|_\infty < \infty, \qquad m, n \geq 0. \qquad \diamondsuit$$

Consequently, if $f \in \mathcal{S}(\mathbf{R})$ then for each choice of m and n there exists a constant C_{mn} such that

$$|f^{(n)}(x)| \leq \frac{C_{mn}}{|x|^m}, \qquad x \neq 0. \tag{9.20}$$

The constants C_{mn} may grow with m or n. We often euphemistically refer to functions satisfying equation (9.20) for all m and n as having *rapid decay* at infinity.

The space $C_c^\infty(\mathbf{R})$ consisting of all infinitely differentiable functions that vanish outside of some finite interval is contained in $\mathcal{S}(\mathbf{R})$. An example of a function in $\mathcal{S}(\mathbf{R})$ that is not compactly supported is the *Gaussian function* $\phi(x) = e^{-\pi x^2}$.

Although we will not prove it, it can be shown that the Schwartz space is invariant under the Fourier transform.

Theorem 9.19. *The Fourier transform maps $\mathcal{S}(\mathbf{R})$ bijectively onto itself.* $\diamondsuit$

As remarked above, we have $C_c^\infty(\mathbf{R}) \subseteq \mathcal{S}(\mathbf{R})$. Here is a similar inclusion formulated "on the Fourier side."

Corollary 9.20. *If $f \in L^1(\mathbf{R})$ and $\widehat{f} \in C_c^\infty(\mathbf{R})$, then $f \in \mathcal{S}(\mathbf{R})$.*

Proof. The function $\widehat{f}$ is integrable by hypothesis, so $f = \big(\widehat{f}\,\big)^\vee$ by the Inversion Formula (Theorem 9.12). However, $\widehat{f} \in \mathcal{S}(\mathbf{R})$, so Theorem 9.19 implies that $\big(\widehat{f}\,\big)^\vee \in \mathcal{S}(\mathbf{R})$ as well. $\square$

Exercises

9.1. Prove that the sinc function $d_\pi(\xi) = \frac{\sin \pi\xi}{\pi\xi}$ does not belong to $L^1(\mathbf{R})$.

9.2. (a) Prove that every function $f \in C_0(\mathbf{R})$ is uniformly continuous, and show that uniform continuity is equivalent to the statement

$$\lim_{a \to 0} \|T_a f - f\|_{L^\infty} = 0. \tag{9.21}$$

Show that equation (9.21) can fail if we only assume $f \in C_b(\mathbf{R})$.

(b) Show that if $1 \leq p < \infty$ and $f \in L^p(\mathbf{R})$, then

$$\lim_{a \to 0} \|T_a f - f\|_{L^p} = 0.$$

Because of this, we say that translation is *strongly continuous* on $L^p(\mathbf{R})$ for finite p.

9.3. Prove that the equalities in equations (9.3) and (9.4) hold for all functions $f \in L^1(\mathbf{R})$.

9.4. Prove Theorem 9.15.

9.4 The Fourier Transform on $L^2(\mathbf{R})$

So far we have defined the Fourier transform as an operator on $L^1(\mathbf{R})$. Now we will see how to extend it to the domain $L^2(\mathbf{R})$.

We let $C_c^m(\mathbf{R})$ denote the space of m-times differentiable functions on $\mathbf{R}$ that are compactly supported. We will need to use the fact that $C_c^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$. The proof of this is quite similar to the solution of Exercise 13.17 in Chapter 13, and will be omitted. In fact, the same argument shows that $C_c^m(\mathbf{R})$ and $C_c^\infty(\mathbf{R})$ are dense in $L^p(\mathbf{R})$ for every $m \in \mathbf{N}$ and $1 \le p < \infty$ (an explicit example of a function in $C_c^\infty(\mathbf{R})$ is constructed in Exercise 11.9).

Theorem 9.21. *If $f \in C_c^2(\mathbf{R})$ then $\widehat{f} \in L^2(\mathbf{R})$ and we have*

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}.$$

Proof. We sketch the proof and assign the details as Exercise 9.5.

Note that $f \in L^1(\mathbf{R})$, so $\widehat{f}$ is a continuous function. Let $\widetilde{f}$ denote the *involution* $\widetilde{f}(x) = \overline{f(-x)}$ of f, whose Fourier transform is

$$(\widetilde{f}\,)^\wedge(\xi) = \overline{\widehat{f}(\xi)}, \qquad \xi \in \mathbf{R}.$$

The *convolution* of f with $\widetilde{f}$ is

$$g(x) = (f * \widetilde{f}\,)(x) = \int_{-\infty}^{\infty} f(y)\, \widetilde{f}(x-y)\, dy.$$

Exercise 9.5 shows that g is continuous and compactly supported. Note that

$$g(0) = \int_{-\infty}^{\infty} f(y)\, \widetilde{f}(-y)\, dy = \int_{-\infty}^{\infty} f(y)\, \overline{f(y)}\, dy = \|f\|_{L^2}^2.$$

Since g is integrable, it has a Fourier transform in the sense of Definition 9.7. Using Exercise 9.5 we see that $\widehat{g}$ has the form

$$\widehat{g}(\xi) = (f * \widetilde{f}\,)^\wedge(\xi) = \widehat{f}(\xi)\,(\widetilde{f}\,)^\wedge(\xi) = |\widehat{f}(\xi)|^2.$$

Exercise 9.5 also shows that $g = f * \widetilde{f}$ is just as smooth as f, so $g \in C_c^2(\mathbf{R})$ and therefore $\widehat{g} \in L^1(\mathbf{R})$ by Corollary 9.17. Hence the Inversion Formula (Theorem 9.12) applies, so $g(x) = (\widehat{g}\,)^\vee(x)$ for every x. Consequently,

$$g(0) = (\widehat{g}\,)^\vee(0) = \int_{-\infty}^{\infty} \widehat{g}(\xi)\, d\xi = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, d\xi = \|\widehat{f}\,\|_{L^2}^2.$$

Therefore $\|\widehat{f}\,\|_{L^2}^2 = g(0) = \|f\|_{L^2}$. $\square$

Convolution is a very important operation, and it will be studied in detail in the setting of the torus in Section 13.3. Very similar results hold for convolution of functions on the real line; see Exercise 9.5.

Theorem 9.21 shows that the mapping $\mathcal{F}\colon f \mapsto \widehat{f}$ is an isometric map of $\big(C_c^2(\mathbf{R}), \|\cdot\|_{L^2}\big)$ into $L^2(\mathbf{R})$. By Exercise 1.72, since $C_c^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$ the mapping $\mathcal{F}$ has a continuous extension to an isometric map $\mathcal{F}$ of $L^2(\mathbf{R})$ into $L^2(\mathbf{R})$. Since the Fourier transform maps the Schwartz space $\mathcal{S}(\mathbf{R})$ onto itself, the range of $\mathcal{F}$ includes the dense space $\mathcal{S}(\mathbf{R})$. As $\mathcal{F}$ is isometric, it has a closed range, so we conclude that $\mathcal{F}$ is an isometric mapping of $L^2(\mathbf{R})$ onto itself. That is, $\mathcal{F}$ is a unitary operator on $L^2(\mathbf{R})$.

Definition 9.22 (Fourier Transform on $L^2(\mathbf{R})$). The unitary mapping $\mathcal{F}\colon L^2(\mathbf{R}) \to L^2(\mathbf{R})$ defined above is the *Fourier transform* on $L^2(\mathbf{R})$. $\diamondsuit$

The definition of the Fourier transform on $L^2(\mathbf{R})$ is not as pleasantly explicit as its definition on $L^1(\mathbf{R})$. In essence, the point is that $\mathcal{F}$ is isometric with respect to the L^2-norm on some space that is dense in both $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$, and this allows us to implicitly extend $\mathcal{F}$ to a unitary mapping on all of $L^2(\mathbf{R})$.

To compute the Fourier transform of $f \in L^2(\mathbf{R})$ using this definition, we must choose functions $f_n \in C_c^2(\mathbf{R})$ such that $f_n \to f$ in $L^2(\mathbf{R})$. Then the Fourier transform of f is the unique function $\widehat{f} \in L^2(\mathbf{R})$ such that $\widehat{f_n} \to \widehat{f}$ in the norm of $L^2(\mathbf{R})$. It can be shown that this definition is independent of the choice of functions $\widehat{f_n}$, so the Fourier transform is uniquely defined on $L^2(\mathbf{R})$, and this definition extends the Fourier transform on $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Once defined in this way, it follows that if $\{f_n\}_{n\in\mathbf{N}}$ is any sequence of functions in $L^2(\mathbf{R})$ such that $f_n \to f$ in L^2-norm, then $\widehat{f_n} \to \widehat{f}$ in L^2-norm.

Since $\mathcal{F}$ is unitary on $L^2(\mathbf{R})$, the inversion formula holds trivially on this space:

$$\forall\, f \in L^2(\mathbf{R}), \quad f \;=\; \big(\widehat{f}\,\big)^{\vee} \;=\; \big(\check{f}\,\big)^{\wedge}. \tag{9.22}$$

In contrast to Theorem 9.12, this is an equality of functions in $L^2(\mathbf{R})$ and hence holds pointwise only in the almost everywhere sense. On the other hand, this equation is the exact analogue for $L^2(\mathbf{R})$ of equation (9.6) for the Fourier transform on $L^2(\mathbf{T})$. Analysis followed by synthesis is the identity, as is synthesis followed by analysis. Just as the Fourier transform for the torus, $\mathcal{F}\colon L^2(\mathbf{T}) \to \ell^2(\mathbf{Z})$, is unitary, so is the Fourier transform $\mathcal{F}\colon L^2(\mathbf{R}) \to L^2(\mathbf{R})$.

Since the Fourier transform is unitary, we have the following equalities.

Theorem 9.23. *The following statements hold for all $f,\, g \in L^2(\mathbf{R})$.*

(a) Plancherel's Equality*:* $\|\widehat{f}\,\|_{L^2} = \|f\|_{L^2}$.

(b) Parseval's Equality: $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\,\rangle$. $\diamondsuit$

The Plancherel and Parseval Equalities are equivalent, and as a consequence these names are often used interchangeably.

Remark 9.24. We emphasize that if $f \in L^1(\mathbf{R})$ then $\widehat{f}$ is a continuous function that is defined everywhere, while if $f \in L^2(\mathbf{R})$ then $\widehat{f} \in L^2(\mathbf{R})$ and so is only defined almost everywhere. These transforms coincide if $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ in the usual sense of identifying the continuous function $\widehat{f} \in C_0(\mathbf{R})$ with the equivalence class of functions that equal it almost everywhere, and it is this equivalence class that we call the function $\widehat{f} \in L^2(\mathbf{R})$. $\diamondsuit$

Many formulas that hold for the Fourier transform on $L^1(\mathbf{R})$ have an analogue on $L^2(\mathbf{R})$, though we must often replace pointwise everywhere statements with pointwise almost everywhere statements. For example, equation (9.17) says that

$$(T_a f)^\wedge = M_{-a}\widehat{f}, \qquad f \in L^1(\mathbf{R}),$$

and we will show how to extend this equality to functions in $L^2(\mathbf{R})$.

Fix $f \in L^2(\mathbf{R})$ and $a \in \mathbf{R}$. Since $C_c^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, there exist functions $f_n \in C_c^2(\mathbf{R})$ such that $f_n \to f$ in L^2-norm. As the Fourier transform is unitary, $\widehat{f_n} \to \widehat{f}$ in L^2-norm. By Theorem A.23, convergence in L^2-norm implies the existence of a subsequence that converges pointwise almost everywhere. Therefore, by passing to a subsequence we can assume that $\widehat{f_n}(\xi) \to \widehat{f}(\xi)$ a.e. Since $f_n \in L^1(\mathbf{R})$ we have $(T_a f_n)^\wedge(\xi) = M_{-a}\widehat{f_n}(\xi)$ for all ξ. Taking limits, it follows that $(T_a f)^\wedge(\xi) = M_{-a}\widehat{f}(\xi)$ for almost every ξ. In particular, this proves that equation (9.2) in Theorem 9.5 holds for $f \in L^2(\mathbf{R})$, and we assign the proof of equations (9.3) and (9.4) in that theorem for $f \in L^2(\mathbf{R})$ as Exercise 9.6.

Exercises

9.5. This exercise provides the details needed for the proof of Theorem 9.21. Let f, $g \in L^1(\mathbf{R})$ be fixed, and prove the following facts.

(a) The Fourier transform of the involution $\widetilde{f}(x) = \overline{f(-x)}$ is

$$(\widetilde{f}\,)^\wedge(\xi) \;=\; \overline{\widehat{f}(\xi)}, \qquad \xi \in \mathbf{R}.$$

(b) The convolution

$$(f * g)(x) \;=\; \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

exists for almost every x, and $f * g \in L^1(\mathbf{R})$ with $\|f * g\|_{L^1} \le \|f\|_{L^1}\, \|g\|_{L^1}$. Further, $f * g = g * f$.

(c) The Fourier transform of $f * g$ is

$$(f * g)^\wedge(\xi) \;=\; \widehat{f}(\xi)\, \widehat{g}(\xi), \qquad \xi \in \mathbf{R}.$$

(d) If f, g are compactly supported, then so is $f * g$.

(e) If g is differentiable and g' is integrable and bounded, then $f * g$ is differentiable and $(f * g)' = f * g'$.

9.6. Use the fact that $C_c^2(\mathbf{R})$ is dense in $L^2(\mathbf{R})$ to show that equations (9.3) and (9.4) hold for $f \in L^2(\mathbf{R})$ in the sense of pointwise almost everywhere equality of functions.

9.7. (a) Prove that if $f \in L^1(\mathbf{R})$ is real and even, then $\widehat{f}$ is real and even.

(b) Show that part (a) also holds if we assume $f \in L^2(\mathbf{R})$.

9.8. (a) Prove that if $f \in L^1(\mathbf{R})$ and $\widehat{f} \in L^1(\mathbf{R})$, then $f^{\wedge\wedge}(\xi) = f(-\xi)$ and $f^{\wedge\wedge\wedge\wedge}(\xi) = f(\xi)$ for all $\xi \in \mathbf{R}$.

(b) Prove that if $f \in L^2(\mathbf{R})$, then $f^{\wedge\wedge}(\xi) = f(-\xi)$ and $f^{\wedge\wedge\wedge\wedge}(\xi) = f(\xi)$ for almost every $\xi \in \mathbf{R}$.

9.5 Absolute Continuity

We saw in Theorems 9.15 and 9.16 that the Fourier transform interchanges smoothness of a function $f \in L^1(\mathbf{R})$ with decay of its Fourier transform $\widehat{f}$, and likewise decay is interchanged with smoothness. There are many ways to quantify the precise way that smoothness is interchanged with decay, and in Chapter 11 we will need the following version for functions in $L^2(\mathbf{R})$. This result formulates smoothness in terms of absolute continuity. Essentially, absolutely continuous functions are those for which the Fundamental Theorem of Calculus is valid.

Definition 9.25. A function $g\colon [a,b] \to \mathbf{C}$ is *absolutely continuous* on $[a,b]$ if g is differentiable at almost every point in $[a,b]$, $g' \in L^1[a,b]$, and

$$g(x) - g(a) \;=\; \int_a^x g'(t)\, dt, \qquad x \in [a,b]. \qquad \diamondsuit$$

For example, let φ be the Cantor–Lebesgue function on $[0,1]$ constructed in Exercise 1.24. This is a continuous, nonzero function that is differentiable almost everywhere but satisfies $\varphi' = 0$ a.e. Hence the Fundamental Theorem of Calculus fails for the Cantor–Lebesgue function, so this function is continuous but not absolutely continuous. All differentiable functions on $[a,b]$ whose derivative is continuous on $[a,b]$ are absolutely continuous, but the converse fails. For example, if f is any function in $L^1[a,b]$ and $g(x) = \int_a^x f(t)\, dt$, then it follows from the Lebesgue Differentiation Theorem (Theorem A.30) that for almost every $x \in (a,b)$ we have

$$g'(x) \;=\; \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt \;=\; f(x).$$

Hence g is absolutely continuous on $[a,b]$ and $g' = f$ a.e. In fact, every absolutely continuous function has precisely this form, since if g is absolutely continuous then Definition 9.25 states that $g' \in L^1[a,b]$ and $g(x) = \int_a^x g'(t)\,dt$.

Remark 9.26. Often, a function g on $[a,b]$ is defined to be absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any finite or countably infinite collection of nonoverlapping subintervals $\{[a_j,b_j]\}_j$ of $[a,b]$, we have

$$\sum_j (b_j - a_j) < \delta \qquad \Longrightarrow \qquad \sum_j |f(b_j) - f(a_j)| < \varepsilon.$$

This can be shown to be equivalent to Definition 9.25, e.g., see [Fol99] or [WZ77]. ♢

One proof of the next result is given in [Heil].

Theorem 9.27. (a) *If $f \in L^2(\mathbf{R})$ and $\int_{-\infty}^{\infty} |x|^2\,|f(x)|^2\,dx < \infty$, then $\widehat{f}$ is absolutely continuous on every finite interval $[a,b]$, $\widehat{f}$ is differentiable at almost every point of $\mathbf{R}$, $\widehat{f}\,' \in L^2(\mathbf{R})$, and $\widehat{f}\,' = \widehat{g}$ a.e. where $g(x) = -2\pi i x f(x)$.*

(b) *If $f \in L^2(\mathbf{R})$ and $\int_{-\infty}^{\infty} |\xi|^2\,|\widehat{f}(\xi)|^2\,d\xi < \infty$, then f is absolutely continuous on every finite interval $[a,b]$, f is differentiable at almost every point of $\mathbf{R}$, $f' \in L^2(\mathbf{R})$, and $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$ a.e.* ♢

One important fact about absolutely continuous functions that we will sometimes need is that integration by parts is valid for such functions. The proof of this result is assigned as Exercise 9.9.

Theorem 9.28 (Integration by Parts). *If f, g are absolutely continuous on $[a,b]$, then*

$$\int_a^b f(x)\,g'(x)\,dx \;=\; f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\,g(x)\,dx. \qquad \diamondsuit$$

Exercises

9.9. Prove Theorem 9.28.

10

Sampling, Weighted Exponentials, and Translations

Now we will apply the machinery of bases and frames to analyze specific types of sequences that arise in many situations in applied harmonic analysis and other areas. This chapter focuses on sequences closely related to the trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$, which forms an orthonormal basis for $L^2(\mathbf{T})$, while Chapter 11 is devoted to Gabor systems and Chapter 12 to wavelets.

We usually think of the exponential function

$$e_n(x) \;=\; e^{2\pi inx}$$

as being a 1-periodic function on $\mathbf{R}$. As such it belongs to the space $L^2(\mathbf{T})$, but it does not belong to $L^2(\mathbf{R})$ because $|e^{2\pi inx}| = 1$ for every x. In Sections 10.1 and 10.2 we take a different approach. Instead of considering the complex exponential to be periodic, we will restrict our attention to a domain of length 1 and extend by zero outside of this domain. In this way we obtain a sequence that belongs to $L^2(\mathbf{R})$. Despite the simplicity of this idea, it yields significant results. It will be convenient in this section to take our domain to be symmetric about the origin, which means that we are considering the functions

$$\epsilon_n \;=\; e_n \cdot \chi_{[-\frac{1}{2},\frac{1}{2}]} \;\in\; L^2(\mathbf{R}).$$

Since the interval $[-\frac{1}{2},\frac{1}{2}]$ has length 1, $\{\epsilon_n\}_{n\in\mathbf{Z}}$ is an orthonormal sequence in $L^2(\mathbf{R})$. Of course, it is not complete in $L^2(\mathbf{R})$. Rather, its closed span is

$$L^2_{[\frac{1}{2},\frac{1}{2}]}(\mathbf{R}) \;=\; \Big\{f \in L^2(\mathbf{R}) : f(x) = 0 \text{ for a.e. } |x| > \frac{1}{2}\Big\}.$$

We will refer to $L^2_{[\frac{1}{2},\frac{1}{2}]}(\mathbf{R})$ as the subspace of $L^2(\mathbf{R})$ consisting of functions that are "timelimited" to the interval $[-\frac{1}{2},\frac{1}{2}]$.

Since the Fourier transform is unitary, the sequence $\{\widehat{\epsilon_n}\}_{n\in\mathbf{Z}}$ is also orthonormal in $L^2(\mathbf{R})$. Letting

$$d_\pi(x) \;=\; \frac{\sin \pi\xi}{\pi\xi}$$

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be the *sinc function* from Example 9.9, we can write $\widehat{\epsilon_n}$ explicitly as a translated sinc function:

$$\widehat{\epsilon_n}(\xi) = \int_{-1/2}^{1/2} e^{2\pi inx}\, e^{-2\pi i\xi x}\, dx = \frac{\sin \pi(\xi - n)}{\pi(\xi - n)} = d_\pi(\xi - n) = T_n d_\pi(\xi).$$

The functions $T_n d_\pi$ are not compactly supported, but each one has a Fourier transform that is nonzero only within $[-\frac{1}{2}, \frac{1}{2}]$. In fact, since $\widehat{f}(\xi) = \check{f}(-\xi)$, we have

$$(T_n d_\pi)^\wedge(\xi) = (\widehat{\epsilon_n})^\wedge(\xi) = (\widehat{\epsilon_n})^\vee(-\xi) = \epsilon_n(-\xi) = e^{-2\pi in\xi}\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi).$$

The closed span of the orthonormal sequence $\{\widehat{\epsilon_n}\}_{n\in\mathbf{Z}} = \{T_n d_\pi\}_{n\in\mathbf{Z}}$ is

$$\mathrm{PW}(\mathbf{R}) = \Big\{ f \in L^2(\mathbf{R}) : \widehat{f}(\xi) = 0 \text{ for a.e. } |\xi| > \frac{1}{2} \Big\}.$$

We call $\mathrm{PW}(\mathbf{R})$ the *Paley–Wiener space* of functions "bandlimited" to the interval $[-\frac{1}{2}, \frac{1}{2}]$.

The fact that $\{T_n d_\pi\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $\mathrm{PW}(\mathbf{R})$ will lead us in Section 10.2 to a proof of the *Classical Sampling Theorem* for the case of critical sampling (see Theorem 10.7). The Sampling Theorem is a ubiquitous result in signal processing that gives an algorithm for reconstructing a bandlimited function f from the countably many "sample values" $f(n)$, $n \in \mathbf{Z}$. Further, by replacing the exponential functions $e^{2\pi inx}$ with $e^{2\pi ibnx}$ we can construct redundant frames for the Paley–Wiener space and thereby obtain the Sampling Theorem for oversampling, which means recovery from the samples $f(bn)$, $n \in \mathbf{Z}$ when $0 < b < 1$ is fixed.

Sections 10.1 and 10.2 are devoted to bandlimited functions and the proof of the Sampling Theorem, both of which deal with functions in $L^2(\mathbf{R})$. In Section 10.3 we return to $L^2(\mathbf{T})$ and modify the trigonometric system in a different way, by introducing a 1-periodic "weight function" φ on $\mathbf{T}$. This gives us a *system of weighted exponentials* $\{e^{2\pi inx}\varphi(x)\}_{n\in\mathbf{Z}}$ in $L^2(\mathbf{T})$. We will completely characterize the functions φ for which this system is a Schauder basis, frame, Bessel sequence, and so forth.

Instead of thinking of $e^{2\pi inx}\varphi(x)$ as being a periodic function, we could simply cut it off outside the interval $[-\frac{1}{2}, \frac{1}{2}]$, giving us the function

$$\varphi_n(x) = e^{2\pi inx}\, \varphi(x)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) \in L^2(\mathbf{R}).$$

In the terminology of Notation 9.4 we can write $\varphi_n = M_n\varphi_0$, i.e., φ_n is a modulation of the timelimited function $\varphi_0 = \varphi\,\chi_{[-\frac{1}{2},\frac{1}{2}]}$. As the Fourier transform converts modulation into translation, $\widehat{\varphi}_n$ is therefore a translation of $\widehat{\varphi}_0$:

$$\widehat{\varphi}_n(\xi) = (M_n\varphi_0)^\wedge(\xi) = T_n\widehat{\varphi}_0(\xi) = \widehat{\varphi}_0(\xi - n).$$

Since the Fourier transform is unitary, the frame and basis properties of the sequence $\{\varphi_n\}_{n\in\mathbf{Z}}$ carry over to the sequence

$$\{\widehat{\varphi}_n\}_{n\in\mathbf{Z}} = \{T_n\widehat{\varphi}_0\}_{n\in\mathbf{Z}} = \{\widehat{\varphi}_0(\xi-n)\}_{n\in\mathbf{Z}}.$$

Therefore we can apply the results of Section 10.3 to understand the properties of the *system of translates* $\{T_n\widehat{\varphi}_0\}_{n\in\mathbf{Z}}$. However, if we replace the bandlimited function $\widehat{\varphi}_0$ by a generic function $g \in L^2(\mathbf{R})$, then it is not nearly so clear what properties the sequence $\{T_ng\}_{n\in\mathbf{Z}}$ will possess. We consider these general systems of integer translations in Section 10.4, and will determine exactly when the system $\{g(x-n)\}_{n\in\mathbf{Z}}$ is a Schauder basis, frame, Riesz basis, Bessel sequence, etc., for its closed span in $L^2(\mathbf{R})$.

10.1 Bandlimited Functions

Our main goal in Sections 10.1 and 10.2 is to prove the Sampling Theorem for bandlimited functions. We break this into two parts. In this section we consider the Paley–Wiener space of bandlimited functions in some detail, and then in the following section we use this knowledge to prove the Sampling Theorem and some related results.

The precise definition of timelimited and bandlimited functions is as follows.

Definition 10.1. Fix $T,\ \Omega > 0$.

(a) A function $f \in L^2(\mathbf{R})$ is *timelimited* to $[-T,T]$ if $\operatorname{supp}(f) \subseteq [-T,T]$ (that is, $f(x) = 0$ for almost every $|x| > T$). We denote the space of functions in $L^2(\mathbf{R})$ timelimited to $[-T,T]$ by

$$L^2_{[-T,T]}(\mathbf{R}) = \Big\{f \in L^2(\mathbf{R}) : \operatorname{supp}(f) \subseteq [-T,T]\Big\}.$$

(b) A function $f \in L^2(\mathbf{R})$ is *bandlimited* to $[-\Omega,\Omega]$ if $\operatorname{supp}(\widehat{f}) \subseteq [-\Omega,\Omega]$ (that is, $\widehat{f}(\xi) = 0$ for almost every $|\xi| > \Omega$). We denote the space of functions in $L^2(\mathbf{R})$ bandlimited to $[-\Omega,\Omega]$ by

$$\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R}) = \big\{f \in L^2(\mathbf{R}) : \operatorname{supp}(\widehat{f}) \subseteq [-\Omega,\Omega]\big\}. \qquad \diamondsuit$$

The spaces $L^2_{[-T,T]}(\mathbf{R})$ and $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$ are closed subspaces of $L^2(\mathbf{R})$ (Exercise 10.3). Further, as the notation suggests, $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$ is the image of $L^2_{[-\Omega,\Omega]}(\mathbf{R})$ under the Fourier transform. In fact, since the interval $[-\Omega,\Omega]$ is symmetric about the origin, $\mathcal{F}$ and $\mathcal{F}^{-1}$ both map $L^2_{[-\Omega,\Omega]}(\mathbf{R})$ unitarily onto $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$, and consequently $\mathcal{F}$ and $\mathcal{F}^{-1}$ each map $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$ unitarily back to $L^2_{[-\Omega,\Omega]}(\mathbf{R})$ (Exercise 10.1).

Many signals encountered in "real life" are bandlimited, or are so close to being bandlimited that we can safely regard them as being so. For example, although a generic sound wave may comprise a large range of frequencies, the human ear is only responsive to a limited range of these frequencies, approximately 20 Hz to 20,000 Hz (Hz is the abbreviation for Hertz, which is frequency measured in cycles per second). The signal produced by the typical telephone speaker encompasses a much smaller range of frequencies, often only 300 Hz to 3000 Hz. Essentially, the Fourier transform of the sound signal coming out of a telephone speaker is supported within $[-3000,-300]\cup[300,3000]$, so this signal belongs to $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$ with $\Omega = 3000$. Interestingly, a nonzero function cannot be *simultaneously* timelimited and bandlimited, see Corollary 10.6.

It will be most convenient for us fix $\Omega = \frac{1}{2}$ throughout this section (by Exercise 10.2, we can always reduce to this case by applying a dilation). We give a special name to the space of functions bandlimited to the interval $[-\frac{1}{2},\frac{1}{2}]$.

Definition 10.2 (Paley–Wiener Space). The *Paley–Wiener space* $\mathrm{PW}(\mathbf{R})$ is the space of functions in $L^2(\mathbf{R})$ whose Fourier transforms are supported within the interval $[-\frac{1}{2},\frac{1}{2}]$:

$$\mathrm{PW}(\mathbf{R}) \;=\; \mathcal{F}L^2_{[\frac{1}{2},\frac{1}{2}]}(\mathbf{R}) \;=\; \Bigl\{f \in L^2(\mathbf{R}) : \mathrm{supp}(\widehat{f}\,) \subseteq [\tfrac{1}{2},\tfrac{1}{2}]\Bigr\}. \qquad \diamondsuit$$

Example 10.3. Consider the sinc function $d_\pi(x) = \frac{\sin \pi\xi}{\pi\xi}$. We implicitly take $d_\pi(0) = 1$, so d_π is infinitely differentiable on $\mathbf{R}$.

The sinc function is bandlimited, although since d_π is not integrable we cannot use the formula $\widehat{f}(\xi) = \int f(x)\, e^{-2\pi i\xi x}\, dx$ to check this. Instead, we note that $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ belongs to both $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$, and its inverse Fourier transform is

$$\bigl(\chi_{[-\frac{1}{2},\frac{1}{2}]}\bigr)^{\vee}(\xi) \;=\; \int_{-1/2}^{1/2} e^{2\pi i\xi x}\, dx \;=\; \frac{\sin \pi\xi}{\pi\xi} \;=\; d_\pi(x).$$

Since $d_\pi \in L^2(\mathbf{R})$ and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are inverse operations on $L^2(\mathbf{R})$, we conclude that

$$\widehat{d_\pi} \;=\; \bigl(\chi_{[-\frac{1}{2},\frac{1}{2}]}\bigr)^{\vee\wedge} \;=\; \chi_{[-\frac{1}{2},\frac{1}{2}]}.$$

Thus $\widehat{d_\pi}$ is supported in $[-\frac{1}{2},\frac{1}{2}]$, so d_π belongs to $\mathrm{PW}(\mathbf{R})$. Note that since $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ is even, its Fourier transform and inverse Fourier transform are equal, and similarly the Fourier and inverse Fourier transforms of d_π coincide.

Since the Fourier transform converts translation into modulation, every translate $T_a d_\pi(x) = d_\pi(x-a)$ of the sinc function also belongs to $\mathrm{PW}(\mathbf{R})$. Explicitly, by applying Theorem 9.3 we see that

$$(T_a d_\pi)^\wedge(\xi) \;=\; M_{-a}\widehat{d_\pi}(\xi) \;=\; e^{2\pi i a x}\,\chi_{[-\frac{1}{2},\frac{1}{2}]}(x). \qquad \diamondsuit$$

We will prove some of the most important properties of the Paley–Wiener space in the next theorem. To motivate statement (d) of Theorem 10.4, recall that the Fourier transform interchanges smoothness with decay (see Theorems 9.15 and 9.16). Qualitatively speaking, the faster that $\widehat{f}$ decays at $\pm\infty$, the smoother that f must be. The ultimate in decay is compact support, since this means the function is zero outside of some finite interval. Hence if f is bandlimited then $\widehat{f}$ has extreme decay, and so we expect that f must be very smooth.

Theorem 10.4. (a) $\mathrm{PW}(\mathbf{R})$ *is a closed subspace of* $L^2(\mathbf{R})$.

(b) $\mathrm{PW}(\mathbf{R})$ *is* translation-invariant*, i.e.,*

$$f \in \mathrm{PW}(\mathbf{R}),\ a \in \mathbf{R} \implies T_a f \in \mathrm{PW}(\mathbf{R}).$$

(c) *If* $f \in \mathrm{PW}(\mathbf{R})$ *then* $\widehat{f} \in L^1(\mathbf{R})$ *and* $f = \big(\widehat{f}\,\big)^{\vee} \in C_0(\mathbf{R})$.

(d) *Every function* $f \in \mathrm{PW}(\mathbf{R})$ *is infinitely differentiable, and* $f^{(n)} \in \mathrm{PW}(\mathbf{R})$ *for all* $n \ge 0$.

(e) $\{T_n d_\pi\}_{n\in\mathbf{Z}}$ *is an orthonormal basis for* $\mathrm{PW}(\mathbf{R})$.

(f) *If* $0 < b < 1$ *then* $\{T_{bn} d_\pi\}_{n\in\mathbf{Z}}$ *is a redundant tight frame for* $\mathrm{PW}(\mathbf{R})$, *with frame bound* $A = B = b^{-1}$.

(g) *If* $b > 1$ *then* $\{T_{bn} d_\pi\}_{n\in\mathbf{Z}}$ *is incomplete in* $\mathrm{PW}(\mathbf{R})$.

Proof. (a) This is Exercise 10.3(a).

(b) If $f \in \mathrm{PW}(\mathbf{R})$ then $\widehat{f}(\xi) = 0$ for almost every $|\xi| > 1/2$. Given $a \in \mathbf{R}$ we have from Theorem 9.3 that

$$(T_a f)^{\wedge}(\xi) = M_{-a}\widehat{f}(\xi) = e^{-2\pi i a\xi}\,\widehat{f}(\xi).$$

Hence $(T_a f)^{\wedge}(\xi) = 0$ for a.e. $|\xi| > 1/2$, so $T_a f \in \mathrm{PW}(\mathbf{R})$.

(c) If $f \in \mathrm{PW}(\mathbf{R})$ then $f \in L^2(\mathbf{R})$, so $f = \big(\widehat{f}\,\big)^{\vee}$ a.e. Applying the Cauchy–Schwarz–Bunyakovski Inequality, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |\widehat{f}(\xi)|\,d\xi &= \int_{-1/2}^{1/2} |\widehat{f}(\xi)|\cdot 1\,d\xi \\ &\le \left(\int_{-1/2}^{1/2} |\widehat{f}(\xi)|^2\,d\xi\right)^{1/2} \left(\int_{-1/2}^{1/2} 1^2\,d\xi\right)^{1/2} = \|\widehat{f}\|_{L^2}^2 < \infty.\end{aligned}$$

Thus $\widehat{f}$ is integrable, so its inverse Fourier transform $\big(\widehat{f}\,\big)^{\vee}$ belongs to $C_0(\mathbf{R})$ by Theorem 9.10. Therefore f is continuous, in the sense that it equals the continuous function $\big(\widehat{f}\,\big)^{\vee}$ almost everywhere. By redefining f on a set of measure zero we can therefore assume that $f(x) = \big(\widehat{f}\,\big)^{\vee}(x)$ for every x.

(d) Given $f \in \mathrm{PW}(\mathbf{R})$, by part (c) we can write f as

$$f(x) \;=\; \bigl(\widehat{f}\bigr)^{\vee}(x) \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi \;=\; \int_{-1/2}^{1/2} \widehat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi. \qquad (10.1)$$

As $\widehat{f}$ is integrable and compactly supported, so the same is true of the function

$$g(\xi) \;=\; 2\pi i\xi \widehat{f}(\xi).$$

In particular, $\operatorname{supp}(g) \subseteq [-\frac{1}{2}, \frac{1}{2}]$. The Fourier or inverse Fourier transform of an integrable function is continuous, so $\overset{\vee}{g}$ is continuous on $\mathbf{R}$. Using equation (10.1) and formally interchanging a limit and an integral, we compute that

$$\begin{aligned} f'(x) \;&=\; \lim_{y\to x} \frac{f(x)-f(y)}{x-y} \\ &=\; \lim_{y\to x} \int_{-1/2}^{1/2} \widehat{f}(\xi)\, \frac{e^{2\pi i\xi x}-e^{2\pi i\xi y}}{x-y}\, d\xi \\ &=\; \int_{-1/2}^{1/2} \widehat{f}(\xi)\, \lim_{y\to x} \frac{e^{2\pi i\xi x}-e^{2\pi i\xi y}}{x-y}\, d\xi \\ &=\; \int_{-1/2}^{1/2} \widehat{f}(\xi)\, \frac{d}{dx} e^{2\pi i\xi x}\, d\xi \\ &=\; \int_{-1/2}^{1/2} \widehat{f}(\xi)\, 2\pi i\xi\, e^{2\pi i\xi x}\, d\xi \\ &=\; \overset{\vee}{g}(\xi). \end{aligned}$$

Because the interval $[-\frac{1}{2}, \frac{1}{2}]$ has finite measure, the Dominated Convergence Theorem can be used to justify the interchange of limit and integral in the calculation above. We assign the details as Exercise 10.3(b).

Thus, f is differentiable at every point and $f' = \overset{\vee}{g}$. Hence $\widehat{f'} = g$ is supported within $[-\frac{1}{2}, \frac{1}{2}]$, so $f' \in \mathrm{PW}(\mathbf{R})$. By induction, we see that f is infinitely differentiable.

(e) Since the trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, it follows that

$$\{\epsilon_n\}_{n\in\mathbf{Z}} \;=\; \bigl\{e^{2\pi inx}\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)\bigr\}_{n\in\mathbf{Z}} \;=\; \bigl\{M_n \chi_{[-\frac{1}{2},\frac{1}{2}]}\bigr\}_{n\in\mathbf{Z}}$$

is an orthonormal basis for $L^2_{[-\frac{1}{2},\frac{1}{2}]}(\mathbf{R})$. Since the Fourier transform maps $L^2_{[\frac{1}{2},\frac{1}{2}]}(\mathbf{R})$ unitarily onto $\mathrm{PW}(\mathbf{R})$ and since

$$\widehat{\epsilon_n} \;=\; \bigl(M_n \chi_{[-\frac{1}{2},\frac{1}{2}]}\bigr)^{\wedge} \;=\; T_n d_\pi,$$

it follows that $\{T_n d_\pi\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $\mathrm{PW}(\mathbf{R})$.

(f) By Exercise 8.9, if $0 < b < 1$ then $\{e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$ is a redundant tight frame for $L^2(\mathbf{T})$ with frame bound $A = B = b^{-1}$. As in the proof of part (d), it follows from this that $\{T_{bn}d_\pi\}_{n\in\mathbf{Z}}$ is a redundant tight frame for $\mathrm{PW}(\mathbf{R})$ with frame bound $A = B = b^{-1}$.

(g) This likewise follows from Exercise 8.9. □

Remark 10.5. Technically, it is an abuse of terminology to say that a bandlimited function is smooth. A bandlimited function is an element of $L^2(\mathbf{R})$ and hence is really an equivalence class of functions that are equal almost everywhere. What we mean when we say that a function $f \in L^2(\mathbf{R})$ is continuous is that there is a representative of this equivalence class that is continuous, and it is this representative that we work with when we speak of the function values $f(x)$ for $x \in \mathbf{R}$. ♢

Part (d) of Theorem 10.4 tells us that every bandlimited function is infinitely differentiable, but even more is true: If $f \in L^2(\mathbf{R})$ and $\widehat{f}$ has compact support then there exists a unique extension of $f\colon \mathbf{R} \to \mathbf{C}$ to a function $f\colon \mathbf{C} \to \mathbf{C}$ that is *analytic* on the complex plane $\mathbf{C}$. This statement is part of the beautiful *Paley–Wiener Theorem*, which provides a fundamental link between harmonic analysis and complex analysis (see [Kat04] for the exact statement and proof of the Paley–Wiener Theorem). One implication of this is that no function can be simultaneously timelimited and bandlimited (also see Exercise 10.4 for a direct proof of Corollary 10.6).

Corollary 10.6. *If $f \in L^2(\mathbf{R})$ and both f and $\widehat{f}$ have compact support, then $f = 0$.*

Proof. Suppose $f \in L^2(\mathbf{R})$ and $\widehat{f}$ has compact support. Then the Paley–Wiener Theorem implies that f can be extended to a function that is analytic on $\mathbf{C}$. Suppose f also has compact support, say $f(x) = 0$ for $|x| > R$. An important property of analytic functions is that they are entirely determined by their values on any line segment, curve, or set in $\mathbf{C}$ that has an accumulation point in $\mathbf{C}$. Hence if f is analytic and $f(z) = 0$ for all $z = x + i0$ with $|x| > R$, then f must be identically zero for all z. □

We will not pursue this interaction with complex analysis further, but we note that it is the basis for much of the analysis of nonharmonic Fourier series and its applications to irregular sampling theory. We refer to the text by Young [You01] for more details on this subject.

Exercises

10.1. Show that the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ each map $L^2_{[-\Omega,\Omega]}(\mathbf{R})$ unitarily onto $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$.

10.2. Show that the dilation operator D_r maps $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$ unitarily onto $\mathcal{F}L^2_{[-r\Omega,r\Omega]}(\mathbf{R})$.

10.3. (a) Prove statement (a) in Theorem 10.4.

(b) Show that $\mathrm{PW}(\mathbf{R}) \subseteq C^\infty(\mathbf{R})$ by justifying the statements made in the proof of Theorem 10.4(d).

10.4. Given $f \in \mathrm{PW}(\mathbf{R})$, prove the following statements.

(a) The nth derivative of f can be written as

$$f^{(n)}(x) \;=\; \int_{-1/2}^{1/2} (2\pi i\xi)^n\, \widehat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi.$$

(b) $\|f^{(n)}\|_\infty \le \pi^n\, \|\widehat{f}\|_{L^1}$ for each $n \ge 0$.

(c) Given $a \in \mathbf{R}$, the Taylor series for f about the point a converges to $f(x)$ for every x, i.e.,

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n, \qquad x \in \mathbf{R}.$$

Remark: This says that f is a *real analytic* function on $\mathbf{R}$. The Paley–Wiener Theorem implies more, in particular f has an extension to a function that is complex analytic on $\mathbf{C}$.

(d) Use part (c) to give another proof of Corollary 10.6.

10.2 The Sampling Theorem

Given $f \in L^2(\mathbf{R})$, the Inversion Formula for the Fourier transform tells us that we can recover a function f from its Fourier transform $\widehat{f}$ by applying the inverse Fourier transform: $f = \bigl(\widehat{f}\bigr)^\vee$. In the case that $f, \widehat{f} \in L^1(\mathbf{R})$, this becomes the pointwise formula from Theorem 9.12:

$$f(x) \;=\; \bigl(\widehat{f}\bigr)^\vee(x) \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi, \qquad x \in \mathbf{R}.$$

That is, if we know the value of $\widehat{f}(\xi)$ for every ξ then, by superimposing the "elementary functions" $e^{2\pi i\xi x}$ multiplied by the amplitudes $\widehat{f}(\xi)$, we can recover f. This is an uncountable superposition in the sense that we must integrate over all $\xi \in \mathbf{R}$, but the amplitude $\widehat{f}(\xi)$ does have a "physical meaning" in the sense that it represents the "amount" of frequency ξ that is present in the signal f.

A frame expansion allows us to recover f via a countable superposition. If $\{f_n\}$ is a frame for $L^2(\mathbf{R})$ then for every $f \in L^2(\mathbf{R})$ we have

$$f = \sum_n \langle f, \widetilde{f}_n \rangle f_n,$$

where $\{\widetilde{f}_n\}$ is the canonical dual frame and the series converges in L^2-norm. Now our "elementary functions" are the frame functions f_n, and they are superimposed with amplitudes given by the frame coefficients $\langle f, \widetilde{f}_n \rangle$. However, in general these frame coefficients need not have an obvious "physical meaning." Is there a way to choose the frame $\{f_n\}$ so that the frame coefficients $\langle f, f_n \rangle$ will reflect "useful" properties of f? What is "useful" may have different meanings depending on the application at hand, so we focus this question further. Can we construct a frame $\{f_n\}$ so that the frame coefficient $\langle f, f_n \rangle$ is an actual *function value* of f, say $f(n)$? We will see that the answer is yes if we restrict our attention to functions that are bandlimited to an appropriate frequency band.

The key is the following computation, which is based on the unitarity of the Fourier transform and the properties of bandlimited functions. Fix a function $f \in \mathrm{PW}(\mathbf{R})$ and a point $a \in \mathbf{R}$. By Theorem 10.4, f is smooth and the Inversion Formula $f(x) = \big(\widehat{f}\,\big)^{\vee}(x)$ holds pointwise for each $x \in \mathbf{R}$. Applying the Parseval Equality and noting that $\widehat{f}\,\chi_{[-\frac{1}{2},\frac{1}{2}]} = \widehat{f}$, we therefore have

$$\begin{aligned}
\langle f, T_a d_\pi \rangle &= \big\langle \widehat{f}, (T_a d_\pi)^\wedge \big\rangle \\
&= \big\langle \widehat{f}, M_{-a} \widehat{d_\pi} \big\rangle \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\, \overline{e^{-2\pi i a \xi}\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)}\, d\xi \\
&= \int_{-1/2}^{1/2} \widehat{f}(\xi)\, e^{2\pi i a \xi}\, d\xi \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i a \xi}\, d\xi \\
&= \big(\widehat{f}\,\big)^{\vee}(a) = f(a).
\end{aligned} \tag{10.2}$$

This observation leads us to the Classical Sampling Theorem (also known as the *Shannon Sampling Theorem*, the *Shannon–Whittaker Sampling Theorem*, the *Nyquist–Shannon Sampling Theorem*, and several other names).

Theorem 10.7 (Classical Sampling Theorem). *If $0 < b \le 1$ then for all $f \in \mathrm{PW}(\mathbf{R})$ we have*

$$f(x) = b \sum_{n \in \mathbf{Z}} f(bn)\, \frac{\sin \pi(\xi - bn)}{\pi(\xi - bn)}, \tag{10.3}$$

where the series converges unconditionally in L^2-norm.

Proof. By Theorem 10.4, we know that $\{T_{bn}d_\pi\}_{n\in\mathbf{Z}}$ is a b^{-1}-tight frame for $\mathrm{PW}(\mathbf{R})$. Therefore, given $f \in \mathrm{PW}(\mathbf{R})$ we have

$$f \;=\; b\sum_{n\in\mathbf{Z}} \langle f, T_{bn}d_\pi\rangle\, T_{bn}d_\pi, \tag{10.4}$$

with unconditional convergence of the series in L^2-norm. Since equation (10.2) tells us that $\langle f, T_{bn}d_\pi\rangle = f(bn)$, equations (10.3) and (10.4) coincide. □

Thus, a bandlimited function f in the Paley–Wiener space is entirely determined by the countably many sample values $f(bn)$, $n \in \mathbf{Z}$, as long as $b \le 1$. Note that there are infinitely many functions (including smooth functions) that have the same sample values as f at the points bn but differ from f at other points. However, by the Paley–Wiener Theorem, a bandlimited function f is the restriction to the real line of a function that is analytic on the complex plane. Analytic functions are very highly constrained, and so it is not so surprising that a bandlimited f should be determined by only countably many sample values. However, we must sample "densely enough." If we take $b > 1$ then $\{T_{bn}d_\pi\}_{n\in\mathbf{Z}}$ is an incomplete sequence in $\mathrm{PW}(\mathbf{R})$, and consequently functions $f \in \mathrm{PW}(\mathbf{R})$ are not completely determined by their sample values $\{f(bn)\}_{n\in\mathbf{Z}}$ in this case (Exercise 10.6). This phenomenon is called *aliasing*. If we take $b = 1$ then $\{T_n d_\pi\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $\mathrm{PW}(\mathbf{R})$, and we refer to this situation as "critical sampling" or "sampling at the Nyquist density." When $0 < b < 1$ the sequence $\{T_{bn}d_\pi\}_{n\in\mathbf{Z}}$ is a redundant tight frame for $\mathrm{PW}(\mathbf{R})$, and we refer to this as "oversampling." We can only reconstruct from samples in the critically sampled or oversampled cases. Oversampling offers many advantages in applications, due to the fact that we have a redundant frame in the background in this case.

Looking at the proof of Theorem 10.7, we see that the key ingredient is that a set of complex exponentials is a frame for the Paley–Wiener space. This gives us the following generalization, whose proof is assigned as Exercise 10.5. In the statement of this result, recall that e_λ denotes the complex exponential function $e_\lambda(x) = e^{2\pi i\lambda x}$.

Theorem 10.8. *Let $\{\lambda_n\}_{n\in\mathbf{N}}$ be any sequence of real numbers such that $\{e_{\lambda_n}\}_{n\in\mathbf{N}}$ is a frame for $L^2(\mathbf{T})$. Let $\{\widetilde{e}_n\}_{n\in\mathbf{N}}$ be the canonical dual frame of $\{e_{\lambda_n}\}_{n\in\mathbf{N}}$ in $L^2(\mathbf{T})$, and define*

$$\widetilde{s}_n \;=\; \bigl(\widetilde{e}_n\, \chi_{[-\frac{1}{2},\frac{1}{2}]}\bigr)^{\vee}.$$

Then for any $f \in \mathrm{PW}(\mathbf{R})$ we have

$$f \;=\; \sum_{n\in\mathbf{Z}} f(\lambda_n)\, \widetilde{s}_n,$$

where the series converges in L^2-norm. ◇

In general, the functions $\widetilde{s}_n$ will not be sinc functions, and indeed it may be difficult to write them explicitly when the frame is not tight. On the other hand, we can often use alternative duals instead of the canonical dual in order to improve on the properties of $\widetilde{s}_n$. These and many other issues are dealt with in irregular sampling theory, and we refer to [You01] for more information.

Of course, Theorem 10.8 begs the question of *which* sequences $\{e^{2\pi i\lambda_n x}\}_{n\in\mathbf{N}}$ are frames for $L^2(\mathbf{T})$. This was a longstanding problem introduced in the very first frame paper by Duffin and Schaeffer [DS52]. A complete solution was finally given by Ortega-Cerdà and Seip in [OS02]. Loosely, the sequence $\{\lambda_n\}_{n\in\mathbf{N}}$ can be assigned a "density" in the real line, and $\{e^{2\pi i\lambda_n x}\}_{n\in\mathbf{N}}$ is a frame when this density is large enough. For the regularly spaced sequence $\{e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$, the "density" of $\{bn\}_{n\in\mathbf{Z}}$ is precisely $1/b$, and we have a frame exactly when $1/b \geq 1$.

Sampling theory is now a major topic in mathematics and signal processing, and there is a vast literature covering both applied and abstract viewpoints. For more details, we refer to the texts [Hig96], [Mar91], and the survey articles and edited volumes [BF01], [Hig85], [Jer77], [Uns00].

Exercises

10.5. Prove Theorem 10.8.

10.6. Show that if $b > 1$ then a function $f \in \mathrm{PW}(\mathbf{R})$ is not uniquely determined by the sequence of sample values $\{f(bn)\}_{n\in\mathbf{Z}}$.

10.3 Frames of Weighted Exponentials

The results of Sections 10.1 and 10.2 were largely driven by the fact that the sequence $\{e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{T})$ whenever $0 < b \leq 1$. By extending these functions by zero outside of $[-\frac{1}{2}, \frac{1}{2}]$ and applying the Fourier transform, we translated this frame property to the sequence $\{T_{bn}d_\pi\}_{n\in\mathbf{Z}}$ in the Paley–Wiener space, which is a subspace of $L^2(\mathbf{R})$.

Now we return to the setting of $L^2(\mathbf{T})$. We consider sequences that are related to the complex exponentials, but which also incorporate a "weighting" function φ.

Definition 10.9. A *lattice system of weighted exponentials* is a sequence in $L^2(\mathbf{T})$ of the form

$$\mathcal{E}(\varphi) = \{e^{2\pi inx}\varphi(x)\}_{n\in\mathbf{Z}} = \{\varphi\, e_n\}_{n\in\mathbf{Z}},$$

where $\varphi \in L^2(\mathbf{T})$ is a fixed 1-periodic function. ◊

More generally, we could consider lattice systems $\{e^{2\pi ibnx}\varphi(x)\}_{n\in\mathbf{Z}}$ where $b > 0$, or even "irregular" systems $\{e^{2\pi i\lambda_n x}\varphi(x)\}_{n\in\mathbf{N}}$ where $\{\lambda_n\}$ is an arbitrary sequence of real numbers. These types of sequences are important in many applications, but we will focus on $b = 1$. In analogy with the Sampling Theorem, this is often referred to as the case of "critical sampling."

Our next theorem will characterize some of the properties of the sequence $\mathcal{E}(\varphi)$ in $L^2(\mathbf{T})$. We let Z_φ denote the zero set of φ:

$$Z_\varphi \;=\; \{x \in \mathbf{T} : \varphi(x) = 0\}.$$

Since φ is 1-periodic, the set Z_φ is a "1-periodic" subset of the real line. Technically, Z_φ is only defined up to sets of measure zero, i.e., if we choose a different representative of φ then we may get a different set Z_φ, but the symmetric difference between any two such sets will have measure zero.

The idea behind the proof of Theorem 10.10 is the simple observation that if $f \in L^2(\mathbf{T})$ then

$$\langle f, \varphi e_n\rangle \;=\; \int_0^1 f(x)\,\overline{\varphi(x)}\,e^{-2\pi inx}\,dx \;=\; \langle f\,\overline{\varphi}, e_n\rangle. \tag{10.5}$$

Thus $\langle f, \varphi e_n\rangle$ is the nth Fourier coefficient of the function $f\,\overline{\varphi}$. If the product $f\,\overline{\varphi}$ belongs to $L^2(\mathbf{T})$ then we can use the fact that $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$ to analyze $f\,\overline{\varphi}$. However, in general we only know that $f, \varphi \in L^2(\mathbf{T})$, which only tells us that $f\,\overline{\varphi} \in L^1(\mathbf{T})$. Still, functions in $L^1(\mathbf{T})$ are completely determined by their Fourier coefficients (Theorem 4.25), so the values $\langle f\,\overline{\varphi}, e_n\rangle$ do determine $f\,\overline{\varphi}$. If we are fortunate enough to know that φ is bounded, then we will have $f\,\overline{\varphi} \in L^2(\mathbf{T})$, in which case we can represent $f\,\overline{\varphi}$ in terms of the orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$.

Theorem 10.10. *Given $\varphi \in L^2(\mathbf{T})$, the following statements hold.*

(a) *$\mathcal{E}(\varphi)$ is complete in $L^2(\mathbf{T})$ if and only if $\varphi(x) \neq 0$ for a.e. x.*

(b) *$\mathcal{E}(\varphi)$ is minimal in $L^2(\mathbf{T})$ if and only if $1/\varphi \in L^2(\mathbf{T})$. In this case $\mathcal{E}(\varphi)$ is exact, and its biorthogonal system is $\mathcal{E}(\widetilde{\varphi})$ where $\widetilde{\varphi} = 1/\overline{\varphi}$.*

(c) *$\mathcal{E}(\varphi)$ is a Bessel sequence in $L^2(\mathbf{T})$ if and only if $\varphi \in L^\infty(\mathbf{T})$. In this case $|\varphi(x)|^2 \le B$ a.e., where B is a Bessel bound.*

(d) *$\mathcal{E}(\varphi)$ is a frame sequence in $L^2(\mathbf{T})$ if and only if there exist $A, B > 0$ such that $A \le |\varphi(x)|^2 \le B$ for a.e. $x \notin Z_\varphi$. In this case the closed span of $\mathcal{E}(\varphi)$ is*

$$H_\varphi \;=\; \bigl\{f \in L^2(\mathbf{T}) : f = 0 \text{ a.e. on } Z_\varphi\bigr\}, \tag{10.6}$$

and A, B are frame bounds for $\mathcal{E}(\varphi)$ as a frame for H_φ.

(e) *$\mathcal{E}(\varphi)$ is an unconditional basis for $L^2(\mathbf{T})$ if and only if there exist $A, B > 0$ such that $A \le |\varphi(x)|^2 \le B$ for a.e. x. In this case $\mathcal{E}(\varphi)$ is a Riesz basis for $L^2(\mathbf{T})$.*

(f) *$\mathcal{E}(\varphi)$ is an orthonormal basis for $L^2(\mathbf{T})$ if and only if $|\varphi(x)| = 1$ for a.e. x.*

Proof. We will prove one direction of the implications in each of statements (a)–(e), and assign the proof of the converse implications, and the proof of statement (f), as Exercise 10.8.

(a) Suppose that $\mathcal{E}(\varphi)$ is complete, and set $f = \chi_{Z_\varphi}$. Then $\langle f, \varphi e_n \rangle = 0$ for every n, so by completeness we have $f = 0$ a.e. Hence $|Z_\varphi| = 0,$ which says that φ is nonzero a.e.

(b) Suppose that $\mathcal{E}(\varphi)$ is minimal in $L^2(\mathbf{T})$. Then it has a biorthogonal system $\{\widetilde{\varphi}_n\}_{n\in\mathbf{Z}}$ in $L^2(\mathbf{T})$. Fix $m \in \mathbf{Z}$, and observe that the product $\widetilde{\varphi}_m\overline{\varphi}$ belongs to $L^1(\mathbf{T})$. Further,

$$\langle \widetilde{\varphi}_m\overline{\varphi}, e_n \rangle \;=\; \int_0^1 \widetilde{\varphi}_m(x)\,\overline{\varphi(x)}\,e^{-2\pi i n x}\,dx \;=\; \langle \widetilde{\varphi}_m, \varphi e_n \rangle \;=\; \delta_{mn}.$$

Although $\{e_n\}_{n\in\mathbf{Z}}$ is not a basis for $L^1(\mathbf{T})$, Theorem 4.25 states that functions in $L^1(\mathbf{T})$ are uniquely determined by their Fourier coefficients. The nth Fourier coefficient of the function $\widetilde{\varphi}_m\overline{\varphi} \in L^1(\mathbf{T})$ is $\langle \widetilde{\varphi}_m\overline{\varphi}, e_n \rangle = \delta_{mn}$, and the nth Fourier coefficient of the function $e_m \in L^1(\mathbf{T})$ is $\langle e_m, e_n \rangle = \delta_{mn}$. Therefore we must have $\widetilde{\varphi}_m\overline{\varphi} = e_m$ a.e. Since e_m is nonzero almost everywhere, this implies that $\varphi \neq 0$ a.e. and $\widetilde{\varphi}_m = e_m/\overline{\varphi}$. In particular, since $e_0 = 1$ we have $1/\overline{\varphi} = e_0/\overline{\varphi} = \widetilde{\varphi}_0 \in L^2(\mathbf{T})$.

(c) Suppose that $\mathcal{E}(\varphi)$ is a Bessel sequence in $L^2(\mathbf{T})$, and let B be a Bessel bound. Set $E = \{x \in \mathbf{T} : |\varphi(x)|^2 > B\}$ and consider χ_E, the characteristic function of E. We have $\chi_E\,\overline{\varphi} \in L^2(\mathbf{T}),$ so

$$\begin{aligned}
\int_E |\varphi(x)|^2\,dx \;=\; \|\chi_E\,\overline{\varphi}\|_{L^2}^2 \;&=\; \sum_{n\in\mathbf{Z}} |\langle \chi_E\,\overline{\varphi}, e_n \rangle|^2 \\
&=\; \sum_{n\in\mathbf{Z}} |\langle \chi_E, \varphi e_n \rangle|^2 \\
&\le\; B\,\|\chi_E\|_{L^2}^2 \;=\; \int_E B\,dx.
\end{aligned}$$

Hence $\int_E \bigl(|\varphi(x)|^2 - B\bigr)\,dx \le 0$. However, $|\varphi(x)|^2 - B$ is strictly positive on E, so this implies that $|E| = 0$ (see Theorem A.15). Therefore $|\varphi|^2 \le B$ a.e.

(d) By Exercise 10.7, the space H_φ defined in equation (10.6) is a closed subspace of $L^2(\mathbf{T})$.

Suppose that $\mathcal{E}(\varphi)$ is a frame sequence in $L^2(\mathbf{T})$, i.e., $\mathcal{E}(\varphi)$ is a frame for its closed span. Let A, B be frame bounds for $\mathcal{E}(\varphi)$ as a frame for its closed span. Since $\mathcal{E}(\varphi)$ is a Bessel sequence, we have $|\varphi|^2 \le B$ a.e. by part (c).

Note that $\overline{\operatorname{span}}(\mathcal{E}(\varphi)) \subseteq H_\varphi$ by the definition of H_φ. To show that $\overline{\operatorname{span}}\bigl(\mathcal{E}(\varphi)\bigr) = H_\varphi$, suppose that $f \in H_\varphi$ satisfies $\langle f, \varphi e_n \rangle = 0$ for every $n \in \mathbf{Z}$. Since φ is bounded, $f\,\overline{\varphi} \in L^2(\mathbf{T})$, and its Fourier coefficients are $\langle f\,\overline{\varphi}, e_n \rangle = \langle f, \varphi e_n \rangle = 0$ for $n \in \mathbf{Z}$. Therefore $f\,\overline{\varphi} = 0$ a.e., and since $f \in H_\varphi$, this implies $f = 0$ a.e. Hence $\mathcal{E}(\varphi)$ is complete in H_φ.

Now fix any $f \in H_\varphi$. Again $f\,\overline{\varphi} \in L^2(\mathbf{T})$ since φ is bounded, so since $\mathcal{E}(\varphi)$ is a frame for H_φ we have

$$\begin{aligned} A \int_0^1 |f(x)|^2\,dx \;=\; A\,\|f\|_{L^2}^2 \;&\le\; \sum_{k\in\mathbf{Z}} |\langle f, \varphi\,e_n\rangle|^2 \\ &=\; \sum_{k\in\mathbf{Z}} |\langle f\,\overline{\varphi}, e_n\rangle|^2 \\ &=\; \|f\,\overline{\varphi}\|_{L^2}^2 \\ &=\; \int_0^1 |f(x)|^2\,|\varphi(x)|^2\,dx. \end{aligned}$$

Since f and φ both vanish on Z_φ, this implies

$$\int_{[0,1]\setminus Z_\varphi} |f(x)|^2\,\bigl(|\varphi(x)|^2 - A\bigr)\,dx \ge 0. \tag{10.7}$$

If $|\varphi(x)|^2 < A$ on any set $E \subseteq [0,1]\setminus Z_\varphi$ of positive measure, then taking $f = \chi_E$ in equation (10.7) leads to a contradiction. Hence we must have $|\varphi(x)|^2 \ge A$ for a.e. $x \notin Z_\varphi$.

(e) Suppose that $\mathcal{E}(\varphi)$ is an unconditional basis for $L^2(\mathbf{T})$. Then since $\|\varphi\,e_n\|_{L^2} = \|\varphi\|_{L^2}$ for every n, it is a bounded unconditional basis, and therefore is a Riesz basis by Theorem 7.13. Every Riesz basis is an exact frame, so by parts (b) and (d) we must have $A \le |\varphi(x)|^2 \le B$ a.e. □

Remark 10.11. (a) The question of when $\mathcal{E}(\varphi)$ is a Schauder basis for $L^2(\mathbf{T})$ is much more subtle. We saw in Example 5.13 that if $\varphi(x) = |x - \frac{1}{2}|^{-\alpha}$ with $0 < \alpha < 1/2$, then $\mathcal{E}(\varphi)$ is a Schauder basis for $L^2(\mathbf{T})$, but it is not a frame or Riesz basis for $L^2(\mathbf{T})$. We stated the Schauder basis characterization of weighted exponentials in Theorem 5.15: $\mathcal{E}(\varphi) = \{\varphi\,e_n\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^2(\mathbf{T})$ with respect to the ordering $\mathbf{Z} = \{0, -1, 1, -2, 2, \dots\}$ if and only if $|\varphi|^2$ is an $\mathcal{A}_2(\mathbf{T})$ weight. The proof of this result is due to Hunt, Muckenhoupt, and Wheeden [HMW73] and will be omitted. However, the material that we will cover in Chapter 14 is a good preparation for the proof, so the interested reader can consult texts such as [Gra04] for details after completing Chapter 14.

(b) If $\mathcal{E}(\varphi)$ is a redundant frame sequence in $L^2(\mathbf{T})$ then it is not a Riesz basis for $L^2(\mathbf{T})$, so Theorem 10.10 implies that Z_φ must have positive measure. But we must also have $A \le |\varphi|^2 \le B$ a.e. on the complement of Z_φ, so φ cannot be continuous in this case. ♢

Finally, we characterize the dual frame of $\mathcal{E}(\varphi)$ when $\mathcal{E}(\varphi)$ is a frame sequence.

Theorem 10.12. *Fix* $\varphi \in L^2(\mathbf{T})$*. If* $\mathcal{E}(\varphi)$ *is a frame sequence in* $L^2(\mathbf{T})$*, then the canonical dual frame within* H_φ *is* $\mathcal{E}(\widetilde{\varphi})$ *where*

$$\widetilde{\varphi}(x) \;=\; \begin{cases} 1/\overline{\varphi(x)}, & x \notin Z_\varphi, \\ 0, & x \in Z_\varphi. \end{cases} \tag{10.8}$$

Proof. Let H_φ be as in Theorem 10.10, and let $S\colon H_\varphi \to H_\varphi$ be the frame operator for $\mathcal{E}(\varphi)$. Then for any $m \in \mathbf{Z}$ we have

$$\begin{aligned} S(f\,e_m) \;&=\; \sum_{n\in\mathbf{Z}} \big\langle f\,e_m,\, \varphi\,e_n \big\rangle\, \varphi\,e_n \\ &=\; \sum_{n\in\mathbf{Z}} \big\langle f,\, \varphi\,e_{n-m} \big\rangle\, \varphi\,e_n \\ &=\; \sum_{n\in\mathbf{Z}} \big\langle f,\, \varphi\,e_n \big\rangle\, \varphi\,e_{n+m} \\ &=\; \sum_{n\in\mathbf{Z}} \big\langle f,\, \varphi\,e_n \big\rangle\, \varphi\,e_n \cdot e_m \;=\; (Sf)\cdot e_m. \end{aligned}$$

That is, S commutes with multiplication by e_m. Consequently S^{-1} commutes with multiplication by e_m as well, and therefore the canonical dual frame is

$$\big\{S^{-1}(\varphi\,e_n)\big\}_{n\in\mathbf{Z}} \;=\; \big\{(S^{-1}\varphi)\,e_n\big\}_{n\in\mathbf{Z}} \;=\; \big\{\psi\,e_n\big\}_{n\in\mathbf{Z}} \;=\; \mathcal{E}(\psi),$$

where $\psi = S^{-1}\varphi$.

Now we must show that ψ has the prescribed form. Let $\widetilde{\varphi}$ be defined by equation (10.8), and note that $\widetilde{\varphi}\,\overline{\varphi} = \chi_{Z_\varphi^{\mathrm{C}}}$, the characteristic function of the complement of Z_φ. Therefore

$$\begin{aligned} S\widetilde{\varphi} \;&=\; \sum_{n\in\mathbf{Z}} \big\langle \widetilde{\varphi},\, \varphi\,e_n \big\rangle\, \varphi\,e_n \\ &=\; \sum_{n\in\mathbf{Z}} \big\langle \widetilde{\varphi}\,\overline{\varphi},\, e_n \big\rangle\, \varphi\,e_n \\ &=\; \sum_{n\in\mathbf{Z}} \big\langle \chi_{Z_\varphi^{\mathrm{C}}},\, e_n \big\rangle\, \varphi\,e_n \\ &=\; \bigg(\sum_{n\in\mathbf{Z}} \big\langle \chi_{Z_\varphi^{\mathrm{C}}},\, e_n \big\rangle\, e_n\bigg)\, \varphi \qquad (10.9) \\ &=\; \chi_{Z_\varphi^{\mathrm{C}}} \cdot \varphi \qquad (10.10) \\ &=\; \varphi. \end{aligned}$$

The factorization in equation (10.9) is allowed because of the fact that φ is bounded (Exercise 10.10), and equation (10.10) follows from equation (10.9) because $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$. Consequently we have $\psi = S^{-1}\varphi = \widetilde{\varphi}$. □

Exercises

10.7. Prove that the subspace H_φ defined in equation (10.6) is a closed subspace of $L^2(\mathbf{T})$.

10.8. Complete the proof of Theorem 10.10.

10.9. Let $\varphi(t) = |t - \frac{1}{2}|^\alpha$ where $0 < \alpha < 1/2$, as in Example 5.13. Show that $\mathcal{E}(\varphi)$ is exact in $L^2(\mathbf{T})$ but cannot be an unconditional basis for $L^2(\mathbf{T})$.

10.10. Justify the equality in equation (10.9).

10.11. Given a sequence $c = (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, let $T_n c$ denote the translated sequence $T_n c = (c_{k-n})_{k\in\mathbf{Z}}$. Characterize those sequences c such that $\{T_n c\}_{n\in\mathbf{Z}}$ is complete, Bessel, minimal, a frame, a Schauder basis, a Riesz basis, or an orthonormal basis for $\ell^2(\mathbf{Z})$.

10.4 Frames of Translates

Frames of translates play important roles in many areas, including wavelet theory and reconstruction of signals from sample values. A lattice system of translates is a sequence in $L^2(\mathbf{R})$ that has the form $\{g(x-ak)\}_{k\in\mathbf{Z}}$ where $g \in L^2(\mathbf{R})$ and $a > 0$ are fixed. Exercise 10.18 shows that such a sequence can never be complete in $L^2(\mathbf{R})$, and therefore can never be a frame or a Riesz basis for all of $L^2(\mathbf{R})$. Instead, what we usually need to know is whether $\{g(x-ak)\}_{k\in\mathbf{Z}}$ is a frame, Riesz basis, etc., *for its closed span* in $L^2(\mathbf{R})$. Since the dilation operator $D_a f(x) = a^{1/2} f(ax)$ is a unitary mapping of $L^2(\mathbf{R})$ onto itself, by making a change of variables it suffices to consider the case $a = 1$ (see Exercise 10.12). Hence in this section we will focus on sequences of integer translates, which we can write as

$$\mathcal{T}(g) \;=\; \bigl\{T_k g\bigr\}_{k\in\mathbf{Z}} \;=\; \bigl\{g(x-k)\bigr\}_{k\in\mathbf{Z}},$$

where T_a is the *translation operator*

$$T_a f(x) \;=\; f(x-a).$$

In order to characterize the properties of $\mathcal{T}(g)$, we will make use of the Fourier transform, which is reviewed in Chapter 9. The most important facts about the Fourier transform $\mathcal{F}f = \widehat{f}$ that we will need are given in Theorems 9.3 and 9.5. Specifically, $\mathcal{F}$ is a unitary mapping of $L^2(\mathbf{R})$ onto itself, and it interchanges translation with modulation according to the rules

$$(T_a f)^\wedge(\xi) \;=\; M_{-a}\widehat{f}(\xi) \;=\; e^{-2\pi i a\xi}\,\widehat{f}(\xi)$$

and

$$(M_b f)^\wedge(\xi) \;=\; T_b\widehat{f}(\xi) \;=\; \widehat{f}(\xi - b).$$

By unitarity, a system of translates $\mathcal{T}(g)$ is a frame for its closed span in $L^2(\mathbf{R})$ if and only if its image $\mathcal{F}(\mathcal{T}(g))$ under the Fourier transform is a frame for its closed span in $L^2(\mathbf{R})$. This image has the form

$$\mathcal{F}(\mathcal{T}(g)) = \{(T_k g)^\wedge\}_{k\in\mathbf{Z}} = \{M_{-k}\widehat{g}\}_{k\in\mathbf{Z}} = \{e^{-2\pi ik\xi}\,\widehat{g}(\xi)\}_{k\in\mathbf{Z}}.$$

This looks very much like the systems of weighted exponentials that we considered in the preceding section, but we must keep in mind that the function $\widehat{g}$ is not 1-periodic. Rather, $\widehat{g}$ is a square integrable function on $\mathbf{R}$, and so does not belong to $L^2(\mathbf{T})$. In order to distinguish between these different types of sequences, we introduce a new notation, and also recall the definitions of $\mathcal{T}(g)$ and $\mathcal{E}(\varphi)$. We choose the sign in the complex exponentials to best match their use in this section.

Definition 10.13. (a) Given $g \in L^2(\mathbf{R})$, the *system of integer translates generated by g* is

$$\mathcal{T}(g) = \{T_k g\}_{k\in\mathbf{Z}} = \{g(x-k)\}_{k\in\mathbf{Z}},$$

and the *system of integer modulates generated by $\widehat{g}$* is

$$\mathcal{M}(\widehat{g}) = \{M_{-k}\widehat{g}\}_{k\in\mathbf{Z}} = \{e^{-2\pi ik\xi}\,\widehat{g}(\xi)\}_{k\in\mathbf{Z}}.$$

(b) Given $\varphi \in L^2(\mathbf{T})$, the *system of weighted exponentials generated by φ* is

$$\mathcal{E}(\varphi) = \{e^{-2\pi ik\xi}\,\varphi(\xi)\}_{k\in\mathbf{Z}}. \qquad \diamondsuit$$

The sequences $\mathcal{T}(g)$ and $\mathcal{M}(\widehat{g})$ are each contained in $L^2(\mathbf{R})$, and the Fourier transform maps $\mathcal{T}(g)$ onto $\mathcal{M}(\widehat{g})$. Therefore $\mathcal{T}(g)$ and $\mathcal{M}(\widehat{g})$ have exactly the same basis or frame properties. If there was a unitary transformation that turned $\mathcal{T}(g)$ or $\mathcal{M}(\widehat{g})$ into a system of weighted exponentials in $L^2(\mathbf{T})$, then we could immediately use the machinery developed in Section 10.3 to analyze them. In order to attempt this, we need the following periodic function associated with g.

Notation 10.14. (a) Given $g \in L^2(\mathbf{R})$, we let Φ_g denote the 1-periodic function

$$\Phi_g(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{g}(\xi+k)|^2, \qquad \xi \in \mathbf{R}. \tag{10.11}$$

We call Φ_g the *periodization* of $|\widehat{g}|^2$.

(b) More generally, given $f,\, g \in L^2(\mathbf{R})$ the *bracket product* of $\widehat{f}$ with $\widehat{g}$ is the 1-periodic function $[\widehat{f},\widehat{g}]$ given by

$$[\widehat{f},\widehat{g}](\xi) = \sum_{k\in\mathbf{Z}} \widehat{f}(\xi+k)\,\overline{\widehat{g}(\xi+k)}, \qquad \xi \in \mathbf{R}. \qquad \diamondsuit \tag{10.12}$$

Since $|g|^2 \in L^1(\mathbf{R})$, it follows from Exercise 10.13 that $\Phi_g \in L^1(\mathbf{T})$, and

$$\int_0^1 \Phi_g(\xi)\,d\xi \;=\; \int_{-\infty}^{\infty} |\widehat{g}(\xi)|^2\,d\xi \;=\; \|\widehat{g}\,\|_{L^2(\mathbf{R})}^2 \;=\; \|g\|_{L^2(\mathbf{R})}^2.$$

Consequently, $\Phi_g^{1/2} = \bigl(\sum_{k\in\mathbf{Z}} |\widehat{g}(\xi+k)|^2\bigr)^{1/2}$ belongs to $L^2(\mathbf{T})$, and

$$\|\Phi_g^{1/2}\|_{L^2(\mathbf{T})} \;=\; \|g\|_{L^2(\mathbf{R})}.$$

Therefore $g \mapsto \Phi_g^{1/2}$ is a norm-preserving map of $L^2(\mathbf{R})$ into $L^2(\mathbf{T})$. Unfortunately this map is neither linear nor surjective, but its isometric nature does lead us to suspect that the properties of $\mathcal{T}(g)$ in $L^2(\mathbf{R})$ may be reflected in the properties of the system of weighted exponentials $\mathcal{E}(\Phi_g^{1/2})$ in $L^2(\mathbf{T})$. In fact, there is one case where we can see this connection directly.

Example 10.15. Suppose that $g \in L^2(\mathbf{R})$ has the property that $\widehat{g}$ is zero a.e. outside of the interval $[0,1]$, i.e., g is bandlimited to $[0,1]$. Restricting our attention to this interval, we can think of

$$\mathcal{M}(\widehat{g}\,) \;=\; \bigl\{e^{-2\pi i k\xi}\,\widehat{g}(\xi)\bigr\}_{k\in\mathbf{Z}}$$

as being a sequence in $L^2[0,1]$. On the other hand, the support property of $\widehat{g}$ implies that $\Phi_g(\xi) = |\widehat{g}(\xi)|^2$ on $[0,1]$, so

$$\mathcal{E}(\Phi_g^{1/2}) \;=\; \bigl\{e^{-2\pi i k\xi}\,|\widehat{g}(\xi)|\bigr\}_{k\in\mathbf{Z}}.$$

The only difference between these two systems is that $\mathcal{M}(\widehat{g}\,)$ is generated by $\widehat{g}$ while $\mathcal{E}(\Phi_g^{1/2})$ is generated by $|\widehat{g}|$. If we examine the hypotheses of Theorem 10.10, we see that the properties of $\mathcal{E}(\varphi)$ given there depend only on $|\varphi|$ and not on the phase of φ. Hence, when $\mathrm{supp}(\widehat{g}\,) \subseteq [0,1]$, we can apply Theorem 10.10 to determine the properties of $\mathcal{E}(\Phi_g^{1/2})$ and hence the properties of $\mathcal{M}(\widehat{g}\,)$ and $\mathcal{T}(g)$. ♢

We will see that by being a little more clever we can define a *linear* operator that maps $\mathcal{T}(g)$ isometrically onto $\mathcal{E}(\Phi_g^{1/2})$. Consequently, we will be able to tie properties of $\mathcal{T}(g)$ to properties of $\mathcal{E}(\Phi_g^{1/2})$ by passing through this unitary map. First we introduce some additional notation that will be used throughout this section.

Notation 10.16. (a) Given $g \in L^2(\mathbf{R})$, we let $V_0(g)$ denote the closed subspace generated by the integer translates of g:

$$V_0(g) \;=\; \overline{\mathrm{span}}(\mathcal{T}(g)) \;=\; \overline{\mathrm{span}}\{T_k g\}_{k\in\mathbf{Z}}. \tag{10.13}$$

As in Theorem 10.10, Z_{Φ_g} denotes the zero set of Φ_g:

$$Z_{\Phi_g} \;=\; \{\xi \in \mathbf{T} : \Phi_g(\xi) = 0\}.$$

We also set

$$H_{\Phi_g} \;=\; \{F \in L^2(\mathbf{T}) : F = 0 \text{ a.e. on } Z_{\Phi_g}\},$$

which is a closed subspace of $L^2(\mathbf{T})$ by Exercise 10.7.

(b) The sequence space $\ell^2(\mathbf{Z})$ will play an important role in this section. This is the space of bi-infinite sequences $c = (\dots, c_{-1}, c_0, c_1, \dots)$ that are square summable. The space of bi-infinite sequences with finitely many nonzero components is denoted by $c_{00}(\mathbf{Z})$. We let δ_k be the kth standard basis vector in $\ell^2(\mathbf{Z})$:

$$\delta_k \;=\; (\delta_{kn})_{n\in\mathbf{Z}}.$$

(c) Given $c = (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, its *Fourier transform* or *Fourier series* is the function $\widehat{c} \in L^2(\mathbf{T})$ defined by

$$\widehat{c}(\xi) \;=\; \sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}. \tag{10.14}$$

Since $\{e^{-2\pi i k\xi}\}_{k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, this series converges unconditionally in $L^2(\mathbf{T})$. The mapping $c \mapsto \widehat{c}$ is a unitary mapping of $\ell^2(\mathbf{Z})$ onto $L^2(\mathbf{T})$. Fourier series will be studied in detail in Chapter 13. It should always be clear from context that $\widehat{f}$ denotes the Fourier transform of a function $f \in L^2(\mathbf{R})$ while $\widehat{c}$ denotes the Fourier series of a sequence $c \in \ell^2(\mathbf{Z})$. $\diamondsuit$

Note that the space $V_0(g)$ defined in equation (10.13) is shift-invariant in the following sense.

Definition 10.17. Let S be a closed subspace of $L^2(\mathbf{R})$.

(a) S is *shift-invariant* if for each $f \in S$ we have $f(x-k) \in S$ for every $k \in \mathbf{Z}$.

(b) S is *translation-invariant* if for each $f \in S$ we have $f(x-a) \in S$ for every $a \in \mathbf{Z}$. $\diamondsuit$

Every translation-invariant space is shift-invariant, but not conversely. The Paley–Wiener space $\mathrm{PW}(\mathbf{R})$ is translation-invariant, and $\mathrm{PW}(\mathbf{R}) = V_0(d_\pi)$ where d_π is the sinc function. On the other hand, the space $V_0(\chi_{[0,1]})$ is shift-invariant but not translation-invariant.

The next lemma defines the isometry that we will use to characterize the properties of systems of translates. The idea of the proof is that we map a function $\sum c_k T_k g$ in $V_0(g) = \overline{\mathrm{span}}(\mathcal{T}(g))$ to the function $\bigl(\sum c_k e_{-k}\bigr)\Phi_g^{1/2}$ in $L^2(\mathbf{T})$. In other words, we are hoping that $T_k g \mapsto e_{-k}\Phi_g^{1/2}$ extends to an isometry on $V_0(g)$. However, we must be careful, because we are not assuming that $\mathcal{T}(g)$ is a basis or a frame for its closed span, so we may not be able to write every function in $V_0(g)$ in the form $\sum c_k T_k g$.

Lemma 10.18. *Given $g \in L^2(\mathbf{R})$, the following statements hold.*

(a) *For any sequence $c = (c_k)_{k\in\mathbf{Z}}$ with finitely many nonzero components, we have*

$$\Big\| \sum_{k\in\mathbf{Z}} c_k\, T_k g \Big\|_{L^2(\mathbf{R})}^2 \;=\; \int_0^1 |\widehat{c}(\xi)|^2\, \Phi_g(\xi)\, d\xi. \tag{10.15}$$

(b) *The mapping $U\colon \operatorname{span}\{T_k g\}_{k\in\mathbf{Z}} \to L^2(\mathbf{T})$ given by*

$$U\Big(\sum_{k\in\mathbf{Z}} c_k\, T_k g\Big) \;=\; \widehat{c}\, \Phi_g^{1/2}, \qquad c = (c_k)_{k\in\mathbf{Z}} \in c_{00}(\mathbf{Z}),$$

is a linear isometry, and it extends to a unitary mapping of $V_0(g)$ onto H_{Φ_g}.

(c) *We have*

$$U(T_k g) \;=\; e_{-k}\, \Phi_g^{1/2}, \qquad k \in \mathbf{Z},$$

and therefore the isometry U maps the sequence $\mathcal{T}(g)$ in $V_0(g)$ to the sequence $\mathcal{E}(\Phi_g^{1/2})$ in H_{Φ_g}.

(d) *If $\mathcal{T}(g)$ is a Bessel sequence, then $\sum_{k\in\mathbf{Z}} c_k\, T_k g$ converges unconditionally and equation* (10.15) *holds for all sequences $c = (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, and*

$$U\Big(\sum_{k\in\mathbf{Z}} c_k\, T_k g\Big) \;=\; \widehat{c}\, \Phi_g^{1/2}, \qquad c = (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z}).$$

Proof. (a) Fix $c = (c_k)_{k\in\mathbf{Z}} \in c_{00}(\mathbf{Z})$. In this case, the series in equation (10.14) defining $\widehat{c}$ is a finite sum. Since $\widehat{c}$ is 1-periodic, by applying the unitarity of the Fourier transform on $L^2(\mathbf{R})$ we compute that

$$\begin{aligned}
\Big\| \sum_{k\in\mathbf{Z}} c_k\, T_k g \Big\|_{L^2(\mathbf{R})}^2 &= \Big\| \sum_{k\in\mathbf{Z}} c_k\, (T_k g)^\wedge \Big\|_{L^2(\mathbf{R})}^2 \\
&= \Big\| \sum_{k\in\mathbf{Z}} c_k\, M_{-k}\widehat{g} \Big\|_{L^2(\mathbf{R})}^2 \\
&= \int_{-\infty}^{\infty} \Big| \sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}\, \widehat{g}(\xi) \Big|^2 d\xi \\
&= \int_{-\infty}^{\infty} |\widehat{c}(\xi)\, \widehat{g}(\xi)\,|^2\, d\xi \qquad (10.16) \\
&= \sum_{j\in\mathbf{Z}} \int_0^1 |\widehat{c}(\xi+j)\, \widehat{g}(\xi+j)\,|^2\, d\xi
\end{aligned}$$

$$= \int_0^1 |\widehat{c}(\xi)|^2 \sum_{j\in\mathbf{Z}} |\widehat{g}(\xi+j)|^2 \, d\xi$$

$$= \int_0^1 |\widehat{c}(\xi)|^2 \, \Phi_g(\xi) \, d\xi$$

$$= \left\|\widehat{c}\, \Phi_g^{1/2}\right\|_{L^2(\mathbf{T})}.$$

The only place in this calculation where the assumption that c is a *finite* sequence is needed is to establish the equality in equation (10.16); all of the other steps are valid for arbitrary sequences $c \in \ell^2(\mathbf{Z})$.

(b) By Exercise 10.14, the sequence $\mathcal{T}(g) = \{T_k g\}_{k\in\mathbf{Z}}$ is finitely independent, so the mapping U is well defined, and equation (10.15) shows that U maps $\mathrm{span}\{T_k g\}_{k\in\mathbf{Z}}$ isometrically into $L^2(\mathbf{T})$. By Exercise 1.72, U has an extension to an isometric map of $V_0(g)$ into $L^2_{\Phi_g}(\mathbf{T})$, which we also call U. By definition we have $\mathrm{range}(U) \subseteq H_{\Phi_g}$, so it only remains to show that the range of U is H_{Φ_g}.

Since U is an isometry, its range is closed. Therefore, if we can find a sequence in range(U) that is complete in H_{Φ_g}, then we must have $\mathrm{range}(U) = H_{\Phi_g}$. Set

$$G_k(\xi) \;=\; U(T_k g)(\xi) \;=\; e^{-2\pi i k\xi}\, \Phi_g(\xi)^{1/2},$$

and suppose that $F \in H_{\Phi_g}$ satisfies $\langle F, G_k\rangle = 0$ for $k \in \mathbf{Z}$. Then for every k we have

$$\left\langle F\Phi_g^{1/2},\, e_{-k}\right\rangle \;=\; \int_0^1 F(\xi)\, \Phi_g(\xi)^{1/2}\, e^{2\pi i k\xi}\, d\xi \;=\; \langle F, G_k\rangle \;=\; 0. \qquad (10.17)$$

Since F and $\Phi_g^{1/2}$ both belong to $L^2(\mathbf{T})$, their product belongs to $L^1(\mathbf{T})$. Although $\{e^{2\pi i k\xi}\}_{k\in\mathbf{Z}}$ is not a basis for $L^1(\mathbf{T})$, Theorem 4.25 states that functions in $L^1(\mathbf{T})$ are uniquely determined by their Fourier coefficients. The kth Fourier coefficient of $F\Phi_g^{1/2}$ is the inner product $\left\langle F\Phi_g^{1/2},\, e_k\right\rangle$, so by equation (10.17) every Fourier coefficient of $F\Phi_g^{1/2}$ is zero. Since every Fourier coefficient of the zero function is also zero, we conclude that $F\Phi_g^{1/2} = 0$ a.e. As $F \in H_{\Phi_g}$, it follows that $F = 0$ a.e. Therefore $\{G_k\}_{k\in\mathbf{Z}}$ is a subset of range(U) that is complete in H_{Φ_g}.

(c) Let δ_k be the delta sequence. Then $\widehat{\delta_k} = e_{-k}$, so by the definition of U we have $U(T_k g) = e_{-k}\, \Phi_g^{1/2}$.

(d) Suppose that $\mathcal{T}(g)$ is a Bessel sequence, and let $(c_k)_{k\in\mathbf{Z}}$ be any sequence in $\ell^2(\mathbf{Z})$. Theorem 7.2 implies that the series $\sum_{n\in\mathbf{Z}} c_k\, T_k g$ converges unconditionally in $L^2(\mathbf{R})$. Since the Fourier transform is unitary, the sequence $\mathcal{M}(\widehat{g}\,) = \{M_{-k}\widehat{g}\,\}_{k\in Z}$ is Bessel, and therefore the series $F = \sum_{k\in\mathbf{Z}} c_k\, M_{-k}\widehat{g}$ converges unconditionally in $L^2(\mathbf{R})$. Also, the series $\widehat{c}(\xi) = \sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}$

converges unconditionally in $L^2(\mathbf{T})$ because $\{e^{-2\pi ik\xi}\}_{k\in\mathbf{Z}}$ is an orthonormal basis for that space. If these series converged pointwise then we could write

$$F(\xi) \;=\; \sum_{k\in\mathbf{Z}} c_k e^{-2\pi ik\xi}\,\widehat{g}(\xi) \;=\; \biggl(\sum_{k\in\mathbf{Z}} c_k e^{-2\pi ik\xi}\biggr)\,\widehat{g}(\xi) \;=\; \widehat{c}(\xi)\,\widehat{g}(\xi).$$

However, we only know that these series converge in the norm of their respective spaces, but even so Exercise 10.15 tells us that the equality $F(\xi) = \widehat{c}(\xi)\,\widehat{g}(\xi)$ holds pointwise almost everywhere. Hence the calculations leading to equation (10.15) extend to arbitrary sequences $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$. □

Now we can characterize the properties of systems of translates. Part (b) of the following theorem was first proved by Benedetto and Li [BL98], and part (f) is due to Nielsen and Šikić [NS07]. Note that, by definition, $\mathcal{T}(g)$ is complete in the space $V_0(g)$ and $\mathcal{E}(\Phi_g^{1/2}) \subseteq H_{\Phi_g}$.

Theorem 10.19. *Given $g \in L^2(\mathbf{R})$, the following statements hold.*

(a) $\mathcal{T}(g)$ *is a Bessel sequence in* $V_0(g)$ *if and only if* $\Phi_g \in L^\infty(\mathbf{T})$, *and in this case* $\Phi_g \le B$ *a.e. where* B *is a Bessel bound.*

(b) $\mathcal{T}(g)$ *is a frame for* $V_0(g)$ *if and only if there exist* A, $B > 0$ *such that*

$$A \;\le\; \Phi_g(\xi) \;\le\; B \qquad a.e.\ \xi \notin Z_{\Phi_g}.$$

In this case A, B *are frame bounds for* $\mathcal{T}(g)$.

(c) $\mathcal{T}(g)$ *is minimal in* $V_0(g)$ *if and only if* $1/\Phi_g \in L^1(\mathbf{T})$, *and in this case* $\mathcal{T}(g)$ *is exact in* $V_0(g)$.

(d) $\mathcal{T}(g)$ *is an unconditional basis for* $V_0(g)$ *if and only if there exist* A, $B > 0$ *such that* $A \le \Phi_g \le B$ *a.e., and in this case it is a Riesz basis for* $V_0(g)$.

(e) $\mathcal{T}(g)$ *is an orthonormal basis for* $V_0(g)$ *if and only if* $\Phi_g(t) = 1$ *a.e.*

(f) *With respect to the ordering* $\{0, -1, 1, -2, 2, \dots\}$ *of the index set* $\mathbf{Z}$, $\mathcal{T}(g)$ *is a Schauder basis for* $V_0(g)$ *if and only if* $\Phi_g \in \mathcal{A}_2(\mathbf{T})$.

Proof. Most of the implications follow directly by combining Lemma 10.18 with Theorem 5.15 or Theorem 10.10, so we will only elaborate on a few details.

(a) Since $U\colon V_0(g) \to H_{\Phi_g}$ is unitary, $\mathcal{T}(g)$ is a Bessel sequence in $V_0(g)$ if and only if $\mathcal{E}(\Phi_g^{1/2})$ is a Bessel sequence in H_{Φ_g}. By Exercise 7.7, being Bessel in H_{Φ_g} is equivalent to being Bessel in $L^2(\mathbf{T})$, and by Theorem 10.10 this happens if and only if Φ_g is bounded.

(b) $\mathcal{T}(g)$ is a frame for $V_0(g)$ if and only if $\mathcal{E}(\Phi_g^{1/2})$ is a frame for H_{Φ_g}. This happens if and only if $\mathcal{E}(\Phi_g^{1/2})$ is a frame sequence in $L^2(\mathbf{T})$, so the result follows from Theorem 10.10.

To illustrate another approach, we give a second proof of one implication, based on the characterization of frames given in Theorem 8.29(e). Assume that $\mathcal{T}(g)$ is a frame for $V_0(g)$ with frame bounds A, B, and let R denote its reconstruction operator. If $c = (c_k)_{k\in\mathbf{Z}} \in \ker(R)$ then $\sum_{k\in\mathbf{Z}} c_k\, T_k g = 0$, so

$$\int_0^1 |\widehat{c}(\xi)|^2\, \Phi_g(\xi)\, d\xi \;=\; \Bigl\|\sum_{k\in\mathbf{Z}} c_k\, T_k g\Bigr\|_{L^2(\mathbf{R})}^2 \;=\; 0.$$

It follows that

$$\ker(R) \;=\; \bigl\{c \in \ell^2(\mathbf{Z}) : \widehat{c}(\xi) = 0 \text{ for a.e. } \xi \notin Z_{\Phi_g}\bigr\},$$

and therefore

$$\begin{aligned} \ker(R)^\perp \;&=\; \bigl\{c \in \ell^2(\mathbf{Z}) : \widehat{c}(\xi) = 0 \text{ for a.e. } \xi \in Z_{\Phi_g}\bigr\} \\ &=\; \bigl\{c \in \ell^2(\mathbf{Z}) : \widehat{c} \in H_{\Phi_g}\bigr\}. \end{aligned} \tag{10.18}$$

Choose any function $F \in H_{\Phi_g}$. Then the characterization of $\ker(R)^\perp$ in equation 10.18 implies that $F = \widehat{c}$ for some sequence $c = (c_k)_{k\in\mathbf{Z}} \in \ker(R)^\perp$. By Lemma 10.18(d), we therefore have

$$\begin{aligned} \int_0^1 |F(\xi)|^2\, \Phi_g(\xi)\, d\xi \;&=\; \Bigl\|\sum_{k\in\mathbf{Z}} c_k\, T_k g\Bigr\|_{L^2(\mathbf{R})}^2 \\ &\geq\; A \sum_{k\in\mathbf{Z}} |c_k|^2 \qquad\qquad (10.19) \\ &=\; A \int_0^1 |F(\xi)|^2\, d\xi, \end{aligned}$$

where the inequality in equation (10.19) follows from Theorem 8.29(e). It follows from this that $\Phi_g(\xi) \geq A$ for a.e. $\xi \notin Z_{\Phi_g}$.

(c) By Theorem 10.10, $\mathcal{E}(\Phi_g^{1/2})$ is minimal if and only if $\Phi_g^{-1/2} \in L^2(\mathbf{T})$, which is equivalent to $1/\Phi_g \in L^1(\mathbf{T})$. □

Next we will deduce the structure of the canonical dual of a frame sequence of translates. For this we will need the *bracket product* function $[\widehat{f}, \widehat{g}\,]$ defined in equation (10.12). Note that if we take $f = g$ then $[\widehat{g}, \widehat{g}\,] = \Phi_g$, the periodization of $|\widehat{g}|^2$ defined in equation (10.11). The bracket product can be viewed as an L^1 function-valued inner product on $L^2(\mathbf{R})$, and more generally it is a special case of an inner product for a Hilbert C^* module. For more information on the bracket product, we refer to the paper [CL03].

The following lemma says that the inner product of f with $T_k g$ is the $(-k)$th Fourier coefficient of $[\widehat{f}, \widehat{g}\,]$. For preciseness, we emphasize in the statement and proof of this lemma which space an inner product is being taken on.

Lemma 10.20. *If $f,\, g \in L^2(\mathbf{R})$ then $[\widehat{f},\widehat{g}\,] \in L^1(\mathbf{T})$ and*

$$\langle f, T_k g\rangle_{L^2(\mathbf{R})} \;=\; \int_0^1 [\widehat{f},\widehat{g}\,](\xi)\, e^{2\pi i k\xi}\, d\xi \;=\; \bigl\langle [\widehat{f},\widehat{g}\,],\, e_{-k}\bigr\rangle_{L^2(\mathbf{T})}, \qquad k \in \mathbf{Z}.$$

Consequently, if $[\widehat{f},\widehat{g}\,] \in L^2(\mathbf{T})$ then

$$[\widehat{f},\widehat{g}\,](\xi) \;=\; \sum_{k\in\mathbf{Z}} \langle f, T_k g\rangle_{L^2(\mathbf{R})}\, e^{-2\pi i k\xi}, \tag{10.20}$$

where the series converges unconditionally in $L^2(\mathbf{T})$.

Proof. Since $\widehat{f},\, \widehat{g} \in L^2(\mathbf{R})$, the product $\widehat{f}\,\overline{\widehat{g}}$ belongs to $L^1(\mathbf{R})$, and therefore its periodization $[\widehat{f},\widehat{g}\,]$ belongs to $L^1(\mathbf{T})$. Applying the unitarity of the Fourier transform on $L^2(\mathbf{R})$ and then periodizing the integral, we compute that

$$\begin{aligned}
\langle f, T_k g\rangle_{L^2(\mathbf{R})} &= \langle \widehat{f}, (T_k g)^\wedge\rangle_{L^2(\mathbf{R})} \\
&= \langle \widehat{f}, M_{-k}\widehat{g}\,\rangle_{L^2(\mathbf{R})} \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i k\xi}\, \overline{\widehat{g}(\xi)}\, d\xi \\
&= \sum_{j\in\mathbf{Z}} \int_0^1 \widehat{f}(\xi+j)\, \overline{\widehat{g}(\xi+j)}\, e^{2\pi i k(\xi+j)}\, d\xi \\
&= \int_0^1 \biggl(\sum_{j\in\mathbf{Z}} \widehat{f}(\xi+j)\, \overline{\widehat{g}(\xi+j)}\biggr)\, e^{2\pi i k\xi}\, d\xi \\
&= \bigl\langle [\widehat{f},\widehat{g}\,],\, e_{-k}\bigr\rangle_{L^2(\mathbf{T})}.
\end{aligned}$$

In the calculations above, we have used the periodicity of the complex exponential to write $e^{2\pi i k(\xi+j)} = e^{2\pi i k\xi}$, and we are allowed to interchange the order of summation and integration by appealing to Fubini's Theorem.

Finally, if $[\widehat{f},\widehat{g}\,]$ belongs to $L^2(\mathbf{T})$ then we have

$$[\widehat{f},\widehat{g}\,] \;=\; \sum_{k\in\mathbf{Z}} \bigl\langle [\widehat{f},\widehat{g}\,],\, e_{-k}\bigr\rangle_{L^2(\mathbf{T})}\, e_{-k},$$

which implies equation (10.20). □

Theorem 10.21. *If $g \in L^2(\mathbf{R})$ and $\mathcal{T}(g)$ is a frame for $V_0(g)$, then the canonical dual frame in $V_0(g)$ is $\mathcal{T}(\widetilde{g})$, where $\widetilde{g} \in V_0(g)$ is the function whose Fourier transform is*

$$\widehat{\widetilde{g}}(\xi) \;=\; \begin{cases} \widehat{g}(\xi)/\Phi_g(\xi), & \xi \in \mathbf{R}\backslash Z_{\Phi_g}, \\ 0, & \xi \in Z_{\Phi_g}. \end{cases} \tag{10.21}$$

Proof. By Exercise 10.17, the frame operator $S\colon V_0(g) \to V_0(g)$ for $\mathcal{T}(g)$ commutes with the translation operator T_k when $k \in \mathbf{Z}$, and as a consequence the canonical dual frame has the form $\mathcal{T}(h)$ for some $h \in V_0(g)$.

Since Φ_g is bounded above and below on the complement of Z_{Φ_g}, there is a function $\widetilde{g} \in L^2(\mathbf{R})$ whose Fourier transform satisfies equation (10.21). If we define

$$m(\xi) \;=\; \begin{cases} 1/\Phi_g(\xi), & \xi \in \mathbf{R}\backslash Z_{\Phi_g}, \\ 0, & \xi \in Z_{\Phi_g}, \end{cases}$$

then $m \in L^2(\mathbf{T})$ and $\widehat{\widetilde{g}} = m\,\widehat{g}$. Consequently, Exercise 10.17 implies that $\widetilde{g} \in V_0(g)$, and our goal is to show that $h = \widetilde{g}$.

For simplicity of notation set $Z = Z_{\Phi_g}$, and let $U\colon V_0(g) \to H_{\Phi_g}$ be the unitary map constructed in Lemma 10.18. Note that the bracket product of $\widehat{\widetilde{g}}$ with $\widehat{g}$ is

$$\bigl[\widehat{\widetilde{g}}, \widehat{g}\bigr](\xi) \;=\; \sum_{j\in\mathbf{Z}} \widehat{\widetilde{g}}(\xi+j)\,\overline{\widehat{g}(\xi+j)} \;=\; \chi_{Z^{\mathrm{C}}}(\xi),$$

the characteristic function of the complement of $Z = Z_{\Phi_g}$. Lemma 10.20 therefore implies that

$$\bigl\langle \widetilde{g},\, T_k g\bigr\rangle_{L^2(\mathbf{R})} \;=\; \bigl\langle \chi_{Z^{\mathrm{C}}},\, e_{-k}\bigr\rangle_{L^2(\mathbf{T})}, \qquad k \in \mathbf{Z}.$$

Setting $c_k = \langle \widetilde{g}, T_k g\rangle_{L^2(\mathbf{R})}$, we have

$$\begin{aligned}\widehat{c} \;=\; \sum_{k\in\mathbf{Z}} c_k\, e_{-k} \;&=\; \sum_{k\in\mathbf{Z}} \langle \widetilde{g}, T_k g\rangle_{L^2(\mathbf{R})}\, e_{-k} \\ &=\; \sum_{k\in\mathbf{Z}} \bigl\langle \chi_{Z^{\mathrm{C}}},\, e_{-k}\bigr\rangle_{L^2(\mathbf{T})}\, e_{-k} \;=\; \chi_{Z^{\mathrm{C}}}.\end{aligned}$$

Applying Lemma 10.18(d), it follows that

$$\begin{aligned}U(S\widetilde{g}) \;&=\; U\biggl(\sum_{k\in\mathbf{Z}} \langle \widetilde{g}, T_k g\rangle_{L^2(\mathbf{R})}\, T_k g\biggr) \\ &=\; U\biggl(\sum_{k\in\mathbf{Z}} c_k\, T_k g\biggr) \\ &=\; \widehat{c}\,\Phi_g^{1/2} \;=\; \chi_{Z^{\mathrm{C}}}\,\Phi_g^{1/2} \;=\; \Phi_g^{1/2} \;=\; U(g).\end{aligned}$$

Thus $S\widetilde{g} = g$, so $\widetilde{g} = S^{-1}g = h$. □

Using a similar approach we can find a function $g^\sharp$ such that $\mathcal{T}(g^\sharp)$ is a Parseval frame for $V_0(g) = \overline{\mathrm{span}}(\mathcal{T}(g))$; see Exercise 10.21. If $\mathcal{T}(g)$ is a Riesz basis for $V_0(g)$, then $\mathcal{T}(g^\sharp)$ will be an orthonormal basis for $V_0(g)$.

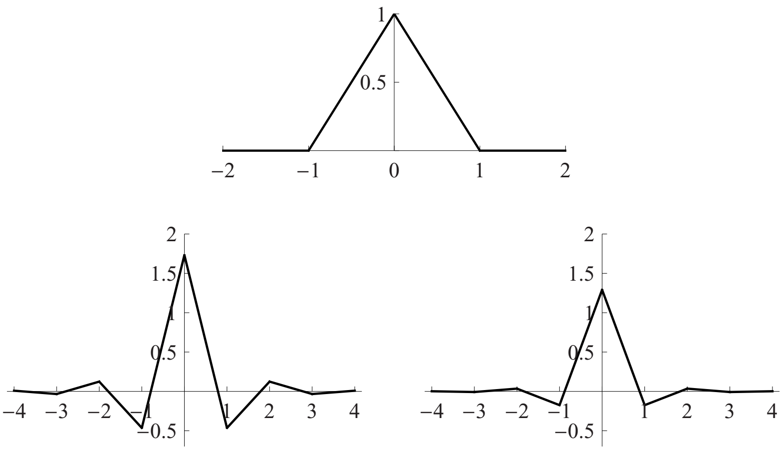

Fig. 10.1. Top: The hat function w. Bottom left: Generator $\widetilde{w}$ of the dual frame. Bottom right: Generator $w^\sharp$ of the associated orthonormal basis (this is the *Battle–Lemarié linear spline scaling function*).

Example 10.22. Let $w(x) = \max\{1 - |x|, 0\}$. This is the "hat function" or "tent function" on $[-1, 1]$, see Figure 10.1. Exercise 10.22 shows that Φ_w is bounded above and below, and therefore $\mathcal{T}(w)$ is a Riesz basis for its closed span $V_0(g)$. Hence Theorem 10.21 gives us a function $\widetilde{w}$ such that $\mathcal{T}(\widetilde{w})$ is the biorthogonal sequence to $\mathcal{T}(w)$. Since $\widetilde{w}$ belongs to $V_0(g)$, we can write it in the basis $\mathcal{T}(w)$ as

$$\widetilde{w} \;=\; \sum_{k\in\mathbf{Z}} \langle \widetilde{w},\, T_k\widetilde{w}\rangle\, T_k w. \tag{10.22}$$

Lemma 10.20 tells us that

$$\big\langle \widetilde{w},\, T_k\widetilde{w}\big\rangle \;=\; \big\langle \Phi_{\widetilde{w}},\, e_{-k}\big\rangle, \tag{10.23}$$

and by Exercise 10.22 we have

$$\Phi_{\widetilde{w}}(\xi) \;=\; \frac{1}{\Phi_w(\xi)} \;=\; \frac{3}{2+\cos 2\pi\xi}.$$

Therefore we can (numerically) compute the inner products appearing in equation (10.23), and find the representation of $\widetilde{w}$ in terms of the Riesz basis $\mathcal{T}(w)$. Since $T_k w$ is nonzero only within an interval of length 2, for any given x the series appearing in equation (10.22) has at most two nonzero terms, so we see that $\widetilde{w}$ is a piecewise linear function. On the other hand, infinitely many of the inner products $\langle \widetilde{w}, T_k\widetilde{w}\rangle$ are nonzero, so $\widetilde{w}$ is not compactly supported. Still, the fact that $\Phi_{\widetilde{w}}(\xi)$ is infinitely differentiable implies that its Fourier coefficients $\big\langle \Phi_{\widetilde{w}},\, e_{-k}\big\rangle$ decay rapidly as $|k| \to \infty$ (see Exercise 13.3). Therefore $\widetilde{w}(\xi)$ decays rapidly as $|\xi| \to \infty$, and in fact it can be shown that $\widetilde{w}$ decays exponentially. We illustrate $\widetilde{w}$ in Figure 10.1.

Additionally, Exercise 10.21 provides us with a function $w^\sharp$ whose integer translates generate an orthonormal basis for $V_0(w) = \overline{\text{span}}(\mathcal{T}(w))$. We can likewise represent $w^\sharp$ in the basis $\mathcal{T}(w)$ and see that $w^\sharp$ is piecewise linear. The function $w^\sharp$, which is pictured in Figure 10.1, is the first of the *Battle–Lemarié scaling functions* [Bat87], [Lem88]. The higher-order Battle–Lemarié scaling functions are constructed similarly, replacing the hat function by a higher-order B-spline function. The nth B-spline is $(n-1)$-times differentiable (Exercise 12.20), so the Battle–Lemarié scaling functions increase in smoothness with n. We refer to [Chr03] for more details on the Battle–Lemarié functions. $\diamondsuit$

It is not so easy to find functions $g \in L^2(\mathbf{R})$ such that $\mathcal{T}(g)$ is a frame but not a Riesz basis for its closed span $V_0(g)$. By Theorem 10.19, if $\mathcal{T}(g)$ is a redundant frame sequence then Z_{Φ_g} must be nontrivial. However, Φ_g must also be bounded away from zero on the complement of Z_{Φ_g}, which implies that Φ_g cannot be continuous. This proves the following lemma.

Lemma 10.23. *Let $g \in L^2(\mathbf{R})$ be such that $\mathcal{T}(g)$ is a frame sequence. Then*

$$\mathcal{T}(g) \text{ is redundant} \implies \Phi_g \text{ is not continuous.} \qquad \diamondsuit$$

If $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ then $\widehat{g}$ is continuous (Theorem 9.10). Consequently, if $\widehat{g}$ decays quickly enough at infinity then the series defining Φ_g will converge uniformly, and hence Φ_g will be continuous. The next result gives a specific criterion quantifying this statement.

Lemma 10.24. *Assume $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and*

$$\sum_{k\in\mathbf{Z}} \|\widehat{g} \cdot \chi_{[k,k+1]}\|_\infty^2 \;<\; \infty. \tag{10.24}$$

Then the following statements hold.

(a) *Φ_g is continuous and $\mathcal{T}(g)$ is a Bessel sequence.*

(b) *$\mathcal{T}(g)$ is a frame for $V_0(g)$ if and only if it is a Riesz basis for $V_0(g)$.*

Proof. (a) By hypothesis, $\widehat{g}$ is continuous, so each function $G_k(x) = |\widehat{g}(\xi+k)|^2$ belongs to $C[0,1]$. By definition, $\Phi_g = \sum_{k\in\mathbf{Z}} G_k$, and equation (10.24) says that this series converges absolutely in $C[0,1]$. Therefore Φ_g is continuous on $[0,1]$, and since it is 1-periodic it is continuous everywhere. Consequently Φ_g is bounded, so Theorem 10.19 implies that $\mathcal{T}(g)$ is a Bessel sequence.

(b) If $\mathcal{T}(g)$ is a frame sequence, then $0 < A \le \Phi_g \le B < \infty$ off the zero set of Φ_g. Since Φ_g is continuous, the only way this can happen is if the zero set of Φ_g is empty. $\square$

Equation (10.24) is an example of an *amalgam space* norm condition. We will study amalgam spaces in more detail in Section 11.4. In the terminology of that section, the hypothesis appearing in equation (10.24) says that $\widehat{g}$ belongs to the space $W(L^\infty, \ell^2)$. Loosely, $\widehat{g}$ is "locally bounded" and has "ℓ^2-type decay" at infinity. This, combined with continuity of $\widehat{g}$, implies continuity of Φ_g, which implies that $\mathcal{T}(g)$ cannot be a redundant frame for $V_0(g)$. This does not tell us that $\mathcal{T}(g)$ will be a Riesz basis for $V_0(g)$, as that depends on whether Φ_g has any zeros.

Here is another sufficient condition on g that implies continuity of Φ_g. In contrast to Lemma 10.24, this condition is formulated in terms of g rather than $\widehat{g}$. This result is due to Jia and Micchelli [JM91].

Theorem 10.25. *Given $g \in L^2(\mathbf{R})$, let*

$$\Theta(x) \;=\; \sum_{k\in\mathbf{Z}} |g(x+k)| \tag{10.25}$$

be the periodization of $|g|$. If $\Theta \in L^2(\mathbf{T})$, i.e.,

$$\int_0^1 \Theta(x)^2\,dx \;=\; \int_0^1 \biggl(\sum_{k\in\mathbf{Z}} |g(x+k)|\biggr)^2 dx \;<\; \infty, \tag{10.26}$$

then the following statements hold.

(a) *Φ_g is continuous and $\mathcal{T}(g)$ is a Bessel sequence.*

(b) *$\mathcal{T}(g)$ is a frame for $V_0(g)$ if and only if it is a Riesz basis for $V_0(g)$.*

Proof. As in the proof of Lemma 10.24, if we prove that Φ_g is continuous then the remaining statements are consequences of Theorem 10.19.

Since each term $|g(x+k)|$ is nonnegative, the series in equation (10.25) defining $\Theta(x)$ converges at almost every x either to a finite nonnegative value or to $+\infty$. Applying Tonelli's Theorem and using the fact that Θ is 1-periodic, we have

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} |\langle g, T_k g\rangle| &= \sum_{k\in\mathbf{Z}} \left| \int_{-\infty}^{\infty} g(x)\,\overline{g(x-k)}\,dx \right| \\
&\le \sum_{k\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} \int_0^1 |g(x-j)\,g(x-j-k)|\,dx \\
&= \sum_{j\in\mathbf{Z}} \int_0^1 |g(x-j)| \sum_{k\in\mathbf{Z}} |g(x-j-k)|\,dx \\
&= \sum_{j\in\mathbf{Z}} \int_0^1 |g(x-j)|\,\Theta(x)\,dx \\
&= \int_0^1 \Theta(x)^2\,dx \;<\; \infty.
\end{aligned}$$

Hence the sequence $\big(\langle g, T_k g\rangle\big)_{k\in\mathbf{Z}}$ belongs to $\ell^1(\mathbf{Z})$. However, by Lemma 10.20, $\langle g, T_k g\rangle = \langle \Phi_g, e_{-k}\rangle$. Thus the sequence of Fourier coefficients of Φ_g belongs to $\ell^1(\mathbf{Z})$. On the other hand we have $\Phi_g \in L^1(\mathbf{T})$, simply because $\widehat{g} \in L^2(\mathbf{R})$. Appealing to a result we will prove in Chapter 13, a function in $L^1(\mathbf{T})$ whose Fourier coefficients belong to $\ell^1(\mathbf{Z})$ can be written as an absolutely convergent Fourier series (see Theorem 13.25):

$$\Phi_g(\xi) \;=\; \sum_{k\in\mathbf{Z}} \langle \Phi_g, e_{-k}\rangle\, e_{-k}(\xi) \;=\; \sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi},$$

where $(c_k) \in \ell^1(\mathbf{Z})$. This is an equality of functions in $L^1(\mathbf{T})$, and hence holds pointwise almost everywhere. However, since $e^{-2\pi i k\xi}$ is continuous on $\mathbf{T}$ and $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$, the series $\sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}$ converges uniformly to a continuous function on $\mathbf{T}$, and therefore Φ_g is continuous on $\mathbf{T}$ (in the usual meaning of equaling a continuous function almost everywhere). $\square$

In short, if $g \in L^2(\mathbf{R})$ is such that $\mathcal{T}(g)$ is a redundant frame for its closed span, then g cannot be a very "nice" function. Some specific examples are given in Exercise 10.20.

Remark 10.26. Note the interesting similarities and differences between the conditions appearing in Lemma 10.24 and Theorem 10.25. Equation (10.24) is an amalgam norm condition on $\widehat{g}$:

$$\sum_{k\in\mathbf{Z}} \left(\sup_{\xi\in[0,1]} |\widehat{g}(\xi+k)|\right)^2 \;<\; \infty,$$

while equation (10.26) is a norm condition on the periodization of g:

$$\int_0^1 \left(\sum_{k\in\mathbf{Z}} |g(x+k|\right)^2 dx \;<\; \infty. \qquad \diamondsuit$$

We close this chapter with an open problem related to systems of translates. By Exercise 10.18, $\mathcal{T}(g)$ must always be incomplete in $L^2(\mathbf{R})$. Surprisingly, if we allow slightly nonregular translations, then we can obtain complete sequences of translates. For example, Olevskii and Ulanovskii have shown that there exists a function $g \in L^2(\mathbf{R})$ and points a_k with $|a_k - k| < \varepsilon$ such that $\{g(x-a_k)\}_{k\in\mathbf{N}}$ is complete in $L^2(\mathbf{R})$ [Ole97], [OU04]. Although such "irregular" systems are considerably more difficult to analyze than the systems $\mathcal{T}(g)$, it has been shown [CDH99] that $\{g(x-a_k)\}_{k\in\mathbf{N}}$ can never be a frame for $L^2(\mathbf{R})$, no matter what function $g \in L^2(\mathbf{R})$ and scalars $a_k \in \mathbf{R}$ that we choose. In an earlier paper, Olson and Zalik proved that $\{g(x-a_k)\}_{k\in\mathbf{N}}$ can never be a Riesz basis for $L^2(\mathbf{R})$, and they made the following conjecture [OZ92].

Conjecture 10.27 (Olson–Zalik Conjecture). *There does not exist a function $g \in L^2(\mathbf{R})$ and scalars $a_k \in \mathbf{R}$ such that $\{g(x-a_k)\}_{k\in\mathbf{N}}$ is a Schauder basis for $L^2(\mathbf{R})$.* $\diamondsuit$

This conjecture is still open as of the time of writing. Currently, the best partial result known is that if $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ then $\{g(x-a_k)\}_{k\in\mathbf{N}}$ cannot be a Schauder basis for $L^2(\mathbf{R})$ [DH00]. We will give a short proof, due to A. Olevskii, of the weaker result that $\{g(x-a_k)\}_{k\in\mathbf{N}}$ can never be a Riesz basis.

Theorem 10.28. *If $g \in L^2(\mathbf{R})$ and $a_k \in \mathbf{R}$ for $k \in \mathbf{N}$ then $\{g(x-a_k)\}_{k\in\mathbf{N}}$ is not a Riesz basis for $L^2(\mathbf{R})$.*

Proof. If $\{g(x-a_k)\}_{k\in\mathbf{N}}$ is a Riesz basis for $L^2(\mathbf{R})$ then, since the Fourier transform is unitary, the sequence $\{e^{-2\pi i a_k \xi}\, \widehat{g}(\xi)\}_{k\in\mathbf{N}}$ is also a Riesz basis for $L^2(\mathbf{R})$. If $\widehat{g}$ vanished on any set E with positive measure then χ_E would be orthogonal to every element of this basis, which is a contradiction. Hence we must have $\widehat{g}(\xi) \neq 0$ for almost every ξ. By Theorem 7.13, there exist constants $A,\, B > 0$ such that for any sequence $(c_k)_{k\in\mathbf{N}} \in \ell^2$ we have

$$A \sum_{k=1}^{\infty} |c_k|^2 \;\le\; \biggl\| \sum_{k=1}^{\infty} c_k\, e^{-2\pi i a_k \xi}\, \widehat{g}(\xi) \biggr\|_{L^2}^2 \;\le\; B \sum_{k=1}^{\infty} |c_k|^2. \tag{10.27}$$

The function $F = \widehat{g} \cdot \chi_{[0,1]}$ belongs to $L^2(\mathbf{R})$, so it can be written in the basis $\{e^{-2\pi i a_k \xi}\, \widehat{g}(\xi)\}_{k\in\mathbf{N}}$ as

$$F(\xi) \;=\; \sum_{k=1}^{\infty} c_k\, e^{-2\pi i a_k \xi}\, \widehat{g}(\xi),$$

where the series converges unconditionally in L^2-norm and the scalars $(c_k)_{k\in\mathbf{N}}$ belong to ℓ^2. Given $r > 0$, define

$$F_r(\xi) \;=\; \sum_{k=1}^{\infty} c_k\, e^{-2\pi i a_k (\xi - r)}\, \widehat{g}(\xi) \;=\; \sum_{k=1}^{\infty} (c_k\, e^{2\pi i a_k r})\, e^{-2\pi i a_k \xi}\, \widehat{g}(\xi).$$

This series converges unconditionally in $L^2(\mathbf{R})$ because $\sum |c_k\, e^{2\pi i a_k r}|^2 < \infty$. Since $\widehat{g}$ is nonzero almost everywhere, we can define $G_r(\xi) = F_r(\xi)/\widehat{g}(\xi)$. Set

$$s_N(\xi) \;=\; \sum_{k=1}^{N} c_k\, e^{-2\pi i a_k (\xi - r)}.$$

Then $s_N(\xi)\, \widehat{g}(\xi) \to F_r(\xi) = G_r(\xi)\, \widehat{g}(\xi)$ in L^2-norm as $N \to \infty$. But we also have

$$s_N(\xi + r)\, \widehat{g}(\xi) \;=\; \sum_{k=1}^{N} c_k\, e^{-2\pi i a_k \xi}\, \widehat{g}(\xi) \;\to\; F(\xi) \;=\; \chi_{[0,1]}(\xi)\, \widehat{g}(\xi).$$

Since convergence in L^2-norm implies the existence of a subsequence that converges pointwise a.e., there exist N_k such that

$$s_{N_k}(\xi)\,\widehat{g}(\xi) \;\to\; F_r(\xi) \;=\; G_r(\xi)\,\widehat{g}(\xi) \;\text{ a.e.}$$

and

$$s_{N_k}(\xi+r)\,\widehat{g}(\xi) \;\to\; F(\xi) \;=\; \chi_{[0,1]}(\xi)\,\widehat{g}(\xi) \;\text{ a.e.}$$

Since $\widehat{g}(\xi) \neq 0$ for a.e. ξ, this implies that

$$G_r(\xi) \;=\; \chi_{[0,1]}(\xi - r) \;=\; \chi_{[r,r+1]}(\xi) \;\text{ a.e.}$$

Consequently $F_r(\xi) = G_r(\xi)\,\widehat{g}(\xi) = \chi_{[r,r+1]}(\xi)\,\widehat{g}(\xi)$, so by applying equation (10.27) we see that

$$\begin{aligned}
\int_0^1 |\widehat{g}(\xi)|^2\,d\xi \;=\; \|F\|_{L^2}^2 \;&\le\; B \sum_{k=1}^{\infty} |c_k|^2 \\
&\le\; \frac{B}{A}\,\|F_r\|_{L^2}^2 \\
&=\; \frac{B}{A} \int_r^{r+1} |\widehat{g}(\xi)|^2\,d\xi \;\to\; 0 \quad \text{as } r \to \infty.
\end{aligned}$$

This implies that $\widehat{g} = 0$ a.e. on $[0,1]$, which is a contradiction. □

Exercises

10.12. Let $g \in L^2(\mathbf{R})$ and $a > 0$ be given, and define $h(x) = g(ax)$. Show that $\{g(x-ak)\}_{k\in\mathbf{Z}}$ is a frame sequence in $L^2(\mathbf{R})$ if and only if $\{h(x-k)\}_{k\in\mathbf{Z}}$ is a frame sequence in $L^2(\mathbf{R})$. What is the relation between the closed spans of these two systems?

10.13. Given a function $f \in L^1(\mathbf{R})$ and given $a > 0$, we call the function

$$\varphi(x) \;=\; \sum_{n\in\mathbf{Z}} f(x+an)$$

the *a-periodization* of f (or simply the *periodization* if $a = 1$). Show that the series defining φ converges absolutely in $L^1[0,a]$, and

$$\int_0^a \varphi(x)\,dx \;=\; \int_{-\infty}^{\infty} f(x)\,dx.$$

In particular, the *bracket product* of $f, g \in L^2(\mathbf{R})$ is the function $[f,g] \in L^1(\mathbf{T})$ defined by

$$[f,g](x) \;=\; \sum_{n\in\mathbf{Z}} f(x+n)\,\overline{g(x+n)}.$$

10.14. Show that if $g \in L^2(\mathbf{R})$ is not the zero function, then $\{T_a g\}_{a\in\mathbf{R}}$ is finitely linearly independent.

10.15. Suppose that $g \in L^2(\mathbf{R})$ is such that $\mathcal{M}(\widehat{g}) = \{e^{-2\pi i k\xi}\widehat{g}(\xi)\}_{k\in\mathbf{Z}}$ is a Bessel sequence in $L^2(\mathbf{R})$. By Theorem 7.2, if $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ then the series $F(\xi) = \sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}\widehat{g}(\xi)$ converges unconditionally in $L^2(\mathbf{R})$. Also, since $\{e^{-2\pi i k\xi}\}_{k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$ the series $\widehat{c}(\xi) = \sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}$ converges unconditionally in $L^2(\mathbf{T})$. Show that $F(\xi) = \widehat{c}(\xi)\,\widehat{g}(\xi)$ a.e.

10.16. Prove the remaining statements in Theorem 10.19.

10.17. Suppose that $g \in L^2(\mathbf{R})$ is such that $\mathcal{T}(g)$ is a frame sequence in $L^2(\mathbf{R})$, and let $S\colon V_0(g) \to V_0(g)$ be its frame operator.

(a) Show that $S(T_k f) = T_k(Sf)$ for each $f \in V_0(g)$. Use this to show that the canonical dual frame in $V_0(g)$ is $\mathcal{T}(h)$ where $h = S^{-1}g$.

(b) Show that

$$f \in V_0(g) \quad\Longleftrightarrow\quad \exists\, m \in L^2(\mathbf{T}) \text{ such that } \widehat{f} = m\,\widehat{g} \text{ a.e.}$$

10.18. (a) Suppose $g \in L^2(\mathbf{R})$ and $\widehat{g}$ is nonzero almost everywhere. Show that if $f \in V_0(g)$ then there is a 1-periodic function p such that $\widehat{f} = p\,\widehat{g}$ a.e.

(b) Given an an arbitrary function g in $L^2(\mathbf{R})$, show that $\mathcal{T}(g)$ is incomplete in $L^2(\mathbf{R})$.

10.19. Suppose that $f,\ g \in L^2(\mathbf{R})$ are such that $\mathcal{T}(f)$ and $\mathcal{T}(g)$ are both orthonormal sequences.

(a) Show that $\mathcal{T}(f)$ and $\mathcal{T}(g)$ have the same closed spans if and only if $\widehat{f}(\xi) = \alpha(\xi)\,\widehat{g}(\xi)$ a.e., where α is 1-periodic and $|\alpha(\xi)| = 1$ a.e.

(b) Show that if f and g are each compactly supported, then $\overline{\text{span}}(\mathcal{T}(f)) = \overline{\text{span}}(\mathcal{T}(g))$ if and only $f = \alpha\, T_k g$ where $k \in \mathbf{Z}$ and α is a scalar with $|\alpha| = 1$.

10.20. Let $g \in L^2(\mathbf{R})$ be given.

(a) Show that if g is compactly supported (zero almost everywhere outside of some finite interval), then $\mathcal{T}(g)$ is a Bessel sequence but cannot be a redundant frame for its closed span $V_0(g)$ (compare Exercise 10.22).

(b) Show that if $\widehat{g}$ is continuous and compactly supported, then $\mathcal{T}(g)$ is a Bessel sequence but cannot be a redundant frame for $V_0(g)$.

(c) Find a function $g \in L^2(\mathbf{R})$ such that $\mathcal{T}(g)$ is a redundant frame for $V_0(g)$.

10.21. Suppose that $g \in L^2(\mathbf{R})$ is such that $\mathcal{T}(g)$ is a frame sequence in $L^2(\mathbf{R})$, and let $g^\sharp \in L^2(\mathbf{R})$ be the function whose Fourier transform is

$$\widehat{g^\sharp}(\xi) \;=\; \begin{cases} \widehat{g}(\xi)\,\Phi_g(\xi)^{-1/2}, & \xi \notin Z_{\Phi_g}, \\ 0, & \xi \in Z_{\Phi_g}. \end{cases}$$

Show that $g^\sharp \in V_0(g) = \overline{\mathrm{span}}(\mathcal{T}(g))$, and $\mathcal{T}(g^\sharp)$ is a Parseval frame for $V_0(g)$. Show further that if $\mathcal{T}(g)$ is a Riesz basis for $V_0(g)$ then $\mathcal{T}(g^\sharp)$ is an orthonormal basis for $V_0(g)$.

10.22. Let $w(x) = \max\{1-|x|, 0\}$ be the "hat" or "tent" function on $[-1,1]$. Use Lemma 10.20 to show that

$$\Phi_w(\xi) \;=\; \frac{2+\cos 2\pi\xi}{3} \;=\; \frac{1+2\cos^2 \pi\xi}{3},$$

and conclude that $\mathcal{T}(w)$ is a Riesz basis for its closed span $V_0(g)$. Find $\widehat{\widetilde{w}}$, $\Phi_{\widetilde{w}}$, $\widehat{w^\sharp}$, and $\Bigl[\widehat{w^\sharp}, \widehat{\widetilde{w}}\Bigr]$.

10.23. Suppose that $g \in L^2(\mathbf{R})$ and $a_k \in \mathbf{R}$ are such that $\{g(x-a_k)\}_{k\in\mathbf{N}}$ is a Schauder basis for $L^2(\mathbf{R})$. Show that the sequence $(a_k)_{k\in\mathbf{N}}$ must be *uniformly separated*, i.e., $\inf_{j\neq k} |a_j - a_k| > 0$.

11

Gabor Bases and Frames

In this chapter we will consider the construction and properties of the class of *Gabor frames* for the Hilbert space $L^2(\mathbf{R})$. The analysis and application of Gabor systems is one part of the field of *time-frequency analysis*, which is more broadly explored in Gröchenig's text [Grö01].

In Chapter 10 we focused on systems of weighted exponentials $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ and systems of translates $\{g(x-k)\}_{k\in\mathbf{Z}}$. Each of these systems is generated by applying a single type of operation (modulation or translation) to a single generating function (φ or g). The resulting sequences have many applications, but their closed spans can only be proper subspaces of $L^2(\mathbf{R})$. In contrast, Gabor systems incorporate both modulations and translations, and can be frames for all of $L^2(\mathbf{R})$.

Gabor systems were briefly introduced in Example 8.10 and are defined precisely as follows.

Definition 11.1. A *lattice Gabor system*, or simply a *Gabor system* for short, is a sequence in $L^2(\mathbf{R})$ of the form

$$\mathcal{G}(g,a,b) \;=\; \{e^{2\pi ibnx} g(x-ak)\}_{k,n\in\mathbf{Z}},$$

where $g \in L^2(\mathbf{R})$ and $a,\, b > 0$ are fixed. We call g the *generator* or the *atom* of the system, and refer to a, b as the *lattice parameters*. ♢

More generally, an "irregular" Gabor system is a sequence of the form $\mathcal{G}(g,\Lambda) = \bigl\{e^{2\pi ibx} g(x-a)\bigr\}_{(a,b)\in\Lambda}$, where Λ is an arbitrary countable set of points in $\mathbf{R}^2$. Lattice Gabor systems have many attractive features and applications, and are much easier to analyze than irregular Gabor systems, so we focus on lattice systems for most of this chapter. For more details on irregular Gabor systems, we refer to [Grö01] or the survey paper [Hei07].

We are especially interested in Gabor systems that form frames or Riesz bases for $L^2(\mathbf{R})$. Naturally, if $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$, then we call it a *Gabor frame*, and if it is a Riesz basis, then we call it a *Gabor Riesz basis* or an *exact Gabor frame*.

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Gabor systems are named after Dennis Gabor (1900–1979), who was awarded the Nobel prize for his invention of holography. In his paper [Gab46], Gabor proposed using the Gabor system $\mathcal{G}(\phi, 1, 1)$ generated by the Gaussian function $\phi(x) = e^{-\pi x^2}$. Von Neumann [vN32, p. 406] had earlier claimed (without proof) that $\mathcal{G}(\phi, 1, 1)$ is complete in $L^2(\mathbf{R})$, i.e., its finite linear span is dense. Gabor conjectured (incorrectly, as we will see) that every function in $L^2(\mathbf{R})$ could be represented in the form

$$f = \sum_{k,n \in \mathbf{Z}} c_{kn}(f) \, M_n T_k \phi \tag{11.1}$$

for some scalars $c_{kn}(f)$; see [Gab46, Eq. 1.29]. This is one reason why general families $\mathcal{G}(g, a, b)$ are named in his honor (see [Jan01] for additional historical remarks and references).

Von Neumann's claim of completeness was proved in [BBGK71], [Per71], and [BGZ75]. However, completeness is a weak property and does not imply the existence of expansions of the form given in equation (11.1). Reading a bit extra into what von Neumann and Gabor actually wrote, possibly they expected that $\mathcal{G}(\phi, 1, 1)$ would be a Schauder basis or a Riesz basis for $L^2(\mathbf{R})$. In fact, $\mathcal{G}(\phi, 1, 1)$ is neither, as it is overcomplete in the sense that any single element may be removed and still leave a complete system. In fact, the excess is precisely 1, because this system becomes incomplete as soon as two elements are removed. However, even with one element removed, the resulting exact system forms neither a Schauder basis nor a Riesz basis; cf. [Fol89, p. 168]. In fact, Janssen proved in [Jan81] that Gabor's conjecture that each $f \in L^2(\mathbf{R})$ has an expansion of the form in equation (11.1) is true, but he also showed that the series converges only in the sense of tempered distributions—not in the norm of L^2—and the coefficients c_{kn} grow with k and n (see also [LS99]).

Today we realize that there are no "good" Gabor Riesz bases $\mathcal{G}(g, a, b)$ for $L^2(\mathbf{R})$. Indeed, the *Balian–Low Theorem*, which we mentioned in Chapter 8 and will consider in detail in Section 11.8, implies that only "badly behaved" atoms g can generate Gabor Riesz bases. On the other hand, redundant Gabor frames with nice generators do exist, and they provide us with useful tools for many applications. We will study the construction and special properties of Gabor frames in this chapter.

11.1 Time-Frequency Shifts

We recall the following operations on functions $f \colon \mathbf{R} \to \mathbf{C}$.

$$\begin{aligned}
\text{Translation:} \quad (T_a f)(x) &= f(x - a), & a &\in \mathbf{R}. \\
\text{Modulation:} \quad (M_b f)(x) &= e^{2\pi i b x} f(x), & b &\in \mathbf{R}. \\
\text{Dilation:} \quad (D_r f)(x) &= r^{1/2} f(rx), & r &> 0.
\end{aligned}$$

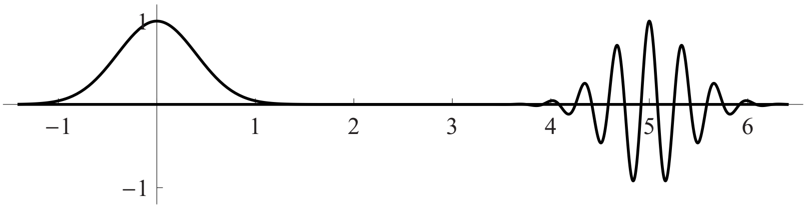

Fig. 11.1. The Gaussian window $\phi(x) = e^{-\pi x^2}$ and the real part of the time-frequency shift $M_3T_5\phi$.

We often think of the independent variable $x \in \mathbf{R}$ as representing time, and hence refer to translation as a *time shift*. We call modulation a *frequency shift*, and say that a composition of translation and modulation is a *time-frequency shift* (see the illustration in Figure 11.1). Thus, a Gabor system $\mathcal{G}(g, a, b)$ is a set of time-frequency shifts of the atom g:

$$\mathcal{G}(g, a, b) \;=\; \bigl\{M_{bn}T_{ak}g\bigr\}_{k,n\in\mathbf{Z}}.$$

Unfortunately, the translation and modulation operators do not commute in general. Being careful with the ordering of composition and evaluation, we compute that

$$\begin{aligned} T_aM_bf(x) \;&=\; (T_a(M_bf))(x) \\ &=\; (M_bf)(x-a) \\ &=\; e^{2\pi ib(x-a)}f(x-a) \\ &=\; e^{-2\pi iab}\,e^{2\pi ibx}f(x-a) \\ &=\; e^{-2\pi iab}\,M_bT_af. \end{aligned}$$

The pesky *phase factor* $e^{-2\pi iab}$ has modulus 1, but we only have $e^{-2\pi iab} = 1$ when $ab \in \mathbf{Z}$. Hence M_b and T_a only commute when the product ab is integer. Even so, by Exercise 11.3, $\{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ is a frame if and only if $\{T_{ak}M_{bn}g\}_{k,n\in\mathbf{Z}}$ is a frame, so in this sense the ordering of T_{ak} and M_{bn} is not important in many circumstances. However, we must still be careful to respect these phase factors in our calculations, as they do create significant difficulties at times (as in Section 11.9).

The product ab of the lattice generators appears in many calculations involving Gabor systems. It is usually the product ab that is important, rather than the individual values of a and b, because by dilating g we can change the value of a at the expense of a complementary change to b. This is made precise in the next lemma.

Lemma 11.2. *Fix $g \in L^2(\mathbf{R})$ and $a,\, b \in \mathbf{R}$. Then given $r > 0$, $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbf{R})$ if and only if $\mathcal{G}(D_rg, a/r, br)$ is a frame for $L^2(\mathbf{R})$.*

Proof. Using the dilation $D_r g(x) = r^{1/2} g(rx)$, we have

$$\begin{aligned} D_r(M_{bn}T_{ak}g)(x) &= r^{1/2}(M_{bn}T_{ak}g)(rx) \\ &= r^{1/2}e^{2\pi ibnrx}g(rx-ak) \\ &= r^{1/2}e^{2\pi ibnrx}g(r(x-ak/r)) \\ &= M_{bnr}T_{ak/r}(D_r g)(x). \end{aligned}$$

Thus $\mathcal{G}(D_r g, a/r, br)$ is the image of $\mathcal{G}(g,a,b)$ under the dilation D_r. The result then follows from the fact that D_r is a unitary mapping of $L^2(\mathbf{R})$ onto itself. □

If $\mathcal{G}(g,a,b)$ is a Gabor frame, then its frame operator is

$$Sf = \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}g.$$

The frame operator commutes with M_{bn} and T_{ak} for k, $n \in \mathbf{Z}$ (Exercise 11.3). A consequence of this is that S^{-1} also commutes with M_{bn} and T_{ak}, so we have $S^{-1}(M_{bn}T_{ak}g) = M_{bn}T_{ak}(S^{-1}g)$. Therefore the canonical dual of $\mathcal{G}(g,a,b)$ is another Gabor frame.

Lemma 11.3. *If $\mathcal{G}(g,a,b)$ is a Gabor frame for $L^2(\mathbf{R})$, then its canonical dual frame is $\mathcal{G}(\widetilde{g},a,b)$ where $\widetilde{g} = S^{-1}g$.* ♢

To each Gabor system $\mathcal{G}(g,a,b)$ we will associate the a-periodic function G_0 defined by

$$G_0(x) = \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 = \sum_{k\in\mathbf{Z}} |T_{ak}g(x)|^2, \qquad x\in\mathbf{R}.$$

Implicitly, G_0 depends on g and a. Note that G_0 is the a-periodization of $|g|^2$ in the sense of Exercise 10.13, and by that exercise we have $G_0 \in L^1[0,a]$ and

$$\int_0^a G_0(x)\,dx = \int_{-\infty}^{\infty} |g(x)|^2\,dx = \|g\|_{L^2}^2. \tag{11.2}$$

Exercises

11.1. Given $g \in L^2(\mathbf{R})$, show that $\{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{R})$ if and only if $\{T_{ak}M_{bn}g\}_{k,n\in\mathbf{Z}}$ is a frame, and in this case their frame operators coincide.

11.2. (a) Use the fact that $T_aM_b = e^{-2\pi iab}M_bT_a$ to show that the set $\{T_aM_b\}_{a,b\in\mathbf{R}}$ of time-frequency shift operators is not closed under compositions, and hence does not form a group.

(b) Define

$$\mathbb{H}_1 = \{e^{2\pi i t} T_a M_b\}_{a,b,t\in\mathbf{R}},$$

and show that $\mathbb{H}_1$ is a nonabelian group under composition of operators.

(c) Define

$$\mathbb{H}_2 = \mathbf{R}^3 = \{(a,b,t)\}_{a,b,t\in\mathbf{R}}.$$

Show that $\mathbb{H}_2$ is a nonabelian group with respect to the operation

$$(a,b,t) * (c,d,u) = (a+c, b+d, t+u+bc).$$

Show further that $\mathbb{H}_2$ is isomorphic to $\mathbb{H}_1$.

(d) Define

$$\mathbb{H}_3 = \left\{ \begin{bmatrix} 1 & b & t \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \right\}_{a,b,t\in\mathbf{R}}.$$

Show that $\mathbb{H}_3$ is a nonabelian group with respect to multiplication of matrices, and $\mathbb{H}_3$ is isomorphic to $\mathbb{H}_1$.

(e) Show that $a\mathbf{Z} \times b\mathbf{Z} \times \{0\} = \{(ak, bn, 0)\}_{k,n\in\mathbf{Z}}$ is not a subgroup of $\mathbb{H}_2$, but the countable subset $a\mathbf{Z} \times b\mathbf{Z} \times ab\mathbf{Z} = \{(ak, bn, abj)\}_{k,n,j\in\mathbf{Z}}$ is a subgroup.

(f) As a set, $\mathbb{H}_2 = \mathbf{R}^3$, and hence has a natural topology. In fact, $\mathbb{H}_2$ is an example of a *locally compact group* (LCG). Every LCG has associated left and right *Haar measures* (and these are unique up to scalar multiples). Show that the left Haar measure for $\mathbb{H}_2$ is $da\,db\,dt$, which means that for every $(c,d,u) \in \mathbb{H}_2$ we have

$$\iiint F\big((c,d,u) * (a,b,t)\big)\,da\,db\,dt = \iiint F(a,b,t)\,da\,db\,dt$$

for every integrable function F on $\mathbb{H}_2 = \mathbf{R}^3$. Show that the right Haar measure is also $da\,db\,dt$. Thus, even though $\mathbb{H}_2$ is nonabelian, its left and Haar right measures coincide (such an LCG is said to be *unimodular*).

Remark: The (isomorphic) groups $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{H}_3$ are called the *Heisenberg group*. The properties of the Heisenberg group should be contrasted with those of the affine group discussed in Exercise 12.2.

11.3. Let $\mathcal{G}(g,a,b)$ be a Gabor frame for $L^2(\mathbf{R})$.

(a) Show that the frame operator S commutes with M_{bn} and T_{ak} for all k, $n \in \mathbf{Z}$, and use this to show that S^{-1} also commutes with M_{bn} and T_{ak}.

(b) Show that the canonical dual frame of $\mathcal{G}(g,a,b)$ is the Gabor frame $\mathcal{G}(\widetilde{g},a,b)$ where $\widetilde{g} = S^{-1}g$.

(c) Suppose that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$. Show that $\mathcal{G}(g,a,b)$ is a Riesz basis if and only if $\langle g, \widetilde{g}\rangle = 1$.

(d) Show that the canonical Parseval frame of a lattice Gabor frame is another lattice Gabor frame. Specifically, if $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ and we set $g^\sharp = S^{-1/2}g$, where S is the frame operator, then $\mathcal{G}(g^\sharp,a,b)$ is a Parseval frame for $L^2(\mathbf{R})$.

11.4. Fix $g \in L^2(\mathbf{R})$ and $a,\, b > 0$. Recall from equations (9.2) and (9.3) that the Fourier transform interchanges translation with modulation. Use this to show that

$$\mathcal{G}(g,a,b) \text{ is a frame} \quad\Longleftrightarrow\quad \mathcal{G}(\widehat{g},b,a) \text{ is a frame.}$$

11.2 Painless Nonorthogonal Expansions

The simplest example of a Gabor frame is

$$\mathcal{G}(\chi_{[0,1]},1,1) \;=\; \bigl\{e^{2\pi inx}\,\chi_{[k,k+1]}(x)\bigr\}_{k,n\in\mathbf{Z}}.$$

If we fix a particular k, then by Example 1.52 we know that the sequence $\bigl\{e^{2\pi inx}\,\chi_{[k,k+1]}(x)\bigr\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[k,k+1]$. Hence the Gabor system $\mathcal{G}(\chi_{[0,1]},1,1)$ is simply the union of orthonormal bases for $L^2[k,k+1]$ over all $k \in \mathbf{Z}$, and consequently $\mathcal{G}(\chi_{[0,1]},1,1)$ is an orthonormal basis for $L^2(\mathbf{R})$.

Unfortunately, this Gabor system is not very useful in practice. The generator $\chi_{[0,1]}$ is very well localized in the time domain in the sense that it is zero outside of a finite interval. However, it is discontinuous, and this means that the expansion of a smooth function in the orthonormal basis $\mathcal{G}(\chi_{[0,1]},1,1)$ will not converge any faster than the expansion of a discontinuous function. From another viewpoint, the problem with the function $g = \chi_{[0,1]}$ is that its Fourier transform is a modulated sinc function:

$$\widehat{g}(\xi) \;=\; e^{-\pi i\xi}\,\frac{\sin\pi\xi}{\pi\xi}.$$

Thus $\widehat{g}$ decays only on the order of $1/|\xi|$ and is not even integrable. We want to find Gabor frames generated by functions that are both smooth and well localized.

We can try to create "better" Gabor systems by using a different atom g or different lattice parameters a, b. If we stick to functions g that are compactly supported in an interval of length $1/b$, then it is quite easy to create Gabor frames $\mathcal{G}(g,a,b)$ for $L^2(\mathbf{R})$, and we can even do so with smooth, compactly supported generators if we choose a and b appropriately. This was first done by Daubechies, Grossmann, and Meyer [DGM86], who referred to these as *Painless Nonorthogonal Expansions.*

Theorem 11.4 (Painless Nonorthogonal Expansions). *Fix* $a,\, b > 0$ *and* $g \in L^2(\mathbf{R})$.

(a) *If* $0 < ab \le 1$ *and* $\operatorname{supp}(g) \subseteq [0,b^{-1}]$, *then* $\mathcal{G}(g,a,b)$ *is a frame for* $L^2(\mathbf{R})$ *if and only if there exist constants* A, $B > 0$ *such that*

$$Ab \;\le\; \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 \;\le\; Bb \text{ a.e.} \tag{11.3}$$

In this case, A, B *are frame bounds for* $\mathcal{G}(g,a,b)$.

(b) *If* $0 < ab < 1$, *then there exist* g *supported in* $[0, b^{-1}]$ *that satisfy equation* (11.3) *and are as smooth as we like (even infinitely differentiable).*

(c) *If* $ab = 1$, *then any* g *that is supported in* $[0, b^{-1}]$ *and satisfies equation* (11.3) *must be discontinuous.*

(d) *If* $ab > 1$ *and* g *is supported in* $[0, b^{-1}]$, *then equation* (11.3) *is not satisfied and* $\mathcal{G}(g, a, b)$ *is incomplete in* $L^2(\mathbf{R})$.

Proof. (a) Suppose that $\mathrm{supp}(g) \subseteq [0, b^{-1}]$ and equation (11.3) holds. Exercise 8.4 tells us that in order to show that $\mathcal{G}(g, a, b)$ is a frame, we need only establish that the frame bounds hold on a dense subset of $L^2(\mathbf{R})$. So, let us consider functions f in the dense subspace $C_c(\mathbf{R})$ (actually, continuity is not needed here, we could just as well restrict our attention to functions that are bounded and compactly supported). Since $g \in L^2(\mathbf{R})$ is supported within $[0, b^{-1}]$, the translated function $T_{ak}g$ belongs to $L^2(I_k)$, where $I_k = [ak, ak + b^{-1}]$. Since f is bounded, the product $f \cdot T_{ak}\overline{g}$ also belongs to $L^2(I_k)$. Now, $\{e^{2\pi i n x}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2[0, 1]$, so by making a change of variables it follows that

$$\{b^{1/2} e_{bn}\}_{n \in \mathbf{Z}} \;=\; \{b^{1/2} e^{2\pi i b n x}\}_{n \in \mathbf{Z}}$$

is an orthonormal basis for $L^2(I_k)$. Applying the Plancherel Equality (and keeping in mind that $T_{ak}g$ is supported in I_k), we therefore have

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)\, g(x - ak)|^2\, dx \;&=\; \int_{ak}^{ak+b^{-1}} \bigl|f(x)\, T_{ak}\overline{g(x)}\bigr|^2\, dx \\
&=\; \bigl\|f \cdot T_{ak}\overline{g}\bigr\|_{L^2(I_k)} \\
&=\; \sum_{n \in \mathbf{Z}} \bigl|\bigl\langle f \cdot T_{ak}\overline{g},\, b^{1/2} e_{bn} \bigr\rangle_{L^2(I_k)}\bigr|^2 \\
&=\; b \sum_{n \in \mathbf{Z}} \left| \int_{ak}^{ak+b^{-1}} f(x)\, \overline{g(x - ak)}\, e^{-2\pi i b n x}\, dx \right|^2 \\
&=\; b \sum_{n \in \mathbf{Z}} \left| \int_{-\infty}^{\infty} f(x)\, \overline{e^{2\pi i b n x}\, g(x - ak)}\, dx \right|^2 \\
&=\; b \sum_{n \in \mathbf{Z}} \bigl|\bigl\langle f,\, M_{bn} T_{ak} g \bigr\rangle\bigr|^2. \qquad (11.4)
\end{aligned}$$

Hence, using Tonelli's Theorem to interchange the sum and integral,

$$\begin{aligned}
\sum_{k,n\in\mathbf{Z}} \left|\langle f,\, M_{bn}T_{ak}g\rangle\right|^2 &= b^{-1}\sum_{k\in\mathbf{Z}}\int_{-\infty}^{\infty} |f(x)\,g(x-ak)|^2\,dx \\
&= b^{-1}\int_{-\infty}^{\infty} |f(x)|^2 \sum_{k\in\mathbf{Z}} |g(x-ak)|^2\,dx \qquad (11.5)\\
&\ge \int_{-\infty}^{\infty} |f(x)|^2\,A\,dx \;=\; A\,\|f\|_{L^2}^2.
\end{aligned}$$

A similar computation shows that the upper frame bound estimate also holds for f. Since $C_c(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, we conclude that $\mathcal{G}(g,a,b)$ is a frame with frame bounds A, B.

We will improve on the converse implication in Theorem 11.6, so we omit the proof here.

(b) Suppose that $0 < ab < 1$, and let g be any continuous function such that $g(x) = 0$ outside of $[0, b^{-1}]$ and $g(x) > 0$ on $(0, b^{-1})$. For example, we could let g be the hat function supported on $[0, b^{-1}]$. Because $a < b^{-1}$, it follows that the a-periodic function $G_0(x) = \sum |g(x-ak)|^2$ is continuous and strictly positive at every point. Consequently, $0 < \inf G_0 \le \sup G_0 < \infty$, so $\mathcal{G}(g,a,b)$ is a frame by part (a).

There are many functions g that satisfy these requirements and are more smooth, even infinitely differentiable. For concrete examples, see Exercise 11.9.

(c) If $ab = 1$ then $a = b^{-1}$. If $\mathrm{supp}(g) \subseteq [0, b^{-1}] = [0,a]$ then $T_{ak}g$ is supported in $[ka, (k+1)a]$. If g is continuous then $g(0) = g(a) = 0$. Since the intervals $[ka, (k+1)a]$ overlap at at most one point, it follows that G_0 is continuous and $G_0(ka) = 0$ for every $k \in \mathbf{Z}$. Part (a) therefore implies that $\mathcal{G}(g,a,b)$ cannot be a frame.

(d) If $ab > 1$ then $a > b^{-1}$. Hence $G_0(x) = \sum |g(x-ak)|^2$ is zero on $[b^{-1}, a]$, so $\mathcal{G}(g,a,b)$ cannot be a frame. In fact, the function $\chi_{[b^{-1},a]}$ is orthogonal to every element of $\mathcal{G}(g,a,b)$, so this Gabor system is incomplete. □

Note that it is the product ab that is important in Theorem 11.4 because, by Lemma 11.2, we can change the value of a at the expense of a complementary change to b. Also, by translating g we can replace $[0, b^{-1}]$ by any interval of length b^{-1}.

Here is a more constructive approach to the proof of Theorem 11.4(b).

Example 11.5. For simplicity, assume that $\frac{1}{2} < ab < 1$. Then for any given x, the series $G_0(x) = \sum |g(x-ak)|^2$ contains at most two nonzero terms. Define a continuous function g supported on $[0, b^{-1}]$ by setting

$$g(x)^2 = \begin{cases} 0, & x < 0, \\ \text{linear}, & x \in [0, b^{-1} - a], \\ 1, & x \in [b^{-1} - a, a], \\ \text{linear}, & x \in [a, b^{-1}], \\ 0, & x > b^{-1}. \end{cases}$$

For this g we have $G_0(x) = 1$ for every $x \in \mathbf{R}$ (see Figure 11.2). Hence $\mathcal{G}(g, a, b)$ is a b^{-1}-tight frame, and by rescaling we can make it a Parseval frame if we wish. By using a smoother g, we can similarly create Parseval Gabor frames with generators that are as smooth as we like (Exercise 11.10). The construction becomes more complicated if $ab < \frac{1}{2}$ because there are more overlaps to consider, but the idea can be extended to any values of a, b with $0 < ab < 1$. ♢

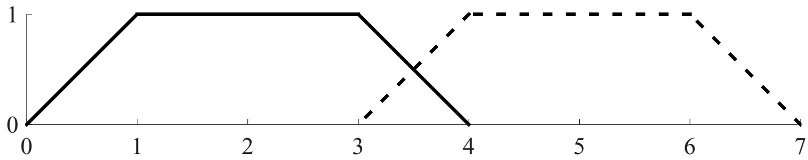

Fig. 11.2. Graphs of $g(x)^2$ and $g(x-a)^2$ from Example 11.5 using $a = 3$ and $b = 1/4$.

We summarize some of the important points in the Painless Nonorthogonal Expansions construction.

- If $0 < ab < 1$ then we can construct nice atoms g (smooth and compactly supported) such that $\mathcal{G}(g, a, b)$ is a frame or even a Parseval frame for $L^2(\mathbf{R})$.
- If $ab = 1$ then there exist Gabor frames $\mathcal{G}(g, a, b)$ for $L^2(\mathbf{R})$, but all of the frames constructed using the methods of this section have generators g that are discontinuous.
- If $ab > 1$ then no Gabor system with $\operatorname{supp}(g) \subseteq [0, b^{-1}]$ can be a frame for $L^2(\mathbf{R})$, and in fact $\mathcal{G}(g, a, b)$ must be incomplete in this case.

Exercise 11.6 refines these observations further, yielding the following additional facts.

- If $0 < ab < 1$ then the frames constructed in this section are redundant (not exact).
- If $ab = 1$ then the frames constructed in this section are exact and hence are Riesz bases for $L^2(\mathbf{R})$.

In the following sections, we will see that the properties listed above apply not only to the "Painless" constructions, but to all Gabor systems $\mathcal{G}(g,a,b)$. The analysis will not be quite as painless and will require new insights, but we will see that there are no "nice" Gabor Riesz bases $\mathcal{G}(g,a,b)$ at all, whereas there are many "well-behaved" redundant Gabor frames. Although it lies outside the scope of this volume, we remark that the utility of redundant Gabor frames extends far beyond the Hilbert space setting. Specifically, if $\mathcal{G}(g,a,b)$ is a Gabor frame that is generated by a function g that has sufficient simultaneous concentration in both time and frequency, then $\mathcal{G}(g,a,b)$ will be a frame not only for $L^2(\mathbf{R})$ but also for an entire range of associated function spaces $M^{p,q}_s(\mathbf{R})$ ($1 \le p,\, q \le \infty,\, s \in \mathbf{R}$) known as *modulation spaces.* These spaces quantify time-frequency concentration of functions (and distributions), and arise naturally in problems that involve both time and frequency. We refer to the text by Gröchenig [Grö01] for a beautiful development of this rich subject.

Exercises

11.5. Show that $\mathcal{G}(\chi_{[0,1]},1,1)$ is an orthonormal basis for $L^2(\mathbf{R})$. Also show that $\mathcal{G}(\chi_{[0,1]},a,1)$ is a frame for $L^2(\mathbf{R})$ if and only if $0 < a \le 1$.

Remark: Amazingly, there is no known explicit characterization of the set of points (a,b) such that $\mathcal{G}(\chi_{[0,1]},a,b)$ is a frame for $L^2(\mathbf{R})$, see [Jan03].

11.6. Assume that the hypotheses of part (a) of Theorem 11.4 are satisfied, i.e., $0 < ab \le 1$, $\mathrm{supp}(g) \subseteq [0,b^{-1}]$, and equation (11.3) holds. Prove the following statements about the frame $\mathcal{G}(g,a,b)$.

(a) The frame operator is pointwise multiplication by $b^{-1}G_0$, i.e., $Sf = b^{-1}G_0 f$ for $f \in L^2(\mathbf{R})$.

(b) The canonical dual frame is $\mathcal{G}(\widetilde{g},a,b)$ where $\widetilde{g} = bg/G_0$.

(c) If $ab = 1$ then $\mathcal{G}(g,a,b)$ is a Riesz basis for $L^2(\mathbf{R})$.

(d) If $0 < ab < 1$ then $\mathcal{G}(g,a,b)$ is a redundant frame for $L^2(\mathbf{R})$.

11.7. Show that if $g \in C_c(\mathbf{R})$ is not the zero function, then there exist some $a,\, b > 0$ such that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$.

11.8. Let $g \in C_c(\mathbf{R})$ satisfy $\mathrm{supp}(g) = [0,b_0^{-1}]$ and $g(x) > 0$ for $x \in (0,b_0^{-1})$. Show that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ for $0 < a < b_0^{-1}$ and $0 < b < b_0$.

11.9. This exercise will construct a compactly supported, infinitely differentiable function on the real line. Define $f(x) = e^{-1/x^2}\,\chi_{(0,\infty)}(x)$.

(a) Show that for every $n \in \mathbf{N}$, there exists a polynomial p_n of degree $3n$ such that

$$f^{(n)}(x) \;=\; p_n(x^{-1})\, e^{-x^{-2}}\, \chi_{(0,\infty)}(x).$$

Conclude that f is infinitely differentiable, every derivative of f is bounded, and $f^{(n)}(x) = 0$ for every $x \le 0$ and $n \ge 0$.

(b) Show that if $a < b$, then $g(x) = f(x-a)\, f(b-x)$ is infinitely differentiable, is zero outside of (a,b), and is strictly positive on (a,b).

11.10. Let $0 < ab < 1$ be fixed. By Exercise 11.9, there exists a function $f \in C_c^\infty(\mathbf{R})$ supported in $[0, b^{-1}]$ such that $f > 0$ on $(0, b^{-1})$.

(a) Set $F_0(x) = \sum_{k\in\mathbf{Z}} |f(x-ak)|^2$ and show that $g = f/F_0^{1/2}$ is infinitely differentiable, compactly supported, and satisfies $\sum_{k\in\mathbf{Z}} |g(x-ak)|^2 = 1$ everywhere.

(b) Show that there exists a function $g \in C_c^\infty(\mathbf{R})$ such that $\mathcal{G}(g,a,b)$ is a Parseval frame for $L^2(\mathbf{R})$.

11.3 The Nyquist Density and Necessary Conditions for Frame Bounds

Theorem 11.4, the Painless Nonorthogonal Expansions construction, gives necessary and sufficient conditions for the existence of Gabor frames $\mathcal{G}(g,a,b)$ when the atom g is supported in an interval of length $1/b$. This equivalence does not extend to general functions in $L^2(\mathbf{R})$. Still, the necessary part of the theorem does extend, as follows.

Theorem 11.6. *If $g \in L^2(\mathbf{R})$ and $a,\, b > 0$ are such that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ with frame bounds $A,\, B > 0$, then we must have $Ab \le G_0 \le Bb$ a.e., i.e.,*

$$Ab \;\le\; \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 \;\le\; Bb \quad a.e. \tag{11.6}$$

In particular, g must be bounded.

Proof. The proof is similar to the proof of part (a) of Theorem 11.4. However, now we do not know the support of g, so instead we restrict our attention to functions f that are bounded and supported in an interval I of length $1/b$. In this case the product $f \cdot T_{ak}\overline{g}$ belongs to $L^2(I)$. Since $\{b^{1/2} e_{bn}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$, it follows, just as in equation (11.4), that

$$b \sum_{n\in\mathbf{Z}} \bigl|\langle f,\, M_{bn}T_{ak}g\rangle\bigr|^2 \;=\; \int_{-\infty}^{\infty} |f(x)\, g(x-ak)|^2\, dx.$$

Applying the lower frame bound for $\mathcal{G}(g,a,b)$, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^2\, G_0(x)\, dx \;&=\; \sum_{k\in\mathbf{Z}} \int_{-\infty}^{\infty} |f(x)\, g(x-ak)|^2\, dx \\ &=\; b \sum_{k,n\in\mathbf{Z}} \bigl|\langle f,\, M_{bn}T_{ak}g\rangle\bigr|^2\end{aligned}$$

$$\geq \; bA \, \|f\|_{L^2}^2$$
$$= \; bA \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

Thus, for every bounded $f \in L^2(I)$ we have

$$\int_{-\infty}^{\infty} |f(x)|^2 \big(G_0(x) - bA\big) \, dx \;\geq\; 0. \tag{11.7}$$

Now, if $G_0(x) < bA$ on some subset E of I that has positive measure, then we could take $f = \chi_E$ and obtain a contradiction to equation (11.7). Therefore we must have $G_0 \geq bA$ a.e. on I, and a similar calculation using the upper frame bound gives $G_0 \leq bB$ a.e. on I. Since I is an arbitrary interval of length $1/b$ and since the real line can be covered by countably many translates of I, we conclude that $bA \leq G_0 \leq bB$ a.e. on $\mathbf{R}$. □

Combining Theorem 11.6 with Exercise 11.3 gives several interesting corollaries for Gabor frames. Note that the statements in the next corollary apply to all Gabor frames $\mathcal{G}(g,a,b)$, not just those with compactly supported atoms g.

Corollary 11.7 (Density and Frame Bounds). *Fix $g \in L^2(\mathbf{R})$ and a, $b > 0$. If $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$ with frame bounds A, B, then the following statements hold.*

(a) $Aab \leq \|g\|_{L^2}^2 \leq Bab$.

(b) *If $\mathcal{G}(g,a,b)$ is a Parseval frame, then $\|g\|_{L^2}^2 = ab$.*

(c) $0 < ab \leq 1$.

(d) $\langle g, \widetilde{g} \rangle = ab$, *where $\widetilde{g} = S^{-1}g$ is the generator of the canonical dual frame.*

(e) *$\mathcal{G}(g,a,b)$ is a Riesz basis if and only if $ab = \langle g, \widetilde{g} \rangle = 1$.*

Proof. (a), (b) Integrating equation (11.6) over the interval $[0,a]$, we have

$$Aab \;=\; \int_0^a Ab \, dx \;\leq\; \int_0^a \sum_{k \in \mathbf{Z}} |g(x-ak)|^2 \, dx \;\leq\; \int_{-\infty}^{\infty} |g(x)|^2 \, dx \;=\; \|g\|_{L^2}^2.$$

A similar calculation shows that $\|g\|_{L^2}^2 \;\leq\; Bab$. If the frame is Parseval then $A = B = 1$.

(c) By Exercise 11.3, if we set $g^\sharp = S^{-1/2}g$ then $\mathcal{G}(g^\sharp, a, b)$ is a Parseval frame. Part (b) therefore implies that $\|g^\sharp\|_{L^2}^2 = ab$. On the other hand, the elements of a Parseval frame can have at most unit norm (see Exercise 7.5), so we must have $\|g^\sharp\|_{L^2}^2 \leq 1$. Hence $ab \leq 1$.

(d) Combining $\|g^\sharp\|_{L^2}^2 = ab$ with the fact that $S^{-1/2}$ is self-adjoint,

$$\langle g, \widetilde{g} \rangle \;=\; \langle g, S^{-1/2}S^{-1/2}g \rangle \;=\; \langle S^{-1/2}g, S^{-1/2}g \rangle \;=\; \|g^\sharp\|_{L^2}^2 \;=\; ab.$$

(e) This follows by combining part (d) with Corollary 8.23 (see also Exercise 11.3). □

Parts (a) and (b) of Corollary 11.7 were proved by Daubechies in her seminal paper [Dau90]. The first proof of part (c) was given by Ramanathan and Steger [RS95] as a special case of their results on irregular Gabor systems. The simple proof of part (c) given here appears to have been first presented by Balan [Bal98], but has been independently discovered several times.

Looking at parts (c) and (e) of Corollary 11.7 a little more closely, we see that the value of ab separates Gabor frames into three categories:

- If $ab > 1$ then $\mathcal{G}(g,a,b)$ is not a frame.
- If $\mathcal{G}(g,a,b)$ is a frame and $ab = 1$ then it is a Riesz basis.
- If $\mathcal{G}(g,a,b)$ is a frame and $0 < ab < 1$ then it is a redundant frame.

We saw in Section 11.2 that this trichotomy held for the Painless Nonorthogonal Expansions, and now we see that it holds for all Gabor systems. The value $1/(ab)$ is called the *density* of the Gabor system $\mathcal{G}(g,a,b)$, because the number of points of $a\mathbf{Z} \times b\mathbf{Z}$ that lie in a given ball in $\mathbf{R}^2$ is asymptotically $1/(ab)$ times the volume of the ball as the radius increases to infinity. We refer to the density $1/(ab) = 1$ as the *critical density* or the *Nyquist density.*

In fact, the trichotomy for the Painless Nonorthogonal Expansions was even more pronounced. We proved in Theorem 11.4(d) that if $ab > 1$ and $g \in L^2(\mathbf{R})$ is supported in $[0, b^{-1}]$ then $\mathcal{G}(g,a,b)$ is *incomplete.* In contrast, Corollary 11.7 only tells us that $\mathcal{G}(g,a,b)$ cannot be a frame, which is a weaker statement. Although it is more difficult to prove, it is true that if g is *any* function in $L^2(\mathbf{R})$ and $ab > 1$ then $\mathcal{G}(g,a,b)$ must be incomplete in $L^2(\mathbf{R})$. The first explicit proof of this fact was given by Baggett [Bag90], using the representation theory of the discrete Heisenberg group. It was also proved by Daubechies for the case that ab is rational [Dau90], and she also pointed out that a proof for general $ab > 1$ can be inferred from results of Rieffel [Rie81] on the coupling constants of C^*-algebras.

There is still a surprise left for us in the case $ab > 1$. Comparing Theorem 11.6 to Theorem 10.19 we see some suspiciously similar equations. Theorem 11.6 tells us that if $\mathcal{G}(g,a,b) = \{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ is a Gabor frame for $L^2(\mathbf{R})$ with frame bounds A, B, then

$$Ab \;\le\; \sum_{k\in\mathbf{Z}} |g(x-ak)|^2 \;\le\; Bb \;\text{ a.e.}$$

After making the appropriate changes of variable (see Exercise 11.12), Theorem 10.19 says that $\mathcal{T}(g) = \{T_{ak}g\}_{k\in\mathbf{Z}}$ is a Riesz basis for its closed span with frame bounds A, B if and only if

$$Aa \;\le\; \sum_{k\in\mathbf{Z}} |\widehat{g}(\xi - \tfrac{k}{a})|^2 \;\le\; Ba \;\text{ a.e.,} \tag{11.8}$$

where $\widehat{g}$ is the Fourier transform of g. Coordinating properly between g and $\widehat{g}$, a and $\frac{1}{a}$, and b and $\frac{1}{b}$, we find that there are Riesz sequences of translates of g and $\widehat{g}$ associated with every Gabor frame, even redundant frames!

Theorem 11.8. *Assume* $\mathcal{G}(g,a,b)$ *is a frame for* $L^2(\mathbf{R})$ *with frame bounds* A, B, *and let* $\widehat{g}$ *be the Fourier transform of* g. *Then the following statements hold.*

(a) $Aa \le \sum_{n\in\mathbf{Z}} |\widehat{g}(\xi - bn)|^2 \le Ba$ *a.e.*

(b) $\{T_{n/b}g\}_{n\in\mathbf{Z}}$ *is a Riesz sequence in* $L^2(\mathbf{R})$ *with frame bounds* Aab, Bab *(as a frame for its closed span).*

(c) $\{T_{k/a}\widehat{g}\}_{k\in\mathbf{Z}}$ *is a Riesz sequence in* $L^2(\mathbf{R})$ *with frame bounds* Aab, Bab *(as a frame for its closed span).*

Proof. (a) Suppose that $\mathcal{G}(g,a,b)$ is a frame. Exercise 11.4 shows that the image of $\mathcal{G}(g,a,b)$ under the Fourier transform is $\mathcal{G}(\widehat{g},b,a)$. Since the Fourier transform is unitary, $\mathcal{G}(\widehat{g},b,a)$ must be a frame with the same frame bounds as $\mathcal{G}(g,a,b)$. Statement (a) then follows by applying Theorem 11.6 to $\mathcal{G}(\widehat{g},b,a)$.

(b) Write part (a) as

$$\frac{Aab}{b} \;\le\; \sum_{n\in\mathbf{Z}} |\widehat{g}(\xi - bn)|^2 \;\le\; \frac{Bab}{b} \quad \text{a.e.}$$

Comparing this to equation (11.8), we see that $\{T_{n/b}g\}_{n\in\mathbf{Z}}$ is a Riesz basis for its closed span, and the frame bounds are Aab, Bab.

(c) This follows by applying part (b) to the frame $\mathcal{G}(\widehat{g},b,a)$. □

Thus, even if $\mathcal{G}(g,a,b)$ is a *redundant* frame (which cannot have a biorthogonal sequence), $\{T_{n/b}g\}_{n\in\mathbf{Z}}$ is a *Riesz sequence* and therefore has a biorthogonal sequence! Although we will not prove it, Theorem 11.8 is actually only a part of a result that seems *very* surprising (at least when first encountered).

Theorem 11.9 (Duality Principle). *Given* $g \in L^2(\mathbf{R})$ *and* a, $b > 0$, *the following statements are equivalent.*

(a) $\mathcal{G}(g,a,b) = \{M_{bn}T_{ak}g\}_{k,n\in\mathbf{Z}}$ *is a frame for* $L^2(\mathbf{R})$, *with frame bounds* A, B.

(b) $\mathcal{G}(g,1/b,1/a) = \{M_{k/a}T_{n/b}g\}_{k,n\in\mathbf{Z}}$ *is a Riesz sequence in* $L^2(\mathbf{R})$, *with frame bounds* Aab, Bab *(as a frame for its closed span).* ♢

Thus, the property of being a frame with respect to the lattice $a\mathbf{Z} \times b\mathbf{Z}$ is dual to the property of being a Riesz sequence with respect to the lattice $\frac{1}{b}\mathbf{Z} \times \frac{1}{a}\mathbf{Z}$ (which is called the *adjoint lattice* to $a\mathbf{Z} \times b\mathbf{Z}$). In spirit, this is similar to the fact that if the rows of a rectangular $m \times n$ matrix span $\mathbf{R}^n$, then its columns are linearly independent vectors in $\mathbf{R}^m$, and conversely.

Independent and essentially simultaneous proofs of Theorem 11.9 were published by Daubechies, H. Landau, and Z. Landau [DLL95], Janssen [Jan95], and Ron and Shen [RS97], each with a completely different technique.

Theorem 11.9 gives us the following addition to the "trichotomy facts" discussed previously.

Corollary 11.10. *If* $\mathcal{G}(g,a,b)$ *is a Riesz sequence in* $L^2(\mathbf{R})$*, then* $ab \ge 1$.

Proof. If $\mathcal{G}(g,a,b)$ is a Riesz sequence, then $\mathcal{G}(g,1/b,1/a)$ is a frame by Theorem 11.9. Corollary 11.7 therefore implies that $\frac{1}{b}\frac{1}{a} \le 1$, so $ab \ge 1$. □

For additional discussion and extensive references related to the material of this section we refer to the survey paper [Hei07].

Exercises

11.11. Fix $g \in L^2(\mathbf{R})$ and $a,\, b > 0$.

(a) Show that $\mathcal{G}(g,a,b)$ is a Riesz basis for $L^2(\mathbf{R})$ if and only if it is a frame and $ab = 1$.

(b) Show that $\mathcal{G}(g,a,b)$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if it is a tight frame, $ab = 1$, and $\|g\|_{L^2} = 1$.

11.12. Given $g \in L^2(\mathbf{R})$, show that $\mathcal{T}(g) = \{T_k g\}_{k\in\mathbf{Z}}$ is a Riesz basis for its closed span with frame bounds A, B if and only if equation (11.8) holds.

11.13. Suppose that $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbf{R})$. Without appealing to Theorem 11.9, show that $\{M_{k/a}\widehat{g}\}_{n\in\mathbf{Z}}$ and $\{M_{n/b}\widehat{g}\}_{n\in\mathbf{Z}}$ are Riesz sequences in $L^2(\mathbf{R})$.

11.14. Assuming Theorem 11.9, show that $\mathcal{G}(g,a,b)$ is a tight frame for $L^2(\mathbf{R})$ if and only if $\mathcal{G}(g,1/b,1/a)$ is an orthogonal sequence in $L^2(\mathbf{R})$.

11.4 Wiener Amalgam Spaces

Now we introduce a family of Banach spaces that will play an important role in our further analysis of Gabor frames. While the L^p spaces are ubiquitous in analysis, one of their limitations is that the L^p-norm is defined by a "global" criterion alone. As the following example shows, we can rearrange functions in many ways that do not change their L^p-norms but do change other properties.

Example 11.11. Recall that the box function $\chi_{[0,1)}$ generates a Gabor system $\mathcal{G}(\chi_{[0,1)},1,1)$ that is an orthonormal basis for $L^2(\mathbf{R})$. Although the box function has the disadvantage of being discontinuous, it at least has the advantage of being well localized in time.

Now let us create a new function by dividing the interval $[0,1)$ into the infinitely many pieces $[0,\frac{1}{2})$, $[\frac{1}{2},\frac{3}{4})$, $[\frac{3}{4},\frac{7}{8})$, ... and then "sending those pieces off to infinity." That is, we define

$$g \;=\; \chi_{[0,\frac{1}{2})} + T_1\chi_{[\frac{1}{2},\frac{3}{4})} + T_2\chi_{[\frac{3}{4},\frac{7}{8})} + \cdots \tag{11.9}$$

$$\;=\; \chi_{[0,\frac{1}{2})} + \chi_{[1+\frac{1}{2},1+\frac{3}{4})} + \chi_{[2+\frac{3}{4},2+\frac{7}{8})} + \cdots . \tag{11.10}$$

Not only is this function discontinuous, but it does not decay at infinity. Even so, it has exactly the same L^p-norm as $\chi_{[0,1)}$, and because we translated the "pieces" by integers it follows that $\mathcal{G}(g,1,1)$ is also an orthonormal basis for $\mathcal{G}(g,1,1)$ (Exercise 11.16). However, we cannot distinguish between the well localized function $\chi_{[0,1)}$ and the poorly localized function g by considering their L^p-norms $\|\chi_{[0,1)}\|_{L^p}$ and $\|g\|_{L^p}$. $\diamondsuit$

The amalgam spaces are determined by a norm which amalgamates, or mixes, a local criterion for membership with a global criterion. Or, it may be more precise to interpret the norm as giving a global criterion for a local property of the function. Special cases were first introduced by Wiener [Wie26], [Wie33]. A more general class of amalgams, named *Wiener amalgam spaces*, was introduced and extensively studied by Feichtinger, with some of the main papers being [FG85], [Fei87], [Fei90]. We refer to [Hei03] for an introductory survey of amalgam spaces with references to the original papers. We will need the following simple amalgams, which mix a local L^p criterion with a global ℓ^q criterion.

Definition 11.12 (Wiener Amalgam Spaces). Given $1 \le p \le \infty$ and $1 \le q < \infty$, the *Wiener amalgam space* $W(L^p, \ell^q)$ consists of those functions $f \in L^p(\mathbf{R})$ for which the norm

$$\|f\|_{W(L^p,\ell^q)} \;=\; \biggl(\sum_{k\in\mathbf{Z}} \|f\cdot\chi_{[k,k+1]}\|_{L^p}^q\biggr)^{1/q}$$

is finite. For $q = \infty$ we substitute the ℓ^∞-norm for the ℓ^q-norm above, i.e.,

$$\|f\|_{W(L^p,\ell^\infty)} \;=\; \sup_{k\in\mathbf{Z}} \|f\cdot\chi_{[k,k+1]}\|_{L^p}.$$

We also define

$$W(C,\ell^q) \;=\; \bigl\{f \in W(L^\infty,\ell^q) : f \text{ is continuous}\bigr\},$$

and we impose the norm $\|\cdot\|_{W(L^\infty,\ell^q)}$ on $W(C,\ell^q)$. $\diamondsuit$

Thus a function in $W(L^p,\ell^q)$ is locally an L^p function, and globally the values $\|f\cdot\chi_{[k,k+1]}\|_{L^p}$ decay in an ℓ^q manner. The space $W(L^\infty,\ell^2)$ made an appearance earlier in this volume; see Lemma 10.24.

Note that $W(L^p,\ell^p) = L^p(\mathbf{R})$. By Exercise 11.15, $W(L^p,\ell^q)$ and $W(C,\ell^q)$ are Banach spaces.

The space $W(L^\infty,\ell^1)$ will be especially important to us in the coming pages. A function g in this space is "locally bounded" and has an "ℓ^1-type decay" at infinity.

Here are some of the properties of $W(L^\infty,\ell^1)$. In particular, part (d) of this result says that the intervals $[k,k+1]$ in the definition of the amalgam norm can be replaced by intervals $[ak,a(k+1)]$ in the sense of giving an equivalent norm on the space. The constants in this norm equivalence will be expressed in terms of the numbers

$$C_a \;=\; \max\bigl\{1+a,\, 2\bigr\}.$$

Theorem 11.13. (a) *$W(L^\infty,\ell^1)$ is contained in $L^p(\mathbf{R})$ for $1 \le p \le \infty$, and is dense in $L^p(\mathbf{R})$ for $1 \le p < \infty$.*

(b) *$W(L^\infty,\ell^1)$ is closed under translations, and for each $b \in \mathbf{R}$ we have*

$$\|T_b f\|_{W(L^\infty,\ell^1)} \;\le\; 2\,\|f\|_{W(L^\infty,\ell^1)}. \tag{11.11}$$

(c) *$W(L^\infty,\ell^1)$ is an ideal in $L^\infty(\mathbf{R})$ with respect to pointwise products, i.e.,*

$$f \in L^\infty(\mathbf{R}),\;\; g \in W(L^\infty,\ell^1) \;\;\Longrightarrow\;\; fg \in W(L^\infty,\ell^1),$$

and

$$\|fg\|_{W(L^\infty,\ell^1)} \;\le\; \|f\|_{L^\infty}\, \|g\|_{W(L^\infty,\ell^1)}. \tag{11.12}$$

(d) *Given $a > 0$,*

$$|||f|||_a \;=\; \sum_{k\in\mathbf{Z}} \|f\cdot\chi_{[ak,a(k+1)]}\|_{L^\infty}$$

is an equivalent norm for $W(L^\infty,\ell^1)$, with

$$\frac{1}{C_{1/a}}\, |||f|||_a \;\le\; \|f\|_{W(L^\infty,\ell^1)} \;\le\; C_a\, |||f|||_a. \tag{11.13}$$

Proof. We will prove the upper inequality in equation (11.13), and assign the remainder of the proof as Exercise 11.17.

Fix $a > 0$, and define

$$\begin{aligned} I_k \;&=\; \bigl\{n \in \mathbf{Z} : [k,k+1] \cap [an, a(n+1)] \ne \emptyset\bigr\},\\ J_n \;&=\; \bigl\{k \in \mathbf{Z} : [k,k+1] \cap [an, a(n+1)] \ne \emptyset\bigr\}. \end{aligned}$$

If $a \ge 1$ then $|J_n| \le 1+a$, while if $0 < a \le 1$ then $|J_n| \le 2$. Hence $|J_n| \le C_a$, independently of n. Therefore

$$\begin{aligned} \|f\|_{W(L^\infty,\ell^1)} \;&=\; \sum_{k\in\mathbf{Z}} \|f\cdot\chi_{[k,k+1]}\|_{L^\infty}\\ &\le\; \sum_{k\in\mathbf{Z}} \sum_{n\in I_k} \|f\cdot\chi_{[an,a(n+1)]}\|_{L^\infty}\\ &=\; \sum_{n\in\mathbf{Z}} \sum_{k\in J_n} \|f\cdot\chi_{[an,a(n+1)]}\|_{L^\infty}\\ &\le\; C_a \sum_{n\in\mathbf{Z}} \|f\cdot\chi_{[an,a(n+1)]}\|_{L^\infty}. \qquad \square \end{aligned}$$

Rewording part of Theorem 11.13(d) gives us the following inequality.

Corollary 11.14. *If* $f \in W(L^\infty, \ell^1)$ *and* $a > 0$, *then*

$$\sum_{k\in\mathbf{Z}} \|T_{ak}f \cdot \chi_{[0,a]}\|_{L^\infty} \;\le\; C_{1/a}\, \|f\|_{W(L^\infty,\ell^1)}.$$

Proof. We simply have to note that

$$\sum_{k\in\mathbf{Z}} \|T_{ak}f \cdot \chi_{[0,a]}\|_{L^\infty} \;=\; \sum_{k\in\mathbf{Z}} \|f \cdot \chi_{[ak,a(k+1)]}\|_{L^\infty}$$

and apply the lower inequality in equation (11.13). □

While the periodization of a generic function in $L^1(\mathbf{R})$ is integrable over a period (Exercise 10.13), the periodization of a function $g \in W(L^\infty, \ell^1)$ is bounded.

Lemma 11.15. *Fix* $a > 1$. *If* $g \in W(L^\infty, \ell^1)$ *then its* a*-periodization*

$$\varphi(x) \;=\; \sum_{n\in\mathbf{Z}} g(x+an) \;=\; \sum_{n\in\mathbf{Z}} T_{an}g(x)$$

is a*-periodic, bounded, and satisfies*

$$\|\varphi\|_{L^\infty} \;=\; \biggl\|\sum_{n\in\mathbf{Z}} T_{an}g\biggr\|_{L^\infty} \;\le\; C_{1/a}\, \|g\|_{W(L^\infty,\ell^1)}. \tag{11.14}$$

Proof. The function φ is a-periodic and integrable by Exercise 10.13. Using the periodicity, we therefore have

$$\|\varphi\|_{L^\infty} \;=\; \|\varphi \cdot \chi_{[0,a]}\|_{L^\infty} \;=\; \biggl\|\sum_{n\in\mathbf{Z}} T_{an}g \cdot \chi_{[0,a]}\biggr\|_{L^\infty} \;\le\; C_{1/a}\, \|g\|_{W(L^\infty,\ell^1)},$$

where the final inequality comes from Corollary 11.14. □

Exercises

11.15. Prove that $W(L^p, \ell^q)$ is a Banach space for each p, q, and $W(C, \ell^q)$ is a closed subspace of $W(L^\infty, \ell^q)$.

11.16. Let g be the function defined in Example 11.11. Show that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbf{R})$, but $g \notin W(L^\infty, \ell^1)$.

11.17. Complete the proof of Theorem 11.13.

11.5 The Walnut Representation

The Painless Nonorthogonal Expansions give us many examples of Gabor frames, but they are limited by the requirement that the atom g be supported in an interval of length $1/b$. This support assumption produces some "miraculous cancellations" that allow us to write the frame condition in very simple terms. Indeed, equation (11.5) tells us that if g is supported in $[0, b^{-1}]$ then

$$\sum_{k,n \in \mathbf{Z}} \bigl|\langle f, M_{bn} T_{ak} g \rangle\bigr|^2 \;=\; b^{-1} \int_{-\infty}^{\infty} |f(x)|^2 \, G_0(x) \, dx.$$

While the left-hand side of this equation is quite complicated, involving both time shifts of g and multiplications by complex exponentials $e^{2\pi i b n x}$, the right-hand side is extremely simple, involving a single multiplication. Even the function G_0 is quite simple, being built purely out of translates of g:

$$G_0(x) \;=\; \sum_{k \in \mathbf{Z}} |g(x - ak)|^2 \;=\; \sum_{k \in \mathbf{Z}} |T_{ak} g(x)|^2.$$

Upon closer examination, what lies behind the miraculous cancellations in the Painless Nonorthogonal Expansions is the Plancherel Equality: g is supported in $[0, b^{-1}]$ and $\{b^{1/2} e^{2\pi i b n x}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2[0, b^{-1}]$. If the support of g is not contained in a single interval of length b^{-1}, then the analysis of the frame condition becomes much more involved. The Walnut Representation [Wal92] is a result of this analysis, and it provides a fundamental characterization of the frame operator for Gabor systems with a much broader class of atoms g. The idea is simply that we break an arbitrary function g into pieces of length b^{-1}, analyze each piece, and paste the pieces back together. In the end we obtain a representation of the frame that is expressed purely in terms of translation operators—no modulations! This representation plays a fundamental role in time-frequency analysis, especially in the extension of the frame properties of Gabor systems from $L^2(\mathbf{R})$ to other function spaces.

A forerunner of the Walnut Representation was used by Daubechies in her paper [Dau90]. Walnut's work appears in [Wal89], [Wal92], [Wal93], and some of it is also summarized in the survey paper [HW89]. We will develop the Walnut Representation in $L^2(\mathbf{R})$, and refer to the text [Grö01] for extensions beyond the Hilbert space setting.

The delicate part of the proof of the Walnut Representation lies in pasting the pieces back together. Here, it becomes necessary to place a mild restriction on g. Specifically, we need g to lie in the Wiener amalgam space $W(L^\infty, \ell^1)$. This excludes functions that have extremely poor decay at infinity, like the one given in equation (11.9), but still leaves us with a very large class of atoms to choose from. Given this restriction, we can define a family of correlation functions associated with g, of which G_0 is only the first member.

Definition 11.16. Given $g \in W(L^\infty, \ell^1)$ and $a,\, b > 0$, we define associated *correlation functions* G_n by

$$G_n(x) \;=\; \sum_{k\in\mathbf{Z}} g(x-ak)\,\overline{g(x-ak-\tfrac{n}{b})}, \qquad n \in \mathbf{Z}. \qquad \diamondsuit$$

In particular, $G_0(x) = \sum_{k\in\mathbf{Z}} |g(x-ak)|^2$.

Note how both the usual lattice $a\mathbf{Z}\times b\mathbf{Z}$ and the adjoint lattice $\frac{1}{b}\mathbf{Z}\times\frac{1}{a}\mathbf{Z}$ from the Duality Principle play a role in the definition of the correlation functions!

It is often useful to write G_n in the forms

$$G_n \;=\; \sum_{k\in\mathbf{Z}} T_{ak}g\cdot T_{ak+\frac{n}{b}}\bar{g} \;=\; \sum_{k\in\mathbf{Z}} T_{ak}\big(g\cdot T_{\frac{n}{b}}\bar{g}\big). \tag{11.15}$$

Thus G_n is the a-periodization of $g\cdot T_{\frac{n}{b}}\bar{g}$. Since g belongs to $W(L^\infty, L^1)$, it is bounded, and therefore the product $g\cdot T_{\frac{n}{b}}\bar{g}$ belongs to $W(L^\infty,\ell^1)$ by Theorem 11.13(c). Applying Lemma 11.15 to this function, we see that G_n is well defined, a-periodic, and bounded. The next lemma shows that the L^∞-norms of the G_n are actually very well controlled.

Lemma 11.17. *If $g \in W(L^\infty,\ell^1)$ then $G_n \in L^\infty(\mathbf{R})$ and*

$$\sum_{n\in\mathbf{Z}} \|G_n\|_{L^\infty} \;\le\; 2\,C_{1/a}\,C_b\,\|g\|^2_{W(L^\infty,\ell^1)}.$$

Proof. By Lemma 11.15, using the form of G_n given in equation (11.15) we see that

$$\|G_n\|_{L^\infty} \;=\; \Big\|\sum_{k\in\mathbf{Z}} T_{ak}\big(g\cdot T_{\frac{n}{b}}\bar{g}\big)\Big\|_{L^\infty} \;\le\; C_{1/a}\,\big\|g\cdot T_{\frac{n}{b}}\bar{g}\big\|_{W(L^\infty,\ell^1)}.$$

Since $|\bar{g}| = |g|$, we therefore have

$$\begin{aligned}
\sum_{n\in\mathbf{Z}} \|G_n\|_{L^\infty} \;&\le\; C_{1/a}\sum_{n\in\mathbf{Z}} \big\|g\cdot T_{\frac{n}{b}}g\big\|_{W(L^\infty,\ell^1)}\\
&=\; C_{1/a}\sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \big\|g\cdot\chi_{[k,k+1]}\cdot T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\big\|_{L^\infty}\\
&\le\; C_{1/a}\sum_{k\in\mathbf{Z}} \|g\cdot\chi_{[k,k+1]}\|_{L^\infty}\bigg(\sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\|_{L^\infty}\bigg).
\end{aligned}$$

The series in parentheses on the last line resembles the $W(L^\infty,\ell^1)$ norm of $T_{\frac{n}{b}}g$, but it is not since the summation is over n instead of k. Instead, after some work similar to that used in the proof of Theorem 11.13(d), we see that

$$\begin{aligned}\sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}} g \cdot \chi_{[k,k+1]}\|_{L^\infty} &= \sum_{n\in\mathbf{Z}} \|g \cdot \chi_{[-\frac{n}{b}+k,-\frac{n}{b}+k+1]}\|_{L^\infty} \\ &\le 2C_b \sum_{m\in\mathbf{Z}} \|g \cdot \chi_{[m,m+1]}\|_{L^\infty} \\ &= 2C_b\, \|g\|_{W(L^\infty,\ell^1)}.\end{aligned}$$

The main issue in the computation above is that an interval of the form $[m, m+1]$ intersects at most $2C_b$ intervals of the form $[-\frac{n}{b}+k, -\frac{n}{b}+k+1]$ with $n \in \mathbf{Z}$. Hence

$$\begin{aligned}\sum_{n\in\mathbf{Z}} \|G_n\|_{L^\infty} &\le 2\, C_{1/a}\, C_b \sum_{k\in\mathbf{Z}} \|g \cdot \chi_{[k,k+1]}\|_{L^\infty}\, \|g\|_{W(L^\infty,\ell^1)} \\ &= 2\, C_{1/a}\, C_b\, \|g\|^2_{W(L^\infty,\ell^1)}. \quad \square\end{aligned}$$

Now we can derive the Walnut Representation. While not every function in $W(L^\infty, \ell^1)$ will generate a Gabor frame, the next theorem tells us that $\mathcal{G}(g, a, b)$ will always be a Bessel sequence, no matter what values of a, $b > 0$ that we choose. Therefore $\mathcal{G}(g, a, b)$ has a well-defined frame operator that maps $L^2(\mathbf{R})$ into itself, and the Walnut Representation realizes this frame operator solely in terms of translations. A simple trick that we will employ several times in the proof is to write

$$\sum_{n\in\mathbf{Z}} \int_0^{b^{-1}} h\bigl(x - \tfrac{n}{b}\bigr)\, dx \;=\; \int_{-\infty}^{\infty} h(x)\, dx \;=\; \int_0^{b^{-1}} \sum_{n\in\mathbf{Z}} h\bigl(x - \tfrac{n}{b}\bigr)\, dx.$$

This is valid for any function $h \in L^1(\mathbf{R})$.

Theorem 11.18 (Walnut Representation). *Let $g \in W(L^\infty, \ell^1)$ and a, $b > 0$ be given. Then $\mathcal{G}(g, a, b)$ is a Bessel sequence, and its frame operator is given by*

$$Sf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} f \cdot G_n, \qquad f \in L^2(\mathbf{R}). \tag{11.16}$$

Proof. Lemma 11.17 implies that the series

$$Lf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} \cdot G_n$$

converges absolutely in $L^2(\mathbf{R})$ for each $f \in L^2(\mathbf{R})$. Moreover,

$$\|Lf\| \;\le\; b^{-1} \sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}} f\|_{L^2}\, \|G_n\|_{L^\infty} \;\le\; B\, \|f\|_{L^2}$$

where

$$B \;=\; \frac{2}{b}\, C_{1/a}\, C_b\, \|g\|^2_{W(L^\infty,\ell^1)}.$$

Hence L is a bounded operator on $L^2(\mathbf{R})$.

By Theorem 7.4, to show that $\mathcal{G}(g,a,b)$ is a Bessel sequence we only need to establish that the Bessel bound holds on a dense subspace of $L^2(\mathbf{R})$. We will show that B is a Bessel bound on the dense subspace $C_c(\mathbf{R})$.

Fix $f \in C_c(\mathbf{R})$ and $k \in \mathbf{Z}$. Then $f \cdot T_{ak}\bar{g}$ is bounded and compactly supported, so its b^{-1}-periodization

$$F_k(x) \;=\; \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}$$

belongs to $L^2[0,b^{-1}]$ (and in fact is bounded). Since F_k is b^{-1}-periodic, we have $F_k\bigl(x-\frac{j}{b}\bigr) = F_k(x)$ for $j \in \mathbf{Z}$.

Using the fact that $\{b^{1/2}e^{2\pi i bnx}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,b^{-1}]$, we compute that

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \\
&\qquad = \sum_{n\in\mathbf{Z}} \left|\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i bnx}\, \overline{g(x-ak)}\, dx\right|^2 \\
&\qquad = \sum_{n\in\mathbf{Z}} \left|\int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\, e^{-2\pi i bn\left(x-\frac{j}{b}\right)}\, \overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}\, dx\right|^2 \\
&\qquad = \sum_{n\in\mathbf{Z}} \left|\int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\, \overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}\, e^{-2\pi i bnx}\, dx\right|^2 \\
&\qquad = \sum_{n\in\mathbf{Z}} \bigl|\bigl\langle F_k,\, e_{bn}\bigr\rangle_{L^2[0,b^{-1}]}\bigr|^2 \\
&\qquad = \bigl\|F_k\bigr\|^2_{L^2[0,b^{-1}]} \\
&\qquad = b^{-1}\int_0^{b^{-1}} |F_k(x)|^2\, dx.
\end{aligned}$$

Assuming that we can interchange the integral and sum as indicated, and using the fact that F_n is b^{-1}-periodic, we therefore have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \\
&\qquad = b^{-1}\sum_{k\in\mathbf{Z}} \int_0^{b^{-1}} F_k(x)\,\overline{F_k(x)}\, dx \\
&\qquad = b^{-1}\sum_{k\in\mathbf{Z}} \int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\, \overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)\, F_k\bigl(x-\tfrac{j}{b}\bigr)}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= b^{-1} \sum_{k\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{g(x-ak)\, F_k(x)}\, dx \\
&= b^{-1} \sum_{k\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{g(x-ak)} \sum_{j\in\mathbf{Z}} \overline{f\big(x-\tfrac{j}{b}\big)}\, g\big(x-ak-\tfrac{j}{b}\big)\, dx \\
&= b^{-1} \sum_{j\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{f\big(x-\tfrac{j}{b}\big)} \sum_{k\in\mathbf{Z}} \overline{g(x-ak)}\, g\big(x-ak-\tfrac{j}{b}\big)\, dx \\
&= b^{-1} \sum_{j\in\mathbf{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{f\big(x-\tfrac{j}{b}\big)\, G_j(x)}\, dx \\
&= \Big\langle f,\, b^{-1} \sum_{j\in\mathbf{Z}} T_{\frac{j}{b}} f \cdot G_j(x) \Big\rangle_{L^2(\mathbf{R})} \\
&= \big\langle f,\, Lf \big\rangle.
\end{aligned}
$$

The interchanges in order can be justified by using Fubini's Theorem (Exercise 11.18). Since T is bounded, we conclude that $\mathcal{G}(g,a,b)$ is a Bessel sequence, and

$$
\begin{aligned}
\langle f, Sf\rangle &= \sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \\
&= \big\langle f,\, Lf \big\rangle \\
&\le \|f\|_{L^2}\, \|Lf\|_{L^2} \\
&\le B\, \|f\|_{L^2}^2.
\end{aligned}
$$

Hence the Bessel bound holds on $C_c(\mathbf{R})$.

This also shows us that $\langle f, Sf\rangle = \langle f, Lf\rangle$ for all $f \in C_c(\mathbf{R})$. Since $C_c(\mathbf{R})$ is dense and both S and L are bounded, we conclude that $\langle f, Sf\rangle = \langle f, Lf\rangle$ for all $f \in L^2(\mathbf{R})$. Since S is self-adjoint, Corollary 2.16 therefore implies that $S = L$. $\square$

We emphasize the contrast between the appearance of the Gabor frame operator in its original form and in the Walnut Representation:

$$
\sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}g \;=\; Sf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} f \cdot G_n.
$$

Aside from the fact that the Walnut Representation contains a single summation, it also contains no complex exponentials. If f and g are real valued then every term on the right-hand side of the line above is real valued, while the terms on the left-hand side need not be.

One of the consequences of Theorem 11.18 is that if $g \in W(L^\infty, \ell^1)$ then $\mathcal{G}(g,a,b)$ will be a frame for all small enough values of a and b [HW89, Thm. 4.1.8].

We end this section by mentioning another fundamental representation of the Gabor frame operator. This is the Janssen Representation (also known as the *Dual Lattice Representation*), which expresses the frame operator as a superposition of time-frequency shift operators [Jan95], [DLL95]. The hypotheses required for the Janssen Representation are slightly different than those of the Walnut Representation. Note the explicit role played by the adjoint lattice in this representation.

Theorem 11.19 (Janssen Representation). *Let $g \in L^2(\mathbf{R})$ and $a, b > 0$ be given. If*

$$\sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \left|\left\langle g,\, M_{\frac{k}{a}} T_{\frac{n}{b}} g\right\rangle\right| \;<\; \infty, \tag{11.17}$$

then

$$S \;=\; \frac{1}{ab} \sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \left\langle g,\, M_{\frac{k}{a}} T_{\frac{n}{b}} g\right\rangle M_{\frac{k}{a}} T_{\frac{n}{b}},$$

where the series converges absolutely in operator norm. ♢

Equation (11.17) is referred to as *Condition A*. It is close but not identical to the requirement that g belong to $W(L^\infty, \ell^1)$. The *Feichtinger algebra* S_0, which equals the modulation space M^1, is a smaller subspace on which both conditions are satisfied simultaneously. The Feichtinger algebra has many other useful properties, e.g., it is closed under both convolution and pointwise products, and in most cases it is the class from which we should choose generators g for Gabor frames [Grö01].

Exercises

11.18. Justify the use of Fubini's Theorem in the proof of Theorem 11.18.

11.19. This exercise gives a perturbation theorem for Gabor frames.

(a) Let $g \in L^2(\mathbf{R})$ and $a, b > 0$ be such that $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbf{R})$. Show that there exists a $\delta > 0$ such that if $h \in L^2(\mathbf{R})$ and $\|g-h\|_{W(L^\infty,\ell^1)} < \delta$, then $\mathcal{G}(h, a, b)$ is a frame for $L^2(\mathbf{R})$.

(b) Does part (a) remain valid if we replace the amalgam norm $\|\cdot\|_{W(L^\infty,\ell^1)}$ by the L^2-norm $\|\cdot\|_{L^2}$?

11.6 The Zak Transform

The *Zak transform* is a fundamental tool for analyzing Gabor frames, especially at the critical density ($ab = 1$). The Zak transform was first introduced by Gel'fand [Gel50]. As with many useful notions, it has been rediscovered many times and goes by a variety of names. Weil [Wei64] defined a Zak transform for locally compact abelian groups, and this transform is often called the *Weil–Brezin map* in representation theory and abstract harmonic analysis, e.g., [Sch84], [AT85]. Zak rediscovered this transform, which he called the *k-q transform*, in his work on quantum mechanics, e.g., [Zak67], [BGZ75]. The terminology "Zak transform" has become customary in applied mathematics and signal processing. For more information, we refer to Janssen's influential article [Jan82] and survey [Jan88], or Gröchenig's text [Grö01, Chap. 8].

In this section we define the Zak transform and examine some of its most interesting properties. In the following sections we will see how the Zak transform can be used to analyze Gabor systems, and how the unusual properties of the Zak transform are related to the Balian–Low Theorem. We will be concentrating in this section on the critical density, $ab = 1$. By dilating g, we can reduce this further to $a = b = 1$, so we simply fix $a = b = 1$ now.

The Gabor system $\mathcal{G}(\chi_{[0,1]}, 1, 1) = \{M_n T_k \chi_{[0,1]}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$. Let

$$Q \;=\; [0,1]^2$$

denote the unit square in $\mathbf{R}^2$, and consider the sequence

$$\{E_{nk}\}_{k,n\in\mathbf{Z}}, \qquad \text{where } E_{nk}(x,\xi) = e^{2\pi i n x}\, e^{-2\pi i k\xi}. \tag{11.18}$$

This sequence is contained in the Hilbert space $L^2(Q)$, whose norm and inner product are given by

$$\|F\|_{L^2(Q)}^2 \;=\; \int_0^1\!\int_0^1 |F(x,\xi)|^2\,dx\,d\xi$$

and

$$\langle F, G\rangle \;=\; \int_0^1\!\int_0^1 F(x,\xi)\,\overline{G(x,\xi)}\,dx\,d\xi.$$

By Theorem B.10 or by direct calculation, $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$. We can define a unitary map by sending the elements of one orthonormal basis to another orthonormal basis, and this is precisely what we do to define the Zak transform (see Exercise 11.20).

Definition 11.20 (Zak Transform). The *Zak transform* is the unique unitary map $Z\colon L^2(\mathbf{R}) \to L^2(Q)$ that satisfies

$$Z(M_n T_k \chi_{[0,1]}) \;=\; E_{nk}, \qquad k, n \in \mathbf{Z}. \qquad \diamondsuit \tag{11.19}$$

Now we give an equivalent formulation of the Zak transform that will help us to extend its domain to spaces other than $L^2(\mathbf{R})$.

Theorem 11.21. *Given* $f \in L^2(\mathbf{R})$, *we have*

$$Zf(x,\xi) \;=\; \sum_{j\in\mathbf{Z}} f(x-j)\, e^{2\pi i j\xi}, \qquad (x,\xi) \in Q, \tag{11.20}$$

where this series converges unconditionally in the norm of $L^2(Q)$.

Proof. A direct calculation shows that if $f \in L^2(\mathbf{R})$ and $j \neq \ell$ then the functions $f(x+j)\,e^{2\pi ij\xi}$ and $f(x+\ell)\,e^{2\pi i\ell\xi}$ are orthogonal elements of $L^2(Q)$. Therefore, if F is any finite subset of $\mathbf{Z}$ then

$$\begin{aligned}\Bigl\|\sum_{j\in F} f(x-j)\,e^{2\pi ij\xi}\Bigr\|_{L^2(Q)}^2 &= \sum_{j\in F} \|f(x-j)\,e^{2\pi ij\xi}\|_{L^2(Q)}^2 \\ &= \sum_{j\in F} \int_0^1\int_0^1 |f(x-j)\,e^{2\pi ij\xi}|^2\,dx\,d\xi \\ &= \sum_{j\in F} \int_0^1 |f(x-j)|^2\,dx. \qquad (11.21)\end{aligned}$$

Since $f \in L^2(\mathbf{R})$, the series $\sum_{j\in\mathbf{Z}} \int_0^1 |f(x-j)|^2\,dx$ converges unconditionally and equals $\|f\|_{L^2}^2$. Consequently, the series appearing on the right-hand side of equation (11.20) converges unconditionally in $L^2(Q)$, and if we set $Uf(x,\xi) = \sum_{j\in\mathbf{Z}} f(x-j)\,e^{2\pi ij\xi}$ then it follows from equation (11.21) that $\|Uf\|_{L^2(Q)} = \|f\|_{L^2}$. This operator U is an isometry, so to show that $U = Z$ we simply have to show that $U(M_nT_k\chi_{[0,1]}) = E_{nk}$ for all k, $n \in \mathbf{Z}$. To see this, note that if $(x,\xi) \in Q$ then $\chi_{[0,1]}(x-j) = 0$ for all $j \neq 0$, so

$$\begin{aligned}U(M_nT_k\chi_{[0,1]})(x,\xi) &= \sum_{j\in\mathbf{Z}} M_nT_k\chi_{[0,1]}(x-j)\,e^{2\pi ij\xi} \\ &= \sum_{j\in\mathbf{Z}} e^{2\pi in(x-j)}\chi_{[0,1]}(x-j-k)\,e^{2\pi ij\xi} \\ &= e^{2\pi in(x+k)}\,e^{-2\pi ik\xi} \;=\; E_{nk}(x,\xi),\end{aligned}$$

where we have used the fact that $e^{2\pi ink} = 1$. □

It will be important for us to consider the Zak transform on domains other than $L^2(\mathbf{R})$, and the correct spaces are precisely the Wiener amalgam spaces $W(L^p,\ell^1)$ introduced in Section 11.4. The next theorem shows that the Zak transform maps $W(L^p,\ell^1)$ into $L^p(Q)$, and maps $W(C,\ell^1)$ into $C(Q)$, the space of continuous functions on $Q = [0,1]^2$.

Theorem 11.22. (a) *If $1 \le p \le \infty$ then for each $f \in W(L^p,\ell^1)$ the series*

$$Zf(x,\xi) \;=\; \sum_{j\in\mathbf{Z}} f(x-j)\,e^{2\pi ij\xi}, \qquad (x,\xi) \in Q, \tag{11.22}$$

converges absolutely in $L^p(Q)$, and Z is a bounded mapping of $W(L^p,\ell^1)$ into $L^p(Q)$.

(b) *For each $f \in W(C,\ell^1)$ the series in equation* (11.22) *converges absolutely in $C(Q)$ with respect to the uniform norm, and Z is a bounded mapping of $W(C,\ell^1)$ into $C(Q)$.*

Proof. (a) If $f \in W(L^p, \ell^1)$ with p finite then

$$\sum_{j\in\mathbf{Z}} \|f(x-j)\, e^{2\pi i j\xi}\|_{L^p(Q)} \;=\; \sum_{j\in\mathbf{Z}} \biggl(\int_0^1\!\int_0^1 |f(x-j)\, e^{2\pi i j\xi}|^p \,dx\,d\xi\biggr)^{1/p}$$

$$= \sum_{j\in\mathbf{Z}} \|f\cdot\chi_{[j,j+1]}\|_{L^p} \;<\; \infty,$$

so the series defining Zf converges absolutely in $L^p(Q)$. A similar calculation holds if $p=\infty$, and these calculations also show that $\|Zf\|_{L^p(Q)} \le \|f\|_{W(L^p,\ell^1)}$.

(b) If $f \in W(C,\ell^1) \subseteq W(L^\infty,\ell^1)$ then we have by part (a) that $Zf \in L^\infty(Q)$, and the series defining Zf converges absolutely in the uniform norm. As each term $f(x-j)\, e^{2\pi i j\xi}$ is continuous on Q and the uniform limit of continuous functions is continuous, Zf is continuous on Q. $\square$

Remark 11.23. In particular, the Zak transform maps $L^1(\mathbf{R}) = W(L^1,\ell^1)$ continuously into $L^1(Q)$, and it is injective by Exercise 11.22. However, that exercise also shows that the range of $Z\colon L^1(\mathbf{R}) \to L^1(Q)$ is a dense but proper subspace of $L^1(Q)$. A consequence of this is that $Z^{-1}\colon \mathrm{range}(Z) \to L^1(\mathbf{R})$ is unbounded, in contrast to the fact that Z is a unitary mapping of $L^2(\mathbf{R})$ onto $L^2(Q)$. Readers familiar with interpolation will recognize that since Z maps $L^1(Q)$ boundedly into itself and $L^2(Q)$ boundedly into itself, it extends to a bounded map of $L^p(Q)$ into itself for each $1 \le p \le 2$. However, if $1 \le p < 2$ then Z is not surjective and Z^{-1} is unbounded. $\diamondsuit$

Given $f \in L^2(\mathbf{R})$, we can extend the domain of Zf from $Q=[0,1]^2$ to all of $\mathbf{R}^2$ in a natural way. In all of the preceding arguments, nothing is changed if we replace the unit square Q with a translated square $Q+z$, where $z \in \mathbf{R}^2$. Moreover, if Q and $Q+z$ overlap then the two definitions of Zf will coincide almost everywhere on $Q\cap(Q+z)$ (and everywhere if Zf is continuous). Hence Zf has a unique extension from Q to the entire plane $\mathbf{R}^2$. This is similar to how a function on $[0,1)$ is extended to a 1-periodic function on $\mathbf{R}$, as in Notation 4.23. However, there is an interesting twist here, because Zf on $\mathbf{R}^2$ is *not* obtained by extending Zf periodically from Q. Instead, Zf satisfies the following rather peculiar *quasiperiodicity relations* (Exercise 11.21).

Theorem 11.24. *If $f \in L^2(\mathbf{R})$ or $f \in W(L^p,\ell^1)$, then for $m,\, n \in \mathbf{Z}$ we have*

$$Zf(x+m,\xi+n) \;=\; e^{2\pi i m\xi}\, Zf(x,\xi),$$

where the equality holds pointwise everywhere on $\mathbf{R}^2$ if Zf is continuous, and almost everywhere otherwise. $\diamondsuit$

Definition 11.25 (Quasiperiodicity). We say that a function F on $\mathbf{R}^2$ that satisfies

$$F(x+m,\xi+n) \;=\; e^{2\pi i m\xi}\,F(x,\xi) \text{ a.e.}, \qquad m,n\in\mathbf{Z}, \tag{11.23}$$

is *quasiperiodic*. We refer to equation (11.23) as the *quasiperiodicity relations* for F. ♢

Quasiperiodicity has a rather unexpected implication: No continuous quasiperiodic function can be nonzero everywhere. Since a complete justification of this statement requires some facts from complex analysis or algebraic geometry, we will be content to appeal to authority for the justification of certain steps in the proof that we present.

Theorem 11.26. *A continuous quasiperiodic function F must vanish at some point of Q.*

Proof. First we give a standard direct argument that proves the theorem but does little to illuminate the mystery of why a zero must exist. Suppose that F was quasiperiodic and continuous on $\mathbf{R}^2$ but everywhere nonzero. Because $\mathbf{R}^2$ is simply connected, there exists a continuous function $\varphi\colon \mathbf{R}^2\to\mathbf{R}$ such that

$$F(x,\xi) \;=\; |F(x,\xi)|\,e^{2\pi i\varphi(x,\xi)}, \qquad (x,\xi)\in\mathbf{R}^2.$$

Students of complex analysis may recognize that this continuous logarithm φ can be constructed directly, and its existence also follows from general topological lifting principles [Grö01]. Applying this logarithm to the quasiperiodicity relations, we see that for each m, $n\in\mathbf{Z}$ there exists an integer $\kappa(m,n)$ such that

$$\varphi(x+m,\xi+n) \;=\; \varphi(x,\xi)+m\xi+\kappa(m,n), \qquad (x,\xi)\in\mathbf{R}^2.$$

Hence

$$\begin{aligned} 0 \;&=\; \big(\varphi(0,0)-\varphi(1,0)\big)+\big(\varphi(1,0)-\varphi(1,1)\big)\\ &\qquad\qquad +\big(\varphi(1,1)-\varphi(0,1)\big)+\big(\varphi(0,1)-\varphi(0,0)\big)\\ &=\; \big(-0-\kappa(1,0)\big)+\big(-0-\kappa(0,1)\big)+\big(1+\kappa(1,0)\big)+\big(0+\kappa(0,1)\big)\\ &=\; 1, \end{aligned}$$

which is a contradiction.

Now we give another argument, due to Janssen [Jan05], that is perhaps more revealing. Suppose that F is continuous, quasiperiodic, and everywhere nonzero on $\mathbf{R}^2$. Then for each fixed $x\in\mathbf{R}$, the function $F_x(\xi)=F(x,\xi)$ is continuous, 1-periodic, and nonzero on $\mathbf{R}$. As ξ varies from 0 to 1, the values $F_x(\xi)$ trace out a closed curve J_x in the complex plane that never intersects the origin. Such a curve has a well-defined *winding number* N_x that is an integer representing the total number of times the curve J_x travels counterclockwise around the origin. Now, since F is continuous, the curves J_x deform continuously as we vary x. Further, since

$$F_1(\xi) = F(1,\xi) = e^{2\pi i\xi}F(0,\xi) = e^{2\pi i\xi}F_0(\xi),$$

the curve J_1 winds one more time around the origin than does J_0. However, there is no way to continuously deform a curve that winds N_0 times around the origin into one that winds $N_1 = N_0 + 1$ times around the origin without having the curve pass through the origin at some time. Hence there must be at least one value of x such that the curve J_x passes through the origin, which says that $F(x,\xi) = 0$ for some ξ. □

Figure 11.3 illustrates the idea of the second proof of Theorem 11.26. We can think of the curve J_x as being a rubber band wound N_x times around the origin. The rubber band is stretched and moved as x varies, but always lies in the complex plane. It can cross itself, but it cannot be cut. The left side of Figure 11.3 shows the curve J_0 for the specific example $F_0(\xi) = 1 + i + e^{2\pi i\xi}$. This curve is a circle that does not contain the origin, and so has winding number $N_0 = 0$. The curve J_1 traced out by the function $F_1(\xi) = e^{2\pi i\xi}F_0(\xi)$ is shown on the right side of Figure 11.3. The point $F_1(\xi)$ is located at the same distance from the origin as $F_0(\xi)$, but has been rotated counterclockwise by an angle of $2\pi\xi$ radians. As a consequence, J_1 makes one extra trip around the origin, so has winding number $N_1 = 1$. There is no way to deform the left-hand rubber band into the right-hand one without passing through the origin in the process.

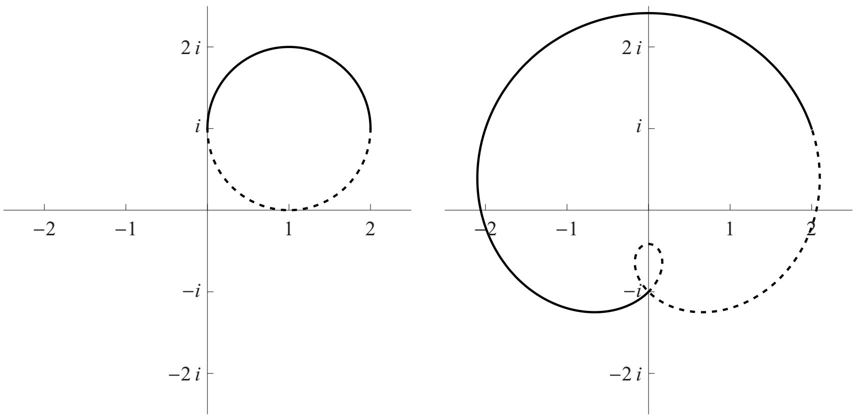

Fig. 11.3. Plots of the complex-valued functions $F_0(\xi) = 1 + i + e^{2\pi i\xi}$ and $F_1(\xi) = e^{2\pi i\xi}\,F_0(\xi)$ for $0 \le \xi \le 1$. The graph is shown as a solid line for $0 \le \xi \le 1/2$, and as a dashed line for $1/2 \le \xi \le 1$. The winding number of the left-hand graph is zero, while it is one for the right-hand graph.

Remark 11.27. Note that the domain of the function F in Theorem 11.26 is the plane $\mathbf{R}^2$, and F is required to be continuous on the entire plane. Applying the quasiperiodicity relations, this is the same as requiring that F

be continuous on the closed square $Q = [0,1]^2$. It is not enough to assume that F is quasiperiodic and continuous on $[0,1)^2$. For example, if we set $F = 1$ on $[0,1)^2$ then we can extend it to a quasiperiodic function on $\mathbf{R}^2$ by defining

$$F(x+m, \xi+n) \;=\; e^{2\pi i m\xi}, \qquad x, \xi \in [0,1),\; m,n \in \mathbf{Z}.$$

This function F is quasiperiodic, but it is not continuous on $\mathbf{R}^2$, and it has no zeros on $\mathbf{R}^2$. ♢

Example 11.28. The third *Jacobi theta function* is

$$\theta_3(z,q) \;=\; 1 \;+\; 2\sum_{k=1}^{\infty} q^{k^2}\cos(4\pi k z) \;=\; \sum_{k=-\infty}^{\infty} q^{k^2} e^{4\pi i k z},$$

where $0 \le q < 1$ and $z \in \mathbf{C}$ [Rai60]. With q fixed, $\theta_3(\cdot, q)$ is analytic on the entire complex plane.

Fix $r > 0$ and let φ_r be the Gaussian function $\varphi_r(x) = e^{-rx^2}$. Since $\varphi_r \in W(C, \ell^1)$, we know that $Z\varphi_r$ is continuous and therefore has a zero. The Zak transform of φ_r is

$$\begin{aligned} Z\varphi_r(x,\xi) &= \sum_{j\in\mathbf{Z}} \varphi_r(x-j)\, e^{2\pi i j\xi} \\ &= \sum_{j\in\mathbf{Z}} e^{-rx^2}\, e^{2rxj}\, e^{-rj^2}\, e^{2\pi i j\xi} \\ &= e^{-rx^2} \sum_{j\in\mathbf{Z}} (e^{-r})^{j^2}\, e^{4\pi i j(\frac{\xi}{2} - \frac{irx}{2\pi})} \\ &= e^{-rx^2}\, \theta_3\big(\tfrac{\xi}{2} - \tfrac{irx}{2\pi},\, e^{-r}\big). \end{aligned}$$

In particular, $Z\varphi_r$ is infinitely differentiable on $\mathbf{R}^2$.

Now, the zeros of $\theta_3(\cdot, q)$ occur precisely at the points

$$z_{mn} \;=\; \frac{1}{4} + \frac{\tau}{4} + \frac{m}{2} + \frac{n\tau}{2},$$

where $q = e^{\pi i \tau}$, $\mathrm{Im}(\tau) > 0$. Since $e^{-r} = e^{\pi i (ir)}$, it follows that $Z\varphi_r(x,\xi) = 0$ if and only if

$$\frac{\xi}{2} - \frac{irx}{2\pi} \;=\; \frac{1}{4} + \frac{ir}{4\pi} + \frac{m}{2} + \frac{irn}{2\pi},$$

i.e., $(x,\xi) = (-n - 1/2, m + 1/2)$. Thus $Z\varphi_r$ has a single zero in the unit square $Q = [0,1]^2$, at the point $(1/2, 1/2)$. ♢

Exercises

11.20. Prove that there is a unique unitary operator that satisfies equation (11.19).

11.21. Prove Theorem 11.24.

11.22. If $f \in L^1(\mathbf{R})$ then $Zf \in W(L^1, \ell^1) = L^1(Q)$ by Theorem 11.22. Prove the following statements.

(a) $f(x) = \int_0^1 Zf(x,\xi)\, d\xi$ for almost every $x \in \mathbf{R}$.

(b) If Zf is continuous, then f is continuous.

(c) Z is an injective mapping of $L^1(\mathbf{R})$ into $L^1(Q)$, and the range of $Z\colon L^1(\mathbf{R}) \to L^1(Q)$ is a proper, dense subspace of $L^1(Q)$.

(d) $Z^{-1}\colon L^1(Q) \to L^1(\mathbf{R})$ is unbounded.

11.23. Suppose that $f \in L^2(\mathbf{R})$ is such that Zf is continuous.

(a) Show that if f is even then $Zf(1/2, 1/2) = 0$.

(b) Show that if f is odd then $Zf(0,0) = Zf(0,1/2) = Zf(1/2,0) = 0$.

(c) Show that if f is real valued then $Zf(x,1/2) = 0$ for some $x \in [0,1]$.

11.7 Gabor Systems at the Critical Density

Now we will use the Zak transform to analyze Gabor systems at the critical density. As before, it suffices to consider $a = b = 1$. In this section we will characterize those Gabor systems $\mathcal{G}(g,1,1)$ that are exact, Riesz bases, or orthonormal bases in terms of the Zak transform of g, and in the next section we will use this characterization to prove some versions of the Balian–Low Theorem.

The utility of the Zak transform is that it converts a Gabor system $\mathcal{G}(g,1,1) = \{M_n T_k g\}_{k,n \in \mathbf{Z}}$ into a system of weighted exponentials on $\mathbf{R}^2$. Recall from equation (11.18) that E_{nk} denotes the two-dimensional complex exponential function $E_{nk}(x,\xi) = e^{2\pi i n x}\, e^{-2\pi i k \xi}$.

Theorem 11.29. *If $g \in L^2(\mathbf{R})$, then*

$$Z(M_n T_k g) \;=\; E_{nk}\, Zg \;\; a.e., \qquad k, n \in \mathbf{Z}.$$

Proof. Using the fact that $e^{-2\pi i n (j-k)} = 1$ for integer j, k, n, we compute that

$$\begin{aligned}
Z(M_n T_k g)(x,\xi) \;&=\; \sum_{j \in \mathbf{Z}} (M_n T_k g)(x-j)\, e^{2\pi i j \xi} \\
&=\; \sum_{j \in \mathbf{Z}} e^{2\pi i n (x-j)}\, g(x-k-j)\, e^{2\pi i j \xi} \\
&=\; \sum_{j \in \mathbf{Z}} e^{2\pi i n (x-j+k)}\, g(x-j)\, e^{2\pi i (j-k) \xi}
\end{aligned}$$

$$= e^{2\pi i n x} \, e^{-2\pi i k \xi} \sum_{j \in \mathbf{Z}} g(x-j) \, e^{2\pi i j \xi}$$

$$= E_{nk}(x,\xi) \, Zg(x,\xi).$$

The series above converge in $L^2(Q)$, not pointwise, but this does not affect the calculation. □

As a consequence, we obtain another proof that a sequence of regular translations $\{g(x-ak)\}_{k\in\mathbf{Z}}$ cannot be complete in $L^2(\mathbf{R})$ (compare Exercise 10.18).

Corollary 11.30. *If* $g \in L^2(\mathbf{R})$ *and* $a > 0$, *then* $\{g(x-ak)\}_{k\in\mathbf{Z}}$ *is incomplete in* $L^2(\mathbf{R})$.

Proof. By dilating g, it suffices to take $a = 1$, so our sequence is $\mathcal{T}(g) = \{T_k g\}_{k\in\mathbf{Z}}$. Taking $n = 0$ in Theorem 11.29, the image of this sequence under the Zak transform is

$$Z\mathcal{T}(g) \;=\; \{E_{0k}\, Zg\}_{k\in\mathbf{Z}} \;=\; \bigl\{e^{-2\pi i k\xi}\, Zg(x,\xi)\bigr\}_{k\in\mathbf{Z}}.$$

Taking finite linear combinations and L^2 limits, it follows that every element of $\overline{\text{span}}(Z\mathcal{T}(g))$ has the form $p(\xi)Zg(x,\xi)$ for some function p. However, not every element of $L^2(Q)$ has this form (why?), so $Z\mathcal{T}(g)$ is incomplete in $L^2(Q)$. Since Z is unitary, $\mathcal{T}(g)$ is therefore incomplete in $L^2(\mathbf{R})$. □

Since the Zak transform is unitary, it preserves basis and frame properties. Consequently, $\mathcal{G}(g,1,1)$ is exact, a frame, a Riesz basis, or an orthonormal basis if and only if the same is true of $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$. Now, the system of unweighted exponentials $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$, and a Riesz basis is the image of an orthonormal basis under a topological isomorphism, so if $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$ is to be a Riesz basis then the mapping that sends E_{nk} to $E_{nk}\, Zg$ must extend to be a topological isomorphism of $L^2(Q)$ onto itself. The only way that the multiplication operation $U(F) = F \cdot Zg$ on $L^2(Q)$ can be a topological isomorphism is if $0 < \inf|Zg| \le \sup|Zg| < \infty$. Extending this idea gives us the following characterization of Gabor systems at the critical density. Note that this result is very much a two-dimensional version of Theorem 10.10!

Theorem 11.31. *Let* $g \in L^2(\mathbf{R})$ *be fixed.*

(a) $\mathcal{G}(g,1,1)$ *is complete in* $L^2(\mathbf{R})$ *if and only if* $Zg \neq 0$ *a.e.*

(b) $\mathcal{G}(g,1,1)$ *is minimal in* $L^2(\mathbf{R})$ *if and only if* $1/Zg \in L^2(Q)$. *In this case,* $\mathcal{G}(g,1,1)$ *is exact and its biorthogonal system is* $\mathcal{G}(\widetilde{g},1,1)$ *where* $\widetilde{g} \in L^2(\mathbf{R})$ *satisfies* $Z\widetilde{g} = 1/\overline{Zg}$.

(c) $\mathcal{G}(g,1,1)$ *is a Bessel sequence in* $L^2(\mathbf{R})$ *if and only if* $Zg \in L^\infty(\mathbf{R})$, *and in this case* $B = \|Zg\|_{L^\infty}^2$ *is a Bessel bound.*

(d) $\mathcal{G}(g,1,1)$ *is a frame for* $L^2(\mathbf{R})$ *if and only if there exist* A, $B > 0$ *such that* $A \le |Zg(x,\xi)|^2 \le B$ *a.e. In this case* $\mathcal{G}(g,1,1)$ *is a Riesz basis and* A, B *are frame bounds.*

(e) $\mathcal{G}(g,1,1)$ *is an orthonormal basis for* $L^2(\mathbf{R})$ *if and only if* $|Zg(x,\xi)| = 1$ *a.e.*

Proof. Much of the proof is similar to the proof of Theorem 10.10. Therefore we will prove some statements and assign the remainder as Exercise 11.24.

(a) Suppose that $Zg \neq 0$ a.e. If we can show that $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$ is complete in $L^2(Q)$, then it follows from the unitarity of Z that $\mathcal{G}(g,1,1)$ is complete in $L^2(\mathbf{R})$.

So, suppose that $F \in L^2(Q)$ is such that $\langle F, E_{nk}\, Zg\rangle_{L^2(Q)} = 0$ for each $k, n \in \mathbf{Z}$. Let $G = F \cdot \overline{Zg}$. Then $G \in L^1(Q)$, and its Fourier coefficients with respect to the orthonormal basis $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ are

$$\begin{aligned} \widehat{G}(n,k) \;=\; \langle G, E_{nk}\rangle_{L^2(Q)} &= \int_0^1\!\!\int_0^1 F(x,\xi)\,\overline{Zg(x,\xi)}\,\overline{E_{nk}(x,\xi)}\,dx\,d\xi \\ &= \langle F, E_{nk}\, Zg\rangle_{L^2(Q)} \\ &= 0. \end{aligned}$$

Although $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is not a basis for $L^1(Q)$, a two-dimensional analogue of Theorem 4.25 implies that functions in $L^1(Q)$ are uniquely determined by their Fourier coefficients. Since the Fourier coefficients of G agree with those of the zero function, we conclude that $G = 0$ a.e. As $G = F \cdot \overline{Zg}$ and $Zg \neq 0$ a.e., it follows that $F = 0$ a.e. Hence $\{E_{nk}\, Zg\}_{k,n\in\mathbf{Z}}$ is complete.

(b) Suppose that $1/Zg \in L^2(Q)$. Then we must have $Zg \neq 0$ a.e., so $\mathcal{G}(g,1,1)$ is complete by statement (a). Also, since Z is surjective, there exists some function $\widetilde{g} \in L^2(Q)$ such that $Z\widetilde{g} = 1/\overline{Zg}$. We compute that

$$\begin{aligned} \langle M_nT_kg,\, M_{n'}T_{k'}\widetilde{g}\rangle &= \langle E_{nk}\, Zg,\, E_{n'k'}\, Z\widetilde{g}\rangle_{L^2(Q)} \\ &= \langle E_{nk}\, Zg,\, E_{n'k'}/\overline{Zg}\rangle_{L^2(Q)} \\ &= \langle E_{nk},\, E_{n'k'}\rangle_{L^2(Q)} \\ &= \delta_{nn'}\,\delta_{kk'}. \end{aligned}$$

Hence $\mathcal{G}(\widetilde{g},1,1)$ is biorthogonal to $\mathcal{G}(g,1,1)$. Thus $\mathcal{G}(g,1,1)$ is both minimal and complete, so it is exact. □

As is the case for the systems of weighted exponentials considered in Theorem 10.10, the characterization of Gabor systems $\mathcal{G}(g,1,1)$ that are Schauder bases for $L^2(\mathbf{R})$ is a more subtle problem. In [HP06] it was shown that if

$\mathcal{G}(g,a,b)$ is a Schauder basis then we must have $ab = 1$ (and therefore can reduce to the case $a = b = 1$), and $\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbf{R})$ if and only if $|Zg|^2$ is a *product* $\mathcal{A}_2$ *weight* for $L^2(Q)$.

From Theorem 11.31 we obtain the following corollary, whose implications will be explored in the next section. We let $C^1(\mathbf{R}^2)$ denote the set of all differentiable functions $F\colon \mathbf{R}^2 \to \mathbf{C}$ whose partial derivatives $\partial F/\partial x$ and $\partial F/\partial \xi$ are both continuous.

Corollary 11.32. *Fix* $g \in L^2(\mathbf{R})$.

(a) *If* Zg *is continuous on* Q *(and hence on* $\mathbf{R}^2$*), then* $\mathcal{G}(g,1,1)$ *not a frame or a Riesz basis for* $L^2(\mathbf{R})$.

(b) *If* $Zg \in C^1(\mathbf{R}^2)$ *then* $\mathcal{G}(g,1,1)$ *not exact in* $L^2(\mathbf{R})$.

Proof. (a) This follows immediately from Theorem 11.31 and the fact that any continuous quasiperiodic function must have a zero.

(b) If Zg is differentiable on $\mathbf{R}^2$ then it is continuous and therefore has at least one zero in Q by Theorem 11.26. For simplicity, assume that this zero is located at the origin. The C^1 hypothesis implies that Zg is Lipschitz on a neighborhood of the origin, i.e., there exist $C > 0$ and $\delta > 0$ such that

$$x^2 + \xi^2 \;<\; \delta \implies |Zg(x,\xi) - Zg(0,0)| \;\le\; C\,|(x,\xi) - (0,0)|,$$

where $|\cdot|$ is the Euclidean norm on $\mathbf{R}^2$. Since $Zg(0,0) = 0$, by switching to polar coordinates we find that the integral of $1/|Zg|^2$ over the open ball $B_\delta(0)$ is

$$\begin{aligned}\iint_{B_\delta(0)} \frac{1}{|Zg(x,\xi)|^2}\,dx\,d\xi \;&\ge\; \frac{1}{C^2}\iint_{B_\delta(0)} \frac{1}{x^2+\xi^2}\,dx\,d\xi \\ &=\; \frac{1}{C^2}\int_0^{2\pi}\!\!\int_0^{\delta} \frac{1}{r^2}\,r\,dr\,d\theta \;=\; \infty.\end{aligned}$$

Hence $1/Zg \notin L^2(Q)$, so $\mathcal{G}(g,1,1)$ is not exact. □

Exercises

11.24. Prove the remaining statements in Theorem 11.31.

11.25. (a) Let $p(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$ be a trigonometric polynomial. Show that if $|p| = 1$ a.e., then $p(x) = c_n e^{2\pi i n x}$ for some n between $-N$ and N.

(b) Suppose that $g \in L^2(\mathbf{R})$ is compactly supported. Show that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|g| = \chi_E$ for some bounded set $E \subseteq \mathbf{R}$ that satisfies $\sum_{k\in\mathbf{Z}} \chi_E(x-k) = 1$ a.e.

11.8 The Balian–Low Theorem

In this section we will prove the two versions of the Balian–Low Theorem given in Theorem 8.12, which state that all Gabor frames at the critical density are "bad" in some sense. We begin with the following simple result, which was proved in [Hei90] and first appeared in journal form in [BHW95]. Recall from Theorem 11.31 that when $a = b = 1$, a Gabor system $\mathcal{G}(g, 1, 1)$ is a frame for $L^2(\mathbf{R})$ if and only if it is a Riesz basis.

Theorem 11.33 (Amalgam BLT). *If $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbf{R})$ then $g \notin W(C, \ell^1)$. Specifically, either*

$$g \text{ is not continuous} \qquad \text{or} \qquad \sum_{k \in \mathbf{Z}} \|g \cdot \chi_{[k,k+1]}\|_{L^\infty} \;=\; \infty.$$

Moreover, we also have $\widehat{g} \notin W(C, \ell^1)$, where $\widehat{g}$ is the Fourier transform of g.

Proof. We have already done the work earlier in the chapter. If $g \in W(C, \ell^1)$ then Theorem 11.22 implies that $Zg \in C(Q)$. Corollary 11.32 therefore implies that $\mathcal{G}(g, 1, 1)$ cannot be a frame for $L^2(\mathbf{R})$, simply because Zg must have a zero. The same reasoning transfers to $\widehat{g}$ by applying Exercise 11.4. □

Thus, if g is to generate a Riesz basis at the critical density, then either g must be discontinuous or it must have poor decay at infinity, and similarly $\widehat{g}$ is either discontinuous or has poor decay.

Example 11.34. We saw in Example 11.28 that the Zak transform of the Gaussian function $\phi(x) = e^{-\pi x^2}$ is continuous, so $\mathcal{G}(\phi, 1, 1)$ cannot be a Riesz basis for $L^2(\mathbf{R})$. Since $Z\phi$ is bounded, $\mathcal{G}(\phi, 1, 1)$ is a Bessel sequence, but it does not have a positive lower frame bound. On the other hand, $\mathcal{G}(\phi, 1, 1)$ is complete since $Z\phi$ has only a single zero in Q and therefore $Z\phi$ is nonzero almost everywhere. Because $Z\phi$ is infinitely differentiable, Corollary 11.32 implies that $\mathcal{G}(\phi, 1, 1)$ is not exact, and therefore it has a positive excess. An argument similar to the one presented in Example 5.9(c) can be used to show that $\mathcal{G}(g, 1, 1)$ is overcomplete by precisely one element. That is, if we remove any single element from $\mathcal{G}(\phi, 1, 1)$ then it will still be complete, but if we remove two elements then it becomes incomplete. In particular,

$$\mathcal{G}(\phi, 1, 1) \setminus \{\phi\} \;=\; \{M_n T_k \phi\}_{(k,n) \neq (0,0)}$$

is exact, but it is not a Schauder basis or a frame for $L^2(\mathbf{R})$ (see [Fol89, p. 168]). ◇

The theorem originally stated by Balian [Bal81] and independently by Low [Low85] quantifies the "unpleasantness" of a Gabor orthonormal basis generator in a different manner than Theorem 11.33. In contrast to the Amalgam BLT, the hypotheses of their theorem do not imply that Zg is continuous, which is why the proof is more difficult. A gap in the original proofs was filled by Coifman, Daubechies, and Semmes in [Dau90]. At the same time, they also extended the proof from orthonormal bases to Riesz bases, yielding the following result that we call the "Classical" Balian–Low Theorem.

Theorem 11.35 (Classical BLT). *If $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbf{R})$ then*

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} |\xi\widehat{g}(\xi)|^2\,d\xi\right) \;=\; \infty. \qquad \diamondsuit \tag{11.24}$$

Before discussing the proof of Theorem 11.35, we make some remarks on what it says qualitatively and how its conclusions compare to those of the Amalgam BLT.

The Fourier transform is a unitary mapping of $L^2(\mathbf{R})$ onto itself, so if g belongs to $L^2(\mathbf{R})$ then so does $\widehat{g}$. The celebrated *Classical Uncertainty Principle* of quantum mechanics takes the following mathematical form: We must always have

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} |\xi\widehat{g}(\xi)|^2\,d\xi\right) \;\geq\; \frac{1}{4\pi}\int_{-\infty}^{\infty} |g(x)|^2\,dx. \tag{11.25}$$

A proof of this inequality is sketched in Exercise 11.32. The left-hand side of equation (11.25) may be finite or infinite, but it can never be smaller than the right-hand side. The Gaussian function $\phi(x) = e^{-\pi x^2}$ achieves equality in equation (11.25), and the only functions that do so are translated and modulated Gaussians of the form

$$c\,e^{2\pi i\xi_0 x}\,e^{-r(x-x_0)^2}, \qquad c \in \mathbf{C},\; r > 0.$$

The Classical BLT states that if $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbf{R})$ then not only do we have the bound given in equation (11.25), but the left-hand side of that equation must actually be infinite. Thus the generator of a Gabor Riesz basis must "maximize uncertainty."

One important feature of the Fourier transform is that it interchanges the roles of smoothness and decay. Roughly speaking, the smoother that g is, the faster that $\widehat{g}$ must decay at infinity, and the faster that g decays, the smoother that $\widehat{g}$ must be. If g decays well at infinity then we should have $\int |xg(x)|^2\,dx < \infty$. For example, if g is bounded and for x large enough we have $|g(x)| \leq C|x|^{-p}$ where $p > 3/2$, then $\int |xg(x)|^2\,dx$ will be finite. Thus, to say that $\int |xg(x)|^2\,dx = \infty$ is to say that g does not decay rapidly, at least in some integrated average sense, and therefore $\widehat{g}$ is not very smooth. Similarly, if $\int |\xi\widehat{g}(\xi)|^2\,d\xi = \infty$ then $\widehat{g}$ does not decay well and hence g is not very smooth. The Classical BLT implies that if $\mathcal{G}(g,1,1)$ is a Riesz basis then at least one of these things must happen, and so g is a "bad function" (at least in terms of Gabor theory).

Qualitatively, the Classical and Amalgam BLTs have similar conclusions: The generator of a Gabor Riesz basis at the critical density is either not smooth or it has poor decay. The two theorems *quantify* this statement in somewhat different ways. While there is a good deal of overlap, neither conclusion implies the other, so the two BLTs are distinct theorems [BHW95].

We will give an elegant proof of Theorem 11.35 due to Battle [Bat88] for the case that $\mathcal{G}(g,1,1)$ is an orthonormal basis. This proof relies on the operator theory that underlies the proof of the Classical Uncertainty Principle, and with some work the proof can be extended to Gabor systems that are Riesz bases, see [DJ93]. For some variations on the proof and more extensive discussion we refer to the survey paper [BHW95].

Proof (of Theorem 11.35 for orthonormal bases). The quantum mechanics operators of position and momentum are, in mathematical terms,

$$Pf(x) \;=\; xf(x) \qquad \text{and} \qquad Mf(x) \;=\; \frac{1}{2\pi i}\, f'(x). \tag{11.26}$$

These operators obviously do not map $L^2(\mathbf{R})$ into itself. We can make them well defined by restricting their domains to appropriate dense subsets of $L^2(\mathbf{R})$, but even if we do this, these operators are unbounded with respect to L^2-norm (Exercise 11.28). Still, these are key operators in harmonic analysis and quantum mechanics.

Suppose that $g \in L^2(\mathbf{R})$ is such that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbf{R})$. In particular, this implies

$$\langle g, M_nT_kg\rangle \;=\; \delta_{0k}\,\delta_{0n}, \qquad k,n \in \mathbf{Z}.$$

If either $\int |xg(x)|^2\,dx \;=\; \infty$ or $\int |\xi\widehat{g}(\xi)|^2\,d\xi \;=\; \infty$ then equation (11.24) holds trivially, so suppose that both of these quantities are finite. In terms of the position operator, this means $Pg \in L^2(\mathbf{R})$ and $P\widehat{g} \in L^2(\mathbf{R})$.

Since g and Pg both belong to $L^2(\mathbf{R})$, for k, $n \in \mathbf{Z}$ we compute that

$$\begin{aligned}
&\big\langle Pg,\, M_nT_kg\big\rangle \qquad (11.27)\\
&= \int_{-\infty}^{\infty} xg(x)\, e^{-2\pi inx}\, \overline{g(x-k)}\,dx\\
&= \int_{-\infty}^{\infty} g(x)\,\overline{e^{2\pi inx}\,(x-k)\,g(x-k)}\,dx + k\int_{-\infty}^{\infty} g(x)\,\overline{e^{2\pi inx}\,g(x-k)}\,dx\\
&= \big\langle g,\, M_nT_kPg\big\rangle \;+\; k\,\big\langle g,\, M_nT_kg\big\rangle\\
&= \big\langle g,\, M_nT_kPg\big\rangle \;+\; k\,\delta_{0k}\,\delta_{0n}\\
&= \big\langle g,\, M_nT_kPg\big\rangle \;+\; 0. \qquad (11.28)
\end{aligned}$$

The adjoint of M_n is M_{-n}, and likewise the adjoint of T_k is T_{-k}. Further, M_n and T_k commute because we are at the critical density. Therefore

$$\big\langle g,\, M_nT_kPg\big\rangle \;=\; \big\langle T_{-k}M_{-n}g,\, Pg\big\rangle \;=\; \big\langle M_{-n}T_{-k}g,\, Pg\big\rangle. \tag{11.29}$$

Combining equations (11.28) and (11.29), we see that

$$\langle Pg, M_nT_kg\rangle = \langle M_{-n}T_{-k}g, Pg\rangle. \tag{11.30}$$

Our next goal is to perform a similar calculation using Mg instead of Pg.

Because the Fourier transform interchanges smoothness with decay, the hypotheses $g, P\widehat{g} \in L^2(\mathbf{R})$ imply that g has a certain amount of smoothness. Specifically, Theorem 9.27(b) states that g is *absolutely continuous* on any finite interval, $g'(x)$ exists a.e., $g' \in L^2(\mathbf{R})$, and

$$\widehat{g'}(\xi) = 2\pi i\xi\,\widehat{g}(\xi) = 2\pi i\,P\widehat{g}(\xi) \text{ a.e.}$$

In particular, $Mg \in L^2(\mathbf{R})$ and

$$(Mg)^\wedge = \Big(\frac{1}{2\pi i}g'\Big)^\wedge = P\widehat{g}.$$

Since the Fourier transform is unitary on $L^2(\mathbf{R})$, we "switch to the Fourier side" and apply equation (11.30) to compute that

$$\begin{aligned}
\langle Mg, M_nT_kg\rangle &= \langle (Mg)^\wedge, (M_nT_kg)^\wedge\rangle \\
&= \langle P\widehat{g}, T_nM_{-k}\widehat{g}\rangle \\
&= \langle P\widehat{g}, M_{-k}T_n\widehat{g}\rangle \\
&= \langle M_kT_{-n}\widehat{g}, P\widehat{g}\rangle \\
&= \langle (T_{-k}M_{-n}g)^\wedge, (Mg)^\wedge\rangle \\
&= \langle T_{-k}M_{-n}g, Mg\rangle \\
&= \langle M_{-n}T_{-k}g, Mg\rangle. \qquad (11.31)
\end{aligned}$$

By expanding Pg and Mg in the orthonormal basis $\{M_nT_kg\}_{k,n\in\mathbf{Z}}$ and applying equations (11.30) and (11.31) we obtain

$$\begin{aligned}
\langle Mg, Pg\rangle &= \Big\langle \sum_{k,n\in\mathbf{Z}} \langle Mg, M_nT_kg\rangle\, M_nT_kg,\ Pg\Big\rangle \\
&= \sum_{k,n\in\mathbf{Z}} \langle Mg, M_nT_kg\rangle\,\langle M_nT_kg, Pg\rangle \\
&= \sum_{k,n\in\mathbf{Z}} \langle M_{-n}T_{-k}g, Mg\rangle\,\langle Pg, M_{-n}T_{-k}g\rangle \\
&= \sum_{k,n\in\mathbf{Z}} \langle Pg, M_nT_kg\rangle\langle M_nT_kg, Mg\rangle \\
&= \langle Pg, Mg\rangle.
\end{aligned}$$

However, we will show that we also have

$$\big\langle Mg,\, Pg\big\rangle \;=\; \langle Pg, Mg\rangle - \frac{1}{2\pi i}, \tag{11.32}$$

which is a contradiction. To see that equation (11.32) holds, first write

$$\big\langle Mg,\, Pg\big\rangle \;=\; \frac{1}{2\pi i}\int_{-\infty}^{\infty} g'(x)\,\overline{xg(x)}\,dx.$$

Integration by parts is valid for absolutely continuous functions (Theorem 9.28). Setting $u(x) = \overline{g(x)}$ and $v(x) = xg(x)$, we therefore compute that

$$\begin{aligned}
&\int_a^b g'(x)\,\overline{xg(x)}\,dx\\
&= \int_a^b \big(xg'(x) + g(x)\big)\,\overline{g(x)}\,dx \;-\; \int_a^b g(x)\,\overline{g(x)}\,dx\\
&= \left(b\,|g(b)|^2 - a\,|g(a)|^2 \;-\; \int_a^b xg(x)\,\overline{g'(x)}\,dx\right) \;-\; \int_a^b |g(x)|^2\,dx.
\end{aligned}$$

If we fix a, then each of the integrals appearing above converges to a finite value as $b \to \infty$. Consequently, $b\,|g(b)|^2$ must converge as $b \to \infty$. However, since g is square integrable, this limit must be zero (Exercise 11.27). A similar argument applies as $a \to -\infty$, so we have

$$\begin{aligned}
\big\langle Mg,\, Pg\big\rangle &= \frac{1}{2\pi i}\lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b g'(x)\,\overline{xg(x)}\,dx\\
&= \frac{1}{2\pi i}\lim_{\substack{a\to-\infty\\ b\to\infty}} \left(-\int_a^b xg(x)\,\overline{g'(x)}\,dx \;-\; \int_a^b |g(x)|^2\,dx\right)\\
&= \int_{-\infty}^{\infty} Pg(x)\,\overline{Mg(x)}\,dx \;-\; \frac{1}{2\pi i}\,\|g\|_{L^2}^2\\
&= \langle Pg, Mg\rangle \;-\; \frac{1}{2\pi i}.
\end{aligned}$$

This gives our contradiction. □

However, this is not the end of the story on bases related to time-frequency shifts. A remarkable construction known as *Wilson bases* yields orthonormal bases for $L^2(\mathbf{R})$ (as well as unconditional bases for the modulation spaces $M^{p,q}_s(\mathbf{R})$) generated by appropriate linear combinations of time-frequency shifts of "nice" functions. For details on this topic we refer to Gröchenig's text [Grö01].

Exercises

11.26. Let $\phi(x) = e^{-\pi x^2}$ be the Gaussian function. Show that $\mathcal{G}(\phi, 1, 1)$ is ℓ^2-independent, i.e., if $c = (c_{kn})_{k,n\in\mathbf{Z}} \in \ell^2(\mathbf{Z}^2)$ and $\sum c_{kn} M_n T_k \phi = 0$, then $c_{kn} = 0$ for every k and n. Note that since $\mathcal{G}(\phi, 1, 1)$ is a Bessel sequence, if we let R denote the synthesis operator for $\mathcal{G}(\phi, 1, 1)$ then ℓ^2-independence is equivalent to the statement that $R\colon \ell^2(\mathbf{Z}^2) \to L^2(\mathbf{R})$ is injective.

11.27. Show that if $g \in L^2(\mathbf{R})$ and $\lim_{x\to\infty} x\,|g(x)|^2$ exists, then this limit must be zero.

11.28. Let P, M be the position and momentum operators introduced in equation (11.26). These operators are not defined on all of $L^2(\mathbf{R})$. Instead, define domains

$$\begin{aligned} D_P &= \{f \in L^2(\mathbf{R}) : xf(x) \in L^2(\mathbf{R})\}, \\ D_M &= \{f \in L^2(\mathbf{R}) : f \text{ is differentiable and } f' \in L^2(\mathbf{R})\}, \end{aligned}$$

which are dense subspaces of $L^2(\mathbf{R})$. Restricted to these domains, P maps D_P into $L^2(\mathbf{R})$ and M maps D_M into $L^2(\mathbf{R})$. Show that P and M are unbounded even when restricted to these domains, i.e.,

$$\sup_{\substack{f\in D_P,\\ \|f\|_{L^2}=1}} \|Pf\|_{L^2} = \infty = \sup_{\substack{f\in D_M,\\ \|f\|_{L^2}=1}} \|Mf\|_{L^2}.$$

11.29. Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space introduced in Definition 9.18. Show that the position and momentum operators map $\mathcal{S}(\mathbf{R})$ into itself, and are self-adjoint when restricted to this domain, i.e.,

$$\langle Pf, g\rangle = \langle f, Pg\rangle \qquad \text{and} \qquad \langle Mf, g\rangle = \langle f, Mg\rangle$$

for all $f \in \mathcal{S}(\mathbf{R})$. (The Schwartz space is a convenient dense subspace of $L^2(\mathbf{R})$, but can be replaced in this problem by some larger subspaces of $L^2(\mathbf{R})$ if desired.)

11.30. The *commutator* of position and momentum is the operator $[P, M] = PM - MP$. Show that $[P, M] = -\frac{1}{2\pi i} I$ in the sense that $[P, M]f = -\frac{1}{2\pi i} f$ for all differentiable functions f. How does this relate to equation (11.32)?

11.31. This exercise will give an abstract operator-theoretic version of the Uncertainty Principle.

Let S be a subspace of a Hilbert space H, and let A, $B\colon S \to H$ be linear but possibly unbounded operators. By replacing S with the smaller space $\mathrm{domain}(AB) \cap \mathrm{domain}(BA)$ if necessary, we may assume that A, B, AB, and BA are all defined on S.

(a) Show that if A, B are self-adjoint in the sense that

$$\forall\, f, g \in S, \qquad \langle Af, g\rangle \;=\; \langle f, Ag\rangle \quad \text{and} \quad \langle Bf, g\rangle \;=\; \langle f, Bg\rangle,$$

then

$$\forall\, f \in S, \quad \|Af\|\,\|Bf\| \;\geq\; \frac{1}{2}\,\big|\big\langle [A,B]f,\, f\big\rangle\big|,$$

where $[A,B] = AB - BA$ is the commutator of A and B.

(b) Show that equality holds in part (a) if and only if $Af = icBf$ for some $c \in \mathbf{R}$.

11.32. Apply Exercises 11.29–11.31 to the position and momentum operators P and M to derive the Classical Uncertainty Principle,

$$\|xg(x)\|_{L^2}\,\|\xi\,\widehat{g}(\xi)\|_{L^2} \;\geq\; \frac{1}{4\pi}\,\|g\|_{L^2}^2, \tag{11.33}$$

for $g \in \mathcal{S}(\mathbf{R})$.

Remark: An extension by density argument can be used to prove that equation (11.33) extends to all $g \in L^2(\mathbf{R})$, or integration by parts for absolutely continuous functions can be used to prove directly that equation (11.33) holds whenever $\|xg(x)\|_{L^2}\,\|\xi\,\widehat{g}(\xi)\|_{L^2}$ is finite, see [Heil].

11.33. Modify Battle's argument to prove the *Weak BLT*: If $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbf{R})$ then

$$\|xg(x)\|_{L^2}\,\|\xi\widehat{g}(\xi)\|_{L^2}\,\|x\widetilde{g}(x)\|_{L^2}\,\|\xi\widehat{\widetilde{g}}(\xi)\|_{L^2} \;=\; \infty,$$

where $\widetilde{g}$ is the dual system generator from Theorem 11.31(b).

Remark: It requires some work, but it can be shown that the Weak BLT implies Theorem 11.35; see [DJ93] or the survey paper [BHW95].

11.9 The HRT Conjecture

In the final section of this chapter we will present an open problem related to Gabor systems that is so very simple to state yet is still unsolved, at least as of the time of writing. This conjecture first appeared in print in 1996 [HRT96]. As this topic is more personal to me than some of the others that appear in this volume, I will often speak more directly to the reader in this section than usual.

In the previous sections we saw many results dealing with Gabor systems $\mathcal{G}(g,a,b)$ that are complete, a frame, exact, a Riesz basis, an orthonormal basis, and so forth. Yet we have not yet asked what may be the most basic questions of all: Are Gabor systems finitely independent? Given any collection of vectors in a vector space, surely one of the very first properties that we would like to determine is whether these vectors are independent or dependent. For *lattice* Gabor systems $\mathcal{G}(g,a,b)$, the answer is known (though the proof is nontrivial!). The next theorem is due to Linnell [Lin99], and partially answers a question first posed in [HRT96].

Theorem 11.36. *If $g \in L^2(\mathbf{R})\backslash\{0\}$ and $a,\ b > 0$, then $\mathcal{G}(g,a,b)$ is finitely linearly independent.* $\diamondsuit$

We will discuss Theorem 11.36 and its proof a little later. Assuming the validity of Theorem 11.36, Exercise 11.36 shows how to extend it a little further, as follows.

Corollary 11.37. *Let A be an invertible 2×2 matrix, choose $z \in \mathbf{R}^2$, and set $\Lambda = A(\mathbf{Z}^2) + z$. Then for any nonzero $g \in L^2(\mathbf{R})$,*

$$\mathcal{G}(g,\Lambda) \;=\; \bigl\{M_b T_a g\bigr\}_{(a,b)\in\Lambda}$$

is finitely linearly independent. $\diamondsuit$

When A is an invertible matrix, we call $A(\mathbf{Z}^2)$ a *full-rank lattice* in $\mathbf{R}^2$. Thus $A(\mathbf{Z}^2) + z$ is a rigid translate of a full-rank lattice. In particular, if we choose any three noncollinear points in $\mathbf{R}^2$, then we can always find A and z so that $\Lambda = A(\mathbf{Z}^2) + z$ contains these three points (Exercise 11.34). Therefore any set of three noncollinear time-frequency shifts of a nonzero $g \in L^2(\mathbf{R})$ is linearly independent, and the collinear case can be addressed by other arguments (Exercise 11.37). Since one point is trivial and two points are always collinear, we obtain the following corollary.

Corollary 11.38. *Let $N = 1,\ 2,$ or 3. If $g \in L^2(\mathbf{R})\backslash\{0\}$ and (p_k,q_k) for $i = 1,\dots,N$ are distinct points in $\mathbf{R}^2$, then*

$$\bigl\{e^{2\pi i q_k x} g(x-p_k) : k = 1,\dots,N\bigr\}$$

is linearly independent. $\diamondsuit$

Thus, any collection of up to three distinct time-frequency shifts of a function $g \in L^2(\mathbf{R})$ is linearly independent. Surely four points cannot be much more difficult—how hard can it be to show that a set of four vectors in a vector space are linearly independent? It is not that hard if we have four specific vectors in hand, but we are asking a somewhat more general question. If we let (p_k,q_k) for $i = 1,2,3,4$ be any set of four distinct points in $\mathbf{R}^2$, we want to know if

$$\bigl\{M_{q_k} T_{p_k} g : k = 1,2,3,4\bigr\} \;=\; \bigl\{e^{2\pi i q_k x} g(x-p_k) : k = 1,2,3,4\bigr\}$$

is linearly independent for every nonzero function $g \in L^2(\mathbf{R})$. *The answer to this question is not known!*

One difficulty is that four noncollinear distinct points in $\mathbf{R}^2$ need not lie on a translate of a full-rank lattice. For example, because the distances between the following points are not rationally related, there is no matrix A and point z so that the four points

$$\bigl\{(0,0),\ (1,0),\ (0,1),\ (\sqrt{2},\sqrt{2})\bigr\}$$

are contained in $A(\mathbf{Z}^2)+z$. Forgetting about generic sets of four points, what about just *this* particular set of points? If $g \in L^2(\mathbf{R})$ is not the zero function, must the set of time-frequency translates of g determined by those four points be independent, i.e., must

$$\bigl\{g(x),\, g(x-1),\, e^{2\pi i x}g(x),\, e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\bigr\}$$

be linearly independent? I don't know, and neither does anyone else.

Conjecture 11.39 (HRT Subconjecture). *If* $g \in L^2(\mathbf{R})\backslash\{0\}$ *then*

$$\bigl\{g(x),\; g(x-1),\; e^{2\pi i x}g(x),\; e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\bigr\} \tag{11.34}$$

is linearly independent.

There's nothing special about $\sqrt{2}$ in this choice of four points; the answer is still unknown if we replace the two instances of $\sqrt{2}$ in equation (11.34) by some other irrational numbers (on the other hand, Ziemowit Rzeszotnik has shown me his unpublished proof that $\bigl\{g(x),\; g(x-1),\; e^{2\pi i x}g(x), e^{2\pi i\sqrt{2}x}g(x)\bigr\}$ is independent for each nonzero $g \in L^2(\mathbf{R})$, and the recent paper [Dem10] addresses the case of any four points that lie on two parallel lines).

The answer to Conjecture 11.39 is known for some special classes of functions $g \in L^2(\mathbf{R})$, and for those functions for which the answer is known the answer is always yes, linear independence holds.

Example 11.40. Suppose that $g \in L^2(\mathbf{R})$ is supported within the halfline $[0,\infty)$, i.e., $g(x) = 0$ for almost every $x < 0$, and suppose also that g is not the zero function on $[0,1]$. If the collection of time-frequency translates in equation (11.34) is dependent then there exist scalars a, b, c, d, not all zero, such that

$$ag(x) + bg(x-1) + ce^{2\pi i x}g(x) + de^{2\pi i\sqrt{2}x}g(x-\sqrt{2}) \;=\; 0 \text{ a.e.} \tag{11.35}$$

Note that the functions $g(x)$ and $e^{2\pi i x}g(x)$ are supported within $[0,\infty)$, while $g(x-1)$ is supported in $[1,\infty)$ and $e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})$ is supported in $[\sqrt{2},\infty)$. Therefore, if we only consider points x between 0 and 1 then equation (11.35) reduces to

$$(a + ce^{2\pi i x})\, g(x) \;=\; 0 \quad \text{for a.e. } x \in [0,1].$$

However, if either a or c is nonzero then $a + ce^{2\pi i x} \neq 0$ for almost every x, so $g(x) = 0$ a.e. on $[0,1]$, which contradicts our assumptions on g. Therefore we must have $a = c = 0$. But then

$$bg(x-1) + de^{2\pi i\sqrt{2}x}g(x-\sqrt{2}) \;=\; 0 \text{ a.e.},$$

which contradicts the fact that any set of two time-frequency translates of g must be independent. ◇

Conjecture 11.39 is a special case of the following conjecture, first made in [HRT96].

Conjecture 11.41 (HRT Conjecture). *If $g \in L^2(\mathbf{R})$ is not the zero function and $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbf{R}^2$, then*

$$\mathcal{G}(g, \Lambda) \;=\; \bigl\{M_{q_k} T_{p_k} g\bigr\}_{k=1}^N$$

is a linearly independent set of functions in $L^2(\mathbf{R})$. $\diamondsuit$

Conjecture 11.41 is also known as the Linear Independence Conjecture for time-frequency shifts. Despite having been worked on by a large number of groups, there is a scarcity of hard results. The main papers specifically dealing with the HRT Conjecture appear to be [HRT96], [Lin99], [Kut02], [Bal08], [BS09], [Dem10], [DG10], [DZ10], and there is also a survey paper on the topic [Hei06].

Some partial results on the HRT Conjecture are known. For example, the idea of Example 11.40 extends to any finite number of points, so independence in the HRT Conjecture is known to hold if we add the extra assumption that g is compactly supported or is only nonzero within a halfline $(-\infty, a]$ or $[a, \infty)$; see Exercise 11.38. On the other hand, it is quite surprising that there are very few partial results based on smoothness or decay conditions on g. In particular, the HRT Conjecture is open even if we impose the extra hypothesis that g lie in the Schwartz class $\mathcal{S}(\mathbf{R})$, i.e., g is infinitely differentiable and $x^m g^{(n)}(x) \to 0$ as $x \to \pm\infty$ for every m, $n \in \mathbf{N}$. While the HRT Conjecture is known to be true for some Schwartz class functions, such as those that are compactly supported, it is not known whether or not it holds for *every* nonzero Schwartz class function.

Let us return to lattice Gabor systems and Theorem 11.36 in particular, and try to illustrate why the proof of that theorem is nontrivial. Consider the case of three specific points in $\mathbf{R}^2$, say

$$\Lambda \;=\; \bigl\{(0,0),\; (a,0),\; (0,1)\bigr\}. \tag{11.36}$$

We will address the "difficult case" where a is irrational.

Example 11.42. Suppose that $g \in L^2(\mathbf{R})\backslash\{0\}$ is such that

$$\mathcal{G}(g, \Lambda) \;=\; \bigl\{g(x),\; g(x-a),\; e^{2\pi i x} g(x)\bigr\}$$

is linearly dependent, where $a > 0$ is irrational. Then there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1 g(x) + c_2 g(x-a) + c_3 e^{2\pi i x} g(x) \;=\; 0 \text{ a.e.}$$

If any one of c_1, c_2, c_3 is zero then we reduce to only two time-frequency shifts, so we assume that c_1, c_2, c_3 are all nonzero. Dividing through by c_2, we can further assume that $c_2 = 1$. Rearranging,

$$g(x-a) \;=\; \bigl(-c_1 - c_3\, e^{2\pi i x}\bigr)\, g(x) \;=\; m(x)\, g(x) \text{ a.e.}, \tag{11.37}$$

where $m(x) = -c_1 - c_3\, e^{2\pi i x}$. Note that m is a 1-periodic trigonometric polynomial. Iterating equation (11.37), for integer $n > 0$ we obtain

$$\begin{aligned} |g(x-na)| \;&=\; |m(x-(n-1)a) \cdots m(x-a)\, m(x)\, g(x)| \\ &=\; |g(x)| \prod_{j=0}^{n-1} |m(x-ja)| \\ &=\; |g(x)|\, e^{n \cdot \frac{1}{n} \sum_{j=0}^{n-1} p(x-ja)} \quad \text{a.e.}, \end{aligned} \tag{11.38}$$

where $p(x) = \ln |m(x)|$. Since g is square integrable, if $g(x-na)$ grows with n then we might hope to obtain a contradiction, although we must be careful since g is only defined almost everywhere.

Now, p is 1-periodic, so $p(x-ja) = p(x-ja \bmod 1)$, where $t \bmod 1$ denotes the fractional part of t. A consequence of the fact that a is irrational is that the points $\{x - ja \bmod 1\}_{j=0}^{\infty}$ form a dense subset of $[0,1)$. In fact, they are "well distributed" in a technical sense due to the fact that $x \mapsto x + a \bmod 1$ is an *ergodic mapping* of $[0,1)$ onto itself (i.e., only subsets of measure 0 or measure 1 can be invariant under this map). Hence the quantity $\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja)$ is like a Riemann sum approximation to $\int_0^1 p(x)\, dx$, except that the rectangles with height $p(x-ja)$ and width $\frac{1}{n}$ are distributed "randomly" around $[0,1)$ instead of uniformly, possibly even with overlaps or gaps (see Figure 11.4). Still, the ergodicity ensures that the Riemann sum analogy is a good one in the limit. Specifically, the *Birkhoff Ergodic Theorem* [Wal82] implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} p(x-ja) \;=\; \int_0^1 p(x)\, dx \;=\; C \text{ a.e.} \tag{11.39}$$

The fact that $C = \int_0^1 p(x)\, dx$ exists and is finite follows from the fact that any singularities of p correspond to zeros of the well-behaved function m (Exercise 11.42). So, if we fix $\varepsilon > 0$, then $\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja) \ge (C-\varepsilon)$ for n large enough. Let us ignore the fact that "large enough" depends on x (or, by applying Egoroff's Theorem, restrict to a subset where the convergence in equation (11.39) is uniform). Substituting into equation (11.38) then yields

$$|g(x-na)| \;\ge\; e^{(C-\varepsilon)n}\, |g(x)|, \qquad n \text{ large}.$$

Considering x in a set of positive measure where g is nonzero and using the fact that $g \in L^2(\mathbf{R})$, we conclude that $C - \varepsilon < 0$. This is true for every $\varepsilon > 0$, so $C \le 0$. A converse argument based on the relation $g(x) = m(x+a)\, g(x+a)$ similarly yields the inequality $C \ge 0$. This still allows the possibility that $C = 0$, but a slightly more subtle argument presented in [HRT96] also based on ergodicity yields the full result. The case a is rational is more straightforward, since then the points $x - ja \bmod 1$ repeat themselves. $\diamondsuit$

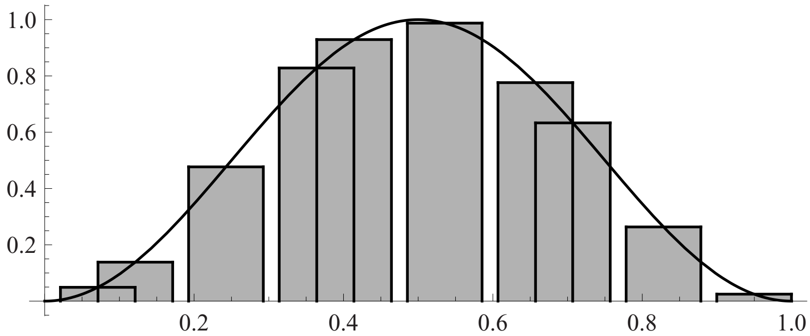

Fig. 11.4. The area of the boxes (counting overlaps) is $\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja)$.

By applying the techniques used in Exercises 11.35 and 11.36, the HRT Conjecture for three noncollinear points can always be reduced to the HRT Conjecture for the three points given in equation (11.36) for some $a \neq 0$. However, the argument given in Example 11.42 is limited to only three points so we have not proved that the HRT Conjecture is valid for all lattice Gabor systems. Still, our argument does suggest why the proof of Theorem 11.36 is nontrivial. There is no obvious way to extend the technique of Example 11.42 to apply to four points in general position. It particular, the argument depends critically on the recurrence relation that appears in equation (11.37), and this recurrence relation is a consequence of the fact that there are only two distinct translations appearing in the collection $\{g(x),\ g(x-a),\ e^{2\pi i x}g(x)\}$. Specifically, $g(x)$ and $e^{2\pi i x}g(x)$ are translated by zero, while $g(x-a)$ is translated by a. As soon as we have three or more distinct translates, the recurrence relation becomes much more complicated (too complicated to use?). Indeed, Linnell's proof takes a quite different approach, relying on the fact that the operators M_{bn}, T_{ak} with $k, n \in \mathbf{Z}$ generate a von Neumann algebra (see [Lin99]).

So, we attack the HRT Conjecture from a different angle. Fix any set $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ of finitely many distinct points in $\mathbf{R}^2$, and define

$$S_\Lambda \;=\; \bigl\{g \in L^2(\mathbf{R}) \;:\; \mathcal{G}(g,\Lambda) \text{ is independent}\bigr\}. \tag{11.40}$$

The HRT Conjecture is that $S_\Lambda = L^2(\mathbf{R})\backslash\{0\}$. While we don't know that this is the case, we do know that S_Λ is dense in $L^2(\mathbf{R})$. For example, S_Λ contains all compactly supported functions in $L^2(\mathbf{R})$ (Exercise 11.38) and all finite linear combinations of Hermite functions (Exercise 11.40), each of which is a dense subset of $L^2(\mathbf{R})$. Perhaps we can apply some kind of perturbation argument to show that S_Λ actually contains all nonzero functions in $L^2(\mathbf{R})$. The next theorem is an attempt in this direction.

Theorem 11.43. *Assume that* $g \in L^2(\mathbf{R})$ *and* $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ *are such that* $\mathcal{G}(g,\Lambda)$ *is linearly independent. Then there exists an* $\varepsilon > 0$ *such that* $\mathcal{G}(h,\Lambda)$ *is independent for any* $h \in L^2(\mathbf{R})$ *with* $\|g-h\|_{L^2} < \varepsilon$.

Proof. Define the linear mapping $T\colon \mathbf{C}^N \to L^2(\mathbf{R})$ by

$$T(c_1,\dots,c_N) \;=\; \sum_{k=1}^N c_k\, M_{q_k} T_{p_k} g.$$

Note that T is injective since $\mathcal{G}(g,\Lambda)$ is independent. Therefore T is a linear bijection of $\mathbf{C}^N$ onto range(T), which is an N-dimensional subspace of $L^2(\mathbf{R})$. Since linear operators on finite-dimensional spaces are continuous, both $T\colon \mathbf{C}^N \to \text{range}(T)$ and $T^{-1}\colon \text{range}(T) \to \mathbf{C}^N$ are bounded. As all norms on $\mathbf{C}^N$ are equivalent, it follows that there exist constants A, $B > 0$ such that

$$A\sum_{k=1}^N |c_k| \;\le\; \biggl\|\sum_{k=1}^N c_k\, M_{q_k} T_{p_k} g\biggr\|_{L^2} \;\le\; B\sum_{k=1}^N |c_k|, \qquad (c_1,\dots,c_N) \in \mathbf{C}^N.$$

Therefore, if $\|g-h\|_{L^2} < A$, then for any $(c_1,\dots,c_N) \in \mathbf{C}^N$ we have

$$\begin{aligned}
\biggl\|\sum_{k=1}^N c_k\, M_{q_k} T_{p_k} h\biggr\|_{L^2} \;&\ge\; \biggl\|\sum_{k=1}^N c_k\, M_{q_k} T_{p_k} g\biggr\|_{L^2} \;-\; \biggl\|\sum_{k=1}^N c_k\, M_{q_k} T_{p_k} (h-g)\biggr\|_{L^2} \\
&\ge\; A\sum_{k=1}^N |c_k| \;-\; \sum_{k=1}^N |c_k|\, \|M_{q_k} T_{p_k}(h-g)\|_{L^2} \\
&=\; (A - \|h-g\|_{L^2}) \sum_{k=1}^N |c_k|.
\end{aligned}$$

Consequently, if $\sum_{k=1}^N c_k\, M_{q_k} T_{p_k} h = 0$ a.e. then $c_k = 0$ for every k. $\square$

Thus, the set S_Λ defined in equation (11.40) is actually an open subset of $L^2(\mathbf{R})$. Plus, we know that it is dense—so isn't it all of $L^2(\mathbf{R})$? No, we can't conclude that. For example, $\mathbf{R}\backslash\{\pi\}$ is an open and dense but proper subset of the real line. Therefore, we still don't know whether the HRT Conjecture is valid for all nonzero $g \in L^2(\mathbf{R})$. On the other hand, this does tell us that any counterexamples are "rare" in some sense.

I've worked hard on the HRT Conjecture but haven't solved it. If you solve it, please let me know! One word of warning—the problem seems to be much harder than it looks. I've produced dozens of incorrect proofs myself, and seen many more. Many of the errors in these proofs are related to the fact that the translation and modulation operators T_a, M_b do not commute for most values of a and b. I hope you enjoy this charming little problem, but beware of the pesky phase factor in the relation $T_a M_b = e^{-2\pi i ab}\, M_b T_a$.

Exercises

11.34. Show that if (p_i, q_i) for $i = 1, 2, 3$ are three noncollinear points in $\mathbf{R}^2$, then there exist an invertible 2×2 matrix A and a point $z \in \mathbf{R}^2$ such that $\Lambda = A(\mathbf{Z}^2) + z$ contains those three points.

11.35. Fix $g \in L^2(\mathbf{R}) \backslash \{0\}$, and let $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ be any set of finitely many distinct points in $\mathbf{R}^2$. Define $\mathcal{G}(g, \Lambda) = \{M_{q_k} T_{p_k} g\}_{k=1}^N$.

(a) Fix $z \in \mathbf{R}^2$. Show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(g, \Lambda + z)$ is linearly independent.

(b) Given $r \in \mathbf{R}$, let $S_r = \left[\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right]$, so multiplication by the matrix S_r is a shear operation on $\mathbf{R}^2$. Define $h(x) = e^{\pi i r x^2} g(x)$, and show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(h, S_r(\Lambda))$ is linearly independent.

(c) Let $R = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, so multiplication by R is a counterclockwise rotation of $\mathbf{R}^2$ by 90 degrees. Show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(\check{g}, R(\Lambda))$ is linearly independent, where $\check{g}$ is the inverse Fourier transform of g.

(d) Given $a \neq 0$, let $D_a = \left[\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right]$, so multiplication by D_a is a dilation by a on the x_1-axis and a corresponding dilation by $1/a$ on the x_2-axis. Define $h(x) = g(x/a)$, and show that $\mathcal{G}(g, \Lambda)$ is linearly independent if and only if $\mathcal{G}(h, D_a(\Lambda))$ is linearly independent.

(e) Show that if A is a 2×2 matrix with $\det(A) = 1$, then A can be written as a product of matrices of the form S_r, R, and D_a.

Remark: This factorization is related to the fact that every 2×2 matrix with determinant 1 is a *symplectic matrix*. In contrast, not every $2d \times 2d$ matrix with determinant 1 is symplectic when $d > 1$. As a consequence, the HRT Conjecture becomes even more intractable in higher dimensions.

11.36. Assuming Theorem 11.36, prove Corollary 11.37.

11.37. Fix $g \in L^2(\mathbf{R}) \backslash \{0\}$.

(a) Show that if $\{q_k\}_{k=1}^N$ is any set of finitely many distinct real numbers, then $\{M_{q_k} g\}_{k=1}^N$ is linearly independent.

(b) Show that if $\{p_k\}_{k=1}^N$ is any set of finitely many distinct real numbers, then $\{T_{p_k} g\}_{k=1}^N$ is linearly independent.

(c) Show that if $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct but collinear points in $\mathbf{R}^2$, then $\mathcal{G}(g, \Lambda) = \{M_{q_k} T_{p_k} g\}_{k=1}^N$ is linearly independent.

11.38. Suppose that $g \in L^2(\mathbf{R}) \backslash \{0\}$ is supported within some halfline, either $(-\infty, a]$ or $[a, \infty)$ where $a \in \mathbf{R}$. Show that if $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbf{R}^2$, then $\mathcal{G}(g, \Lambda) = \{M_{q_k} T_{p_k} g\}_{k=1}^N$ is linearly independent.

11.39. The nth *Hermite function* H_n is

$$H_n(x) \;=\; e^{\pi x^2}\, D^n e^{-2\pi x^2}, \qquad n \ge 0,$$

where D^n denotes the nth derivative operator.

(a) Prove that

$$H_{n+1}(x) \;=\; H_n'(x) - 2\pi x\, H_n(x), \qquad n \ge 0. \tag{11.41}$$

(b) Use equation (11.41) to show that $H_n(x) = p_n(x)\, e^{-\pi x^2}$, where p_n is a polynomial of degree n whose leading coefficient is $(-4\pi)^n$. Consequently, each H_n is infinitely differentiable and has exponential decay at infinity, and $\mathrm{span}\{H_n\}_{n\ge 0} = \bigl\{p(x)\, e^{-\pi x^2} : p \text{ is a polynomial}\bigr\}$.

Remark: It can be shown that $\{H_n\}_{n\ge 0}$ is an orthogonal (but not orthonormal) basis for $L^2(\mathbf{R})$. Hence $\mathrm{span}\{H_n\}_{n\ge 0}$ is a dense subspace of $L^2(\mathbf{R})$.

11.40. Let $\phi(x) = e^{-\pi x^2}$ be the Gaussian function, and let $h(x) = p(x)\, e^{-\pi x^2}$ where p is any nontrivial polynomial. Show that if $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbf{R}^2$, then $\mathcal{G}(h, \Lambda) = \{M_{q_k} T_{p_k} h\}_{k=1}^N$ is linearly independent.

11.41. Assume that $g \in L^2(\mathbf{R})$ and $\Lambda = \{(p_k, q_k)\}_{k=1}^N$ are such that $\mathcal{G}(g, \Lambda)$ is linearly independent. Show that there exists $\varepsilon > 0$ such that $\mathcal{G}(g, \Lambda')$ is independent for any set $\Lambda' = \{(\alpha_k', \beta_k')\}_{k=1}^N$ with $|\alpha_k - \alpha_k'|$, $|\beta_k - \beta_k'| < \varepsilon$ for $k = 1, \dots, N$.

11.42. Suppose that m is differentiable and $m(0) = 0$. Set $p(x) = \ln|m(x)|$, and show that $\int_{-\delta}^{\delta} p(x)\, dx$ exists and is finite if $\delta > 0$ is small enough.

12

Wavelet Bases and Frames

The Gabor systems that we studied in the preceding chapter are generated from time-frequency shifts of a single function. A time-frequency shift is a composition of the two simple operations of translation and modulation. Wavelets are likewise generated by two simple operations, this time translation and dilation. A composition of translation and dilation is called a time-scale shift (see the illustration in Figure 12.1), and so a wavelet system is a collection of time-scale shifts of a single function.

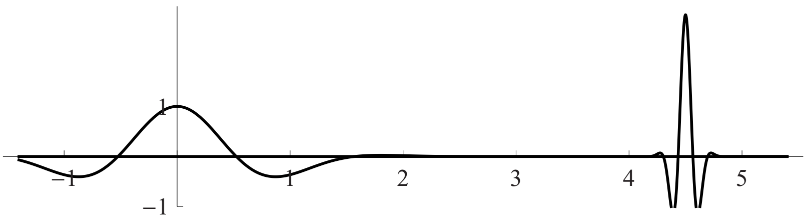

Fig. 12.1. The function $\psi(x) = e^{-\pi x^2} \cos 3x$ and a time-scale shift $D_{2^3} T_{36} \psi(x) = 8^{1/2} \psi(8x - 36)$.

Definition 12.1. A *wavelet system* is a sequence in $L^2(\mathbf{R})$ of the form

$$\mathcal{W}(\psi, a, b) \;=\; \{a^{n/2} \psi(a^n x - bk)\}_{k,n \in \mathbf{Z}},$$

where $\psi \in L^2(\mathbf{R})$ and $a > 1$, $b > 0$ are fixed. If we use the translation and dilation operators introduced in Notation 9.4, we can write this wavelet system as

$$\mathcal{W}(\psi, a, b) \;=\; \{D_{a^n} T_{bk} \psi\}_{k,n \in \mathbf{Z}}.$$

We call ψ the *generator* or the *atom* of the system, and refer to a, b as the *parameters* of the system. A typical choice is $a = 2$ and $b = 1$. We set

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$$\mathcal{W}(\psi) \;=\; \mathcal{W}(\psi, 2, 1) \;=\; \{D_{2^n} T_k \psi\}_{k,n \in \mathbf{Z}} \;=\; \{2^{n/2} \psi(2^n x - k)\}_{k,n \in \mathbf{Z}},$$

and call $\mathcal{W}(\psi)$ a *dyadic wavelet system*. A wavelet system that forms a frame is called a *wavelet frame*, etc. ♢

The simplest example of a wavelet orthonormal basis was introduced by Haar in 1910 [Haa10]. The original Haar system is an orthonormal basis for $L^2[0,1]$, but there are natural ways to extend it to form an orthonormal basis for $L^2(\mathbf{R})$. Setting $\chi = \chi_{[0,1)}$ and $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, each of

$$\bigl\{\chi(x-k)\bigr\}_{k \in \mathbf{Z}} \;\cup\; \bigl\{2^{n/2} \psi(2^n x - k)\bigr\}_{n \geq 0,\, k \in \mathbf{Z}} \tag{12.1}$$

and

$$\mathcal{W}(\psi) \;=\; \bigl\{2^{n/2} \psi(2^n x - k)\bigr\}_{n,k \in \mathbf{Z}} \tag{12.2}$$

forms an orthonormal basis for $L^2(\mathbf{R})$. Each of these is often referred to as the *Haar system* for $L^2(\mathbf{R})$.

Unfortunately, the function $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, which we call the *Haar wavelet*, is not smooth, and this limits its utility. We faced this same issue in Chapter 11 when we considered the Gabor orthonormal basis $\mathcal{G}(\chi_{[0,1]}, 1, 1)$ generated by the box function. We saw in that chapter that the Balian–Low Theorem implies that there is no way to find a "nice" generator g such that the Gabor system $\mathcal{G}(g, a, b)$ is a Riesz basis for $L^2(\mathbf{R})$. On the other hand, we can find very nice generators g such that $\mathcal{G}(g, a, b)$ forms a redundant frame for $L^2(\mathbf{R})$.

In this chapter we explore the construction of wavelet bases and frames. We will see that while there are many similarities to Gabor systems, there are also many fundamental, and surprising, differences. In particular, there exist orthonormal wavelet bases $\mathcal{W}(\psi)$ for $L^2(\mathbf{R})$ that are generated by functions ψ that are both very smooth and decay rapidly at infinity. For a more detailed introduction to wavelet theory in $L^2(\mathbf{R})$ we refer to the text by Daubechies [Dau92].

Wavelet systems have many uses outside of the Hilbert space setting. In particular, a wavelet basis for $L^2(\mathbf{R})$ generated by a smooth wavelet ψ is also an unconditional basis for an entire associated family of function spaces, the *Besov spaces* $B^{p,q}_s(\mathbf{R})$ and the *Triebel–Lizorkin spaces* $F^{p,q}_s(\mathbf{R})$ (which include $L^p(\mathbf{R})$ for $1 < p < \infty$). These are classical function spaces that quantify smoothness-related properties of functions and distributions. They are widely used in problems in analysis and other areas. In this chapter, our focus will be on the Hilbert space $L^2(\mathbf{R})$. For details on the extension of the basis properties of wavelet systems to other function spaces we refer to the texts by Meyer [Mey92] or Hernández and Weiss [HW96].

12.1 Some Basic Facts

Interestingly, the operations of translation and modulation on which Gabor systems are based "almost commute." We have $T_a M_b = e^{-2\pi i a b} M_b T_a$, and while the scalar $e^{-\pi iab}$ will not be 1 in general, it does have modulus 1. The situation for wavelets is quite different, as translation and dilation are "highly noncommutative." Being careful with the ordering of composition and evaluation, we compute that

$$\begin{aligned} D_a T_b f(x) &= (D_a(T_b f))(x) \\ &= a^{1/2}(T_b f)(ax) \\ &= a^{1/2} f(ax-b) \\ &= a^{1/2} f(a(x-b/a)) \\ &= T_{b/a} D_a f(x). \end{aligned}$$

Thus the translation parameter is dramatically affected when we interchange the order of translation and dilation. One consequence of this is that the canonical dual of a wavelet frame need not itself be a wavelet frame. The problem is that if $\mathcal{W}(\psi, a, b)$ is a wavelet frame, then its frame operator

$$Sf \;=\; \sum_{n\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \langle f, D_{a^n} T_{bk}\psi\rangle \, D_{a^n} T_{bk}\psi$$

will commute with dilations D_{a^n}, but it need not commute with translations T_{bk}. Indeed, we have

$$\begin{aligned} S(T_b f) &= \sum_{n\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \bigl\langle T_b f,\, D_{a^n} T_{bk}\psi \bigr\rangle \, D_{a^n} T_{bk}\psi \\ &= \sum_{n\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \bigl\langle f,\, T_{-b} D_{a^n} T_{bk}\psi \bigr\rangle \, D_{a^n} T_{bk}\psi \\ &= \sum_{n\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \bigl\langle f,\, D_{a^n} T_{-a^n b} T_{bk}\psi \bigr\rangle \, D_{a^n} T_{bk}\psi, \end{aligned}$$

but since n ranges through $\mathbf{Z}$, the translation parameters $a^n b$ cannot all be integer multiples of b. In practice, the dual of a wavelet frame often does turn out to be another wavelet frame, but this is not assured. Of course, Parseval frames and orthonormal bases are self-dual, and this is the situation on which we primarily concentrate in this chapter.

Exercises

12.1. Show that if $\mathcal{W}(\psi, a, b)$ is a wavelet frame for $L^2(\mathbf{R})$, then its frame operator S commutes with D_{a^n} for $n \in \mathbf{Z}$. What does this imply about the structure of the canonical dual frame?

12.2. Define

$$\mathbb{A}_1 \;=\; \{D_a T_b\}_{a>0,b\in\mathbf{R}}, \quad \mathbb{A}_2 \;=\; \left\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right\}_{a>0,b\in\mathbf{R}}, \quad \mathbb{A}_3 \;=\; \bigl\{(a,b)\bigr\}_{a>0,b\in\mathbf{R}}.$$

In particular, $\mathbb{A}_1$ is the set of all time-scale shift operators, and so is a subset of $\mathcal{B}(L^2(\mathbf{R}))$. Define the following operations: On $\mathbb{A}_1$ it is composition of operators, on $\mathbb{A}_2$ it is matrix multiplication, and on $\mathbb{A}_3$ it is

$$(a,b)*(c,d) \;=\; (ac, bc+d).$$

(a) Show that $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ are isomorphic groups with respect to these operations.

(b) Given $a>1$, $b>0$, show that $\{D_{a^n} T_{bk}\}_{k,n\in\mathbf{Z}}$ is *not* a subgroup of $\mathbb{A}_1$.

(c) With $a>1$ and $b>0$ fixed, let G be the subgroup of $\mathbb{A}_1$ generated by D_a and T_b, i.e., it is the intersection of all subgroups of $\mathbb{A}_1$ that contain both D_a and T_b. Show that G contains $D_{a^n} T_{(a^m j+k)b}$ for every m, n, j, $k \in \mathbf{Z}$. As $\bigl\{(a^m j+k)b\bigr\}_{m,j,k\in\mathbf{Z}}$ is dense in $\mathbf{R}$, we conclude that there are no discrete "separated" subgroups of $\mathbb{A}_1$ that contain both dilations and translations.

(d) As a set, $\mathbb{A}_3 = (0,\infty)\times\mathbf{R}$, and hence has a natural topology. In fact, $\mathbb{A}_3$ is an example of a *locally compact group* (LCG). Every LCG has associated left and right *Haar measures* (and these are unique up to scalar multiples). Show that the left Haar measure for $\mathbb{A}_3$ is $\frac{da}{a}\,db$, which means that for every $(u,v)\in\mathbb{A}_3$ we have

$$\int_{-\infty}^{\infty}\int_0^{\infty} F\bigl((u,v)*(a,b)\bigr)\,\frac{da}{a}\,db \;=\; \int_{-\infty}^{\infty}\int_0^{\infty} F(a,b)\,\frac{da}{a}\,db$$

for every function F on $\mathbb{A}_3$ that is integrable with respect to $\frac{da}{a}\,db$. What is the right Haar measure?

Remark: The (isomorphic) groups $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ are called the *affine group*. The properties of the affine group should be contrasted with those of the Heisenberg affine group discussed in Exercise 11.2.

12.2 Wavelet Frames and Wavelet Sets

We saw in Section 11.2 that it is easy to construct Gabor frames with compactly supported generators. Moreover, we can construct redundant Gabor frames with generators that are as smooth as we like. These "Painless Nonorthogonal Expansions" are due to Daubechies, Grossmann, and Meyer [DGM86]. In that same paper they also observed that those techniques can be used to construct wavelet frames for $L^2(\mathbf{R})$.

The key to the construction is the fact that if I is an interval of length b^{-1} then $\{b^{1/2}e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$. Now, a Gabor system $\mathcal{G}(g,a,b) = \{e^{2\pi ibnx}g(x-ak)\}_{k,n\in\mathbf{Z}}$ explicitly includes exponential functions as part of its definition, but exponential functions do not appear in the definition of the wavelet system $\mathcal{W}(\psi,a,b) = \{a^{n/2}\psi(a^nx-bk)\}_{k,n\in\mathbf{Z}}$. However, the Fourier transform converts translations into modulations and dilations into reciprocal dilations, so if we move to the "Fourier side" then we will see exponential functions. Letting $\widehat{\mathcal{W}}(\psi,a,b)$ denote the image of $\mathcal{W}(\psi,a,b)$ under the Fourier transform, we have

$$
\begin{aligned}
\widehat{\mathcal{W}}(\psi,a,b) &= \big\{(D_{a^n}T_{bk}\psi)^\wedge\big\}_{k,n\in\mathbf{Z}} \\
&= \big\{D_{a^{-n}}M_{-bk}\widehat{\psi}\big\}_{k,n\in\mathbf{Z}} \\
&= \big\{a^{-n/2}e^{-2\pi ibka^{-n}\xi}\,\widehat{\psi}(a^{-n}\xi)\big\}_{k,n\in\mathbf{Z}} && (12.3)\\
&= \big\{a^{n/2}e^{2\pi ibka^{n}\xi}\,\widehat{\psi}(a^{n}\xi)\big\}_{k,n\in\mathbf{Z}}. && (12.4)
\end{aligned}
$$

Since the Fourier transform is unitary, $\mathcal{W}(\psi,a,b)$ is a frame if and only if $\widehat{\mathcal{W}}(\psi,a,b)$ is a frame.

Example 12.2. Since the Fourier transform maps $L^2(\mathbf{R})$ onto itself, there is a function $\psi \in L^2(\mathbf{R})$ such that $\widehat{\psi} = \chi_{[1,2]}$. Explicitly, ψ is the modulated sinc function $\psi(x) = e^{3\pi ix}\,(\sin\pi x)/(\pi x)$. By equation (12.3), the image of the dyadic wavelet system $\mathcal{W}(\psi) = \mathcal{W}(\psi,2,1)$ under the Fourier transform is

$$\widehat{\mathcal{W}}(\psi) = \big\{2^{-n/2}e^{-2\pi ik2^{-n}\xi}\,\chi_{[1,2]}(2^{-n}\xi)\big\}_{k,n\in\mathbf{Z}}.$$

With n fixed,

$$\big\{2^{-n/2}e^{-2\pi ik2^{-n}\xi}\,\chi_{[1,2]}(2^{-n}\xi)\big\}_{k\in\mathbf{Z}}$$

is an orthonormal basis for $L^2[2^n,2^{n+1}]$. Since $\cup_{n\in\mathbf{Z}}[2^n,2^{n+1}] = (0,\infty)$, we conclude that $\widehat{\mathcal{W}}(\psi)$ is an orthonormal basis for $L^2[0,\infty)$. We were hoping for an orthonormal basis for $L^2(\mathbf{R})$ but didn't quite get it. On the other hand, if we set $\psi_1 = \psi$ and let ψ_2 be the function such that $\widehat{\psi_2} = \chi_{[-2,-1]}$ then $\widehat{\mathcal{W}}(\psi_2)$ is an orthonormal basis for $L^2(-\infty,0]$, and hence $\widehat{\mathcal{W}}(\psi_1)\cup\widehat{\mathcal{W}}(\psi_2)$ is an orthonormal basis for $L^2(\mathbf{R})$. Applying the unitary of the Fourier transform, we conclude that the union of the two dyadic wavelet systems $\mathcal{W}(\psi_1)$ and $\mathcal{W}(\psi_2)$ forms an orthonormal basis for $L^2(\mathbf{R})$. Individually, $\mathcal{W}(\psi_1)$ is an orthonormal basis for the closed subspace

$$H^2_+(\mathbf{R}) = \big\{f\in L^2(\mathbf{R}) : \operatorname{supp}(\widehat{f}\,)\subseteq[0,\infty)\big\},$$

while $\mathcal{W}(\psi_2)$ is an orthonormal basis for

$$H^2_-(\mathbf{R}) = \big\{f\in L^2(\mathbf{R}) : \operatorname{supp}(\widehat{f}\,)\subseteq(-\infty,0]\big\}. \qquad \diamondsuit$$

The Painless Nonorthogonal Expansions have a very similar construction except that we allow the supports of the Fourier transforms of our functions to overlap. Exercise 12.3 shows, for example, that if we choose a generator $\psi_1 \in L^2(\mathbf{R})$ such that

$$\operatorname{supp}(\widehat{\psi}_1) \subseteq [1, 1+b^{-1}] \qquad \text{and} \qquad \sum_{n\in\mathbf{Z}} |\widehat{\psi}_1(a^n\xi)|^2 = b \text{ a.e.}, \tag{12.5}$$

then $\widehat{\mathcal{W}}(\psi_1, a, b) = \{a^{n/2} e^{2\pi i bka^n\xi}\, \widehat{\psi}_1(a^n\xi)\}_{k,n\in\mathbf{Z}}$ is a Parseval frame for $L^2[0,\infty)$. Figure 12.2 illustrates one possible choice of $\widehat{\psi}_1$ corresponding to the parameters $a = 2$ and $b = 1/2$. Combining this with a similar function ψ_2 whose Fourier transform is supported on the negative halfline, we can create a Parseval frame for $L^2(\mathbf{R})$ of the form $\widehat{\mathcal{W}}(\psi_1, a, b) \cup \widehat{\mathcal{W}}(\psi_2, a, b)$. Since the Fourier transform is unitary, $\mathcal{W}(\psi_1, a, b) \cup \mathcal{W}(\psi_2, a, b)$ is therefore a Parseval frame for $L^2(\mathbf{R})$. We can do this with functions ψ_1, ψ_2 whose Fourier transforms are both compactly supported and as smooth as we like. Since the Fourier transform interchanges smoothness and decay, the smoother that $\widehat{\psi}_1$, $\widehat{\psi}_2$ are, the faster that ψ_1, ψ_2 will decay at infinity.

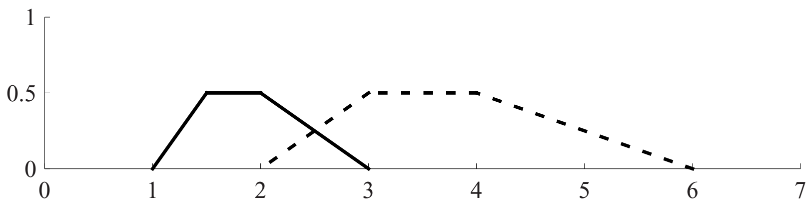

Fig. 12.2. Graphs of $\widehat{\psi}_1(\xi)^2$ and $\widehat{\psi}_1(2\xi)^2$ for $a = 2$ and $b = 1/2$. We have $\sum_{n\in\mathbf{Z}} |\widehat{\psi}(2^n\xi)|^2 = 1/2$ on $(0,\infty)$.

Except that we needed to use two generators instead of one to achieve a frame for $L^2(\mathbf{R})$, this "painless" construction of a wavelet frame is very similar to that of a Gabor frame. The construction even suggests that there may be a Nyquist density for wavelet frames. For, in order for equation (12.5) to hold, we must have $a \le 1+b^{-1}$. Must this same restriction apply to generic wavelet frames? To see that the answer is no, fix any function $\psi \in L^2(\mathbf{R})$, parameters $a > 1$ and $b > 0$, and scale $j \in \mathbf{Z}$. If we define $\phi(x) = D_{a^{-j}}\psi(x) = a^{-j/2}\psi(a^{-j}x)$ then

$$\begin{aligned}
\mathcal{W}(\phi, a, a^j b) &= \big\{a^{n/2}\phi(a^n x - a^j bk)\big\}_{k,n\in\mathbf{Z}} \\
&= \big\{a^{n/2}a^{-j/2}\psi\big(a^{-j}(a^n x - a^j bk)\big)\big\}_{k,n\in\mathbf{Z}} \\
&= \big\{a^{(n-j)/2}\psi(a^{n-j}x - bk)\big\}_{k,n\in\mathbf{Z}} \\
&= \mathcal{W}(\psi, a, b).
\end{aligned}$$

The new wavelet system $\mathcal{W}(\phi, a, a^j b)$ has exactly the same closed span as the original system $\mathcal{W}(\psi, a, b)$, but has a different translation parameter. Moreover, $a^j b$ can be as large or as small as we like. The following theorem due to Dai, Larson, and Speegle [DLS97] is even more surprising.

Theorem 12.3. *Given any $a > 1$ and $b > 0$, there exists a function $\psi \in L^2(\mathbf{R})$ such that $\widehat{\psi}$ is the characteristic function of a compact set and the wavelet system $\mathcal{W}(\psi, a, b)$ is an orthonormal basis for $L^2(\mathbf{R})$.*

More generally, let A be any expansive *$d \times d$ matrix (i.e., every eigenvalue λ of A satisfies $|\lambda| > 1$), and let $b > 0$ be any positive number. Then there exists a compact set $E \subseteq \mathbf{R}^d$ such that if we let $\psi \in L^2(\mathbf{R}^d)$ be the function satisfying $\widehat{\psi} = \chi_E$, then*

$$\big\{ |\det(A)|^{n/2} \psi(A^n x - bk) \big\}_{n \in \mathbf{Z}, k \in \mathbf{Z}^d}$$

is an orthonormal basis for $L^2(\mathbf{R}^d)$. $\diamondsuit$

In contrast, if $\mathcal{G}(g, a, b)$ is a Gabor Riesz basis for $L^2(\mathbf{R})$ then necessarily $ab = 1$, and if $\mathcal{G}(g, a, b)$ is a frame then $ab \le 1$.

A set E of the type appearing in Theorem 12.3 is called a *wavelet set*. Exercise 12.4 shows that, for the one-dimensional case, the wavelet sets are precisely those subsets of $\mathbf{R}$ that "simultaneously tile by translation and dilation." A similar characterization holds for higher dimensions. However, it is *not obvious* that such sets exist, especially in higher dimensions.

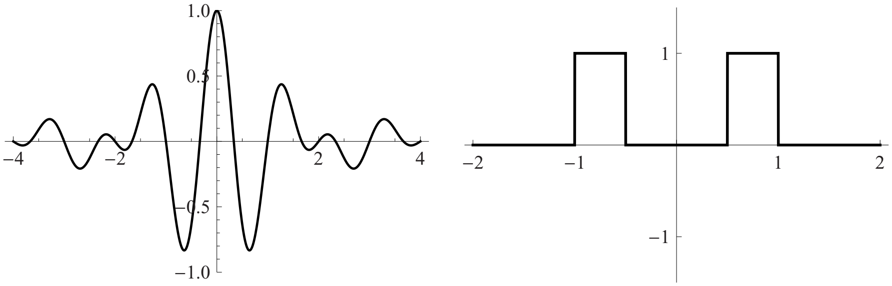

Fig. 12.3. The Shannon wavelet ψ (left) and its Fourier transform $\widehat{\psi}$ (right).

Example 12.4 (The Shannon Wavelet). The simplest example of a wavelet set in one dimension is $E = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. This set tiles by integer translations, i.e., the integer translates of E cover $\mathbf{R}$ and the overlaps of these translates have measure zero:

$$\bigcup_{k \in \mathbf{Z}} (E + k) \;=\; \mathbf{R} \qquad \text{and} \qquad |(E + j) \cap (E + k)| \;=\; 0 \text{ if } j \neq k.$$

Moreover, this set tiles by dyadic dilations:

$$\bigcup_{n\in\mathbf{Z}} (2^n E) \;=\; \mathbf{R}\backslash\{0\} \qquad \text{and} \qquad |(2^m E)\cap(2^n E)| \;=\; 0 \text{ if } m \neq n.$$

Exercise 12.4 therefore implies that $\mathcal{W}(\psi) = \{D_{2^n} T_k \psi\}_{k,n\in\mathbf{Z}}$ is a dyadic orthonormal basis for $L^2(\mathbf{R})$, where $\psi \in L^2(\mathbf{R})$ satisfies $\widehat{\psi} = \chi_E$. This function ψ is called the *Shannon wavelet*, and is given explicitly by

$$\psi(x) \;=\; \frac{\sin 2\pi x}{\pi x} - \frac{\sin \pi x}{\pi x}. \tag{12.6}$$

The Shannon wavelet ψ and its Fourier transform $\widehat{\psi} = \chi_E$ are shown in Figure 12.3. ◇

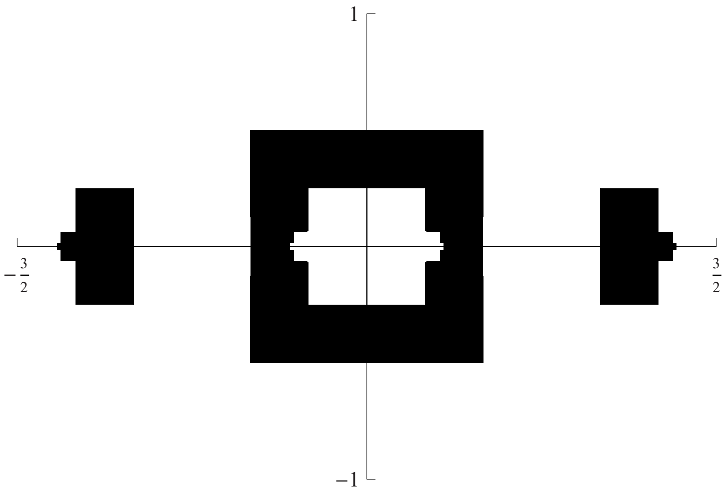

Fig. 12.4. The wedding cake wavelet set.

The wavelet set E for the Shannon wavelet is fairly nice, but because of the dilation/translation tiling properties of wavelet sets, they often exhibit a kind of fractal or self-similar appearance. Figure 12.4 displays a wavelet set corresponding to the dilation matrix $A = 2I$ and translation parameter $b = 1$. This set, discovered by Dai, Larson, and Speegle, tiles by translations and dilations in the sense that

$$\bigcup_{k\in\mathbf{Z}^2} (E+k) \;=\; \mathbf{R}^2 \qquad \text{and} \qquad \bigcup_{n\in\mathbf{Z}} (2^n E) \;=\; \mathbf{R}^2,$$

both with overlaps of measure zero. This set E is disconnected, with three separate connected components. It is also fractal-like; turned sideways the left-hand and right-hand components have infinitely many "tiers" that decrease rapidly in size, reminiscent of a wedding cake. Consequently, E is known as the "wedding cake" wavelet set. Many other wavelet sets are now known, some "complicated" and some "simple." We refer to [DLS97], [BMM99], [BS06], [Mer08] for more details and examples.

As fascinating as wavelet sets are, they have the disadvantage that their Fourier transforms are characteristic functions and hence are discontinuous. The Haar system has a similar disadvantage, although its discontinuities occur on the time side rather than the Fourier side. This suggests that while there may not be a Nyquist density for wavelets, there may still be some type of Balian–Low Theorem. Perhaps all wavelet Riesz bases must be "bad" in some sense? We will address this question in Section 12.4 by taking a completely different approach to the construction of wavelet bases, and we will see that the situation for wavelet bases is very different from that for Gabor bases.

There are many other interesting issues involving wavelets in higher dimensions that we will not be able to pursue. We will give just one example of a Haar-like wavelet basis for $L^2(\mathbf{R}^2)$, and then return to the one-dimensional setting for the remainder of the chapter.

Example 12.5 (The Twin Dragon). A *quincunx* was a Roman coin worth five-twelfths of the standard bronze coin *as*. Today the word *quincunx* refers to the geometrical pattern of five dots that represents the number five on a playing die. The *quincunx matrix* is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{12.7}$$

Note that A is not only expansive in the sense of Theorem 12.3, but also has integer entries. Consequently A maps the set $\mathbf{Z}^2$ into itself. In particular, the five points

$$(0,0), \quad (1,0), \quad (0,1), \quad (-1,0), \quad (0,-1)$$

are mapped by A to the five points

$$(0,0), \quad (1,1), \quad (1,-1), \quad (-1,-1), \quad (-1,1),$$

which form a quincunx pattern. The set $A(\mathbf{Z}^2)$ can be viewed as an infinite repetition of this quincunx pattern. By a stretch of the imagination, the quincunx lends its name to the matrix A. Since the matrix

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{12.8}$$

satisfies $B(\mathbf{Z}^2) = A(\mathbf{Z}^2)$, it is also called the quincunx matrix.

It can be shown that there is a unique compact set Q that satisfies

$$Q = A^{-1}(Q) \cup A^{-1}(Q+d) \tag{12.9}$$

where $d = (1,0)$. The set Q is shown in Figure 12.5, and the sets $A^{-1}(Q)$ and $A^{-1}(Q+d)$ can be seen in Figure 12.6. In the language of fractals, Q

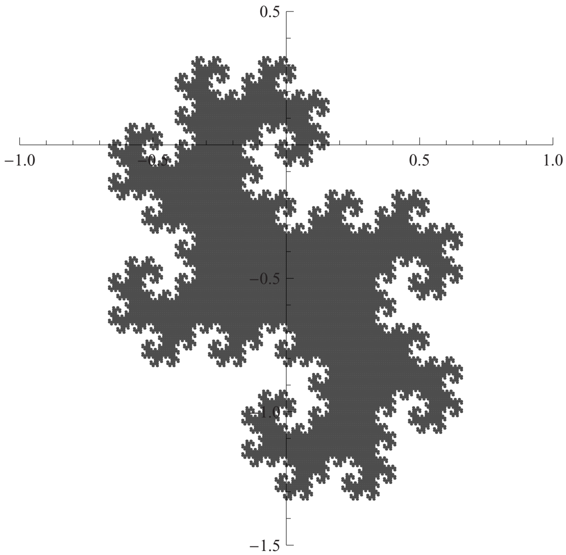

Fig. 12.5. The twin dragon.

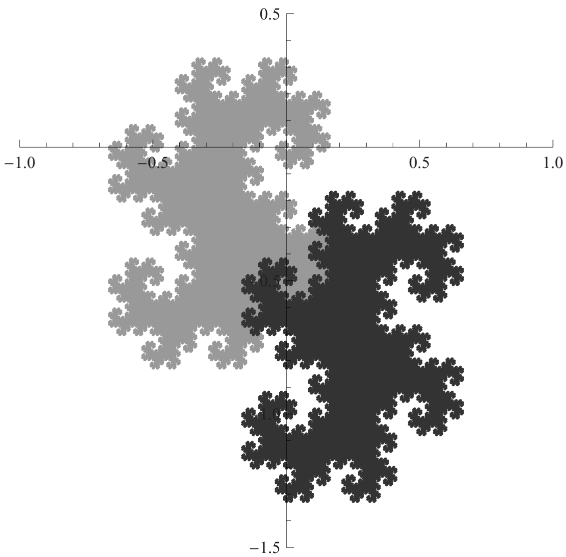

Fig. 12.6. The twin dragon wavelet ψ takes the value 1 on the light region and -1 on the dark region.

is the *attractor* of the iterated function system generated by the two maps $w_1(x) = A^{-1}x$ and $w_2(x) = A^{-1}(x + d)$. This is a two-dimensional analogue of the fact that the unit interval $[0, 1]$ is the unique compact set in $\mathbf{R}$ that satisfies

$$[0,1] \;=\; \frac{1}{2}\big([0,1]\big) \,\cup\, \frac{1}{2}\big([0,1]+1\big).$$

The set Q satisfying (12.9) is a well-known fractal called the *twin dragon*. The two smaller sets $A^{-1}(Q)$ and $A^{-1}(Q+d)$ are shrunken, rotated versions of Q that intersect only along their boundaries (which have measure zero). Thus Q is self-similar in the sense that it can be built out of two smaller copies of itself. For more on fractals and iterated function systems, we refer to [Hut81], [YHK97], [Fal03].

The self-similarity of the set Q can also be expressed in the functional equation

$$\chi_Q(x) \;=\; \chi_Q(Ax) + \chi_Q(Ax-d) \text{ a.e.},$$

which should be compared to the equation

$$\chi_{[0,1]}(x) \;=\; \chi_{[0,1]}(2x) + \chi_{[0,1]}(2x-1) \text{ a.e.}$$

that is satisfied by the box function $\chi_{[0,1]}$. Just as the Haar wavelet

$$\psi(x) \;=\; \chi_{[0,1]}(2x) - \chi_{[0,1]}(2x-1)$$

generates a wavelet orthonormal basis for $L^2(\mathbf{R})$, the function

$$\psi(x) \;=\; \chi_Q(Ax) - \chi_Q(Ax-d)$$

generates an orthonormal basis for $L^2(\mathbf{R}^2)$ of the form

$$\{2^{n/2}\psi(A^n x - k)\}_{n\in\mathbf{Z},k\in\mathbf{Z}^2}.$$

In contrast to the wavelet sets described in Theorem 12.3, here it is the function ψ rather than its Fourier transform $\widehat{\psi}$ that is compactly supported. ◇

A *dilation matrix* is an expansive $d \times d$ matrix A that has integer entries. Gröchenig and Madych [GM92] proved that there are Haar-like orthonormal wavelet bases associated with "most" dilation matrices. The issue of exactly which dilation matrices for which this is true involves interesting number-theoretic and other issues; see [LagW95].

Exercises

12.3. Fix $a > 1$, $b > 0$, $c > 0$, and let $\psi \in L^2(\mathbf{R})$ be such that $\mathrm{supp}(\widehat{\psi}) \subseteq [c, c+b^{-1}]$ and $\sum_{n\in\mathbf{Z}} |\widehat{\psi}(a^n\xi)|^2 = b$ for almost every $\xi \ge 0$.

(a) Show that $\widehat{\mathcal{W}}(\psi,a,b)$ is a Parseval frame for $L^2[0,\infty)$, and $\mathcal{W}(\psi,a,b)$ is a Parseval frame for $H^2_+(\mathbf{R}) = \{f \in L^2(\mathbf{R}) : \mathrm{supp}(\widehat{f}) \subseteq [0,\infty)\}$.

(b) Show that if a, b are chosen correctly, then we can choose ψ so that $\widehat{\psi}$ is continuous. Can we choose ψ so that $\widehat{\psi}$ is infinitely differentiable?

(c) Find ψ_1, ψ_2 with continuous Fourier transforms so that $\mathcal{W}(\psi_1) \cup \mathcal{W}(\psi_2)$ is a Parseval frame for $L^2(\mathbf{R})$.

12.4. This exercise will consider wavelet sets in one dimension. For simplicity we fix $b = 1$, but allow any dilation parameter $a > 1$.

Let E be a measurable subset of $\mathbf{R}$. We say that E *tiles by translation* if $\cup_{k\in\mathbf{Z}}\,(E+k) = \mathbf{R}$ up to a set of measure zero and the overlaps $(E+j)\cap(E+k)$ have measure zero for $j \neq k$. Similarly, E *tiles by dilation* if $\cup_{n\in\mathbf{Z}}\,(a^n)E = \mathbf{R}$ up to a set of measure zero and $(a^m E)\cap(a^n E)$ has measure zero when $m \neq n$.

(a) Prove that if E tiles by translation then $\{e^{2\pi ikx}\}_{k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(E)$.

(b) Show that if E tiles both by translation and by dilation and $\widehat{\psi} = \chi_E$, then $\{D_{a^n}T_k\psi\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$. Therefore E is a wavelet set in this case.

(c) Suppose that $\{D_{a^n}T_k\psi\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$, where $\widehat{\psi} = \chi_E$. Show that E tiles by translation and by dilation.

(d) Show that $E = [-1,-\frac{1}{2}]\cup[\frac{1}{2},1]$ is a wavelet set if we take $a = 2$. Verify the explicit formula for the Shannon wavelet ψ given in equation (12.6).

12.5. Let $\mathcal{H}(\mathbf{R}^d)$ denote the collection of all compact, nonempty subsets of $\mathbf{R}^d$. Given $B \in \mathbf{R}^d$ define

$$\mathrm{dist}(x,B) \;=\; \inf\bigl\{|x-y| : y \in B\bigr\} \quad \text{and} \quad B_\varepsilon \;=\; \bigl\{x \in \mathbf{R}^d : \mathrm{dist}(x,B) < \varepsilon\bigr\}.$$

Show that

$$\mathrm{d}(B,C) \;=\; \inf\bigl\{\varepsilon > 0 : B \subseteq C_\varepsilon \text{ and } C \subseteq B_\varepsilon\bigr\}$$

is a metric on $\mathcal{H}(\mathbf{R}^d)$.

Remark: In fact, $\mathcal{H}(\mathbf{R}^d)$ is complete with respect to this metric [Hut81].

12.6. Let A be the quincunx matrix given in equation (12.7) and let $d = (1,0)$. Then $f(K) = A^{-1}(K) \cup A^{-1}(K+d)$ maps $\mathcal{H}(\mathbf{R}^2)$ into itself, and it can be shown that f is contractive with respect to the metric given in Exercise 12.5. This implies that f has a unique fixed point Q, and given any set $K_0 \in \mathcal{H}(\mathbf{R}^2)$ the iteration $K_{n+1} = f(K_n)$ converges to Q. Use this to plot the twin dragon Q (take $K_0 = \{0\}$).

12.7. What is the attractor Q if we replace the matrix A in Exercise 12.6 by the quincunx matrix B defined in equation (12.8)?

12.3 Frame Bounds and the Admissibility Condition

We constructed a variety of wavelet frames and bases in the preceding section. Some of these, like the Haar system or the Shannon wavelet system, are orthonormal bases for $L^2(\mathbf{R})$. Others, such as the Painless Nonorthogonal Expansions, combine a frame for the space $H^2_+(\mathbf{R})$ with another frame for $H^2_-(\mathbf{R})$ in order to obtain a frame for $L^2(\mathbf{R})$ (see Example 12.2). In any case, we will see in this section that we can derive interesting connections between the wavelet system parameters a, b, the frame bounds A, B, and a new quantity called the *admissibility constant* for the generator ψ. We will focus on wavelet frames for $L^2(\mathbf{R})$ that require a single generator, but the results can be modified to apply to frames for $H^2_\pm(\mathbf{R})$.

The following result should be compared to Theorem 11.6 and Corollary 11.7 for Gabor frames. The proof we give is similar in spirit to the proof of those results. However, we have to be considerably more careful because while $\{b^{1/2}a^{n/2}e^{2\pi ibka^n\xi}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(I)$ where I is an interval of length $b^{-1}a^{-n}$, these lengths depend on n. The argument given here is due to Chui and Shi [CS93].

Theorem 12.6. *If $\psi \in L^2(\mathbf{R})$ and $a > 1$, $b > 0$ are such that $\mathcal{W}(\psi, a, b)$ is a frame for $L^2(\mathbf{R})$ with frame bounds A, B, then the following statements hold.*

(a) *$\widehat{\psi}$ is bounded, and*

$$Ab \;\le\; \sum_{n\in\mathbf{Z}} |\widehat{\psi}(a^n\xi)|^2 \;\le\; Bb \;\; a.e. \tag{12.10}$$

(b) *We have*

$$Ab\ln a \;\le\; \int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\,d\xi,\; \int_{-\infty}^0 \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\,d\xi \;\le\; Bb\ln a.$$

(c) *If $\mathcal{W}(\psi, a, b)$ is a Parseval frame for $L^2(\mathbf{R})$, then*

$$\int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\,d\xi \;=\; \int_{-\infty}^0 \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\,d\xi \;=\; b\ln a.$$

(d) *$\mathcal{W}(\psi, a, b)$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if it is a Parseval frame and $\|\psi\|_{L^2} = 1$.*

Proof. (a) For this proof, given $E \subseteq \mathbf{R}$ we will consider $L^2(E)$ to be a subspace of $L^2(\mathbf{R})$ by regarding each function $f \in L^2(E)$ to be zero a.e. outside of E.

We will work on the Fourier side. By equation (12.4), our hypothesis is that

$$\widehat{\mathcal{W}}(\psi, a, b) \;=\; \{a^{n/2}e^{2\pi ibka^n\xi}\,\widehat{\psi}(a^n\xi)\}_{k,n\in\mathbf{Z}}$$

is a frame for $L^2(\mathbf{R})$ with frame bounds A, B. That is, for every $f \in L^2(\mathbf{R})$ we have

$$A\,\|f\|_{L^2}^2 \;\le\; \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f(\xi),\, a^{n/2}e^{2\pi ibka^n\xi}\,\widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \;\le\; B\,\|f\|_{L^2}^2.$$

Define the following series for $M < N \in \mathbf{Z}$:

$$W_{M,N}(\xi) \;=\; b^{-1} \sum_{n=M}^{N} |\widehat{\psi}(a^n\xi)|^2,$$

$$W_N(\xi) \;=\; b^{-1} \sum_{n=-\infty}^{N} |\widehat{\psi}(a^n\xi)|^2,$$

$$W(\xi) \;=\; b^{-1} \sum_{n\in\mathbf{Z}} |\widehat{\psi}(a^n\xi)|^2.$$

Note that the function $W_{M,N}$ is integrable on $\mathbf{R}$ since $\widehat{\psi} \in L^2(\mathbf{R})$. While W will not be integrable on $\mathbf{R}$, it is "dilationally periodic" in the sense that $W_0(a\xi) = W_0(\xi)$.

Step 1. Given $N \in \mathbf{N}$, let I be any closed interval in $(0,\infty)$ whose length satisfies $|I| \le b^{-1}a^{-N}$. Then for $n \le N$ the interval I is contained in an interval J of length $b^{-1}a^{-n}$. If $g \in L^2(I)$ then $\|g\|_{L^2(I)} = \|g\|_{L^2(J)}$, so the Plancherel Equality implies that for every $n \le N$ and $g \in L^2(I)$ we have

$$\|g\|_{L^2}^2 \;=\; \sum_{k\in\mathbf{Z}} \bigl|\bigl\langle g,\, b^{1/2}a^{n/2}e^{2\pi ibka^n\xi}\bigr\rangle\bigr|^2. \tag{12.11}$$

Now let f be any bounded function supported within I. Since $f(\xi)\,\overline{\widehat{\psi}(a^n\xi)}$ belongs to $L^2(I)$, we apply equation (12.11) to this function and compute that, for any $M < N$,

$$\begin{aligned}
&\sum_{n=M}^{N}\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f(\xi),\, a^{n/2}e^{2\pi ibka^n\xi}\,\widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \\
&\quad=\; \sum_{n=M}^{N} b^{-1} \sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f(\xi)\,\overline{\widehat{\psi}(a^n\xi)},\, b^{1/2}a^{n/2}e^{2\pi ibka^n\xi}\bigr\rangle\bigr|^2 \\
&\quad=\; b^{-1}\sum_{n=M}^{N} \bigl\|f(\xi)\,\overline{\widehat{\psi}(a^n\xi)}\bigr\|_{L^2}^2 \\
&\quad=\; b^{-1}\sum_{n=M}^{N} \int_I |f(\xi)|^2\,|\widehat{\psi}(a^n\xi)|^2\,d\xi \\
&\quad=\; \int_I |f(\xi)|^2\,W_{M,N}(\xi)\,d\xi.
\end{aligned} \tag{12.12}$$

Step 2. Fix $M < N$ and $\xi_0 > 0$, and let $h > 0$ be small enough that the interval $I_h = [\xi_0, \xi_0 + h]$ satisfies $h = |I_h| \le b^{-1}a^{-N}$. Applying equation (12.12) to the function $f_h = h^{-1/2}\chi_{I_h}$ and taking the upper frame bound into consideration, we see that

$$\begin{aligned}\frac{1}{h}\int_{\xi_0}^{\xi_0+h} W_{M,N}(\xi)\,d\xi \;&=\; \sum_{n=M}^{N}\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f_h(\xi),\, a^{n/2}e^{2\pi ibka^n\xi}\,\widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \\ &\le\; \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f_h(\xi),\, a^{n/2}e^{2\pi ibka^n\xi}\,\widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \\ &\le\; B\,\|f_h\|_{L^2}^2 \;=\; B.\end{aligned}$$

Since $W_{M,N}$ is integrable, if we let $h \to 0$ then the Lebesgue Differentiation Theorem (Theorem A.30) tells us that for almost every $\xi_0 > 0$ we have

$$W_{M,N}(\xi_0) \;=\; \lim_{h\to 0}\frac{1}{h}\int_{\xi_0}^{\xi_0+h} W_{M,N}(\xi)\,d\xi \;\le\; B.$$

Thus $W_{M,N} \le B$ a.e. on $(0,\infty)$. Letting $M \to -\infty$ and $N \to \infty$, it follows that $W \le B$ a.e. on $(0,\infty)$, and a similar argument shows that this inequality also holds almost everywhere on $(-\infty,0)$. Consequently we have proved that the upper inequality in equation (12.10) holds. The lower inequality will take more effort.

Step 3. Fix $\xi_0 > 0$ and $\varepsilon > 0$. Then there exists some $N \in \mathbf{Z}$ such that

$$\int_{a^N\xi_0}^{\infty} |\widehat{\psi}(\xi)|^2\,d\xi \;<\; \varepsilon.$$

Again set $I_h = [\xi_0, \xi_0 + h]$, this time with $h > 0$ small enough that

$$h \;=\; |I_h| \;\le\; b^{-1}a^{-N} \qquad \text{and} \qquad \xi_0 + h \;\le\; a\xi_0.$$

Taking $f_h = h^{-1/2}\chi_{I_h}$ we have, just as in equation (12.12),

$$\sum_{n=-\infty}^{N}\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f_h(\xi),\, a^{n/2}e^{2\pi ibka^n\xi}\,\widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \;=\; \frac{1}{h}\int_{\xi_0}^{\xi_0+h} W_N(\xi)\,d\xi. \quad (12.13)$$

We need to estimate the series analogous to the one appearing in equation (12.13), but with n running from $N+1$ to infinity.

Fix $n > N$. Since $f_h(a^{-n}\xi)\,\overline{\widehat{\psi}(\xi)}$ is integrable on $\mathbf{R}$, its b^{-1}-periodization

$$F(\xi) \;=\; \sum_{j\in\mathbf{Z}} f_h\bigl(a^{-n}(\xi + b^{-1}j)\bigr)\,\overline{\widehat{\psi}\bigl(\xi + b^{-1}j\bigr)} \qquad (12.14)$$

belongs to $L^1[0,b^{-1}]$ (see Exercise 10.13). Since I_h has length h, given ξ there can be at most $C = ba^nh + 1$ values of $j \in \mathbf{Z}$ such that $a^{-n}(\xi + b^{-1}j) \in I_h$. Substituting equation (12.14) and applying the Cauchy–Bunyakovski–Schwarz Inequality,

$$\int_0^{b^{-1}} |F(\xi)|^2\,d\xi \le \int_0^{b^{-1}} \Bigl(\sum_{j\in\mathbf{Z}} \bigl|f_h\bigl(a^{-n}(\xi+b^{-1}j)\bigr)\bigr|^2\Bigr) \Bigl(\sum_{j\in\mathbf{Z}} \bigl|\widehat{\psi}\bigl(\xi+b^{-1}j\bigr)\bigr|^2\Bigr)\,d\xi$$

$$\le \frac{C}{h}\int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} \bigl|\widehat{\psi}\bigl(\xi+b^{-1}j\bigr)\bigr|^2\,d\xi$$

$$= \frac{C}{h}\,\|\widehat{\psi}\|_{L^2}^2 \;<\; \infty.$$

Thus $F \in L^2[0,b^{-1}]$. Hence, by the Plancherel Equality,

$$\|F\|_{L^2}^2 \;=\; \sum_{j\in\mathbf{Z}} \bigl|\bigl\langle F(\xi),\, b^{1/2}e^{2\pi i bj\xi}\bigr\rangle\bigr|^2,$$

where the norm and inner product in the preceding equation are taken on the interval $[0,b^{-1}]$. Using this, the fact that $\mathrm{supp}(f) \subseteq [0,\infty)$, and the periodicity of the functions $e^{2\pi i b\xi}$, we compute that

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f_h(\xi),\, a^{n/2}e^{2\pi i bka^n\xi}\,\widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \\
&\quad= \sum_{k\in\mathbf{Z}} \biggl|\int_0^\infty f_h(\xi)\,a^{n/2}\,e^{2\pi i bka^n\xi}\,\overline{\widehat{\psi}(a^n\xi)}\,d\xi\biggr|^2 \\
&\quad= \sum_{k\in\mathbf{Z}} \biggl|\int_0^\infty f_h(a^{-n}\xi)\,a^{-n/2}\,e^{2\pi i bk\xi}\,\overline{\widehat{\psi}(\xi)}\,d\xi\biggr|^2 \qquad \text{(change variables)} \\
&\quad= a^{-n}\sum_{k\in\mathbf{Z}} \biggl|\sum_{j\in\mathbf{Z}}\int_0^{b^{-1}} f_h\bigl(a^{-n}(\xi+b^{-1}j)\bigr)\,\overline{\widehat{\psi}\bigl(\xi+b^{-1}j\bigr)}\,e^{2\pi i bk(\xi+b^{-1}j)}\;d\xi\biggr|^2 \\
&\quad= a^{-n}\sum_{k\in\mathbf{Z}} \biggl|\int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} f_h\bigl(a^{-n}(\xi+b^{-1}j)\bigr)\,\overline{\widehat{\psi}\bigl(\xi+b^{-1}j\bigr)}\,e^{2\pi i bk\xi}\;d\xi\biggr|^2 \\
&\quad= b^{-1}a^{-n}\sum_{k\in\mathbf{Z}} \bigl|\bigl\langle F(\xi),\, b^{1/2}e^{2\pi i bk\xi}\bigr\rangle\bigr|^2 \\
&\quad= b^{-1}a^{-n}\|F\|_{L^2}^2.
\end{aligned} \tag{12.15}$$

It is important now that we recall that F is b^{-1}-periodic. Using this periodicity and the definition of f_h, we have

$$\begin{aligned}
\|F\|_{L^2}^2 \;&=\; \int_0^{b^{-1}} |F(\xi)|\,|F(\xi)|\,d\xi \\
&\le \int_0^{b^{-1}} \sum_{j\in\mathbf{Z}} \bigl|f_h\bigl(a^{-n}(\xi+b^{-1}j)\bigr)\bigr|\,\bigl|\widehat{\psi}\bigl(\xi+b^{-1}j\bigr)\bigr|\,\bigl|F(\xi+b^{-1}j)\bigr|\,d\xi
\end{aligned}$$

$$= \int_0^\infty |f_h(a^{-n}\xi)|\,|\widehat{\psi}(\xi)|\,|F(\xi)|\,d\xi$$

$$= h^{-1/2} \int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|\,|F(\xi)|\,d\xi$$

$$\leq h^{-1/2}\left(\int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|^2\,d\xi\right)^{1/2} \left(\int_{a^n\xi_0}^{a^n(\xi_0+h)} |F(\xi)|^2\,d\xi\right)^{1/2}. \quad (12.16)$$

As $|F|^2$ is b^{-1}-periodic, it has the same integral on any interval of length b^{-1}. The interval $[a^n\xi_0, a^n(\xi_0+h)]$ has length $a^n h$, so it can be covered by $ba^n h+1$ intervals of length b^{-1}. Therefore

$$\int_{a^n\xi_0}^{a^n(\xi_0+h)} |F(\xi)|^2\,d\xi \;\leq\; (ba^n h+1)\int_0^{b^{-1}} |F(\xi)|^2\,d\xi \;=\; (ba^n h+1)\,\|F\|_{L^2}^2.$$

Combining this with equation (12.16), we see that

$$\|F\|_{L^2}^2 \;\leq\; h^{-1/2}\left(\int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|^2\,d\xi\right)^{1/2} \Big((ba^n h+1)\,\|F\|_{L^2}^2\Big)^{1/2}.$$

Simplifying and dividing both sides by $\|F\|_{L^2}$ yields

$$\|F\|_{L^2} \;\leq\; \left(\frac{ba^n h+1}{h}\right)^{1/2} \left(\int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|^2\,d\xi\right)^{1/2}.$$

Squaring, we can continue equation (12.15) as follows:

$$b^{-1}a^{-n}\,\|F\|_{L^2}^2 \;\leq\; \big(1+b^{-1}a^{-n}h^{-1}\big)\int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|^2\,d\xi. \quad (12.17)$$

Now, since $\xi_0+h < a\xi_0$, the intervals $[a^n\xi_0, a^n(\xi_0+h)]$ are disjoint. Therefore, combining equations (12.16) and (12.17) and summing over $n > N$, we find that

$$\sum_{n=N+1}^{\infty}\sum_{k\in\mathbf{Z}} \big|\big\langle f_h(\xi),\, a^{n/2}e^{2\pi i bka^n\xi}\,\widehat{\psi}(a^n\xi)\big\rangle\big|^2$$

$$\leq \sum_{n=N+1}^{\infty}\int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|^2\,d\xi + \sum_{n=N+1}^{\infty} b^{-1}a^{-n}h^{-1}\int_{a^n\xi_0}^{a^n(\xi_0+h)} |\widehat{\psi}(\xi)|^2\,d\xi$$

$$\leq \int_{a^N\xi_0}^{\infty} |\widehat{\psi}(\xi)|^2\,d\xi \;+\; b^{-1}h^{-1}\sum_{n=N+1}^{\infty}\int_{\xi_0}^{\xi_0+h} |\widehat{\psi}(a^n\xi)|^2\,d\xi$$

$$< \varepsilon \;+\; \frac{1}{h}\int_{\xi_0}^{\xi_0+h} b^{-1}\sum_{n=N+1}^{\infty} |\widehat{\psi}(a^n\xi)|^2\,d\xi$$

$$< \varepsilon \;+\; \frac{1}{h}\int_{\xi_0}^{\xi_0+h} (W-W_N)(\xi)\,d\xi. \quad (12.18)$$

Finally, combining equations (12.13) and (12.18) and using the facts that $\widehat{\mathcal{W}}(\psi, a, b)$ is a frame and $\|f_h\|_{L^2} = 1$, we have

$$\begin{aligned} A \;=\; A\,\|f_h\|_{L^2}^2 \;&\le\; \sum_{n=-\infty}^{N} \sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f_h(\xi),\, a^{n/2} e^{2\pi i b k a^n \xi}\, \widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \\ &\qquad + \sum_{n=N+1}^{\infty} \sum_{k\in\mathbf{Z}} \bigl|\bigl\langle f_h(\xi),\, a^{n/2} e^{2\pi i b k a^n \xi}\, \widehat{\psi}(a^n\xi)\bigr\rangle\bigr|^2 \\ &\le\; \frac{1}{h}\int_{\xi_0}^{\xi_0+h} W_N(\xi)\,d\xi \;+\; \varepsilon \;+\; \frac{1}{h}\int_{\xi_0}^{\xi_0+h} (W-W_N)(\xi)\,d\xi. \\ &=\; \frac{1}{h}\int_{\xi_0}^{\xi_0+h} W(\xi)\,d\xi \;+\; \varepsilon. \end{aligned}$$

This is valid for all h small enough. Letting $h \to 0$, the Lebesgue Differentiation Theorem therefore implies that for almost every choice of $\xi_0 > 0$ we have

$$A \;\le\; \lim_{h\to 0} \frac{1}{h}\int_{\xi_0}^{\xi_0+h} W(\xi)\,d\xi \;+\; \varepsilon \;=\; W(\xi_0) + \varepsilon.$$

Since ε is arbitrary, we obtain $A \le W(\xi_0)$ for almost every $\xi_0 > 0$, and a similar argument applies for $\xi_0 < 0$. This establishes the lower inequality in equation (12.10).

(b) Integrating the result from part (a) and using the change of variables $\eta = a^n\xi$, we have

$$\begin{aligned} Ab\ln a \;=\; \int_1^a \frac{Ab}{\xi}\,d\xi \;&\le\; \int_1^a \sum_{n\in\mathbf{Z}} \frac{|\widehat{\psi}(a^n\xi)|^2}{\xi}\,d\xi \\ &=\; \sum_{n\in\mathbf{Z}} \int_{a^n}^{a^{n+1}} \frac{|\widehat{\psi}(\eta)|^2}{\eta}\,d\eta \\ &=\; \int_0^{\infty} \frac{|\widehat{\psi}(\eta)|^2}{\eta}\,d\eta. \end{aligned}$$

Combining this with a similar computation using the upper frame bound and estimates for the interval $(-\infty, 0]$ gives the result.

(c), (d) These follow immediately from part (b). □

As an example, let ψ be the Shannon wavelet (see Example 12.4). Then $\widehat{\psi} = \chi_E$ where $E = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. Since E is a wavelet set, $\mathcal{W}(\psi) = \mathcal{W}(\psi, 2, 1)$ is a dyadic wavelet orthonormal basis for $L^2(\mathbf{R})$. Considering Figure 12.3, we see that $\sum_{n\in\mathbf{Z}} |\widehat{\psi}(2^n\xi)|^2 = 1$ a.e., which is in agreement with Theorem 12.6(a).

The quantity appearing in part (b) of Theorem 12.6 is very important for wavelet frames.

Definition 12.7 (Admissibility Constant). The *admissibility constant* of a function $\psi \in L^2(\mathbf{R})$ is

$$C_\psi \;=\; \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi.$$

The admissibility constant could be infinite. We say that ψ *is admissible* if $C_\psi < \infty$. $\diamondsuit$

By Theorem 12.6, if $\mathcal{W}(\psi, a, b)$ is a frame, then ψ must be admissible. However, the converse fails in general. Note that if $\widehat{\psi}$ is continuous then admissibility requires $\widehat{\psi}(0) = 0$. In particular, if a function $\psi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is admissible then $\widehat{\psi}$ is continuous (Theorem 9.10) and so we must have

$$\int_{-\infty}^{\infty} \psi(x)\, dx \;=\; \int_{-\infty}^{\infty} \psi(x)\, e^{-2\pi i 0 \xi}\, dx \;=\; \widehat{\psi}(0) \;=\; 0.$$

Thus, an admissible integrable ψ must oscillate in some sense. This fact is sometimes given as the reason for the terminology "wavelet," though the original motivation for the name seems to have been different.

Note that in Corollary 11.7 for Gabor systems, we were able to go a little further and obtain the Nyquist density condition for Gabor frames. The key there was that the canonical dual frame and the canonical Parseval frame associated with a Gabor frame are each themselves Gabor frames. This need not be the case for wavelet frames. However, even if we consider Parseval wavelet frames, we are faced with the admissibility constant in Theorem 12.6 rather than $\|\psi\|_{L^2}^2$. Since all the elements of $\mathcal{W}(\psi, a, b)$ have the same norm, we know that if it is a Parseval frame then it is an orthonormal basis if and only if $\|\psi\|_{L^2}^2 = 1$, but this does not tell us anything directly about the admissibility constant. From the point of view of abstract group representations, it can be shown that the admissibility constant for the generator g of a Gabor system is simply its norm, whereas for wavelets we have a distinction between the norm and the admissibility constant. This is a consequence of the "highly noncommutative" nature of translations and dilations in comparison to translations and modulations. Daubechies's book [Dau92] or the survey [HW89] are sources for additional information on this topic.

The appearance of the admissibility constant in Theorem 12.6 also suggests that our discussion in Section 12.2 on the lack of a Nyquist density for wavelet frames may be incomplete, since we did not take the value of the admissibility constant into account. However, Balan [Bal97] has shown that this is not the case (see also [Dau90, Thm. 2.10]). But even this is not the end of the story on the relationship between density and frame properties for wavelets. See [HK03], [SZ03], [SZ04] [HK07], [Kut07] for more information.

Exercises

12.8. Show that $\big\{\psi \in L^2(\mathbf{R}) : \psi \text{ is admissible}\big\}$ is dense in $L^2(\mathbf{R})$.

12.9. Show that if $\psi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is admissible, then $\widehat{\psi}(0) = 0$.

12.4 Multiresolution Analysis

In this section we will focus on dyadic wavelet orthonormal bases $\mathcal{W}(\psi) = \mathcal{W}(\psi, 2, 1)$. If we keep $b = 1$ then most of the ideas of this section have generalizations to integer dilation factors $a \in \mathbf{N}$, but the fact that the dilation factor a is integer is quite important.

The idea of multiresolution analysis is that a dyadic wavelet orthonormal basis $\{D_{2^n} T_k \psi\}_{k,n \in Z}$ naturally divides $L^2(\mathbf{R})$ into subspaces with different "resolution levels." If we fix a particular value of n, then the functions $T_k\psi$ for $k \in \mathbf{Z}$ all have the same "size," but have different "centers" on the real line due to the translation by k. As n increases, these functions become more and more compressed, while as n decreases they become more and more stretched out. If we define closed subspaces

$$W_n = \overline{\mathrm{span}}\{D_{2^n} T_k \psi\}_{k \in \mathbf{Z}}, \qquad n \in \mathbf{Z}, \tag{12.19}$$

and let Q_n denote the orthogonal projection of $L^2(\mathbf{R})$ onto W_n, then we can write the basis representation of a function $f \in L^2(\mathbf{R})$ as

$$f = \sum_{n \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} \langle f, D_{2^n} T_k \psi \rangle D_{2^n} T_k \psi = \sum_{n \in \mathbf{Z}} Q_n f.$$

The space W_n is generated by functions $D_{2^n} T_k \psi$ that all have the same "detail size." Projecting f onto this space will therefore, in some sense, give us the information that is present in f specifically at "detail size n." Putting together all the information at the different detail sizes allows us to recapture f.

Now define

$$V_n = \overline{\mathrm{span}}\{D_{2^m} T_k \psi\}_{m<n,\, k \in \mathbf{Z}}.$$

The projection $P_n f$ of f onto V_n gives us a "blurry" picture of f. In some sense $P_n f$ is an approximation to f at "resolution level n." We move from resolution level to resolution level by adding "details" from W_n:

$$P_{n+1} f = P_n f + Q_n f.$$

By definition of orthonormal basis, $P_n f$ converges to f as n increases, and we accomplish this by adding information with smaller and smaller details as n increases. The space V_0 can be considered a central space in this scheme, essentially consisting of functions that have details of at most unit size ("resolution level zero").

To start making these comments about resolution level more precise, note some of the properties that these subspaces V_n possess:

- the V_n are nested, i.e., $V_n \subseteq V_{n+1}$,

- V_{n+1} is the dilation of V_n by a factor of 2, i.e.,

$$V_{n+1} = D_2(V_n) = \{f(2x) : f \in V_n\},$$

- $\cup_{n\in\mathbf{Z}} V_n$ is dense in $L^2(\mathbf{R})$,
- $\cap_{n\in\mathbf{Z}} V_n = \{0\}$, and
- $V_n \oplus W_n = V_{n+1}$, where $\oplus$ denotes the orthogonal direct sum of subspaces.

These facts are all predicated on the assumption that we have a dyadic wavelet orthonormal basis $\{D_{2^n} T_k \psi\}_{k,n\in Z}$ in hand. What if we don't have an orthonormal basis but rather want to construct one? Multiresolution analysis turns the above discussion into a construction algorithm for wavelet bases by focusing on the subspaces V_n rather than the wavelet ψ.

Definition 12.8 (Multiresolution Analysis). A *multiresolution analysis* (MRA) for $L^2(\mathbf{R})$ is a sequence $\{V_n\}_{n\in\mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R})$ such that:

(a) $V_n \subseteq V_{n+1}$ for each $n \in \mathbf{Z}$,

(b) $V_{n+1} = D_2(V_n)$ for each $n \in \mathbf{Z}$,

(c) $\cup_{n\in\mathbf{Z}} V_n$ is dense in $L^2(\mathbf{R})$,

(d) $\cap_{n\in\mathbf{Z}} V_n = \{0\}$,

(e) there exists a function $\varphi \in V_0$ such that $\{T_k \varphi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_0.

We call φ a *scaling function* for the MRA. $\diamondsuit$

MRAs were introduced by Mallat and developed by Mallat and Meyer (see [Mal89a], [Mal89b]). The introduction of MRAs sparked an enormous surge of interest in wavelet theory. Many of the influential early papers in wavelet theory are reprinted in the volume [HW96].

The aspect of MRAs that we have not encountered before is the scaling function φ. The scaling function is not the wavelet that generates the orthonormal basis for $L^2(\mathbf{R})$, but we will see that if we can find a scaling function φ, then we can construct an associated wavelet ψ such that $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2(\mathbf{R})$. Not every wavelet orthonormal basis is associated with an MRA, but every MRA does have an associated orthonormal wavelet basis.

Remark 12.9. (a) Since the space V_0 in an MRA is the closed span of $\mathcal{T}(\varphi) = \{T_k \varphi\}_{k\in\mathbf{Z}}$, we could emphasize this fact by writing $V_0(\varphi)$, as we did in Chapter 10. However, in this chapter we will follow wavelet tradition and just write V_0.

(b) Requirements (a)–(e) in Definition 12.8 are not independent. For example, requirement (d) is implied by the other hypotheses [Mad92].

(c) We could create a more general definition of MRA by requiring only that $\{T_k\varphi\}_{k\in Z}$ be a Riesz basis or a frame for V_0. By Exercise 10.21, if $\{T_k\varphi\}_{k\in Z}$ is a frame for V_0 then there exists a function $\varphi^\sharp \in V_0$ such that $\{T_k\varphi^\sharp\}_{k\in\mathbf{Z}}$ is a Parseval frame for V_0, and if $\{T_k\varphi\}_{k\in Z}$ is a Riesz basis for V_0 then $\{T_k\varphi^\sharp\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_0. However, the mapping $\varphi \mapsto \varphi^\sharp$ will usually not preserve desirable properties of φ such as compact support (see Example 12.38). Hence, it is often better to work directly with φ instead of replacing it by $\varphi^\sharp$. Some references on MRAs using Riesz bases or frames include [CDF92] and [BL98].

(d) Even more generally, there are situations where, instead of assuming that a scaling function exists, we only assume that the space V_0 is *shift-invariant* (invariant under integer translations). However, we will restrict our attention to MRAs of the form given in Definition 12.8, and refer to [BM99] for an introduction to *generalized MRAs*. $\diamondsuit$

The next lemma gives some implications of the definition of an MRA (we assign the proof as Exercise 12.11). We say that a subspace S of $L^2(\mathbf{R})$ is *a-shift-invariant* if $T_{ak}S \subseteq S$ for $k \in \mathbf{Z}$, or, equivalently,

$$f \in S \quad \Longrightarrow \quad f(x - ak) \in S \quad \text{for } k \in \mathbf{Z}.$$

If $a = 1$ then we simply say that S is *shift-invariant.*

Lemma 12.10. *Suppose that $\{V_n\}_{n\in\mathbf{Z}}$ is an MRA for $L^2(\mathbf{R})$ and let P_n denote the orthogonal projection of $L^2(\mathbf{R})$ onto V_n. Then the following statements hold.*

(a) $V_n = D_{2^n}(V_0) = \{f(2^n x) : f \in V_n\}$.

(b) $\{D_{2^n}T_k\varphi\}_{k\in\mathbf{Z}}$ *is an orthonormal basis for* V_n.

(c) V_0 *is* shift-invariant*, and* V_n *is* 2^{-n}*-shift-invariant.*

(d) $P_nf \to f$ *in* $L^2(\mathbf{R})$ *as* $n \to \infty$ *for every* $f \in L^2(\mathbf{R})$.

(e) $P_nf \to 0$ *in* $L^2(\mathbf{R})$ *as* $n \to -\infty$ *for every* $f \in L^2(\mathbf{R})$. $\diamondsuit$

Lemma 12.10(a) tells us that the spaces V_n in an MRA are completely determined by the base space V_0. Therefore, if we want to build an MRA then we can focus on the space V_0 and the scaling function φ. Once we have these, we know what the space V_n has to be, and the issue is whether properties (a), (c), and (d) in the definition of an MRA are satisfied.

Let us see now how the scaling function has been hiding in some of the examples of wavelet orthonormal bases that we have already discussed. This will also give some insight into how to proceed from an MRA to an orthonormal wavelet basis.

Example 12.11 (MRA for the Haar System). An MRA begins with subspaces and a scaling function, whereas we have so far begun with a wavelet and the wavelet system that it generates. In this example we will start with the MRA that is associated with the Haar wavelet, and see how the Haar wavelet is produced from this MRA.

The "base space" V_0 for the Haar MRA is the space of all step functions in $L^2(\mathbf{R})$ that are constant on intervals $[k, k+1)$:

$$V_0 \;=\; \biggl\{\sum_{k\in\mathbf{Z}} c_k \chi_{[k,k+1)} : (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})\biggr\}.$$

By Lemma 12.10, if we are to have an MRA then the space V_n must consist of all dilations by 2^n of functions in V_0. For the Haar MRA, V_n is therefore the space of step functions in $L^2(\mathbf{R})$ that are constant on intervals $\bigl[\frac{k}{2^n}, \frac{k+1}{2^n}\bigr)$. Hence, the projection $P_n f$ of a function $f \in L^2(\mathbf{R})$ onto V_n is the best approximation (in L^2-norm) to f by a step function with nodes at the points $k/2^n$. In this sense, $P_n f$ is a picture of f at "resolution level n."

If we set $\chi = \chi_{[0,1)}$, then $\{T_k\chi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_0. Hence the box function χ is a scaling function for the Haar MRA.

To show that we actually have an MRA, it remains to show that properties (a), (c), and (d) in Definition 12.8 are satisfied. Property (a) is the nestedness requirement $V_n \subseteq V_{n+1}$, and this is certainly satisfied. For example, functions in V_0 are step functions constant on each interval $[k, k+1)$, while functions in V_1 are step functions constant on each interval $\bigl[\frac{k}{2}, \frac{k+1}{2}\bigr)$. Hence $V_0 \subseteq V_1$, and since $V_n = D_{2^n} V_0$, it follows that $V_n \subseteq V_{n+1}$ for every n.

Here is an alternative way to see the nestedness property. The orthonormal basis $\{T_k\chi\}_{k\in\mathbf{Z}}$ for V_0 is generated from integer translates of χ. This function χ has a self-similarity property in the sense that it is a sum of two smaller shifted copies of itself. Specifically, since $\chi(2x) = \chi_{[0,\frac{1}{2})}$ and $\chi(2x-1) = \chi_{[\frac{1}{2},1)}$, the box function χ satisfies the *refinement equation*

$$\chi(x) \;=\; \chi(2x) + \chi(2x-1). \tag{12.20}$$

This refinement equation is illustrated in Figure 12.7. We will study more general refinement equations in Section 12.5. The important point now is that each of $\chi(2x)$ and $\chi(2x-1)$ belong to V_1, so $\chi \in V_1$ as well. Since V_1 is shift-invariant (in fact, it is invariant under half-integer translations), we have $T_k\chi \in V_1$ for every $k \in \mathbf{Z}$. Therefore, since every element of the basis $\{T_k\chi\}_{k\in\mathbf{Z}}$ for V_0 belongs to the closed subspace V_1, we have $V_0 \subseteq V_1$.

Now suppose that $f \in L^2(\mathbf{R})$ belongs to every subspace V_n. Then f must be constant on every interval $\bigl[\frac{k}{2^n}, \frac{k+1}{2^n}\bigr)$ for all $k,\, n \in \mathbf{Z}$. In particular, f is constant on $[0, 2^n)$ for every $n \in \mathbf{N}$, which implies f is constant on $[0,\infty)$, and similarly it is constant on $(-\infty, 0]$. Since $f \in L^2(\mathbf{R})$, this implies that $f = 0$. Hence property (d) in Definition 12.8 is satisfied.

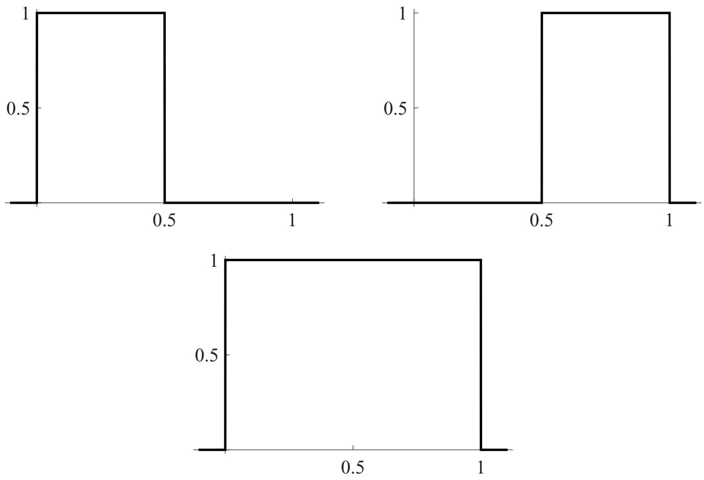

Fig. 12.7. The refinement equation for the box function. Top left: $\chi(2x)$. Top right: $\chi(2x-1)$. Bottom: $\chi(x) = \chi(2x) + \chi(2x-1)$.

Finally, $\cup_{n\in\mathbf{Z}} V_n$ is dense because the projection $P_n f$ of f onto V_n converges to f as $n \to \infty$. In fact,

$$P_n f \;=\; \sum_{k\in\mathbf{Z}} \langle f, D_{2^n} T_k \chi \rangle \, D_{2^n} T_k \chi \;=\; \sum_{k\in\mathbf{Z}} c_{k,n} \, \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})}, \tag{12.21}$$

where

$$c_{k,n} \;=\; 2^n \, \langle f, \chi_{[\frac{k}{2}, \frac{k+1}{2})} \rangle \;=\; 2^n \int_{k/2^n}^{(k+1)/2^n} f(x)\, dx \tag{12.22}$$

is simply the average of f on the interval $\bigl[\frac{k}{2^n}, \frac{k+1}{2^n}\bigr)$. Property (c) in Definition 12.8 is satisfied, so $\{V_n\}_{n\in\mathbf{Z}}$ is an MRA for $L^2(\mathbf{R})$.

But what does this have to do with the Haar wavelet $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$? While the subspaces V_n are "resolution levels," the Haar wavelet ψ determines "detail spaces" W_n that move us from one resolution level to another. In particular, define

$$W_0 \;=\; \overline{\operatorname{span}}\{T_k \psi\}_{k\in\mathbf{Z}},$$

and note that $\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0. Since the Haar wavelet ψ is orthogonal to the scaling function χ, the subspaces V_0 and W_0 are orthogonal. Now, just as equation (12.20) implies that $V_0 \subseteq V_1$, the fact that

$$\psi(x) \;=\; \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x) \;=\; \chi(2x) - \chi(2x-1) \;\in\; V_1$$

implies that $W_0 \subseteq V_1$. Hence the orthogonal direct sum of V_0 and W_0 is contained in V_1:

$$V_0 \oplus W_0 \;=\; \{f+g : f \in V_0,\, g \in W_0\} \;\subseteq\; V_1,$$

and we claim that equality holds. Indeed, if $h \in V_1$ then, as in equations (12.21) and (12.22), we have

$$h \;=\; \sum_{k\in\mathbf{Z}} c_k\, \chi_{[\frac{k}{2},\frac{k+1}{2})}$$

where c_k is the average of h on $\chi_{[\frac{k}{2},\frac{k+1}{2})}$. If we let a_k be the average of h on $\chi_{[k,k+1)}$, then a_k is the average of c_{2k} and c_{2k+1}. Hence

$$h(x) \;=\; \sum_{k\in\mathbf{Z}} a_k\, \chi(x-k) \;+\; \sum_{k\in\mathbf{Z}} b_k\, \psi(x-k) \;\in\; V_0 \oplus W_0,$$

where $b_k = c_{2k} - a_k$ (see the "proof by picture" in Figure 12.8). Thus we have $V_1 = V_0 \oplus W_0$. In particular, if we let $P_1 f$, $P_0 f$, $Q_0 f$ denote the orthogonal projections of a function $f \in L^2(\mathbf{R})$ onto V_1, V_0, and W_0, respectively, then

$$P_1 f \;=\; P_0 f + Q_0 f.$$

The function $P_0 f$ is an approximation to f at "resolution level 0." Adding $Q_0 f$, we obtain $P_1 f$, the approximation at "resolution level 1."

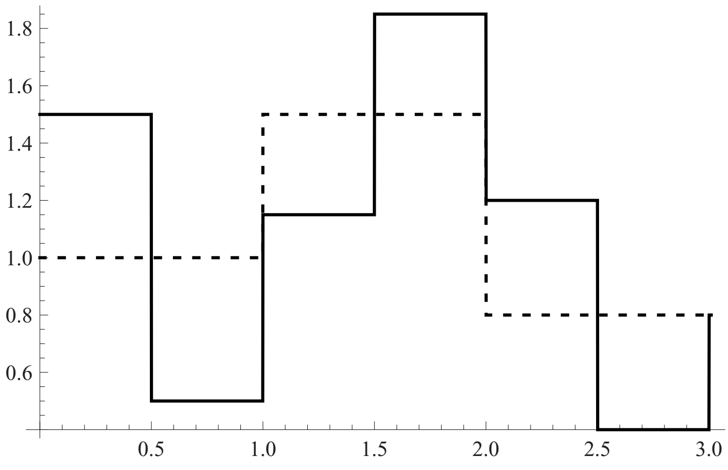

Fig. 12.8. A function h in V_1 (solid line) is a sum of a function f in V_0 (dashed line) and a function g in W_0 (difference between h and f).

Now define W_n as in equation (12.19), i.e.,

$$W_n \;=\; \overline{\operatorname{span}}\{D_{2^n} T_k \psi\}_{k\in\mathbf{Z}}.$$

The W_n are orthogonal subspaces, and for any $n > 0$ we have

$$V_{n+1} = V_n \oplus W_n.$$

Iterating,

$$V_n = V_0 \oplus W_0 \oplus \cdots \oplus W_{n-1}.$$

As $\{T_k\chi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_0 and $\{D_{2^m}T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_m, it follows that

$$P_nf = \sum_{k\in\mathbf{Z}} a_kT_k\chi + \sum_{m=0}^{n-1}\sum_{k\in\mathbf{Z}} b_{k,m}D_{2^m}T_k\psi$$

for some appropriate scalars a_k and $b_{k,m}$. Since $P_nf \to f$, we therefore have

$$f = \sum_{k\in\mathbf{Z}} a_kT_k\chi + \sum_{m=0}^{\infty}\sum_{k\in\mathbf{Z}} b_{k,m}D_{2^m}T_k\psi.$$

This shows that the Haar system given in equation (12.1) is an orthonormal basis for $L^2(\mathbf{R})$. If we write

$$V_n = V_{-n} \oplus W_{-n} \oplus \cdots \oplus W_{n-1}$$

and recall that $P_{-n}f \to 0$ as $n \to \infty$, then we can similarly show that the Haar system given in equation (12.2) is an orthonormal basis for $L^2(\mathbf{R})$. ◇

Strictly speaking, we went through some extra steps in Example 12.11, but the ideas presented there are a good introduction to the general procedure for constructing MRAs and wavelet orthonormal bases from scaling functions.

Here is another interesting MRA.

Example 12.12 (MRA for the Shannon Wavelet). The Shannon wavelet was introduced in Example 12.4. If we set $E = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$, then the Shannon wavelet is the function $\psi \in L^2(\mathbf{R})$ whose Fourier transform is $\widehat{\psi} = \chi_E$. Let $B = [-\frac{1}{2}, \frac{1}{2}]$. The scaling function for the Shannon MRA is the *sinc function*

$$\varphi(x) = d_\pi(x) = \frac{\sin \pi x}{\pi x},$$

whose Fourier transform is

$$\widehat{\varphi} = \chi_B = \chi_{[-\frac{1}{2},\frac{1}{2}]};$$

see Figure 12.9

The sinc function played an important role in our discussion of sampling theory in Chapter 10. We saw in Theorem 10.4 that $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for the Paley–Wiener space $\mathrm{PW}(\mathbf{R})$. Hence the base space for this MRA is

$$V_0 = \mathrm{PW}(\mathbf{R}) = \Big\{f \in L^2(\mathbf{R}) : \mathrm{supp}(\widehat{f}\,) \subseteq [-\tfrac{1}{2}, \tfrac{1}{2}]\Big\}.$$

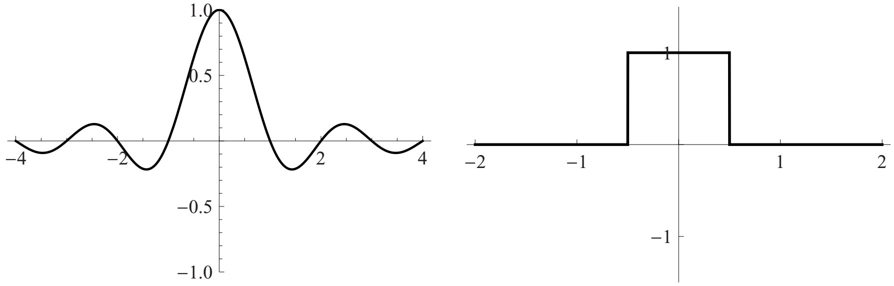

Fig. 12.9. The Shannon scaling function φ (left) and its Fourier transform $\widehat{\varphi}$ (right).

Dilating, the space V_n contains the functions in $L^2(\mathbf{R})$ that are bandlimited to $[-2^{n-1}, 2^{n-1}]$:

$$V_n \;=\; \Big\{f \in L^2(\mathbf{R}) : \operatorname{supp}(\widehat{f}\,) \subseteq [-2^{n-1}, 2^{n-1}]\Big\}. \tag{12.23}$$

Consequently $V_n \subseteq V_{n+1}$, and it also follows from equation (12.23) that $\cup_{n\in\mathbf{Z}} V_n$ is dense in $L^2(\mathbf{R})$ and $\cap_{n\in\mathbf{Z}} V_n = \{0\}$ (we assign the verification of these facts as Exercise 12.10). Hence $\{V_n\}_{n\in\mathbf{Z}}$ is an MRA.

The detail spaces are

$$\begin{aligned} W_n \;&=\; \overline{\operatorname{span}}\{D_{2^n} T_k \psi\}_{k\in\mathbf{Z}} \\ &=\; \Big\{f \in L^2(\mathbf{R}) : \operatorname{supp}(\widehat{f}\,) \subseteq [-2^n, -2^{n-1}] \cup [2^{n-1}, 2^n]\Big\}. \end{aligned}$$

These spaces W_n are mutually orthogonal and we have $V_{n+1} = V_n \oplus W_n$ for each $n \in \mathbf{Z}$. Just as in Example 12.11, it follows that

$$\{T_k\varphi\}_{k\in\mathbf{Z}} \cup \{D_{2^n} T_k \psi\}_{n\geq 0,\, k\in\mathbf{Z}} \qquad \text{and} \qquad \mathcal{W}(\psi) \;=\; \{D_{2^n} T_k \psi\}_{n,k\in\mathbf{Z}}$$

are each orthonormal bases for $L^2(\mathbf{R})$. $\diamondsuit$

The two examples above are perhaps not quite as convincing as we might like, because in each of them we had a function in hand that we knew generated a wavelet orthonormal basis. In the following sections we will see how to use MRAs to construct new examples of wavelet bases.

Exercises

12.10. Prove that the Shannon MRA $\{V_n\}_{n\in\mathbf{Z}}$ constructed in Example 12.12 is indeed an MRA.

12.11. Prove Lemma 12.10.

12.12. (a) Suppose that $\{V_n\}_{n\in\mathbf{Z}}$ is an MRA for $L^2(\mathbf{R})$ with scaling function φ. Show that $\{D_{2^n} T_k \varphi\}_{k,n\in\mathbf{Z}}$ is complete in $L^2(\mathbf{R})$.

(b) Show that $\{D_{2^n} T_k \chi_{[0,1]}\}_{k,n\in\mathbf{Z}}$ is complete in $L^2(\mathbf{R})$.

12.5 All About the Scaling Function, I: Refinability

The key to using an MRA to construct a wavelet orthonormal basis is the scaling function φ. The scaling function determines V_0 and hence V_n, and these determine the detail spaces W_n and ultimately the wavelet ψ. In this section and the next we focus on the construction and properties of the scaling function, and then in Section 12.7 we will see how to construct an MRA and a wavelet ψ from the scaling function φ.

We begin by examining some of the properties that a scaling function for an MRA must possess. By Definition 12.8, a first requirement is that the scaling function φ must have orthonormal integer translates. We characterized this requirement in Section 10.4 in terms of the periodization Φ_φ of $|\widehat{\varphi}|^2$, and we give here another direct proof of this condition.

Lemma 12.13. *Given $\varphi \in L^2(\mathbf{R})$,*

$$\{T_k\varphi\}_{k\in\mathbf{Z}} \text{ is orthonormal} \quad \Longleftrightarrow \quad \Phi_\varphi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi+k)|^2 = 1 \text{ a.e.}$$

Moreover, in case these hold, $f \in \overline{\operatorname{span}}\{T_k\varphi\}_{k\in\mathbf{Z}}$ if and only if

$$\widehat{f}(\xi) = m(\xi)\,\widehat{\varphi}(\xi) \text{ a.e. for some } m \in L^2(\mathbf{T}). \tag{12.24}$$

Proof. The Fourier transform is unitary and interchanges translation with modulation, so

$$\langle T_n\varphi, \varphi\rangle = \langle (T_n\varphi)^\wedge, \widehat{\varphi}\rangle = \langle M_{-n}\widehat{\varphi}, \widehat{\varphi}\rangle.$$

Since $|\widehat{\varphi}|^2$ is integrable, its periodization $\Phi_\varphi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi+k)|^2$ belongs to $L^1(\mathbf{T})$ (see Exercise 10.13). By Theorem 4.25, functions in $L^1(\mathbf{T})$ are uniquely determined by their Fourier coefficients. The Fourier coefficients of the constant function 1 are the delta sequence, $\widehat{1}(n) = \delta_{0n}$ for $n \in \mathbf{Z}$. Using the fact that $e^{-2\pi i n\xi}$ is 1-periodic, the Fourier coefficients of Φ_φ are

$$\begin{aligned}
\widehat{\Phi_\varphi}(n) &= \int_0^1 \Phi_\varphi(\xi)\, e^{-2\pi i n\xi}\, d\xi \\
&= \int_0^1 \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi+k)|^2\, e^{-2\pi i (n+k)\xi}\, d\xi \\
&= \int_{-\infty}^{\infty} |\widehat{\varphi}(\xi)|^2\, e^{-2\pi i n\xi}\, d\xi \\
&= \int_{-\infty}^{\infty} e^{-2\pi i n\xi}\, \widehat{\varphi}(\xi)\, \overline{\widehat{\varphi}(\xi)} d\xi = \langle M_{-n}\widehat{\varphi}, \widehat{\varphi}\rangle = \langle T_n\varphi, \varphi\rangle.
\end{aligned}$$

Therefore φ has orthonormal translates if and only if the Fourier coefficients of Φ_φ are the delta sequence, which happens if and only if $\Phi_\varphi = 1$ a.e.

The proof that $f \in \overline{\operatorname{span}}\{T_k\varphi\}_{k\in\mathbf{Z}}$ is equivalent to equation (12.24) follows from Exercise 10.17. To sketch one direction of the implication, suppose that φ has orthonormal translates and $f \in \overline{\operatorname{span}}\{T_k\varphi\}_{k\in\mathbf{Z}}$. Then

$$f \;=\; \sum_{k\in\mathbf{Z}} \langle f, T_k\varphi\rangle\, T_k\varphi,$$

where the series converges unconditionally in L^2-norm. Applying the Fourier transform,

$$\begin{aligned}\widehat{f}(\xi) \;&=\; \sum_{k\in\mathbf{Z}} \langle f, T_k\varphi\rangle\, M_{-k}\widehat{\varphi}(\xi)\\ &=\; \Bigl(\sum_{k\in\mathbf{Z}} \langle f, T_k\varphi\rangle\, e^{-2\pi i k\xi}\Bigr)\widehat{\varphi}(\xi) \qquad (12.25)\\ &=\; m(\xi)\,\widehat{\varphi}(\xi).\end{aligned}$$

The function $m(\xi) \sum_{k\in\mathbf{Z}} \langle f, T_k\varphi\rangle\, e^{-2\pi i k\xi}$ belongs to $L^2(\mathbf{T})$ because we have $\sum_{k\in\mathbf{Z}} |\langle f, T_k\varphi\rangle|^2 < \infty$. The factoring performed in equation (12.25) does need justification, and this follows from Exercise 10.15. $\square$

Once we have in hand a function φ that has orthonormal integer translates, we can define

$$V_0 \;=\; \overline{\operatorname{span}}\{T_k\varphi\}_{k\in\mathbf{Z}} \;=\; \Bigl\{\sum_{k\in\mathbf{Z}} c_k\, \varphi(x-k) : (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})\Bigr\} \qquad (12.26)$$

and

$$V_n \;=\; D_{2^n} V_0, \qquad n \in \mathbf{Z}, \qquad (12.27)$$

and then check to see whether $\{V_n\}_{n\in\mathbf{Z}}$ is an MRA. In general this will not happen. The nestedness requirement $V_n \subseteq V_{n+1}$ is particularly restrictive and will usually not be satisfied.

Since $V_{n+1} = D_2(V_n)$, the nestedness requirement $V_n \subseteq V_{n+1}$ is equivalent to the single inclusion $V_0 \subseteq V_1$. As the scaling function φ belongs to V_0, if we are to have an MRA then we must have

$$\varphi \;\in\; V_0 \;\subseteq\; V_1 \;=\; \Bigl\{\sum_{k\in\mathbf{Z}} c_k\, \varphi(2x-k) : (c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})\Bigr\}.$$

Hence in order for the nestedness requirement to be satisfied, there must exist some sequence of scalars $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ such that

$$\varphi(x) \;=\; \sum_{k\in\mathbf{Z}} c_k\, \varphi(2x-k).$$

Somehow we need to find a function φ with orthonormal translates that *also* satisfies an equation of this form.

Definition 12.14 (Refinable Function). We say that a function $\varphi \in L^p(\mathbf{R})$ is *refinable* in $L^p(\mathbf{R})$ if there exists a sequence of scalars $(c_k)_{k\in\mathbf{Z}}$ such that the series $\sum_{k\in\mathbf{Z}} c_k\, \varphi(2x-k)$ converges in $L^p(\mathbf{R})$ and we have

$$\varphi(x) \;=\; \sum_{k\in\mathbf{Z}} c_k\, \varphi(2x-k) \text{ a.e.} \tag{12.28}$$

We refer to equation (12.28) as a *refinement equation*, *dilation equation*, or *two-scale difference equation*. The scalars c_k are called the *refinement coefficients*. $\diamondsuit$

We are usually interested in the case $p = 2$, but often it is convenient to consider refinable functions in $L^1(\mathbf{R})$ or $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. In the statements of our theorems and exercises we specify what domain we are considering, usually whatever is most convenient for that result. The difficult problem of completely characterizing those sequences $(c_k)_{k\in\mathbf{Z}}$ which have an L^p solution to the corresponding refinement equation has been studied in detail, especially when $(c_k)_{k\in\mathbf{Z}}$ has only finitely many nonzero components. We refer to the extensive bibliography in [CHM04] for references on this subject. For a short survey of refinement equations in wavelet theory, see the survey paper [Str89] by Strang.

The box function $\chi = \chi_{[0,1)}$, which is the scaling function for the Haar MRA, is refinable since it satisfies the refinement equation given in equation (12.20). Note that since we only require the equality in the refinement equation to hold almost everywhere, the function $\chi_{[0,1]}$ satisfies the same refinement equation as $\chi_{[0,1)}$ (of course, both of these functions define the same element of $L^2(\mathbf{R})$).

The Shannon scaling function is another example of a refinable function (Exercise 12.15). However, not every function is refinable and not every refinement equation has a solution. Further, not every refinable function has orthonormal translates, e.g., consider the hat function (see Figure 12.10 and Exercise 12.14).

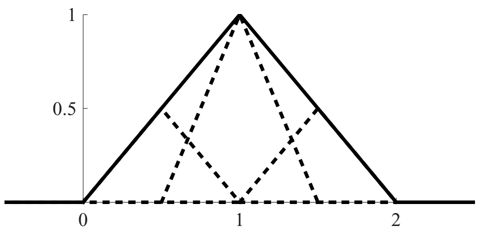

Fig. 12.10. The hat function h on the interval $[0,2]$ is refinable: $h(x) = \frac{1}{2}h(2x) + h(2x-1) + \frac{1}{2}h(2x-2)$.

Refinable functions have been extensively studied because they play key roles in many areas other than just wavelet theory. In many of those contexts there is no need to seek functions with orthonormal translates. For example, subdivision schemes are widely used in computer graphics to represent smooth surfaces as a limit of polygonal meshes, and these schemes are closely related to refinement equations [CDM91].

The refinement equation takes the following form in the Fourier domain.

Lemma 12.15. *Suppose* $\varphi \in L^2(\mathbf{R})$ *is refinable with refinement coefficients* $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$. *Then*

$$\widehat{\varphi}(\xi) \;=\; m_0(\xi/2)\,\widehat{\varphi}(\xi/2) \;\; a.e., \tag{12.29}$$

where $m_0 \in L^2(\mathbf{T})$ *is given by*

$$m_0(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}. \tag{12.30}$$

Proof. Note that the series defining m_0 in equation (12.30) converges unconditionally in $L^2(\mathbf{T})$ since $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$.

In operator notation, the refinement equation is

$$\varphi \;=\; \sum_{k\in\mathbf{Z}} 2^{-1/2} c_k\, D_2 T_k \varphi. \tag{12.31}$$

Technically, while the hypothesis that φ is refinable requires that the series in equation (12.31) converge, we cannot assume that it converges unconditionally. We only know that there is some ordering of the index set $\mathbf{Z}$ with respect to which the partial sums of this series will converge. Implicitly taking series with respect to this ordering and recalling from equations (9.2)–(9.4) that the Fourier transform interchanges translation with modulation and dilation with a reciprocal dilation, we compute that

$$\begin{aligned}
\widehat{\varphi}(\xi) &= \sum_{k\in\mathbf{Z}} 2^{-1/2} c_k\,(D_2 T_k\varphi)^{\wedge}(\xi) \\
&= \sum_{k\in\mathbf{Z}} 2^{-1/2}\, c_k\, D_{1/2} M_{-k} \widehat{\varphi}(\xi) \\
&= \sum_{k\in\mathbf{Z}} \Big(2^{-1/2}\, c_k\; 2^{-1/2}\, e^{-2\pi i k(\xi/2)}\; \widehat{\varphi}(\xi/2)\Big) \\
&= \frac{1}{2}\Big(\sum_{k\in\mathbf{Z}} c_k\, e^{-2\pi i k(\xi/2)}\Big)\, \widehat{\varphi}(\xi/2) \qquad (12.32) \\
&= m_0(\xi/2)\,\widehat{\varphi}(\xi/2).
\end{aligned}$$

This is an equality of functions in $L^2(\mathbf{R})$, so pointwise we have $\widehat{\varphi}(\xi) = m_0(\xi/2)\,\widehat{\varphi}(\xi/2)$ a.e. The factoring performed at equation (12.32) does need to be justified. This justification is very similar to the proof of Exercise 10.15, but slightly different since we do not have unconditional convergence of all series. We assign the details as Exercise 12.17. □

For partial converses to Lemma 12.15, see Exercise 12.18.

Notation 12.16. Given refinement coefficients $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, we let m_0 denote the function

$$m_0(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi} \;\in\; L^2(\mathbf{T}). \tag{12.33}$$

We call m_0 the *symbol* of the refinement equation $\varphi(x) = \sum c_k\varphi(2x-k)$. ◇

Except for the multiplicative factor 1/2 appearing on the right-hand side of equation (12.33), in the language of Chapter 13 the symbol m_0 is the *Fourier transform* of the sequence $(c_k)_{k\in\mathbf{Z}}$. To avoid the issues with conditional convergence that arose in the proof of Lemma 12.15, we will usually need to impose some conditions on the refinement coefficients. Typically, we will assume at minimum that $(c_k)_{k\in\mathbf{Z}}$ is summable, i.e., it belongs to $\ell^1(\mathbf{Z})$.

Now we give some examples of refinable functions. For the most part we will only sketch ideas and not give complete proofs here, although some of these will receive a more rigorous treatment in the exercises and following sections. We refer to sources such as [Dau92], [DL91], [DL92], [CH94], [HC94] for complete proofs and discussion of these examples.

The "simplest" refinement equations are those with only finitely many nonzero refinement coefficients. So, we consider refinement equations of the form

$$\varphi(x) \;=\; \sum_{k=0}^{N} c_k\,\varphi(2x-k). \tag{12.34}$$

We set $c_k = 0$ for $k < 0$ and $k > N$. The number $N+1$ is the *length* of this refinement equation, or, in engineering parlance, the number of *taps* in the equation. Up to multiplication by a scalar, there is at most one integrable solution to equation (12.34) (see Corollary 12.26). If an integrable solution φ exists, then φ is compactly supported and $\operatorname{supp}(\varphi) \subseteq [0,N]$ (see Exercise 12.36). If all the c_k are real, then φ is real valued. The smoothness of a solution is limited by N; Exercise 12.21 shows that a solution φ can have at most $N-2$ continuous derivatives.

There are no integrable solutions to one-term refinement equations (i.e., those with $N=0$), although there do exist solutions in the sense of distributions or "generalized functions" (see Exercise 12.34). To obtain interesting solutions to the refinement equation we need to consider larger N, and we usually also need to impose some extra conditions on the c_k, most importantly the normalization

$$\sum_{k\in\mathbf{Z}} c_k \;=\; 2.$$

Typically, we must further refine this by imposing the following *minimal accuracy* condition:

$$\sum_{k\in\mathbf{Z}} c_{2k} \;=\; 1 \;=\; \sum_{k\in\mathbf{Z}} c_{2k+1}. \tag{12.35}$$

This is a necessary condition if φ is to have orthonormal integer translates, although it is not sufficient. One consequence of equation (12.35) is that if an integrable solution φ to the refinement equation exists, then its periodization $\sum_{j\in\mathbf{Z}} \varphi(x+j)$ is constant a.e. (Exercise 12.29). Exact representation of polynomials is important because a function that is smooth at a point x can be well approximated in a neighborhood of x by a polynomial. The smoother that f is at x, the higher-order polynomial we can use in the approximation. At least intuitively, the more polynomials that we can exactly reproduce, the better we will be able to approximate f using translates of φ (hence the name "accuracy condition").

If we take $N = 1$ and assume the minimal accuracy requirement, then we are looking at the refinement equation

$$\varphi(x) \;=\; \varphi(2x) + \varphi(2x-1).$$

Up to scale, the unique integrable solution is the box function $\chi_{[0,1)}$.

Example 12.17 (B-Splines). There is an interesting family of refinable functions known as *B-splines* that are related to the box function. As discussed in Exercise 12.20, the B-splines are defined recursively by $B_0 = \chi_{[0,1]}$ and $B_{n+1} = B_n * B_0$, the *convolution* of B_n with B_0. Each B_n satisfies a refinement equation with $N = n+1$, and B_n increases in smoothness with n. The spline B_1 is the hat function on $[0,2]$, which is pictured in Figure 12.10, and the spline B_2 is the piecewise quadratic function shown in Figure 12.11. Unfortunately for our purposes, except for B_0 the spline B_n does not have orthonormal integer translates. ◇

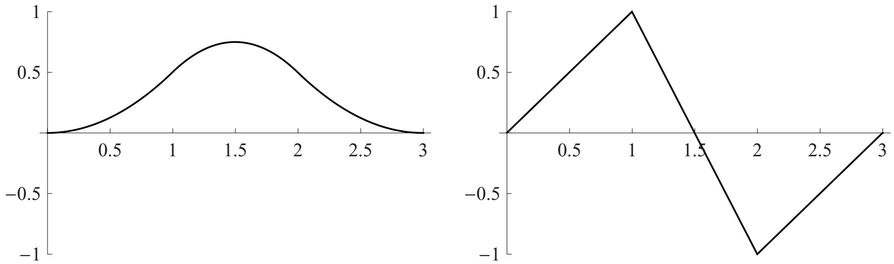

Fig. 12.11. Left: The spline B_2, which satisfies the refinement equation $B_2(x) = \frac{1}{4}\,B_2(2x) + \frac{3}{4}\,B_2(2x-1) + \frac{3}{4}\,B_2(2x-2) + \frac{1}{4}\,B_2(2x-3)$. Right: The first derivative B_2' of B_2.

To obtain more examples, we fix $N = 3$ and assume that the minimal accuracy condition in equation (12.35) is satisfied. This means that our refinement equation has the form

$$\varphi(x) = c_0\,\varphi(2x) + c_1\,\varphi(2x-1) + c_2\,\varphi(2x-2) + c_3\,\varphi(2x-3), \qquad (12.36)$$

with the minimal accuracy constraint

$$c_0 + c_2 = c_1 + c_3 = 1.$$

Hence we have only two degrees of freedom in the choice of the coefficients c_0, c_1, c_2, c_3, and we arbitrarily select the independent variables to be c_0, c_3. Each choice of (c_0, c_3) gives us a different refinement equation to examine. Thus we can consider our refinement equations to be parametrized by the (c_0, c_3) plane. Restricting our attention to real-valued coefficients, this is the plane pictured in Figure 12.12. We will discuss the geometric objects appearing in this figure below, and we refer to [CH94], [HC94] for more detailed discussions.

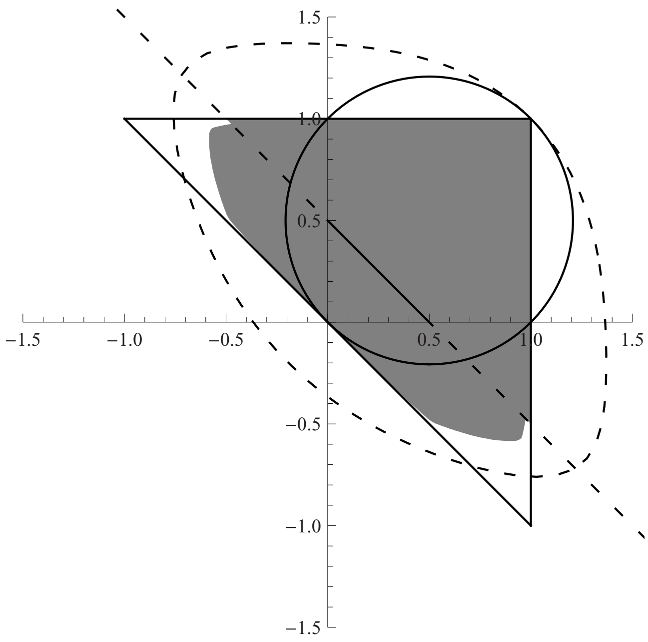

Fig. 12.12. The (c_0, c_3) plane, including the "curve of existence," "circle of orthogonality," "triangle of exclusion," "line of smoothness," and the "region of continuity."

In the following examples we will examine some of the properties of φ as we let (c_0, c_3) vary throughout this plane.[1] Some particular points in the (c_0, c_3) plane that we have already seen (or will soon encounter) and their corresponding refinable functions are given in Table 12.1.

[1]For an interactive version of the (c_0, c_3) plane, see Wim Swelden's wavelet applet at `http://cm.bell-labs.com/who/wim/cascade/`.

Table 12.1. Some particular points in the (c_0, c_3) plane.

Point	Description	Illustration
$(1,0)$	Box function $\chi_{[0,1]}$	Figure 12.7
$(0,0)$	Box function $\chi_{[1,2]}$	
$(0,1)$	Box function $\chi_{[2,3]}$	
$(1,1)$	Stretched box function $\chi_{[0,3]}$	Example 12.21
$(\frac{1}{2},0)$	Hat function on $[0,2]$	Figure 12.10
$(0,\frac{1}{2})$	Hat function on $[1,3]$	
$(\frac{1}{4},\frac{1}{4})$	B-spline B_2	Figure 12.11
$(\frac{1+\sqrt{3}}{4},\frac{1-\sqrt{3}}{4})$	Daubechies D_4 function	Example 12.22

The fact that the objects appearing in Figure 12.12 are symmetric with respect to the line $c_3 = c_0$ is due to the fact that if $\varphi(x)$ is the scaling function corresponding to the point (c_0, c_3), then $\varphi(3-x)$ is the scaling function corresponding to the point (c_3, c_0). Thus, for example, the point $(\frac{1-\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4})$ corresponds to a time-reversed version of the Daubechies D_4 function.

Example 12.18 (L^2 Existence). Theorem 12.29, which we prove in the next section, implies that there is an L^2 solution to the refinement equation for every point on the circle that appears in Figure 12.12, and Exercise 12.27 extends this to the interior of the circle. However, this is not a sharp result. It is possible to give an exact mathematical characterization of those points (c_0, c_3) for which a compactly supported refinable function $\varphi \in L^2(\mathbf{R})$ exists [Eir92], [LauW95], [Vil92]. This characterization is in terms of the spectral radius of a single associated finite matrix whose entries are determined by the refinement coefficients c_k (the spectral radius is the maximum modulus of the eigenvalues of the matrix). A numerical computation of this spectral radius shows that all points in the interior of the ellipse-like dashed curve in Figure 12.12 have L^2 solutions to their corresponding refinement equations. ◇

Example 12.19 (Continuity). We prefer our refinable functions to be as smooth as possible. Suppose that there is a continuous solution to the 4-tap refinement equation. Taking into account that φ is supported in $[0,3]$, we see that for $x \in [0, \frac{1}{2}]$ the refinement equation reduces to

$$\varphi(x) \;=\; c_0\varphi(2x), \qquad x \in [0, \tfrac{1}{2}]. \tag{12.37}$$

Since $\varphi(0) = 0$, for any $x > 0$ we have

$$0 \;=\; \varphi(0) \;=\; \lim_{n\to\infty} \varphi(2^{-n}x) \;=\; \lim_{n\to\infty} c_0^n\,\varphi(x).$$

Consequently we must have $|c_0| < 1$, and a similar argument shows that $|c_3| < 1$. Hence continuous solutions are restricted to the interior of the square $[-1,1] \times [-1,1]$ in the (c_0, c_3) plane. A more refined analysis shows that continuous solutions are restricted to the interior of the triangle pictured in Figure 12.12.

It is possible to give a mathematical characterization of those points (c_0, c_3) that have continuous solutions in terms of the *joint spectral radius* of two matrices whose entries are determined by the coefficients c_k [DL92]. However, whereas the spectral radius of a single matrix is simply the maximum of the moduli of its eigenvalues, the evaluation of the *joint* spectral radius of two matrices can be very challenging (see the references in [Jun09], [CHM04]). The shaded region in Figure 12.12 is a numerical approximation to the set of points for which this joint spectral radius is strictly less than 1, which corresponds to continuous solutions. Out of these points, those lying on the *solid* line segment correspond to differentiable φ. For example, the spline B_2 corresponds to the point $(1/4, 1/4)$. ◇

Example 12.20 (Graphing). A continuous refinable function φ is easy to plot to any desired level of resolution. If we know the values of $\varphi(j)$ for j integer then we know the value of $\varphi(j/2)$ by applying the refinement equation:

$$\varphi(j/2) \;=\; \sum_{k=0}^{3} c_k\, \varphi(j-k).$$

Iterating, we can obtain $\varphi(j/2^\ell)$ for any $j \in \mathbf{Z}$ and $\ell \in \mathbf{N}$. In particular, if φ is not the zero function then we cannot have $\varphi(j) = 0$ for every integer j.

So, we just need to find $\varphi(j)$ for $j \in \mathbf{Z}$. Since $\operatorname{supp}(\varphi) \subseteq [0,3]$, out of these values only $\varphi(1)$ and $\varphi(2)$ can be nonzero. Applying the refinement equation and taking the support of φ into consideration, we have

$$\begin{aligned}\varphi(1) \;&=\; \sum_{k=0}^{3} c_k\, \varphi(2-k) \;=\; c_0\, \varphi(2) + c_1\, \varphi(1),\\ \varphi(2) \;&=\; \sum_{k=0}^{3} c_k\, \varphi(4-k) \;=\; c_2\, \varphi(2) + c_3\, \varphi(1).\end{aligned}$$

Hence $\bigl(\varphi(1), \varphi(2)\bigr)^{\mathrm{T}}$ is an eigenvector of the matrix

$$M \;=\; \begin{bmatrix} c_1 & c_0 \\ c_3 & c_2 \end{bmatrix},$$

with eigenvalue 1. The minimal accuracy condition implies that $\sum_{j\in\mathbf{Z}} \varphi(x+j)$ is constant. Scaling so that this constant is 1, we obtain $\varphi(1) + \varphi(2) = 1$. Combined with the eigenvector condition above, this completely determines the values of $\varphi(j)$ for j integer. As above we can then precisely compute $\varphi(j/2^\ell)$. As long as φ is continuous, this gives us an accurate picture of its graph.

Another approach to graphing, as well as to proofs of existence or other properties, is to note that a refinable function φ is a fixed point of the operator

$$Tf(x) = \sum_{k\in\mathbf{Z}} c_k\, f(2x-k).$$

A fixed point can often be computed via the iteration $f_{i+1} = Tf_i$ for a suitable starting function f_0. This iteration, called the *Cascade Algorithm*, is an important tool in the study of refinable functions. $\diamondsuit$

Example 12.21 (Orthonormal Translates). Given (c_0, c_3), suppose that φ is such that $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal sequence in $L^2(\mathbf{R})$. Set $c_k = 0$ for $k < 0$ and $k > 3$. Since only finitely many c_0 are nonzero, we can manipulate the order of the series in the following calculation as we like:

$$\begin{aligned}
\delta_{0,n} = \langle T_n\varphi, \varphi\rangle &= \int_{-\infty}^{\infty} \varphi(x-n)\,\overline{\varphi(x)}\,dx \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_j\,\overline{c_k} \int_{-\infty}^{\infty} \varphi(2x-2n-j)\,\overline{\varphi(2x-k)}\,dx \\
&= \frac{1}{2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_j\,\overline{c_k} \int_{-\infty}^{\infty} \varphi(x-2n-j)\,\overline{\varphi(x-k)}\,dx \\
&= \frac{1}{2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_j\,\overline{c_k}\,\big\langle T_{2n+j}\varphi,\, T_k\varphi\big\rangle \\
&= \frac{1}{2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_j\,\overline{c_k}\,\delta_{2n+j,k} \\
&= \frac{1}{2}\sum_{j=-\infty}^{\infty} c_j\,\overline{c_{2n+j}}.
\end{aligned} \tag{12.38}$$

Only c_0, c_1, c_2, c_3 can be nonzero, so this reduces to the two equations

$$\begin{aligned}
c_0^2 + c_1^2 + c_2^2 + c_3^2 &= 2, \\
c_0c_2 + c_1c_3 &= 0.
\end{aligned}$$

By the minimal accuracy assumption we have $c_2 = 1 - c_0$ and $c_3 = 1 - c_1$, so these two equations reduce yet further to

$$\Big(c_0 - \frac{1}{2}\Big)^2 + \Big(c_3 - \frac{1}{2}\Big)^2 = \frac{1}{2}. \tag{12.39}$$

Restricting to real-valued coefficients, this one-parameter family corresponds to the circle that appears in Figure 12.12. Any φ that has orthonormal integer translates must lie on this circle and, with a single exception, all of the functions on this circle do have orthonormal translates. The exception is the point $(1,1)$, which corresponds to the refinement equation

$$\varphi(x) \;=\; \varphi(2x) + \varphi(2x-3).$$

The solution is the "stretched box" $\varphi = \chi_{[0,3]}$, which does not have orthonormal integer translates. $\diamondsuit$

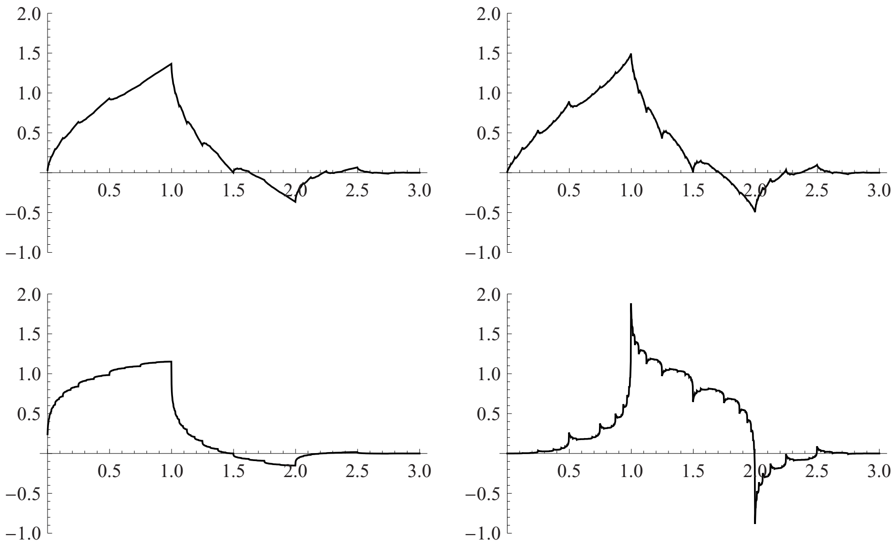

Fig. 12.13. The functions discussed in Example 12.22.

Example 12.22 (Examples). Figure 12.13 shows continuous refinable functions corresponding to four particular points that lie on the circle in Figure 12.12. On the top left we see the *Daubechies* D_4 *scaling function*, which corresponds to the point

$$(c_0, c_3) \;=\; \left(\frac{1+\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right) \;\approx\; (0.683013, -0.183013),$$

and on the top right we see the refinable function φ that corresponds to the "nearby" point

$$(c_0, c_3) \;=\; \left(\frac{3}{5}, -\frac{1}{5}\right) \;=\; (0.6, -0.2),$$

which has the computational advantage of being rational. The bottom row shows two points on the circle that are more "distant" from D_4. On the bottom left we see

$$(c_0, c_3) \;=\; \left(\frac{2+\sqrt{2}}{4}, \frac{2-\sqrt{6}}{4}\right) \;\approx\; (0.853553, -0.112372).$$

Note that this point is "close" to $(1, 0)$, which corresponds to the box function $\chi_{[0,1]}$. On the bottom right we see

$$(c_0, c_3) \;=\; \left(\frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{6}}{4}\right) \;\approx\; (0.146447, -0.112372),$$

which lies close to $(0, 0)$, whose corresponding refinable function is $\chi_{[1,2]}$. While it may not be obvious from their graphs, each of these four functions is continuous and has orthonormal integer translates.

These functions are continuous, but they are not differentiable. The refinement equation tells us that the graph of a refinable function has a certain kind of self-similarity, in the sense that φ equals a sum of translated, dilated, and rescaled copies of itself. A refinable function φ that is continuous but not differentiable is Hölder continuous in the sense of Exercise 1.23, and the graphs of such functions typically exhibit the fractal-like appearance that we see in Figure 12.13. Indeed, for x in the range $0 \le x \le 1/2$ we have precise self-similarity in the sense that $\varphi(x) = c_0\,\varphi(2x)$. It can be shown that the function D_4 is Hölder continuous precisely for exponents α in the range $0 < \alpha < -\log_2(1+\sqrt{3})/4 \approx 0.550\ldots$ [Dau92]. It is actually much more difficult to compute the analogous range for the function φ corresponding to the point $(0.6, -0.2)$, but numerical computations show that it is globally slightly smoother, in the sense that it is Hölder continuous for exponents in the range $0 < \alpha < -\log_2 0.660 \approx 0.600\ldots$ [CH94], [HC94].

Of all of the refinable functions corresponding to points on the "circle of orthogonality," D_4 is in some sense the "best." While D_4 is not differentiable, the point for D_4 in Figure 12.12 lies at the intersection of the circle and the dashed line (the other intersection point corresponds to the time-reversed function $D_4(3-x)$). The significance of the dashed line is that all points on it satisfy *two* accuracy conditions, namely,

$$\sum_{k\in\mathbf{Z}} (-1)^k\, k^j\, c_k \;=\; 0, \qquad j = 0, 1.$$

As a consequence, D_4 not only reproduces the constant function exactly, but also reproduces linear functions. Specifically, Exercise 12.30 shows that there exist scalars a_j such that

$$\sum_{j\in\mathbf{Z}} a_j\, D_4(x+j) \;=\; x, \qquad x \in \mathbf{R}.$$

Note that for any particular x, only finitely many terms in this series are nonzero. Higher-order accuracy is desirable for many reasons. It is necessary, though not sufficient, for φ to be smooth. In light of Taylor expansions, it is important to have as many polynomials as possible representable by translates of φ. The function D_4 corresponds to the point on the circle of orthogonality that satisfies the greatest number of accuracy conditions. $\diamondsuit$

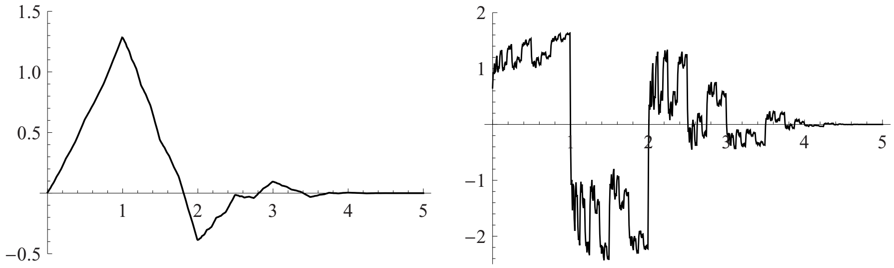

Fig. 12.14. The Daubechies function D_6 and its first derivative D_6'.

By taking N larger we can construct many more refinable functions, having more smoothness, satisfying more accuracy conditions, or having some other desirable property. However, exhaustively characterizing these properties, as we have attempted to do for $N = 3$, becomes increasingly difficult as N increases. Instead of trying to characterize all refinable functions, another approach is to create families which have particular desirable properties, such as the B-splines considered in Example 12.17. Unfortunately, the B-splines do not have orthonormal translates. The *Daubechies scaling functions* [Dau92] are a family of refinable functions D_{2N} that satisfy refinement equations of length $2N$, have orthonormal integer translates, satisfy N accuracy conditions, and increase in smoothness with N (though not as quickly as the B-splines). The function D_2 is the box function, and D_4 is the function discussed in Example 12.21. The function D_6, pictured in Figure 12.14, satisfies the refinement equation

$$D_6(x) \;=\; \sum_{k=0}^{5} c_k\, D_6(2x-k)$$

where

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} \;=\; \frac{1}{16} \begin{bmatrix} 1+\sqrt{10}+\sqrt{5+2\sqrt{10}} \\ 5+\sqrt{10}+3\sqrt{5+2\sqrt{10}} \\ 10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}} \\ 10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}} \\ 5+\sqrt{10}-3\sqrt{5+2\sqrt{10}} \\ 1+\sqrt{10}-\sqrt{5+2\sqrt{10}} \end{bmatrix}.$$

This function is differentiable, but only "barely" so, in the sense that its first derivative is Hölder continuous only for exponents α that lie in the range $0 < \alpha < 0.087833\ldots$, see [DL92]. For complete details on the construction of the Daubechies family of scaling functions, we refer to [Dau92], [Wal02].

We close this section by remarking that a refinement equation of finite length is an expression of finite linear dependence among the time-scale shifts of φ. In contrast, the HRT Conjecture discussed in Section 11.9 is that every function in $L^2(\mathbf{R})$ has finitely linearly independent time-frequency shifts. Thus time-scale and time-frequency shifts behave quite differently in this regard. But why is there such a difference? There seems to be no satisfying answer to this question at present.

More abstractly, time-scale shifts are associated with a representation of the affine group (Exercise 12.2), while time-frequency shifts are associated with a representation of the Heisenberg group (Exercise 11.2). Formulated in an abstract group setting, Linnell has observed that the HRT Conjecture is related to zero divisor issues [Lin99]. In this context, the HRT Conjecture has a flavor similar to the Zero Divisor Conjecture of Higman [Hig40], which has been open since 1940.

Conjecture 12.23 (Zero Divisor Conjecture). *The group algebra FG of a torsion-free group G over a field F is a domain.* $\diamondsuit$

We return in the next section to analysis of general refinement equations and their solutions, especially those that have orthonormal integer translates.

Exercises

12.13. Suppose that $\varphi \in L^2(\mathbf{R})$ is refinable, with refinement coefficients $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$. Show that $T_m\varphi$ is refinable for every $m \in \mathbf{Z}$, and the refinement coefficients for $T_m\varphi$ are $(c_{k+m})_{k\in\mathbf{Z}}$.

12.14. The hat function on $[0,2]$ is $B_1(x) = \max\{1-|x-1|, 0\}$. Show directly that B_1 is refinable.

12.15. The Shannon scaling function is the sinc function $\varphi(x) = \frac{\sin \pi x}{\pi x}$. Show that φ is refinable, and show that the symbol for the refinement equation is $m_0 = \chi_{[-\frac{1}{4},\frac{1}{4}]}$ (extended 1-periodically to $\mathbf{R}$).

12.16. The following refinement equation (based on dilation by 3) has a continuous solution:

$$\varphi(x) \;=\; \frac{1}{2}\varphi(3x) + \frac{1}{2}\varphi(3x-1) + \varphi(3x-2) + \frac{1}{2}\varphi(3x-3) + \frac{1}{2}\varphi(3x-4).$$

What function satisfies this refinement equation?

12.17. Justify the factorization performed in equation (12.32) in the proof of Lemma 12.15.

12.18. Fix $\varphi \in L^2(\mathbf{R})$ and $m(\xi) = \frac{1}{2}\sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}$ with $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, and suppose that

$$\widehat{\varphi}(\xi) \;=\; m(\xi/2)\,\widehat{\varphi}(\xi/2) \text{ a.e.}$$

Show that if either:

(a) $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is a Bessel sequence, or

(b) $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$,

then φ is refinable.

12.19. (a) The *convolution* of sequences $c = (c_k)_{k\in\mathbf{Z}}$ and $d = (d_k)_{k\in\mathbf{Z}}$ in $\ell^1(\mathbf{Z})$ is the sequence $c * d$ whose components are

$$(c*d)_k \;=\; \sum_{j\in\mathbf{Z}} c_j\, d_{k-j}.$$

Show that this series converges for every k, and $c * d \in \ell^1(\mathbf{Z})$. Show further that if c and d are both finite sequences, then so is $c * d$.

(b) The *convolution* of functions $\varphi,\, \psi \in L^1(\mathbf{R})$ is the function $\varphi * \psi$ given by

$$(\varphi * \psi)(x) \;=\; \int_{-\infty}^{\infty} \varphi(y)\,\psi(x-y)\,dy.$$

Show that this integral exists for almost every x, and we have $\varphi * \psi \in L^1(\mathbf{R})$. Show also that if φ and ψ are each compactly supported, then so is $\varphi * \psi$.

(c) Suppose that $\varphi,\, \psi \in L^1(\mathbf{R})$ satisfy the refinement equations

$$\varphi(x) \;=\; \sum_{k\in\mathbf{Z}} c_k\,\varphi(2x-k) \qquad \text{and} \qquad \psi(x) \;=\; \sum_{k\in\mathbf{Z}} d_k\,\psi(2x-k),$$

where $c = (c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$ and $d = (d_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$. Show that their convolution $\varphi * \psi$ is refinable. Let $m_c(x) = \frac{1}{2}\sum c_k e^{-2\pi i k\xi}$ be the symbol for φ and $m_d(x) = \frac{1}{2}\sum c_k e^{-2\pi i k\xi}$ be the symbol for ψ, and show that the symbol for $\varphi * \psi$ is $m_{c*d}(\xi) = m_c(\xi)\, m_d(\xi)$.

12.20. Set $B_0 = \chi_{[0,1]}$, and recursively define the nth *B-spline* B_n by

$$B_n \;=\; B_{n-1} * \chi_{[0,1]},$$

where $*$ denotes the convolution operation defined in Exercise 12.19.

(a) Show that $B_1 = \chi_{[0,1]} * \chi_{[0,1]}$ is the hat function on the interval $[0,2]$, and find explicit formulas for B_2 and B_2'.

(b) Show that B_n is refinable, integrable, and compactly supported. Show that the symbol for this refinement equation is

$$m_0(\xi) \;=\; \left(\frac{1+e^{-2\pi i\xi}}{2}\right)^{n+1}.$$

(c) Find an explicit formula for $\widehat{B_n}$, and show that $\widehat{B_n} \in L^1(\mathbf{R})$ for all $n > 0$.

(d) Prove that $B_n'(x) = B_{n-1}(x-1) - B_{n-1}(x)$ for $n > 1$.

(e) Show that $B_n \in C^{n-1}(\mathbf{R})$ for $n > 0$, and $B_n^{(n-1)}$ is piecewise linear.

12.21. Suppose that $\varphi \in L^2(\mathbf{R})$ is a compactly supported solution to the finite length refinement equation

$$\varphi(x) \;=\; \sum_{k=0}^{N} c_k\, \varphi(2x-k).$$

Prove the following statements.

(a) $\operatorname{supp}(\varphi) \subseteq [0,N]$, i.e., $\varphi(x) = 0$ for almost every $x \notin [0,N]$.

(b) If φ is continuous then the vector $\big(\varphi(1),\dots,\varphi(N-1)\big)$ is an eigenvector of the $(N-1)\times(N-1)$ matrix

$$M \;=\; \begin{bmatrix} c_1 & c_0 & 0 & \cdots & 0 & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_N & c_{N-1} \end{bmatrix} \;=\; [c_{2i-j}]_{i,j=1,\dots,N}$$

for the eigenvalue 1. Note the double-shift in the rows of M; for this reason M is called a *two-slanted matrix*.

(c) If φ is differentiable then φ' is refinable.

(d) If $\varphi \in C^1(\mathbf{R})$ then 1 and 1/2 are both eigenvalues of M.

(e) φ cannot be infinitely differentiable, and in fact can have at most $N-2$ continuous derivatives.

12.22. Suppose $\varphi \in L^2(\mathbf{R})$ is refinable with refinement coefficients $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$. Show that if φ is compactly supported and $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is orthonormal, then only finitely many of the refinement coefficients are nonzero.

12.6 All About the Scaling Function, II: Existence

We cannot choose coefficients $(c_k)_{k\in\mathbf{Z}}$ at random and expect that a solution φ to the refinement equation

$$\varphi(x) \;=\; \sum_{k\in\mathbf{Z}} c_k\,\varphi(2x-k)$$

will exist. At minimum, we usually assume $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$ and require $\sum c_k = 2$. To motivate this normalization, suppose that φ is refinable and $\widehat{\varphi}(\xi)$ is continuous. Then we can take $\xi = 0$ in equation (12.29), and we obtain $\widehat{\varphi}(0) = m_0(0)\,\widehat{\varphi}(0)$. As we will see in Corollary 12.26, if φ is not the zero function then we must have $\widehat{\varphi}(0) \neq 0$, and therefore $m_0(0) = 1$. This is equivalent to the requirement that $\sum c_k = 2$. We will see later that other conditions, such as the minimal accuracy condition imposed in most of the examples in Section 12.5, are also important.

Iterating equation (12.29), we see that

$$\begin{aligned}\widehat{\varphi}(\xi) \;&=\; m_0(\xi/2)\,\widehat{\varphi}(\xi/2)\\ &=\; m_0(\xi/2)\,m_0(\xi/4)\,\widehat{\varphi}(\xi/4)\\ &\;\;\vdots\\ &=\; \left(\prod_{j=1}^{n} m_0(2^{-j}\xi)\right)\widehat{\varphi}(2^{-n}\xi).\end{aligned} \tag{12.40}$$

If m_0 is continuous then the normalization $m_0(0) = 1$ implies that

$$\lim_{n\to\infty} m_0(2^{-n}\xi) \;=\; m_0(0) \;=\; 1.$$

This suggests taking a limit in equation (12.40). An infinite product is implicated, and we suspect that $\widehat{\varphi}$ will be given by

$$\widehat{\varphi}(\xi) \;=\; C\prod_{j=1}^{\infty} m_0(2^{-j}\xi), \tag{12.41}$$

where C is the constant $\widehat{\varphi}(0)$.

The calculations above assume that a refinable function φ exists and has certain properties, whereas what we really want to do is to start with refinement coefficients $(c_k)_{k\in\mathbf{Z}}$ and show that a solution to the corresponding refinement equation exists. The function m_0 is determined by the c_k, and, in light of the discussion above, a good start would be to try to choose c_k so that the infinite product

$$P(\xi) \;=\; \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$$

converges. In this case, equation (12.41) suggests that we might be able to define a refinable function φ by declaring that its Fourier transform is $\widehat{\varphi}(\xi) = P(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$. Since the Fourier transform is unitary on $L^2(\mathbf{R})$, if we have $P \in L^2(\mathbf{R})$ then there will exist a function $\varphi \in L^2(\mathbf{R})$ such that $\widehat{\varphi} = P$. This function φ is likely to be the refinable function that we seek.

The remainder of this section essentially makes this approach precise. First we will find conditions on the c_k that ensure that the infinite product converges, then we impose further conditions that imply that P is square integrable, and finally we obtain our refinable function φ. In so doing, we will find some necessary and some sufficient conditions for the existence of a solution to the refinement equation, but we will not find conditions that are both necessary and sufficient.

We must impose some decay conditions on the scalars c_k in order to obtain results about the convergence of the infinite product or the existence of a scaling function. Typical hypotheses and their relation to properties of m_0 and the refinement equation are given next.

Example 12.24. (a) If $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, then the series defining m_0 converges unconditionally in $L^2(\mathbf{T})$. In this case, m_0 is square integrable over a period, although it is only defined almost everywhere. On the other hand, with this hypothesis on the c_k we cannot even be sure that the series $\sum_{k\in\mathbf{Z}} c_k\,\varphi(2x-k)$ in the refinement equation will converge, and therefore it will be very difficult to determine if the refinement equation has any solutions. Even if this series does converge, we may be left with unpleasant issues about conditional versus unconditional convergence, as in the proof of Lemma 12.15.

(b) If $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$, then the series $m_0(\xi) = (1/2)\sum_{k\in\mathbf{Z}} c_k e^{-2\pi i k\xi}$ converges absolutely in $L^p(\mathbf{T})$ for every p. In particular, $p=\infty$ corresponds to uniform convergence, and since each function $e^{2\pi i k\xi}$ is continuous it follows that m_0 is continuous. Also, this hypothesis implies that

$$\sum_{k\in\mathbf{Z}} \|c_k\,\varphi(2x-k)\|_{L^2} \;=\; 2^{-1/2}\sum_{k\in\mathbf{Z}} |c_k|\,\|\varphi\|_{L^2} \;<\; \infty,$$

so the series $\sum_{k\in\mathbf{Z}} c_k\,\varphi(2x-k)$ converges absolutely in $L^2(\mathbf{R})$. This still does not imply that there is a solution to the refinement equation, but at least it gives us something to work with.

(c) If $\sum_{k\in\mathbf{Z}} |kc_k| < \infty$, then m_0 is not only continuous, but is differentiable and has a continuous derivative (Exercise 12.28), and we will see in Theorem 12.25 that this hypothesis implies that the infinite product discussed above converges.

(d) If $(c_k)_{k\in\mathbf{Z}}$ is a finite sequence then m_0 is a trigonometric polynomial and hence is infinitely differentiable and has at most finitely many zeros in any finite interval. $\diamondsuit$

The next result, whose proof is adapted from [DL91], shows that if we assume the normalization condition $m_0(0) = 1$ and sufficient decay on the coefficients c_k, then the infinite product in equation (12.41) will converge.

Theorem 12.25. *Suppose that we have*
(a) *decay:* $\sum_{k\in\mathbf{Z}} |kc_k| < \infty$, *and*
(b) *normalization:* $m_0(0) = \frac{1}{2}\sum_{k\in\mathbf{Z}} c_k = 1$.

Define $P_n(\xi) = \prod_{j=1}^{n} m_0(2^{-j}\xi)$. *Then the infinite product*

$$P(\xi) \;=\; \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \;=\; \lim_{n\to\infty} P_n(\xi)$$

converges uniformly on compact subsets of $\mathbf{R}$ *to a continuous function* P *that satisfies* $P(0) = 1$.

Proof. By Exercise 12.28, the symbol m_0 is differentiable and has a continuous derivative. It therefore follows from the Mean Value Theorem that m_0 is Lipschitz (see Exercise 1.23). In particular, since $m_0(0) = 1$, there is a constant C such that

$$|m_0(\xi) - 1| \;\le\; C|\xi|, \qquad \xi \in \mathbf{T}.$$

Combining this with the fact that $1 + x \le e^x$, we obtain the estimate

$$\begin{aligned} |P_n(\xi)| \;&=\; \prod_{j=1}^{n} \big|\big(m_0(2^{-j}\xi) - 1\big) + 1\big| \\ &\le\; \prod_{j=1}^{n} \big(C\,2^{-j}\,|\xi| + 1\big) \\ &\le\; \prod_{j=1}^{n} e^{C\,2^{-j}\,|\xi|} \\ &=\; e^{C\sum_{j=1}^{n} 2^{-j}|\xi|} \\ &\le\; e^{C|\xi|}. \end{aligned}$$

Note that this upper bound is independent of n. If we fix $R > 0$ then

$$\begin{aligned} \sup_{\xi\in[-R,R]} |P_n(\xi) - P_{n-1}(\xi)| \;&=\; \sup_{\xi\in[-R,R]} |m_0(2^{-n}\xi) - 1|\,|P_{n-1}(\xi)| \\ &\le\; C\,2^{-n}\,R\,e^{CR}. \end{aligned}$$

Consequently $\{P_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $L^\infty[-R,R]$. Therefore, since each P_n is continuous there exists a continuous function q_R on $[-R,R]$ such that $P_n \to q_R$ uniformly on $[-R,R]$ as $n \to \infty$. Clearly any two of these functions q_R must coincide on the domain where they are both defined, so there is a single continuous function P defined on the real line that equals q_R on $[-R,R]$ for every R. Further, $P(0) = 1$ since $P_n(0) = 1$ for every n. □

As a consequence, we obtain the following facts about the existence and uniqueness of a solution to the refinement equation.

Corollary 12.26. *Assume that the hypotheses of Theorem 12.25 are satisfied, and let P be as in that theorem.*

(a) *If $\varphi \in L^1(\mathbf{R})$ satisfies the refinement equation* (12.28) *then $\widehat{\varphi}$ is a scalar multiple of P. Specifically, if such an integrable φ exists then*

$$\widehat{\varphi}(\xi) \;=\; \widehat{\varphi}(0)\,P(\xi).$$

Consequently, if φ is not the zero function then $\int_{-\infty}^{\infty} \varphi(x)\,dx = \widehat{\varphi}(0) \neq 0$.

(b) *If $P \in L^2(\mathbf{R})$ then there exists a function $\varphi \in L^2(\mathbf{R})$ such that $\widehat{\varphi} = P$, and this function φ satisfies the refinement equation* (12.28).

Proof. (a) If $\varphi \in L^1(\mathbf{R})$, then its Fourier transform $\widehat{\varphi}$ is a continuous function (Theorem 9.10). By Lemma 12.15, if φ satisfies the refinement equation then $\widehat{\varphi}(\xi) = m_0(\xi/2)\,\widehat{\varphi}(\xi/2)$. Iterating this equation and applying Theorem 12.25, we see that

$$\widehat{\varphi}(\xi) \;=\; \biggl(\prod_{j=1}^{n} m_0(2^{-j}\xi)\biggr)\,\widehat{\varphi}(2^{-n}\xi) \;=\; P_n(\xi)\,\widehat{\varphi}(2^{-n}\xi) \;\to\; P(\xi)\,\widehat{\varphi}(0)$$

as $n \to \infty$.

(b) If $P \in L^2(\mathbf{T})$, then it has an inverse Fourier transform $\varphi = \check{P} \in L^2(\mathbf{T})$, and this is the unique function that satisfies $\widehat{\varphi} = P$. Since $P(\xi) = m_0(\xi/2)\,P(\xi/2)$, it follows from Exercise 12.18 that φ is refinable. □

While there do exist refinable functions φ that are not integrable, they are usually not of much use to us. Thus, Corollary 12.26 tells us that, in most practical situations, if a solution to the refinement equation exists at all then it is unique (up to a scaling factor). Corollary 12.26 also tells us that if we can ensure that $P \in L^2(\mathbf{R})$, then a solution to the refinement equation will exist. Unfortunately, the hypotheses of Theorem 12.25 and Corollary 12.26 do not in general imply that P will be square integrable. On the other hand, the next result gives a necessary condition for a refinable function to have orthonormal integer translates, and in Theorem 12.29 we will see that this necessary condition actually implies that we have $P \in L^2(\mathbf{R})$. As usual, we let $\Phi_\varphi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi+k)|^2$ denote the periodization of $|\widehat{\varphi}|^2$.

Theorem 12.27. *If $\varphi \in L^2(\mathbf{R})$ is refinable and satisfies the refinement equation* (12.28) *with coefficients $(c_k)_{k\in\mathbf{Z}} \in \ell^2$, then the following statements hold.*

(a) $\Phi_\varphi(\xi) = \bigl|m_0\bigl(\frac{\xi}{2}\bigr)\bigr|^2\,\Phi_\varphi\bigl(\frac{\xi}{2}\bigr) \;+\; \bigl|m_0\bigl(\frac{\xi}{2}+\frac{1}{2}\bigr)\bigr|^2\,\Phi_\varphi\bigl(\frac{\xi}{2}+\frac{1}{2}\bigr)$ *a.e.*

(b) *If $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal sequence, then*

$$|m_0(\xi)|^2 + |m_0(\xi+\tfrac{1}{2})|^2 \;=\; 1 \text{ a.e.} \tag{12.42}$$

(c) *If* $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$, *then equation* (12.42) *is equivalent to*

$$\sum_{k\in\mathbf{Z}} \overline{c_k}\, c_{k+2n} \;=\; 2\delta_{0n}, \qquad n \in \mathbf{Z}. \tag{12.43}$$

Proof. (a) By Lemma 12.15, $\widehat{\varphi}(\xi) = m_0(\xi/2)\,\widehat{\varphi}(\xi/2)$ a.e. Applying the refinement equation and the periodicity of m_0, we therefore have

$$\begin{aligned}
\Phi_\varphi(\xi) &= \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi+k)|^2 \\
&= \sum_{k\in\mathbf{Z}} \left|m_0\left(\tfrac{\xi+k}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\tfrac{\xi+k}{2}\right)\right|^2 \\
&= \sum_{k\in\mathbf{Z}} \left|m_0\left(\tfrac{\xi+2k}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\tfrac{\xi+2k}{2}\right)\right|^2 \;+\; \sum_{k\in\mathbf{Z}} \left|m_0\left(\tfrac{\xi+2k+1}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\tfrac{\xi+2k+1}{2}\right)\right|^2 \\
&= \left|m_0\left(\tfrac{\xi}{2}\right)\right|^2 \sum_{k\in\mathbf{Z}} \left|\widehat{\varphi}\left(\tfrac{\xi}{2}+k\right)\right|^2 \;+\; \left|m_0\left(\tfrac{\xi+1}{2}\right)\right|^2 \sum_{k\in\mathbf{Z}} \left|\widehat{\varphi}\left(\tfrac{\xi+1}{2}+k\right)\right|^2 \\
&= \left|m_0\left(\tfrac{\xi}{2}\right)\right|^2 \Phi_\varphi\left(\tfrac{\xi}{2}\right) \;+\; \left|m_0\left(\tfrac{\xi}{2}+\tfrac{1}{2}\right)\right|^2 \Phi_\varphi\left(\tfrac{\xi}{2}+\tfrac{1}{2}\right).
\end{aligned}$$

(b) Lemma 12.13 implies that if φ has orthonormal integer translates then $\Phi_\varphi(\xi) = 1$ a.e. It therefore follows from part (a) that $|m_0(\frac{\xi}{2})|^2 + |m_0(\frac{\xi}{2}+\frac{1}{2})|^2 = 1$ a.e., so the result follows by replacing ξ with 2ξ.

(c) We assign the proof of the equivalence of equations (12.42) and (12.43) as Exercise 12.23. □

Note that equation (12.43) is precisely the necessary condition for orthonormal translates that we derived in equation (12.38) for the special case of 4-tap refinement equations.

The converse of Theorem 12.27 does not hold in general. On the other hand, it can be shown that counterexamples like the one following are "rare" in some sense (see [Law90]).

Example 12.28. The stretched box $\chi_{[0,3]}$ satisfies the refinement equation

$$\chi_{[0,3]}(x) \;=\; \chi_{[0,3]}(2x) + \chi_{[0,3]}(2x-3),$$

which has refinement coefficients $c_0 = c_3 = 1$ and all other $c_k = 0$. A direct calculation shows that equation (12.43) is satisfied, yet $\chi_{[0,3]}$ does not have orthonormal integer translates. As discussed in Example (12.21), out of all the refinement equations of length 4 that satisfy equation (12.43) only the stretched box fails to have orthonormal translates. ♢

In signal processing terminology, equation (12.42) is the *antialiasing condition.* Moreover, the equivalent form given in equation (12.43) is almost a convolution condition. Let $\widetilde{c}$ denote the sequence $\widetilde{c} = \left(\overline{c_{-k}}\right)_{k\in\mathbf{Z}}$. The *convolution* of $\widetilde{c}$ with c is the sequence $\widetilde{c} * c$ whose components are

$$(\widetilde{c} * c)_n = \sum_{k\in\mathbf{Z}} \widetilde{c}_k \, c_{n-k}, = \sum_{k\in\mathbf{Z}} \overline{c_{-k}} \, c_{n-k}, = \sum_{k\in\mathbf{Z}} \overline{c_k} \, c_{k+n}, \qquad n \in \mathbf{Z}.$$

Thus, the equivalent form of the antialiasing condition given in equation (12.43) says that the *downsampled* sequence $\left((\widetilde{c} * c)_{2n}\right)_{n\in\mathbf{Z}}$ is twice the delta sequence. For more on convolution see Exercise 12.19 and Section 13.3.

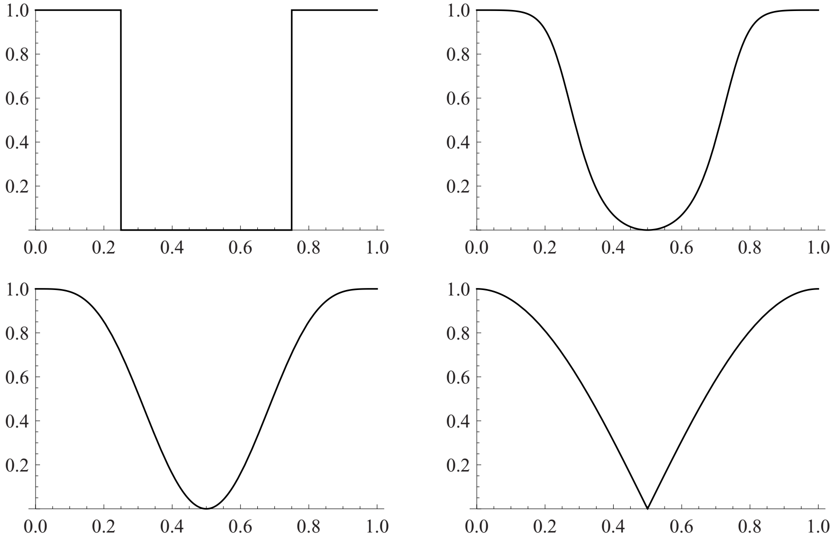

Fig. 12.15. Symbols for some refinable functions. Top left: Shannon scaling function. Top right: Linear spline Battle–Lemarié scaling function (see Example 12.38). Bottom left: Daubechies D_4 function (absolute value of the symbol). Bottom right: Box function $\chi_{[0,1]}$ (absolute value of the symbol).

If the antialiasing condition holds and m_0 is continuous, then $m_0(0) = 1$ and so $m_0(\xi)$ will be close to 1 for ξ close to zero (and, by periodicity, for ξ close to any integer n). These are the "low frequencies" in $\mathbf{T}$, so antialiasing roughly corresponds to m_0 being close to 1 for low frequencies and close to 0 for high frequencies. Thus m_0 is a *low-pass filter*. The "ideal" low-pass filter would be the 1-periodic extension of $\chi_{[-\frac{1}{4},\frac{1}{4}]}$. Indeed, this is the symbol for the Shannon scaling function. Unfortunately, the refinement coefficients for the Shannon scaling function decay very slowly. If we impose faster decay on the c_k then m_0 will be smoother, but by the same token it will be farther from the "ideal" low-pass filter. In Figure 12.15 we show the symbols m_0 for four particular refinable functions that satisfy the antialiasing requirement.

Now we show that when the decay, normalization, and antialiasing conditions all hold simultaneously, there will exist a solution to the refinement equation. This argument is due to Mallat [Mal89b].

Theorem 12.29. *If we have*

(a) *decay:* $\sum_{k\in\mathbf{Z}} |kc_k| < \infty$,

(b) *normalization:* $m_0(0) = 1$, *and*

(c) *antialiasing:* $|m_0(\xi)|^2 + |m_0(\xi+\frac{1}{2})|^2 = 1$ *for* $\xi \in \mathbf{T}$,

then the function $\varphi \in L^2(\mathbf{R})$ *whose Fourier transform is*

$$\widehat{\varphi}(\xi) \;=\; \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$$

satisfies the refinement equation $\varphi(x) = \sum_{k\in\mathbf{Z}} c_k\,\varphi(2x-k)$. *Moreover,* $\widehat{\varphi}$ *is continuous and* $\widehat{\varphi}(0) = 1$.

Proof. The decay hypothesis implies that m_0 is continuous (which is why the statement of hypothesis (c) is for all ξ rather than almost every ξ). Set

$$P_n(\xi) \;=\; \prod_{j=1}^{n} m_0(2^{-j}\xi) \qquad \text{and} \qquad p_n \;=\; P_n \cdot \chi_{[-2^{n-1},2^{n-1}]}.$$

Then P_n is continuous, and p_n is continuous on the interval $[-2^{n-1},2^{n-1}]$. By Theorem 12.25, the infinite product

$$P(\xi) \;=\; \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \;=\; \lim_{n\to\infty} P_n(\xi)$$

converges uniformly on compact sets, and therefore p_n converges pointwise to the continuous function P. Since $m_0(2^{-j}\xi)$ is 2^j-periodic, the function P_n is 2^n-periodic. Further, for $n > 1$ we have

$$P_n(\xi) \;=\; m_0(2^{-n}\xi)\,P_{n-1}(\xi).$$

Since p_n is supported on $[-2^{n-1},2^{n-1}]$, we therefore compute that

$$\begin{aligned}
\|p_n\|_{L^2}^2 \;&=\; \int_{-2^{n-1}}^{2^{n-1}} |P_n(\xi)|^2\,d\xi \\
&=\; \int_0^{2^{n-1}} |P_n(\xi)|^2\,d\xi \;+\; \int_0^{2^{n-1}} |P_n(\xi+2^{n-1})|^2\,d\xi \\
&=\; \int_0^{2^{n-1}} |m_0(2^{-n}\xi)|^2\,|P_{n-1}(\xi)|^2\,d\xi \\
&\qquad +\; \int_0^{2^{n-1}} |m_0(2^{-n}\xi+\tfrac{1}{2})|^2\,|P_{n-1}(\xi+2^{n-1})|^2\,d\xi
\end{aligned}$$

$$= \int_0^{2^{n-1}} \left(|m_0(2^{-n}\xi)|^2 + |m_0(2^{-n}\xi + \tfrac{1}{2})|^2\right) |P_{n-1}(\xi)|^2 \, d\xi \qquad (12.44)$$

$$= \int_0^{2^{n-1}} |P_{n-1}(\xi)|^2 \, d\xi$$

$$= \int_{-2^{n-2}}^{2^{n-2}} |P_{n-1}(\xi)|^2 \, d\xi \qquad (12.45)$$

$$= \|p_{n-1}\|_{L^2}^2,$$

where at equations (12.44) and (12.45) we have used the fact that P_{n-1} is 2^{n-1}-periodic.

Thus $\|p_n\|_{L^2}$ is independent of n. Applying Fatou's Lemma (Theorem A.19), we obtain

$$\begin{aligned} \|P\|_{L^2}^2 \;=\; \int_{-\infty}^{\infty} |P(\xi)|^2 \, d\xi \;&=\; \int_{-\infty}^{\infty} \lim_{n\to\infty} |p_n(\xi)|^2 \, d\xi \\ &\le\; \lim_{n\to\infty} \int_{-\infty}^{\infty} |p_n(\xi)|^2 \, d\xi \\ &=\; \|p_1\|_{L^2}^2 \;<\; \infty. \end{aligned}$$

Therefore $P \in L^2(\mathbf{R})$, so by Corollary 12.26 the function $\varphi \in L^2(\mathbf{R})$ that satisfies $\widehat{\varphi} = P$ is refinable. Finally, $\widehat{\varphi}(0) = P(0) = 1$. □

This brings us close to a method for constructing MRAs. The preceding results tell us that if we can find coefficients $(c_k)_{k\in\mathbf{Z}}$ that satisfy the decay, normalization, and antialiasing conditions of Theorem 12.29, then there will exist a function $\varphi \in L^2(\mathbf{R})$ that satisfies the refinement equation with coefficients $(c_k)_{k\in\mathbf{Z}}$. The antialiasing condition is necessary but not sufficient to ensure that φ will have orthonormal integer translates, but "most" of the time this will be the case. We will see in the next section that if φ does have orthonormal integer translates then it generates an MRA, and from this MRA we can construct a wavelet ψ such that $\mathcal{W}(\psi)$ is a dyadic wavelet orthonormal basis for $L^2(\mathbf{R})$.

Before proceeding to the wavelet, we make some remarks about accuracy conditions. If we combine the antialiasing condition with the normalization $m_0(0) = 1$ we obtain $m_0(1/2) = 0$. In terms of the refinement coefficients, this says that

$$\sum_{k\in\mathbf{Z}} c_k \;=\; 2 \qquad \text{and} \qquad \sum_{k\in\mathbf{Z}} (-1)^k c_k \;=\; 0.$$

We call this the "minimal accuracy condition." It is a consequence of (but not equivalent to) the normalization and antialiasing hypotheses on the refinement coefficients. It is an "accuracy" condition in that it implies that the periodization of φ is constant almost everywhere (Exercise 12.29). More precisely, if there is an integrable solution φ to the refinement equation then the minimal accuracy condition implies that $\sum_{j\in\mathbf{Z}} \varphi(x+j) = C$, a constant, almost everywhere. This constant must be nonzero if the integer translates of φ are independent in an appropriate sense.

To motivate why this is interesting, recall that we are hoping to build an MRA from φ. In particular, the space V_0 is generated from the integer translates of φ. Since the constant function does not belong to $L^2(\mathbf{R})$ we know that the series $\sum_{k\in\mathbf{Z}} \varphi(x+k) = C$ does not converge in L^2-norm, and the constant function C cannot literally belong to V_0. But still in some sense we have "in spirit" that constants belong to V_0, and higher-order accuracy conditions,

$$\sum_{k\in\mathbf{Z}} (-1)^k\, k^j\, c_k \;=\; 0, \qquad j = 0, \dots, p, \tag{12.46}$$

correspond to polynomials up to degree p being representable by integer translates of φ (see Exercise 12.30). Considering Taylor expansions suggests that the more polynomials that "belong" to V_0, the better we will be able to approximate the smooth parts of functions by elements of V_0. Another reason to impose accuracy conditions is that high smoothness requires high accuracy, although the converse does not hold in general.

Exercises

12.23. Show that equations (12.42) and (12.43) are equivalent when $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$.

12.24. Show that the conclusion of Theorem 12.25 remains valid if we replace the decay hypothesis on the refinement coefficients by $\sum_{k\in\mathbf{Z}} |k|^\delta |c_k| < \infty$ for some $\delta > 0$.

12.25. Suppose that $c = (\dots, 0, 0, c_0, \dots, c_N, 0, 0, \dots)$ is a finite sequence with $c_0, c_N \neq 0$, Show that if the antialiasing requirement (12.42) is satisfied then N is odd.

12.26. Let $\chi = \chi_{[0,1)}$ be the box function, which satisfies the refinement equation $\chi(x) = \chi(2x) + \chi(2x-1)$.

(a) Show that $\widehat{\chi}(\xi) = e^{\pi i \xi}\, \frac{\sin \pi\xi}{\pi\xi}$.

(b) Show that $|m_0(\xi)|^2 + |m_0(\xi + \frac{1}{2})|^2 = 1$.

(c) Give a direct proof of *Viète's formula*:

$$\prod_{j=1}^{\infty} \cos(2^{-j}\pi\xi) \;=\; \frac{\sin \pi\xi}{\pi\xi}.$$

(d) Explain how Viète's formula relates to Theorem 12.29.

Remark: Viète proved the special case $\xi = 1/2$, which leads to the formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

12.27. Show that Theorem 12.29 continues to hold if instead of the antialiasing condition we only require that $|m_0(\xi)|^2 + |m_0(\xi+\frac{1}{2})|^2 \le 1$ for $\xi \in \mathbf{T}$.

12.28. (a) Show that if $\sum_{k\in\mathbf{Z}} |k^j c_k| < \infty$, then m_0 is j-times differentiable and its jth derivative is the continuous function

$$m_0^{(j)}(\xi) = \frac{(-2\pi i)^j}{2} \sum_{k\in\mathbf{Z}} k^j\, c_k\, e^{-2\pi i k x}.$$

Conclude that

$$m_0^{(j)}(1/2) = 0 \quad\Longleftrightarrow\quad \sum_{k\in\mathbf{Z}} (-1)^k\, k^j\, c_k = 0,$$

and compare this to the accuracy conditions given in equation (12.46).

(b) Let B_n be the nth B-spline defined in Exercise 12.20. Show that the *accuracy condition* $m_0^{(j)}(1/2) = 0$ holds for $j = 0, \dots, n$.

12.29. Suppose that $\varphi \in L^1(\mathbf{R})$ is refinable, with refinement coefficients $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$ that satisfy $\sum_{k\in\mathbf{Z}} c_k = 2$. Show that if the "minimal accuracy" condition $\sum_k (-1)^k\, c_k = 0$ holds then the periodization of φ is equal almost everywhere to a constant, specifically

$$\sum_{j\in\mathbf{Z}} \varphi(x+j) = \widehat{\varphi}(0) \quad \text{a.e.}$$

(This is easier to do if φ is assumed to be continuous.)

12.30. This exercise extends Exercise 12.29, but to avoid issues of convergence we assume that φ is a compactly supported solution of a refinement equation with finitely many nonzero coefficients c_k. Suppose that $\sum_{k\in\mathbf{Z}} c_k = 2$ and the accuracy conditions $\sum_{k\in\mathbf{Z}} (-1)^k\, k^j\, c_k = 0$ hold for $j = 0$ and $j = 1$. Show that

$$\sum_{j\in\mathbf{Z}} (j-2a)\,\varphi(x+j) = x \quad \text{a.e.},$$

where

$$a = \sum_{k\in\mathbf{Z}} 2k\, c_{2k} = \sum_{k\in\mathbf{Z}} (2k+1)\, c_{2k+1}.$$

12.31. The *Hilbert transform* of $f \in L^2(\mathbf{R})$ is the function Hf whose Fourier transform is $(Hf)^\wedge(\xi) = -i\,\mathrm{sign}(\xi)\,\widehat{f}(\xi)$, where $\mathrm{sign}(\xi) = 1, 0,$ or -1 according to whether $\xi > 0$, $\xi = 0$, or $\xi < 0$, respectively.

(a) Show that the Hilbert transform is a unitary mapping of $L^2(\mathbf{R})$ onto itself, but it does not map $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ onto itself. In particular, if $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ satisfies $\widehat{f}(0) \neq 0$, then $Hf \notin L^1(\mathbf{R})$.

(b) Suppose that $\varphi \in L^2(\mathbf{R})$ satisfies a refinement equation with coefficients $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$. Show that $H\varphi$ satisfies the same refinement equation.

12.32. Show that if the hypotheses of Theorem 12.25 are satisfied, then the function P has at most polynomial growth at infinity, i.e., there exists some $C > 0$ and integer $M > 0$ such that

$$|P(\xi)| \;\le\; C\,|\xi|^M, \qquad |\xi| \ge 1.$$

12.33. Assume $(c_k)_{k\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$ satisfies $|m_0(0)| = \frac{1}{2}\left|\sum c_k\right| < 1$. Show that there are no nontrivial integrable solutions to the corresponding refinement equation.

Note: Exercises 12.34–12.36 require some knowledge of distributions, tempered distributions, and the distributional Fourier transform.

12.34. The dilation of a distribution $\mu \in \mathcal{D}'(\mathbf{R})$ is defined by duality:

$$\langle f, D_2\mu\rangle \;=\; \langle D_{1/2}f, \mu\rangle, \qquad f \in C_c^\infty(\mathbf{R}).$$

We say that a distribution μ satisfies the refinement equation $\varphi(x) = 2\,\varphi(2x)$ in a distributional sense if $\mu = 2^{1/2}D_2\mu$.

(a) Show that the function $1/x$ satisfies the one-term refinement equation $\varphi(x) = 2\,\varphi(2x)$. However, $1/x$ is not an integrable function and does not determine a tempered distribution.

(b) The δ distribution is the linear functional on $C_c^\infty(\mathbf{R})$ defined by $\langle f, \delta\rangle = f(0)$ for $f \in C_c^\infty(\mathbf{R})$. Show that δ satisfies the equation $\varphi(x) = 2\,\varphi(2x)$ in a distributional sense.

(c) The jth distributional derivative $\delta^{(j)}$ of δ is defined by $\langle f, \delta^{(j)}\rangle = (-1)^j f^{(j)}(0)$ for $f \in C_c^\infty(\mathbf{R})$. Show that $\delta^{(j)}$ satisfies $\varphi(x) = 2^{j+1}\,\varphi(2x)$ in a distributional sense.

(d) The *principal value* of $1/x$ is the linear functional $\mathrm{pv}(1/x)$ on $C_c^\infty(\mathbf{R})$ given by

$$\bigl\langle f, \mathrm{pv}(1/x)\bigr\rangle \;=\; \lim_{T\to\infty} \int_{\frac{1}{T}<|t|<T} \frac{f(t)}{t}\,dt, \qquad f \in C_c^\infty(\mathbf{R}).$$

Show that $\mathrm{pv}(1/x)$ satisfies $\varphi(x) = 2\,\varphi(2x)$ in a distributional sense.

Remark: δ and $\mathrm{pv}(1/x)$ are each *tempered* distributions, and therefore have distributional Fourier transforms. The Fourier transform of δ is identified with the constant function, i.e., $\widehat{\delta} = 1$. The Fourier transform of $\mathrm{pv}(1/x)$ is identified with the function

$$\big(\mathrm{pv}(1/x)\big)^{\wedge}(\xi) \;=\; -\pi i\,\mathrm{sign}(\xi),$$

where $\mathrm{sign}(\xi) = 1$, 0, or -1 according to whether $\xi > 0$, $\xi = 0$, or $\xi < 0$, respectively. Up to scale, δ is the unique distributional solution to $\varphi(x) = 2\,\varphi(2x)$ whose Fourier transform is continuous.

12.35. A tempered distribution $\mu \in \mathcal{S}'(\mathbf{R})$ satisfies the refinement equation in the sense of distributions if we have $\mu = \sum 2^{-1/2} c_k\, D_2 T_k \mu$, where

$$\big\langle f,\, D_2 T_k \mu \big\rangle \;=\; \big\langle T_{-k} D_{1/2} f,\, \mu \big\rangle, \qquad f \in \mathcal{S}(\mathbf{R}).$$

(a) Show that if the hypotheses of Theorem 12.25 are satisfied, then $\mu = \check{P}$ (the inverse distributional Fourier transform of P) is a tempered distribution that satisfies the refinement equation in the sense of distributions.

(b) Show that, up to scale, μ is the unique tempered distribution that satisfies the refinement equation and has a Fourier transform that is a continuous function.

(c) Let θ be any bounded function such that $\theta(2\xi) = \theta(\xi)$ for all ξ. Show that $\widehat{\nu}(\xi) = P(\xi)\,\theta(\xi)$ defines a tempered distribution that satisfies the refinement equation in the sense of distributions.

(d) Show that a function θ that satisfies $\theta(2\xi) = \theta(\xi)$ and is continuous at $\xi = 0$ is a constant function. Give examples of nonconstant functions θ such that $\theta(2\xi) = \theta(\xi)$ for all ξ.

12.36. This exercise considers tempered distributions that are solutions to the refinement equation

$$\varphi(x) \;=\; \sum_{k=0}^{N} c_k\, \varphi(2x-k), \qquad \text{where} \quad \sum_{k=0}^{N} c_k = 2. \tag{12.47}$$

(a) Let δ_k be the point mass at k distribution, i.e., $\langle f, \delta_k \rangle = f(k)$ for $f \in \mathcal{S}(\mathbf{R})$. Set $\nu = \frac{1}{2} \sum_{k=0}^{N} c_k\, \delta_k$, and show that the distributional Fourier transform of ν is $\widehat{\nu} = m_0$.

(b) Set

$$\nu_n \;=\; \frac{1}{2} \sum_{k=0}^{N} c_k\, \delta_{2^{-n}k} \qquad \text{and} \qquad \mu_n = \nu_1 * \cdots * \nu_n.$$

Show that

$$\widehat{\mu_n}(\xi) = P_n(\xi) = \prod_{j=1}^{n} m_0(2^{-j}\xi)$$

and

$$\operatorname{supp}(\mu_n) \subseteq [0, \tfrac{1}{2}N] + \cdots + [0, 2^{-n}N] \subseteq [0, N].$$

(c) By Exercise 12.35(a), there is a distributional solution μ to the refinement equation that satisfies $\widehat{\mu} = P$. By Theorem 12.25, $\widehat{\mu_n} = P_n$ converges to $P = \widehat{\mu}$ uniformly on compact sets. Show that $\mu_n \xrightarrow{\text{w*}} \varphi$, i.e.,

$$\lim_{n\to\infty} \langle f, \mu_n \rangle = \langle f, \varphi \rangle, \qquad f \in \mathcal{S}(\mathbf{R}),$$

and use this to show that $\operatorname{supp}(\varphi) \subseteq [0, N]$.

(d) Show that if there exists an integrable solution φ to the refinement equation (12.47), then φ is compactly supported and $\operatorname{supp}(\varphi) \subseteq [0, N]$.

(e) Show that if the antialiasing condition is also satisfied, then there exists a compactly supported solution $\varphi \in L^2(\mathbf{R})$ to the refinement equation (which is therefore integrable by Cauchy–Bunyakovski–Schwarz).

12.7 All About the Wavelet

The preceding section showed us how to construct refinable functions that are likely to have orthonormal translates. Now we show that if such a function φ does have orthonormal translates then it generates an MRA. We will then be able to use this to construct dyadic wavelet orthonormal bases for $L^2(\mathbf{R})$.

The proof of the next result, which is due to Cohen [Coh90], requires that φ be integrable. For example, if the refinement coefficients are a finite sequence then φ is both square integrable and compactly supported and hence is automatically integrable.

Theorem 12.30. *Assume $\varphi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is a refinable function and $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal sequence. If we set*

$$V_0 = \overline{\operatorname{span}}\{T_k\varphi\}_{k\in\mathbf{Z}} \qquad \text{and} \qquad V_n = D_{2^n}(V_0), \; n \in \mathbf{Z},$$

then $\{V_n\}$ is an MRA for $L^2(\mathbf{R})$.

Proof. The hypotheses imply that statements (a), (b), and (e) in the definition of an MRA (Definition 12.8) are satisfied. So, it remains to show that $\cap V_n = \{0\}$ and $\cup V_n$ is dense in $L^2(\mathbf{R})$.

For each $n \in \mathbf{Z}$ let P_n denote the orthogonal projection of $L^2(\mathbf{R})$ onto V_n. By Lemma 12.10, for each fixed n the sequence $\{D_{2^n}T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_n. Therefore

$$P_n f \;=\; \sum_{k\in\mathbf{Z}} \langle f,\, D_{2^n}T_k\varphi\rangle\, D_{2^n}T_k\varphi,$$

and we have

$$\|P_n f\|_{L^2}^2 \;=\; \sum_{k\in\mathbf{Z}} \bigl|\langle f,\, D_{2^n}T_k\varphi\rangle\bigr|^2.$$

Since the subspaces V_n are nested, to show that $\cap V_n = \{0\}$ we need only show that

$$\forall\, f \in L^2(\mathbf{R}), \qquad \lim_{n\to-\infty} \|P_n f\|_{L^2} \;=\; 0.$$

Moreover, it suffices to establish this limit for f contained in a complete subset of $L^2(\mathbf{R})$. By Exercise 12.12, $\{D_{2^j}T_\ell \chi\}_{j,\ell\in\mathbf{Z}}$ is complete in $L^2(\mathbf{R})$, where $\chi = \chi_{[0,1]}$. With $j,\, \ell \in \mathbf{Z}$ fixed, define

$$E_n \;=\; \bigcup_{k\in\mathbf{Z}} \bigl[2^{n-j}\ell + k,\, 2^{n-j}(\ell+1) + k\bigr], \qquad n \in \mathbf{Z}.$$

Then

$$\begin{aligned}
\bigl\|P_n(D_{2^j}T_\ell\chi)\bigr\|_{L^2}^2 \;&=\; \sum_{k\in\mathbf{Z}} \bigl|\bigl\langle D_{2^j}T_\ell\chi,\, D_{2^n}T_k\varphi\bigr\rangle\bigr|^2 \\
&=\; \sum_{k\in\mathbf{Z}} \bigl|\bigl\langle D_{2^{j-n}}T_\ell\chi,\, T_k\varphi\bigr\rangle\bigr|^2 \\
&=\; \sum_{k\in\mathbf{Z}} \left|2^{(j-n)/2} \int_{\ell\, 2^{n-j}}^{(\ell+1)\, 2^{n-j}} \varphi(x-k)\, dx\right|^2 \qquad (12.48) \\
&\le\; \sum_{k\in\mathbf{Z}} \int_{\ell\, 2^{n-j}}^{(\ell+1)\, 2^{n-j}} |\varphi(x-k)|^2\, dx \qquad (12.49) \\
&=\; \int_{E_n} |\varphi(x)|^2\, dx,
\end{aligned}$$

where the inequality in equation (12.49) follows from Cauchy–Bunyakovski–Schwarz. Since $\varphi \in L^2(\mathbf{R})$, the Lebesgue Dominated Convergence Theorem therefore implies that

$$\lim_{n\to-\infty} \bigl\|P_n(D_{2^j}T_\ell\chi)\bigr\|_{L^2} \;=\; \lim_{n\to-\infty} \int_{E_n} |\varphi(x)|^2\, dx \;=\; 0.$$

This establishes that $\cap V_n = \{0\}$.

To show that $\cup V_n$ is dense in $L^2(\mathbf{R})$, it suffices to show that

$$\forall\, f \in L^2(\mathbf{R}), \qquad \lim_{n\to\infty} \|f - P_n f\|_{L^2} \;=\; 0.$$

By orthogonality, $\|f\|_{L^2}^2 = \|P_n f\|_{L^2}^2 + \|f - P_n f\|_{L^2}^2$, so we can equivalently formulate our goal as

$$\forall\, f \in L^2(\mathbf{R}), \qquad \lim_{n\to\infty} \|P_n f\|_{L^2} \;=\; \|f\|_{L^2}.$$

Moreover, as above it suffices to establish that this equality holds for the functions $D_{2^j} T_\ell \chi$ with $j,\ \ell \in \mathbf{Z}$. Using the computations leading to equation (12.48), we have

$$\|P_n \chi\|_{L^2}^2 \;=\; \sum_{k\in\mathbf{Z}} \left| 2^{-n/2} \int_0^{2^n} \varphi(x-k)\,dx \right|^2. \tag{12.50}$$

Also, if $n \ge j$ then

$$\begin{aligned} \|P_n(D_{2^j}T_\ell\chi)\|_{L^2}^2 \;&=\; \sum_{k\in\mathbf{Z}} \left| 2^{(j-n)/2} \int_{\ell\, 2^{n-j}}^{(\ell+1)\, 2^{n-j}} \varphi(x-k)\,dx \right|^2 \\ &=\; \sum_{k\in\mathbf{Z}} \left| 2^{(j-n)/2} \int_0^{2^{n-j}} \varphi(x-k)\,dx \right|^2 \;=\; \|P_{n-j}\chi\|_{L^2}^2. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \|P_n(D_{2^j}T_\ell\chi)\|_{L^2} \;=\; \lim_{n\to\infty} \|P_{n-j}\chi\|_{L^2} \;=\; \lim_{n\to\infty} \|P_n\chi\|_{L^2}.$$

Since $\|D_{2^j}T_\ell\chi\|_{L^2} = \|\chi\|_{L^2} = 1$, ultimately it suffices to establish the single equality

$$\lim_{n\to\infty} \|P_n\chi\|_{L^2} \;=\; \|\chi\|_{L^2} \;=\; 1. \tag{12.51}$$

To estimate $\|P_n\chi\|_{L^2}$, we will divide the summation in equation (12.50) into three parts, indexed by sets I_1, I_2, I_3 that we will define below. We fix $\varepsilon > 0$, and let T be an integer large enough that

$$\int_{|x|>T} |\varphi(x)|\,dx \;<\; \varepsilon.$$

Since we are interested in the limit as $n \to \infty$, we can consider the interval $[k, k+2^n]$ to be much longer than the interval $[-T, T]$.

Our first index set I_1 contains those k such that $[-T,T]$ is entirely contained within $[k, k+2^n]$:

$$I_1 \;=\; \bigl\{k \in \mathbf{Z} : [-T,T] \subseteq [k, k+2^n]\bigr\} \;=\; \bigl\{T-2^n, \dots, -T\bigr\}.$$

Note that I_1 contains $2^n - 2T + 1$ elements.

The second index set I_2 consists of those k such that $[-T,T]$ intersects the interval $[k, k+2^n]$ but is not contained within it. Since n is large, I_2 contains $4T$ elements.

The third index set is

$$I_3 \;=\; \mathbf{Z} \setminus (I_1 \cup I_2) \;=\; \bigl\{k \in \mathbf{Z} : [k, k+2^n] \subseteq \mathbf{R}\backslash[-T,T]\bigr\}.$$

Recall that $\widehat{\varphi}(0) = \int \varphi(x)\,dx$. For $k \in I_1$ we have that

$$\left|\int_k^{k+2^n} \varphi(x)dx \;-\; \widehat{\varphi}(0)\right| \;=\; \left|\int_{x\notin[k,k+2^n]} \varphi(x)dx\right| \;\le\; \int_{x\notin[-T,T]} |\varphi(x)|dx \;<\; \varepsilon.$$

Hence

$$\begin{aligned} S_1(n) \;&=\; \sum_{k\in I_1} 2^{-n} \left|\int_k^{k+2^n} \varphi(x)\,dx\right|^2 \\ &\le\; \sum_{k\in I_1} 2^{-n} \big(|\widehat{\varphi}(0)| + \varepsilon\big)^2 \\ &=\; \frac{2^n - 2T + 1}{2^n} \big(|\widehat{\varphi}(0)| + \varepsilon\big)^2, \end{aligned}$$

so

$$\limsup_{n\to\infty} S_1(n) \;\le\; \big(|\widehat{\varphi}(0)| + \varepsilon\big)^2.$$

A similar estimate from below shows that

$$\liminf_{n\to\infty} S_1(n) \;\ge\; \big(|\widehat{\varphi}(0)| - \varepsilon\big)^2.$$

For the second sum, we use the gross estimate

$$S_2(n) \;=\; \sum_{k\in I_2} 2^{-n} \left|\int_k^{k+2^n} \varphi(x)\,dx\right|^2 \;\le\; \sum_{k\in I_2} 2^{-n}\, \|\varphi\|_{L^1}^2 \;=\; \frac{4T}{2^n}\, \|\varphi\|_{L^1}^2,$$

which implies that

$$\lim_{n\to\infty} S_2(n) \;=\; 0.$$

For the third sum, note that the intervals $[k, k+2^n]$ cover $\mathbf{R}\backslash[-T,T]$ about 2^n times. More precisely, for $|x| > T$ we have

$$1 \;\le\; \sum_{k\in I_3} \chi_{[k,k+2^n]}(x) \;\le\; 2^n \quad \text{a.e.}$$

Also, since ε is small, for $k \in I_3$ the number $\left|\int_k^{k+2^n} \varphi(x)\,dx\right|$ is less than 1, and hence its square is even smaller. Therefore,

$$\begin{aligned} S_3(n) \;&=\; \sum_{k\in I_3} 2^{-n} \left|\int_k^{k+2^n} \varphi(x)\,dx\right|^2 \\ &\le\; 2^{-n} \sum_{k\in I_3} \int_k^{k+2^n} |\varphi(x)|\,dx \\ &\le\; 2^{-n}\, 2^n \int_{|x|>T} |\varphi(x)|\,dx \;<\; \varepsilon. \end{aligned}$$

Consequently,

$$0 \le \limsup_{n\to\infty} S_3(n) \le \varepsilon.$$

By equation (12.50) and the definition of I_1, I_2, I_3,

$$\|P_n\chi\|_{L^2}^2 = S_1(n) + S_2(n) + S_3(n).$$

Combining all the preceding estimates we therefore obtain

$$\big(|\widehat{\varphi}(0)| - \varepsilon\big)^2 \le \liminf_{n\to\infty} \|P_n\chi\|_{L^2}^2 \le \limsup_{n\to\infty} \|P_n\chi\|_{L^2}^2 \le \big(|\widehat{\varphi}(0)| + \varepsilon\big)^2,$$

and since ε is arbitrary, it follows that

$$\lim_{n\to\infty} \|P_n\chi\|_{L^2}^2 = |\widehat{\varphi}(0)|^2 = 1.$$

This establishes that equation (12.51) holds, and completes the proof. □

Now that we have an MRA, we need to construct the associated wavelet function. As discussed in Section 12.4, projection of a function f onto V_n gives us a “blurry” picture of f, essentially containing that information in f that is visible at “resolution level n”. The purpose of the wavelet is to represent the details of size n that we need to add in order to move to “resolution level $n+1$”. More precisely, we have $V_n \subseteq V_{n+1}$, and the wavelet space W_n is the space that we need to combine with V_n in order to obtain V_{n+1}. Since we are interested in orthogonal decompositions, we need W_n to be the orthogonal complement of V_n within V_{n+1}. That is, W_n is the space such that

$$V_n \oplus W_n = V_{n+1}.$$

Moreover, because $V_n = D_{2^n}(V_0)$, if we can just find the space W_0 such that $V_0 \oplus W_0 = V_1$ then the spaces W_n for $n \ne 0$ will be given by $W_n = D_{2^n}(W_0)$. We summarize this in the next lemma, but note that while this gives us the “wavelet spaces” W_n, the real issue will be finding an orthonormal basis for the space W_0.

Lemma 12.31. *Let $\{V_n\}_{n\in\mathbf{Z}}$ be an MRA for $L^2(\mathbf{R})$ with scaling function φ. Let W_0 be the orthogonal complement of V_0 within the space V_1, and set $W_n = D_{2^n}W_0$. Let P_n denote the orthogonal projection of $L^2(\mathbf{R})$ onto V_n, and Q_n the orthogonal projection of $L^2(\mathbf{R})$ onto W_n. Then the following statements hold.*

(a) $V_n \oplus W_n = V_{n+1}$.

(b) $P_{n+1} = P_n + Q_n$.

(c) *The spaces W_n are mutually orthogonal, i.e., $W_m \perp W_n$ whenever $m \ne n$.*

(d) *If* $f \in L^2(\mathbf{R})$, *then* $\{Q_n f\}_{n\in\mathbf{Z}}$ *is an orthogonal sequence, and*

$$f = \sum_{n\in\mathbf{Z}} Q_n f.$$

(e) $L^2(\mathbf{R}) = \oplus_{n\in\mathbf{Z}} W_n$.

Proof. (a) We have $V_0 \oplus W_0 = V_1$ by construction. Combining this with the fact that $V_n = D_{2^n}(V_0)$ and $W_n = D_{2^n}(W_0)$, we obtain $V_n \oplus W_n = V_{n+1}$.

(b) This is again an immediate consequence of the fact that W_0 is the orthogonal complement of V_0 as a subspace of V_1.

(c) The space W_1 is the orthogonal complement of V_1 within V_2. However, W_0 is contained within V_1, so the space W_0 is orthogonal to W_1. By induction, W_m is orthogonal to W_n whenever $m \neq n$.

(d) The orthogonality of the sequence $\{Q_n f\}_{n\in\mathbf{Z}}$ follows from part (c). Consequently, if the series $\sum Q_n f$ converges, then it converges unconditionally. Iterating part (b), we have

$$P_n f - P_{-n} f = \sum_{k=-n}^{n-1} \bigl(P_{k+1} f - P_k f\bigr) = \sum_{k=-n}^{n-1} Q_k f.$$

Lemma 12.10 tells us that $P_n f \to f$ and $P_{-n} f \to 0$ as $n \to \infty$, so, as limits in L^2-norm,

$$f = \lim_{n\to\infty} \bigl(P_n f - P_{-n} f\bigr) = \lim_{n\to\infty} \sum_{k=-n}^{n-1} Q_k f = \sum_{k\in\mathbf{Z}} Q_k f.$$

(e) This is a restatement of part (d). □

As a consequence, if we can find an orthonormal basis of translates for W_0, then we have found an orthonormal dyadic wavelet basis for $L^2(\mathbf{R})$.

Corollary 12.32. *Using the same notation as Lemma 12.31, if there exists a function* $\psi \in W_0$ *such that* $\{T_k\psi\}_{k\in\mathbf{Z}}$ *is an orthonormal basis for* W_0, *then the wavelet system*

$$\mathcal{W}(\psi) = \bigl\{D_{2^n} T_k \psi\bigr\}_{k,n\in\mathbf{Z}}$$

is an orthonormal basis for $L^2(\mathbf{R})$.

Proof. By scaling, if $\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0, then for each fixed n we have that $\{D_{2^n} T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_n. Combining this with parts (d) or (e) of Lemma 12.31, we see that $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2(\mathbf{R})$. □

So, our goal reduces to finding a single function ψ such that $\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0. If such a function exists, then $\psi \in W_0 \subseteq V_1$. We know that $\{D_2T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_1, so ψ must be representable in terms of this basis. Consequently, ψ satisfies an equation much like the refinement equation that φ satisfies, namely,

$$\psi \;=\; \sum_{k\in\mathbf{Z}} \langle \psi, D_2T_k\varphi\rangle\, D_2T_k\varphi, \tag{12.52}$$

or

$$\psi(x) \;=\; \sum_{k\in\mathbf{Z}} d_k\, \varphi(2x-k), \tag{12.53}$$

where $d_k = 2^{1/2}\,\langle \psi, D_2T_k\varphi\rangle$. This is not actually a refinement equation since ψ is not represented in terms of dilated and translated copies *of itself*. Still, it tells us that once we have the scaling function φ, we can hope to build the wavelet ψ from the scaling function. We just need an explicit formula for the scalars d_k.

As in the proof of Lemma 12.15, equation (12.53) translates on the Fourier side to

$$\widehat{\psi}(\xi) \;=\; m_1(\xi/2)\,\widehat{\varphi}(\xi/2),$$

where

$$m_1(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} d_k\, e^{-2\pi i k\xi}. \tag{12.54}$$

We can work with the scalars c_k, d_k or with the functions m_0, m_1, whichever is more convenient. Unfortunately, by themselves equations (12.52), (12.53), and (12.54) do not tell us how to find the scalars d_k or the function m_1. We need to use more information, namely that W_0 is orthogonal to V_0. For this, we adapt a motivational discussion from [Dau90].

If we had a function ψ such that $\{T_k\psi\}_{k\in\mathbf{Z}}$ was an orthonormal basis for W_0 then, by Lemma 12.13, the Fourier transform of a generic function $f \in W_0$ would have the form

$$\widehat{f}(\xi) \;=\; m(\xi)\,\widehat{\psi}(\xi) \quad \text{a.e.}, \tag{12.55}$$

where $m \in L^2(\mathbf{T})$. We do not yet know that such a wavelet ψ exists, but W_0 is a well-defined space (being the orthogonal complement of V_0 within V_1). Therefore, our goal is to create a function $\psi \in W_0$ so that every function $f \in W_0$ can be represented in the form given in equation (12.55).

Now, if we take an arbitrary function $f \in W_0$, then f belongs to the larger space V_1 and therefore it can be written as

$$f \;=\; \sum_{k\in\mathbf{Z}} \langle f, D_2T_k\varphi\rangle\, D_2T_k\varphi.$$

An argument similar to the one used to prove Lemma 12.15 shows that

$$\widehat{f}(\xi) \;=\; m_f(\xi/2)\,\widehat{\varphi}(\xi/2) \quad \text{a.e.}, \tag{12.56}$$

where

$$m_f(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} \langle f, D_2T_k\varphi\rangle\, e^{-2\pi i k\xi} \;\in\; L^2(\mathbf{T}).$$

The function $m_f(\xi/2)$ is 2-periodic, not 1-periodic, so equation (12.56) is not in the form that we desire. However, we will manipulate it until we reach something that has the form given in equation (12.55).

Since $f \in W_0$, $\varphi \in V_0$, and $W_0 \perp V_0$, we have $\langle f, T_k\varphi\rangle = 0$ for every $k \in \mathbf{Z}$. Lemma 10.20 tells us that the bracket product function

$$[\widehat{f}, \widehat{\varphi}](\xi) \;=\; \sum_{k\in\mathbf{Z}} \widehat{f}(\xi+k)\,\overline{\widehat{\varphi}(\xi+k)}$$

belongs to $L^1(\mathbf{T})$. The same lemma also tells us that the Fourier coefficients of $[\widehat{f}, \widehat{\varphi}]$ are

$$\big\langle [\widehat{f}, \widehat{g}],\, e_k\big\rangle_{L^2(\mathbf{T})} \;=\; \langle f, T_{-k}\varphi\rangle \;=\; 0, \qquad k \in \mathbf{Z}.$$

As functions in $L^1(\mathbf{T})$ are uniquely determined by their Fourier coefficients (Theorem 4.25), this implies that $[\widehat{f}, \widehat{\varphi}] = 0$ a.e. Therefore, by using the refinement equation and equation (12.56) and arguing similarly as in the proof of Theorem 12.27, we find that

$$\begin{aligned}
0 \;=\; [\widehat{f}, \widehat{\varphi}](\xi) \;&=\; \sum_{k\in\mathbf{Z}} \widehat{f}(\xi+k)\,\overline{\widehat{\varphi}(\xi+k)} \\
&=\; \sum_{k\in\mathbf{Z}} m_f\big(\tfrac{\xi+k}{2}\big)\,\widehat{\varphi}\big(\tfrac{\xi+k}{2}\big)\,\overline{m_0\big(\tfrac{\xi+k}{2}\big)\,\widehat{\varphi}\big(\tfrac{\xi+k}{2}\big)} \\
&=\; m_f\big(\tfrac{\xi}{2}\big)\,\overline{m_0\big(\tfrac{\xi}{2}\big)}\,\sum_{k\in\mathbf{Z}} \big|\widehat{\varphi}\big(\tfrac{\xi}{2}+k\big)\big|^2 \\
&\qquad +\; m_f\big(\tfrac{\xi+1}{2}\big)\,\overline{m_0\big(\tfrac{\xi+1}{2}\big)}\,\sum_{k\in\mathbf{Z}} \big|\widehat{\varphi}\big(\tfrac{\xi+1}{2}+k\big)\big|^2 \\
&=\; m_f\big(\tfrac{\xi}{2}\big)\,\overline{m_0\big(\tfrac{\xi}{2}\big)} \;+\; m_f\big(\tfrac{\xi}{2}+\tfrac{1}{2}\big)\,\overline{m_0\big(\tfrac{\xi}{2}+\tfrac{1}{2}\big)}.
\end{aligned} \tag{12.57}$$

Replacing ξ by 2ξ, we conclude that

$$m_f(\xi)\,\overline{m_0(\xi)} \;+\; m_f(\xi+\tfrac{1}{2})\,\overline{m_0(\xi+\tfrac{1}{2})} \;=\; 0 \quad \text{a.e.} \tag{12.58}$$

Since φ has orthonormal translates, the antialiasing condition

$$|m_0(\xi)|^2 + |m_0(\xi+\tfrac{1}{2})|^2 \;=\; 1 \quad \text{a.e.}$$

holds. Therefore $m_0(\xi)$ and $m_0(\xi+\frac{1}{2})$ cannot vanish simultaneously on any set with positive measure. Applying this fact to equation (12.58), we can write

$$m_f(\xi) \;=\; \lambda(\xi)\,\overline{m_0(\xi+\tfrac{1}{2})} \quad \text{a.e.}, \tag{12.59}$$

where

$$\lambda(\xi) = \begin{cases} -m_f(\xi + \frac{1}{2})/\overline{m_0(\xi)}, & m_0(\xi) \neq 0, \\ m_f(\xi)/\overline{m_0(\xi + \frac{1}{2})}, & \text{otherwise.} \end{cases}$$

Suppose $m_0(\xi)$ and $m_0(\xi+\frac{1}{2})$ are both nonzero. Then by applying periodicity and equation (12.58) we have

$$\begin{aligned} \lambda(\xi) &= -m_f(\xi + \tfrac{1}{2})/\overline{m_0(\xi)} \\ &= m_f(\xi)/\overline{m_0(\xi + \tfrac{1}{2})} \\ &= m_f\big((\xi + \tfrac{1}{2}) + \tfrac{1}{2}\big)/\overline{m_0(\xi + \tfrac{1}{2})} = -\lambda(\xi + \tfrac{1}{2}), \end{aligned}$$

and the same equality $\lambda(\xi+\frac{1}{2}) = -\lambda(\xi)$ also holds if either $m_0(\xi)$ or $m_0(\xi+\frac{1}{2})$ is zero. While this does not say that λ is $\frac{1}{2}$-periodic, it does imply that the function $e^{-2\pi i\xi}\lambda(\xi)$ is $\frac{1}{2}$-periodic. Therefore, if we define

$$m(\xi) = e^{\pi i\xi}\lambda(\tfrac{\xi}{2}) \tag{12.60}$$

then m is a 1-periodic function and

$$\begin{aligned} \widehat{f}(\xi) = m_f(\tfrac{\xi}{2})\,\widehat{\varphi}(\tfrac{\xi}{2}) &= \lambda(\tfrac{\xi}{2})\,\overline{m_0(\tfrac{\xi}{2} + \tfrac{1}{2})}\,\widehat{\varphi}(\tfrac{\xi}{2}) \\ &= m(\xi)\,e^{-\pi i\xi}\,\overline{m_0(\tfrac{\xi+1}{2})}\,\widehat{\varphi}(\tfrac{\xi}{2}) \end{aligned} \tag{12.61}$$

Comparing equation (12.61) to our hoped-for equation (12.55), we are led to guess that ψ will be the function whose Fourier transform is

$$\widehat{\psi}(\xi) = e^{-\pi i\xi}\,\overline{m_0(\tfrac{\xi+1}{2})}\,\widehat{\varphi}(\tfrac{\xi}{2}). \tag{12.62}$$

Note that we have not proved that this choice will work. In particular, a major issue is that we have not proved that the function m belongs to $L^2(\mathbf{T})$. Also, even if we do prove that this formula for $\widehat{\psi}$ gives us a function ψ that generates an orthonormal basis for W_0, it is not unique. For example, W_0 is shift-invariant, so we can always replace ψ by an integer translate $T_j\psi$. More generally, Exercise 10.19 implies that if $\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0, then so is $\{T_k\eta\}_{k\in\mathbf{Z}}$ where $\widehat{\eta}(\xi) = \alpha(\xi)\,\widehat{\psi}(\xi)$ and α is a 1-periodic function such that $|\alpha(\xi)| = 1$ a.e.

Still, we will proceed to verify our guess that $\widehat{\psi}$ defined by equation (12.62) gives us a function ψ such that $\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0. With $\widehat{\psi}$ defined in this way, the function m_1 given in equation (12.54) has the form

$$\begin{aligned} m_1(\xi) &= e^{-2\pi i\xi}\,\overline{m_0(\xi + \tfrac{1}{2})} \\ &= e^{-2\pi i\xi}\,\frac{1}{2}\sum_{k\in\mathbf{Z}} \overline{c_k}\,e^{2\pi ik(\xi+\frac{1}{2})} \end{aligned}$$

$$= \frac{1}{2} \sum_{k\in\mathbf{Z}} \overline{c_k}\, e^{\pi ik}\, e^{2\pi i(k-1)\xi}$$

$$= \frac{1}{2} \sum_{k\in\mathbf{Z}} (-1)^{k-1}\, \overline{c_{1-k}}\, e^{-2\pi ik\xi}.$$

Hence the coefficients d_k are

$$d_k = (-1)^{k-1}\, \overline{c_{1-k}}.$$

Since

$$|m_0(\xi)|^2 + |m_1(\xi)|^2 = |m_0(\xi)|^2 + |m_0(\xi+\tfrac{1}{2})|^2 = 1 \quad \text{a.e.},$$

m_1 is a *high-pass filter* in the same sense that m_0 is a *low-pass filter*.

Now we will make this motivational discussion precise. The next theorem shows that if we have an MRA and we define ψ as above, then ψ generates a wavelet orthonormal basis for $L^2(\mathbf{R})$. Combining this with Theorem 12.30, we see that from any refinable function $\varphi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ that has orthonormal translates we can construct a wavelet orthonormal basis for $L^2(\mathbf{R})$.

Theorem 12.33. *Assume that $\{V_n\}_{n\in\mathbf{Z}}$ is an MRA for $L^2(\mathbf{R})$ with scaling function φ, and let $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ be the refinement coefficients for φ. Then the following statements hold.*

(a) *The series*

$$\psi(x) = \sum_{k\in\mathbf{Z}} (-1)^{k-1}\, \overline{c_{1-k}}\, \varphi(2x-k)$$

converges unconditionally in L^2-norm.

(b) *The Fourier transform of ψ is*

$$\widehat{\psi}(\xi) = m_1(\xi/2)\, \varphi(\xi/2) \quad \text{where} \quad m_1(\xi) = e^{-2\pi i\xi}\, \overline{m_0(\xi+\tfrac{1}{2})}.$$

(c) *$\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0, and $T_k\psi$ is orthogonal to $T_j\varphi$ for every j, $k \in \mathbf{Z}$.*

(d) *$\mathcal{W}(\psi) = \{2^{n/2}\psi(2^n x - k)\}_{k,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$.*

Proof. We will prove statements (c) and (d), and assign the proofs of statements (a) and (b) as Exercise 12.37.

(c) With $m_1(\xi) = e^{-2\pi i\xi}\, \overline{m_0(\xi+\frac{1}{2})}$, an argument similar to the one in the proof of Theorem 12.27 shows that

$$\Phi_\psi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\psi}(\xi+k)|^2$$

$$= |m_1(\xi)|^2 + |m_1(\xi+\tfrac{1}{2})|^2$$

$$= |m_0(\xi+\tfrac{1}{2})|^2 + |m_0(\xi)|^2 = 1 \quad \text{a.e.} \tag{12.63}$$

By Lemma 12.13 we conclude that $\{T_k\psi\}_{k\in\mathbf{Z}}$ is an orthonormal sequence.

As in equation (12.57), we have

$$m_1(\xi)\,\overline{m_0(\xi)} \;+\; m_1(\xi+\tfrac{1}{2})\,\overline{m_0(\xi+\tfrac{1}{2})} \;=\; 0 \quad \text{a.e.},$$

and from this it follows that

$$\big[\widehat{\psi},\widehat{\varphi}\big](\xi) \;=\; \sum_{k\in\mathbf{Z}} \widehat{\psi}(\xi+k)\,\overline{\widehat{\varphi}(\xi+k)} \;=\; 0 \quad \text{a.e.} \tag{12.64}$$

Lemma 10.20 therefore implies that $T_j\psi$ is orthogonal to $T_k\varphi$ for every j, $k\in\mathbf{Z}$.

We must still show that $\{T_k\psi\}_{k\in\mathbf{Z}}$ is complete in W_0. So, suppose that f is an arbitrary function in W_0. The argument that proceeds from equation (12.56) through equation (12.61) does not involve ψ, but only uses the fact that $f\in W_0$ and $W_0\perp V_0$. Consequently all of the argument in equations (12.56)–(12.61) is valid. Using the notation from that argument, we have

$$\widehat{f}(\xi) \;=\; m(\xi)\,e^{-\pi i\xi}\,\overline{m_0(\tfrac{\xi+1}{2})}\,\widehat{\varphi}(\tfrac{\xi}{2}) \;=\; m(\xi)\,\widehat{\psi}(\xi).$$

If we can show that $m\in L^2(\mathbf{T})$, then it follows from Lemma 12.13 that $f\in\overline{\mathrm{span}}\{T_k\psi\}_{k\in\mathbf{Z}}$ and the proof will be complete.

By equation (12.60), the function m is square integrable if and only if λ is, so our task reduces to showing that $\lambda\in L^2(\mathbf{T})$. We know that $m_f\in L^2(\mathbf{T})$, and by combining equation (12.59) with the fact that $|\lambda(\xi+\frac{1}{2})|=|\lambda(\xi)|$, we see that

$$\begin{aligned}
&\int_0^1 |m_f(\xi)|^2\,d\xi\\
&= \int_0^1 |\lambda(\xi)|^2\,|m_0(\xi+\tfrac{1}{2})|^2\,d\xi\\
&= \int_0^{1/2} |\lambda(\xi)|^2\,|m_0(\xi+\tfrac{1}{2})|^2\,d\xi \;+\; \int_0^{1/2} |\lambda(\xi+\tfrac{1}{2})|^2\,|m_0(\xi+1)|^2\,d\xi\\
&= \int_0^{1/2} |\lambda(\xi)|^2\,\Big(|m_0(\xi+\tfrac{1}{2})|^2+|m_0(\xi)|^2\Big)\,d\xi\\
&= \int_0^{1/2} |\lambda(\xi)|^2\,d\xi.
\end{aligned}$$

Since $m_f\in L^2(\mathbf{R})$, we therefore have $\lambda\in L^2(\mathbf{T})$.

(d) This follows from Corollary 12.32. □

Corollary 12.34. *Assume that $\varphi\in L^1(\mathbf{R})\cap L^2(\mathbf{R})$ is a refinable function such that $\{T_k\varphi\}_{k\in\mathbf{Z}}$ is an orthonormal sequence. Then the spaces V_n defined in Theorem 12.30 form an MRA for $L^2(\mathbf{R})$, and the function ψ defined in Theorem 12.33 generates a dyadic wavelet orthonormal basis $\mathcal{W}(\psi)$ for $L^2(\mathbf{R})$.* ◇

Theorem 12.29 tells us that if we assume decay, normalization, and antialiasing conditions on φ then a square integrable solution to the refinement equation will exist. With rare exceptions, such a refinable function will have orthonormal translates, and hence can be used to generate a wavelet orthonormal basis for $L^2(\mathbf{R})$. Necessary and sufficient conditions for φ to have orthonormal translates can be given (this is called *Cohen's condition*). We refer to [Dau92] for more details on this issue.

Exercises

12.37. Prove statements (a) and (b) in Theorem 12.33. Also, justify equations (12.63) and (12.64) in the proof of statement (c) of that theorem.

12.38. Suppose that $\varphi \in L^2(\mathbf{R})$ is the scaling function for an MRA, and the wavelet ψ is defined as in Theorem 12.33. Prove the following statements.

(a) $|\widehat{\varphi}(\xi)|^2 + |\widehat{\psi}(\xi)|^2 = |\widehat{\varphi}(\xi/2)|^2$ a.e.

(b) $\sum_{n=1}^{\infty} |\widehat{\psi}(2^n\xi)|^2 = |\widehat{\varphi}(\xi)|^2$ a.e.

(c) $\sum_{n\in\mathbf{Z}} |\widehat{\psi}(2^n\xi)|^2 = 1$ a.e. (try to do this without appealing to Theorem 12.6).

Remark: This problem is easier if we assume that $\varphi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, for then $\widehat{\varphi} \in C_0(\mathbf{R})$ by Theorem 9.10.

12.8 Examples

Before applying the results of the preceding sections we note that not every orthonormal wavelet basis is associated with an MRA.

Example 12.35 (The Journé Wavelet). The *Journé wavelet* is the function ψ whose Fourier transform $\widehat{\psi}$ is the characteristic function of the set

$$E \;=\; \left[-\tfrac{16}{7}, -2\right] \cup \left[-\tfrac{1}{2}, -\tfrac{2}{7}\right] \cup \left[\tfrac{2}{7}, \tfrac{1}{2}\right] \cup \left[2, \tfrac{16}{7}\right]. \qquad (12.65)$$

Exercise 12.4 can be used to show that E is a wavelet set, which implies that the dyadic wavelet system $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2(\mathbf{R})$. However, Exercise 12.39 sketches a proof that there is *no* MRA whose associated wavelet is the function ψ. $\diamondsuit$

In some sense the Journé wavelet is "pathological," but in another sense this pathology is simply a reflection of our preference for MRA wavelets. Some wavelet sets are associated with MRAs; for example, the Shannon wavelet is a wavelet set example and is associated with an MRA (see Example 12.4).

However, "most" wavelet sets behave more like the Journé wavelet in the sense that they are not associated with MRAs.

For the remainder of this section we will focus on wavelets that are associated with MRAs. We begin by applying Corollary 12.34 to refinable functions that satisfy finite length refinement equations of the form

$$\varphi(x) = \sum_{k=0}^{N} c_k\, \varphi(2x-k).$$

Setting $c_k = 0$ for $k < 0$ or $k > N$, the decay condition $\sum |kc_k| < \infty$ is automatically satisfied. We assume that the normalization condition $\sum c_k = 2$ and the antialiasing condition

$$\sum_{k\in\mathbf{Z}} \overline{c_k}\, c_{k+2n} = 2\delta_{0n}, \qquad n \in \mathbf{Z},$$

both hold. This implies that N is odd (Exercise 12.25), and it also implies that the minimal accuracy condition

$$\sum_{k\in\mathbf{Z}} c_{2k} = 1 = \sum_{k\in\mathbf{Z}} c_{2k+1}$$

is satisfied. Further, by Theorem 12.29 there exists a square integrable solution φ to the refinement equation, and Exercise 12.36 shows that this solution is compactly supported (and hence integrable). If this function φ has orthonormal translates, then the wavelet

$$\psi(x) = \sum_{k=1-N}^{1} (-1)^{k-1}\, \overline{c_{1-k}}\, \varphi(2x-k) \tag{12.66}$$

generates a wavelet orthonormal basis $\mathcal{W}(\psi)$ for $L^2(\mathbf{R})$. While Theorem 12.29 does not imply that $\{T_k\varphi\}_{k\in\mathbf{Z}}$ *must* be an orthonormal system, this is usually the case in practice. In a sense that can be made precise, out of all the coefficients $c_0, \dots, c_N$ that satisfy the hypotheses of Theorem 12.29, only a set of measure zero yields functions φ that do not have orthonormal integer translates [Law90].

In any case, since $\mathrm{supp}(\varphi) \subseteq [0, N]$, the wavelet ψ given by equation (12.66) is supported within the interval $[(1-N)/2, (N+1)/2]$. However, since any integer translate of the wavelet generates the same space W_0, when working with finite length refinement equations we often define the wavelet by

$$\psi(x) = \sum_{k=0}^{N} (-1)^k\, c_{N-k}\, \varphi(2x-k), \tag{12.67}$$

which is supported in the same interval $[0, N]$ in which the scaling function φ is supported.

Example 12.36 (The Haar Wavelet). The scaling function for the Haar MRA is the box function $\chi = \chi_{[0,1]}$, which satisfies the refinement equation

$$\chi(x) \;=\; \chi(2x) + \chi(2x-1).$$

Since χ has orthonormal translates, the Haar wavelet

$$\psi \;=\; \chi(2x) - \chi(2x-1)$$

generates a wavelet orthonormal basis for $L^2(\mathbf{R})$ (compare this to the direct proof given in Example 1.54). $\diamondsuit$

Example 12.37 (The Daubechies Wavelet). By going to refinement equations with more coefficients, we can create scaling functions and wavelets that are continuous (or smoother) and compactly supported. We mentioned the family of Daubechies scaling functions D_{2N} in Section 12.5, and described the function D_4 in detail there. The D_4 function satisfies the refinement equation

$$D_4(x) \;=\; \sum_{k=0}^{3} c_k\, D_4(2x-k),$$

where

$$c_0 = \frac{1+\sqrt{3}}{4}, \qquad c_1 = \frac{3+\sqrt{3}}{4}, \qquad c_2 = \frac{1+\sqrt{3}}{4}, \qquad c_3 = \frac{1-\sqrt{3}}{4}.$$

As D_4 has orthonormal translates, Theorem 12.33 implies that

$$W_4(x) \;=\; \sum_{k=0}^{3} (-1)^k\, c_{3-k}\, D_4(2x-k)$$

generates a wavelet orthonormal basis for $L^2(\mathbf{R})$. Since W_4 is a finite linear combination of translated and dilated copies of D_4, it has the same smoothness as D_4, as can be seen in the plot in Figure 12.16. Specifically, W_4 is Hölder continuous, but it is not differentiable. Though it does not seem obvious by looking at its graph, the function W_4 has the remarkable property that it is orthogonal to every dyadic dilation and translation $W_4(2^n x - k)$ of itself, and the collection $\mathcal{W}(\psi) = \bigl\{2^{n/2} W_4(2^n x - k)\bigr\}_{n,k\in\mathbf{Z}}$ forms an orthonormal basis for $L^2(\mathbf{R})$. $\diamondsuit$

More generally, each scaling function φ corresponding to a point (c_0, c_3) lying on the circle in Figure 12.12 has orthonormal translates, with the single exception of the point $(1,1)$ whose scaling function is the stretched box $\varphi = \chi_{[0,3]}$. Each scaling function that has orthonormal translates determines a wavelet ψ and an orthonormal wavelet basis $\mathcal{W}(\psi)$ for $L^2(\mathbf{R})$. One such wavelet is shown in Figure 12.16 alongside W_4.

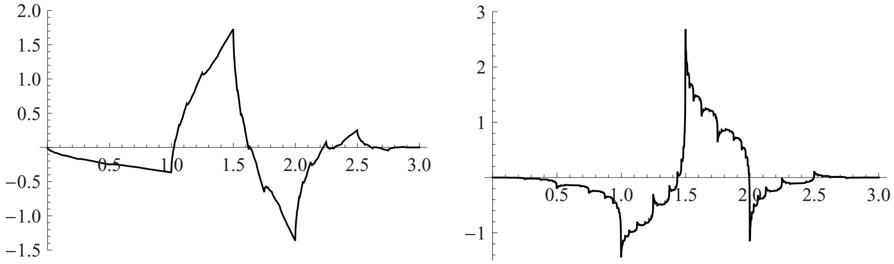

Fig. 12.16. Left: The Daubechies wavelet W_4. Right: Wavelet corresponding to the bottom right scaling function in Figure 12.13.

Some of the scaling functions and wavelets corresponding to points on the (c_0, c_3) circle are continuous and some are discontinuous, but none are differentiable. By going to longer refinement equations, we can create compactly supported wavelets that have as many continuous derivatives as we like (though by Exercise 12.21 we can never create a compactly supported infinitely differentiable wavelet by this method). In particular, the Daubechies wavelets W_{2N} are supported in $[0, 2N]$ and their smoothness increases linearly with N (for large N we have D_{2N}, $W_{2N} \in C^{\mu N}(\mathbf{R})$ with $\mu \approx 0.2$, see [Dau92]). For example, the function D_6 is differentiable (see Figure 12.14). In contrast to the situation for wavelets, the Balian–Low Theorems (Theorems 11.33 and 11.35) tell us that the generator of a Gabor Riesz basis for $L^2(\mathbf{R})$ can never be both smooth and decay well at infinity!

Now we turn to some non-compactly supported wavelets. For these types of functions, it is often more convenient to work directly with m_0 and m_1 than with the refinement coefficients.

Example 12.38 (The Battle–Lemarié Wavelet). Let $w(x) = \max\{1 - |x|, 0\}$ be the hat function on $[-1, 1]$. This is a continuous refinable function, but it does not have orthonormal translates. On the other hand, its integer translates form a Riesz basis for the space $V_0 = \overline{\operatorname{span}}\{T_k w\}_{k \in \mathbf{Z}}$. In Example 10.22 we showed how to construct a function $w^\sharp$ that has orthonormal translates and generates the same space V_0 as the hat function,

$$V_0 \;=\; \overline{\operatorname{span}}\{T_k w\}_{k \in \mathbf{Z}} \;=\; \overline{\operatorname{span}}\{T_k w^\sharp\}_{k \in \mathbf{Z}}.$$

This function $w^\sharp$ is the *linear spline Battle–Lemarié scaling function*, and it is pictured in Figure 12.17. It is continuous, piecewise linear, and has rapid decay at infinity. Its Fourier transform is given explicitly by

$$\widehat{w^\sharp}(\xi) \;=\; \widehat{w}(\xi)\, \Phi_w(\xi)^{-1/2} \;=\; \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2 \left(\frac{3}{2 + \cos 2\pi\xi}\right)^{1/2}. \qquad (12.68)$$

The function Φ_w is continuous, 1-periodic, and everywhere nonzero, so the same is true of $\Phi_w^{-1/2}$.

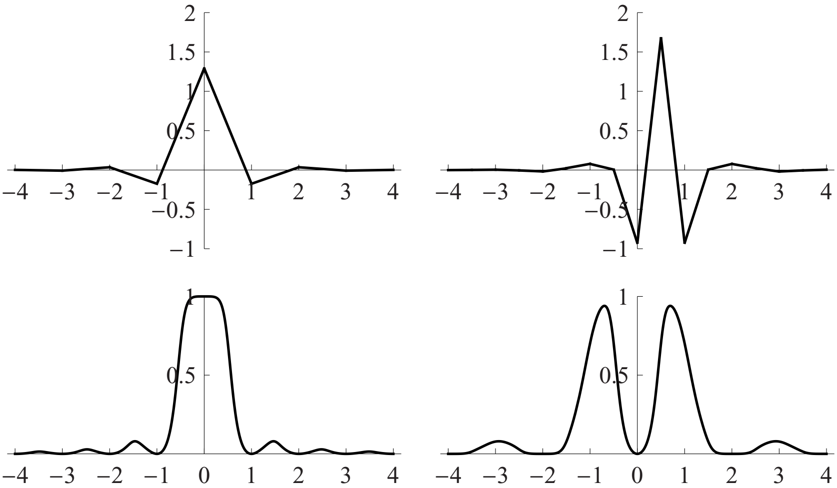

Fig. 12.17. Top: Linear spline Battle–Lemarié scaling function (left) and wavelet (right). Bottom: Absolute values of the corresponding Fourier transforms.

Equation (12.68) expresses in the Fourier domain the fact that $w^\sharp$ belongs to V_0. While it is easy to see that w is refinable, it is not quite so clear that $w^\sharp$ is refinable. To show this, we need to find a function $m_0 \in L^2(\mathbf{T})$ such that $\widehat{w^\sharp}(\xi) = m_0(\xi/2)\,\widehat{w^\sharp}(\xi/2)$. Since $\widehat{w^\sharp}$ is nonzero almost everywhere, we can define

$$m_0(\xi) \;=\; \frac{\widehat{w^\sharp}(2\xi)}{\widehat{w^\sharp}(\xi)} \;=\; \frac{1}{4}\left(\frac{\sin 2\pi\xi}{\sin \pi\xi}\right)^2 \left(\frac{2+\cos 2\pi\xi}{2+\cos 4\pi\xi}\right)^{1/2}$$

$$= \;(\cos^2 \pi\xi)\left(\frac{2+\cos 2\pi\xi}{2+\cos 4\pi\xi}\right)^{1/2}.$$

This is a square integrable, 1-periodic function (see Figure 12.15). Exercise 12.18 therefore implies that $w^\sharp$ is refinable. The high-pass filter is

$$m_1(\xi) \;=\; e^{-2\pi i\xi}\,(\sin^2 \pi\xi)\left(\frac{2-\cos 2\pi\xi}{2+\cos 4\pi\xi}\right)^{1/2}.$$

The refinement coefficients c_k decay quickly, so the wavelet

$$\psi(x) \;=\; \sum_{k\in\mathbf{Z}} (-1)^{k-1}\, c_{1-k}\, w^\sharp(2x-k)$$

has rapid decay as well (in fact, it can be shown that φ and ψ have exponential decay at infinity). A plot of this wavelet, obtained by numerically computing the above quantities, is shown in Figure 12.17. Note that since ψ belongs to the space V_1, it is piecewise linear. By replacing the hat function with higher-order B-splines, we can create wavelets that have greater smoothness and still decay quickly at infinity, see [Chr03] for details. $\diamondsuit$

We end this chapter by discussing the Meyer wavelet, though it may be more precise to refer to "Meyer wavelets," since the same construction produces an entire family of wavelets with similar properties. An important property of the Meyer wavelet is that its Fourier transform is compactly supported. The Shannon wavelet is another example of a wavelet with this property, but while the Fourier transform of the Shannon wavelet is discontinuous, the Fourier transform of the Meyer wavelet is smooth. In fact, we can construct examples that are as smooth as we like, even infinitely differentiable.

Example 12.39 (The Meyer Wavelet). To create the Meyer wavelet we first construct a scaling function. Let $\widehat{\varphi}$ be any continuous "bell function" satisfying the following properties:

(a) $\operatorname{supp}(\widehat{\varphi}) = [-\frac{2}{3}, \frac{2}{3}]$,

(b) $\widehat{\varphi} > 0$ on $(-\frac{2}{3}, \frac{2}{3})$,

(c) $\widehat{\varphi} = 1$ on $(-\frac{1}{3}, \frac{1}{3})$,

(d) $\sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi + k)|^2 = 1$.

For example, if we take $a = 1$ and $b = 3/4$ in Exercise 11.10 then we obtain a function g that almost has the right properties. That function is supported on $[0, \frac{4}{3}]$, so to obtain $\widehat{\varphi}$ we just have to translate left: $\widehat{\varphi}(x) = g(x + \frac{2}{3})$. More explicit constructions can also be given, see [Dau92], [Wal02], and Exercise 12.40.

In any case, since $\widehat{\varphi}$ is continuous and compactly supported, it is square integrable, and therefore its inverse Fourier transform φ belongs to $L^2(\mathbf{R})$. Further, requirement (d) above implies that φ has orthonormal integer translates. We call φ the *Meyer scaling function*, see Figure 12.18. Note that since $\widehat{\varphi}$ is both real and even, the function φ is real and even as well (Exercise 9.7).

Of course, in order to properly call φ a scaling function, we must show that it is refinable. Rather than trying to use the approach we used for the Battle–Lemarié wavelet, we take advantage of the fact that $\widehat{\varphi}$ is compactly supported and constant on the interval $[-\frac{1}{3}, \frac{1}{3}]$. We want to find m_0 so that $\widehat{\varphi}(\xi) = m_0(\xi/2)\,\widehat{\varphi}(\xi/2)$, and we observe that

$$\widehat{\varphi}(\xi) \;=\; \widehat{\varphi}(\xi)\,\widehat{\varphi}(\xi/2),$$

since $\widehat{\varphi}(\xi/2) = 1$ on $[-\frac{2}{3}, \frac{2}{3}]$. This suggests taking $m_0(\xi/2) = \widehat{\varphi}(\xi)$, or $m_0(\xi) = \widehat{\varphi}(2\xi)$. But we need m_0 to be 1-periodic, so what we do is define $m_0(\xi)$ to be the 1-periodic extension of $\widehat{\varphi}(2\xi)$ to the real line. Since $\widehat{\varphi}(2\xi)$ is supported in $[-\frac{1}{3}, \frac{1}{3}]$, which has length less than 1, we can write m_0 explicitly as

$$m_0(\xi) \;=\; \sum_{k\in\mathbf{Z}} \widehat{\varphi}(2(\xi + k)).$$

This function belongs to $L^2(\mathbf{T})$, and we have

$$\widehat{\varphi}(\xi) \;=\; m_0(\xi/2)\,\widehat{\varphi}(\xi/2), \qquad \xi \in \mathbf{R}.$$

Therefore φ is refinable, so Corollary 12.34 implies the existence of a corresponding wavelet ψ, which we call the *Meyer wavelet*. As m_0 is real valued, the high-pass filter is

$$m_1(\xi) \;=\; e^{-2\pi i\xi}\, m_0(\xi+\tfrac{1}{2}) \;=\; e^{-2\pi i\xi} \sum_{k\in\mathbf{Z}} \widehat{\varphi}(2\xi+2k+1).$$

Taking the support of $\widehat{\varphi}$ into account, the Fourier transform of the wavelet ψ is

$$\begin{aligned}\widehat{\psi}(\xi) \;=\; m_1(\xi/2)\,\widehat{\varphi}(\xi/2) \;&=\; e^{-\pi i\xi} \sum_{k\in\mathbf{Z}} \widehat{\varphi}(\xi+2k+1)\,\widehat{\varphi}(\xi/2) \\ &=\; e^{-\pi i\xi} \big(\widehat{\varphi}(\xi-1)+\widehat{\varphi}(\xi+1)\big)\,\widehat{\varphi}(\xi/2).\end{aligned}$$

Note that the function $\widehat{g}(\xi) = \big(\widehat{\varphi}(\xi-1)+\widehat{\varphi}(\xi+1)\big)\,\widehat{\varphi}(\xi/2)$ is real and even, so its inverse Fourier transform g is real and even. Since $\widehat{\psi} = e^{-\pi i\xi}\,\widehat{g}(\xi)$ we have $\psi(x) = g(x-\frac{1}{2})$, so ψ is real valued and is symmetric about $x = \frac{1}{2}$.

Illustrations of the Meyer scaling function and wavelet and their Fourier transforms appear in Figure 12.18. These illustrations are based on the bell function $\widehat{\varphi}$ that is constructed in Exercise 12.40. $\diamondsuit$

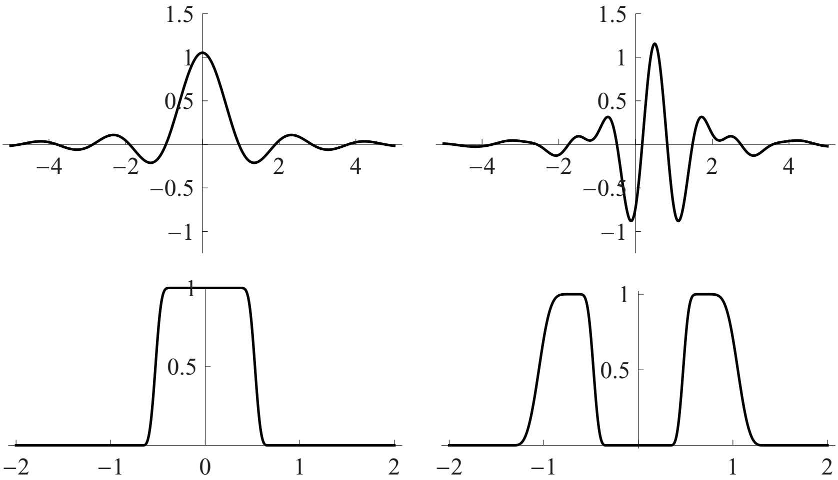

Fig. 12.18. Top left: Meyer scaling function. Top right: Meyer wavelet. Bottom: Absolute values of the corresponding Fourier transforms.

One of the many interesting things about the Meyer wavelet is that if we take our starting bell function to be infinitely differentiable then $\widehat{\varphi}$ and $\widehat{\psi}$

are both infinitely differentiable and compactly supported. As a consequence, Corollary 9.20 implies that φ and ψ both belong to the Schwartz space $\mathcal{S}(\mathbf{R})$, and therefore they decay faster at infinity than any polynomial (though they do not have exponential decay). This is again in remarkable contrast to the situation for Gabor bases, and leads us to a final historical remark.

As we have noted, the first orthonormal wavelet basis was the Haar system, constructed in 1910. However, the functions comprising the Haar system are discontinuous. A wavelet basis generated by a k-times differentiable function ψ was constructed by Strömberg in 1982 [Str83], but unfortunately was mostly overlooked at the time. While trying to prove that an analogue of the Balian–Low Theorem holds for wavelet bases, Meyer (who was unaware of Strömberg's construction) instead found his wavelet, which manifestly shows that the BLT does not hold for wavelets. Meyer's construction in [Mey85] did not have the short proof given above. Multiresolution analysis had not yet been invented, and Meyer did not arrive at his construction through the use of a scaling function. Instead, he defined the wavelet ψ explicitly, and directly proved that $\mathcal{W}(\psi)$ was an orthonormal basis for $L^2(\mathbf{R})$, via an argument that can only be described as relying upon "miraculous cancellations"! Other constructions, including those of Battle [Bat87] and Lemarié [Lem88] followed shortly after, and the need for miracles was soon removed when Mallat and Meyer developed the framework of multiresolution analysis. Though Meyer's 1986 lectures at the University of Torino on multiresolution analysis were not available to me, one of my very first encounters with wavelet theory was with a xerox copy of his handwritten notes for the Zygmund lecture he gave at the University of Chicago a year later [Mey87]. These notes were eventually typed and appear in the reprint volume [HW96].

Exercises

12.39. (a) Show that the Journé set E defined in equation (12.65) is a wavelet set (use Exercise 12.4).

(b) Define $\widehat{\psi} = \chi_E$, and suppose that ψ was associated with an MRA. Let φ be the scaling function for this MRA. Use Exercise 12.38 to show that $|\widehat{\varphi}| = \chi_F$, where

$$F = \left[-\tfrac{8}{7}, -1\right] \cup \left[-\tfrac{4}{7}, -\tfrac{1}{2}\right] \cup \left[-\tfrac{2}{7}, \tfrac{2}{7}\right] \cup \left[\tfrac{1}{2}, \tfrac{4}{7}\right] \cup \left[1, \tfrac{8}{7}\right].$$

(c) Use the equation $\widehat{\varphi}(\xi) = m_0(\xi/2)\, \widehat{\varphi}(\xi/2)$ to show that $|m_0(\xi)| = 1$ for $\xi \in \left[-\frac{2}{7}, \frac{2}{7}\right]$. Since m_0 is 1-periodic, this also shows that $|m_0(\xi)| = 1$ for $\xi \in \left[\frac{5}{7}, \frac{9}{7}\right]$. Show that if $\xi \in \left[1, \frac{8}{7}\right]$ then $\varphi(2\xi) = 0$ but $|m_0(\xi)\, \varphi(\xi)| = 1$. Conclude that ψ cannot be associated with an MRA.

(d) For an alternative proof that ψ cannot be associated with an MRA, compute $\Phi_\varphi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi + k)|^2$, and use this to obtain a contradiction.

12.40. (a) Define

$$\nu(x) = \begin{cases} 0, & x < 0, \\ x^4\,(35 - 84x + 70x^2 - 20x^3), & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

Show that $\nu \in C^3(\mathbf{R})$ and $0 \le \nu(x) \le 1$ for all x. We call ν a C^3-*sigmoid function*.

(b) Define

$$\widehat{\varphi}(\xi) = \begin{cases} 0, & \xi \le -\frac{2}{3} \text{ or } \xi \ge \frac{2}{3}, \\ \sin\bigl(\frac{\pi}{2}\,\nu(3x+2)\bigr), & -\frac{2}{3} < \xi < -\frac{1}{3}, \\ 1, & -\frac{1}{3} \le \xi \le \frac{1}{3}, \\ \cos\bigl(\frac{\pi}{2}\,\nu(3x+2)\bigr), & \frac{1}{3} < \xi < \frac{2}{3}. \end{cases}$$

Show that $\widehat{\varphi}$ is a *bell function* in the sense of Example 12.39.

Part IV

Fourier Series

13

Fourier Series

This chapter and the next are devoted to the trigonometric system $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ in the spaces $L^p(\mathbf{T})$ and $C(\mathbf{T})$. In this chapter we will develop some new tools, including convolution and approximate identities, and then in Chapter 14 we apply these tools to determine the basis properties of the trigonometric system in $L^p(\mathbf{T})$ and $C(\mathbf{T})$. Sources for additional information on the material in these chapters include [Ben97], [Kat04], [Gra04], or [Heil].

13.1 Notation and Terminology

Most of the functions that we will encounter in this chapter will be defined on the domain $\mathbf{T}$, which means that they are 1-periodic functions on the real line. We have discussed this domain before, but it will be useful to recall some terminology precisely here.

As a set, we think of $\mathbf{T}$ as being the real line, but functions in $L^p(\mathbf{T})$ or $C(\mathbf{T})$ are required to be 1-periodic, and the L^p-norm of a function in $L^p(\mathbf{T})$ is computed over one period of the function, e.g., on any interval of length 1 (typically the interval $[0,1)$, but the interval $[-\frac{1}{2},\frac{1}{2})$ is often convenient as well). A function specified on the interval $[0,1)$ has a unique extension to a 1-periodic function on $\mathbf{R}$, and we usually work interchangeably with the function on $[0,1)$ and its 1-periodic extension. Since single points have zero Lebesgue measure, as far as L^p is concerned it does not matter if we define a function on $[0,1)$ or $[0,1]$ and then extend it periodically to $\mathbf{R}$. However, for continuous functions it is important to distinguish between being continuous on $[0,1)$ and having a continuous 1-periodic extension to $\mathbf{R}$. Functions in $C(\mathbf{T})$ satisfy $f(0)=f(1)$, and we often do not distinguish between functions in $C(\mathbf{T})$ and continuous functions on the closed interval $[0,1]$ that satisfy $f(0)=f(1)$.

A trivial, but important, fact about 1-periodic functions is that if $f\in L^1(\mathbf{T})$ and $y\in\mathbf{R}$, then we have

$$\int_0^1 f(x-y)\,dx \;=\; \int_0^1 f(x)\,dx. \tag{13.1}$$

C. Heil, *A Basis Theory Primer: Expanded Edition*, Applied and Numerical Harmonic Analysis,
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Thus, integrals on $\mathbf{T}$ are invariant under the change of variable $x \mapsto x - y$. A less trivial but equally important fact is that $C(\mathbf{T})$ is dense in $L^p(\mathbf{T})$ for each $1 \le p < \infty$, see Lemma A.27.

In addition to the spaces $L^p(\mathbf{T})$ and $C(\mathbf{T})$, we will use the following spaces of m-times differentiable or infinitely differentiable functions on $\mathbf{T}$:

$$C^m(\mathbf{T}) = \bigl\{f \in C(\mathbf{T}) : f, f', \dots, f^{(m)} \in C(\mathbf{T})\bigr\},$$
$$C^\infty(\mathbf{T}) = \bigl\{f \in C(\mathbf{T}) : f, f', \dots \in C(\mathbf{T})\bigr\}.$$

The sequence spaces that appear in this chapter will usually be on the integers $\mathbf{Z}$ rather than the natural numbers. We denote these spaces by $\ell^p(\mathbf{Z})$, $c_0(\mathbf{Z})$, $c(\mathbf{Z})$, etc. For example, $c_0(\mathbf{Z})$ consists of those bi-infinite sequences $x = (x_n)_{n\in\mathbf{Z}}$ for which $x_n \to 0$ as $n \to \pm\infty$. It will be convenient in this chapter to write sequences in either of the forms $x = (x_n)$ or $x = \bigl(x(n)\bigr)$.

The standard basis vectors will be indexed by the integers $\mathbf{Z}$. That is, in this chapter we set

$$\delta_n = (\delta_{nk})_{k\in\mathbf{Z}}, \qquad n \in \mathbf{Z},$$

and refer to $\{\delta_n\}_{n\in\mathbf{Z}}$ as the *standard basis*. The standard basis is an unconditional basis for $\ell^p(\mathbf{Z})$ for each $1 \le p < \infty$, and it is also an unconditional basis for $c_0(\mathbf{Z})$.

Throughout this chapter, e_n will denote the 1-periodic complex exponential function

$$e_n(x) = e^{2\pi i n x}, \qquad x \in \mathbf{T}.$$

A finite linear combination

$$\sum_{n=-N}^{N} c_n e^{2\pi i n x} \in \operatorname{span}\{e_n\}_{n\in\mathbf{Z}}$$

is called a *trigonometric polynomial*.

Because the functions e_n belong simultaneously to the Hilbert space $L^2(\mathbf{T})$ and the Banach spaces $L^p(\mathbf{T})$, it will be convenient in this chapter to use a notation for linear functionals that extends the inner product on $L^2(\mathbf{T})$. Specifically, when f, g are functions such that their product fg is integrable, we will write

$$\langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\,dx.$$

Using this convention, the sequence $\{e_n\}_{n\in\mathbf{Z}} \subseteq L^p(\mathbf{T})$ is biorthogonal to the sequence $\{e_n\}_{n\in\mathbf{Z}} \subseteq L^{p'}(\mathbf{T})$ when $1 \le p \le \infty$. Hence $\{e_n\}_{n\in\mathbf{Z}}$ is minimal in $L^p(\mathbf{T})$ for each $1 \le p \le \infty$, and it is likewise minimal in $C(\mathbf{T})$. Our goal is to determine the completeness and basis properties of this sequence. By direct calculation, $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal sequence in $L^2(\mathbf{T})$. We will show in Section 13.6 that $\{e_n\}_{n\in\mathbf{Z}}$ is complete in $L^p(\mathbf{T})$ for $1 \le p < \infty$ and is also complete in $C(\mathbf{T})$. Combined with orthonormality, this tells us that the trigonometric system is an orthonormal basis for $L^2(\mathbf{T})$. Addressing the basis properties in other spaces will take more work, but we will ultimately show that $\{e_n\}_{n\in\mathbf{Z}}$ is a basis for $L^p(\mathbf{T})$ for each $1 < p < \infty$, and is exact but not a basis for $L^1(\mathbf{T})$ and $C(\mathbf{T})$. We already know from Exercise 6.5 that the trigonometric system cannot be an unconditional basis in $L^p(\mathbf{T})$ when $p \neq 2$, so we conclude that it is a conditional basis for $1 < p < 2$ and $2 < p < \infty$.

We will need the translation and modulation operators, acting on functions on $\mathbf{T}$ or sequences on $\mathbf{Z}$.

Definition 13.1. We define the following operations on functions $f\colon \mathbf{T} \to \mathbf{C}$.

$$\begin{aligned} &\text{Translation:} \quad (T_a f)(x) \;=\; f(x-a), && a \in \mathbf{R}.\\ &\text{Modulation:} \quad (M_\theta f)(x) \;=\; e^{2\pi i\theta x} f(x), && \theta \in \mathbf{R}. \end{aligned}$$

Analogous operations on sequences $c = \big(c(n)\big)_{n\in\mathbf{Z}}$ are defined as follows.

$$\begin{aligned} &\text{Translation:} \quad (T_m c)(n) \;=\; c(n-m), && m \in \mathbf{Z}.\\ &\text{Modulation:} \quad (M_\theta c)(n) \;=\; e^{2\pi i\theta n} c(n), && \theta \in \mathbf{R}. \qquad \diamondsuit \end{aligned}$$

Exercises

13.1. (a) Prove that every function in $C(\mathbf{T})$ is uniformly continuous, and use this to prove that translation is *strongly continuous* on $C(\mathbf{T})$, i.e.,

$$\forall\, f \in C(\mathbf{T}), \quad \lim_{a\to 0} \|T_a f - f\|_\infty \;=\; 0.$$

(b) Use the fact that $C(\mathbf{T})$ is dense in $L^p(\mathbf{T})$ to prove that translation is *strongly continuous* on $L^p(\mathbf{T})$ when $1 \le p < \infty$, i.e.,

$$\forall\, 1 \le p < \infty, \quad \forall\, f \in L^p(\mathbf{T}), \quad \lim_{a\to 0} \|T_a f - f\|_{L^p} \;=\; 0.$$

13.2 Fourier Coefficients and Fourier Series

We begin with the Fourier coefficients of an integrable function on $\mathbf{T}$. Thinking of $\{e_n\}_{n\in\mathbf{Z}}$ in its role as the biorthogonal system, the Fourier coefficients of f are simply its "inner products" with the biorthogonal system $\{e_n\}_{n\in\mathbf{Z}}$.

Definition 13.2 (Fourier Coefficients). Given $f \in L^1(\mathbf{T})$, its *Fourier coefficients* are

$$\widehat{f}(n) \;=\; \langle f, e_n\rangle \;=\; \int_0^1 f(x)\, e^{-2\pi i n x}\, dx, \qquad n \in \mathbf{Z}.$$

We set

$$\widehat{f} = \bigl(\widehat{f}(n)\bigr)_{n\in\mathbf{Z}},$$

and refer to the sequence $\widehat{f}$ as the *Fourier transform* of f. When we wish to emphasize the role of the Fourier transform as an operator, we write $\mathcal{F}f = \widehat{f}$.

We also define

$$\check{f}(n) = \langle f, e_{-n}\rangle = \int_0^1 f(x)\, e^{2\pi inx}\, dx, \qquad n \in \mathbf{Z},$$

and call the sequence

$$\check{f} = \bigl(\check{f}(n)\bigr)_{n\in\mathbf{Z}}$$

the *inverse Fourier transform* of f. We often write $\mathcal{F}^{-1}f = \check{f}$. ♢

The reason for the terminology "inverse" will become clear later. For notational clarity, we sometimes write $f^\wedge$ or $(f)^\wedge$ instead of $\widehat{f}$.

If $f \in L^1(\mathbf{T})$ then

$$|\widehat{f}(n)| \le \int_0^1 |f(x)\, e^{2\pi inx}|\, dx = \int_0^1 |f(x)|\, dx = \|f\|_{L^1}, \tag{13.2}$$

so $\widehat{f} \in \ell^\infty(\mathbf{Z})$. Hence $\mathcal{F}\colon L^1(\mathbf{T}) \to \ell^\infty(\mathbf{Z})$ and

$$\|\widehat{f}\|_{\ell^\infty} \le \|f\|_{L^1},$$

so $\mathcal{F}$ is bounded with operator norm $\|\mathcal{F}\| \le 1$.

Example 13.3. Given $m \in \mathbf{N}$, we have by biorthogonality that the Fourier coefficients of e_m are

$$\widehat{e_m}(n) = \langle e_m, e_n\rangle = \delta_{mn}, \qquad n \in \mathbf{Z}.$$

That is, $\widehat{e_m} = \delta_m$. In particular, $\|\widehat{e_m}\|_{\ell^\infty} = 1 = \|e_m\|_{L^1}$, so the operator norm of $\mathcal{F}\colon L^1(\mathbf{T}) \to \ell^\infty(\mathbf{Z})$ is $\|\mathcal{F}\| = 1$ (compare equation (9.15)). ♢

Note that $L^1(\mathbf{T})$ is the largest space in the universe of function spaces that we are considering, in the sense that $L^1(\mathbf{T})$ contains $C(\mathbf{T})$ and $L^p(\mathbf{T})$ for each $1 \le p \le \infty$. Since we know that $\{e_n\}_{n\in\mathbf{Z}}$ is minimal and the biorthogonal sequence is $\{e_n\}_{n\in\mathbf{Z}}$, if it is a basis for $L^p(\mathbf{T})$ or $C(\mathbf{T})$ then the basis representation of a function f would be

$$f(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e_n(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi inx}, \tag{13.3}$$

where the series converges in norm with respect to some fixed ordering of $\mathbf{Z}$. Given $f \in L^1(\mathbf{T})$, it is customary to refer to the formal series

$$\sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi inx} \tag{13.4}$$

as the *Fourier series* of f. There is no guarantee that this series will converge in any sense, and one of the main issues that we will address in Chapter 14 is the question of when and in what sense equation (13.3) holds.

We have a symmetric definition of the Fourier transform of a sequence in $\ell^1(\mathbf{Z})$. Note that, in contrast to the L^p spaces, $\ell^1(\mathbf{Z})$ is the smallest sequence space in our universe, as $\ell^1(\mathbf{Z}) \subseteq \ell^p(\mathbf{Z})$ for each $1 \le p \le \infty$ and $\ell^1(\mathbf{Z}) \subseteq c_0(\mathbf{Z})$.

Definition 13.4. If $c = (c_n)_{n\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$, then its *Fourier transform* is the function

$$\widehat{c}(x) \;=\; \sum_{n\in\mathbf{Z}} c_n \overline{e_n(x)} \;=\; \sum_{n\in\mathbf{Z}} c_n e^{-2\pi i n x}, \qquad x \in \mathbf{T}. \tag{13.5}$$

The *inverse Fourier transform* of $c \in \ell^1(\mathbf{Z})$ is

$$\check{c}(x) \;=\; \sum_{n\in\mathbf{Z}} c_n \overline{e_{-n}(x)} \;=\; \sum_{n\in\mathbf{Z}} c_n e^{2\pi i n x}, \qquad x \in \mathbf{T}. \qquad \diamondsuit \tag{13.6}$$

Since each term $c_n e_n$ is continuous and $\|e_n\|_\infty = 1$, the series in equation (13.5) converges absolutely with respect to the uniform norm. Therefore $\widehat{c}$ is a continuous, 1-periodic function when $c \in \ell^1(\mathbf{Z})$, and similarly $\check{c} \in C(\mathbf{T})$. That is,

$$c \in \ell^1(\mathbf{Z}) \implies \widehat{c},\, \check{c} \in C(\mathbf{T}).$$

For example, the Fourier transform of the sequence δ_m is the function $\widehat{\delta_m}(x) = e_m(x) = e^{2\pi i m x}$.

Ignoring questions of convergence, we have at least formally that the Fourier series of f given in equation (13.4) is the Fourier transform of f followed by the inverse Fourier transform of the sequence $\widehat{f}$:

$$\bigl(\widehat{f}\,\bigr)^{\vee}(x) \;=\; \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x}.$$

Thus the question of whether the Fourier series of f converges to f is the question of whether the *Inversion Formula* $f = \bigl(\widehat{f}\,\bigr)^{\vee}$ is valid, and in what sense it is valid.

We often think of the variable $x \in \mathbf{T}$ as representing time, and therefore we often say that a function f on $\mathbf{T}$ lives "in the time domain," while its Fourier transform $\widehat{f}$ lives "in the frequency domain." The next result is one example of how an operation in the time domain is converted by the Fourier transform into a dual operation in the frequency domain. Specifically, the Fourier transform interchanges translation in time with modulation in frequency and vice versa, both for functions on $\mathbf{T}$ and for sequences on $\mathbf{Z}$.

Lemma 13.5. *Given $f \in L^1(\mathbf{T})$, $a \in \mathbf{R}$, and $m \in \mathbf{Z}$ we have for all $n \in \mathbf{Z}$:*

(a) $(T_a f)^\wedge(n) = (M_{-a}\widehat{f}\,)(n) = e^{-2\pi i n a}\widehat{f}(n)$,

(b) $(M_m f)^\wedge(n) = (T_m \widehat{f}\,)(n) = \widehat{f}(n-m)$, *Given* $c \in \ell^1(\mathbf{Z})$, $m \in \mathbf{Z}$, *and* $a \in \mathbf{R}$ *we have for all* $x \in \mathbf{T}$:

(c) $(T_m c)^\wedge(x) = (M_{-m}\widehat{c}\,)(x) = e^{-2\pi i m x}\,\widehat{c}(x)$,

(d) $(M_a c)^\wedge(x) = (T_a\widehat{c}\,)(x) = \widehat{c}(x-a)$.

Proof. We compute that

$$\begin{aligned}(M_m f)^\wedge(n) \;&=\; \int_0^1 (M_m f)(x)\, e^{-2\pi i n x}\, dx \\ &=\; \int_0^1 f(x)\, e^{-2\pi i (n-m)x}\, dx \;=\; \widehat{f}(n-m).\end{aligned}$$

The remainder of the proof is assigned as Exercise 13.2. □

Exercises

13.2. Prove Lemma 13.5, and also derive analogous formulas for the inverse Fourier transforms $(T_a f)^\vee$, $(M_m f)^\vee$, $(T_m c)^\vee$, and $(M_a c)^\vee$.

13.3. This exercise will show how the Fourier transform transforms smoothness of a function f into decay of its Fourier coefficients. Show that if $f \in C^1(\mathbf{T})$ then $\widehat{f'}(n) \;=\; 2\pi i n \widehat{f}(n)$ for $n \in \mathbf{Z}$, and use this to show that

$$\widehat{f}(n)| \;\le\; \frac{\|f'\|_{L^1}}{2\pi\,|n|}, \qquad n \neq 0.$$

Extend to higher derivatives, and compare Theorem 9.16.

13.4. This exercise will show how the Fourier transform transforms decay of a sequence c into smoothness of its Fourier transform. Show that if $c = (c_n)_{n\in\mathbf{Z}}$ satisfies $\sum_{n\in\mathbf{Z}} |nc_n| < \infty$, then $\widehat{c}(\xi) = \sum_{n\in\mathbf{Z}} c_n e^{-2\pi i n x}$ is differentiable and

$$\widehat{c}\,'(\xi) \;=\; -2\pi i \sum_{n\in\mathbf{Z}} n c_n e^{-2\pi i n x} \;=\; \widehat{d}(\xi),$$

where $d = (-2\pi i n c_n)_{n\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$. Extend to higher derivatives, and compare Theorem 9.15.

13.3 Convolution

Convolution is a fundamental tool that we will use extensively in our analysis of Fourier series. Therefore we spend this section developing some of the basic properties of this tool.

Although $L^1(\mathbf{T})$ is not closed under pointwise products (Exercise 13.6), we will see that convolution is a multiplication-like operation under which $L^1(\mathbf{T})$ is closed.

Definition 13.6 (Convolution of Periodic Functions). Let f and g be measurable 1-periodic functions. The *convolution of f with g* is the function $f * g$ given by

$$(f * g)(x) \;=\; \int_0^1 f(y)\, g(x-y)\, dy, \tag{13.7}$$

whenever this integral is well defined. ◇

We have a corresponding discrete version of convolution.

Definition 13.7 (Convolution of Sequences). Let $a = (a_k)_{k\in\mathbf{Z}}$ and $b = (b_k)_{k\in\mathbf{Z}}$ be sequences of complex scalars. Then the *convolution of a with b* is the sequence $a * b = \big((a*b)_k\big)_{k\in\mathbf{Z}}$ given by

$$(a*b)_k \;=\; \sum_{j\in\mathbf{Z}} a_j\, b_{k-j}, \tag{13.8}$$

whenever this series converges. ◇

More generally, convolution can be defined on any locally compact group, of which $\mathbf{T}$ and $\mathbf{Z}$ are two examples. We briefly encountered convolution of functions on $\mathbf{R}$ a few times earlier in this volume; see in particular Exercises 9.5, 12.19, and 12.20.

The definition of convolution may seem rather ad hoc at first. We will give some motivation for it in Section 13.4, but for now let us show that convolution is well defined in many cases.

Theorem 13.8 (Young's Inequality). *Fix $1 \le p \le \infty$. If $f \in L^p(\mathbf{T})$ and $g \in L^1(\mathbf{T})$ then $f * g \in L^p(\mathbf{T})$, and we have*

$$\|f*g\|_{L^p} \;\le\; \|f\|_{L^p}\, \|g\|_{L^1}. \tag{13.9}$$

Proof. First we show that $f*g$ exists and is measurable if $f, g \in L^1(\mathbf{T})$. Since g is 1-periodic, for any y we have $\int_0^1 |g(x-y)|\, dx = \int_0^1 |g(x)|\, dx = \|g\|_{L^1}$, and therefore

$$\begin{aligned}
\int_0^1 \int_0^1 |f(y)\, g(x-y)|\, dy\, dx \;&=\; \int_0^1 \bigg(\int_0^1 |g(x-y)|\, dx\bigg)\, |f(y)|\, dy \\
&=\; \int_0^1 \|g\|_{L^1}\, |f(y)|\, dy \\
&=\; \|g\|_{L^1}\, \|f\|_{L^1} \;<\; \infty.
\end{aligned}$$

Hence, it follows from Fubini's Theorem that $(f*g)(x) = \int_0^1 f(y)\, g(x-y)\, dy$ exists for almost every x and is an integrable function of x. Since g is 1-periodic, $f*g$ is 1-periodic as well.

Now suppose that $1 < p < \infty$, and choose $f \in L^p(\mathbf{T})$ and $g \in L^1(\mathbf{T})$. Since $L^p(\mathbf{T}) \subseteq L^1(\mathbf{T})$, the above work tells us that $f * g$ exists. Applying Hölder's Inequality with exponents p and p' and using the change-of-variable formula given in equation (13.1), we have

$$\begin{aligned}
|(f*g)(x)| &\le \int_0^1 |f(y)\,g(x-y)|\,dy \\
&= \int_0^1 \Bigl(\bigl|f(y)\bigr|\,\bigl|g(x-y)\bigr|^{1/p}\Bigr)\,\bigl|g(x-y)\bigr|^{1/p'}\,dy \\
&\le \left(\int_0^1 |f(y)|^p\,|g(x-y)|^{p/p}\,dy\right)^{1/p} \left(\int_0^1 |g(x-y)|^{p'/p'}\,dy\right)^{1/p'} \\
&= \left(\int_0^1 |f(y)|^p\,|g(x-y)|\,dy\right)^{1/p} \left(\int_0^1 |g(y)|\,dy\right)^{1/p'} \\
&= \|g\|_{L^1}^{1/p'} \left(\int_0^1 |f(y)|^p\,|g(x-y)|\,dy\right)^{1/p}.
\end{aligned}$$

Note that

$$1 + \frac{p}{p'} \;=\; 1 + \frac{p(p-1)}{p} \;=\; 1 + p - 1 \;=\; p.$$

Therefore, interchanging integrals by Tonelli's Theorem,

$$\begin{aligned}
\|f*g\|_{L^p}^p &= \int_0^1 |(f*g)(x)|^p\,dx \\
&\le \|g\|_{L^1}^{p/p'} \int_0^1\!\int_0^1 |f(y)|^p\,|g(x-y)|\,dy\,dx \\
&= \|g\|_{L^1}^{p/p'} \int_0^1 |f(y)|^p \left(\int_0^1 |g(x-y)|\,dx\right) dy \\
&= \|g\|_{L^1}^{p/p'} \int_0^1 |f(y)|^p \left(\int_0^1 |g(x)|\,dx\right) dy \\
&= \|g\|_{L^1}^{p/p'} \int_0^1 |f(y)|^p\,\|g\|_{L^1}\,dy \\
&= \|g\|_{L^1}^{1+\frac{p}{p'}}\,\|f\|_{L^p}^p \\
&= \|g\|_{L^1}^p\,\|f\|_{L^p}^p,
\end{aligned}$$

so the result follows upon taking pth roots.

The cases $p = 1$ and $p = \infty$ are assigned as Exercise 13.7. □

In particular, $L^1(\mathbf{T})$ is closed under convolution, and we have

$$\|f * g\|_{L^1} \;\le\; \|f\|_{L^1}\,\|g\|_{L^1}. \tag{13.10}$$

In another language, the fact that $L^1(\mathbf{T})$ is a Banach space that has an operation $*$ that satisfies equation (13.10) says that $L^1(\mathbf{T})$ is a *Banach algebra* with respect to the operation $*$.

There is a corresponding discrete version of Young's Inequality, and as a consequence $\ell^1(\mathbf{Z})$ is also a Banach algebra with respect to convolution (Exercise 13.7).

Theorem 13.9 (Young's Inequality). *Fix* $1 \le p \le \infty$. *If* $f \in \ell^p(\mathbf{Z})$ *and* $g \in \ell^1(\mathbf{Z})$ *then* $f * g \in \ell^p(\mathbf{Z})$, *and we have*

$$\|f * g\|_{\ell^p} \;\le\; \|f\|_{\ell^p}\,\|g\|_{\ell^1}. \qquad \diamondsuit \tag{13.11}$$

One of the most important facts about convolution is that the Fourier transform interchanges the operation of convolution with pointwise multiplication.

Theorem 13.10. (a) *If* $f,\, g \in L^1(\mathbf{T})$ *then* $(f * g)^\wedge$ *is the sequence*

$$(f * g)^\wedge \;=\; \bigl(\widehat{f}(n)\,\widehat{g}(n)\bigr)_{n\in\mathbf{Z}}.$$

(b) *If* $c,\, d \in \ell^1(\mathbf{Z})$ *then* $(c * d)^\wedge$ *is the function*

$$(c * d)^\wedge(x) \;=\; \sum_{n\in\mathbf{Z}} c_n d_n e^{-2\pi i n x}.$$

Proof. Fix $f,\, g \in L^1(\mathbf{T})$. Appealing to Exercise 13.9 for justification of the interchange of the order of integration, we apply Fubini's Theorem to compute that

$$\begin{aligned}
(f * g)^\wedge(n) &= \int_0^1 (f * g)(x)\, e^{-2\pi i n x}\, dx \\
&= \int_0^1 \int_0^1 f(y)\, g(x-y)\, dy\, e^{-2\pi i n x}\, dx \\
&= \int_0^1 f(y)\, e^{-2\pi i n y} \left(\int_0^1 g(x-y)\, e^{-2\pi i n (x-y)}\, dx \right) dy \\
&= \int_0^1 f(y)\, e^{-2\pi i n y} \left(\int_0^1 g(x)\, e^{-2\pi i n x}\, dx \right) dy \\
&= \int_0^1 f(y)\, e^{-2\pi i n y}\, \widehat{g}(n)\, dy \\
&= \widehat{f}(n)\,\widehat{g}(n).
\end{aligned}$$

The proof of part (b) is Exercise 13.9. $\square$

Some of the algebraic properties of convolution are given in the next result, whose proof is Exercise 13.10.

Lemma 13.11. *The following facts hold for* $f,\ g,\ h \in L^1(\mathbf{T})$.

(a) *Commutativity:* $f * g = g * f$.

(b) *Associativity:* $(f * g) * h = f * (g * h)$.

(c) *Distributive laws:* $f * (g + h) = f * g + f * h$.

(d) *Commutativity with translations:*

$$f * (T_a g) \;=\; (T_a f) * g \;=\; T_a(f * g), \qquad a \in \mathbf{R}. \qquad \diamondsuit$$

Exercises

13.5. Let $p(x) = \sum_{k=0}^{M} a_k x^k$ and $q(x) = \sum_{k=0}^{N} b_k x^k$ be two polynomials. Their pointwise product is a polynomial of the form $p(x)q(x) = \sum_{n=0}^{M+N} c_n x^n$. Find an explicit formula for c_n, and explain its relation to the convolution of the sequences $(a_k)_{k\in\mathbf{Z}}$ and $(b_k)_{k\in\mathbf{Z}}$, where we take $a_k = 0$ if $k \ne 0, \dots, M$ and similarly for b_k.

13.6. Show that $f,\ g \in L^1(\mathbf{T})$ does not imply $fg \in L^1(\mathbf{T})$.

13.7. Prove Theorem 13.8 for the cases $p = 1$ and $p = \infty$.

13.8. Prove Theorem 13.9.

13.9. Justify the use of Fubini's Theorem in the proof of part (a) of Theorem 13.10, and prove part (b) of that theorem.

13.10. Prove Lemma 13.11, and establish analogous properties for the convolution of sequences.

13.11. Given $f \in L^1(\mathbf{T})$, set $f^*(x) = f(-x)$ and show that $\widehat{f^*}(n) = \widehat{f}(-n) = \check{f}(n)$. Conclude that if f is even then $\widehat{f}$ is even and $\widehat{f} = \check{f}$.

13.12. Let $\delta = \delta_0 = \big(\delta_{0n}\big)_{n\in\mathbf{Z}}$. Show that δ is an identity for convolution on $\ell^p(\mathbf{Z})$, i.e., $c * \delta = c$ for every $c \in \ell^p(\mathbf{Z})$, where $1 \le p \le \infty$.

Remark: We will see in Corollary 13.15 that there is no element of $L^1(\mathbf{T})$ that is an identity for convolution on $L^p(\mathbf{T})$.

13.13. (a) Show that if $f \in L^1(\mathbf{T})$ and $g \in C(\mathbf{T})$ then $f * g \in C(\mathbf{T})$.

(b) Prove that convolution commutes with differentiation in the following sense: If $f \in L^1(\mathbf{T})$ and $g \in C^1(\mathbf{T})$ then $f * g \in C^1(\mathbf{T})$, and $(f * g)' = f * g'$. Extend to $g \in C^m(\mathbf{T})$ and $g \in C^\infty(\mathbf{T})$.

13.14. Fix $1 \le p \le \infty$. Show that if $f \in L^p(\mathbf{T})$ and $g \in L^{p'}(\mathbf{T})$ then $f * g \in C(\mathbf{T})$.

13.4 Approximate Identities

One motivation for convolution is to think of it as a kind of weighted averaging operator. For example, consider the functions

$$\chi_T \;=\; \frac{1}{2T}\chi_{[-T,T]},$$

extended 1-periodically to the real line (we assume $0 < T < 1/2$). Given $f \in L^1(\mathbf{T})$,

$$(f * \chi_T)(x) \;=\; \int_0^1 f(y)\,\chi_T(x-y)\,dy \;=\; \frac{1}{2T}\int_{x-T}^{x+T} f(y)\,dy \;=\; \mathrm{Avg}_T f(x),$$

the average of f on the interval $[x-T, x+T]$ (see Figure 13.1).

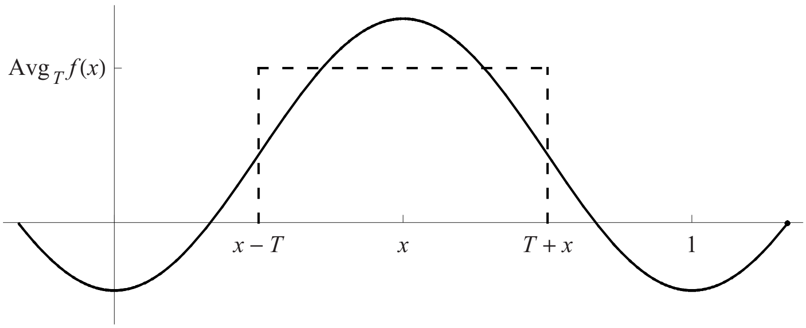

Fig. 13.1. The area of the dashed box equals $\int_{x-T}^{x+T} f(y)\,dy$, which is the area under the graph of f between $y = x - T$ and $y = x + T$.

For a general function g, the mapping $f \mapsto f * g$ can be regarded as a weighted averaging of f, with g weighting some parts of the domain more than others. Technically, it may be better to think of the function $g^*(x) = g(-x)$ as the weighting function, since g^* is the function being translated when we compute

$$(f * g)(x) \;=\; \int_0^1 f(y)\,g^*(y-x)\,dy \;=\; \mathrm{Avg}_{g*} f(x).$$

In any case, $(f * g)(x)$ is a weighted average of f around the point x. Alternatively, since convolution is commutative, we can equally view it as an averaging of g using the weighting corresponding to $f^*(x) = f(-x)$.

Now consider what happens to the convolution $f * \chi_T = \mathrm{Avg}_T f$ as $T \to 0$. The function χ_T becomes a taller and taller "spike" centered at the origin (and since it is 1-periodic, there are corresponding spikes centered at each integer point). The height of the spike is chosen so that the integral of χ_T over a period is always 1. As T decreases, we are averaging f over smaller and smaller intervals. Intuitively, this should give values $(f * \chi_T)(x)$ that are closer and closer to the value $f(x)$. This intuition is made precise in Lebesgue's Differentiation Theorem (Theorem A.30), which states that if $f \in L^1(\mathbf{T})$ then for almost every x we have

$$f(x) \;=\; \lim_{T \to 0} (f * \chi_T)(x) \;=\; \lim_{T \to 0} \operatorname{Avg}_T f(x).$$

Thus $f \approx f * \chi_T$ when T is small. We will see later (Theorem 13.14) that there is no identity element for convolution on $L^1(\mathbf{T})$. That is, there is no function $g \in L^1(\mathbf{R})$ such that $f * g = f$ for all $f \in L^1(\mathbf{R})$. Still, the function χ_T is *approximately* an identity for convolution, and this approximation becomes better and better the smaller T becomes.

A similar phenomenon occurs for convolution with functions other than χ_T. That is, even though there is no single function $g \in L^1(\mathbf{T})$ such that $f * g = f$ for all $f \in L^1(\mathbf{T})$, we can create many different sequences of functions k_N such that $f * k_N \to f$ as $N \to \infty$. Moreover, by designing the k_N appropriately, we can make $f * k_N$ converge to f in different senses. Essentially, what we need are 1-periodic functions k_N that become more and more "spike-like" as N increases. The following definition specifies the exact features that we need the functions k_N to possess.

Definition 13.12. An *approximate identity* or a *summability kernel* on $\mathbf{T}$ is a family $\{k_N\}_{N \in \mathbf{N}}$ of functions in $L^1(\mathbf{T})$ such that

(a) $\int_0^1 k_N(x)\,dt \;=\; 1$ for every N,

(b) $\sup \|k_N\|_{L^1} \;<\; \infty$, and

(c) for every $0 < \delta < 1/2$ we have

$$\lim_{N \to \infty} \int_{\delta \le |x| < \frac{1}{2}} |k_N(x)|\,dx \;=\; 0. \qquad \diamondsuit$$

Property (a) says that each function k_N has the same total "signed mass" in the sense that its integral over a period is 1, and property (c) says that most of this mass is being squeezed into smaller and smaller intervals around the origin (and hence, by periodicity, around each integer point). Property (b) requires that the "absolute mass" of the k_N be uniformly bounded. Some of the elements of one particular approximate identity are displayed in Figure 13.4, and we can see there how those functions k_N become more spike-like as N increases.

One way to create an approximate identity is to take a single function k that satisfies $\int_0^1 k = 1$ and is only nonzero on a small interval around the origin and then mimic the shrinking process that created the functions χ_T, e.g., define $k_N(x) = Nk(Nx)$ near the origin, followed by a 1-periodic extension to $\mathbf{R}$. The dilation $Nk(Nt)$ ensures that $\int_0^1 k_N = \int_0^1 k$ for all N, and also makes the functions k_N more and more concentrated around integer points (see Exercise 13.15). On the other hand, in many circumstances the functions k_N are imposed upon us, so to check that $\{k_N\}$ forms an approximate identity we must verify that the conditions given in Definition 13.12 hold. We will see some particularly important approximate identities in the next section, but for now let us see what we can do with them.

Theorem 13.13. *Let $\{k_N\}_{N\in\mathbf{N}}$ be an approximate identity.*

(a) *If $1 \le p < \infty$ and $f \in L^p(\mathbf{T})$, then $f * k_N \to f$ in L^p-norm as $N \to \infty$.*

(b) *If $f \in C(\mathbf{T})$, then $f * k_N \to f$ uniformly as $N \to \infty$.*

Proof. Consider the case $p = 1$. Fix $f \in L^1(\mathbf{T})$. Noting that $\int_0^1 k_N = 1$, we compute that

$$\begin{aligned}
\|f - f * k_N\|_{L^1} &= \int_0^1 |f(x) - (f * k_N)(x)|\, dx \\
&= \int_0^1 \left| f(x) \int_0^1 k_N(t)\, dt \;-\; \int_0^1 f(x-t)\, k_N(t)\, dt \right| dx \\
&\le \int_0^1 \int_0^1 |f(x) - f(x-t)|\, |k_N(t)|\, dt\, dx \\
&= \int_0^1 \int_0^1 |f(x) - f(x-t)|\, |k_N(t)|\, dx\, dt \\
&= \int_0^1 |k_N(t)| \int_0^1 |f(x) - T_t f(x)|\, dx\, dt \\
&= \int_0^1 |k_N(t)|\, \|f - T_t f\|_{L^1}\, dt, \qquad (13.12)
\end{aligned}$$

where the interchange in the order of integration is permitted by Tonelli's Theorem since the integrands are nonnegative. We want to show that the quantity above tends to zero as $N \to \infty$.

Choose $\varepsilon > 0$. Appealing to the *strong continuity of translation* proved in Exercise 13.1, there exists a $\delta > 0$ such that

$$|t| < \delta \implies \|f - T_t f\|_{L^1} < \varepsilon.$$

By definition of approximate identity, $K = \sup \|k_N\|_{L^1} < \infty$ and there exists some N_0 such that

$$N > N_0 \implies \int_{\delta \le |t| < 1/2} |k_N(t)|\, dt < \varepsilon.$$

Taking the domain of a period to be $[-\frac{1}{2}, \frac{1}{2}]$, for $N > N_0$ we can continue the estimate in equation (13.12) as follows:

$$\begin{aligned}
&\|f - f * k_N\|_{L^1} \\
&\quad\le \int_{|t|<\delta} |k_N(t)|\,\|f - T_t f\|_{L^1}\,dt \;+\; \int_{\delta\le|t|<1/2} |k_N(t)|\,\|f - T_t f\|_{L^1}\,dt \\
&\quad\le \int_{|t|<\delta} |k_N(t)|\,\varepsilon\,dt \;+\; \int_{\delta\le|t|<1/2} |k_N(t)|\,\big(\|f\|_{L^1} + \|T_t f\|_{L^1}\big)\,dt \\
&\quad\le \varepsilon \int_{-1/2}^{1/2} |k_N(t)| \;+\; 2\|f\|_{L^1} \int_{\delta\le|t|<1/2} |k_N(t)|\,dt \\
&\quad\le \varepsilon K \;+\; 2\|f\|_{L^1}\,\varepsilon.
\end{aligned}$$

Thus $\|f - f * k_N\|_{L^1} \to 0$ as $N \to \infty$. The remainder of the proof is assigned as Exercise 13.16. □

We end this section with some applications of the strong continuity of translation. Compare the next result to Theorem 9.10.

Theorem 13.14 (Riemann–Lebesgue Lemma). *If $f \in L^1(\mathbf{T})$, then $\widehat{f} \in c_0(\mathbf{Z})$, i.e., $\widehat{f}(n) \to 0$ as $n \to \pm\infty$.*

Proof. Since $e^{-\pi i} = -1$, we have for $n \ne 0$ that

$$\begin{aligned}
\widehat{f}(n) &= \int_0^1 f(x)\,e^{-2\pi i n x}\,dx \\
&= -\int_0^1 f(x)\,e^{-2\pi i n x}\,e^{-2\pi i n(\frac{1}{2n})}\,dx \\
&= -\int_0^1 f(x)\,e^{-2\pi i n(x+\frac{1}{2n})}\,dx \\
&= -\int_0^1 f\Big(x - \frac{1}{2n}\Big)\,e^{-2\pi i n x}\,dx.
\end{aligned}$$

Averaging the first and last lines in the equalities above, we obtain

$$\widehat{f}(n) \;=\; \frac{1}{2}\int_0^1 \Big(f(x) - f\Big(x - \frac{1}{2n}\Big)\Big)\,e^{-2\pi i n x}\,dx. \tag{13.13}$$

Using the strong continuity of translation proved in Exercise 13.1, we therefore have that

$$|\widehat{f}(n)| \;\le\; \frac{1}{2}\int_0^1 \Big|f(x) - f\Big(x - \frac{1}{2n}\Big)\Big|\,dx \;=\; \frac{1}{2}\,\|f - T_{\frac{1}{2n}} f\|_{L^1} \;\to\; 0$$

as $|n| \to \infty$. □

As a corollary, we find that there is no identity element for convolution in $L^1(\mathbf{T})$. In contrast, Exercise 13.12 shows that there does exist an identity for convolution in $\ell^1(\mathbf{Z})$.

Corollary 13.15. *There is no function $g \in L^1(\mathbf{T})$ such that $f * g = f$ for all $f \in L^1(\mathbf{T})$.*

Proof. Suppose that such a function g existed. Then by Theorem 13.10, for every $f \in L^1(\mathbf{T})$ and $n \in \mathbf{Z}$ we would have

$$\widehat{f}(n)\,\widehat{g}(n) \;=\; (f*g)^\wedge(n) \;=\; \widehat{f}(n).$$

In particular, if we fix n and take $f = e_n$ then $\widehat{f} = \delta_n$ and therefore $\widehat{f}(n) = 1$, so we must have $\widehat{g}(n) = 1$. This contradicts the fact that $\widehat{g} \in c_0(\mathbf{Z})$. □

Exercises

13.15. Design an approximate identity $\{k_N\}$ such that $k_N \in C(\mathbf{T})$, $k_N \ge 0$ everywhere, and $k_N = 0$ on $\bigl[\frac{1}{N}, 1 - \frac{1}{N}\bigr]$. How smooth can you make k_N?

13.16. Prove Theorem 13.13 for $1 < p < \infty$ and for $C(\mathbf{T})$.

13.17. The existence of infinitely differentiable, compactly supported functions is demonstrated in Exercise 11.9. Combine Exercises 13.13 and 13.15 with Theorem 13.13 to show that $C^\infty(\mathbf{T})$ is dense in $L^p(\mathbf{T})$ for $1 \le p < \infty$, and is dense in $C(\mathbf{T})$ with respect to the uniform norm. Conclude that $C^m(\mathbf{T})$ is dense in these spaces for each integer $m \ge 0$.

13.18. Fix $0 < \alpha < 1$. Show that if $f \in C(\mathbf{T})$ is Hölder continuous with exponent α (see Exercise 1.23) then

$$|\widehat{f}(n)| \;\le\; \frac{1}{2}\left(\frac{1}{2|n|}\right)^{\alpha}, \qquad n \neq 0.$$

13.5 Partial Sums and the Dirichlet Kernel

Given $f \in L^1(\mathbf{T})$, let $S_N f$ denote the Nth symmetric partial sum of the formal Fourier series in equation (13.4), i.e.,

$$S_N f(x) \;=\; \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x}, \qquad x \in \mathbf{T}.$$

This is a well-defined element of $L^1(\mathbf{T})$, and in fact $S_N f \in C(\mathbf{T})$. Our question is whether $S_N f$ will converge to f in any sense. Let

$$d_N(x) \;=\; \sum_{n=-N}^{N} e^{2\pi i n x}, \tag{13.14}$$

so we have $d_N \in C(\mathbf{T})$ for each N. Then

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x} \\ &= \sum_{n=-N}^{N} \left(\int_0^1 f(t)\, e^{-2\pi i n t}\, dt \right) e^{2\pi i n x} \\ &= \int_0^1 f(t) \sum_{n=-N}^{N} e^{2\pi i n (x-t)}\, dt \\ &= \int_0^1 f(t)\, d_N(x-t)\, dt \\ &= (f * d_N)(x). \end{aligned} \tag{13.15}$$

Now we see exactly why we spent so much time discussing convolution and approximate identities: The symmetric partial sums of the Fourier series of f are given by the convolutions $S_N f = f * d_N$. If $\{d_N\}$ was an approximate identity, then we would know that $f * d_N$ converges to f in $L^p(\mathbf{T})$, $1 \le p < \infty$, and in $C(\mathbf{T})$. Alas, $\{d_N\}$ is *not* an approximate identity, and this is precisely what makes the question of convergence of Fourier series so delicate.

Definition 13.16 (Dirichlet Kernel). The *Dirichlet kernel* for $\mathbf{T}$ is the family $\{d_N\}_{N \in \mathbf{N}}$, where d_N is the function defined in equation (13.14). ♢

The next lemma writes the elements of the Dirichlet kernel in several alternative forms (see Exercise 13.19).

Lemma 13.17. *Let χ_N denote the discrete characteristic function of the set $\{-N, \dots, N\}$, i.e.,*

$$\chi_N(n) = \sum_{m=-N}^{N} \delta_m(n) = \begin{cases} 1, & -N \le n \le N, \\ 0, & |n| > N. \end{cases}$$

Then for each $N \in \mathbf{N}$ we have

$$d_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin{(2N+1)\pi x}}{\sin \pi x} = \widehat{\chi_N}(x) = \check{\chi}_N(x). \qquad ♢$$

Note that χ_N is a sequence, and so its Fourier transform as defined in equation (13.4) is a continuous 1-periodic function. Also, the equality $\widehat{\chi_N} = \check{\chi}_N$ follows from the fact that χ_N is even, compare Exercise 13.11.

Each function d_N belongs to $L^1(\mathbf{T})$, and its graph does appear to become more like a "1-periodic spike train" as $N \to \infty$ (see Figure 13.2). Unfortunately, the oscillations of d_N only decay slowly with N, and as a consequence the sequence $\{d_n\}$ is not bounded above in L^1-norm. While we do

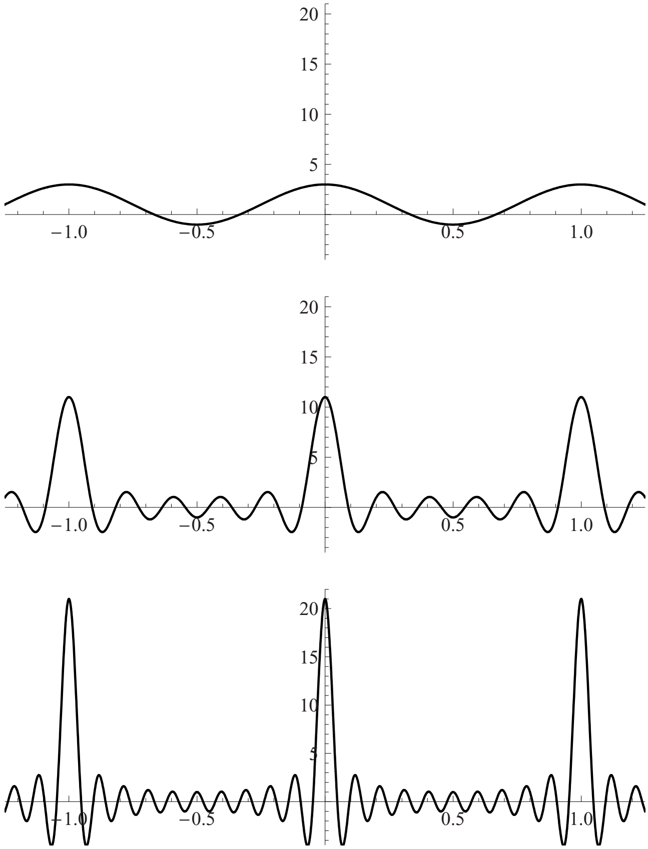

Fig. 13.2. Three elements of the Dirichlet kernel. Top: d_1. Middle: d_5. Bottom: d_{10}.

have $\int_0^1 d_N = 1$ for every N, we achieve this only because the large oscillations of d_N produce "miraculous" cancellations in this integral.

The proof of the next result is Exercise 13.20.

Theorem 13.18. *$d_N \in L^1(\mathbf{T})$ and $\int_0^1 d_N = 1$ for each $N \in \mathbf{N}$, and for $N > 1$ we have*

$$\frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \;\le\; \|d_N\|_{L^1} \;\le\; 3 \;+\; \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k}. \qquad \diamondsuit$$

By the Integral Test,

$$\ln N \;\le\; \sum_{k=1}^{N} \frac{1}{k} \;\le\; 1 + \ln N,$$

so $\sup_N \|d_N\|_{L^1} = \infty$, and therefore the Dirichlet kernel does not form an approximate identity on $\mathbf{T}$.

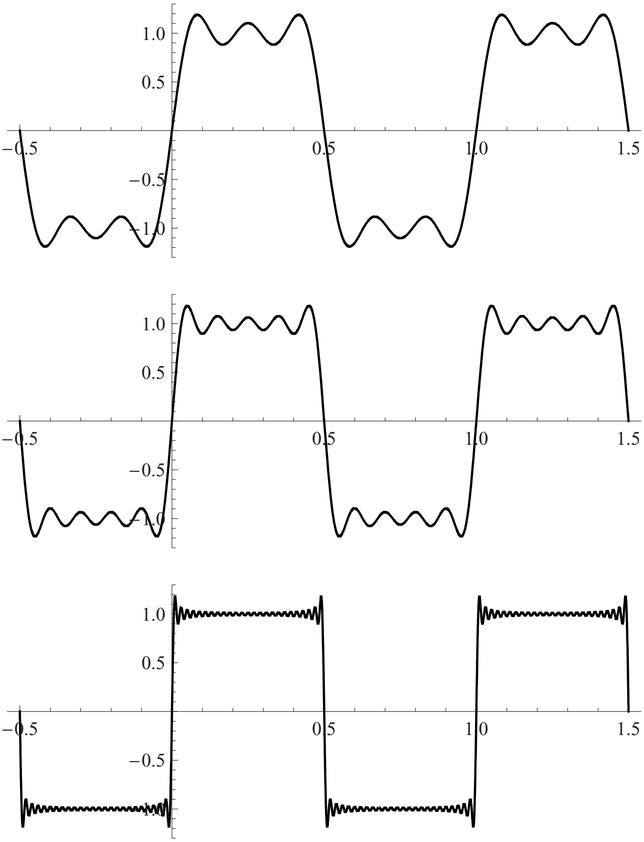

Fig. 13.3. Convolution of the square wave f with elements of the Dirichlet kernel. Top: $f * d_5$. Middle: $f * d_{10}$. Bottom: $f * d_{50}$.

Let $f = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ be the square wave function, extended 1-periodically to $\mathbf{R}$. Figure 13.3 shows $f * d_5$, $f * d_{10}$, and $f * d_{50}$. We can see *Gibbs's phenomenon* in this figure: $f * d_N$ does not converge uniformly to f. Instead, $f * d_N$ always overshoots f by an amount (about 9%) that does not decrease with N (see [DM72] for proof).

Exercises

13.19. Prove Lemma 13.17.

13.20. Prove Theorem 13.18.

13.6 Cesàro Summability and the Fejér Kernel

When considering series whose convergence properties are unknown, or which may even diverge, it is often useful to consider the *averages* of the partial sums of the series instead of the partial sums themselves. These *Cesàro*, or *arithmetic*, means are usually much "better behaved" than the partial sums themselves. For the case of the Fourier series of $f \in L^1(\mathbf{T})$, these means are

$$\begin{aligned} \sigma_N f(x) &= \frac{S_0 f(x) + \cdots + S_N f(x)}{N+1} \\ &= \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, \widehat{f}(n)\, e^{2\pi i n x}, \end{aligned} \tag{13.16}$$

where the second equality is proved in Exercise 13.21. Define sequences

$$W_N = \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, \delta_n,$$

and note that the components of W_N are

$$W_N(k) = \max\Bigl\{1 - \frac{|k|}{N+1},\, 0\Bigr\}, \qquad k \in \mathbf{Z}.$$

Thus W_N is a "discrete hat function," as compared to the "discrete characteristic function" χ_N. Set

$$\begin{aligned} w_N(x) = \widehat{W_N}(x) &= \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, \widehat{\delta_n}(x) \\ &= \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, e^{2\pi i n x}. \end{aligned} \tag{13.17}$$

Note that $\widehat{W_N} = (W_N)^\vee$ since W_N is even.

Definition 13.19 (Fejér Kernel). The *Fejér kernel* for $\mathbf{T}$ is the family $\{w_N\}_{N \in \mathbf{N}}$ where w_N is the function defined in equation (13.17). $\diamondsuit$

The letter "w" is for "Weiss," which was Fejér's surname at birth.

Appealing to Exercise 13.22, a calculation similar to the one in equation (13.15) shows that the Cesàro means have the form

$$\sigma_N f = f * w_N. \tag{13.18}$$

Some elements of the Fejér kernel are shown in Figure 13.4. These functions appear to be more concentrated than the Dirichlet kernel, and hence it seems

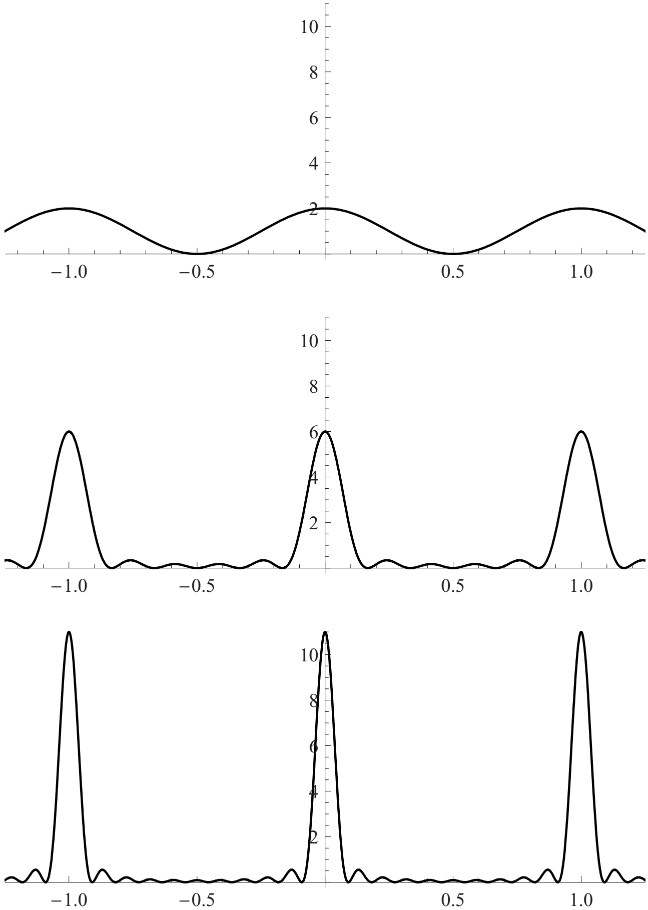

Fig. 13.4. Three elements of the Fejér kernel. Top: w_1. Middle: w_5. Bottom: w_{10}.

reasonable to expect that $\{w_N\}$ will form an approximate identity. If this is true then the Cesàro means $\sigma_N f = f * w_N$ will converge to f in $L^p(\mathbf{T})$ and in $C(\mathbf{T})$ even though the partial sums $S_N f$ may not converge. With some work, we can write w_N in closed form and show that the Fejér kernel does indeed form an approximate identity (see Exercise 13.23).

Lemma 13.20. (a) *Given* $N \in \mathbf{N}$ *we have*

$$w_N(x) \;=\; \frac{1}{N+1}\left(\frac{\sin{(N+1)\pi x}}{\sin \pi x}\right)^2.$$

(b) $\|w_N\|_{L^1} = \int_0^1 w_N = 1$ *for every* $N \in \mathbf{N}$.

(c) *The Fejér kernel* $\{w_N\}_{N\in\mathbf{N}}$ *forms an approximate identity on* $\mathbf{T}$.

(d) *If* $1 \le p < \infty$ *and* $f \in L^p(\mathbf{T})$*, then* $\sigma_N f = f * w_N \to f$ *in* L^p*-norm as* $N \to \infty$*.*

(e) *If* $f \in C(\mathbf{T})$*, then* $\sigma_N f = f * w_N \to f$ *uniformly as* $N \to \infty$*.* ◊

To contrast convergence using the Fejér kernel with that using the Dirichlet kernel, let $f = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ be the square wave function. Figure 13.3 shows $f * d_N$ for various N, while Figure 13.5 shows $f * w_N$ for the same N. We can see that $f * w_N$ appears to be a much better approximation to f than does $f * d_N$.

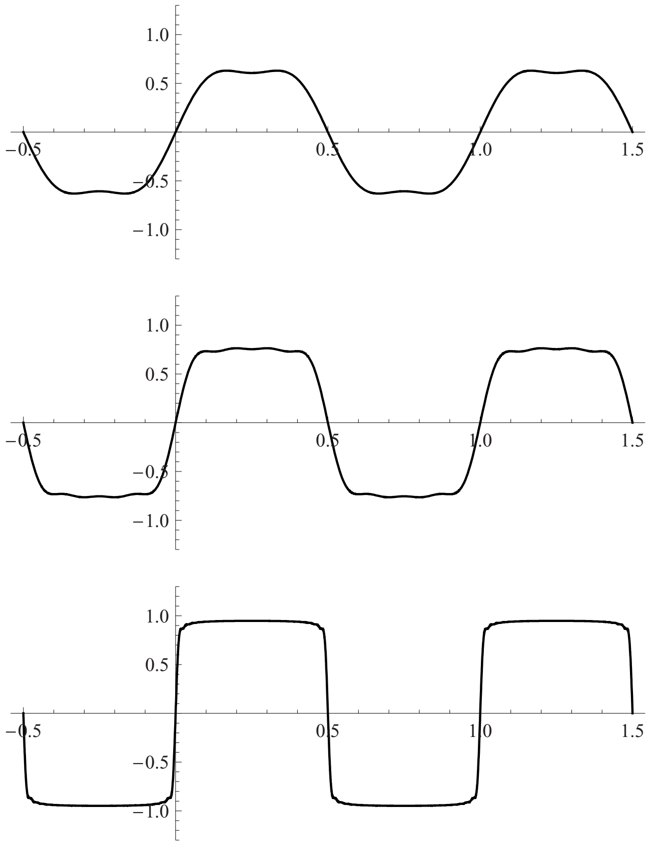

Fig. 13.5. Convolution of the square wave f with elements of the Fejér kernel. Top: $f * w_5$. Middle: $f * w_{10}$. Bottom: $f * w_{50}$.

The fact that the Fejér kernel is an approximate identity has many important implications, which we will explore in the remainder of this section and in the following sections.

Our first observation is that by combining equation (13.16) with the fact that $\sigma_N f = f * w_N$, we see that

$$(f * w_N)(x) = \sigma_N f(x) = \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr) \widehat{f}(n)\, e^{2\pi i n x}$$
$$= \sum_{n=-N}^{N} W_N(n)\, \widehat{f}(n)\, e_n(x),$$

which is a finite linear combination of the exponential functions e_n. Hence $\sigma_N f \in \operatorname{span}\{e_n\}_{n\in\mathbf{Z}}$. Since $\{w_N\}_{N\in\mathbf{N}}$ is an approximate identity, $\sigma_N f \to f$ in L^p-norm if p is finite and uniformly if $f \in C(\mathbf{T})$. Consequently, the set of trigonometric polynomials, $\operatorname{span}\{e_n\}_{n\in\mathbf{Z}}$, is dense in these spaces, which tells us that $\{e_n\}_{n\in\mathbf{Z}}$ is complete.

Theorem 13.21 (Completeness). *The trigonometric system $\{e_n\}_{n\in\mathbf{Z}}$ is complete in $C(\mathbf{T})$ and in $L^p(\mathbf{T})$ for each $1 \le p < \infty$.* $\diamondsuit$

The completeness of $\{e_n\}_{n\in\mathbf{Z}}$ in $C(\mathbf{T})$ is usually given a special name and worded as follows.

Corollary 13.22 (Weierstrass Approximation Theorem). *If $f \in C(\mathbf{T})$, then given any $\varepsilon > 0$, there exists a trigonometric polynomial $p(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$ such that $\|f - p\|_\infty < \varepsilon$.* $\diamondsuit$

Since every trigonometric polynomial is infinitely differentiable, this gives another proof that $C^\infty(\mathbf{T})$ is dense in $L^p(\mathbf{T})$ for p finite and also in $C(\mathbf{T})$; compare Exercise 13.13.

As we have observed in earlier chapters, completeness alone is a rather weak property. However, in a Hilbert space we have the wonderful fact that a complete orthonormal sequence is an orthonormal basis. Since we already know that the trigonometric system is orthonormal and we have completeness from Theorem 13.21, we immediately obtain the following important fact.

Theorem 13.23 (Orthonormal Basis of Exponentials in $L^2(\mathbf{T})$). *The trigonometric system $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$. Consequently, the following statements hold.*

(a) *For each $f \in L^2(\mathbf{T})$ we have*

$$f(x) = \bigl(\widehat{f}\,\bigr)^{\vee}(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x}, \tag{13.19}$$

where the series converges unconditionally in L^2-norm.

(b) Plancherel's Equality *and* Parseval's Equality *hold for $f, g \in L^2(\mathbf{T})$:*

$$\|f\|_{L^2}^2 = \sum_{n\in\mathbf{Z}} |\widehat{f}(n)|^2 \qquad \text{and} \qquad \langle f, g\rangle = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\,\overline{\widehat{g}(n)} = \langle \widehat{f}, \widehat{g}\,\rangle.$$

(c) $\mathcal{F}\colon f \mapsto \widehat{f}$ *is a unitary map of* $L^2(\mathbf{T})$ *onto* $\ell^2(\mathbf{Z})$. ◇

We also have a dual result for the Fourier transform of sequences.

Theorem 13.24. *For each* $c = (c_n) \in \ell^2(\mathbf{Z})$, *the series*

$$\widehat{c}(x) \;=\; \sum_{n\in\mathbf{Z}} c_n e^{-2\pi i n x}$$

converges unconditionally in L^2*-norm, and* Plancherel's Equality *and* Parseval's Equality *hold for* $c,\, d \in \ell^2(\mathbf{Z})$:

$$\|\widehat{c}\,\|_{L^2}^2 \;=\; \sum_{n\in\mathbf{Z}} |c_n|^2 \qquad \textit{and} \qquad \langle \widehat{c}, \widehat{d}\,\rangle \;=\; \sum_{n\in\mathbf{Z}} c_n \overline{d_n}.$$

Consequently $\mathcal{F}\colon c \mapsto \widehat{c}$ *is a unitary map of* $\ell^2(\mathbf{Z})$ *onto* $L^2(\mathbf{T})$. ◇

Thus Theorem 13.24 extends the definition of the Fourier transform of sequences from $\ell^1(\mathbf{Z})$ to the larger space $\ell^2(\mathbf{Z})$. However, while the Fourier transform of a sequence $c \in \ell^1(\mathbf{Z})$ is a continuous function, the Fourier transform of $c \in \ell^2(\mathbf{Z})$ is a function in $L^2(\mathbf{T})$ and therefore may only be defined almost everywhere.

Exercises

13.21. Given a sequence of scalars $a = (a_k)_{k\in\mathbf{Z}}$, let $s_N = \sum_{k=-N}^{N} a_k$.

(a) Let $\sigma_N = (s_0 + \cdots + s_N)/(N+1)$ denote the Cesàro means, and show that

$$\sigma_N \;=\; \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) a_n.$$

(b) Show that if the partial sums s_N converge, then the Cesàro means σ_N converge to the same limit, i.e.,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) a_n \;=\; \lim_{N\to\infty} s_N \;=\; \sum_{n=-\infty}^{\infty} a_n.$$

(c) Set $a_n = (-1)^n$ for $n \ge 0$ and $a_n = 0$ for $n < 0$. Show that the series $\sum_{n\in\mathbf{Z}} a_n$ is Cesàro summable even though the partial sums do not converge, and find the limit of the Cesàro means.

13.22. Prove the equality appearing in equation (13.18).

13.23. Prove Lemma 13.20.

13.7 The Inversion Formula for $L^1(\mathbf{T})$

The trigonometric system is not a basis for $L^1(\mathbf{T})$. However, we will prove in this section that if $f \in L^1(\mathbf{T})$ and we also have $\widehat{f} \in \ell^1(\mathbf{Z})$, then the Fourier series of f converges uniformly to f. In fact, with these hypotheses the Fourier series of f converges absolutely in the uniform norm, and therefore also converges unconditionally.

Theorem 13.25 (Inversion Formula). *If $f \in L^1(\mathbf{T})$ and $\widehat{f} \in \ell^1(\mathbf{Z})$, then f is continuous and*

$$f(x) \;=\; \big(\widehat{f}\,\big)^{\vee}(x) \;=\; \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x}, \qquad x \in \mathbf{T},$$

where the series converges absolutely with respect to the L^∞-norm on $\mathbf{T}$.

Proof. Since $\widehat{f} \in \ell^1(\mathbf{Z})$, we have

$$\sum_{n\in\mathbf{Z}} \|\widehat{f}(n)\, e_n\|_\infty \;=\; \sum_{n\in\mathbf{Z}} |\widehat{f}(n)| \;<\; \infty.$$

Hence the series $\big(\widehat{f}\,\big)^{\vee} = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e_n$ converges absolutely with respect to $\|\cdot\|_\infty$. Since this is a uniformly convergent series of continuous functions, we conclude that $\big(\widehat{f}\,\big)^{\vee}$ is a continuous function. Our task is to show that this function equals f.

We know that $f * w_N \to f$ in L^1-norm, and furthermore

$$(f * w_N)(x) \;=\; \sigma_N f(x) \;=\; \sum_{n\in\mathbf{Z}} W_N(n)\, \widehat{f}(n)\, e_n(x).$$

Fix x, and note that for each n we have

$$\lim_{N\to\infty} W_N(n)\, \widehat{f}(n)\, e_n(x) \;=\; \widehat{f}(n)\, e_n(x).$$

Further, $|W_N(n)\, \widehat{f}(n)\, e_n(x)| \le |\widehat{f}(n)|$ and $\widehat{f} \in \ell^1(\mathbf{Z})$. Therefore we can apply the series version of the Dominated Convergence Theorem (Theorem A.25) to obtain

$$\begin{aligned}\lim_{N\to\infty} (f * w_N)(x) \;&=\; \lim_{N\to\infty} \sum_{n\in\mathbf{Z}} W_N(n)\, \widehat{f}(n)\, e^{2\pi i n x} \\ &=\; \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x} \;=\; \big(\widehat{f}\,\big)^{\vee}(x).\end{aligned}$$

On the other hand, $f * w_N \to f$ in L^1-norm, so there is a subsequence such that $(f * w_{N_k})(x) \to f(x)$ for almost every x. Therefore $\big(\widehat{f}\,\big)^{\vee}(x) = f(x)$ a.e. By redefining f on a set of measure zero, we therefore have that f equals the continuous function $\big(\widehat{f}\,\big)^{\vee}$ pointwise everywhere. □

The *Fourier algebra* or *Wiener algebra* for the torus is the set of all absolutely convergent Fourier series:

$$A(\mathbf{T}) \;=\; \bigl\{\widehat{c} : c \in \ell^1(\mathbf{Z})\bigr\} \;=\; \biggl\{\sum_{n\in\mathbf{Z}} c_n e^{-2\pi i n x} : \sum_{n\in\mathbf{Z}} |c_n| < \infty\biggr\}.$$

Since each element $\widehat{c}$ of $A(\mathbf{T})$ is integrable, the Inversion Formula implies that

$$A(\mathbf{T}) \;=\; \bigl\{f \in L^1(\mathbf{T}) : \widehat{f} \in \ell^1(\mathbf{Z})\bigr\}.$$

As a consequence of the Inversion Formula, we can show that the Fourier transform is injective on $L^1(\mathbf{T})$.

Corollary 13.26 (Uniqueness Theorem). *If $f \in L^1(\mathbf{T})$ then*

$$\widehat{f}(n) = 0 \text{ for all } n \in \mathbf{Z} \quad\Longleftrightarrow\quad f = 0 \text{ a.e.}$$

Proof. If $f \in L^1(\mathbf{T})$ and $\widehat{f} = 0$ then we have $\widehat{f} \in \ell^1(\mathbf{Z})$. Hence the Inversion Formula tells us that $f = \bigl(\widehat{f}\bigr)^{\vee} = 0$. □

Exercises

13.24. (a) Show that if $f \in L^1(\mathbf{T})$ and $\widehat{f} \in \ell^2(\mathbf{Z})$, then $f \in L^2(\mathbf{T})$.

(b) Use part (a) to show that the Plancherel Equality remains true if we assume that f belongs to $L^1(\mathbf{T})$ instead of the smaller space $L^2(\mathbf{T})$. In other words, show that if $f \in L^1(\mathbf{T})$, then we have

$$\sum_{n\in\mathbf{Z}} |\widehat{f}(n)|^2 \;=\; \|f\|_{L^2}^2,$$

in the sense that one side is finite if and only if the other side is finite, and in this case they are equal; otherwise both sides are infinite.

13.25. Let $f(x) = \pi^2\,(x^2 - x + \frac{1}{6})$ for $x \in [0,1)$. When extended 1-periodically, f is a continuous function on $\mathbf{T}$. Compute $\widehat{f}$ and show that $\widehat{f} \in \ell^1(\mathbf{Z})$. Use this to show that

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{n^2} \;=\; \pi^2\Bigl(x^2 - x + \frac{1}{6}\Bigr), \qquad x \in [0,1], \tag{13.20}$$

where the series converges uniformly on $[0,1]$. What does the series converge to for other x? Take $x = 0$ to obtain *Euler's formula*:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}.$$

Compare Exercise 1.55.

13.26. Prove the following facts about the Fourier algebra $A(\mathbf{T})$.

(a) $A(\mathbf{T})$ is a Banach space with respect to the norm $\|f\|_A = \|\widehat{f}\,\|_{\ell^1}$, and $\mathcal{F}\colon f \mapsto \widehat{f}$ is an isometric isomorphism of $A(\mathbf{T})$ onto $L^1(\mathbf{T})$.

(b) $A(\mathbf{T})$ is closed under pointwise products, and $\|fg\|_A \le \|f\|_A\,\|g\|_A$ (hence $A(\mathbf{T})$ is a *Banach algebra* with respect to pointwise products).

(c) $C^2(\mathbf{T}) \subseteq A(\mathbf{T})$, and therefore $A(\mathbf{T})$ is dense in $C(\mathbf{T})$.

(d) $L^2(\mathbf{T}) * L^2(\mathbf{T}) = A(\mathbf{T})$.

(e) $A(\mathbf{T})$ is a meager subset of $C(\mathbf{T})$.

14

Basis Properties of Fourier Series

In this chapter we will prove that the trigonometric system $\{e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^p(\mathbf{T})$ for each $1 < p < \infty$. The proof will combine the tools of convolution and approximate identities developed in Chapter 13 with the theory of bases developed in Part II of this volume.

14.1 The Partial Sum Operators

Since we know that the trigonometric system is both minimal and complete in $L^p(\mathbf{T})$ and $C(\mathbf{T})$, to determine if it is a basis we must determine the norms of the partial sum operators. If these are uniformly bounded, then $\{e_n\}_{n\in\mathbf{Z}}$ has a finite basis constant and is therefore a basis by Theorem 5.12. However, in order to do this we must choose an ordering of the index set $\mathbf{Z}$, and consider the partial sums corresponding to that ordering. We impose a "natural" ordering on $\mathbf{Z}$, namely,

$$\mathbf{Z} \;=\; \{0,-1,1,-2,2,-3,3,\dots\}. \tag{14.1}$$

With respect to this ordering, the partial sums corresponding to the exact system $\{e_n\}_{n\in\mathbf{Z}}$ are the *symmetric partial sums*

$$S_N f(x) \;=\; \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi inx},$$

interleaved with the *asymmetric partial sums*

$$S_N^{\mathrm{a}} f(x) \;=\; \sum_{n=-N}^{N-1} \widehat{f}(n)\, e^{2\pi inx}.$$

That is, to say that the series $f(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi inx}$ converges with respect to the ordering in equation (14.1) is to say that the sequence of partial sums

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$$S_0 f,\ S_1^{\mathrm{a}} f,\ S_1 f,\ S_2^{\mathrm{a}} f,\ S_2 f,\ \dots$$

converges to f in norm. Therefore, by Theorem 5.12, the trigonometric system is a basis for $L^p(\mathbf{T})$ with respect to this ordering if and only if its *basis constant*

$$\mathcal{C}_p \;=\; \sup\bigl\{\|S_N\|_{L^p\to L^p},\ \|S_N^{\mathrm{a}}\|_{L^p\to L^p}\bigr\}_{N\in\mathbf{N}}$$

is finite (we write $\|S_N\|_{L^p\to L^p}$ to emphasize the dependence of the operator norm on p). Similarly, $\{e_n\}_{n\in\mathbf{Z}}$ is a basis for $C(\mathbf{T})$ if and only if

$$\mathcal{C}_\infty \;=\; \sup\bigl\{\|S_N\|_{C\to C},\ \|S_N^{\mathrm{a}}\|_{C\to C}\bigr\}_{N\in\mathbf{N}}$$

is finite.

While we have discussed the symmetric partial sums in some detail in the preceding sections, the lack of any prior discussion of the asymmetric partial sums now stands out conspicuously. The reason for this is that our next theorem will show that the symmetric partial sums are uniformly bounded in operator norm if and only if the asymmetric partial sums are uniformly bounded. Hence it is safe to simply ignore the asymmetric partial sums (and indeed, they are rarely even mentioned). More surprising, at least at first glance, is that we can also replace the symmetric partial sums with the *one-sided partial sums*

$$S_N^{\mathrm{o}} f(x) \;=\; \sum_{n=0}^{2N} \widehat{f}(n)\, e^{2\pi i n x},$$

and even with the *twisted partial sums*

$$S_N^{\mathrm{t}} f(x) \;=\; -i \sum_{n=-2N}^{2N} \mathrm{sign}(n)\, \widehat{f}(n)\, e^{2\pi i n x},$$

where $\mathrm{sign}(n)$ is the sign function (1, 0, or -1 according to whether $n>0$, $n=0$, or $n<0$, respectively). To prove this, it is useful to note that we can write $S_N f$ and $S_N^{\mathrm{o}} f$ on the Fourier side as

$$(S_N f)^\wedge \;=\; \widehat{f}\cdot \chi_{[-N,N]} \qquad \text{and} \qquad (S_N^{\mathrm{o}} f)^\wedge \;=\; \widehat{f}\cdot \chi_{[0,2N]},$$

where we have slightly abused notation by writing $\chi_{[-N,N]}$ instead of the more cumbersome $\chi_{\{-N,\dots,N\}}$. Now, multiplying $\widehat{f}$ by $\chi_{[0,2N]}$ is exactly the same as translating $\widehat{f}$ left by N units, multiplying by $\chi_{[-N,N]}$, and translating the result right by N units, i.e.,

$$\widehat{f}\cdot \chi_{[0,2N]} \;=\; T_{-N}\Bigl(\bigl(T_N\widehat{f}\,\bigr)\cdot \chi_{[-N,N]}\Bigr).$$

Since the Fourier transform turns translations into modulations and vice versa, we obtain the following relation between $S_N f$ and $S_N^{\mathrm{o}} f$ (see Exercise 14.1).

Lemma 14.1. *Given $f \in L^1(\mathbf{T})$ and $N \in \mathbf{N}$, we have*

$$S_N^{\text{o}} f \;=\; M_N S_N M_{-N} f. \qquad \diamondsuit$$

Since modulation is an isometry on $L^p(\mathbf{T})$ and on $C(\mathbf{T})$, we see that the operator norms of S_N and S_N^{o} are identical. The relationships among the operator norms of the other partial sum operators are more complicated, but nonetheless we obtain the following connections between them.

Theorem 14.2. *If $1 \le p < \infty$, then the following statements are equivalent.*

(a) *$\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^p(\mathbf{T})$ with respect to the ordering of $\mathbf{Z}$ given in equation* (14.1).

(b) $\sup \|S_N\|_{L^p\to L^p} < \infty$.

(c) $\sup \|S_N^{\text{a}}\|_{L^p\to L^p} < \infty$.

(d) $\sup \|S_N^{\text{o}}\|_{L^p\to L^p} < \infty$.

(e) $\sup \|S_N^{\text{t}}\|_{L^p\to L^p} < \infty$.

Analogous equivalences also hold if $L^p(\mathbf{T})$ is replaced everywhere by $C(\mathbf{T})$.

Proof. We prove some implications for $L^p(\mathbf{T})$, and assign the remaining implications and the extension to $C(\mathbf{T})$ as Exercise 14.2.

(b) $\Rightarrow$ (c) If $f \in L^p(\mathbf{T})$ and $N > 0$, then, using equation (13.2) and Exercise 1.13, we compute that

$$\begin{aligned}
\|S_N^{\text{a}} f\|_{L^p} &= \|S_N f - \widehat{f}(N)\, e^{2\pi iNx}\|_{L^p} \\
&\le \|S_N f\|_{L^p} + |\widehat{f}(N)| \\
&\le \|S_N f\|_{L^p} + \|f\|_{L^1} \\
&\le \|S_N f\|_{L^p} + \|f\|_{L^p}.
\end{aligned}$$

Hence $\|S_N^{\text{a}}\|_{L^p\to L^p} \le \|S_N\|_{L^p\to L^p} + 1$.

(d) $\Rightarrow$ (e). Suppose that statement (d) holds, and fix $f \in L^p(\mathbf{T})$. By Exercise 13.11, the function $f^*(x) = f(-x)$ satisfies $(f^*)^\wedge(n) = \widehat{f}(-n)$ for $n \in \mathbf{Z}$. Therefore,

$$\begin{aligned}
S_N^{\text{o}} f(x) - S_N^{\text{o}} f^*(-x) &= \sum_{n=0}^{2N} \widehat{f}(n)\, e^{2\pi inx} \;-\; \sum_{n=0}^{2N} \widehat{f}(-n)\, e^{-2\pi inx} \\
&= \sum_{n=0}^{2N} \widehat{f}(n)\, e^{2\pi inx} \;-\; \sum_{n=-2N}^{0} \widehat{f}(n)\, e^{2\pi inx}
\end{aligned}$$

$$= \sum_{n=-2N}^{2N} \operatorname{sign}(n)\, \widehat{f}(n)\, e^{2\pi i n x}$$

$$= i\, S_N^{\mathrm{t}} f(x).$$

Consequently,

$$\begin{aligned} \|S_N^{\mathrm{t}} f\| &\le \|S_N^{\mathrm{o}} f\|_{L^p} + \|S_N^{\mathrm{o}} f^*\|_{L^p} \\ &\le \|S_N^{\mathrm{o}}\|_{L^p \to L^p}\, \|f\|_{L^p} \;+\; \|S_N^{\mathrm{o}}\|_{L^p \to L^p}\, \|f^*\|_{L^p} \\ &= 2\, \|S_N^{\mathrm{o}}\|_{L^p \to L^p}\, \|f\|_{L^p}. \qquad \square \end{aligned}$$

We will return to the twisted and one-sided sums shortly, but for the moment let us concentrate on the symmetric partial sums. Since $S_N f = f * d_N$, we have that $\|S_N f\|_{L^p} \le \|f\|_{L^p}\, \|d_N\|_{L^1}$, and therefore

$$\|S_N\|_{L^p \to L^p} \;\le\; \|d_N\|_{L^1}. \tag{14.2}$$

Similarly, $\|S_N\|_{C \to C} \;\le\; \|d_N\|_{L^1}$. The numbers $\|d_N\|_{L^1}$ are called the *Lebesgue constants*. By Exercise 13.20, they tend to infinity as $N \to \infty$. Therefore, if equality holds in equation (14.2), then the exponentials cannot form a Schauder basis for $L^p(\mathbf{T})$.

Theorem 14.3. $\|S_N\|_{L^1 \to L^1} \;=\; \|d_N\|_{L^1} \;=\; \|S_N\|_{C \to C}$. *Consequently, while* $\{e^{2\pi i n x}\}_{n \in \mathbf{Z}}$ *is exact in both* $L^1(\mathbf{T})$ *and* $C(\mathbf{T})$, *it is not a Schauder basis for either space.*

Proof. Consider $L^1(\mathbf{T})$ first. Since the Fejér kernel $\{w_m\}_{m \in \mathbf{N}}$ is an approximate identity, we have

$$\lim_{m \to \infty} \|S_N w_m\|_{L^1} \;=\; \lim_{m \to \infty} \|w_m * d_N\|_{L^1} \;=\; \|d_N\|_{L^1}.$$

Since $\|w_m\|_{L^1} = 1$, it follows that $\|S_N\|_{L^1 \to L^1} \ge \|d_N\|_{L^1}$.

Now we turn to $C(\mathbf{T})$. Consider $\operatorname{sign} d_N$, the sign function of d_N, which takes only the values 1, 0, or -1. The function $\operatorname{sign} d_N$ is piecewise constant, with $2N$ zero crossings in $[0,1]$. Fix $\varepsilon > 0$, and let g_N be any continuous function such that $g_N(x) = \operatorname{sign} d_N(x)$ except for a set E_N of measure at most $\varepsilon/(4N+2)$. We can do this in such a way that $-1 \le g_N \le 1$ everywhere, so we have $\|g_N\|_\infty = 1$. Since $\|d_N\|_\infty = 2N+1$,

$$\begin{aligned} &\left| \int_0^1 |d_N(x)|\, dx \;-\; \int_0^1 g_N(x)\, d_N(x)\, dx \right| \\ &\qquad = \int_{E_N} |d_N(x)|\, dx \;-\; \int_{E_N} g_N(x)\, d_N(x)\, dx \\ &\qquad \le 2\, |E_N|\, \|d_N\|_\infty \\ &\qquad \le \varepsilon. \end{aligned}$$

Consequently,

$$\begin{aligned}\|S_N g_N\|_\infty \;\ge\; |S_N g_N(0)| \;&=\; |(g_N * d_N)(0)| \\ &=\; \left|\int_0^1 g_N(x)\, d_N(x)\, dx\right| \\ &\ge\; \int_0^1 |d_N(x)|\, dx \;-\; \varepsilon \\ &=\; \|d_N\|_{L^1} - \varepsilon.\end{aligned}$$

Since g_N is a unit vector, it follows that $\|S_N\|_{C\to C} \ge \|d_N\|_{L^1} - \varepsilon$. Since ε is arbitrary, the result follows. □

Consequently, there exist integrable functions whose Fourier series do not converge in L^1-norm, and continuous functions whose Fourier series do not converge uniformly. In fact, we show next that the proof of Theorem 14.3 implies the existence of continuous functions whose Fourier series diverge at a point. Although the following proof is nonconstructive, explicit examples of such functions are known.

Corollary 14.4. *Given $x \in \mathbf{T}$, there exists a function $f \in C(\mathbf{T})$ such that $S_N f(x)$ does not converge as $N \to \infty$.*

Proof. By replacing f with a 1-periodic translation of f, it suffices to consider $x = 0$. For each $N \in \mathbf{N}$ define a linear functional $\mu_N \colon C(\mathbf{T}) \to \mathbf{C}$ by $\langle f, \mu_N\rangle = S_N f(0)$. The proof of Theorem 14.3 shows that μ_N is bounded and $\|\mu_N\| = \|d_N\|_{L^1}$. If $\lim_{N\to\infty} \langle f, \mu_N\rangle$ exists for each $f \in C(\mathbf{T})$, then the Banach–Steinhaus Theorem (Theorem 2.23) implies that $\sup_N \|\mu_N\| < \infty$, which contradicts the fact that $\sup_N \|d_N\|_{L^1} = \infty$. □

Even more surprising, given any set $E \subseteq \mathbf{T}$ of measure zero, there exists a continuous function f such that $S_N f(x)$ diverges for each $x \in E$ [Kat04, Thm. 3.5]. In particular, there exist continuous functions whose Fourier series diverge for every x in a dense subset of $\mathbf{T}$.

Exercises

14.1. Prove Lemma 14.1.

14.2. Finish the proof of Theorem 14.2.

14.3. (a) Suppose that $f \in C(\mathbf{T})$, $f(0) = 0$, and f is differentiable at $x = 0$. Show that $g(x) = f(x)/(e^{-2\pi i x} - 1) \in C(\mathbf{T})$ and

$$\sum_{k=-N}^{N} \widehat{f}(k) \;=\; \widehat{g}(N+1) - \widehat{g}(-N).$$

Conclude that the symmetric partial sums of the Fourier series for f converge at the point $x = 0$ to $f(0)$.

(b) Show that if $f \in C(\mathbf{T})$ is differentiable at a point $x \in \mathbf{T}$, then

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{f}(k)\, e^{2\pi i k x} \;=\; f(x).$$

14.2 The Conjugate Function

Typically, the analysis of the convergence of Fourier series is worded in terms of the *conjugate function* $\widetilde{f}$, which is the formal limit of the twisted partial sums of the Fourier series of f. The mapping $f \mapsto \widetilde{f}$ is closely related to the *Hilbert transform* of functions on $\mathbf{R}$, and hence we will denote it by H (we refer to [Gra04] for more details on the Hilbert transform).

Definition 14.5 (Conjugate Function). Let $f \in L^1(\mathbf{T})$ be given. If there exists a function $g \in L^1(\mathbf{T})$ such that

$$\widehat{g}(n) \;=\; -i\,\mathrm{sign}(n)\,\widehat{f}(n), \qquad n \in \mathbf{Z},$$

then g is called the *conjugate function* of f. We denote the conjugate function by $\widetilde{f}$ or Hf.

We say that a subspace $B \subseteq L^1(\mathbf{T})$ *admits conjugation* if each function $f \in B$ has a conjugate function Hf that belongs to B. $\diamondsuit$

Formally, the conjugate function of f is

$$Hf(x) \;=\; -i \sum_{n\in\mathbf{Z}} \mathrm{sign}(n)\,\widehat{f}(n)\,e^{2\pi i n x}, \tag{14.3}$$

although it is important to note that there is no guarantee that this series will converge in general. One special case is functions f that have absolutely convergent Fourier series. In this case the series in equation (14.3) also converges absolutely. This leads to the next result, whose proof is Exercise 14.5.

Theorem 14.6. *The Fourier algebra $A(\mathbf{T})$ admits conjugation, and if $f \in A(\mathbf{T})$, then the twisted partial sums $S_N^{\mathrm{t}} f$ converge to Hf in L^p-norm for each $1 \le p \le \infty$:*

$$\forall\, f \in A(\mathbf{T}), \quad \forall\, 1 \le p \le \infty, \quad \lim_{N\to\infty} \|Hf - S_N^{\mathrm{t}} f\|_{L^p} \;=\; 0. \qquad \diamondsuit \tag{14.4}$$

Thus, conjugation is well defined on $A(\mathbf{T})$, which is a dense subspace of $L^p(\mathbf{T})$ for $1 \le p < \infty$ (see Exercise 13.26). Typically, an operator that is defined on a dense subspace of a Banach space can only be extended to the entire space if it is bounded. Combining this with our previous results relating boundedness of the various types of partial sums to Schauder basis properties, we obtain the following theorem for $L^p(\mathbf{T})$.

Theorem 14.7. *Given $1 \le p \le \infty$, the following statements are equivalent.*

(a) *The trigonometric system $\{e_n\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^p(\mathbf{T})$ with respect to the ordering of $\mathbf{Z}$ given in equation* (14.1).

(b) *$L^p(\mathbf{T})$ admits conjugation.*

(c) *Conjugation is a bounded mapping of some dense subspace of $L^p(\mathbf{T})$ into $L^p(\mathbf{T})$.*

(d) *$L^p(\mathbf{T})$ admits conjugation, and conjugation is a bounded mapping of $L^p(\mathbf{T})$ into itself.*

Proof. We prove some implications, and assign the rest as Exercise 14.6.

(a) $\Rightarrow$ (c). Assume that $\{e_n\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^p(\mathbf{T})$. Then by Theorem 14.2, $C = \sup \|S_N^{\mathrm{t}}\|_{L^p\to L^p} < \infty$. Therefore, using equation (14.4), for each $f \in A(\mathbf{T})$ we have

$$\|Hf\|_{L^p} \;=\; \lim_{N\to\infty} \|S_N^{\mathrm{t}} f\|_{L^p} \;\le\; \limsup_{N\to\infty} \|S_N^{\mathrm{t}}\|_{L^p\to L^p}\, \|f\|_{L^p} \;\le\; C\,\|f\|_{L^p}.$$

Hence H is a bounded mapping of $\bigl(A(\mathbf{T}), \|\cdot\|_{L^p}\bigr)$ into $L^p(\mathbf{T})$, so statement (c) holds.

(d) $\Rightarrow$ (a). Assume that H is a bounded mapping of $L^p(\mathbf{T})$ into itself. Since $|\widehat{f}(0)| \le \|f\|_{L^1} \le \|f\|_{L^p}$, it follows that the *Riesz projection operator*

$$Rf \;=\; \frac{f + iHf}{2} + \frac{\widehat{f}(0)}{2}, \qquad f \in L^p(\mathbf{T}),$$

is also a bounded mapping of $L^p(\mathbf{T})$ into itself. Note that

$$(Rf)^\wedge \;=\; \widehat{f}\cdot \chi_{[0,\infty)} \qquad \text{and} \qquad (M_{2N+1}RM_{-2N-1}f)^\wedge \;=\; \widehat{f}\cdot\chi_{[2N+1,\infty)},$$

and therefore

$$S_N^{\mathrm{o}} f \;=\; Rf - M_{2N+1}RM_{-2N-1}f.$$

Since modulation is an isometry on $L^p(\mathbf{T})$, it follows that

$$\|S_N^{\mathrm{o}}\|_{L^p\to L^p} \;\le\; 2\,\|R\|_{L^p\to L^p}.$$

The right-hand side is a constant independent of N, so Theorem 14.2 implies that $\{e_n\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^p(\mathbf{T})$. $\square$

Since we know that the trigonometric system does not form a Schauder basis for $L^1(\mathbf{T})$ or $C(\mathbf{T})$, we conclude that these two spaces do not admit conjugation. We next consider $L^p(\mathbf{T})$ when $1 < p < \infty$, beginning with the case where p is an even integer.

Theorem 14.8. *If $k \in \mathbf{N}$, then the trigonometric system $\{e_n\}_{n\in\mathbf{Z}}$ is a Schauder basis for $L^{2k}(\mathbf{T})$ with respect to the ordering of $\mathbf{Z}$ given in equation* (14.1).

Proof. Let $S = \operatorname{span}\{e_n\}_{n\in\mathbf{Z}}$. This is the set of all trigonometric polynomials, and it is dense in $L^{2k}(\mathbf{T})$ by Theorem 13.21. By Theorem 14.7, it suffices to show that conjugation is a bounded mapping of S into $L^{2k}(\mathbf{T})$.

Step 1. Suppose that $f(x) = \sum_{n=-N}^{N} c_n\, e^{2\pi i n x}$ is a nonzero real-valued trigonometric polynomial that satisfies $c_0 = \widehat{f}(0) = 0$. Then f has a conjugate function Hf, and Hf is also real valued by Exercise 14.4. Since

$$f(x) + i\, Hf(x) \;=\; 2\sum_{n=1}^{N} c_n\, e^{2\pi i n x},$$

function $f + iHf$ is a trigonometric polynomial that contains only positive frequencies. Therefore $g = (f + iHf)^{2k}$ is also a trigonometric polynomial that contains only positive frequencies. Since f and Hf are both real valued, expanding by the Binomial Theorem and considering real parts, it follows that

$$\begin{aligned}
0 \;=\; \widehat{g}(0) \;&=\; \int_0^1 \bigl(f(x) + iHf(x)\bigr)^{2k}\, dx \\
&=\; \operatorname{Re}\biggl(\sum_{j=0}^{2k} i^{2k-j} \binom{2k}{j} \int_0^1 f(x)^j\, Hf(x)^{2k-j}\, dx\biggr) \\
&=\; \sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{2j} \int_0^1 f(x)^{2j}\, Hf(x)^{2k-2j}\, dx.
\end{aligned}$$

Solving for the $j = 0$ term and applying Hölder's Inequality with exponents $p_j = (2k)/(2j)$ and $p_j' = (2k)/(2k-2j)$, we find that

$$\begin{aligned}
\|Hf\|_{2k}^{2k} \;&=\; \int_0^1 Hf(x)^{2k}\, dx \\
&=\; \sum_{j=1}^{k} (-1)^{j+1} \binom{2k}{2j} \int_0^1 f(x)^{2j}\, Hf(x)^{2k-2j}\, dx
\end{aligned}$$

$$\le \sum_{j=1}^{k} (-1)^{j+1} \binom{2k}{2j} \left(\int_0^1 |f(x)|^{2j \cdot \frac{2k}{2j}} \, dx \right)^{\frac{2j}{2k}}$$
$$\times \left(\int_0^1 |Hf(x)|^{(2k-2j) \cdot \frac{2k}{2k-2j}} \, dx \right)^{\frac{2k-2j}{2k}}$$
$$= \sum_{j=1}^{k} \binom{2k}{2j} \|f\|_{2k}^{2j} \, \|Hf\|_{2k}^{2k-2j}.$$

If we set $t = \|Hf\|_{2k}/\|f\|_{2k}$ and rearrange the preceding inequality, we obtain

$$1 \le \sum_{j=1}^{k} \binom{2k}{2j} t^{-2j}. \tag{14.5}$$

Define

$$r(s) = \sum_{j=1}^{k} \binom{2k}{2j} s^{-2j}, \qquad s > 0.$$

This is a decreasing function of s, and $\lim_{s \to \infty} r(s) = 0$. Hence

$$C = \sup\bigl\{ s > 0 : r(s) \ge 1 \bigr\} < \infty.$$

By equation (14.5) we have $r(t) \ge 1$, so we must have $t \le C$, and therefore $\|Hf\|_{2k} \le C \, \|f\|_{2k}$.

Step 2. Now suppose that f is any real-valued trigonometric polynomial. Then $\widehat{f}(0) = \int_0^1 f(x) \, dx$ is real valued, so $g(x) = f(x) - \widehat{f}(0)$ is a real-valued trigonometric polynomial that satisfies $\widehat{g}(0) = 0$. Further, $Hg = Hf$, so

$$\begin{aligned} \|Hf\|_{2k} = \|Hg\|_{2k} &\le C \, \|g\|_{2k} \\ &\le C \, \|f\|_{2k} + C \, |\widehat{f}(0)| \\ &\le C \, \|f\|_{2k} + C \, \|f\|_{L^1} \\ &\le (C+1) \, \|f\|_{2k}. \end{aligned}$$

Step 3. If f is an arbitrary trigonometric polynomial, then we can write $f = g + ih$, where g and h are real-valued trigonometric polynomials. Since $Hf = Hg + iHh$, by applying Step 2 to Hg and Hh we obtain $\|Hf\|_{2k} \le (2C+2) \, \|f\|_{2k}$. Therefore conjugation is a bounded mapping on the dense subspace S, so the result follows by Theorem 14.7. □

Thus, we have proved that conjugation is a bounded mapping on $L^2(\mathbf{T})$, $L^4(\mathbf{T})$, $L^6(\mathbf{T})$, etc. If we could somehow infer boundedness for the "in-between" values of p, we could conclude that the trigonometric system is a Schauder basis for $L^p(\mathbf{T})$ for each $2 \le p < \infty$. This is a classic situation, and *interpolation theory* comes to our rescue here. In particular, the *Riesz–Thorin Interpolation Theorem* implies that conjugation is bounded on $L^p(\mathbf{T})$ for each $p \in [2k, 2k+2]$, and hence for all $2 \le p < \infty$. For details on interpolation theory in this context, we refer to [Kat04], and for a general reference on interpolation theory we refer to [BL76].

Finally, if f and g are sufficiently nice functions, e.g., $f, g \in A(\mathbf{T})$, then we have by the Parseval Equality that

$$\begin{aligned}\langle Hf, g\rangle \;=\; \langle (Hf)^\wedge, \widehat{g}\,\rangle \;&=\; -i\sum_{n\in\mathbf{Z}} \operatorname{sign}(n)\,\widehat{f}(n)\,\overline{\widehat{g}(n)}\\ &=\; -\langle \widehat{f}, (Hg)^\wedge\rangle \;=\; -\langle f, Hg\rangle.\end{aligned}$$

Hence conjugation is *skew-adjoint*, and this can be used to give an argument "by duality" that boundedness of H on $L^p(\mathbf{T})$ for $2 \le p < \infty$ implies boundedness for $1 < p \le 2$ (see Exercise 14.7).

Thus, with considerably more work than for the case $p = 2$, we see that the trigonometric system forms a Schauder basis for $L^p(\mathbf{T})$ for each $1 < p < \infty$. Moreover, Exercise 6.5 tells us that this basis is conditional if $p \ne 2$. The partial sums of the Fourier series of $f \in L^p(\mathbf{T})$ converge to f in L^p-norm if we follow the ordering $\mathbf{Z} = \{0, -1, 1, 2, -2, \dots\}$. However, if $p \ne 2$ then there exists some $f \in L^p(\mathbf{T})$ such that the partial sums of its Fourier series do not converge with respect to some other ordering of $\mathbf{Z}$.

Exercises

14.4. (a) Suppose that $g \in L^1(\mathbf{T})$. Show that g is real valued if and only if $\overline{\widehat{g}(n)} = \widehat{g}(-n)$ for $n \in \mathbf{Z}$.

(b) Show that if $f \in A(\mathbf{T})$ is real valued, then its conjugate function Hf is also real valued.

14.5. Prove Theorem 14.6.

14.6. Finish the proof of Theorem 14.7.

14.7. Given $1 < p < \infty$, show that conjugation is bounded on $L^p(\mathbf{T})$ if and only if it is bounded on $L^{p'}(\mathbf{T})$.

14.3 Pointwise Almost Everywhere Convergence

We have concentrated on norm convergence of Fourier series. One of the deepest results in Fourier analysis is the following theorem on pointwise almost everywhere convergence of Fourier series, which we state without proof. This theorem was proved by Lennart Carleson for the case $p = 2$ in [Car66] and extended to $1 < p < \infty$ by Richard Hunt in [Hun68].

Theorem 14.9 (Carleson–Hunt Theorem). *If* $1 < p < \infty$, *then for each* $f \in L^p(\mathbf{T})$, *the partial sums* $S_N f$ *converge to* f *pointwise a.e. That is,*

$$f(x) \;=\; \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x} \quad \textit{a.e.},$$

in the sense of convergence of the symmetric partial sums. ◊

The restriction to $p > 1$ is necessary, as there exist functions $f \in L^1(\mathbf{T})$ whose Fourier series diverge at almost every x [Gra04, Thm. 3.4.2].

Part V

Appendices

A

Lebesgue Measure and Integration

In this appendix we give a brief review, without proofs, of Lebesgue measure and integration on subsets of $\mathbf{R}^d$. Details and proofs can be found in texts on real analysis, such as [Fol99] or [WZ77].

A.1 Exterior Lebesgue Measure

For compactness of notation, we will refer to rectangular parallelepipeds in $\mathbf{R}^d$ whose sides are parallel to the coordinate axes simply as "boxes."

Definition A.1. (a) A *box* in $\mathbf{R}^d$ is a set of the form

$$Q \;=\; [a_1, b_1] \times \cdots \times [a_d, b_d] \;=\; \prod_{i=1}^{d} [a_i, b_i].$$

The *volume* of this box is

$$\mathrm{vol}(Q) \;=\; (b_1 - a_1) \cdots (b_d - a_d) \;=\; \prod_{i=1}^{d} (b_i - a_i).$$

(b) The *exterior Lebesgue measure* or *outer Lebesgue measure* of a set $E \subseteq \mathbf{R}^d$ is

$$|E|_e \;=\; \inf \Bigl\{ \sum_k \mathrm{vol}(Q_k) \Bigr\},$$

where the infimum is taken over all *finite or countable* collections of boxes Q_k such that $E \subseteq \bigcup_k Q_k$. $\diamondsuit$

Thus, every subset of $\mathbf{R}^d$ has a uniquely defined exterior measure that lies in the range $0 \le |E|_e \le \infty$. Here are some of the basic properties of exterior measure.

Theorem A.2. (a) *If Q is a box in $\mathbf{R}^d$, then $|Q|_e = \mathrm{vol}(Q)$.*

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(b) Monotonicity: *If* $E \subseteq F \subseteq \mathbf{R}^d$, *then* $|E|_e \le |F|_e$.

(c) Countable subadditivity: *If* $E_k \subseteq \mathbf{R}^d$ *for* $k \in \mathbf{N}$, *then*

$$\left|\bigcup_{k=1}^{\infty} E_k\right|_e \le \sum_{k=1}^{\infty} |E_k|_e.$$

(d) Translation invariance: *If* $E \subseteq \mathbf{R}^d$ *and* $h \in \mathbf{R}^d$, *then* $|E+h|_e = |E|_e$, *where* $E + h = \{t + h : t \in E\}$.

(e) Regularity: *If* $E \subseteq \mathbf{R}^d$ *and* $\varepsilon > 0$, *then there exists an open set* $U \supseteq E$ *such that* $|U|_e \le |E|_e + \varepsilon$, *and hence*

$$|E|_e = \inf\{|U|_e : U \text{ open, } U \supseteq E\}. \qquad \diamondsuit$$

A.2 Lebesgue Measure

Definition A.3. A set $E \subseteq \mathbf{R}^d$ is *Lebesgue measurable*, or simply *measurable*, if

$$\forall\, \varepsilon > 0, \quad \exists \text{ open } U \supseteq E \text{ such that } |U \backslash E|_e \le \varepsilon. \tag{A.1}$$

If E is Lebesgue measurable, then its *Lebesgue measure* is its exterior Lebesgue measure and is denoted by $|E| = |E|_e$. $\diamondsuit$

Note that equation (A.1) does not follow from Theorem A.2(e). One consequence of the Axiom of Choice is that there exist subsets of $\mathbf{R}^d$ that are not measurable.

The following result summarizes some of the properties of measurable sets.

Theorem A.4. (a) *The class of measurable subsets of* $\mathbf{R}^d$ *is a* σ*-algebra, meaning that*:

i. $\emptyset$ *and* $\mathbf{R}^d$ *are measurable,*

ii. *if* $E_1, E_2, \dots$ *are measurable, then* $\cup E_k$ *is measurable,*

iii. *if* E *is measurable, then* $\mathbf{R}^d \backslash E$ *is measurable.*

(b) *Every open and every closed subset of* $\mathbf{R}^d$ *is measurable.*

(c) *Every subset* E *of* $\mathbf{R}^d$ *with* $|E|_e = 0$ *is measurable.* $\diamondsuit$

Since measurability is preserved under complements and countable unions, it is also preserved under countable intersections.

We give some equivalent formulations of measurability.

Definition A.5. (a) A set $H \subseteq \mathbf{R}^d$ is a G_δ*-set* if there exist finitely or countably many open sets U_k such that $H = \cap U_k$.

(b) A set $H \subseteq \mathbf{R}^d$ is an F_σ*-set* if there exist finitely or countably many closed sets F_k such that $H = \cup F_k$. $\diamondsuit$

Theorem A.6. *Let $E \subseteq \mathbf{R}^d$ be given. Then the following statements are equivalent.*

(a) *E is measurable.*

(b) *For every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E \backslash F|_e \le \varepsilon$.*

(c) *$E = H \backslash Z$ where H is a G_δ-set and $|Z| = 0$.*

(d) *$E = H \cup Z$ where H is an F_σ-set and $|Z| = 0$.* ♢

Now we list some properties of Lebesgue measure.

Theorem A.7. *Let E and E_k be measurable subsets of $\mathbf{R}^d$.*

(a) Countable additivity: *If $E_1, E_2, \dots$ are disjoint measurable subsets of $\mathbf{R}^d$, then*

$$\left|\bigcup_{k=1}^{\infty} E_k\right| = \sum_{k=1}^{\infty} |E_k|.$$

(b) *If $E_1 \subseteq E_2$ and $|E_1| < \infty$, then $|E_2 \backslash E_1| = |E_2| - |E_1|$.*

(c) Continuity from below: *If $E_1 \subseteq E_2 \subseteq \cdots$, then $\left|\cup E_k\right| = \lim_{k\to\infty} |E_k|$.*

(d) Continuity from above: *If $E_1 \supseteq E_2 \supseteq \cdots$ and $|E_1| < \infty$, then $\left|\cap E_k\right| = \lim_{k\to\infty} |E_k|$.*

(e) Translation invariance: *If $h \in \mathbf{R}^d$, then $|E + h| = |E|$, where $E + h = \{x + h : x \in E\}$.*

(f) Linear changes of variable: *If $T \colon \mathbf{R}^d \to \mathbf{R}^d$ is linear, then $T(E)$ is measurable and $|T(E)| = |\det(T)|\,|E|$.*

(g) Cartesian products: *If $E \subseteq \mathbf{R}^m$ and $F \subseteq \mathbf{R}^n$ are measurable, then $E \times F \subseteq R^{m+n}$ is measurable and $|E \times F| = |E|\,|F|$.* ♢

We end this section with some terminology.

Definition A.8. A property that holds except possibly on a set of measure zero is said to hold *almost everywhere*, abbreviated a.e. ♢

For example, if C is the classical Cantor middle-thirds set, then $|C| = 0$. Hence, the characteristic function χ_C of C satisfies $\chi_C(t) = 0$ except for those t that belong to the zero measure set C. Therefore we say that $\chi_C(t) = 0$ for almost every t, or $\chi_C = 0$ a.e. for short.

The essential supremum of a function is an example of a quantity that is defined in terms of a property that holds almost everywhere.

Definition A.9 (Essential Supremum). The *essential supremum* of a function $f \colon E \to \mathbf{R}$ is

$$\operatorname*{ess\,sup}_{t\in E} f(t) = \inf\{M : f \le M \text{ a.e.}\}.$$

We say that f is *essentially bounded* if $\operatorname{ess\,sup}_{t\in E} |f(t)| < \infty$. ♢

A.3 Measurable Functions

Now we define the class of measurable functions on subsets of $\mathbf{R}^d$.

Definition A.10 (Real-Valued Measurable Functions). Fix a measurable set $E \subseteq \mathbf{R}^d$, and let $f\colon E \to \mathbf{R}$ be given. Then f is a *Lebesgue measurable function*, or simply a *measurable function*, if $f^{-1}(\alpha, \infty) = \{t \in E : f(t) > \alpha\}$ is a measurable subset of $\mathbf{R}^d$ for each $\alpha \in \mathbf{R}$. ♢

In particular, every continuous function $f\colon \mathbf{R}^d \to \mathbf{R}$ is measurable. However, a measurable function need not be continuous.

Measurability is preserved under most of the usual operations, including addition, multiplication, and limits. Some care does need to be taken with compositions, but if we compose a measurable function with a continuous function in the correct order, then measurability will be assured.

Theorem A.11. *Let $E \subseteq \mathbf{R}^d$ be measurable.*

(a) *If $f\colon E \to \mathbf{R}$ is measurable and $g = f$ a.e., then g is measurable.*

(b) *If $f,\, g\colon E \to \mathbf{R}$ are measurable, then so is $f + g$.*

(c) *If $f\colon E \to \mathbf{R}$ is measurable and $\varphi\colon \mathbf{R} \to \mathbf{R}$ is continuous, then $\varphi \circ f$ is measurable. Consequently, $|f|$, f^2, f^+, f^-, and $|f|^p$ for $p > 0$ are all measurable.*

(d) *If $f,\, g\colon E \to \mathbf{R}$ are measurable, then so is fg.*

(e) *If $f_n\colon E \to \mathbf{R}$ are measurable for $n \in \mathbf{N}$, then so are $\sup f_n$, $\inf f_n$, $\limsup f_n$, and $\liminf f_n$.*

(f) *If $f_n\colon E \to \mathbf{R}$ are measurable for $n \in \mathbf{N}$ and $f(t) = \lim_{n\to\infty} f_n(t)$ exists for a.e. t, then f is measurable.* ♢

Definition A.12 (Complex-Valued Measurable Functions). Given a measurable domain $E \subseteq \mathbf{R}^d$ and a complex-valued function $f\colon E \to \mathbf{C}$, write f in real and imaginary parts as $f = f_r + i f_i$. Then we say that f is *measurable* if both f_r and f_i are measurable. ♢

Egoroff's Theorem says that pointwise convergence of measurable functions is uniform convergence on "most" of the set.

Theorem A.13 (Egoroff's Theorem). *Let $E \subseteq \mathbf{R}^d$ be measurable with $|E| < \infty$. If $f_n,\, f\colon E \to \mathbf{C}$ are measurable functions and $f_n(t) \to f(t)$ for a.e. $t \in E$, then for every $\varepsilon > 0$ there exists a measurable set $A \subseteq E$ such that $|A| < \varepsilon$ and f_n converges uniformly to f on $E\backslash A$, i.e.,*

$$\lim_{n\to\infty} \Bigl(\sup_{t\notin A} |f(t) - f_n(t)|\Bigr) \;=\; 0. \qquad ♢$$

A.4 The Lebesgue Integral

To define the Lebesgue integral of a measurable function, we first begin with "simple functions" and then extend to nonnegative functions, real-valued functions, and complex-valued functions.

Definition A.14. Let $E \subseteq \mathbf{R}^d$ be measurable.

(a) A *simple function* on E is a function $\phi\colon E \to \mathbf{F}$ of the form

$$\phi = \sum_{k=1}^{N} a_k \, \chi_{E_k}, \tag{A.2}$$

where $N > 0$, $a_k \in \mathbf{F}$, and the E_k are measurable subsets of E.

(b) If $a_1, \dots, a_N \in \mathbf{F}$ are the distinct values assumed by a simple function ϕ and we set $E_k = \{t \in E : \phi(t) = a_k\}$, then ϕ has the form given in equation (A.2) and the sets $E_1, \dots, E_N$ form a partition of E. We call this the *standard representation* of ϕ.

(c) If ϕ is a nonnegative simple function on E with standard representation $\phi = \sum_{k=1}^{N} a_k \, \chi_{E_k}$, then the *Lebesgue integral of* ϕ *over* E is

$$\int_E \phi = \int_E \phi(t)\, dt = \sum_{k=1}^{N} a_k \, |E_k|.$$

(d) If $f\colon E \to [0,\infty)$ is a measurable function, then the *Lebesgue integral of* f *over* E is

$$\int_E f = \int_E f(t)\, dt = \sup\biggl\{\int_E \phi : 0 \le \phi \le f,\ \phi \text{ simple}\biggr\}.$$

If A is a measurable subset of E, then we write $\int_A f = \int_E f\, \chi_A$. ♢

Following are some of the basic properties of integrals of nonnegative functions.

Theorem A.15. *Let* $E \subseteq \mathbf{R}^d$ *and* $f,\, g\colon E \to [0,\infty)$ *be measurable.*

(a) *If* ϕ *is a simple function on* E, *then the integrals of* ϕ *given in parts* (c) *and* (d) *of Definition* A.14 *coincide.*

(b) *If* $f \le g$ *then* $\int_E f \le \int_E g$.

(c) Tchebyshev's Inequality: *If* $\alpha > 0$, *then* $|\{t \in E : f(t) > \alpha\}| \le \frac{1}{\alpha} \int_E f$.

(d) $\int_E f = 0$ *if and only if* $f = 0$ *a.e.* ♢

The definition of $\int_E f$ given in Definition A.14 is often cumbersome to implement. One application of the next result (which is also known as the *Beppo Levi Theorem*) is that the integral of f can be obtained as a limit instead of a supremum of integrals of simple functions. We say that a sequence of real-valued functions $\{f_n\}$ is *monotone increasing* if $f_1(t) \le f_2(t) \le \cdots$ for all t. We write $f_n \nearrow f$ to mean that $\{f_n\}$ is monotone increasing and $f_n(t) \to f(t)$ pointwise.

Theorem A.16 (Monotone Convergence Theorem). *Let $E \subseteq \mathbf{R}^d$ be measurable, and assume $\{f_n\}$ are nonnegative measurable functions on E such that $f_n \nearrow f$. Then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f. \qquad \diamondsuit$$

Theorem A.17. *If $E \subseteq \mathbf{R}^d$ and $f \colon E \to [0,\infty)$ are measurable then there exist simple functions ϕ_n such that $\phi_n \nearrow f$, and consequently $\int_E \phi_n \nearrow \int_E f$.* $\diamondsuit$

Corollary A.18. *Let $\{f_n\}$ be a sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbf{R}^d$. Then*

$$\int_E \biggl(\sum_{n=1}^\infty f_n\biggr) = \sum_{n=1}^\infty \int_E f_n.$$

In particular, if $f \colon E \to [0,\infty)$ is measurable, $A_1, A_2, \dots$ are disjoint and measurable, and $A = \cup A_k$, then

$$\int_A f = \sum_k \int_{A_k} f. \qquad \diamondsuit$$

If we have functions f_n that are not monotone increasing, then we may not be able to interchange a limit with an integral. The following result states that as long as the f_n are all nonnegative, we do at least have an inequality.

Theorem A.19 (Fatou's Lemma). *If $\{f_n\}$ is a sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbf{R}^d$, then*

$$\int_E \Bigl(\liminf_{n\to\infty} f_n\Bigr) \le \liminf_{n\to\infty} \int_E f_n. \qquad \diamondsuit$$

We define the integral of a general real-valued function by writing it as a difference of two nonnegative functions, and that of a complex-valued function by splitting it into real and imaginary parts.

Definition A.20. Let $E \subseteq \mathbf{R}^d$ be measurable.

(a) Given a measurable function $f \colon E \to \mathbf{R}$ define

$$f^+(t) = \max\bigl\{f(t),\, 0\bigr\}, \qquad f^-(t) = \max\bigl\{-f(t),\, 0\bigr\}.$$

Then $f^+,\ f^- \ge 0$, and we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$. The *Lebesgue integral of f on E* is

$$\int_E f = \int_E f^+ - \int_E f^-,$$

as long as this does not have the form $\infty - \infty$ (in that case, the integral is undefined).

(b) Given a measurable function $f\colon E \to \mathbf{C}$, write the real and imaginary parts of f as $f = f_r + if_i$. If $\int_E f_r$ and $\int_E f_i$ both exist and are finite, then the *Lebesgue integral of f on E* is

$$\int_E f \;=\; \int_E f_r \,+\, i\int_E f_i. \qquad \diamondsuit$$

Theorem A.21. *Let f be a measurable function on a measurable set $E \subseteq \mathbf{R}^d$. Then $\int_E f$ exists and is a finite scalar if and only if $\int_E |f| < \infty$, and in this case $\bigl|\int_E f\bigr| \le \int_E |f|$.* $\diamondsuit$

A.5 L^p Spaces and Convergence

Let E be a measurable subset of $\mathbf{R}^d$. Given $1 \le p < \infty$, for each measurable function $f\colon E \to \mathbf{C}$ we define the L^p-norm of f to be

$$\|f\|_{L^p} \;=\; \biggl(\int_E |f(t)|^p\,dt\biggr)^{1/p}.$$

$L^p(E)$ is the space of all functions for which $\|f\|_{L^p}$ is finite. Technically, $\|\cdot\|_{L^p}$ is only a seminorm on $L^p(E)$ because any function f satisfying $f = 0$ a.e. will have $\|f\|_{L^p} = 0$. However, if we identify functions that are equal almost everywhere, i.e., we consider them as defining the same element of $L^p(E)$, then $\|\cdot\|_{L^p}$ is a norm on $L^p(E)$. Further, it can be shown that $L^p(E)$ is complete with respect to this norm, and hence is a Banach space.

For $p = \infty$ we define the L^∞-norm of f to be

$$\|f\|_{L^\infty} \;=\; \operatorname*{ess\,sup}_{t\in E} |f(t)| \;=\; \inf\bigl\{M \ge 0 : |f(t)| \le M \text{ a.e.}\bigr\}.$$

Then $L^\infty(E)$ is a Banach space with respect to this norm if we again identify functions that are equal almost everywhere.

Remark A.22. (a) Technically, an element of $L^p(E)$ is an equivalence class of functions that are equal almost everywhere rather than a single function. We can usually safely ignore the distinction between a function and the equivalence class of functions that are equal to it a.e., but on occasion some care needs to be taken. One such situation arises when dealing with continuous functions. Every function in $C_b(\mathbf{R})$ is continuous and bounded, so we often write $C_b(\mathbf{R}) \subseteq L^\infty(\mathbf{R})$. However, in doing so we are really identifying $C_b(\mathbf{R})$ with its image in $L^\infty(\mathbf{R})$ under the equivalence relation of equality almost everywhere. That is, if $f \in C_b(\mathbf{R})$ then it determines an equivalence class $\widetilde{f}$ of functions that are equal to it almost everywhere, and it is this equivalence class $\widetilde{f}$ that belongs to $L^\infty(\mathbf{R})$. Conversely, if we are given $f \in L^\infty(\mathbf{R})$ (really an equivalence class $\widetilde{f}$ of functions) and there is a representative of this equivalence class that belongs to $C_b(\mathbf{R})$, then we write $f \in C_b(\mathbf{R})$, meaning that there is a representative of f that belongs to $C_b(\mathbf{R})$.

(b) The two statements "f is continuous a.e." and "f equals a continuous function a.e." are distinct. The first means that $\lim_{y\to x} f(y) = f(x)$ for almost every x, while the second means that there exists a continuous function g such that $f(x) = g(x)$ for almost every x. Only in the latter case can we say that there is a representative of f that is a continuous function. The function $\chi_{[0,1]}$ is an example of a function that is continuous a.e. but does not equal any continuous function a.e. $\diamondsuit$

Convergence in L^p-norm is not equivalent to pointwise convergence of functions, but we do have the following important fact.

Theorem A.23. *Let $E \subseteq \mathbf{R}^d$ be measurable and fix $1 \le p \le \infty$. If f_n, $f \in L^p(E)$ and $f_n \to f$ in L^p-norm, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbf{N}}$ such that $f_{n_k}(t) \to f(t)$ for almost every $t \in E$.* $\diamondsuit$

The Dominated Convergence Theorem is one of the most important convergence theorems for integrals.

Theorem A.24 (Lebesgue Dominated Convergence Theorem). *Assume $\{f_n\}$ is a sequence of measurable functions on a measurable set $E \subseteq \mathbf{R}^d$ such that:*

(a) *$f(t) = \lim_{n\to\infty} f_n(t)$ exists for a.e. $t \in E$, and*

(b) *there exists $g \in L^1(E)$ such that $|f_n(t)| \le g(t)$ a.e. for every n.*

Then f_n converges to f in L^1-norm, i.e.,

$$\lim_{n\to\infty} \|f - f_n\|_{L^1} \;=\; \lim_{n\to\infty} \int_E |f - f_n| \;=\; 0,$$

and, consequently,

$$\lim_{n\to\infty} \int_E f_n = \int_E f. \qquad \diamondsuit$$

There is also a series version of the Dominated Convergence Theorem.

Theorem A.25 (Dominated Convergence Theorem for Series). *Assume $(a_{mn})_{m,n\in\mathbf{N}}$ is a sequence of complex scalars such that:*

(a) *$a_m = \lim_{n\to\infty} a_{mn}$ exists for all $m \in \mathbf{N}$, and*

(b) *there exists a sequence $b = (b_m) \in \ell^1$ such that $|a_{mn}| \le b_m$ for every m and n.*

Then

$$\lim_{n\to\infty} \sum_m a_{mn} = \sum_m a_m. \qquad \diamondsuit$$

It is often useful to know that we can approximate a given L^p function by functions that have some special properties. For example, combining the Lebesgue Dominated Convergence Theorem with Theorem A.17 shows that the set of L^p simple functions is dense in $L^p(E)$, and we can restrict further to simple functions with compact support.

Theorem A.26. *Let $E \subseteq \mathbf{R}^d$ be Lebesgue measurable. Then the set S consisting of all compactly supported simple functions is dense in $L^p(E)$ for each $1 \le p < \infty$.* $\diamondsuit$

Here are some other examples of dense subspaces of L^p.

Lemma A.27. *$C_c(\mathbf{R}^d)$ is dense in $L^p(\mathbf{R}^d)$ for each $1 \le p < \infty$. If $K \subseteq \mathbf{R}^d$ is compact, then $C(K)$ is dense in $L^p(K)$ for each $1 \le p < \infty$.* $\diamondsuit$

Lemma A.28. *$\{\chi_{[a,b]} : -\infty < a < b < \infty\}$ is complete in $L^p(\mathbf{R})$ for each $1 \le p < \infty$.* $\diamondsuit$

Lemma A.29. *$\{\chi_{E\times F} : E, F \subseteq \mathbf{R}\}$ is complete in $L^p(\mathbf{R}^2)$ for each index $1 \le p < \infty$.* $\diamondsuit$

An important property of integrable functions is given in the next theorem.

Theorem A.30 (Lebesgue Differentiation Theorem). *Fix $f \in L^1[a,b]$. Then for almost every $x \in (a,b)$,*

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(y)\,dy \;=\; \lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(y)\,dy \;=\; f(x).$$

Consequently, the indefinite integral of f,

$$F(x) \;=\; \int_a^x f(y)\,dy,$$

is differentiable a.e., and $F' = f$ a.e. $\diamondsuit$

In fact, the intervals $[x, x+h]$ or $[x-h, x+h]$ can be replaced by any collection of sets $\{S_h\}_{h>0}$ that *shrink regularly* to x, which means that $\operatorname{diam}(S_h) \to 0$, and there exists a constant $C > 0$ such that if Q_h is the smallest interval centered at x that contains S_h, then $|Q_h| \le C\,|S_h|$. The Lebesgue Differentiation Theorem can also be generalized to higher dimensions.

A.6 Repeated Integration

Let $E \subseteq \mathbf{R}^m$ and $F \subseteq \mathbf{R}^n$ be measurable. If f is a measurable function on $E \times F$ then there are three natural integrals of f over $E \times F$. First, there is the integral of f over the set $E \times F \subseteq \mathbf{R}^{m+n}$, which we write as the *double integral*

$$\iint_{E\times F} f \;=\; \iint_{E\times F} f(x,y)\,(dx\,dy).$$

Second, for each fixed y we can integrate $f(x,y)$ as a function of x, and then integrate the result in y, obtaining the *iterated integral*

$$\int_F \left(\int_E f(x,y)\,dx\right) dy.$$

Third, we also have the iterated integral

$$\int_E \left(\int_F f(x,y)\,dy\right) dx.$$

In general these three integrals need not be equal, even if they all exist. The theorems of Fubini and Tonelli give sufficient conditions under which we can exchange the order of integration. We begin with Tonelli's Theorem, which states that interchange is allowed if f is nonnegative.

Theorem A.31 (Tonelli's Theorem). *Let E be a measurable subset of $\mathbf{R}^m$ and F a measurable subset of $\mathbf{R}^n$. If $f\colon E \times F \to [0,\infty)$ is measurable, then the following statements hold.*

(a) *$f_x(y) = f(x,y)$ is measurable on F for each $x \in E$.*

(b) *$f^y(x) = f(x,y)$ is measurable on E for each $y \in F$.*

(c) *$g(x) = \int_F f_x(y)\,dy$ is a measurable function on E.*

(d) *$h(y) = \int_E f^y(x)\,dx$ is a measurable function on F.*

(e) *We have*

$$\begin{aligned}\iint_{E\times F} f(x,y)\,(dx\,dy) &= \int_F \left(\int_E f(x,y)\,dx\right) dy \\ &= \int_E \left(\int_F f(x,y)\,dy\right) dx,\end{aligned}$$

in the sense that either all three of the quantities above are finite and equal, or all are infinite. $\diamondsuit$

As a corollary, we obtain the useful fact that to test whether a given function belongs to $L^1(E \times F)$ we can simply show that any one of three possible integrals is finite.

Corollary A.32. *Let E be a measurable subset of $\mathbf{R}^m$ and F a measurable subset of $\mathbf{R}^n$. If f is a measurable function on $E \times F$, then (as nonnegative real numbers or as infinity):*

$$\iint_{E\times F} |f(x,y)|\,(dx\,dy) = \int_F \left(\int_E |f(x,y)|\,dx\right) dy = \int_E \left(\int_F |f(x,y)|\,dy\right) dx.$$

Consequently, if any one of these three integrals is finite, then f belongs to $L^1(E\times F)$. $\diamondsuit$

Fubini's Theorem allows the interchange of integrals if f is integrable.

Theorem A.33 (Fubini's Theorem). *Let E be a measurable subset of $\mathbf{R}^m$ and F a measurable subset of $\mathbf{R}^n$. If $f \in L^1(E\times F)$, then the following statements hold.*

(a) *$f_x(y) = f(x,y)$ is measurable and integrable on F for almost every $x\in E$.*

(b) *$f^y(x) = f(x,y)$ is measurable and integrable on E for almost every $y\in F$.*

(c) *$g(x) = \int_F f_x(y)\,dy$ is a measurable and integrable function on E.*

(d) *$h(y) = \int_E f^y(x)\,dx$ is a measurable and integrable function on F.*

(e) *We have*

$$\iint_{E\times F} f(x,y)(dx\,dy) = \int_F \left(\int_E f(x,y)\,dx\right) dy = \int_E \left(\int_F f(x,y)\,dy\right) dx,$$

where each of these quantities is a finite scalar. $\diamondsuit$

There are also corresponding discrete versions of Fubini's Theorem and Tonelli's Theorem for series.

Theorem A.34. (a) *Given a sequence $(a_{mn})_{m,n\in\mathbf{N}}$ with all $a_{mn}\ge 0$, we have*

$$\sum_m \sum_n a_{mn} \;=\; \sum_n \sum_m a_{mn},$$

in the sense that either both are finite and equal, or both are infinite.

(b) *Given a sequence $(a_{mn})_{m,n\in\mathbf{N}}$ with $\sum_m\sum_n |a_{mn}| < \infty$, we have*

$$\sum_m \sum_n a_{mn} \;=\; \sum_n \sum_m a_{mn}. \qquad \diamondsuit$$

An entirely similar result holds for interchanging an integral with a series.

B

Compact and Hilbert–Schmidt Operators

Although compact operators play an important role in many areas of analysis, they appear only rarely in this volume. In particular, we use them in the proof of Theorem 5.26, and the special type of compact operators known as Hilbert–Schmidt operators play a role in some exercises that deal with tensor products. This appendix provides a brief review of the properties of compact and Hilbert–Schmidt operators that are relevant to our uses in this volume. A few proofs are included, and some others are sketched in the Exercises for this appendix.

B.1 Compact Sets

Strictly speaking, we can define compact operators without needing to know the meaning of a compact set, but for completeness we recall the definition and basic properties of compact sets in normed spaces.

Definition B.1 (Compact Set). A subset K of a normed space X is *compact* if every covering of K by open sets has a finite subcovering. More precisely, K is compact if it is the case that whenever

$$K \subseteq \bigcup_{i \in I} U_i,$$

where $\{U_i\}_{i \in I}$ is any collection of open subsets of X, there exist *finitely* many $i_1, \dots, i_N \in I$ such that

$$K \subseteq \bigcup_{k=1}^{N} U_{i_k}. \qquad \diamondsuit$$

If we replace "normed space" by "topological space," then Definition B.1 is the abstract definition of a compact set in a general topological space. However, we are most interested in Banach spaces. In this setting compactness can be equivalently reformulated in the following ways.

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Theorem B.2. *If K is a subset of a Banach space X, then the following statements are equivalent.*

(a) *K is compact.*

(b) *K is* sequentially compact, *i.e., every sequence $\{x_n\}$ of points of K contains a convergent subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ whose limit belongs to K.*

(c) *K is closed and* totally bounded, *i.e., for every $r > 0$, there exist finitely many $x_1, \dots, x_N \in X$ such that*

$$K \subseteq \bigcup_{k=1}^{N} B_r(x_k). \qquad \diamondsuit$$

Theorem B.2 carries over without change if X is a complete metric space instead of a Banach space. If X is an arbitrary normed or metric space, then Theorem B.2 remains valid if we replace the word "closed" in statement (c) by "complete," i.e., every Cauchy sequence in K converges to an element of K.

A compact subset of a normed space is both closed and bounded, and in a finite-dimensional normed space the converse holds as well. However, the converse fails in every infinite-dimensional normed space. In particular, if X is an infinite-dimensional normed space, then Exercise 1.44 shows that the closed unit ball $D = \{x \in X : \|x\| \le 1\}$ in X is not compact.

B.2 Compact Operators

Now we give the definition of a compact operator in the setting of Banach spaces.

Definition B.3. Let X, Y be Banach spaces. A linear operator $T\colon X \to Y$ is *compact* if for every sequence $\{x_n\} \subseteq X$ with $\|x_n\| \le 1$ for every n, there exists a subsequence of $\{Tx_n\}$ that converges in Y. $\diamondsuit$

Equivalently, T is compact if the closure in Y of $\{Tx : \|x\| \le 1\}$ is a compact subset of Y.

Here are some examples of compact and non-compact operators, and more examples are given below.

Example B.4. (a) By Exercise 1.44, the closed unit ball $D = \{x : \|x\| \le 1\}$ is a compact subset of a normed space X if and only if X is finite dimensional. Hence the identity operator $I\colon X \to X$ is a compact operator if and only if X is finite dimensional.

(b) Suppose that $T\colon X \to Y$ is a bounded linear operator with finite-dimensional range. Then range(T) is a finite-dimensional normed space, where the norm on range(T) is inherited from Y. Hence $\{Tx : \|x\| \le 1\}$ is a closed and bounded subset of a finite-dimensional space, and therefore is compact. Hence all bounded linear operators with finite-dimensional ranges are compact. Since the dimension of the range is the *rank* of an operator, an operator with finite-dimensional range is called a *finite-rank* operator.

(c) A compact operator need not have finite rank. For example, if $\lambda = (\lambda_k) \in \ell^\infty$ then we can define a bounded operator $M_\lambda \colon \ell^2 \to \ell^2$ by

$$M_\lambda(x_1, x_2, \dots) \;=\; (\lambda_1 x_1, \lambda_2 x_2, \dots).$$

Exercise B.3 shows that M_λ is compact if and only if $\lambda \in c_0$. However, M_λ has finite rank if and only if λ is a finite sequence. If $\lambda \in c_0$ and $\lambda_n \neq 0$ for every n then M_λ is compact and injective and range(M_λ) is a dense subspace of ℓ^2, so M_λ is certainly not finite rank in this case. ◇

In essence, although compact operators need not have finite-dimensional ranges, they are still in some sense "close" to being finite rank.

The next result summarizes some of the basic properties of compact operators (see Exercise B.1).

Theorem B.5. *Let X, Y, Z be Banach spaces, and let $T \colon X \to Y$ and $S \colon Y \to Z$ be linear operators.*

(a) *If T is compact, then T is bounded.*

(b) *If T is bounded and has finite rank, then T is compact.*

(c) *If $T_n \colon X \to Y$ are compact operators and $\|T - T_n\| \to 0$, then T is compact.*

(d) *If T is compact and S is bounded, or if T is bounded and S is compact, then ST is compact.*

(e) *If $T \colon X \to X$ is compact and $\lambda \in \mathbf{F}$, then* $\ker(T - \lambda I)$ *is finite dimensional.* ◇

Restating part (e) of Theorem B.5 another way, if λ is an eigenvalue of a compact operator T, then the corresponding λ-eigenspace is finite dimensional.

We will also need the following fact, which we state without proof. This theorem is a special case of the *Fredholm Alternative*, e.g., see [Meg98, Thm. 3.2.24].

Theorem B.6. *Let X be a Banach space. If $T \in \mathcal{B}(X)$ is compact and* $\ker(I - T) = \{0\}$, *then $I - T$ is a topological isomorphism of X onto itself.* ◇

Exercises

B.1. Prove Theorem B.5.

B.2. Let H, K be Hilbert spaces. Show that if $T \colon H \to K$ is compact and $\{e_n\}$ is an orthonormal sequence in H, then $Te_n \to 0$.

B.3. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Fix $\lambda \in \ell^\infty$, and let $M_\lambda \colon H \to H$ be the operator defined in Exercise 1.66.

(a) Show that M_λ is compact if and only if $\lambda \in c_0$.

(b) Show that if $\lambda \in c_0$ and $\lambda_n \neq 0$ for every n, then $\bigl\{M_\lambda x : \|x\| \le 1\bigr\}$ is not closed in H.

B.3 Hilbert–Schmidt Operators

Hilbert–Schmidt operators are a special type of compact operator on Hilbert spaces. The definition can be extended to arbitrary Hilbert spaces, but we will restrict our attention to separable spaces.

Definition B.7. Given separable Hilbert spaces H, K, we say that a bounded linear operator $T\colon H \to K$ is *Hilbert–Schmidt* if there exists an orthonormal basis $\{e_n\}$ for H such that $\sum \|Te_n\|^2 < \infty$. The space of Hilbert–Schmidt operators mapping H into K is denoted

$$\mathcal{B}_2(H,K) \;=\; \bigl\{T \in \mathcal{B}(H,K) \;:\; T \text{ is Hilbert–Schmidt}\bigr\},$$

and we write $\mathcal{B}_2(H) = \mathcal{B}(H,H)$. $\diamondsuit$

The next theorem states some of the properties of Hilbert–Schmidt operators (see Exercise B.4).

Theorem B.8. *Let H, K be separable Hilbert spaces.*

(a) *Given $T \in \mathcal{B}(H,K)$, the quantity*

$$\|T\|_{\mathrm{HS}}^2 \;=\; \sum_n \|Te_n\|^2$$

is independent of the choice of orthonormal basis $\{e_n\}$ for H (i.e., it is either finite and equal for all orthonormal bases, or infinite for every orthonormal basis). Further,

$$\|T^*\|_{\mathrm{HS}} \;=\; \|T\|_{\mathrm{HS}} \qquad \text{and} \qquad \|T\| \le \|T\|_{\mathrm{HS}}.$$

(b) *$\|\cdot\|_{\mathrm{HS}}$ is a norm on $\mathcal{B}_2(H,K)$, and $\mathcal{B}_2(H,K)$ is complete with respect to this norm. Further, $\mathcal{B}_2(H,K)$ is a Hilbert space with respect to the inner product*

$$\langle T,U\rangle_{\mathrm{HS}} \;=\; \sum_n \langle Te_n, Ue_n\rangle, \qquad T,U \in \mathcal{B}_2(H,K),$$

where $\{e_n\}$ is any orthonormal basis for H.

(c) *If $T \in \mathcal{B}_2(H,K)$, $A \in \mathcal{B}(H)$, and $B \in \mathcal{B}(K)$, then BT, $TA \in \mathcal{B}_2(H,K)$, and we have*

$$\|BT\|_{\mathrm{HS}} \;\le\; \|B\|\,\|T\|_{\mathrm{HS}}, \qquad \|TA\|_{\mathrm{HS}} \;\le\; \|A\|\,\|T\|_{\mathrm{HS}}.$$

(d) *All bounded linear finite-rank operators on H are Hilbert–Schmidt, and all Hilbert–Schmidt operators are compact. Moreover,*

$$\mathcal{B}_{00}(H,K) \;=\; \bigl\{L \in \mathcal{B}(H,K) : L \text{ has finite rank}\bigr\}$$

is dense in $\mathcal{B}_2(H,K)$ with respect to $\|\cdot\|_{\mathrm{HS}}$. $\diamondsuit$

We call $\|T\|_{\mathrm{HS}}$ the *Hilbert–Schmidt norm* of T.

Exercises

B.4. Prove Theorem B.8.

B.4 Finite-Rank Operators and Tensor Products

We examine the finite-rank operators a little more closely, and show how they can be written in terms of tensor product operators.

Definition B.9 (Tensor Product Operator). Let H, K be Hilbert spaces. The *tensor product* of $x \in H$ with $y \in K$ is the *operator* $x \otimes y \in \mathcal{B}(H, K)$ defined by

$$(x \otimes y)(z) \;=\; \langle z, x\rangle\, y, \qquad z \in H. \qquad \diamondsuit \tag{B.1}$$

Note that $x \otimes y$ is a bounded linear operator with finite rank, and so is Hilbert–Schmidt. If $x = 0$ or $y = 0$ then $x \otimes y$ is the zero operator. Otherwise the range of $x \otimes y$ is $\operatorname{span}\{y\}$, the line through y, and hence $x \otimes y$ has rank one.

These tensor product operators are "elementary building blocks" for $\mathcal{B}_2(H, K)$ in the following sense.

Theorem B.10. *Let H, K be separable Hilbert spaces.*

(a) *$T \in \mathcal{B}(H, K)$ has rank one if and only if $T = x \otimes y$ for some $x \in H$, $y \in K$ not both zero.*

(b) *$T \in \mathcal{B}(H, K)$ has finite rank if and only if $T = \sum_{k=1}^{N} x_k \otimes y_k$ for some $N \in \mathbf{N}$, $x_k \in H$, and $y_k \in K$.*

(c) *$\{x \otimes y : x \in H,\, y \in K\}$ is complete in $\mathcal{B}_2(H, K)$.*

(d) *If $x_1, x_2 \in H$ and $y_1, y_2 \in K$, then*

$$\big\langle x_1 \otimes y_1,\, x_2 \otimes y_2 \big\rangle_{\mathrm{HS}} \;=\; \langle x_2, x_1\rangle_H\, \langle y_1, y_2\rangle_K.$$

(e) *If $\{e_n\}$ is an orthonormal basis for H and $\{f_n\}$ is an orthonormal basis for K, then $\{e_m \otimes f_n\}_{m,n\in\mathbf{N}}$ is an orthonormal basis for $\mathcal{B}_2(H, K)$.*

(f) *If $T \in \mathcal{B}(H, K)$, then $\langle T, e_m \otimes f_n\rangle = \langle Te_m, f_n\rangle$ for every $m, n \in \mathbf{N}$.*

Proof. We prove part (e) and assign the remainder as Exercise B.5.

If $\{e_n\}$ and $\{f_n\}$ are orthonormal bases, then by part (d) we have

$$\big\langle e_m \otimes f_n,\, e_{m'} \otimes f_{n'} \big\rangle_{\mathrm{HS}} \;=\; \langle f_n, f_{n'}\rangle\, \langle e_{m'}, e_m\rangle \;=\; \delta_{mm'}\, \delta_{nn'}.$$

Therefore $\{e_m \otimes f_n\}_{m,n\in\mathbf{N}}$ is an orthonormal sequence in $\mathcal{B}_2(H, K)$.

Fix $T \in \mathcal{B}_2(H, K)$. Applying the Plancherel Equality, we have

$$\sum_{m,n} |\langle Te_m, f_n\rangle|^2 \;=\; \sum_m \Bigl(\sum_n |\langle Te_m, f_n\rangle|^2\Bigr) \;=\; \sum_m \|Te_m\|^2 \;=\; \|T\|_{\mathrm{HS}}^2 \;<\; \infty.$$

Since $\{e_m \otimes f_n\}_{m,n\in\mathbf{N}}$ is orthonormal, this implies that the series

$$U \;=\; \sum_{m,n} \langle Te_m, f_n\rangle \, (e_m \otimes f_n)$$

converges in $\mathcal{B}_2(H,K)$. If we can show that $T = U$ then we can conclude that $\{e_m \otimes f_n\}_{m,n\in\mathbf{N}}$ is complete, and hence is an orthonormal basis.

To see this, choose any $x \in H$. Then

$$\begin{aligned}
Ux \;&=\; \sum_{m,n} \langle Te_m, f_n\rangle \, (e_m \otimes f_n)(x) \\
&=\; \sum_n \Bigl(\sum_m \langle Te_m, f_n\rangle \, \langle x, e_m\rangle\Bigr) f_n \\
&=\; \sum_n \Bigl(\sum_m \langle x, e_m\rangle \, \langle e_m, T^* f_n\rangle\Bigr) f_n \\
&=\; \sum_n \langle x, T^* f_n\rangle \, f_n \\
&=\; \sum_n \langle Tx, f_n\rangle \, f_n \;=\; Tx,
\end{aligned}$$

so $T = U$ as desired. □

By part (d) of Theorem B.10, $\mathcal{B}_2(H,K)$ is isomorphic to the completion of $\operatorname{span}\{x \otimes y : x \in H,\, y \in K\}$. This is a specific example of the abstract construction of the tensor product $H \otimes K$ of Hilbert spaces H, K. Likewise, there exists an abstract notion of the tensor product of Banach spaces. We refer to the text by Ryan [Rya02] for more details. In this language $H \otimes K$ is isometrically isomorphic to $\mathcal{B}_2(H,K)$, and so for us it will be sufficient to define the tensor product of H and K to be

$$H \otimes K \;=\; \mathcal{B}_2(H,K).$$

Exercise B.9 is a useful illustration of how an operator $U \in \mathcal{B}(H)$ and an operator $V \in \mathcal{B}(K)$ can be combined to give a *tensor product operator* $U \otimes V \in \mathcal{B}(H \otimes K)$.

To give a concrete example of a tensor product of Hilbert spaces, let us consider $L^2(E) \otimes L^2(F) = \mathcal{B}_2(L^2(E), L^2(F))$. We will characterize the operators in this space as special types of integral operators.

Definition B.11 (Integral Operator). Let E, F be measurable subsets of $\mathbf{R}$, and let k be a fixed measurable function on $E \times F$. Then the *integral operator* L_k *with kernel* k is formally defined by

$$L_k f(y) \;=\; \int_E k(x,y)\, f(x)\, dx, \qquad y \in F. \qquad \diamondsuit \qquad \text{(B.2)}$$

The use of the word *kernel* in this definition should not be confused with its use as meaning the nullspace of the operator. It should always be clear from context which meaning of "kernel" is intended. Also, "formally" means that there is no guarantee that the integral in equation (B.2) will exist in general. We must determine conditions on k that imply that $L_k f$ is well defined.

An integral operator is a generalization of ordinary matrix-vector multiplication. Let A be an $m \times n$ matrix with entries a_{ij} and let $u \in \mathbf{F}^n$ be given. Then $Au \in \mathbf{F}^m$, and its components are

$$(Au)_i \;=\; \sum_{j=1}^{n} a_{ij}\, u_j, \qquad i = 1, \dots, m.$$

Thus, the function values $k(x,y)$ are analogous to the entries a_{ij} of the matrix A, and the values $L_k f(x)$ are analogous to the entries $(Au)_i$ (although for convenience we have ordered x, y in $k(x,y)$ differently than i, j in a_{ij}).

Let us examine some integral operators with especially simple kernels.

Example B.12 (Tensor Product Kernels). Fix $g \in L^2(E)$ and $h \in L^2(F)$. We call the function $g \otimes h$ on $E \times F$ defined by

$$(g \otimes h)(x,y) \;=\; \overline{g(x)}\, h(y), \qquad (x,y) \in E \times F, \qquad \text{(B.3)}$$

the *tensor product* of g and h. Sometimes the complex conjugate is omitted in the definition of tensor product, and sometimes $g \otimes h$ is defined as $(g \otimes h)(x,y) = g(x)\, \overline{h(y)}$, but it will be convenient for our purposes to place the complex conjugate on g.

Of course, we have already declared that the symbols $g \otimes h$ denote the *operator* mapping $L^2(E)$ into $L^2(F)$ defined by

$$(g \otimes h)(f) \;=\; \langle f, g \rangle\, h, \qquad f \in L^2(E).$$

The reason we use this ambiguous notation is that the *operator* $g \otimes h$ is precisely the integral operator whose kernel is the *function* $g \otimes h$. For, if we fix any $f \in L^2(E)$ then

$$\begin{aligned} L_{g\otimes h} f(y) \;&=\; \int_E (g \otimes h)(x,y)\, f(x)\, dx \\ &=\; \int_E \overline{g(x)}\, h(y)\, f(x)\, dx \\ &=\; \langle f, g \rangle\, h(y) \;=\; (g \otimes h)(f)(y), \end{aligned}$$

in the sense of almost everywhere equality. Thus $L_{g\otimes h} = g \otimes h$. The distinction between $g \otimes h$ as a function and as an operator is usually clear from context. $\diamondsuit$

Exercises

B.5. Complete the proof of Theorem B.10.

B.6. Let H, K be separable Hilbert spaces. Prove the following facts about tensor products.

(a) $\|x \otimes y\|_{\mathrm{HS}} = \|x\|_H \, \|y\|_K$.

(b) $(ax) \otimes y = \bar{a}\,(x \otimes y)$ and $x \otimes (by) = b\,(x \otimes y)$.

(c) $(x+w) \otimes y = (x \otimes y) + (w \otimes y)$ and $x \otimes (y+z) = (x \otimes y) + (x \otimes z)$.

(d) $\|x \otimes y - w \otimes z\|_{\mathrm{HS}} \le \|x - w\|_H \, \|y\|_K + \|w\|_H \, \|y - z\|_K$.

B.7. (a) Given $x_k \in H$ and $y_k \in K$, show that the adjoint of $T = \sum_{k=1}^{M} x_k \otimes y_k$ is $T^* = \sum_{k=1}^{M} y_k \otimes x_k$

(b) Given $T \in \mathcal{B}(H, K)$, show that T is compact if and only if T^* is compact.

B.8. Let E, F be measurable subsets of $\mathbf{R}$, and let $\{e_m\}_{m \in \mathbf{N}}$ and $\{f_n\}_{n \in \mathbf{N}}$ be orthonormal bases for $L^2(E)$ and $L^2(F)$, respectively. Let $e_m \otimes e_n$ denote the tensor product function defined in equation (B.3), and show that the family $\{e_m \otimes f_n\}_{m,n \in \mathbf{N}}$ is an orthonormal basis for $L^2(E \times F)$.

B.9. Let H, K be separable Hilbert spaces. Fix $U \in \mathcal{B}(H)$ and $V \in \mathcal{B}(K)$ and prove the following statements.

(a) If $x \in H$ and $y \in K$ then $V(x \otimes y)U^* = Ux \otimes Vy$.

(b) There exists a unique bounded operator $T \in \mathcal{B}(H \otimes K)$ such that $\|T\| \le \|U\| \, \|V\|$ and $T(g \otimes h) = Tg \otimes Th$ for all $g \in H$ and $h \in K$. We call T the *tensor product* of U and V, and write $T = U \otimes V$.

(c) $(U_1 \otimes V_1)(U_2 \otimes V_2) = U_1 U_2 \otimes V_1 V_2$.

(d) If U and V are each topological isomorphisms, then so is $T = U \otimes V$.

(e) Specialize to $H = L^2(E)$ and $K = L^2(F)$. In particular, show that if $U \in \mathcal{B}(L^2(E))$ and $V \in \mathcal{B}(L^2(R))$ then there exists a unique operator $U \otimes V \in \mathcal{B}(L^2(E \times F))$ such that $(U \otimes V)(g \otimes h) = Ug \otimes Vh$, where $g \otimes h$ and $Ug \otimes Vh$ are the functions defined by equation (B.3).

B.5 The Hilbert–Schmidt Kernel Theorem

Now we will show that $L^2(E \times F) = L^2(E) \otimes L^2(F)$ in the sense that there is an isometric isomorphism between $L^2(E \times F)$ and $L^2(E) \otimes L^2(F) = \mathcal{B}_2(L^2(E), L^2(F))$.

Theorem B.13 (Hilbert–Schmidt Kernel Theorem).

(a) *If* $k \in L^2(E \times F)$, *then the integral operator* L_k *with kernel* k *is Hilbert–Schmidt, and* $\|L_k\|_{\mathrm{HS}} = \|k\|_{L^2}$.

(b) $k \mapsto L_k$ *is an isometric isomorphism of* $L^2(E \times F)$ *onto* $L^2(E) \otimes L^2(F)$.

Proof. (a) Fix $k \in L^2(\mathbf{R}^2)$. Then by the Cauchy–Bunyakovski–Schwarz Inequality we have

$$\begin{aligned}
\|L_k f\|_{L^2}^2 &= \int_F |L_k f(y)|^2 \, dy \\
&\le \int_F \biggl(\int_E |k(x,y)| \, |f(x)| \, dx \biggr)^2 dy \\
&\le \int_F \biggl(\int_E |k(x,y)|^2 \, dx \biggr) \biggl(\int_E |f(x)|^2 \, dx \biggr) dy \\
&= \int_F \int_E |k(x,y)|^2 \, dx \, \|f\|_{L^2}^2 \, dy \\
&= \|k\|_{L^2}^2 \, \|f\|_{L^2}^2 .
\end{aligned}$$

Hence $L_k f \in L^2(F)$, and L_k is a bounded mapping of $L^2(E)$ into $L^2(F)$.

Now let $\{e_m\}_{m \in \mathbf{N}}$ be an orthonormal basis for $L^2(E)$ and let $\{f_n\}_{n \in \mathbf{N}}$ be an orthonormal basis for $L^2(F)$. By Exercise B.8, $\{e_m \otimes e_n\}_{m,n \in \mathbf{N}}$ is an orthonormal basis (of functions) for $L^2(E \times F)$. Consequently, the norm of $k \in L^2(E \times F)$ is

$$\|k\|_{L^2}^2 = \sum_{m,n} |\langle k, e_m \otimes e_n \rangle|^2 .$$

Now, since the product $k \cdot (e_m \otimes f_n)$ is an integrable function, Fubini's Theorem (Theorem A.33) allows us to interchange integrals in the following calculation:

$$\begin{aligned}
\langle k, e_m \otimes f_n \rangle &= \int_F \int_E k(x,y) \, e_m(x) \, \overline{f_n(y)} \, dx \, dy \\
&= \int_F \biggl(\int_E k(x,y) \, e_m(x) \, dx \biggr) \overline{f_n(y)} \, dy \\
&= \int_F L_k e_m(y) \, \overline{f_n(y)} \, dy \\
&= \langle L_k e_m, f_n \rangle .
\end{aligned}$$

Therefore

$$\begin{aligned}
\|L_k\|_{\mathrm{HS}}^2 = \sum_m \|L_k e_m\|_{L^2}^2 &= \sum_m \sum_n |\langle L_k e_m, f_n \rangle|^2 \\
&= \sum_{m,n} |\langle k, e_m \otimes e_n \rangle|^2 = \|k\|_{L^2}^2 ,
\end{aligned}$$

so L_k is Hilbert–Schmidt and the mapping $k \mapsto L_k$ is isometric.

(b) Suppose that T is a Hilbert–Schmidt operator mapping $L^2(E)$ into $L^2(F)$. Let $\{e_n\}$, $\{f_n\}$ be an orthonormal bases for $L^2(E)$ and $L^2(F)$, respectively. Then

$$\sum_{m,n} |\langle Te_m, f_n\rangle|^2 \;=\; \sum_m \|Te_m\|^2 \;=\; \|T\|_{\mathrm{HS}}^2 \;<\; \infty. \tag{B.4}$$

Since $\{e_m \otimes f_n\}_{m,n\in\mathbf{N}}$ is an orthonormal basis (of functions) for $L^2(E \times F)$, the series

$$k \;=\; \sum_{m,n} \langle Te_m, f_n\rangle\,(e_m \otimes f_n)$$

therefore converges in $L^2(E \times F)$. Hence $k \in L^2(E \times F)$ and the integral operator L_k is Hilbert–Schmidt by part (a). In fact, since $k \mapsto L_k$ is an isometry and the operator $e_m \otimes f_n$ is the integral operator whose kernel is the function $e_m \otimes f_n$, we conclude that

$$L_k \;=\; \sum_{m,n} \langle Te_m, f_n\rangle\,(e_m \otimes e_n), \tag{B.5}$$

where this series converges in Hilbert–Schmidt norm (note that $e_m \otimes e_n$ is an operator in equation (B.5)). Since the Hilbert–Schmidt norm dominates the operator norm, the series defining L_k above also converges in operator norm, so given $f \in L^2(E)$ we have that

$$\begin{aligned} L_k f \;&=\; \sum_{m,n} \langle Te_m, f_n\rangle\,\langle f, e_m\rangle\, e_n \\ &=\; \sum_m \langle f, e_m\rangle \biggl(\sum_n \langle Te_m, f_n\rangle\, f_n\biggr) \\ &=\; \sum_m \langle f, e_m\rangle\, Te_m \\ &=\; T\biggl(\sum_m \langle f, e_m\rangle\, e_m\biggr) \;=\; Tf. \end{aligned}$$

Therefore $T = L_k$. □

Hints for Exercises

Chapter 1

1.3 (a) Fix p and show that $|\cdot|_p$ is equivalent to $|\cdot|_\infty$.

(b) Let $\{x_n\}$ be a Cauchy sequence in $\mathbf{F}^d$. Each x_n is a vector in $\mathbf{F}^d$. Write the components of x_n as $x_n = (x_n(1), \dots, x_n(d))$. Fix $1 \le k \le d$ and show that the sequence of scalars $(x_n(k))_{k\in\mathbf{N}}$ is Cauchy (with respect to absolute value). Since $\mathbf{F}$ is complete, this sequence must converge. Set $y(k) = \lim_{n\to\infty} x_n(k)$, and let $y = (y(1), \dots, y(d))$. Show that $\lim_{n\to\infty} |y - y_n|_p = 0$. In fact, by part (a) it suffices to consider just one value of p, say $p = 1$ or $p = \infty$.

1.8 For each n, let x_n be a rational number such that $\pi < x_n < \pi + 1/n$. Then (x_n) is a Cauchy sequence in $\mathbf{Q}$, but it does not converge *in the space* $\mathbf{Q}$. It does converge in the larger space $\mathbf{R}$, but since the limit does not belong to $\mathbf{Q}$, it is not convergent in $\mathbf{Q}$.

1.9 (a), (c) Consider $x = \delta_1$ and $y = \delta_2$ (the first two standard basis vectors).

(b) Assume $0 < p < 1$. Show that $(1+t)^p \le 1 + t^p$ for $t > 0$, and use this to show that $(a+b)^p \le a^p + b^p$ for $a, b \ge 0$.

1.12 To show strict inclusion, consider $x_n = (n \log^2 n)^{-1/q}$.

1.13 Suppose $1 \le p < q < \infty$, and note that $1 < q/p < \infty$. Given $f \in L^q(E)$, apply Hölder's Inequality to $\int_E |f|^p \cdot 1$ using exponents q/p and $(q/p)'$.

One way to show strict inclusion is use the fact that there exist disjoint sets $E_n \subseteq E$ such that $|E_n| = 2^{-n}|E|$, and consider $f = \sum 2^{n/q} \chi_{E_n}$.

1.19 Suppose that $f \in C_b(\mathbf{R})$, and let $M = \|f\|_{L^\infty}$. Suppose that there is a point x where $|f(x)| > M$. Then since $|f|$ is continuous, there must be an open interval (a, b) containing x such that $|f(y)| > M$ for $y \in (a, b)$. But then $|f| > M$ on a set with positive measure, i.e., it is not true that $|f| \le M$ a.e., which contradicts the definition of M. Hence we must have $|f(x)| \le M$ for all x, so $\|f\|_\infty \le \|f\|_{L^\infty}$.

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1.20 (c) Let $x = (1/n)_{n \in \mathbf{N}}$ and $x_n = (1, 1/2, \dots, 1/n, 0, 0, \dots)$. Show that $\|x - x_n\|_{\ell^\infty} \to 0$, but $x \notin c_{00}$. A similar idea can be used to show that *every* element $x \in c_0$ is a limit point of c_{00}, so c_{00} is dense in c_0.

1.21 (c) Show that $g(x) = e^{-x^2}$ is a limit point of $C_0(\mathbf{R})$ but does not belong to $C_c(\mathbf{R})$.

1.25 (c) Given $[x_n] \in \widetilde{X}$, for each $m \in \mathbf{N}$ let $Y_m = T(x_m)$ and show that $Y_m \to [x_n]$ in $\widetilde{X}$ as $m \to \infty$.

(d) Given a Cauchy sequence $\{X_N\}_{N \in \mathbf{N}}$ in $\widetilde{X}$, by part (c) there exists some $y_N \in X$ such that $\|X_N - T(y_N)\|_{\widetilde{X}} < 1/N$. Show that $\{y_N\}$ is Cauchy in X and let $Y = [y_N]$. Show that $T(y_N) \to Y$ in $\widetilde{X}$, and use this to show that $X_N \to Y$ in $\widetilde{X}$.

(e) Given $A \in \widetilde{X}$, there exist $x_n \in X$ such that $T(x_n) \to A$. Show that $\{U(x_n)\}$ is Cauchy in Y, and so there exists some $B \in Y$ such that $U(x_n) \to B$. Define $V(A) = B$, and show that V is a well-defined isometric isomorphism of $\widetilde{X}$ onto Y.

1.30 (b) Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbf{F}^n$, and consider the matrix $A = [\langle e_i, e_j \rangle]_{j,i=1}^n$.

(c) All positive definite matrices are diagonalizable and have strictly positive eigenvalues, and there exists an orthonormal basis for $\mathbf{C}^n$ consisting of eigenvectors of A.

1.32 Show that the Parallelogram Law is not satisfied.

1.37 (a) $\Rightarrow$ (b). Let p be the point in M closest to x, and let $e = p - x$. Given $m \in M$ and $\lambda \in \mathbf{C}$, show that

$$\|x - p\|^2 \;\le\; \|x - (p + \lambda m)\|^2 \;=\; \|x - p\|^2 - 2\,\mathrm{Re}(\lambda \langle m, e \rangle) + |\lambda|^2\, \|m\|^2.$$

Consider $\lambda = t > 0$ to show that $\mathrm{Re}(\langle m, e \rangle) \ge 0$, and then consider other λ to obtain $\langle f, e \rangle = 0$.

(c) $\Rightarrow$ (b). Suppose $x = p + e$ where e is the orthogonal projection of x onto $M^\perp$. Then $p \in (M^\perp)^\perp$ by the equivalence of statements (a) and (b). Write $p = q + f$ where $q \in M$ and $f \in M^\perp$. Then $\langle p, f \rangle = 0 = \langle q, f \rangle$. Show that $\|f\|^2 = \langle f, f \rangle = 0$, so $f = 0$ and $p = q \in M$.

1.44 (a) Choose any $u \in X \setminus M$. Since M is closed, $a = \mathrm{dist}(u, M) > 0$. Fix $\delta > 0$ small enough that $\frac{a}{a+\delta} > 1 - \varepsilon$. Then there exists $v \in M$ such that $a \le \|u - v\| < a + \delta$. Set $x = (u - v)/\|u - v\|$. Given $y \in M$ we have $z = v + \|u - v\|\, y \in M$. Show that $\|x - y\| = \|u - z\|/\|u - v\| > 1 - \varepsilon$.

(b) If X is infinite dimensional, repeatedly apply part (a) to find vectors x_n such that $\|x_n - x_m\| > 1/2$ for all $m < n$.

1.45 Inductively apply Exercise 1.43 using $M = \mathrm{span}\{y_1, \dots, y_N\}$ and $x = x_{N+1}$.

1.48 The Fourier coefficients are $\widehat{f}(n) = 0$ for n even and $\widehat{f}(n) = -(2i)/(\pi n)$ for n odd.

1.53 Let $\{f_n\}$ be an orthonormal basis obtained by taking an orthonormal basis for M and extending it to an orthonormal basis for H. Use Plancherel to write $\sum_n \|Pe_n\|^2 = \sum_n \sum_m |\langle Pe_n, f_m\rangle|^2$, and interchange summations.

1.54 Show that the Plancherel Equality holds for the functions $\chi_{[a,x]}$.

1.56 Use Exercise 1.54.

1.57 Let m be a bounded function such that $m(x) \neq 0$ for almost every x and such that $f/m \notin L^2[a,b]$.

1.71 Choose an orthonormal basis $\{x_n\}$ for H and an orthonormal basis $\{y_n\}$ for K. Define $L\colon H \to K$ by $Lx = \sum \langle x, x_n\rangle\, y_n$.

1.72 Fix $x \in X$. Since Y is dense in X, there exist $y_n \in Y$ such that $y_n \to x$. Show that $\{Ly_n\}_{n\in\mathbf{N}}$ is Cauchy in Z, so there exists a vector $z \in Z$ such that $Lg_n \to z$. Show that $\widetilde{L}x = z$ is well defined and has the required properties.

1.79 Use Exercises 1.44 and 1.74.

Chapter 2

2.3 If X^* is separable, then by rescaling the elements of a countable dense subset of X^* we can find a countable set $\{\lambda_n\}_{n\in\mathbf{N}}$ that is dense in the closed unit sphere $D^* = \{\mu \in X^* : \|\mu\|_{X^*} = 1\}$ in X^*. Since $\|\lambda_n\|_{X^*} = 1$, there must exist some $x_n \in X$ with $|\langle x_n, \lambda_n\rangle| \geq 1/2$. Then $M = \overline{\operatorname{span}}\{x_n\}_{n\in\mathbf{N}}$ is a closed subspace of X, and it is separable by Theorem 1.27.

If $M \neq X$, choose $x_0 \in X \backslash M$ with $d = \operatorname{dist}(x_0, M) = 1$. Then by Hahn–Banach, there exists some $\lambda \in X^*$ such that $\lambda|_M = 0$, $\langle x_0, \lambda\rangle = 1$, and $\|\lambda\|_{X^*} = 1$. Show that $\|\lambda - \lambda_n\|_{X^*} \geq 1/2$ for every n, contradicting the fact that $\{\lambda_n\}_{n\in\mathbf{N}}$ is dense in D^*.

2.6 Apply Corollary 2.4 to $\bar{S}$.

2.18 To construct noncommutating self-adjoint operators, consider orthogonal projections.

2.23 Show that $T_{n+1} - T_n = \frac{1}{2}\,(T_n - T_{n-1})\,(T_n + T_{n+1})$. Therefore, if $T_n - T_{n-1}$ is a polynomial in B with all coefficients nonnegative, then so is $T_{n+1} - T_n$.

2.24 (a) By Theorem 2.18, there exists at least one positive operator S such that $S^2 = A$.

(b) Suppose that B is also a positive operator such that $B^2 = A$. Fix x and let $y = (S - B)x$. Show that $\langle Sy, y\rangle + \langle By, y\rangle = 0$. Since B, $S \geq 0$, this implies $\langle Sy, y\rangle = 0 = \langle By, y\rangle$. Show that $\|Sx - Bx\|^2 = \langle Sy, x\rangle - \langle By, x\rangle$ and use part (a).

2.27 Consider $F_n = \{x \in \mathbf{R} : f^{(n)}(x) = 0\}$. Show that F_n is closed and apply the Baire Category Theorem.

2.28 Hint: Given $n \in \mathbf{N}$, let F_n consist of all functions $f \in C[0,1]$ for which there exists some $x_0 \in [0,1)$ such that $|f(x) - f(x_0)| < n\,(x - x_0)$ for all $x_0 < x < 1$. Show that F_n is closed and has empty interior.

2.32 For $p = \infty$, work directly instead of applying the Uniform Boundedness Principle.

2.33 (a) Bounded means bounded in norm, i.e., $\sup_{x^* \in S} \|x^*\| < \infty$.

(b) Apply part (a) to $S^{**} = \{\pi(x) : x \in S\}$, where π is the natural embedding of X into X^{**}.

2.34 Let $a_i = (a_{ij})_{j \in \mathbf{N}}$. Then, by hypothesis, $(Ax)_i = \langle x, a_i \rangle = \sum_j x_j a_{ij}$ converges for every $x \in \ell^p$, so Theorem 2.24 implies that $a_i \in \ell^{p'}$. Define $A_N x = \bigl(\langle x, a_1 \rangle, \dots, \langle x, a_N \rangle, 0, 0, \dots\bigr)$. For $q < \infty$, apply the Banach–Steinhaus Theorem. For $q = \infty$, use the Uniform Boundedness Principle directly.

2.35 Write $\mathrm{range}(A) = \cup_k A(B_k^X(0))$. If $\mathrm{range}(A)$ is nonmeager, then some set $\overline{A(B_k^X(0))}$ must contain an open ball. Apply Lemma 2.26 to conclude that $\mathrm{range}(A)$ contains an open ball.

2.38 Suppose $y_n = Lx_n \in \mathrm{range}(L)$ and $y_n \to y$ in Y. Show that $\{x_n\}$ is a Cauchy sequence in X.

2.45 (c) Apply the Cauchy–Bunyakovski–Schwarz identity to $(\cdot,\cdot)$ as follows:

$$\|x\|^4 = (A^{-1}x, x) \le |||A^{-1}x|||^2\, |||x|||^2 = \langle x, A^{-1}x \rangle\, |||x|||^2 \le \|A^{-1}\|\, \|x\|^2\, |||x|||^2.$$

Alternatively, let $A^{1/2}$ be the positive square root of A. Then $|||x||| = \|A^{1/2}x\|^2$, so A^{-1} is bounded on $(H, |||\cdot|||)$ because

$$\sup_{|||x|||=1} |||A^{-1}x||| = \sup_{\|A^{1/2}x\|=1} \|A^{-1/2}x\| = \sup_{\|y\|=1} \|A^{-1}x\| = \|A^{-1}\|.$$

2.47 Given $x \in \ell^p$, by considering $(\alpha_n x_n)$ where $\alpha_n x_n y_n = |x_n y_n|$, show that $Tx = (x_n y_n) \in \ell^1$. Then use the Closed Graph Theorem to prove that T is continuous, and finally prove that $x \mapsto \sum x_n y_n$ is a continuous linear functional on ℓ^p.

2.49 Fix $x \in \mathbf{R}$, and show that $\int_0^x f_n'(x)\,dx \to \int_0^x g(t)\,dt$ and $\int_0^x f_n'(x)\,dx \to f(x) - f(0)$.

2.51 (a) Weakly convergent sequences are bounded by Theorem 2.38. For the index $p = 1$, weak convergence implies boundedness and componentwise convergence, but the converse fails.

Chapter 3

3.3 $\Rightarrow$. Let $s_N = \sum_{n=1}^{N} x_n$, and show that the sequence of partial sums $\{s_N\}_{N\in\mathbf{N}}$ is Cauchy in X.

$\Leftarrow$. Suppose that every absolutely convergent series is convergent. Let $\{x_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in X. Show that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ such that $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$ for every k. Then $\sum_k (x_{n_{k+1}} - x_{n_k})$ is absolutely convergent, hence converges, say to x. Show that $\{x_n\}_{n\in\mathbf{N}}$ has a subsequence that converges (consider the partial sums of $\sum_k (x_{n_{k+1}} - x_{n_k})$). Show that $\{x_n\}_{n\in\mathbf{N}}$ converges.

3.8 (a) Apply the Uniform Boundedness Principle or the Closed Graph Theorem.

(b) Apply Theorem 3.10(f).

(c) Consider $X = c_0$ and the standard basis vectors $\{\delta_n\}$.

3.12 If $\sum |c_n|^2 < \infty$, then $g = \sum c_n f_n$ converges in L^2-norm. Use the fact that a sequence that converges in L^p-norm has a subsequence that converges pointwise a.e. to show that $g = f$ a.e.

3.14 Use Lemma 3.3 and the fact that all norms on a finite-dimensional normed space are equivalent.

Chapter 4

4.2 (a) Consider $F_N = \mathrm{span}\{x_1, \dots, x_N\}$ and apply the Baire Category Theorem.

(c) Let $F_N = \{f \in C_c(\mathbf{R}) : f(x) = 0 \text{ for } x \notin (-N, N)\}$. Then F_N is a closed subspace of $C_0(\mathbf{R})$, and $C_c(\mathbf{R}) = \cup F_N$. Apply Baire Category to show that $C_c(\mathbf{R})$ is meager in $C_0(\mathbf{R})$. However, if $C_c(\mathbf{R})$ had a countable Hamel basis, then F_N would have one as well, contradicting part (a).

4.5 Use the idea of Example 4.2 to create a discontinuous function $f\colon \mathbf{R} \to \mathbf{R}$ that is $\mathbf{Q}$-linear, i.e., $f(ax + by) = af(x) + bf(y)$ for x, $y \in \mathbf{R}$ and a, $b \in \mathbf{Q}$.

4.8 (b) Consider $\delta_0 = (1, 1, 1, \dots)$.

(c) Given $\mu \in c^*$, define $T(\mu) = (y_n)_{n\geq 0}$ where $y_0 = \langle \delta_0, \mu\rangle - \sum_{n=1}^{\infty} \langle \delta_n, \mu\rangle$ and $y_n = \langle \delta_n, \mu\rangle$ for $n \in \mathbf{N}$. Show that $T\colon c^* \to \ell^1$ is an isometric isomorphism.

Alternatively, given $y = (y_n)_{n\geq 0} \in \ell^1$, define $\mu_y \in c^*$ by $\langle x, \mu\rangle = \sum_{n=0}^{\infty} x_n y_n$ where $x_0 = \lim_{n\to\infty} x_n$, and show that $y \mapsto \mu_y$ is an isometric isomorphism.

4.15 (d) By Theorem 1.66, every linear operator on a finite-dimensional domain is continuous. Therefore, S_N *restricted to* $\mathrm{span}\{x_1, \dots, x_{N+1}\}$ is continuous. The vectors $S_{N+1}y_n$ and z_{N+1} belong to this domain.

4.16 (a) Since X is a metric space, if $K \subseteq X$ is compact then it is totally bounded, i.e., given $\varepsilon > 0$ there exist finitely many $y_1, \dots, y_M$ such that $K \subseteq \bigcup_{k=1}^{M} B_\varepsilon(y_k)$. Then there exists a single N_0 such that $\|y_k - S_N y_k\| \le \varepsilon$ for all $N \ge N_0$ and $k = 1, \dots, N$.

(b) If T is compact, then $D = \overline{T(B)}$ is a compact subset of Y, where $B = \{x \in X : \|x\| \le 1\}$ is the closed unit ball in X.

4.24 (d) Consider t^α.

Chapter 5

5.3 (a) Show that $\{e_n + e_{n+1}\}_{n \in \mathbf{N}}$ is complete.

5.4 The biorthogonal system is $\{ne_n - (n+1)e_{n+1}\}_{n \in \mathbf{N}}$.

5.7 (b) $\Rightarrow$ (a). Use the continuity of a_m to show that the representation $x = \sum \langle x, a_n \rangle x_n$ is unique.

(d) $\Rightarrow$ (e). Uniform Boundedness Principle.

5.8 (a) The finite-rank operators are dense in $H \otimes K = \mathcal{B}_2(H, K)$, so if $T \in H \otimes K$ then there exists an operator of the form $L = \sum_{k=1}^{N} p_k \otimes q_k$ such that $\|T - L\|_{\mathrm{HS}} < \varepsilon$. Exercise B.6 may be useful in showing that L can be approximated by a finite linear combination of operators of the form $x_m \otimes y_n$.

(b) By the uniqueness statement in Exercise B.9(b), to show that $S_{N^2}^{Z} = S_N^X \otimes S_N^Y$ it suffices to check that equality holds when these operators are applied to "simple tensors" $x \otimes y$.

5.9 (a) Suppose that $f \in L^2(\mathbf{T})$ and $\langle f, |t - \frac{1}{2}|^{-\alpha} e^{2\pi i n t} \rangle = 0$ for every n. Then by rearranging the integral, the function $g(t) = f(t)\,|t - \frac{1}{2}|^{-\alpha}$ satisfies $\langle g, e^{2\pi i n t} \rangle = 0$ for every n. Although g need not belong to $L^2(\mathbf{T})$, it does belong to $L^1(\mathbf{T})$. Use the fact (see Theorem 4.25) that functions in $L^1(\mathbf{T})$ are uniquely determined by their Fourier coefficients to conclude that $g = 0$.

5.10 This is harder than it looks, because the hypothesis that the series $x = \sum (a_n + ib_n)\, x_n$ converges *does not* imply that $\sum a_n x_n$ converges (compare Theorem 3.10).

$\Rightarrow$. Given a basis $\{x_n\}$ for X, consider Theorem 5.17.

$\Leftarrow$. Show that if $\{x_n, ix_n\}$ is a basis for $X_{\mathbf{R}}$ then $\{x_n\}$ is minimal in X and every $x \in X$ has some representation of the form $x = \sum c_n x_n$.

5.13 $\Leftarrow$. Suppose that $x_1, \dots, x_N \in \mathrm{range}(P_N)$ have been chosen. Since $\mathrm{range}(P_N) \subseteq \mathrm{range}(P_{N+1})$ and $P_N^2 = P_N$, the operator P_N maps the $(N+1)$-dimensional space $\mathrm{range}(P_{N+1})$ onto the N-dimensional space $\mathrm{range}(P_N)$. Hence there must exist some unit vector $x_{N+1} \in \mathrm{range}(P_{N+1})$ such that $P_N x_{N+1} = 0$.

5.16 (b) The same conclusion holds for those n and k such that

$$\mathrm{supp}(\psi_{n,k}) \;=\; \big[2^{-n}k, 2^{-n}(k+1)\big] \;\subseteq\; (x-\delta, x+\delta).$$

In essence, the conclusion holds for all n large enough and for those k such that $\psi_{n,k}$ is "localized at x."

5.17 Consider Example 5.13.

Chapter 6

6.8 Let $c_N = h_N(\mu_N)$, and show that $c_N = \frac{c_{N-1}+c_{N-2}}{2} + 1$ when N is odd and $c_N = \frac{c_{N-1}+c_{N-2}}{2} - 1$ when N is even. Use this to prove that $1 \le c_N \le 2$ for N odd and $-1 \le c_N \le 0$ for N even. Use the fact that h_N is piecewise linear to show that its global maximum must occur at $t = \mu_n$ for some $1 \le n \le N$, and likewise its global minimum will occur at $t = \mu_m$ for some $1 \le m \le N$.

Chapter 7

7.1 (c) Consider unions of orthonormal bases.

(d) Would a union of infinitely many orthonormal bases work?

(f) Consider Example 5.13.

(g) Consider the dual system in Example 5.13, which is $\{f_n\}_{n\in\mathbf{Z}}$ where $f_n(t) = |t-\frac{1}{2}|^{-\alpha} e^{2\pi i n t}$. Set $g_r = \chi_{[\frac{1}{2},\frac{1}{2}+r]}$, and use the fact that $\{e^{2\pi i n t}\}_{n\in\mathbf{Z}}$ is an orthonormal basis to show that there is no finite constant B such that $\sum_{n\in\mathbf{Z}} |\langle g_r, f_n\rangle|^2 \le B\,\|g_r\|_{L^2}^2$ for all $r > 0$.

7.2 (b) Write

$$\Bigg\|\sum_{n=M+1}^{N} c_n x_n\Bigg\| \;=\; \sup_{\|y\|=1} \Bigg|\Bigg\langle \sum_{n=M+1}^{N} c_n x_n,\; y\Bigg\rangle\Bigg| \;\le\; \sup_{\|y\|=1} \sum_{n=M+1}^{N} |c_n|\,|\langle x_n, y\rangle|,$$

and apply the Cauchy–Bunyakovski–Schwarz Inequality to the series. The same argument can be applied to $\{x_{\sigma(n)}\}$ for any permutation σ of $\mathbf{N}$.

7.3 (b) $\Rightarrow$ (a). Given $x \in H$, fix $N \in \mathbf{N}$, choose $y_k \in E$ such that $y_k \to x$, and consider $\sum_{n=1}^{N} |\langle y_k, x_n\rangle|^2$.

(d) $\Rightarrow$ (e). Statement (d) implies that $Tc = \sum c_n x_n$ is a linear map of ℓ^2 into H. Given $N \in \mathbf{N}$, define $T_N c = \sum_{n=1}^{N} c_n x_n$. Show that $T_N \colon \ell^2 \to H$ is bounded, and apply the Banach–Steinhaus Theorem.

(e) $\Rightarrow$ (a). Consider R^*.

7.7 If P is the orthogonal projection of K onto H, then $\sum |\langle x, Px_n\rangle|^2 = \sum |\langle Px, x_n\rangle|^2$.

7.8 Consider $\{ne_n\}$ where $\{e_n\}$ is an orthonormal basis.

7.9 (a) Let G be the Gram matrix for $\{x_n\}$, and fix $c \in c_{00}$. Note that

$$\|Gc\|_{\ell^2}^2 \;\le\; \sum_m \biggl|\sum_n \langle x_n, x_m\rangle\, c_n\biggr|^2,$$

and apply the Cauchy–Bunyakovski–Schwarz Inequality to the inner series.

Remark: Under these hypotheses G is a special type of compact operator on ℓ^2 known as a *Hilbert–Schmidt operator*, compare Exercise B.4.

(b) This is a special case of *Schur's Test* for boundedness of operators on ℓ^2. Write

$$\|Gc\|_{\ell^2}^2 \;\le\; \sum_m \biggl(\sum_n \bigl(|\langle x_n, x_m\rangle|^{1/2}\bigr)\,\bigl(|\langle x_n, x_m\rangle|^{1/2}\,|c_n|\bigr)\biggr)^2,$$

and apply Cauchy–Bunyakovski–Schwarz to the factors appearing in the inner summation.

Remark: By applying Hölder's Inequality with proper exponents, the same idea shows that under these hypotheses G defines a bounded map of ℓ^p into itself for each $1 \le p \le \infty$.

7.10 (b) Use the biorthogonality to write

$$\sum_{n=1}^N |c_n|^2 \;=\; \sum_{n=1}^N \biggl|\biggl\langle \sum_{m=1}^N c_m y_m,\; x_n\biggr\rangle\biggr|^2,$$

and apply the fact that B is a Bessel bound.

7.18 Use Theorem 5.24.

7.20 (b) Write

$$\begin{aligned} e^{2\pi i n x} - e^{2\pi i \lambda_n x} \;&=\; e^{2\pi i n x}\bigl(1 - e^{2\pi i(\lambda_n - \lambda)x}\bigr) \\ &=\; -e^{2\pi i n x} \sum_{k=1}^\infty \frac{\bigl(2\pi i(\lambda_n - n)\bigr)^k}{k!}\, x^k. \end{aligned}$$

(c) With λ as in Exercise 7.19, show that $\lambda \le e^{\pi\delta} - 1$.

Chapter 8

8.3 A subset of a frame need not be a frame sequence, consider Exercise 8.2.

8.5 $S - AI$ is self-adjoint. The norm of a self-adjoint operator T can be computed using $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

8.9 (a) Show that every function in $\overline{\text{span}}\{e^{2\pi ibnx}\}_{n\in\mathbf{Z}}$ is $1/b$-periodic.

(b) $\{b^{1/2}\, e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,1/b]$, which properly contains $L^2[0,1]$. The frame bound is $A = B = 1/b$.

8.10 (a) Use the Triangle Inequality and the Reverse Triangle Inequality in ℓ^2. Frame bounds for $\{y_n\}$ are $(A^{1/2} - K^{1/2})^2$, $(B^{1/2} + K^{1/2})^2$.

(c) Using the estimate $|e^{it} - 1| \le |t|$, it follows that $\|e_{bn} - e_{\lambda_n}\|_{L^2}^2 \le \frac{4\pi^2}{3}\,|bn - \lambda_n|^2$. Hence if $\sum |bn - \lambda_n|^2 < \frac{3}{4\pi^2 b}$ then $\{e_{\lambda_n}\}_{n\in\mathbf{Z}}$ will be a frame.
Remark: This result is not even close to being optimal! For example, for $b = 1$ the *Kadec* $\frac{1}{4}$*-Theorem* states that if $\sup |n - \lambda_n| \le L < 1/4$, then $\{e_{\lambda_n}\}_{n\in\mathbf{Z}}$ is an exact frame for $L^2[0,1]$ (see [You01]).

8.11 Since S is a positive $d \times d$ matrix, there is an orthonormal basis $\{w_1, \dots, w_d\}$ for $\mathbf{F}^d$ consisting of eigenvectors of S, with corresponding eigenvalues $0 \le \lambda_1 \le \cdots \le \lambda_d$.

8.15 (a) Use Theorem 2.15.

(b) Use a Neumann series argument (see Exercise 2.40) to show that $B^{-1}S$ or $2(A+B)^{-1}S$ is a topological isomorphism.

8.16 (a) Show first that $y = \sum \langle y, x_n\rangle\, x_n$ for every $y \in H$.

(b) Show that $\widetilde{C}C^*\widetilde{C} = \widetilde{C}$, and use the fact that $\text{range}(C) = \text{range}(\widetilde{C})$ to infer that $\widetilde{C}C^*C = C$ and $C^* = C^*C\widetilde{C}^*$.

8.17 (b) $\Leftarrow$. Show that if $PS = SP$, then $S^{-1}P = PS^{-1}$. Let T be the frame operator for $\{Px_n\}$ as a frame for M, and show that $TP = SP$ on H, so $T = S|_M$ as operators on M.

8.18 (b) Show that $Pc = c$ for $c \in \text{range}(C)$ and $Pc = 0$ for $c \in \text{range}(C)^\perp$.

8.19 (a) $\Rightarrow$ (b). Let V be the synthesis operator for $\{y_n\}$.

(b) $\Rightarrow$ (a). $RV^* = C^*V^* = (CV)^* = I$.

8.22 (b) Using the lower frame bound, $A\,\|\widetilde{x}_m\|^2 \le \sum_n |\langle \widetilde{x}_m, x_n\rangle|^2$. Now apply the fact that $\{x_n\}$ and $\{\widetilde{x}_n\}$ are biorthogonal.

8.23 If $\{x_n\}$ is exact, then it is biorthogonal to its canonical dual frame.

8.26 (b) Let T be the synthesis operator restricted to $\ker(R)^\perp = \text{range}(C)$. Then T is a topological isomorphism of $\text{range}(C)$ onto H, and it follows from the construction of the pseudoinverse (Theorem 2.33) that $R^\dagger = T^{-1}\colon H \to \text{range}(C)$. Keeping in mind that $\widetilde{R}C = I = R\widetilde{C}$, show that $T^{-1} = \widetilde{C}$.

8.30 Suppose that $\{x_n\}$ is a frame for H. Let $M = \text{range}(C)$ and let $T\colon M \to H$ be the topological isomorphism from Corollary 8.33. Let $K = H \times M^\perp$. Then $U = (TP_M c, P_{M^\perp} c)$ is a topological isomorphism of ℓ^2 onto K, so $\{U\delta_n\}$ is a Riesz basis for K. Identify H with $H \times \{0\} \subseteq K$.

8.32 (d) Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H and consider the frames $\{e_1, e_1, e_2, e_3, e_4, \cdots\}$ and $\{e_1, -e_1, e_2, e_3, e_4, \cdots\}$.

8.33 $\Rightarrow$. Suppose that $\{x_n\}$ is a Parseval frame for H. By Corollary 8.33, there exists a Hilbert space $K \supseteq H$ and an orthonormal basis $\{e_N\}$ for K such that $x_n = Pe_n$ for every n. Set $y_n = (I - P)e_n$, so $\{y_n\}$ is a Parseval frame for $H^\perp$. Show directly that $\{(x_n, y_n)\}$ is a Parseval frame for $H \times H^\perp$ and $\|(x_n, y_n)\|_{H\times H^\perp} = 1$ for every n (see Exercise 1.40 for the definition of the inner product and norm on a Cartesian product of Hilbert spaces).

8.34 (a) Let $T\colon H \to M$ be an isometric isomorphism, and consider $\{T^{-1}P_M\delta_n\}$.

8.35 (a) The inner product and norm on $H \times K$ are constructed in Exercise 1.40.

(c) To create a nontrivial example, let M be a closed subspace of ℓ^2 such that both M and $M^\perp$ are infinite dimensional, and let P_M, $P_{M^\perp}$ be the orthogonal projections of ℓ^2 onto M and $M^\perp$. Consider $\{P_M\delta_n\}$ and $\{P_{M^\perp}\delta_n\}$, and show that $\{(P_M\delta_n, P_{M^\perp}\delta_n)\}$ is an orthonormal basis for $M \times M^\perp$. Note that even though $M \times M^\perp$ is isomorphic to ℓ^2, this frame is quite different from the frame $\{P_M\delta_n\} \cup \{P_{M^\perp}\delta_n)\}$ for ℓ^2, which is tight but is not a basis.

8.36 Exercise B.9 implies that $\{x_m \otimes y_n\}_{m,n\in\mathbf{N}}$ is Bessel. Show that its frame operator is $S_X \otimes S_Y$ where S_X, S_Y are the frame operators for $\{x_m\}$, $\{y_n\}$ respectively.

8.37 If we set $\psi_k = \sum_{n=N_k+1}^{N_{k+1}} k^{-1/2} \langle \phi_{m_k}, \delta_n\rangle P\delta_n$ then we have $\|\psi_k\|_{\ell^2} \le k^{-1/2}\, 2^{-k}$. Show that we always have $\big\|\sum_{n=s}^{t} \langle \phi_{m_j}, \delta_n\rangle P\delta_n\big\|_{\ell^2} \le 1$. Given $N_1 + 1 \le M < N < \infty$, estimate $\big\|\sum_{n=M}^{N} c_n\, P\delta_n\big\|_{\ell^2}$ by writing $\sum_{n=M}^{N} = \sum_{n=M}^{N_{j+1}} + \sum_{k=j+1}^{\ell} \sum_{n=N_k+1}^{N_{k+1}} + \sum_{n=N_\ell+1}^{N}$, where j is as small as possible and ℓ is as large as possible.

8.40 Suppose statement (a) holds. By Exercise 8.24, since $S^{-1/2}$ is a topological isomorphism and $\|S^{1/2}\|^2 \le B$, the frame $\{S^{-1/2}x_n\}_{n\ne n_k}$ has a lower bound of L/B. However, since $\{S^{-1}x_n\}$ is a Parseval frame, Exercise 8.6 implies that its optimal lower bound is $1 - \|S^{-1/2}y_n\|^2$. Apply Theorem 8.44 to $\{S^{-1}x_n\}$.

8.42 (b) $\{e^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is a b^{-1}-tight frame, so $\{be^{2\pi ibnt}\}_{n\in\mathbf{Z}}$ is Parseval. Apply Theorem 8.44.

Chapter 9

9.2 (a) The function $f(x) = \sin x^2$ is continuous and bounded, but is not uniformly continuous.

(b) First prove the result for functions in $C_c(\mathbf{R})$, and then use the fact that $C_c(\mathbf{R})$ is dense in $L^p(\mathbf{R})$.

9.4 $|e^{i\theta} - 1| \le |\theta|$ for all θ.

9.5 (b) Write $\int |(f * g)(x)|\, dx$ as an iterated integral, and apply Fubini's Theorem.

(c) Write $(f * g)^{\wedge}(\xi)$ as an iterated integral, and apply Fubini's Theorem.

(d) Suppose that $f(x) = 0$ for a.e. $x \notin [-R, R]$ and $g(x) = 0$ for a.e. $x \notin [-S, S]$. Show that $(f * g)(x) = 0$ for a.e. $x \notin [-R-S, R+S]$.

(e) Use the Mean Value Theorem and the Lebesgue Dominated Convergence Theorem to justify the following calculation:

$$\begin{aligned}(f * g)'(x) &= \lim_{h \to 0} \frac{(f * g)(x+h) - (f * g)(x)}{h} \\ &= \lim_{h \to 0} \int_{-\infty}^{\infty} f(y)\, \frac{g(x+h-y) - g(x-y)}{h}\, dy \\ &= \int_{-\infty}^{\infty} f(y)\, g'(x-y)\, dy \;=\; (f * g')(x).\end{aligned}$$

9.9 Let $E = \{(x, y) \in [a, b]^2 : x \le y\}$. By Fubini's Theorem, the two iterated integrals

$$\int_a^b \int_a^b \chi_E(x, y)\, f'(x)\, g'(y)\, dx\, dy \;=\; \int_a^b \biggl(\int_a^y f'(x)\, dx\biggr)\, g'(y)\, dy$$

and

$$\int_a^b \int_a^b \chi_E(x, y)\, f'(x)\, g'(y)\, dy\, dx \;=\; \int_a^b f'(x)\, \biggl(\int_x^b g'(y)\, dy\biggr)\, dx$$

are equal.

Chapter 10

10.3 Every sequence that converges in L^2-norm has a subsequence that converges pointwise almost everywhere.

10.4 (a) Setting $g(\xi) = 2\pi i \xi \widehat{f}(\xi)$, the proof of Theorem 10.4(d) shows that $f' = \check{g}$. Hence

$$f'(x) \;=\; \check{g}(x) \;=\; \int_{-\infty}^{\infty} 2\pi i \xi \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi \;=\; \int_{-1/2}^{1/2} 2\pi i \xi \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

(c) Part (b) implies that the Taylor series converges absolutely for each x, but consider the remainder term in order to show that it converges to $f(x)$.

10.8 (d) Show that the frame inequalities hold for all $f \in H_\varphi \cap L^\infty(\mathbf{T})$, which is a dense subspace of H_φ, and apply Exercise 8.4.

10.11 Since $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, if we set $\widehat{c}(\xi) = \sum_{k\in\mathbf{Z}} c_k\, e^{-2\pi ikx}$ then $c \mapsto \widehat{c}$ is a unitary mapping of $\ell^2(\mathbf{Z})$ onto $L^2(\mathbf{T})$. Show that $(T_nc)^\wedge(\xi) = e^{-2\pi inx}\,\widehat{c}(\xi)$, and conclude that $\{T_nc\}_{n\in\mathbf{Z}} = \mathcal{E}(\widehat{c}\,)$ is the system of weighted exponentials generated by the function $\widehat{c} \in L^2(\mathbf{T})$. Theorem 10.10 tells us exactly when this system is complete, Bessel, etc., in $L^2(\mathbf{T})$, so $\{T_nc\}_{n\in\mathbf{Z}}$ must have exactly the same properties in $\ell^2(\mathbf{Z})$. For example,

$$\{T_nc\}_{n\in\mathbf{Z}} \text{ is complete} \quad\Longleftrightarrow\quad \widehat{c}(\xi) \neq 0 \text{ a.e.}$$

10.14 Take the Fourier transform of both sides, and use the fact that a nontrivial trigonometric polynomial $m(\xi) = \sum_{k=1}^N e^{2\pi i a_k\xi}$ can have only countably many zeros.

10.15 Set $m_N(\xi) = \sum_{k=-N}^N c_k e^{-2\pi ik\xi}$. Show that there exists some subsequence of $\{m_N\}$ that converges pointwise a.e., say $m_{N_k}(\xi) \to \widehat{c}(\xi)$ a.e. as $k \to \infty$. Hence $m_{N_k}(\xi)\,\widehat{g}(\xi) \to \widehat{c}(\xi)\,\widehat{g}(\xi)$ a.e. However, by hypothesis we have that $m_N\,\widehat{g} \to F$ in $L^2(\mathbf{R})$ as $N \to \infty$.

10.17 (b) Suppose that $\widehat{f} = m\,\widehat{g}$ where $m \in L^2(\mathbf{T})$. Then

$$m(\xi) \;=\; \widehat{c}(\xi) \;=\; \sum_{k\in\mathbf{Z}} c_k e^{-2\pi ik\xi}$$

for some sequence $c \in \ell^2(\mathbf{Z})$. Show that $\sum_{k=-N}^N c_k\, e^{-2\pi ik\xi}\,\widehat{g}(\xi)$ converges to $\widehat{f}$ in the norm of $L^2(\mathbf{R})$. Applying the inverse Fourier transform, this implies that $f = \sum_{k\in\mathbf{Z}} c_k\, T_k g \in V_0$.

10.19 (b) In this case α is a trigonometric polynomial such that $|\alpha(\xi)|^2 = 1$. Let $\alpha(\xi) = \sum_{k\in\mathbf{Z}} \alpha_k\, e^{-2\pi i\alpha_k\xi}$, where only finitely many α_k are nonzero. Write out $|\alpha(\xi)|^2 = \alpha(\xi)\,\overline{\alpha(\xi)}$, and use the linear independence of the complex exponentials to show that all but one α_k must be zero.

10.22 $\Phi_{\widetilde{w}}(\xi) = 3/(2+\cos 2\pi\xi)$ and $\Big[\widehat{w^\sharp}, \widehat{\widetilde{w}}\Big](\xi) = 3^{1/2}/(2+\cos 2\pi\xi)^{1/2}$.

10.23 Translation is *strongly continuous* in $L^p(\mathbf{R})$ for $1 \le p < \infty$. That is, if $g \in L^p(\mathbf{R})$ then $\lim_{a\to 0} \|g(x)-g(x-a)\|_{L^p} = 0$. Let S_j denote the partial sum operators and consider $S_j(f_{jk})$ for $j < k$, where $f_{jk}(x) = g(x-a_j)-g(x-a_k)$.

Chapter 11

11.3 (b) Using part (a), $S(M_{bn}T_{ka}f) = M_{bn}T_{ka}Sf$ for every $f \in L^2(\mathbf{R})$. Since S is a topological isomorphism, we can replace f by $S^{-1}f$.

(c) Apply part (b) and Corollary 8.23.

(d) By Theorem 2.18, $S^{-1/2}$ commutes with every operator that commutes with S^{-1}.

11.6 (d) Use the fact that $\widetilde{g} = bg/G_0$ to show that $\langle g, \widetilde{g}\rangle = 1$ if and only if $g(x - ak) = 0$ a.e. on $[0, b^{-1}]$ for all $k \neq 0$.

11.13 Use Theorem 11.8.

11.22 (c) Since $Z(M_n T_k \chi_{[0,1]}) = E_{nk}$, the range of Z contains the finite span of $\{E_{nk}\}_{k,n\in\mathbf{Z}}$. The two-dimensional analogue of Theorem 4.25 implies that $\{E_{nk}\}_{k,n\in\mathbf{Z}}$ is complete in $L^1(Q)$, so range(Z) is dense.

If $Z\colon L^1(\mathbf{R}) \to L^1(Q)$ was surjective, then given $f \in L^2(\mathbf{R})$ we would have $Zf \in L^2(Q) \subseteq L^1(Q)$, and therefore $Zg = Zf$ for some $g \in L^1(Q)$. Show that $g = f$ a.e. and conclude that $L^2(\mathbf{R}) \subseteq L^1(\mathbf{R})$, which is a contradiction.

Here is one way to do this. The series $Zf(x,\xi) = \sum_{j\in\mathbf{Z}} f(x-j)e^{2\pi i j\xi}$ converges in $L^2(Q)$. Use this to show that

$$\bigl\langle Zf,\, E_{nk}\bigr\rangle \;=\; \int_0^1 f(x+k)\, e^{-2\pi i n x}\, dx \;=\; \bigl(f\,\chi_{[k,k+1]}\bigr)^{\wedge}(n),$$

the nth Fourier coefficient of $f\,\chi_{[k,k+1]}$. Show that a similar formula holds for g. Since $f\,\chi_{[k,k+1]}$ and $g\,\chi_{[k,k+1]}$ both belong to $L^1[k,k+1]$, and functions in $L^1[k,k+1]$ are uniquely determined by their Fourier coefficients (see Theorem 4.25), it follows that $f\,\chi_{[k,k+1]} = g\,\chi_{[k,k+1]}$ a.e.

(d) If $Z^{-1}\colon \text{range}(Z) \to L^1(\mathbf{R})$ was continuous, then it would have an extension to a continuous map of $L^1(Q)$ to $L^1(\mathbf{R})$.

11.25 (a) Let $c_k = 0$ for $|k| > N$ and show that

$$1 \;=\; |p(x)|^2 \;=\; \sum_{k\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} c_j\, \overline{c_{k+j}}\, e^{2\pi i k x} \text{ a.e.}$$

Then apply the fact that $\{e^{2\pi i n x}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$ to conclude that $\sum_{j\in\mathbf{Z}} c_j\, \overline{c_{k+j}} = \delta_{0k}$ for $k \in \mathbf{Z}$.

(b) If $\mathcal{G}(g,1,1)$ is an orthonormal basis then we have $|Zg|^2 = 1$ a.e. and $\sum |g(x-j)|^2 = 1$ a.e. If g is compactly supported then

$$Zg(x,\xi) \;=\; \sum_{j=-N}^{N} g(x-j)\, e^{2\pi i j\xi}$$

for some finite N. Apply part (a) to $Zg(x,\cdot)$.

11.31 (a) Show that $\bigl\langle [A,B]f,\, f\bigr\rangle = 2i\,\mathrm{Im}\bigl\langle Bf,\, Af\bigr\rangle$, and apply Cauchy–Bunyakovski–Schwarz.

11.36 Since A is invertible, it is a scalar multiple of a matrix B with determinant 1. By Exercise 11.35(e), the matrix B can be written as a product of matrices of the form S_r, R, and D_a. Combine Theorem 11.36 with parts (a)–(d) of Exercise 11.35.

11.37 (a) A function of the form $m(x) = \sum_{k=1}^{N} c_k e^{2\pi i q_k x}$ is called a (nonharmonic) trigonometric polynomial. If we extend the domain of m to $x \in \mathbf{C}$ then m is an analytic function on $\mathbf{C}$. Therefore, if m is not the zero function then it cannot vanish on any set that has an accumulation point. In particular, m cannot vanish on any subset of $\mathbf{R}$ that has positive measure.

(b) Suppose that $\sum_{k=1}^{N} c_k T_{p_k} g = 0$ a.e., apply the Fourier transform, and use part (a).

(c) By applying Exercise 11.35(a), it suffices to assume that the collection $\Lambda = \{(p_k, q_k)\}_{k=1}^{N}$ is contained in a line that passes through the origin. If this line is vertical, apply part (a). Otherwise, by choosing $r \in \mathbf{R}$ correctly, the set $\mathcal{G}(h, S_{-r}(\Lambda))$ will have the form $\{T_{p_k} h\}_{k=1}^{N}$, where S_{-r} is the shear matrix considered in Exercise 11.35(b).

11.38 A nontrivial trigonometric polynomial cannot vanish on any set of positive measure.

Chapter 12

12.1 Define $\widetilde{\psi}_k = S^{-1}(T_{bk}\psi)$ for $k \in \mathbf{Z}$, Then the canonical dual has the form $\{D_{a^n} \widetilde{\psi}_k\}_{k,n \in \mathbf{Z}}$.

12.2 (c) $D_{a^n} = (D_a)^n$ and $T_{bk} = (T_b)^k$ belong to G for every $k,\, n \in \mathbf{Z}$. Consider compositions of $D_{a^n} T_{bk}$ with $D_{a^m} T_{b\ell}$.

(d) The right Haar measure is $\frac{da}{a^2}\, db$.

12.4 (a) The trigonometric system $\{e^{2\pi i n x}\}_{n \in \mathbf{Z}}$ forms an orthonormal basis for $L^2[0,1]$ and each function $e^{2\pi i n x}$ is 1-periodic.

12.7 The attractor Q is the parallelogram with vertices at $(0,0)$, $(1,0)$, $(2,1)$, and $(1,1)$.

12.11 (e) Show that it suffices to show that $\|P_n f\|_{L^2} \to 0$ as $n \to -\infty$ for all f in a dense subspace of $L^2(\mathbf{R})$. Then let f be bounded and compactly supported, say $\operatorname{supp}(f) \subseteq [-R, R]$. Use the Cauchy–Bunyakovski–Schwarz Inequality to show that

$$\begin{aligned}
\|P_n f\|_{L^2}^2 &= \sum_{k \in \mathbf{Z}} \bigl|\langle f,\, D_{2^n} T_k \varphi \rangle\bigr|^2 \\
&= \sum_{k \in \mathbf{Z}} 2^{-n} \left| \int_{-2^n R}^{2^n R} f(2^{-n}x)\, \overline{\varphi(x-k)}\, dx \right|^2
\end{aligned}$$

$$\leq \sum_{k\in\mathbf{Z}} 2^{-n}\,\|f\|_{L^\infty}^2 \left(\int_{-2^nR}^{2^nR} dx\right)\left(\int_{-2^nR+k}^{2^nR+k} |\varphi(x)|^2\,dx\right),$$

and consider what happens as $n \to -\infty$.

12.16 Plot the function, and compare Figure 1.1.

12.17 The argument is similar to Exercise 10.15. By hypothesis, the refinement equation converges with respect to some ordering of $\mathbf{Z}$. Hence there exist some finite subsets F_n of $\mathbf{Z}$ such that

$$\varphi \;=\; \lim_{n\to\infty} \sum_{k\in F_n} 2^{-1/2} c_k\, D_2 T_k \varphi.$$

By applying the unitarity of the Fourier transform we obtain

$$\widehat{\varphi}(\xi) \;=\; \frac{1}{2} \lim_{n\to\infty} \sum_{k\in F_n} c_k e^{-2\pi i k(\xi/2)}\, \widehat{\varphi}(\xi/2).$$

This is a limit in L^2-norm, but by passing to a subsequence of the sets F_n it holds pointwise a.e. On the other hand, the series defining m_0 converges unconditionally in L^2-norm, so

$$m_0(\xi) \;=\; \frac{1}{2} \lim_{n\to\infty} \sum_{k\in F_n} c_k\, e^{-2\pi i k \xi}.$$

This is convergence in $L^2(\mathbf{T})$, but by again passing to a subsequence it holds pointwise a.e.

12.18 The hypotheses imply that the series defining m converges absolutely in $L^2(\mathbf{T})$. Also, since $\|D_2T_k\varphi\|_{L^2} = \|\varphi\|_{L^2}$, the series $\sum c_k\, D_2T_k\varphi$ converges absolutely in $L^2(\mathbf{R})$, and hence $\sum c_k\,(D_2T_k\varphi)^\wedge$ converges absolutely in $L^2(\mathbf{R})$ as well.

12.19 (a) Write out $\sum_k |(c*d)_k|$ as an iterated series, and use Fubini's Theorem to interchange the order of summation.

(b) See Exercise 9.5.

(c) Show that

$$(\varphi * \psi)(x) \;=\; \frac{1}{2} \sum_{k\in\mathbf{Z}} (c*d)_k\, (\varphi*\psi)(2x-k).$$

12.20 (b) Apply Exercise 12.19 (see also Exercise 9.5). In particular, $L^1(\mathbf{R})$ is closed under convolution, so $B_n = B_{n-1} * \chi_{[0,1]} \in L^1(\mathbf{R})$ by induction.

(c) By equation (9.12), the Fourier transform of $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ is the sinc function. Since $\chi_{[0,1]} = T_{\frac{1}{2}}\chi_{[-\frac{1}{2},\frac{1}{2}]}$, its Fourier transform is

$$(\chi_{[0,1]})^\wedge(\xi) \;=\; M_{-\frac{1}{2}}(\chi_{[-\frac{1}{2},\frac{1}{2}]})^\wedge(\xi) \;=\; e^{-\pi i\xi}\,\frac{\sin \pi\xi}{\pi\xi}.$$

(d) Either compute directly, or show that $(T_1 B_n - B_n)^\wedge = \widehat{B_n'}$ and apply the Uniqueness Theorem (Corollary 9.14). By Exercise 9.5(c), the Fourier transform converts convolution into multiplication.

(e) Use Theorem 9.15 and the Inversion Formula (Theorem 9.12) to show that since $\widehat{B_n}$ has fast decay, B_n must be smooth. To show that $B_n^{(n-1)}$ is piecewise linear, use the relation proved in part (d).

12.23 Set $a_n = \sum_{k\in\mathbf{Z}} c_{k-2n}\,\overline{c_k}$, and show that $a = (a_n)_{n\in\mathbf{Z}}$ belongs to $\ell^1(\mathbf{Z})$. Then show that $|m_0(\xi)|^2 + |m_0(\xi+\frac{1}{2})|^2 = \frac{1}{2}\sum_{n\in\mathbf{Z}} a_n e^{2\pi i n\xi}$.

12.24 Write

$$\begin{aligned} |m_0(\xi) - 1| \;&=\; \left|\frac{1}{2}\sum_{k\in\mathbf{Z}} c_k e^{2\pi i k\xi} - \frac{1}{2}\sum_{k\in\mathbf{Z}} c_k\right| \\ &\le\; \frac{1}{2}\sum_{k\in\mathbf{Z}} |e^{2\pi i k\xi}-1|^{\delta}|e^{2\pi i k\xi}-1|^{1-\delta}\,|c_k| \end{aligned}$$

and use the fact that $|e^{i\theta}-1| \le \min\{|\theta|, 2\}$.

12.28 (a) Use the Dominated Convergence Theorem for series (Theorem A.25) to show that

$$\begin{aligned} \lim_{h\to 0}\frac{m_0(\xi+h)-m_0(\xi)}{h} \;&=\; \lim_{h\to 0}\frac{1}{2}\sum_{k\in\mathbf{Z}} c_k\,\frac{e^{-2\pi i k(\xi+h)} - e^{-2\pi i k\xi}}{h} \\ &=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} c_k\,\lim_{h\to 0}\frac{e^{-2\pi i k(\xi+h)} - e^{-2\pi i k\xi}}{h}. \end{aligned}$$

(b) Let M_n be the symbol for B_n. Exercises 12.19 and 12.20 imply that $M_n(\xi) = M_{n-1}(\xi)\,M_0(\xi)$. Apply induction and the product rule to show that $M_n^{(j)}(1/2) = 0$ for $j = 0, \dots, n+1$.

12.29 Since $\varphi \in L^1(\mathbf{R})$, Exercise 10.13 implies that its periodization $p(x) = \sum_{k\in\mathbf{Z}} \varphi(x+k)$ belongs to $L^1(\mathbf{T})$. Note that $p(x+1) = p(x)$ a.e. by definition, and use the refinement equation to show that $p(2x) = p(x)$ a.e. If p was continuous, then these two conditions on p can be used directly to show that p is constant. One way to prove that this still holds a.e. when we only assume p is integrable is to use the fact that $\tau x = 2x \bmod 1$ is an *ergodic mapping* of $[0,1)$ onto itself (i.e., any subset of $[0,1)$ that is invariant under this map must either have measure 0 or measure 1). The Birkhoff Ergodic Theorem [Wal82, Thm. 1.14] implies that if $f \in L^1(\mathbf{T})$ then

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k x) \;\to\; \int_0^1 f(t)\,dt \;\; \text{a.e.} \qquad \text{as } n\to\infty.$$

Note that $p(\tau x) = p(x)$.

12.30 The idea is similar to that of Exercise 12.29, although the details are more difficult. Let

$$h(x) \;=\; \sum_{j\in\mathbf{Z}} (j-2a)\,\varphi(x+j) \;-\; x,$$

and show that

$$h(x+1) = h(x) \qquad \text{and} \qquad h(x) = \frac{h(x)}{2}$$

for almost all x. Apply the Birkhoff Ergodic Theorem to show that $h = 0$ a.e.

12.31 (a) The Fourier transform of any function $f \in L^1(\mathbf{R})$ is continuous.

(b) The function $\theta(\xi) = -i\,\mathrm{sign}(\xi)$ satisfies $\theta(2\xi) = \theta(\xi)$. Use the fact that $\widehat{\varphi}(\xi) = m_0(\xi/2)\,\widehat{\varphi}(\xi/2)$ a.e. to show that $(H\varphi)^\wedge(\xi) \;=\; m_0(\xi/2)\,(H\varphi)^\wedge(\xi/2)$ a.e., and then apply Exercise 12.18.

12.32 Fix $R > \|m_0\|_\infty$ large enough so that $\alpha = \log_2 R > 0$, and let $C = \|P \cdot \chi_{[-1,1]}\|_\infty$. Given $2^{n-1} \le |\xi| < 2^n$ with $n > 1$, show that

$$|P(\xi)| \;\le\; CR^n \;=\; CR\,2^{(n-1)\alpha} \;\le\; CR\,|\xi|^\alpha.$$

12.33 Since $|m_0(0)| < 1$, there exists some $\delta > 0$ such that $|m_0(\xi)| \le r < 1$ for all $|\xi| < \delta$. The refinement equation on the Fourier side is $\widehat{\varphi}(\xi) = m_0(\xi/2)\,\widehat{\varphi}(\xi/2)$. Iterate to show that $\widehat{\varphi} = 0$.

12.36 (c) Show that P_n and P obey the same growth estimate, i.e., there exist C and M independent of n such that $|P_n(\xi)| \le C\,|\xi|^M$ for all $|\xi| \ge 1$. If $f \in \mathcal{S}(\mathbf{R})$, then

$$|f(\xi)| \;\le\; \frac{D}{|\xi|^{M+2}}, \qquad |\xi| \ge 1,$$

where $D = \|\xi^{M+2}\,\widehat{f}(\xi)\|_\infty$. Estimate

$$|\langle f, \mu - \mu_n\rangle| \;=\; \bigl|\langle \widehat{f}, \widehat{\mu} - \widehat{\mu_n}\rangle\bigr| \;=\; \int_{-\infty}^{\infty} |\widehat{f}(\xi)|\,|P(\xi) - P_n(\xi)|\,d\xi$$

by breaking the integral into the regions $|\xi| \le T$ and $|\xi| > T$ for some appropriate T.

12.38 (b) Use part (a). By the Riemann–Lebesgue Lemma, $\widehat{\varphi}(\xi) \to 0$ as $|\xi| \to \infty$.

Chapter 13

13.4 Except for a scaling factor, this is the same as Exercise 12.28.

13.13 (b) Write $\frac{1}{h}\big((f*g)(x+h)-(f*g)(x)\big)$ as an integral, and apply the Lebesgue Dominated Convergence Theorem.

13.14 Translation is strongly continuous on $L^p(\mathbf{T})$ when $1 \le p < \infty$ (see Exercise 13.1).

13.16 Show that $\|f - f*k_N\|_{L^p}^p$ is bounded by

$$\int_0^1 \left(\int_0^1 |f(x)-f(x-t)|\, |k_N(t)|^{1/p}\, |k_N(t)|^{1/p'}\, dt\right)^p dx.$$

Then apply Hölder's Inequality in a similar fashion to how it is used to prove Young's Inequality.

13.18 Consider equation (13.13).

13.19 $\overset{\vee}{\chi}_N(x)$ is a geometric series in the variable $\omega = e^{2\pi i x}$.

13.20 To obtain the lower estimate, use the fact that $|\sin x| \le |x|$ and make a change of variables to write

$$\begin{aligned}\frac{1}{2}\,\|d_N\|_{L^1} &\ge \int_0^{1/2} \frac{|\sin(2N+1)\pi x|}{\pi|x|}\,dx \\ &= \int_0^{N+\frac{1}{2}} \frac{|\sin \pi x|}{\pi|x|}\,dx \\ &\ge \sum_{k=0}^{N-1}\int_k^{k+1} \frac{|\sin \pi x|}{\pi|x|}\,dx.\end{aligned}$$

For the upper estimate, note that

$$f(x) = \frac{1}{\sin \pi x} - \frac{1}{\pi x}$$

is odd and increasing on $[-1/2, 1/2]$. Consequently,

$$\frac{1}{|\sin \pi x|} \le \frac{1}{\pi|x|} + \left(1 - \frac{2}{\pi}\right), \qquad |x| \le \frac{1}{2}.$$

Hence

$$\begin{aligned}\frac{1}{2}\,\|d_N\|_{L^1} &\le \int_0^{1/2} \frac{|\sin(2N+1)\pi x|}{\pi|x|}\,dx + \left(1-\frac{2}{\pi}\right)\int_0^{1/2} |\sin(2N+1)\pi x|\,dx \\ &\le \int_0^{N+\frac{1}{2}} \frac{|\sin \pi x|}{\pi|x|}\,dx + \left(1-\frac{2}{\pi}\right)\frac{1}{2} \\ &\le \int_0^1 \frac{\sin \pi x}{\pi x}\,dx + \sum_{k=1}^{N}\int_k^{k+1} \frac{|\sin \pi x|}{\pi|x|}\,dx + \left(\frac{1}{2}-\frac{1}{\pi}\right).\end{aligned}$$

The number $\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \ln N\right)$ is called *Euler's constant*, and it has the numerical value $\gamma \approx 0.57721566\ldots$.

13.23 (a) Note that $\check{\chi}_N(x) = \sum_{n=-N}^{N} e^{2\pi inx}$. Use Exercise 13.21 to write

$$\check{W}_N(x) \;=\; \sum_{n=-N}^{N}\Big(1 - \frac{|n|}{N+1}\Big)\, e^{2\pi inx} \;=\; \frac{\check{\chi}_0(x) + \cdots + \check{\chi}_N(x)}{N+1}.$$

Now substitute

$$\check{\chi}_n(x) \;=\; \frac{\sin(2n+1)\pi x}{\sin \pi x} \;=\; \frac{e^{(2n+1)\pi ix} - e^{-(2n+1)\pi ix}}{e^{\pi ix} - e^{-\pi ix}},$$

and simplify the resulting geometric series.

(b) To show $\int w_N = 1$, use the form

$$w_N(x) \;=\; \sum_{n=-N}^{N}\big(1 - \frac{|n|}{N+1}\big)\, e^{2\pi inx}.$$

To show requirement (c) in the definition of approximate identity, use the form

$$w_N(x) \;=\; \frac{1}{N+1}\Big(\frac{\sin(N+1)\pi x}{\sin \pi x}\Big)^2.$$

13.26 (a) Apply Exercise 13.3.

(b) Suppose that $f \in A(\mathbf{T})$. For each $n \in \mathbf{Z}$, let g_n be any complex number such that $g_n^2 = \widehat{f}(n)$. Then $(g_n)_{n\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, so $g(x) = \sum_{n\in\mathbf{Z}} g_n\, e^{2\pi inx}$ belongs to $L^2(\mathbf{T})$. Show that $g * g = f$.

(c) For each $N \in \mathbf{N}$, define $F_N = \big\{f \in A(\mathbf{T}) : \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| \le N\big\}$, so $A(\mathbf{T}) = \cup F_N$. Show that each F_N is a closed subset of $C(\mathbf{T})$. Since $A(\mathbf{T})$ is a dense subspace of $C(\mathbf{T})$, it contains no open subsets of $C(\mathbf{T})$. Therefore, F_N contains no interior points, so is nowhere dense, and consequently $A(\mathbf{T})$ is meager.

Chapter 14

14.2 (e) $\Rightarrow$ (d). $2\, S_N^{\mathrm{o}} f = S_{2N} f + i S_N^{\mathrm{t}} f + \widehat{f}(0) = -S_N^{\mathrm{t}} S_N^{\mathrm{t}} f + i S_N^{\mathrm{t}} f + 2\,\widehat{f}(0)$.

14.4 (a) Suppose that $g \in L^1(\mathbf{T})$ satisfies $\overline{\widehat{g}(n)} = \widehat{g}(-n)$ for every n. Show that $\overline{(g * w_N)^\wedge(n)} = (g * w_N)^\wedge(-n)$ for every n. Apply the Inversion Formula to $g * w_N$ to show that $g * w_N$ is real valued. Since $g * w_N \to g$ in L^1-norm, there is a subsequence that converges to g pointwise a.e.

14.5 If $f \in A(\mathbf{T})$ then $f \in L^1(\mathbf{T})$ and $\widehat{f} \in \ell^1(\mathbf{Z})$. The Inversion Formula implies that $f(x) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\, e^{2\pi i n x}$, with absolute convergence of this series in L^p-norm.

14.6 (b) $\Rightarrow$ (d). Apply the Closed Graph Theorem.

(c) $\Rightarrow$ (d). Use an extension by density argument similar to the one used to solve Exercise 1.72.

Appendix B

B.1 (b) Every closed and bounded subset of a finite-dimensional vector space is compact (this is the *Heine–Borel Theorem*).

(c) Use the following Cantor-type diagonalization argument. Suppose that $\{x_n\}_{n\in\mathbf{N}}$ is a bounded sequence in X. Since T_1 is compact, there exists a subsequence $\{x_n^{(1)}\}_{n\in\mathbf{N}}$ of $\{x_n\}_{n\in\mathbf{N}}$ such that $\{T_1 x_n^{(1)}\}_{n\in\mathbf{N}}$ converges. Then since T_2 is compact, there exists a subsequence $\{x_n^{(2)}\}_{n\in\mathbf{N}}$ of $\{x_n^{(1)}\}_{n\in\mathbf{N}}$ such that $\{T_2 x_n^{(2)}\}_{n\in\mathbf{N}}$ converges (and note that $\{T_1 x_n^{(2)}\}_{n\in\mathbf{N}}$ also converges!). Continue to construct subsequences in this way, and then show that the "diagonal subsequence" $\{T x_n^{(n)}\}_{n\in\mathbf{N}}$ converges (use the fact that there exists a k such that $\|T - T_k\| < \varepsilon$). Therefore T is compact.

B.2 Use Exercise 1.6.

B.3 Consider the finite-rank operators $T_N f = \sum_{n=1}^{N} \lambda_n \langle f, e_n\rangle\, e_n$, and apply Theorem B.5(c).

B.4 (b) If $\{T_n\}$ is Cauchy with respect to $\|\cdot\|_{\mathrm{HS}}$ then it is Cauchy with respect to $\|\cdot\|$ by part (a). Hence there exists some $T \in \mathcal{B}(H)$ such that T_n converges to T in operator norm. Show that $\|T - T_n\|_{\mathrm{HS}} \to 0$.

(d) Let $\{e_n\}$ be an orthonormal basis for H and for each $N \in \mathbf{N}$ define $T_N x = \sum_{n=1}^{N} \langle x, e_n\rangle\, T e_n$. Each T_N is continuous and has finite rank, and so is compact. Show that $\|T - T_N\|_{\mathrm{HS}} \to 0$, so the finite-rank operators are dense in $\mathcal{B}_2(H,K)$. Also, since the operator norm is dominated by the Hilbert–Schmidt norm, Theorem B.5 implies that T is compact.

Index of Symbols

c_{00}	Finite sequences	Example 1.18
$\widehat{c}$	Fourier transform of a sequence c	Definition 13.4
$\check{c}$	Inverse Fourier transform of a sequence c	Definition 13.4
$c * d$	Convolution of sequences	Definition 13.7
$\mathbf{C}$	Complex plane	General Notation
C	Analysis (coefficient) operator	Definition 7.3
$C_b(\mathbf{R})$	Bounded continuous functions	Example 1.19
$C_0(\mathbf{R})$	Continuous functions vanishing at infinity	Example 1.19
$C_c(\mathbf{R})$	Continuous, compactly supported functions	Example 1.19
$C^m(\mathbf{R})$	m-times differentiable functions	General Notation
$C_b^m(\mathbf{R})$	Bounded functions in $C^m(\mathbf{R})$	Exercise 1.22
$C_c^m(\mathbf{R})$	Compactly supported functions in $C^m(\mathbf{R})$	General Notation
$C^\infty(\mathbf{R})$	Infinitely differentiable functions	General Notation
$C_c^\infty(\mathbf{R})$	Compactly supported functions in $C^\infty(\mathbf{R})$	General Notation
$C[a,b]$	Continuous functions on $[a,b]$	Example 1.19
$C(\mathbf{T})$	Continuous functions on $\mathbf{T}$	Notation 4.23
$C^m(\mathbf{T})$	m-times differentiable functions on $\mathbf{T}$	Section 13.1
$C^\infty(\mathbf{T})$	Infinitely differentiable functions on $\mathbf{T}$	Section 13.1
D_r	Dilation operator	Notation 9.4
D_4, D_6, D_{2N}	Daubechies scaling functions	Section 12.5
e_λ	Complex exponential $e^{2\pi i\lambda x}$	Notation 9.2
$\overline{E}$	Closure of a set E	Definition 1.15
$\|E\|$	Lebesgue measure of $E \subseteq \mathbf{R}$	General Notation
$\mathcal{E}(\varphi)$	System of weighted exponentials	Definition 10.9
ess sup	Essential supremum	Equation (1.4)
$f \perp g$	Orthogonal vectors	Section 1.5
$\mathbf{F}$	Generic scalar field	General Notation
$\mathcal{F}$	Fourier transform operator for $\mathbf{R}$	Definition 9.7
$\widehat{f}$	Fourier transform of $f \in L^1(\mathbf{R})$	Definition 9.7
$\check{f}$	Inverse Fourier transform of $f \in L^1(\mathbf{R})$	Definition 9.11
$\mathcal{F}$	Fourier transform operator for $\mathbf{T}$	Definition 13.2
$\widehat{f}$	Fourier transform of $f \in L^1(\mathbf{T})$	Definition 13.2
$\check{f}$	Inverse Fourier transform of $f \in L^1(\mathbf{T})$	Definition 13.2
$f * g$	Convolution of functions	Definition 13.6
$\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbf{R})$	Functions bandlimited to $[-\Omega,\Omega]$	Definition 10.1
G	Gram matrix	Definition 7.3

$\mathcal{G}(g,a,b)$	Gabor system	Definition 11.1
G_0	a-periodization of $\|g\|^2$	Section 11.1
G_n	Correlation function	Section 11.5
$\ker(T)$	Kernel (nullspace) of an operator T	Definition 1.58
$\ell^p(I)$	p-summable sequences	Example 1.9
$L^p(E)$	Lebesgue space of p-integrable functions	Example 1.10
$L^p(\mathbf{T})$	p-integrable functions on $\mathbf{T}$	Notation 4.23
$L^2_{[-T,T]}(\mathbf{R})$	Functions timelimited to $[-T,T]$	Definition 10.1
m_0	Symbol of a refinement equation	Notation 12.16
$M^\perp$	Orthogonal complement of a subspace M	Definition 1.41
$M+N$	Direct sum of subspaces	Definition 1.45
$M \oplus N$	Orthogonal direct sum of subspaces	Definition 1.45
M_b	Modulation operator on $\mathbf{R}$	Notation 9.4
M_b	Modulation operator on $\mathbf{T}$ or $\mathbf{Z}$	Definition 13.1
$\mathcal{M}(\widehat{g}\,)$	System of integer modulates of $\widehat{g}$	Definition 10.13
$\mathbf{N}$	Natural numbers, $\{1,2,3,\dots\}$	General Notation
$\mathrm{PW}(\mathbf{R})$	Paley–Wiener space	Definition 10.2
$\mathbf{Q}$	Set of rational numbers	General Notation
$\mathbf{R}$	Real line	General Notation
R	Synthesis (reconstruction) operator	Definition 7.3
S	Frame operator	Definition 7.3
$\mathcal{S}(\mathbf{R})$	Schwartz space	Definition 9.18
$\mathrm{sign}(x)$	Sign function	Section 14.1
$\mathrm{span}(E)$	Finite linear span of E	Definition 1.25
$\overline{\mathrm{span}}(E)$	Closed linear span of E	Definition 1.25
$\mathbf{T}$	Torus, domain of 1-periodic functions	Notation 4.23
T_a	Translation operator on $\mathbf{R}$	Notation 9.4
T_a	Translation operator on $\mathbf{T}$ or $\mathbf{Z}$	Definition 13.1
$\mathcal{T}(g)$	System of integer translates of g	Definition 10.13
$\mathcal{W}(\psi,a,b)$	Wavelet system	Definition 12.1
$\mathcal{W}(\psi)$	Dyadic wavelet system	Definition 12.1
$W(L^p,\ell^q)$	Wiener amalgam space	Definition 11.12
$W(C,\ell^q)$	Wiener amalgam space	Definition 11.12
W_4, W_6, W_{2N}	Daubechies wavelets	Section 12.5
X^*	Dual space of X	Definition 1.71
x^*	Generic element of X^*	Notation 1.72

$\widetilde{x}_n$	Dual frame element	Notation 8.15
$x_n \xrightarrow{\mathrm{w}} x$	Weak convergence	Definition 2.36
$\mu_n \xrightarrow{\mathrm{w}^*} \mu$	Weak* convergence	Definition 2.36
$\mathbf{Z}$	Integers, $\{\dots, -1, 0, 1, \dots\}$	General Notation
Z	Zak transform	Definition 11.20

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