9

Regularity of Mass-Minimizing Currents

In the last chapter we proved the existence of solutions to certain variational problems in the context of integer-multiplicity rectifiable currents. In this chapter, we address the question of whether such solutions are in fact smooth surfaces. Such a question is quite natural: Indeed, Hilbert's 19th problem asked [Hil 02], "Are the solutions of regular problems in the calculus of variations always necessarily analytic?"

While Hilbert proposed his famous problems in 1900, the earliest precursors of currents as a tool for solving variational problems are the generalized curves of Laurence Chisholm Young (1905–2000) [You 37]. So of course, Hilbert could not have been been referring to variational problems in the context of integer-multiplicity currents.

Sets of finite perimeter are essentially equivalent to codimension-one integer-multiplicity rectifiable currents. It was Ennio de Giorgi (1928–1996) [DGi 61a], [DGi 61b] who first proved the existence and almost-everywhere regularity of area-minimizing sets of finite perimeter. Subsequently, Ernst Robert Reifenberg (1928–1964) [Rei 64a], [Rei 64b] proved the almost-everywhere regularity of area-minimizing surfaces in higher codimensions.

Later work of W. Fleming [Fle 62], E. De Giorgi [DGi 65], Frederick Justin Almgren, Jr. (1933–1997) [Alm 66], J. Simons [Sis 68], E. Bombieri, E. De Giorgi, and E. Giusti [BDG 69], and H. Federer [Fed 70], led to the definitive result that states that, in $\mathbb{R}^N$, an $(N-1)$-dimensional mass-minimizing integer-multiplicity current is a smooth, embedded manifold in its interior, except for a singular set of Hausdorff dimension at most $N-8$.

The extension of the regularity theory to general elliptic integrands was made by Almgren [Alm 68]. His result is that an integer-multiplicity current that minimizes the integral of an elliptic integrand is regular on an open dense set. Later work of Almgren, R. Schoen, and L. Simon [SSA 77] gave a stronger result in codimension one.

In our exposition, we will limit the scope of what we prove in favor of including more detail. Specifically, we will limit our attention to the area integrand and to codimension-one surfaces. An advantage of this approach is that we can include a complete derivation of the needed a priori estimates. Our exposition is based on the direct argument of R. Schoen and L. Simon [SS 82].

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9.1 Preliminaries

Notation 9.1.1.

(1) We let M be a positive integer, $M \geq 2$.
(2) We identify $\mathbb{R}^{M+1}$ with $\mathbb{R}^M \times \mathbb{R}$ and let $\mathbf{p}$ be the projection onto $\mathbb{R}^M$ and $\mathbf{q}$ be the projection onto $\mathbb{R}$.
(3) We let $\mathbb{B}^M(y, \rho)$ denote the open ball in $\mathbb{R}^M$ of radius ρ, centered at y. The closed ball of radius ρ, centered at y, will be denoted by $\overline{\mathbb{B}}^M(y, \rho)$.
(4) The cylinder $\mathbb{B}^M(y, \rho) \times \mathbb{R}$ will be denoted by $\mathrm{C}(y, \rho)$ and its closure by $\overline{\mathrm{C}}(y, \rho)$.
(5) Recall that $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{M+1}$ is the standard basis for $\mathbb{R}^{M+1}$, and $dx_1, dx_2, \ldots, dx_{M+1}$ is the dual basis in $\bigwedge^1 \mathbb{R}^{M+1}$.
(6) As basis elements for $\bigwedge_M \mathbb{R}^{M+1}$ we will use

$$\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \ldots, \mathbf{e}_{\widehat{M+1}}, \tag{9.1}$$

where

$$\mathbf{e}_{\hat{\imath}} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_{i-1} \wedge \mathbf{e}_{i+1} \wedge \cdots \wedge \mathbf{e}_{M+1} .$$

Since the M-dimensional subspace associated with $\mathbf{e}_{\widehat{M+1}}$ will play a special role in what follows, we will also use the notation

$$\mathbf{e}^M = \mathbf{e}_{\widehat{M+1}} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_M .$$

(7) We will identify $\bigwedge^M \mathbb{R}^{M+1}$ and the dual space of $\bigwedge_M \mathbb{R}^{M+1}$ using the standard isomorphism. Thus we will write $\langle \phi, \eta \rangle$ and $\phi(\eta)$ interchangeably when $\eta \in \bigwedge_M \mathbb{R}^{M+1}$ and $\phi \in \bigwedge^M \mathbb{R}^{M+1} \simeq \left[\bigwedge_M \mathbb{R}^{M+1}\right]'$.
(8) We set

$$dx_{\hat{\imath}} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{M+1} \tag{9.2}$$

for $i = 1, 2, \ldots, M+1$. We will also use the notation

$$dx^M = dx_{\widehat{M+1}} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_M . \tag{9.3}$$

Definition 9.1.2.

(1) According to the definition given in Example 8.3.6(1), the M-dimensional area integrand on $\mathbb{R}^{M+1}$ is a function on $\mathbb{R}^{M+1} \times \bigwedge_M \mathbb{R}^{M+1}$, but a function that is in fact independent of the first component of the argument. For simplicity of notation, we will consider the M-dimensional area integrand to be a function only on $\bigwedge_M \mathbb{R}^{M+1}$, so that

$$A : \bigwedge_M \mathbb{R}^{M+1} \to \mathbb{R}$$

is given by

$$A(\xi) = |\xi|$$

for $\xi \in \bigwedge_M \mathbb{R}^{M+1}$.

(2) The M-dimensional area functional $\mathbf{A}$ is defined by setting

$$\mathbf{A}(S) = \int A\left(\overrightarrow{S}(x)\right) d\|S\|(x)$$

whenever S is an M-dimensional current representable by integration. We also have $\mathbf{A}(S) = \mathbf{M}(S) = \|S\|(\mathbb{R}^{M+1})$. Of course, the area integrand is called that because, when S is the current associated with a classical M-dimensional surface, then $\mathbf{A}(S)$ equals the area of that surface.

Next we will calculate the first and second derivatives of the area integrand and note some important identities.

Using the basis (9.1), we find that if $\xi = \sum_{i=1}^{M} \xi_i \mathbf{e}_{\widehat{\imath}}$, then

$$A(\xi) = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_{M+1}^2}\,; \tag{9.4}$$

so the derivative of the area integrand, DA, is represented by the 0-by-$(M+1)$ matrix

$$DA(\xi) = \left(\xi_1/|\xi|,\ \xi_2/|\xi|, \ldots, \ \xi_{M+1}/|\xi|\right). \tag{9.5}$$

That is,

$$\langle\, DA(\xi),\ \eta\,\rangle = (\xi \cdot \eta)/|\xi| \tag{9.6}$$

holds for $\xi, \eta \in \bigwedge_M \mathbb{R}^{M+1}$, or equivalently, we have

$$DA(\xi) = |\xi|^{-1} \sum_{i=1}^{M+1} \xi_i \, dx_{\widehat{\imath}}\,. \tag{9.7}$$

In particular, we have

$$DA(\mathbf{e}_{\widehat{\imath}}) = dx_{\widehat{\imath}}\,. \tag{9.8}$$

We see that the second derivative of the area integrand, $D^2 A$, is represented by the Hessian matrix

$$D^2 A(\xi) = |\xi|^{-1} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

$$- |\xi|^{-3} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \ldots & \xi_1\xi_{M+1} \\ \xi_2\xi_1 & \xi_2^2 & \ldots & \xi_2\xi_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{M+1}\xi_1 & \xi_{M+1}\xi_2 & \ldots & \xi_{M+1}^2 \end{pmatrix}. \tag{9.9}$$

Equivalently, for the partial derivatives $\partial^2 A/\partial\xi_i\partial\xi_j = D_{\xi_i\,\xi_j} A$, we have

$$D_{\xi_i \xi_j} A(\xi) = |\xi|^{-3} \left(|\xi|^2 \delta_{i\,j} - \xi_i\, \xi_j\right), \tag{9.10}$$

where $\delta_{i\,j}$ is the Kronecker delta.[1]

Using (9.10), we can compute the Hilbert–Schmidt norm of $D^2 A$ as follows:

$$\begin{aligned}
|D^2 A(\xi)|^2 &= \sum_{i,j=1}^{M+1} [D_{\xi_i \xi_j} A(\xi)]^2 \\
&= |\xi|^{-6} \sum_{i,j=1}^{M+1} \left[|\xi|^2 \delta_{i\,j} - \xi_i\, \xi_j\right]^2 \\
&= |\xi|^{-6} \sum_{i,j=1}^{M+1} \left[|\xi|^4 \delta_{i\,j} - 2\,|\xi|^2\, \xi_i\, \xi_j\, \delta_{i\,j} + \xi_i^2\, \xi_j^2\right] \\
&= |\xi|^{-6} \left[(M+1)\,|\xi|^4 - 2\,|\xi|^4 + |\xi|^4\right] \\
&= M\,|\xi|^{-2}\,.
\end{aligned}$$

So we have

$$|D^2 A| = \sqrt{M}/|\xi|\,. \tag{9.11}$$

We note that

$$\frac{1}{2}\,|\xi - \eta|^2 = A(\eta) - \langle\, DA(\xi),\ \eta\,\rangle,\ \text{ for } |\xi| = |\eta| = 1\,. \tag{9.12}$$

Equation (9.12) follows because

$$\begin{aligned}
\frac{1}{2}|\xi - \eta|^2 &= \frac{1}{2}\left(|\xi|^2 - 2\xi\cdot\eta + |\eta|^2\right) \\
&= 1 - \xi\cdot\eta \\
&= |\eta| - (\xi\cdot\eta)/|\xi| \\
&= A(\eta) - \langle\, DA(\xi),\ \eta\,\rangle\,,
\end{aligned}$$

where the last equality follows from (9.6).

Equation (9.12) will play an important role in the regularity theory, but it is the inequality

$$\frac{1}{2}\,|\xi - \eta|^2 \le A(\eta) - \langle\, DA(\xi),\ \eta\,\rangle,\ \text{ for } |\xi| = |\eta| = 1, \tag{9.13}$$

[1] Leopold Kronecker (1823–1891).

that is essential. Any inequality of the form (9.13) (but with $\frac{1}{2}$ possibly replaced by another positive constant) is called a *Weierstrass condition*. Ellipticity of an integrand is equivalent to the integrand satisfying a Weierstrass condition (see [Fed 75, Section 3]).

Definition 9.1.3. We say that the M-dimensional integer-multiplicity current T is *mass-minimizing* if

$$\mathbf{A}(T) \le \mathbf{A}(S) \tag{9.14}$$

holds whenever $S \in \mathcal{D}_M(\mathbb{R}^{M+1})$ is integer-multiplicity with $\partial S = \partial T$.

When a current is projected into a plane, the mass of the projection is less than or equal to the mass of the original current. The difference between the two masses is the "excess" (see Figure 9.1). The fundamental quantity used in the regularity theory is the "cylindrical excess," which is the excess of the part of a current in a cylinder, normalized to account for the radius of the cylinder. We give the precise definition next.

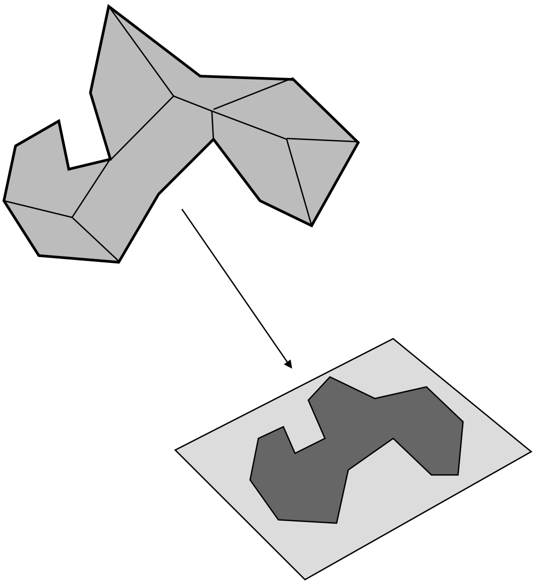

Fig. 9.1. The excess.

Definition 9.1.4. For an integer-multiplicity $T \in \mathcal{D}_M(\mathbb{R}^{M+1})$, $y \in \mathbb{R}^M$, and $\rho > 0$, the *cylindrical excess* $E(T, y, \rho)$ is defined by

$$E(T, y, \rho) = \frac{1}{2}\,\rho^{-M} \int_{\mathrm{C}(y,\rho)} |\overrightarrow{T} - \mathbf{e}^M|^2 \, d\|T\| \,, \tag{9.15}$$

where we recall that

$$T = \|T\| \wedge \overrightarrow{T} .$$

The next lemma shows the connection between equation (9.15), which defines the excess, and the more heuristic description of the excess given before the definition.

Lemma 9.1.5. *Suppose that $T \in \mathcal{D}_M(\mathbb{R}^{M+1})$ is integer-multiplicity, $y \in \mathbb{R}^M$, ℓ is a positive integer, and $\rho > 0$. If*

$$\mathbf{p}_\#(T \llcorner \mathrm{C}(y,\rho)) = \ell\, \mathbf{E}^M \llcorner \mathbb{B}^M(y,\rho)$$

and $\operatorname{spt} \partial T \subseteq \mathbb{R}^{M+1} \setminus \mathrm{C}(y,\rho)$*, then it holds that*

$$\begin{aligned} E(T,y,\rho) &= \rho^{-M} \Big(\|T\|(\mathrm{C}(y,\rho)) - \|\mathbf{p}_\# T\|(\mathbb{B}^M(y,\rho)) \Big) \\ &= \rho^{-M} \left(\|T\|(\mathrm{C}(y,\rho)) - \ell\, \Omega_M\, \rho^M \right) . \end{aligned} \tag{9.16}$$

Proof. Since $|\overrightarrow{T}| = |\mathbf{e}^M| = 1$, we have

$$\begin{aligned} |\overrightarrow{T} - \mathbf{e}^M|^2 &= |\overrightarrow{T}|^2 + |\mathbf{e}^M|^2 - 2\left(\overrightarrow{T} \cdot \mathbf{e}^M\right) \\ &= 2 - 2\left(\overrightarrow{T} \cdot \mathbf{e}^M\right) . \end{aligned}$$

So we have

$$\begin{aligned} \frac{1}{2}\int_{\mathrm{C}(y,\rho)} |\overrightarrow{T} - \mathbf{e}^M|^2 \, d\|T\| &= \int_{\mathrm{C}(y,\rho)} 1 - \left(\overrightarrow{T} \cdot \mathbf{e}^M\right) d\|T\| \\ &= \|T\|(\mathrm{C}(y,\rho)) - \|\mathbf{p}_\# T\|(\mathbb{B}^M(y,\rho)) \\ &= \|T\|(\mathrm{C}(y,\rho)) - \ell\, \Omega_M\, \rho^M . \end{aligned}$$ □

We now give two corollaries of the lemma. The first is an immediate consequence of the proof of Lemma 9.1.5 and the second shows us the effect of an isometry on the excess.

Corollary 9.1.6. *Suppose that $T \in \mathcal{D}_M(\mathbb{R}^{M+1})$ is integer-multiplicity, $y \in \mathbb{R}^M$, ℓ is a positive integer, and $\rho > 0$. If*

$$\mathbf{p}_\#(T \llcorner \mathrm{C}(y,\rho)) = \ell\, \mathbf{E}^M \llcorner \mathbb{B}^M(y,\rho)$$

and $\operatorname{spt} \partial T \subseteq \mathbb{R}^{M+1} \setminus \mathrm{C}(y,\rho)$*, then for any $\mathcal{L}^M$-measurable $B \subseteq \mathbb{B}^M(y,\rho)$, it holds that*

$$\|T\|(B \times \mathbb{R}) \le \frac{1}{2}\int_{B\times\mathbb{R}} |\overrightarrow{T} - \mathbf{e}^M|^2 \, d\|T\| + \ell\, \mathcal{L}^M(B) . \tag{9.17}$$

Proof. The corollary is an immediate consequence of the proof of the lemma. □

Corollary 9.1.7. *Suppose that* $T \in \mathcal{D}_M(\mathbb{R}^{M+1})$ *is integer-multiplicity,* $\rho > 0$,

$$\mathbf{p}_\#(T \llcorner \mathrm{C}(0,\rho)) = \ell\, \mathbf{E}^M \llcorner \mathbb{B}^M(0,\rho)\,,$$

and $\operatorname{spt} \partial T \subseteq \mathbb{R}^{M+1} \setminus \mathrm{C}(0,\rho)$.

If $1 < \lambda < \infty$, $\mathbf{j} : \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$ *is an isometry,* $0 < \rho' < \rho$, *and*

$$\operatorname{spt} \mathbf{j}_\# T \llcorner \mathrm{C}(0,\rho') \subseteq \mathbf{j}\Big(\operatorname{spt} T \llcorner \mathrm{C}(0,\rho)\Big)\,,$$

then

$$\begin{aligned} E(\mathbf{j}_\# T,\ 0,\ \rho') &\le \lambda\,(\rho/\rho')^M\, E(T,\ 0,\ \rho) \\ &\quad + \frac{\lambda}{2(\lambda-1)} \cdot (\rho/\rho')^M \cdot \ell \cdot \|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\|^{2M} \cdot E(T,\ 0,\ \rho) \\ &\quad + \frac{\lambda\,\ell\,\Omega_M}{2(\lambda-1)} \cdot (\rho/\rho')^M \cdot \|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\|^{2M}\,. \end{aligned}$$

Proof. Using

$$\Big|\textstyle\bigwedge_M \mathbf{j}\left(\overrightarrow{T}\right) - \mathbf{e}^M\Big| \le \Big|\textstyle\bigwedge_M \mathbf{j}\left(\overrightarrow{T}\right) - \textstyle\bigwedge_M \mathbf{j}\left(\mathbf{e}^M\right)\Big| + \Big|\textstyle\bigwedge_M \mathbf{j}\left(\mathbf{e}^M\right) - \mathbf{e}^M\Big|$$

and

$$\begin{aligned} (\,|\alpha| + |\beta|\,)^2 &= \lambda\,\alpha^2 + \frac{\lambda}{\lambda-1}\,\beta^2 - \left(\sqrt{\lambda-1}\,|\alpha| - |\beta|/\sqrt{\lambda-1}\right)^2 \\ &\le \lambda\,\alpha^2 + \frac{\lambda}{\lambda-1}\,\beta^2\,, \end{aligned}$$

we obtain

$$\begin{aligned} E(\mathbf{j}_\# T,\ 0,\ \rho') &\le \frac{1}{2}\,(\rho')^{-M} \int_{\mathrm{C}(0,\rho)} \Big|\textstyle\bigwedge_M \mathbf{j}\left(\overrightarrow{T}\right) - \mathbf{e}^M\Big|^2\, d\|T\| \\ &\le \frac{\lambda}{2}(\rho')^{-M} \int_{\mathrm{C}(0,\rho)} \Big|\textstyle\bigwedge_M \mathbf{j}\left(\overrightarrow{T}\right) - \textstyle\bigwedge_M \mathbf{j}\left(\mathbf{e}^M\right)\Big|^2\, d\|T\| \\ &\qquad + \frac{\lambda}{2(\lambda-1)}(\rho')^{-M} \int_{\mathrm{C}(0,\rho)} \Big|\textstyle\bigwedge_M \mathbf{j}\left(\mathbf{e}^M\right) - \mathbf{e}^M\Big|^2\, d\|T\| \\ &= \frac{\lambda}{2}(\rho')^{-M} \int_{\mathrm{C}(0,\rho)} \Big|\overrightarrow{T} - \mathbf{e}^M\Big|^2\, d\|T\| \\ &\qquad + \frac{\lambda}{2(\lambda-1)}(\rho')^{-M} \int_{\mathrm{C}(0,\rho)} \Big|\textstyle\bigwedge_M \mathbf{j}\left(\mathbf{e}^M\right) - \mathbf{e}^M\Big|^2\, d\|T\| \end{aligned}$$

$$\leq \frac{\lambda}{2}(\rho')^{-M}\int_{\mathrm{C}(0,\rho)}\left|\overrightarrow{T}-\mathbf{e}^M\right|^2 d\|T\|$$

$$+\frac{\lambda}{2(\lambda-1)}(\rho')^{-M}\|\,\mathbf{j}-\mathbf{I}_{\mathbb{R}^{M+1}}\,\|^{2M}\,\|T\|\mathrm{C}(0,\rho)\,,$$

and the result follows from Lemma 9.1.5. □

Notation 9.1.8. Certain hypotheses will occur frequently in what follows, so we collect them here (with labels) for easy reference:

(H1) $\operatorname{spt}\partial T\subseteq\mathbb{R}^{M+1}\setminus\mathrm{C}(y,\rho)$,
(H2) $\mathbf{p}_{\#}[T\,\llcorner\,\mathrm{C}(y,\rho)]=\mathbf{E}^M\,\llcorner\,\mathbb{B}^M(y,\rho)$,
(H3) $\Omega_M\,r^M\leq\|T\|\{X\in\mathbb{R}^{M+1}:|X-Y|<r\}$ holds whenever $Y\in\operatorname{spt}T$ and $\{X\in\mathbb{R}^{M+1}:|X-Y|<r\}\cap\operatorname{spt}\partial T=\emptyset$,
(H4) $E(T,y,\rho)<\epsilon$,
(H5) T is mass-minimizing.

Here ρ and ϵ are positive and $y\in\mathbb{R}^M$.

Note that the constancy theorem, i.e., Proposition 7.3.1, implies that if $\operatorname{spt}T\subseteq\mathbb{R}^{M+1}\setminus\mathrm{C}(y,\rho)$, then, because $\partial\mathbf{p}_{\#}T=\mathbf{p}_{\#}\partial T$, we have

$$\mathbf{p}_{\#}(T\,\llcorner\,\mathrm{C}(y,\rho))=\ell\,\mathbf{E}^M\,\llcorner\,\mathbb{B}^M(y,\rho)\,,\tag{9.18}$$

where ℓ is an integer. So in (H2) we are making the simplifying assumption that $\ell=1$.

Note that (H5) allows us to apply Theorem 8.4.3 to obtain (H3), so (H3) is, in fact, a consequence of (H5).

9.2 The Height Bound and Lipschitz Approximation

We begin this section with the height bound lemma. The proof we give is simplified by using hypothesis (H3). While the height bound lemma remains true for currents minimizing the integral of an integrand other than area, the proof is more difficult because the lower bound on mass that they satisfy (see Theorem 8.4.5) is weaker than that in (H3).

Lemma 9.2.1 (Height bound). *For each σ with $0<\sigma<1$, there are $\epsilon_0=\epsilon_0(M,\sigma)$ and $c_1=c_1(M,\sigma)$ such that the hypotheses* (H1–H4), *with $\epsilon=\epsilon_0$ in* (H4), *imply*

$$\sup\Big\{|\mathbf{q}\,(X_1)-\mathbf{q}\,(X_2)|:X_1,X_2\in\operatorname{spt}T\cap\mathrm{C}(y,\sigma\rho)\Big\}$$

$$\leq\,c_1\,\rho\,\Big(E(T,y,\rho)\Big)^{\frac{1}{2M}}\,.$$

Proof. By using a translation and homothety if need be, we may assume that $y = 0$ and $\rho = 1$. We write

$$E = E(T, 0, 1)\,.$$

Set

$$r_0 = \tfrac{1}{2}(1 - \sigma) \tag{9.19}$$

and

$$\epsilon_0 = 2^{-M}\,\Omega_M\,(1 - \sigma)^M\,. \tag{9.20}$$

First we consider points whose projections onto $\mathbb{B}^M(0, 1)$ are separated by a distance less than $2\,r_0$. So suppose that $X_1, X_2 \in \operatorname{spt} T \cap \mathrm{C}(0, \sigma)$ are such that

$$\tfrac{1}{2}\Big|\mathbf{p}\,(X_1) - \mathbf{p}\,(X_2)\Big| < r_0\,.$$

We set

$$r = \tfrac{1}{2}\Big|\mathbf{p}\,(X_1) - \mathbf{p}\,(X_2)\Big|\,, \qquad h = \tfrac{1}{2}\Big|\mathbf{q}\,(X_1) - \mathbf{q}\,(X_2)\Big|\,.$$

Then we have

$$\Big|X_1 - X_2\Big| = 2\sqrt{r^2 + h^2}\,.$$

We set

$$s = \min\{\sqrt{r^2 + h^2} - r\,,\; r_0\}\,.$$

Then we have

$$\mathbb{B}(X_1, r + s) \bigcap \mathbb{B}(X_2, r + s) = \emptyset$$

and

$$\mathbb{B}(X_1, r + s) \bigcup \mathbb{B}(X_2, r + s) \subseteq \mathrm{C}(0, 1)\,.$$

Setting

$$x^* = \tfrac{1}{2}\,(\mathbf{p}\,(X_1) + \mathbf{p}\,(X_2))\;,$$

so that

$$\Big|\mathbf{p}\,(X_1) - x^*\Big| = \Big|\mathbf{p}\,(X_2) - x^*\Big| = r\,,$$

we see (Figure 9.2) that

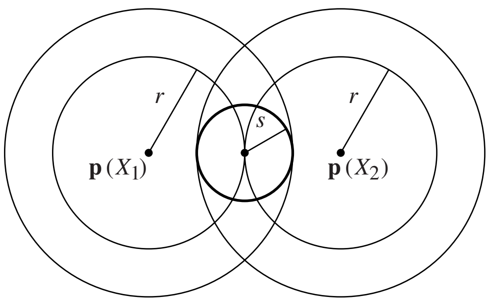

Fig. 9.2. The projections of the balls.

$$\mathbb{B}^M(x^*, s) \subseteq \mathbf{p}\,(\mathbb{B}(X_1, r+s)) \bigcap \mathbf{p}\,(\mathbb{B}(X_2, r+s))$$

and thus that

$$\mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_1, r+s)) \bigcap \mathbf{p}\,(\mathbb{B}(X_2, r+s))\,\Big] \geq \Omega_M\, s^M\,.$$

By (H3) we have

$$\begin{aligned}\|T\|\,\mathbb{B}(X_1, r+s) + \|T\|\,\mathbb{B}(X_2, r+s) &\geq 2\,\Omega_M\,(r+s)^M \\ &= \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_1, r+s))\,\Big] + \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_2, r+s))\,\Big].\end{aligned}$$

Thus we have

$$\begin{aligned}E &\geq \|T\|\,\Big[\,\mathbb{B}(X_1, r+s) \bigcup \mathbb{B}(X_2, r+s)\,\Big] \\ &\qquad - \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_1, r+s)) \bigcup \mathbf{p}\,(\mathbb{B}(X_2, r+s))\,\Big] \\ &\geq \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_1, r+s))\,\Big] + \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_2, r+s))\,\Big] \\ &\qquad - \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_1, r+s)) \bigcup \mathbf{p}\,(\mathbb{B}(X_2, r+s))\,\Big] \\ &= \mathcal{L}^M\Big[\,\mathbf{p}\,(\mathbb{B}(X_1, r+s)) \bigcap \mathbf{p}\,(\mathbb{B}(X_2, r+s))\,\Big] \geq \Omega_M\, s^M\,.\end{aligned}$$

We now consider two possibilities.

Case 1. $s = r_0$,
Case 2. $s = \sqrt{r^2 + h^2} - r < r_0$.

In Case 1, by the definition of r_0, i.e., (9.19), the definition of ϵ_0, i.e., (9.20), and (H4), we have

$$E \geq \Omega_M\, s^M = \Omega_M\, r_0^M = 2^{-M}\,\Omega_M\,(1-\sigma)^M = \epsilon_0 > E\,,$$

a contradiction. Thus we may assume that Case 2 holds.

In Case 2, we note that

$$\begin{aligned}h &\leq \sqrt{r^2 + h^2} \\ &\leq (\sqrt{r^2 + h^2} - r) + r_0 \\ &\leq 2\, r_0\,.\end{aligned}$$

Then it follows that

$$\begin{aligned}E &\geq \Omega_M\, s^M \\ &= \Omega_M\, (\sqrt{r^2+h^2} - r)^M \\ &= \Omega_M\, \left(\frac{(r^2+h^2)-r^2}{\sqrt{r^2+h^2}+r}\right)^M \\ &\geq \Omega_M\, \left(\frac{h^2}{\sqrt{r_0^2+4r_0^2}+r_0}\right)^M \\ &\geq \Omega_M\, 2^{-M}\,(1-\sigma)^{-M}\,h^{2M}\,,\end{aligned}$$

where we obtain the last inequality by using the definition of r_0, i.e., (9.19), and, for simplicity, we have replaced $\sqrt{5}+1$ by the larger number 4.

We have shown that any two points in $\operatorname{spt} T \cap \mathrm{C}(0,\sigma)$ whose projections onto $\mathbb{B}^M(0,1)$ are separated by a distance less than $2\,r_0$ will have their projections by $\mathbf{q}$ separated by less than

$$2^{1/2}\,\Omega_M^{-1/(2M)}\,(1-\sigma)^{1/2}\,E^{1/(2M)}\,.$$

But any two points x_1 and x_2 in $\mathbb{B}^M(0,\sigma)$ are separated by a distance less than $2\,\sigma$, so if the two points are separated by more than $2\,r_0 = (1-\sigma)$, then we can form a sequence of points $z_1 = x_1, z_2, \ldots, z_M = x_2$ such that $|z_{i+1} - z_i| \leq (1-\sigma) = 2r_0$. We can take L to be the smallest integer exceeding $2\,\sigma/(1-\sigma)$. Thus we have

$$L \leq 1 + \frac{2\,\sigma}{1-\sigma} = \frac{1+\sigma}{1-\sigma} < \frac{2}{1-\sigma}\,.$$

Hence we may set

$$\begin{aligned}c_1(M,\sigma) &= L\cdot 2^{1/2}\,\Omega_M^{-1/(2M)}\,(1-\sigma)^{1/2} \\ &\leq 2^{3/2}\,\Omega_M^{-1/(2M)}\,(1-\sigma)^{-1/2}\,.\end{aligned}$$ □

Lemma 9.2.2 (Lipschitz approximation). *Let γ with $0 < \gamma \leq 1$ be given. There exist constants c_2, c_3, and c_4 such that the following holds:*

If the hypotheses (H1–H4) *are satisfied with $\epsilon = \epsilon_0(M, 2/3)$ in* (H4), *where $\epsilon_0(M, 2/3)$ is as in Lemma* 9.2.1, *then there is a Lipschitz function $g : \mathbb{B}^M(y, \rho/4) \to \mathbb{R}$ satisfying the following conditions:*

$$\operatorname{Lip} g \leq \gamma, \tag{9.21}$$

$$\sup\Big\{|g(z) - g(y)| : z \in \mathbb{B}^M(y, \rho/4)\Big\} \leq c_2\,\rho\,\Big(E(T, y, \rho)\Big)^{\frac{1}{2M}}, \tag{9.22}$$

$$\mathcal{L}^M\Big[\,\mathbb{B}^M(y,\rho/4)\setminus\Big\{z\in\mathbb{B}^M(y,\rho/4):\mathbf{p}^{-1}(z)\cap\operatorname{spt}T=\{(z,g(z))\}\Big\}\Big]$$

$$\leq\ \rho^M\,c_3\,\gamma^{-2M}\,E(T,y,\rho),\tag{9.23}$$

$$\|T-T^g\|\mathrm{C}(y,\rho/4)\leq\rho^M\,c_4\,\gamma^{-2M}\,E(T,y,\rho)\,,\tag{9.24}$$

where

$$T^g=G_\#\Big(\mathbf{E}^M\llcorner\mathbb{B}^M(y,\rho/4)\Big),\tag{9.25}$$

with $G:\mathbb{B}^M(y,\rho/4)\to\mathrm{C}(y,\rho/4)$ *defined by*

$$G(x)=(x,g(x))\,,\qquad \textit{for } x\in\mathbb{B}^M(y,\rho/4)\,.$$

Proof. Fix the choice of $0<\gamma\leq 1$ and specify a value of ϵ_0 for which the conclusion of Lemma 9.2.1 holds with σ chosen to equal $2/3$. That is, if the hypotheses (H1–H4) hold with $\epsilon=\epsilon_0$ and with z and δ in place of y and ρ, respectively, then

$$\sup\Big\{|\mathbf{q}\,(X_1)-\mathbf{q}\,(X_2)|:X_1,\ X_2\in\operatorname{spt}T\cap\mathrm{C}(z,2\delta/3)\Big\}$$

$$\leq\ c_1\,\delta\,\Big(E(T,z,\delta)\Big)^{\frac{1}{2M}}\,.\tag{9.26}$$

Consider η with

$$0<\eta<\epsilon_0\,.\tag{9.27}$$

Set

$$A=\Big\{z\in\mathbb{B}^M(y,\rho/4)\ :\ E(T,z,\delta)\leq\eta\text{ for all }\delta\text{ with }0<\delta<3\rho/4\Big\}\,,\tag{9.28}$$

and set

$$B=\mathbb{B}^M(0,\rho/4)\setminus A\,.$$

For each $b\in B$ there exists $\delta(b)$ with $0<\delta(b)<3\rho/4$ such that the excess $E(T,b,\delta(b))$ is greater than η, that is,

$$\frac{1}{2}\int_{\mathrm{C}(b,\delta(b))}|\overrightarrow{T}-\mathbf{e}^M|^2\,d\|T\|=\delta(b)^M\cdot E(T,b,\delta(b))>\eta\cdot\delta(b)^M\,.\tag{9.29}$$

Applying the Besicovitch covering theorem (i.e., Theorem 4.2.12) to the family of closed balls

$$\mathcal{B}=\Big\{\overline{\mathbb{B}}^M(b,\delta(b))\ :\ b\in B\Big\}\,,$$

we obtain the subfamilies $\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_K$ of $\mathcal{B}$ such that each $\mathcal{B}_i$ consists of pairwise disjoint balls and

$$B\subseteq\bigcup_{i=1}^K B_i\,,$$

where

$$B_i = \bigcup_{\overline{\mathbb{B}}^M(b,\delta(b))\in\mathcal{B}_i} \overline{\mathbb{B}}^M(b, \delta(b)) .$$

Here K is a number that depends only on the dimension M. Using (9.29), we see that, for each $i = 1, 2, \ldots, K$, we have

$$\begin{aligned}
\eta\,\mathcal{L}^M(B_i) &= \eta \sum_{\overline{\mathbb{B}}^M(b,\delta(b))\in\mathcal{B}_i} \Omega_M \Big[\,\delta(b)\,\Big]^M \\
&< \Omega_M \sum_{\overline{\mathbb{B}}^M(b,\delta(b))\in\mathcal{B}_i} \delta(b)^M\; E(T, b, \delta(b)) \\
&= \frac{1}{2}\,\Omega_M \int_{B_i} |\overrightarrow{T} - \mathbf{e}^M|^2\, d\|T\| \\
&\le \frac{1}{2}\,\Omega_M \int_{\mathrm{C}(y,\rho)} |\overrightarrow{T} - \mathbf{e}^M|^2\, d\|T\| .
\end{aligned}$$

We conclude that

$$\begin{aligned}
\eta\,\mathcal{L}^M(B) &\le \sum_{i=1}^{K} \eta\,\mathcal{L}^M\Big(\bigcup_i B_i\Big) \\
&\le \frac{K}{2}\,\Omega_M \int_{\mathrm{C}(y,\rho)} |\overrightarrow{T} - \mathbf{e}^M|^2\, d\|T\| \\
&= c_5\,\rho^M\, E(T, y, \rho) . \qquad (9.30)
\end{aligned}$$

If $x_1, x_2 \in \mathbb{B}^M(0, \rho/4) \cap A$, and if X_1, X_2 are points with

$$X_i \in \operatorname{spt} T \cap \mathbf{p}^{-1}(x_i),\ \ i = 1, 2,$$

then

$$|x_1 - x_2| < \rho/2 ,$$

so we can apply (9.26) with $z = x_1$ and with δ chosen to satisfy

$$3\,|x_1 - x_2|/2 < \delta < 3\rho/4 . \qquad (9.31)$$

Letting δ in (9.31) decrease to $3\,|x_1 - x_2|/2$, we conclude that

$$|\mathbf{q}\,(X_1) - \mathbf{q}\,(X_2)| \le c_6\,\eta^{1/(2M)}\,|x_1 - x_2| , \qquad (9.32)$$

where we set

$$c_6 = \max\{\,3/2,\ (3/2)\,c_1,\ \epsilon_0^{-1}\,\} . \qquad (9.33)$$

Thus we may choose

$$\eta = \gamma^{2M} c_6^{-2M} \le c_6^{-2M} < c_6^{-1} \le \epsilon_0 , \tag{9.34}$$

so that $c_6 \, \eta^{1/(2M)} = \gamma$ holds, and consequently we have

$$|\mathbf{q}(X_1) - \mathbf{q}(X_2)| \le \gamma \, |x_1 - x_2| \tag{9.35}$$

for any points

$$x_1, x_2 \in \mathbb{B}^M(0, \rho/4) \bigcap A ,$$

where

$$X_1 \in \operatorname{spt} T \bigcap \mathbf{p}^{-1}(x_1) \text{ and } X_2 \in \operatorname{spt} T \bigcap \mathbf{p}^{-1}(x_2) .$$

In particular, (9.35) shows that, for any $x \in A \cap \mathbb{B}^M(0, \rho/4)$, there is exactly one $X \in \mathbf{p}^{-1}(x) \bigcap \operatorname{spt} T$. Thus, we can define $g^* : A \bigcap \mathbb{B}^M(0, \rho/4) \to \mathbb{R}$ by requiring

$$\Big\{ (x, g^*(x)) \Big\} = \mathbf{p}^{-1}(x) \bigcap \operatorname{spt} T, \quad \text{whenever } x \in A \bigcap \mathbb{B}^M(0, \rho/4) .$$

Inequality (9.35) tells us that $\operatorname{Lip}(g^*) \le \gamma$ holds on $A \bigcap \mathbb{B}^M(y, \rho/4)$, so by Kirszbraun's extension theorem (see [KPk 99, Theorem 5.2.2]) g^* extends to $g^{**} : \mathbb{B}^M(y, \rho/4) \to \mathbb{R}$ with the same Lipschitz constant.

By Lemma 9.2.1, if we set

$$g = \min \Big\{ \alpha, \ \max\{ \beta, \ g^{**} \} \Big\} ,$$

where

$$\alpha = g(y) - c_1 \, E^{1/(2M)}(T, y, \rho) \, \rho, \quad \beta = g(y) + c_1 \, E^{1/(2M)}(T, y, \rho) \, \rho ,$$

then

$$\Big\{ (x, g(x)) \Big\} = \mathbf{p}^{-1}(x) \bigcap \operatorname{spt} T \text{ whenever } x \in A \bigcap \mathbb{B}^M(0, \rho/4)$$

and

$$\sup \Big\{ |g(x) - g(y)| : \mathbb{B}^M(y, \rho/4) \Big\} \le c_1 \, E^{1/(2M)}(T, y, \rho) \, \rho$$

will both hold.

Using (9.17), (9.30), and (9.34), we see that

$$\begin{aligned}
&\|T\| \Big[(\mathbb{B}^M(y, \rho/4) \setminus A) \times \mathbb{R} \Big] \\
&\quad = \mathcal{L}^M \Big[\mathbb{B}^M(y, \rho/4) \setminus A \Big] + \frac{1}{2} \int_{(\mathbb{B}^M(y,\rho/4)\setminus A)\times\mathbb{R}} |\overrightarrow{T} - \mathbf{e}^M|^2 \, d\|T\| \\
&\quad \le \mathcal{L}^M[B] + \frac{1}{2} \int_{\mathrm{C}(y,\rho)} |\overrightarrow{T} - \mathbf{e}^M|^2 \, d\|T\| \\
&\quad \le (\eta^{-1} c_5 + 1) \, \rho^M \, E(T, y, \rho) \\
&\quad = (c_5 \, c_6^{2M} \, \gamma^{-2M} + 1) \, \rho^M \, E(T, y, \rho) \\
&\quad \le (c_5 \, c_6^{2M} + 1) \, \gamma^{-2M} \, \rho^M \, E(T, y, \rho) .
\end{aligned}$$

So we conclude that (9.23) holds with $c_3 = c_5\, c_6^{2M} + 1$.

Finally, we have

$$\begin{aligned}\|T - T^g\|\mathrm{C}(y, \rho/4) &\le \|T\|\Big[\, (\mathbb{B}^M(y, \rho/4) \setminus A) \times \mathbb{R}\,\Big] \\ &\qquad + \|T^g\|\Big[\, (\mathbb{B}^M(y, \rho/4) \setminus A) \times \mathbb{R}\,\Big] \\ &\le \|T\|\Big[\, (\mathbb{B}^M(y, \rho/4) \setminus A) \times \mathbb{R}\,\Big] + \gamma\, \mathcal{L}^M[B] \\ &\le 2\,(c_5\, c_6^{2M} + 1)\, \gamma^{-2M}\, \rho^M\, E(T, y, \rho)\,,\end{aligned}$$

so we see that (9.24) holds with $c_4 = 2\,(c_5\, c_6^{2M} + 1)$. □

9.3 Currents Defined by Integrating over Graphs

Currents obtained by integration over the graph of a function are particularly nice and are helpful to our intuitive understanding of the concepts being developed here. We will show how the cylindrical excess of such a current relates to a familiar quantity from analysis, namely the Dirichlet integral (see Corollary 9.3.7).

Notation 9.3.1. Let $f : \mathbb{B}^M(0, \sigma) \to \mathbb{R}$ be Lipschitz.

(1) We use the notation F for the function from $\mathbb{B}^M(0, \sigma)$ to $\mathbb{R}^{M+1}$ given by $F(x) = (x, f(x))$.
(2) We use the notation G_F for the M-dimensional current that is defined by integration over the graph of f, that is,

$$G_F = F_\#(\mathbf{E}^M \llcorner \mathbb{B}^M(0, \sigma))\,.$$

Writing

$$J_F(x) = \langle \textstyle\bigwedge_M (DF(x)),\, \mathbf{e}^M \rangle\,,$$

we have

$$G_F[\psi] = \int_{\mathbb{B}^M(0,\sigma)} \langle\, \psi(x, f(x)),\, J_F(x)\, \rangle\, d\mathcal{L}^M(x) \tag{9.36}$$

for any differential M-form ψ defined on $\mathrm{C}(0, \sigma)$.

Lemma 9.3.2. *If $f : \mathbb{B}^M(0, \sigma) \to \mathbb{R}$ is Lipschitz, then we have*

$$\overrightarrow{G}_F(F(x)) = (1 + |Df|^2)^{-1/2} \left(\mathbf{e}^M + \sum_{i=i}^{M} \frac{\partial f}{\partial x_i}\, \mathbf{e}_{\hat{\imath}} \right), \tag{9.37}$$

$$DA(\overrightarrow{G}_F) = (1 + |Df|^2)^{-1/2} \left(dx^M + \sum_{i=1}^{M} \left(\frac{\partial f}{\partial x_i} \right) dx_{\hat{\imath}} \right), \tag{9.38}$$

$$DA(\overrightarrow{G}_F) - DA(\mathbf{e}^M) =$$

$$(1 + |Df|^2)^{-1/2} \left(dx^M + \sum_{i=1}^{M} \left(\frac{\partial f}{\partial x_i} \right) dx_{\hat{\imath}} \right) - dx^M \,. \tag{9.39}$$

Proof. By definition, we have

$$\langle \bigwedge_M (DF(x)),\ \mathbf{e}^M \rangle = \bigwedge_{i=1}^{M} \left(\mathbf{e}_i + \frac{\partial f}{\partial x_i} \mathbf{e}_{M+1} \right) .$$

So

$$J_F = \mathbf{e}^M + \sum_{i=i}^{M} \frac{\partial f}{\partial x_i} \mathbf{e}_{\hat{\imath}} . \tag{9.40}$$

We obtain (9.37) from (9.40) by dividing by the norm of J_F. Equation (9.38) follows from (9.37) and (9.7). Equation (9.39) follows from (9.38) and (9.8). □

For the record, we note that the coefficient of dx^M in (9.39) is

$$(1 + |Df|^2)^{-1/2} - 1 \,.$$

Lemma 9.3.3. *Define a map from* $\mathbb{R}^M$ *to* $\mathbb{R}^{M+1}$ *by*

$$x = (x_1, x_2, \ldots, x_M) \longmapsto X = (1 + |x|^2)^{-1/2} \,(1, x_1, x_2, \ldots, x_M) \,.$$

If A and B are the images of a and b under this map then

(1) $|A - B| \le |a - b|$;
(2) *for each* $0 < c < \infty$*, it holds that*
$|a|, |b| \le c$ *implies* $|a - b| \le (1 + c^2)^2 \,|A - B|$.

Proof. The mapping $x \mapsto X$ is the composition of two mappings: the distance-preserving map

$$x = (x_1, x_2, \ldots, x_k) \longmapsto (1, x_1, x_2, \ldots, x_k)$$

followed by the radial projection onto the unit sphere

$$y = (y_1, y_2, \ldots, y_{k+1}) \longmapsto |y|^{-1} \,(y_1, y_2, \ldots, y_{k+1}) \,.$$

Part (1) follows from the fact that the radial projection does not increase the distance between points that are outside of the open unit ball.

To prove (2), we note that

$$|1 + a \cdot b| \le (1 + |a|^2)^{1/2} \,(1 + |b|^2)^{1/2}$$

holds, with equality if and only if $a = b$. Thus

$$0 < (1 + |a|^2)^{1/2}\,(1 + |b|^2)^{1/2} + (1 + a \cdot b)$$

always holds, so we may compute

$$\begin{aligned}
&(1 + |a|^2)^{1/2}\,(1 + |b|^2)^{1/2}\,|A - B|^2 \\
&\qquad = 2\Big[\,(1 + |a|^2)^{1/2}\,(1 + |b|^2)^{1/2} - (1 + a \cdot b)\,\Big] \\
&\qquad = 2\Big[\,(1 + |a|^2)^{1/2}\,(1 + |b|^2)^{1/2} + (1 + a \cdot b)\,\Big]^{-1} \\
&\qquad\qquad \cdot\Big[\,(1 + |a|^2)\,(1 + |b|^2) - (1 + a \cdot b)^2\,\Big] \\
&\qquad = 2\Big[\,(1 + |a|^2)^{1/2}\,(1 + |b|^2)^{1/2} + (1 + a \cdot b)\,\Big]^{-1} \\
&\qquad\qquad \cdot\Big[\,|a - b|^2 + |a|^2\,|b|^2 - (a \cdot b)^2\,\Big] \\
&\qquad \geq 2\Big[\,(1 + |a|^2)^{1/2}\,(1 + |b|^2)^{1/2} + (1 + a \cdot b)\,\Big]^{-1}\,|a - b|^2\,.
\end{aligned}$$

The estimate in (2) now follows readily. □

Proposition 9.3.4. *We have*

$$\left|\overrightarrow{G}_F(F(x)) - \overrightarrow{G}_F(F(y))\right| \leq |Df(x) - Df(y)| \tag{9.41}$$

and, provided $|Df(x)|, |Df(y)| \leq c$*, we have*

$$|Df(x) - Df(y)| \leq (1 + c^2)^2\,\left|\overrightarrow{G}_F(F(x)) - \overrightarrow{G}_F(F(y))\right|\,. \tag{9.42}$$

Proof. This result follows immediately from Lemma 9.3.3 and (9.37). □

We leave the easy proof of the next lemma to the reader.

Lemma 9.3.5. *For* $t \in \mathbb{R}$ *we have*

$$0 \leq 1 - (1 + t^2)^{-1/2} \leq \min\{\tfrac{1}{2}t^2\,,\ |t|\}\,. \tag{9.43}$$

If additionally $|t| \leq C < \infty$ *holds, then we have*

$$\frac{t^2}{2(1 + C^2)} \leq 1 - (1 + t^2)^{-1/2}\,. \tag{9.44}$$

Proposition 9.3.6. *It holds that*

$$[1 + \mathrm{Lip}\,(f)]^{-2}\ |Df|^2 \leq \left|\overrightarrow{G}_F - \mathbf{e}^M\right|^2 \leq \min\Big\{\,|Df|^2,\ 2|Df|\,\Big\}\,. \tag{9.45}$$

Proof. By (9.37) we have

$$\overrightarrow{G}_F - \mathbf{e}^M = (1+|Df|^2)^{-1/2}\left[(1-(1+|Df|^2)^{1/2})\mathbf{e}^M + \sum_{i=1}^{M} \frac{\partial f}{\partial x_i}\, \mathbf{e}_{\hat{\imath}} \right],$$

so

$$\begin{aligned} |\overrightarrow{G}_F - \mathbf{e}^M|^2 &= (1+|Df|^2)^{-1}\Big[1 - 2(1+|Df|^2)^{1/2} + (1+|Df|^2) + |Df|^2 \Big] \\ &= (1+|Df|^2)^{-1}\Big[2(1+|Df|^2) - 2(1+|Df|^2)^{1/2} \Big] \\ &= 2\Big[1 - (1+|Df|^2)^{-1/2} \Big]. \end{aligned}$$

The upper bound follows from (9.43), while the lower bound follows from (9.44). □

Corollary 9.3.7. *It holds that*

$$\begin{aligned} 2^{-1}\,[1+\mathrm{Lip}\,(f)]^{-2}\,\sigma^{-M} \int_{\mathbb{B}^M(0,\sigma)} |Df|^2\, d\mathcal{L}^M &\le E(G_F, 0, \sigma) \\ &\le\; 2^{-1}\,\sigma^{-M} \int_{\mathbb{B}^M(0,\sigma)} |Df|^2\, d\mathcal{L}^M . \end{aligned}$$

Proof. The corollary is an immediate consequence of Proposition 9.3.6 and the definition of the cylindrical excess, i.e., Definition 9.1.4. □

Proposition 9.3.8. *We have*

$$\left| DA(\overrightarrow{G}_F) - DA(\mathbf{e}^M) \right| \le \min\left\{ |Df|^2,\ 2\,|Df| \right\}. \tag{9.46}$$

Proof. By (9.39), we have

$$\begin{aligned} &DA(\overrightarrow{G}_F) - DA(\mathbf{e}^M) \\ &\quad = (1+|Df|^2)^{-1/2}\left[(1-(1+|Df|^2)^{1/2} dx^M + \sum_{i=1}^{M} \left(\frac{\partial f}{\partial x_i}\right) dx_{\hat{\imath}} \right], \end{aligned}$$

so we can proceed as in the proof of Proposition 9.3.6 and apply (9.43). □

9.4 Estimates for Harmonic Functions

The heuristic behind the regularity theory for area-minimizing surfaces is that, at a point where an area-minimizing surface is horizontal, the closer you look at the surface, the more it looks like the graph of a harmonic function. This is made plausible by the fact that an area-minimizing graph is given by a function u that minimizes the integral of the area integrand

$$\sqrt{1+|Du|^2}\,,$$

while a harmonic function u minimizes the integral of

$$\frac{1}{2}|Du|^2\,.$$

Since the area integrand $\sqrt{1+|Du|^2}$ has the expansion

$$1+\frac{1}{2}|Du|^2+\sum_{k=2}^{\infty}\binom{1/2}{k}|Du|^{2k}\,,$$

we see that, at a point where the graph is horizontal, minimizing $\frac{1}{2}|Du|^2$ must be nearly the same as minimizing $\sqrt{1+|Du|^2}$.

To turn the heuristic discussion above into a useful estimate, we will need to investigate the boundary regularity of solutions for the Dirichlet problem[2] for Laplace's equation[3] on the unit ball. To obtain a sharp result we must use the Lipschitz spaces that we introduce next.

Notation 9.4.1. Let B denote the open unit ball in $\mathbb{R}^M$ and let Σ denote the unit sphere.

(1) For $g : \Sigma \to \mathbb{R}$, we say that g is *differentiable at* $x \in \Sigma$ if G defined by

$$G(z) = g(z/|z|) \qquad (z \neq 0)$$

is differentiable at x. This definition exploits the special structure of Σ, but it is easily seen to be equivalent to the usual definition of differentiability for a function defined on a surface (for example, see [Hir 76, pp. 15ff.]).

(2) If $g : \Sigma \to \mathbb{R}$ is differentiable at $x \in \Sigma$ and if v a unit vector, then the *directional derivative of* g *at* x *in the direction* v is defined by

$$\frac{\partial g}{\partial v}(x) = \langle\, DG(x),\; v\,\rangle\,. \tag{9.47}$$

We will also use (9.47) as the definition of $\partial g/\partial v$ when v is not a unit vector.

(3) For δ with $1 < \delta < 2$, we say that $g : \Sigma \to \mathbb{R}$ is *Lipschitz of order* δ, written $g \in \Lambda_\delta(\Sigma)$, if g is differentiable at every point of Σ, $\frac{\partial g}{\partial v}(x)$ is a continuous function of x for each unit vector v, and there exists $C < \infty$ such that for each unit vector v,

$$\left|\frac{\partial g}{\partial v}(x_1) - \frac{\partial g}{\partial v}(x_0)\right| \le C\,|x_1 - x_0|^{\delta-1}$$

holds for $x_0, x_1 \in \Sigma$.

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[2] Johann Peter Gustav Lejeune Dirichlet (1805–1859).

[3] Pierre-Simon Laplace (1749–1827).

(4) If $g : \Sigma \to \mathbb{R}$ is Lipschitz of order δ on Σ ($1 < \delta < 2$), then we set

$$\|g\|_{\Lambda_\delta} = \sup_{\substack{x\in\Sigma \\ |v|=1}} \left|\frac{\partial g}{\partial v}(x)\right| + \sup_{\substack{x_0,x_1\in\Sigma,\, x_0\neq x_1 \\ |v|=1}} |x_1 - x_0|^{1-\delta} \left|\frac{\partial g}{\partial v}(x_1) - \frac{\partial g}{\partial v}(x_0)\right| . \tag{9.48}$$

The number $\|g\|_{\Lambda_\delta}$ defines a seminorm on $\Lambda_\delta(\Sigma)$. Had we wished to define a norm, we could have done so by including the term $\sup_{x\in\Sigma} |g(x)|$ as an additional summand on the right-hand side of (9.48).

We have defined the Lipschitz spaces $\Lambda_\delta(\Sigma)$ for δ in the limited range $1 < \delta < 2$ because those are the only spaces we will need in this section. For a comprehensive study of Lipschitz spaces, the reader should see [Kra 83].

Lemma 9.4.2. *For δ with $1 < \delta < 2$ there exists a constant $c_7 = c_7(\delta)$ with the following property:*

If $g \in \Lambda_\delta(\Sigma)$ and if $u \in C^0(\overline{B}) \bigcap C^2(B)$ satisfies

$$\begin{aligned} \triangle u &= 0 \ \ on \ \ B\,, \\ u &= g \ \ on \ \ \Sigma\,, \end{aligned} \tag{9.49}$$

then the Hilbert–Schmidt norm of the Hessian matrix of u (i.e., the square root of the sum of the squares of the entries in the matrix) is bounded by

$$\Big|\text{Hess}\,[u(x)]\Big| \le c_7 \cdot \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2} . \tag{9.50}$$

Here, of course, $\triangle$ denotes the Laplacian $\sum_{i=1}^M \partial^2/\partial x_i^2$.

Proof. Our proof will be based on the fact that the function u solving (9.49) is given by the Poisson integral formula.[4] Recall (see [CH 62, pp. 264ff.], [Kra 99, p. 186], or [Kra 05, p. 143]) that the Poisson kernel for the unit ball in $\mathbb{R}^M$ is given by

$$P(x, y) = \frac{\Gamma(M/2)}{2\pi^{M/2}} \cdot \frac{1 - |x|^2}{|x - y|^M} \tag{9.51}$$

$$= \frac{\Gamma(M/2)}{2\pi^{M/2}} \cdot \frac{\varrho(x)\,(2 - \varrho(x))}{|x - y|^M} , \tag{9.52}$$

where

$$\varrho(x) = 1 - |x|$$

[4] Siméon Denis Poisson (1781–1840).

is the distance from $x \in B$ to Σ. The solution to the Dirichlet problem (9.49) is given by

$$u(x) = \int_{\Sigma} P(x, y)\, g(y)\, d\mathcal{H}^{M-1}(y)\,. \tag{9.53}$$

Interior estimate. Observe that if $x \in B$ stays at least a fixed positive distance away from Σ, then each $|\partial P/\partial x_i|$ (and all higher derivatives of P as well) will be bounded above. Thus we can obtain estimates for the derivatives of u by differentiating the right-hand side of (9.53) under the integral and estimating the resulting integral. Thus we have (9.50) for $x \in \mathbb{B}^M(0, 1/2)$.

Notation. For $v \in \mathbb{R}^M$ a unit vector, $\partial f/\partial v$ will denote the *directional derivative* of the function f in the direction v. Here f may be real-valued or vector-valued.

Of particular interest are the directional derivatives of the Poisson kernel $P(x, y)$. Since P depends on the two arguments $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^M$, we will augment our notation for directional derivatives to indicate the variable with respect to which the differentiation is to be performed. The notation $\partial P/\partial_x v$ will mean that the directional derivative of $P(x, y)$ in the direction v is to be computed by differentiating with respect to x while treating y as a parameter. We have

$$\frac{\partial P}{\partial_x v} = \sum_{i=1}^{M} v_i \, \frac{\partial P}{\partial x_i}\,. \tag{9.54}$$

On the other hand, when we wish to differentiate $P(x, y)$ as a function of y while treating x as a parameter, we will write $\partial P/\partial_y v$. We have

$$\frac{\partial P}{\partial_y v} = \sum_{i=1}^{M} v_i \, \frac{\partial P}{\partial y_i}\,. \tag{9.55}$$

Equations (9.54) and (9.55) remain meaningful when v is not a unit vector, and later we will have occasion to apply (9.55) in such a circumstance.

Estimates for derivatives of *P*. Fix a point $x \in B \setminus \{0\}$. Let y be a point on Σ. Using (9.51), we compute the derivatives of $P(x, y)$ as follows: Let v be a unit vector. Since

$$\frac{\partial x}{\partial v} = v$$

(that is, the directional derivative, in the direction v, of the map $x \mapsto x$ is v itself), we have

$$\frac{\partial P}{\partial_x v}(x, y) = \frac{\Gamma(M/2)}{2\,\pi^{M/2}} \cdot \left(-\frac{2\, x \cdot v}{|x-y|^M} - \frac{M\,(1-|x|^2)\,(x-y)\cdot v}{|x-y|^{M+2}} \right).$$

Similarly, we find that

$$\frac{\partial P}{\partial_y v}(x, y) = \frac{\Gamma(M/2)}{2\,\pi^{M/2}} \cdot \frac{M\,(1-|x|^2)\,(x-y)\cdot v}{|x-y|^{M+2}} = M\,\frac{(x-y)\cdot v}{|x-y|^2}\, P(x, y)\,.$$

If we consider $v = \tau$, where τ is a unit vector tangent at x to the sphere of radius $|x|$ centered at the origin, then we have $x \cdot \tau = 0$. We conclude that

$$\frac{\partial P}{\partial_x \tau}(x, y) = \frac{\Gamma(M/2)}{2\pi^{M/2}} \cdot \frac{-M\,(1-|x|^2)\,(x-y)\cdot\tau}{|x-y|^{M+2}} = -M\,\frac{(x-y)\cdot\tau}{|x-y|^2}\,P(x, y) \tag{9.56}$$

and that

$$\frac{\partial P}{\partial_x \tau}(x, y) = -\frac{\partial P}{\partial_y \tau}(x, y)\,. \tag{9.57}$$

(Note that the vector τ is the same vector on both sides of (9.57). The subscript y in the notation $\frac{\partial P}{\partial_y \tau}(x, y)$ on the right-hand side of (9.57) merely tells us to differentiate with respect to y while treating x as a constant; the subscript in no way implies that τ is tangent to Σ at y.) From (9.56), we also obtain the estimate

$$\left|\frac{\partial P}{\partial_x \tau}(x, y)\right| \le M\,|x-y|^{-1}\,P(x, y)\,. \tag{9.58}$$

Similarly, if $\widehat{\tau}$ is also a unit vector tangent at x to the sphere of radius $|x|$ centered at the origin, we have

$$\frac{\partial^2 P}{\partial_x \tau\,\partial_x \widehat{\tau}}(x, y) = -\frac{\partial^2 P}{\partial_y \tau\,\partial_x \widehat{\tau}}(x, y)\,. \tag{9.59}$$

For the vector v, which here need not be a unit vector, we find that

$$\frac{\partial^2 P}{\partial_y v\,\partial_x \tau}(x, y) = M\,\frac{v\cdot\tau}{|x-y|^2}\,P(x, y)$$
$$-(2M+M^2)\,\frac{[\,(x-y)\cdot\tau\,]\,[\,(x-y)\cdot v\,]}{|x-y|^4}\,P(x, y)\,,$$

and we obtain the estimate

$$\left|\frac{\partial^2 P}{\partial_y v\,\partial_x \tau}(x, y)\right| \le (3M+M^2)\,|v|\,|x-y|^{-2}\,P(x, y)\,. \tag{9.60}$$

Suppose $x \in B \setminus \{0\}$ and let $\nu = x/|x|$ be the outward unit normal vector at x to the sphere of radius $|x|$ centered at the origin. We compute

$$\frac{\partial P}{\partial_x \nu}(x, y) = \frac{\Gamma(M/2)}{2\pi^{M/2}} \cdot \left(-\frac{2\,x\cdot\nu}{|x-y|^M} - \frac{M\,(1-|x|^2)\,(x-y)\cdot\nu}{|x-y|^{M+2}}\right).$$

We obtain the estimate

$$\left|\frac{\partial P}{\partial_x \nu}(x, y)\right| \le \frac{\Gamma(M/2)}{2\pi^{M/2}} \cdot \frac{1-|x|^2}{|x-y|^M}\left(\frac{2\,|x\cdot\nu|}{1-|x|^2} + M\,\frac{|(x-y)\cdot\nu|}{|x-y|^2}\right)$$

$$
\begin{aligned}
&\le \frac{\Gamma(M/2)}{2\,\pi^{M/2}} \cdot \frac{1-|x|^2}{|x-y|^M} \left(\frac{2\,|x|}{\varrho(x)\,(2-\varrho(x))} + M\,\frac{|x-y|}{|x-y|^2} \right) \\
&\le P(x,y)\,(2\,\varrho(x)^{-1} + M\,|x-y|^{-1}) \\
&\le P(x,y) \cdot (M+2) \cdot \varrho(x)^{-1}\,, \qquad (9.61)
\end{aligned}
$$

where we have used the fact that $\varrho(x) \le |x-y|$ (which holds because $y \in \Sigma$), thus implying

$$
\frac{1}{|x-y|} \le \varrho(x)^{-1}\,. \qquad (9.62)
$$

In the remainder of the proof, we will use the identity (9.59) for tangential derivatives and the estimates for the derivatives of P to obtain estimates for the second derivatives of u.

Estimates for tangential second derivatives of u. Fix a point $x \in B \setminus \{0\}$. Let τ and $\widehat{\tau}$ be unit vectors tangent at x to the sphere of radius $|x|$ centered at the origin.

Since Hess $[\,u(x)\,]$ is unaffected by adding a constant to g, we may suppose for convenience that

$$
g(\zeta(x)) = 0\,, \qquad (9.63)
$$

where $\zeta(x) = x/|x|$ is the radial projection of x into Σ. It also will be convenient to use "C" to denote a generic constant, the specific value of which may vary from line to line.

We compute

$$
\begin{aligned}
\left| \frac{\partial^2 u}{\partial \tau\, \partial \widehat{\tau}} \right| &= \left| \int_\Sigma \frac{\partial^2 P}{\partial_x \tau\, \partial_x \widehat{\tau}}(x,y)\, g(y)\, d\mathcal{H}^{M-1}(y) \right| \\
&= \left| \int_\Sigma -\frac{\partial^2 P}{\partial_y \tau\, \partial_x \widehat{\tau}}(x,y)\, g(y)\, d\mathcal{H}^{M-1}(y) \right| \\
&= \left| \int_\Sigma \frac{\partial P}{\partial_x \widehat{\tau}}(x,y)\, \frac{\partial g}{\partial_y \tau}(y)\, d\mathcal{H}^{M-1}(y) \right. \\
&\qquad \left. - \int_\Sigma \frac{\partial}{\partial_y \tau} \left(\frac{\partial P}{\partial_x \widehat{\tau}}(x,y)\, g(y) \right) d\mathcal{H}^{M-1}(y) \right| \\
&\le \left| \int_\Sigma \frac{\partial P}{\partial_x \widehat{\tau}}(x,y) \left[\frac{\partial g}{\partial_y \tau}(y) - \frac{\partial g}{\partial_y \tau}(\zeta(x)) \right] d\mathcal{H}^{M-1}(y) \right| \\
&\qquad + \left| \int_\Sigma \frac{\partial}{\partial_y \tau} \left(\frac{\partial P}{\partial_x \widehat{\tau}}(x,y)\, g(y) \right) d\mathcal{H}^{M-1}(y) \right| \\
&= I + II\,.
\end{aligned}
$$

Here we have also used the fact that

$$\int_{\Sigma} \frac{\partial P}{\partial_x \widehat{\tau}}(x, y)\, d\mathcal{H}^{M-1}(y) = 0\,. \tag{9.64}$$

Equation (9.64) holds because

$$\int_{\Sigma} P(x, y)\, d\mathcal{H}^{M-1}(y) \equiv 1 \tag{9.65}$$

implies

$$0 = \frac{\partial}{\partial \widehat{\tau}} \int_{\Sigma} P(x, y)\, d\mathcal{H}^{M-1}(y) = \int_{\Sigma} \frac{\partial P}{\partial_x \widehat{\tau}}(x, y)\, d\mathcal{H}^{M-1}(y)\,.$$

Set

$$S_1 = \Big\{ y \in \Sigma \ :\ |y - \zeta(x)| \le \varrho(x) \Big\}\,, \tag{9.66}$$

$$S_2 = \Big\{ y \in \Sigma \ :\ |y - \zeta(x)| > \varrho(x) \Big\} \tag{9.67}$$

(see Figure 9.3).

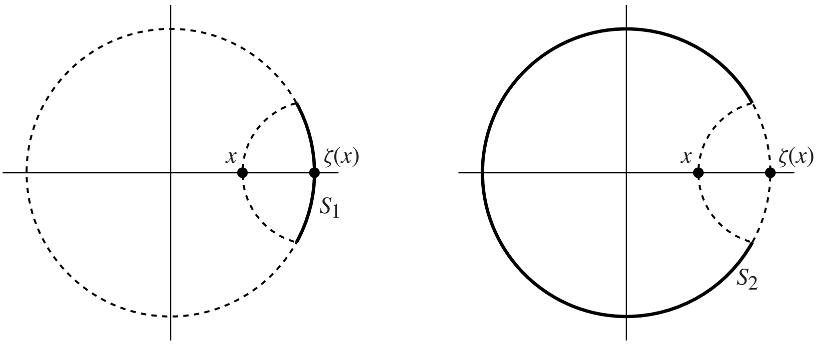

Fig. 9.3. The regions S_1 and S_2 in Σ.

Using (9.58), we can estimate that I is bounded by

$$\begin{aligned} M \int_{\Sigma} \frac{1}{|x-y|}\, P(x, y)\, \|g\|_{\Lambda_\delta}\, |y - \zeta(x)|^{\delta - 1}\, d\mathcal{H}^{M-1}(y) \\ = M \int_{S_1} \frac{1}{|x-y|}\, P(x, y)\, \|g\|_{\Lambda_\delta}\, |y - \zeta(x)|^{\delta - 1}\, d\mathcal{H}^{M-1}(y) \\ + M \int_{S_2} \frac{1}{|x-y|}\, P(x, y)\, \|g\|_{\Lambda_\delta}\, |y - \zeta(x)|^{\delta - 1}\, d\mathcal{H}^{M-1}(y) \\ = I_1 + I_2\,. \end{aligned}$$

We estimate I_1 by using (9.62), (9.65), the nonnegativity of P, and the fact that on S_1, it holds that

$$|y - \zeta(x)|^{\delta-1} \leq \varrho(x)^{\delta-1}$$

because $\delta - 1 > 0$. We have

$$\begin{aligned} I_1 &\leq \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{-1} \int_{S_1} P(x, y)\, |y - \zeta(x)|^{\delta-1}\, d\mathcal{H}^{M-1}(y) \\ &\leq \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{-1} \int_{S_1} P(x, y)\, \varrho(x)^{\delta-1}\, d\mathcal{H}^{M-1}(y) \\ &= \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2} \int_{S_1} P(x, y)\, d\mathcal{H}^{M-1}(y) \\ &\leq \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2} \int_{\Sigma} P(x, y)\, d\mathcal{H}^{M-1}(y) = \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2}. \end{aligned}$$

To estimate I_2, we first note that

$$|y - \zeta(x)| \leq |y - x| + |\zeta(x) - x| = |y - x| + \varrho(x) \leq 2|y - x|, \tag{9.68}$$

which implies that

$$\frac{1}{|x - y|} \leq 2|y - \zeta(x)|^{-1}.$$

Also we note that on S_2, it holds that

$$|y - \zeta(x)|^{\delta-2} \leq \varrho(x)^{\delta-2}$$

because $\delta - 2 < 0$. We estimate

$$\begin{aligned} I_2 &\leq 2\, \|g\|_{\Lambda_\delta} \int_{S_2} P(x, y)\, |y - \zeta(x)|^{\delta-2}\, d\mathcal{H}^{M-1}(y) \\ &\leq 2\, \|g\|_{\Lambda_\delta} \int_{S_2} P(x, y)\, \varrho(x)^{\delta-2}\, d\mathcal{H}^{M-1}(y) \\ &= 2\, \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2} \int_{S_2} P(x, y)\, d\mathcal{H}^{M-1}(y) \\ &\leq 2\, \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2} \int_{\Sigma} P(x, y)\, d\mathcal{H}^{M-1}(y) = 2\, \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta-2}. \end{aligned}$$

To obtain an estimate for II, suppose without loss of generality that $\zeta(x) = \mathbf{e}_1$ and $\tau = \mathbf{e}_2$. Setting

$$T = T(y) = (y_1^2 + y_2^2)^{-1/2}\, (-y_2\, \mathbf{e}_1 + y_1\, \mathbf{e}_2),$$

for each $y = (y_1, y_2, \ldots, y_M) \in \Sigma$, with $(y_1, y_2) \neq (0, 0)$, and applying the fundamental theorem of calculus, we see that

$$\int_\Sigma \frac{\partial}{\partial_y T}\left(\frac{\partial P}{\partial_x \widehat{\tau}}(x, y)\, g(y)\right) d\mathcal{H}^{M-1}(y) = 0\,;$$

more specifically, we parametrize the sphere by

$$\left(r\cos\theta, r\sin\theta, y', \pm\sqrt{1 - r^2 - |y'|^2}\right),$$

where $0 < r < 1$, $0 < \theta < 2\pi$, $y' \in \mathbb{R}^{M-3}$, with $0 < |y'| < \sqrt{1 - r^2}$, and integrate first with respect to θ.

Setting $v = v(y) = \tau - T(y)$ and using (9.63), we have

$$\begin{aligned} II &= \left|\int_\Sigma \left(\frac{\partial}{\partial_y \tau} - \frac{\partial}{\partial_y T}\right)\left(\frac{\partial P}{\partial_x \widehat{\tau}}(x, y)\, g(y)\right) d\mathcal{H}^{M-1}(y)\right| \\ &= \left|\int_\Sigma \frac{\partial}{\partial_y v}\left(\frac{\partial P}{\partial_x \widehat{\tau}}(x, y)\, g(y)\right) d\mathcal{H}^{M-1}(y)\right| \\ &\leq \left|\int_\Sigma \frac{\partial^2 P}{\partial_y v\, \partial_x \widehat{\tau}}(x, y)\,[\,g(y) - g(\zeta(x))\,]\, d\mathcal{H}^{M-1}(y)\right| \\ &\quad + \left|\int_\Sigma \frac{\partial P}{\partial_x \widehat{\tau}}(x, y)\, \frac{\partial g}{\partial_y v}(y)\, d\mathcal{H}^{M-1}(y)\right| \\ &= II_1 + II_2\,, \end{aligned}$$

where we have used the assumption that $g(\zeta(x)) = 0$.

Consider $y = (y_1, y_2, \ldots, y_M) \in \Sigma$ and write $(y_1, y_2) = (r\cos\theta, r\sin\theta)$, where $0 \leq r \leq 1$. It is easy to check that $1 - \cos\theta \leq 2(1 - r\cos\theta)$ holds for $0 \leq r \leq 1$. The law of cosines tells us that $|\tau - T(y)| = \sqrt{2(1 - \cos\theta)}$ and that $|(y_1, y_2) - (1, 0)| = \sqrt{2(1 - r\cos\theta)}$, so we have

$$|\tau - T(y)| \;\leq\; \sqrt{2}\,|y - \zeta(x)| \;\overset{(9.68)}{\leq}\; 2\sqrt{2}\,|y - x|\,. \tag{9.69}$$

Observe that $|g(y) - g(\zeta(x))|$ is bounded by $\|g\|_{\Lambda_\delta}$ multiplied by the distance from y to $\zeta(x)$ *measured along the sphere.* Thus we have

$$|g(y) - g(\zeta(x))| \leq C \cdot \|g\|_{\Lambda_\delta} \cdot |y - \zeta(x)| \leq 2C \cdot \|g\|_{\Lambda_\delta} \cdot |y - x|\,.$$

Using (9.60) and (9.69), we may estimate

$$\begin{aligned} II_1 &\leq C \int_\Sigma \frac{|\tau - T|}{|x - y|^2}\, P(x, y) \cdot \|g\|_{\Lambda_\delta} \cdot |y - x|\, d\mathcal{H}^{M-1}(y) \\ &\leq C \cdot \|g\|_{\Lambda_\delta}\,. \end{aligned}$$

Next, observe that

$$\left| \frac{\partial g}{\partial_y v}(y) \right| \le |v| \cdot \|g\|_{\Lambda_\delta} ,$$

so, by (9.58) and (9.69), we see that

$$\begin{aligned} II_2 &\le C \int_\Sigma |x-y|^{-1} P(x,y) \cdot \|g\|_{\Lambda_\delta} \cdot |\tau - T| \, d\mathcal{H}^{M-1}(y) \\ &\le C \cdot \|g\|_{\Lambda_\delta} . \end{aligned}$$

Thus we have

$$\left| \frac{\partial^2 u}{\partial \tau \, \partial \widehat{\tau}} \right| \le C \cdot \|g\|_{\Lambda_\delta} \cdot \varrho(x)^{\delta - 2} , \tag{9.70}$$

for $x \in B \setminus \{0\}$ and unit vectors $\tau, \widehat{\tau}$ with $\tau \cdot x = \widehat{\tau} \cdot x = 0$.

Mixed normal and tangential second derivatives. Fix a point $x \in B \setminus \{0\}$, let τ be a unit vector tangent at x to the sphere of radius $|x|$ centered at the origin, and let $\nu = x/|x|$ be the outward unit normal vector at x to the sphere of radius $|x|$.

We have

$$\begin{aligned} \frac{\partial^2 u}{\partial \nu \, \partial \tau} &= \int_\Sigma \frac{\partial^2 P}{\partial \nu \, \partial \tau}(x,y) \, g(y) \, d\mathcal{H}^{M-1}(y) \\ &= \int_\Sigma \frac{\partial P}{\partial_x \nu}(x,y) \, \frac{\partial g}{\partial_y \tau}(y) \, d\mathcal{H}^{M-1}(y) \\ &= \int_\Sigma \frac{\partial P}{\partial_x \nu}(x,y) \left[\frac{\partial g}{\partial_y \tau}(y) - \frac{\partial \, (g \circ \zeta)}{\partial_y \tau}(g \circ \zeta)(x) \right] d\mathcal{H}^{M-1}(y) . \end{aligned} \tag{9.71}$$

We can proceed as before, with S_1 and S_2 defined as in (9.66) and (9.67), to estimate

$$\begin{aligned} \left| \frac{\partial^2 u}{\partial \nu \, \partial \tau} \right| &\le \|g\|_{\Lambda_\delta} \int_\Sigma \left| \frac{\partial P}{\partial_x \nu}(x,y) \right| \, |y - \zeta(x)|^{\delta-1} \, d\mathcal{H}^{M-1}(y) \\ &= \|g\|_{\Lambda_\delta} \int_{S_1} \left| \frac{\partial P}{\partial_x \nu}(x,y) \right| \, |y - \zeta(x)|^{\delta-1} \, d\mathcal{H}^{M-1}(y) \\ &\quad + \|g\|_{\Lambda_\delta} \int_{S_2} \left| \frac{\partial P}{\partial_x \nu}(x,y) \right| \, |y - \zeta(x)|^{\delta-1} \, d\mathcal{H}^{M-1}(y) \\ &= III + IV . \end{aligned}$$

We use (9.61) to estimate

$$III \le \|g\|_{\Lambda_\delta} \cdot (M+2) \cdot \varrho(x)^{\delta-2} .$$

Estimating IV is more complicated. We use the estimate (9.61) to see that

$$\begin{aligned}\left|\frac{\partial P}{\partial_x \nu}(x,y)\right| &\le (M+2)\cdot \varrho(x)^{-1}\cdot P(x,y)\\
&= (M+2)\cdot \varrho(x)^{-1}\cdot \frac{\Gamma(M/2)}{2\,\pi^{M/2}}\cdot \frac{\varrho(x)\,(2-\varrho(x))}{|x-y|^M}\\
&= (M+2)\cdot \frac{\Gamma(M/2)}{2\,\pi^{M/2}}\cdot \frac{2-\varrho(x)}{|x-y|^M}\\
&\le \frac{(M+2)\,\Gamma(M/2)}{\pi^{M/2}}\cdot \frac{1}{|x-y|^M}\,.\end{aligned}$$

Then, using the estimate $|y-x|^{-1}\le 2|y-\zeta(x)|^{-1}$, we obtain

$$IV \le C\cdot\ \|g\|_{\Lambda_\delta}\int_{S_2}|y-\zeta(x)|^{\delta-1-M}\,d\mathcal{H}^{M-1}(y)\,.$$

To estimate this last integral, we suppose without loss of generality that $\zeta(x) = (1,0,\dots,0)$. We write

$$(y_1,y_2,\dots,y_M) = (y',y'',\eta)\ \text{ with }\ y'=y_1,\ \ y''=(y_2,y_3,\dots,y_{M-1}),\ \ \eta = y_M\,,$$

so that Σ can be parametrized by

$$\eta = \pm(1-y'^2-|y''|^2)^{1/2}$$

with

$$d\mathcal{H}^{M-1}(y) = (1-y'^2-|y''|^2)^{-1/2}\,d\mathcal{L}^{M-1}(y',y'')\,.$$

We have $|y-\zeta(x)| = (2-2y')^{1/2}$, so

$$IV \le C\,\|g\|_{\Lambda_\delta}\int_{-1}^{1-\varrho(x)^2/2}\int_{|y''|=\sqrt{1-y'^2}}\frac{(2-2y')^{(\delta-1-M)/2}}{(1-y'^2-|y''|^2)^{1/2}}\,d\mathcal{L}^{M-2}(y'')\,d\mathcal{L}(y')\,.$$

We note that the integral

$$\int_{|y''|=\sqrt{1-y'^2}}(1-y'^2-|y''|^2)^{-1/2}\,d\mathcal{L}^{M-2}(y'')$$

equals the $(M-2)$-dimensional area of the upper hemisphere of radius $\sqrt{1-y'^2}$ in $\mathbb{R}^{M-1}$. Thus we have

$$\begin{aligned}IV &= C\,\|g\|_{\Lambda_\delta}\int_{-1}^{1-\varrho(x)^2/2}(2-2y')^{(\delta-1-M)/2}\,(1-y'^2)^{(M-2)/2}\,d\mathcal{L}(y')\\
&\le C\,\|g\|_{\Lambda_\delta}\int_{-1}^{1-\varrho(x)^2/2}(1-y')^{(\delta-3)/2}\,d\mathcal{L}(y')\\
&\le C\,\|g\|_{\Lambda_\delta}\,2^{(M+\delta-1)/2}/(\delta-1)\,,\end{aligned}$$

and we conclude that

$$\left|\frac{\partial^2 u}{\partial\nu\,\partial\tau}\right| \le C\cdot\frac{1}{\delta-1}\cdot\|g\|_{\Lambda_\delta}\cdot\varrho(x)^{\delta-2}\,. \tag{9.72}$$

The second normal derivative. Fix a point $x \in B\setminus\{0\}$ and let $\nu = x/|x|$ be the outward unit normal vector to the sphere of radius $|x|$ centered at the origin.

If $\tau_1, \tau_2, \ldots, \tau_{M-1}$ are pairwise orthogonal unit vectors, all tangent at x to the sphere of radius $|x|$, then

$$\frac{\partial^2 u}{\partial\nu^2} = -\sum_{i=1}^{M-1}\frac{\partial^2 u}{\partial\tau_i^2}\,,$$

so that

$$\left|\frac{\partial^2 u}{\partial\nu^2}\right| \le C\cdot\|g\|_{\Lambda_\delta}\cdot\varrho(x)^{\delta-2}\,. \tag{9.73}$$

Summary. For $x \in B\setminus\{0\}$, we can make an orthogonal change of basis such that $x/|x|$ coincides with one of the standard basis vectors. Then (9.70), (9.72), and (9.73) give us the required bound for the Hilbert–Schmidt norm of the Hessian matrix for u at x. □

Lemma 9.4.3. *Fix* $0<\delta<1$ *and* $1<\hat\sigma<2$. *There is a constant* $c_8 = c_8(\delta)$ *such that if*

$$g : \mathbb{B}^M(0,\hat\sigma)\to\mathbb{R}$$

is smooth and $u \in C^0(\overline{B})\bigcap C^2(B)$ *satisfies*

$$\triangle u = 0 \;\; on \;\; B\,,$$

$$u = g \;\; on \;\; \Sigma\,,$$

then

(1) $$\sup\Big\{|x-z|^{-\delta}\,|Du(x)-Du(z)| : x,z\in B,\; x\ne z\Big\} + \sup_B |Du|$$
$$\le c_8\cdot\Big(\sup\Big\{|x-z|^{-\delta}\,|Dg(x)-Dg(z)| : x,z\in\mathbb{B}^M(0,\hat\sigma),\; x\ne z\Big\} + \sup_{\mathbb{B}^M(0,\hat\sigma)}|Dg|\Big)\,,$$

(2) $$\sup_{\mathbb{B}^M(0,1/2)}\Big|\mathrm{Hess}\,[u(x)]\Big| \le c_8\left(\int_B \Big|\mathrm{Hess}\,[u(x)]\Big|^2 d\mathcal{L}^M\right)^{1/2},$$

(3) $$\sup_{x\in\mathbb{B}^M(0,\hat\eta)}|Du(x)-Du(0)|^2 \le c_8\,\hat\eta^2\int_B\Big|\mathrm{Hess}\,[u(x)]\Big|^2 d\mathcal{L}^M\,,$$
for each $0<\hat\eta<1/2$.

Proof.
(1) Since

$$\sup_B |Du| \le \sup_\Sigma |Dg|$$

holds by the maximum principle, it suffices to estimate

$$\sup \left\{ |x-z|^{-\delta} \, |Du(x) - Du(z)| : x, z \in B, \; x \neq z \right\}.$$

We do so by comparing

$$|Du(x_1) - Du(x_0)|$$

to h^δ, where $x_0, x_1 \in B$ and $h = |x_1 - x_0|$. We need only consider h small, and again by the maximum principle, we need to consider only x_0 near Σ.

Set $\hat{\delta} = 1 + \delta$. We will apply Lemma 9.4.2 with δ replaced by $\hat{\delta}$. By that lemma, we have

$$\Big| \text{Hess}\,[u(x)] \Big| \le c_7 \cdot \|g\|_{\Lambda_{\hat{\delta}}} \cdot \varrho(x)^{\hat{\delta}-2}$$

for $x \in B$, where $\varrho(x) = 1 - |x|$. Note that

$$\|g\|_{\Lambda_{\hat{\delta}}} \le \sup \left\{ |x-z|^{-\delta} \, |Dg(x) - Dg(z)| : x, z \in \mathbb{B}^M(0, \hat{\sigma}), \; x \neq z \right\} + \sup_{\mathbb{B}^M(0,\hat{\sigma})} |Dg|$$

holds. In what follows, C will denote a generic positive, finite constant incorporating the value of c_7.

We need to estimate $|Du(x_1) - Du(x_0)|$. The proximity of the boundary Σ makes it difficult to obtain the needed estimate. Rather than proceeding directly, we replace each point x_i by a point $\widetilde{x_i}$ that is at distance h farther away from Σ (see Figure 9.4). Remarkably, it is then feasible to estimate the individual terms $|Du(\widetilde{x_0}) - Du(x_0)|$, $|Du(\widetilde{x_1}) - Du(x_1)|$, and $|Du(\widetilde{x_0}) - Du(\widetilde{x_1})|$.

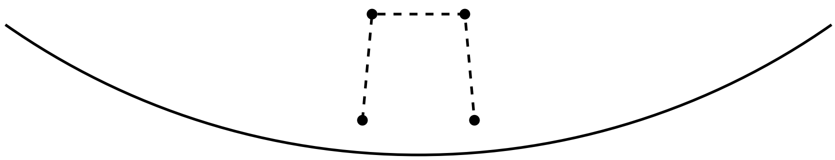

Fig. 9.4. Moving the points away from the boundary.

Let $\widetilde{x_i}$ be such that

$$\zeta(\widetilde{x_i}) = \zeta(x_i)\,,$$

$$|\widetilde{x_i}| = |x_i| - h\,;$$

then we have

$$\begin{aligned}|Du(x_1) - Du(x_0)| &\le |Du(x_1) - D(\widetilde{x_1})| \\ &\quad + |Du(\widetilde{x_1}) - Du(\widetilde{x_0})| \\ &\quad + |Du(\widetilde{x_0}) - Du(x_0)| \\ &= I + II + III\,.\end{aligned}$$

Set $\nu = x_0/|x_0|$. We have

$$\begin{aligned} III &\le \int_0^h \left| \frac{\partial(Du)}{\partial\nu}(x_0 - t\nu)\right| \, d\mathcal{L}^1(t) \\ &\le \int_0^h \Big| \text{Hess}\,[u(x_0 - t\nu)] \Big| \, d\mathcal{L}^1(t) \\ &\le C\,\|g\|_{\Lambda_{\hat\delta}} \int_0^h \varrho(x_0 - t\nu)^{\hat\delta - 2}\, d\mathcal{L}^1(t) \\ &\le C\,\|g\|_{\Lambda_{\hat\delta}} \int_0^h [\varrho(x_0) + t]^{\hat\delta - 2}\, d\mathcal{L}^1(t) \\ &= C\,\|g\|_{\Lambda_{\hat\delta}} \Big([\varrho(x_0) + h]^{\hat\delta-1} - \varrho(x_0)^{\hat\delta - 1}\Big) \\ &\le C\, h^{\hat\delta - 1} = C\, h^{\delta}\,, \end{aligned}$$

if $\varrho(x_0)$ is small. (Note that $\hat\delta - 1 > 0$.)

Likewise, we estimate

$$I \le C\,\|g\|_{\Lambda_{\hat\delta}}\, h^{\hat\delta - 1}\,.$$

To estimate II, we note that

$$II \le \int_0^h h \,\Big| \text{Hess}\,[u(\widetilde{x_0} + \xi)] \Big| \, d\mathcal{L}^1(t)\,, \tag{9.74}$$

where $\widetilde{x_0} + \xi$ is a point on the segment between $\widetilde{x_0}$ and $\widetilde{x_1}$. The right-hand side of (9.74) is bounded above by

$$\begin{aligned} C\,\|g\|_{\Lambda_{\hat\delta}}\, h \int_0^h \varrho(\widetilde{x_0} + \xi)^{\hat\delta - 2}\, d\mathcal{L}^1(t) &\le C\,\|g\|_{\Lambda_{\hat\delta}}\, h \int_0^h h^{\hat\delta - 2}\, d\mathcal{L}^1(t) \\ &\le C\,\|g\|_{\Lambda_{\hat\delta}}\, h^{\hat\delta}\,. \end{aligned}$$

(2) Fix $i, j \in \{1, 2, \ldots, M\}$ and $x \in \mathbb{B}^M(0, 1/2)$. For $0 < r < 1/2$, by the mean value property of harmonic functions, we have

$$\frac{\partial^2 u}{\partial x_i\, \partial x_j}(x) = C \cdot r^{1-M} \int_{\{y : |y| = r\}} \frac{\partial^2 u}{\partial x_i\, \partial x_j}(x + y)\, d\mathcal{H}^{M-1}(y)\,.$$

But then

$$\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x)\right| = C\left|\int_{1/4}^{1/2} r^{1-M}\int_{\{y:|y|=r\}}\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x+y)\,d\mathcal{H}^{M-1}(y)\,d\mathcal{L}^1(r)\right|$$

$$\leq C\left|\int_{\mathbb{B}^M(x,1/2)}\frac{\partial^2 u}{\partial x_i\,\partial x_j}(z)\,d\mathcal{L}^M(z)\right|$$

$$\leq C\left(\int_B\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^2 d\mathcal{L}^M\right)^{1/2}$$

holds and the result follows.

(3) Fix $i \in \{1, 2, \ldots, M\}$ and $x \in \mathbb{B}^M(0, 1/2) \setminus \{0\}$. Set $\nu = x/|x|$ and

$$\psi(t) = \frac{\partial u}{\partial x_i}(t\nu)$$

for $-1 < t < 1$. Thus $\psi'(t)$ is the directional derivative of $\partial u/\partial x_i$ in the direction ν at the point $t\nu$. It follows that $|\psi'(t)|$ is bounded by the operator norm of the Hessian matrix for u at $t\nu$. Hence $|\psi'(t)|$ is bounded by a multiple of $\Big|\text{Hess}\,[u(t\nu)]\Big|$.

Using the fundamental theorem of calculus, we estimate

$$\left|\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(0)\right|^2 = \left|\int_0^{|x|}\psi'(t)\,d\mathcal{L}^1(t)\right|^2$$

$$\leq |x|^2\cdot\sup\Big\{|\psi'(t)|^2 : 0 \leq t \leq |x|\Big\}$$

$$\leq |x|^2\cdot\sup_{y\in\mathbb{B}^M(0,1/2)}\Big|\text{Hess}\,[u(y)]\Big|^2,$$

so we see that conclusion (3) follows from conclusion (2). □

9.5 The Main Estimate

The next lemma is the main tool in the regularity theory. The lemma tells us that once the cylindrical excess (see Definition 9.1.4) of an area-minimizing surface is small enough, then the excess on a smaller cylinder can be made even smaller by appropriately rotating the surface.

Lemma 9.5.1. *There exist constants*

$$0 < \theta < 1/8\,, \quad 0 < \epsilon_* \leq (\theta/4)^{2M}\,, \tag{9.75}$$

depending only on M, with the following property:

If $0 \in \operatorname{spt} T$*, if* $T_0 = T \llcorner \mathrm{C}(0, \rho/2)$*, and if the hypotheses* (H1–H5) (*see page* 262) *hold with*

$$y = 0\,, \quad \epsilon = \epsilon_*\,,$$

then

$$\sup_{X \in \operatorname{spt} T_0} |\mathbf{q}\,(X)| \le \rho/8 \tag{9.76}$$

holds and there exists a linear isometry $\mathbf{j} : \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$ *with*

$$\theta^{-2M}\, E(\,T,\ 0,\ \rho\,) \le 1/64\,, \tag{9.77}$$

$$\|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\|^2 \le \theta^{-2M}\, E(\,T,\ 0,\ \rho\,)\,, \tag{9.78}$$

$$E(\,\mathbf{j}_\# T_0,\ 0,\ \theta\rho\,) \le \theta\, E(\,T,\ 0,\ \rho\,)\,. \tag{9.79}$$

Here $\mathbf{I}_{\mathbb{R}^{M+1}}$ *is the identity map on* $\mathbb{R}^{M+1}$.

Proof. Since we may change scale if need be, it will be sufficient to prove the lemma with $\rho = 1$. We ultimately will choose

$$\epsilon_* < \epsilon_0\,, \tag{9.80}$$

where ϵ_0 is as in Lemmas 9.2.1 and 9.2.2 (in particular, Lemma 9.2.1 is invoked with $\sigma = 2/3$), so we will assume that $0 \in \operatorname{spt} T$ and that the hypotheses (H1–H5) hold with $y = 0$, $\rho = 1$, and with $\epsilon = \epsilon_0$, where ϵ_0 is as in Lemma 9.2.1.

We set

$$\delta = \frac{1}{9M^2}\,,$$

$$E = E(\,T,\ 0,\ 1\,)\,.$$

Lipschitz approximations. We can apply Lemma 9.2.2 to obtain a Lipschitz function whose graph approximates $\operatorname{spt} T$. In fact, there are two such approximating functions that will be of interest:

- We let $g_\delta : \mathbb{B}^M(0, 1/4) \to \mathbb{R}$ be a Lipschitz function as in Lemma 9.2.2 corresponding to the choice
$$\gamma = E^{2\,\delta}\,.$$
- We let $h : \mathbb{B}^M(0, 1/4) \to \mathbb{R}$ be a Lipschitz function as in Lemma 9.2.2 corresponding to the choice $\gamma = 1$.

Smoothing g_δ. Let $\varphi \in C^\infty(\mathbb{R}^M)$ be a mollifier as in Definition 5.5.1 with N replaced by M. As usual, for $0 < \nu$,

- set
$$\varphi_\nu(z) = \nu^{-M}\, \varphi(\nu^{-1} z);$$

- let $f * \varphi_\nu$ denote convolution of f with φ_ν.

Let $0 < c_9 < \infty$ satisfy

$$\sup |\varphi| \le c_9 \,,$$

$$\sup |D\varphi| \le c_9 \,,$$

$$\sup_{x \neq z} |x - z|^{-\delta} \, |D\varphi(x) - D\varphi(z)| \le c_9 \,.$$

Defining

$$\widetilde{g}_\delta = g_\delta * \varphi_E \,, \tag{9.81}$$

we obtain the following standard estimates:

$$\sup_{\mathbb{B}^M(0,1/8)} |D\widetilde{g}_\delta| \le \sup_{\mathbb{B}^M(0,1/4)} |Dg_\delta| \le E^{2\delta} \le E^\delta, \tag{9.82}$$

$$\sup_{\mathbb{B}^M(0,1/8)} |\widetilde{g}_\delta - g_\delta| \le E \sup_{\mathbb{B}^M(0,1/4)} |Dg_\delta| \le E^{1+\delta}, \tag{9.83}$$

$$\begin{aligned} \sup\{\, |x - z|^{-\delta} \, |D\widetilde{g}_\delta(x) - D\widetilde{g}_\delta(z)| \;:\; x, z \in \mathbb{B}^M(0, 1/8), \ x \neq z \,\} \\ \le \sup_{\mathbb{B}^M(0,1/4)} |Dg_\delta| \cdot \sup_{x \neq z} |x - z|^{-\delta} \, |\phi(E^{-1}x) - \phi(E^{-1}z)| \\ \le \ E^{2\delta} \cdot E^{-\delta} \cdot \sup_{x \neq z} |x - z|^{-\delta} \, |\phi(x) - \phi(z)| \\ \le c_9 \, E^\delta \,. \end{aligned} \tag{9.84}$$

The graph of $\widetilde{g}_\delta$. We next define

$$\widetilde{S} = \widetilde{G}_\#(\mathbf{E}^M \llcorner \mathbb{B}^M(0, 1/8) \,) \,, \tag{9.85}$$

where $\widetilde{G} : \mathbb{B}^M(0, 1/8) \to \mathrm{C}(0, 1/8)$ is defined by

$$\widetilde{G}(x) = (x, \widetilde{g}_\delta(x)) \,.$$

Choosing σ. For each $0 < \sigma < 1/8$ we let

$$T_\sigma = T \llcorner \mathrm{C}(0, \sigma), \qquad \widetilde{S}_\sigma = \widetilde{S} \llcorner \mathrm{C}(0, \sigma) \,.$$

We wish to show that there is a finite positive constant c_{10} such that there are infinitely many choices of $1/16 < \sigma < 1/8$ for which the following inequalities all hold:

$$\mathcal{H}^{M-1}\Big\{ x \in \partial\mathbb{B}^M(0, \sigma) : g_\delta(x) \neq h(x) \Big\} \le c_{10} \, E^{1-4M\delta} \,, \tag{9.86}$$

$$\|\partial T_\sigma\|(\mathbb{R}^{M+1}) \le c_{10}, \tag{9.87}$$

$$\|\partial T_\sigma\|\Big\{ X : |P(X) - X| > E^{1+\delta} \Big\} \le c_{10} \, E^{1-4M\delta} \,, \tag{9.88}$$

where P is the "vertical retraction" of $C(0, 1/8)$ onto the graph of $\widetilde{g}_\delta$. That is, for $X \in C(0, 1/8)$ we have

$$P(X) = (\mathbf{p}(X), \widetilde{g}_\delta(\mathbf{p}(X))) .$$

Notice that $P_\# T_\sigma = \widetilde{S}_\sigma$ by (9.18) and the definition of $\widetilde{S}$.

- First, by (9.23) and by Theorem 5.2.1, i.e., the coarea formula, we have

$$\int_{1/16}^{1/8} \mathcal{H}^{M-1}\Big\{ x \in \partial\mathbb{B}^M(0,\sigma) : g_\delta(x) \neq h(x) \Big\}\, d\mathcal{L}^1(\sigma)$$

$$\leq \mathcal{L}^M\Big(\mathbb{B}^M(y, 1/4) \setminus \Big\{ z \in \mathbb{B}^M(y, 1/4) : \mathbf{p}^{-1}(z) \bigcap \operatorname{spt} T = \{(x, h(x))\} \Big\}\Big)$$

$$+ \mathcal{L}^M\Big(\mathbb{B}^M(y, 1/4) \setminus \Big\{ z \in \mathbb{B}^M(y, 1/4) : \mathbf{p}^{-1}(z) \bigcap \operatorname{spt} T = \{(x, g_\delta(x))\} \Big\}\Big)$$

$$\leq c_3\,(1 + E^{-4\delta})\, E \leq 2\, c_3\, E^{1-4\delta} .$$

- Because ∂T has its support outside the cylinder of radius 1, we can identify ∂T_σ with the slice $\langle T, r, \sigma+\rangle$, where r is the distance from the axis of the cylinder. We conclude that

$$\int_{1/16}^{1/8} \|\partial T_\sigma\|(\mathbb{R}^{M+1})\, d\mathcal{L}^1(\sigma) \leq \int_{C(0,1/8)} d\|T\|$$

holds.

- Third, by (9.83), if $X = (x, g_\delta(x))$ coincides with the point $\mathbf{p}^{-1}(x) \bigcap \operatorname{spt} T$, then X and $P(X)$ are separated by a distance not exceeding $E^{1+\delta}$. So we use (9.24) to estimate

$$\int_{1/16}^{1/8} \|\partial T_\sigma\|\{\, X \;:\; |P(X) - X| > E^{1+\delta} \,\}\, d\mathcal{L}^1(\sigma)$$

$$= \int_{1/16}^{1/8} \|\langle T, r, \sigma+\rangle\|\{\, X \;:\; |P(X) - X| > E^{1+\delta} \,\}\, d\mathcal{L}^1(\sigma)$$

$$= \int_{1/16}^{1/8} \|\langle T - \widetilde{S}, r, \sigma+\rangle\| C(y, 1/4)\, d\mathcal{L}^1(\sigma)$$

$$\leq \|T - \widetilde{S}\| C(y, 1/4) \leq c_4\, E^{-4M\delta}\, E ,$$

where we note that, in the notation of Lemma 9.2.2, $\widetilde{S}$ corresponds to T^{g_δ}.

The homotopy between T_σ and $\widetilde{S}_\sigma$. Let $H : [0, 1] \times C(0, 1/8) \to \mathbb{R}^{M+1}$ be defined by $H(t, x) = tP(X) + (1 - t)X$. By the homotopy formula (7.22), we have

$$\partial V = \partial T_\sigma - \partial \widetilde{S}_\sigma , \tag{9.89}$$

where

$$V = H_{\#}([\![0,1]\!] \times \partial T_\sigma)\,.$$

By (7.23) and Lemma 9.2.2 applied with $\gamma = E^{2\delta}$ (in particular, using (9.21) and (9.23)), and by (9.83), (9.86), and (9.88), we have

$$\begin{aligned}
&\|V\|(\mathbb{R}^{M+1}) \\
&\qquad \le 2\int |P(X) - X|\, d\|T_\sigma\| \\
&\qquad \le 2\Big(\sup_{X \in \operatorname{spt} \partial T_\sigma} |P(X) - X| \Big) \cdot \|\partial T_\sigma\| \Big\{ X : |X - P(X)| > E^{1+\delta} \Big\} \\
&\qquad\qquad + c_{10}\, E^{1+\delta} \\
&\qquad \le c_{11}\, E^{1+1/(2M)-4M\delta} + c_{10}\, E^{1+\delta} \\
&\qquad \le c_{12}\, E^{1+\delta}\,,
\end{aligned} \tag{9.90}$$

where we have made use of the fact that $\delta = (9M^2)^{-1}$.

The approximating harmonic function. The aim is to show that with $1/16 < \sigma < 1/8$ chosen such that (9.86), (9.87), and (9.88) hold, $T \llcorner \mathrm{C}(0,\sigma)$ can be very closely approximated by the graph of a harmonic function.

Let $1/16 < \sigma < 1/8$ be such that (9.86), (9.87), and (9.88) (and consequently (9.90)) hold. Let $u : \overline{\mathbb{B}}^M(0,\sigma) \to \mathbb{R}$ be continuous and satisfy

$$\left.\begin{aligned}
\triangle u &= 0 \ \text{ on } \mathbb{B}^M(0,\sigma), \\
u &= \widetilde{g}_\delta \ \text{ on } \partial\mathbb{B}^M(0,\sigma),
\end{aligned}\right\} \tag{9.91}$$

where $\widetilde{g}_\delta$ is as in (9.81), so (9.82) and (9.84) will hold.

Recall that (9.82) and (9.84) are the estimates

$$\sup_{\mathbb{B}^M(0,1/8)} |D\widetilde{g}_\delta| \le E^\delta$$

and

$$\sup\{\, |x-z|^{-\delta}\, |D\widetilde{g}_\delta(x) - D\widetilde{g}_\delta(z)| \ :\ x, z \in \mathbb{B}^M(0,1/8),\ x \ne z\} \le c_9\, E^\delta\,.$$

By applying Lemma 9.4.3 with $\hat{\sigma} = 1/(8\sigma)$, $g(x) = \widetilde{g}_\delta(x/\sigma)$, and $\hat{\eta} = \eta/\sigma$, we see that there exist constants c_{13} and c_{14} such that if u is as in (9.91), then the following estimates hold:

$$\begin{aligned}
&\sup\{\, |x-z|^{-\delta}\, |Du(x) - Du(z)| \ :\ x, z \in \mathbb{B}^M(0,\sigma),\ x \ne z\} \\
&\qquad + \sup_{\mathbb{B}^M(0,\sigma)} |Du| \le c_{13}\, E^\delta\,,
\end{aligned} \tag{9.92}$$

$$\sup_{x \in \mathbb{B}^M(0,\eta)} |Du(x) - Du(0)|^2 \le c_{14}\, \eta^2 \int_{\mathbb{B}^M(0,\sigma)} |Du|^2\, d\mathcal{L}^M\,, \tag{9.93}$$

for each $0 < \eta < \sigma/2$.

The comparison surface and the first use of the minimality of T. Define $G : \mathbb{B}^M(0, \sigma) \to C(0, \sigma)$ by setting $G(x) = (x, u(x))$ and set

$$S = G_{\#}(\mathbf{E}^M \llcorner \mathbb{B}^M(0, \sigma)) .$$

We have $\partial S = \partial \widetilde{S}_\sigma$, where we recall that $\widetilde{S}_\sigma = \widetilde{S} \llcorner C(0, \sigma)$ and that $\widetilde{S}$ is defined in (9.85). Consequently, we have

$$\partial(V + S - T_\sigma) = 0 , \tag{9.94}$$

by (9.89). This last equation tells us that

$$\partial(V + S) = \partial T_\sigma ,$$

so we can use $V + S$ as a comparison surface for the area-minimizing surface T_σ. Since it is true for any V and S that

$$\mathbf{A}[V] + \mathbf{A}[S] \geq \mathbf{A}[V + S] ,$$

we have

$$\mathbf{A}[V] + \mathbf{A}[S] \geq \mathbf{A}[V + S] \geq \mathbf{A}[T_\sigma] , \tag{9.95}$$

because T_σ is area-minimizing.

The first calculation of the difference between T_σ and S. We extend $\overrightarrow{S}$ to all of $C(0, \sigma)$ by setting

$$\overrightarrow{S}(X) = \overrightarrow{S}\Big(\mathbf{p}(X), u(\mathbf{p}(X))\Big) . \tag{9.96}$$

Using the extension of $\overrightarrow{S}$ in (9.96) and noting that $\overrightarrow{T_\sigma} = \overrightarrow{T}$ holds $\|T_\sigma\|$-almost everywhere, we get

$$\begin{aligned}
\mathbf{A}[T_\sigma] - \mathbf{A}[S] &= \int A(\overrightarrow{T})\, d\|T_\sigma\| - \int A(\overrightarrow{S})\, d\|S\| \\
&= \int \Big(A(\overrightarrow{T}) - \Big\langle DA(\overrightarrow{S}), \overrightarrow{T} \Big\rangle \Big) d\|T_\sigma\| \\
&\quad + \int \Big\langle DA(\overrightarrow{S}), \overrightarrow{T} \Big\rangle d\|T_\sigma\| - \int A(\overrightarrow{S})\, d\|S\| \\
&= \int \Big(A(\overrightarrow{T}) - \Big\langle DA(\overrightarrow{S}), \overrightarrow{T} \Big\rangle \Big) d\|T_\sigma\| \\
&\quad + \int \Big\langle DA(\overrightarrow{S}), \overrightarrow{T} \Big\rangle d\|T_\sigma\| - \int \Big\langle DA(\overrightarrow{S}), \overrightarrow{S} \Big\rangle d\|S\| ,
\end{aligned} \tag{9.97}$$

where we have also used (9.6) to conclude that $A(\overrightarrow{S}) = \Big\langle DA(\overrightarrow{S}), \overrightarrow{S} \Big\rangle$.

By (9.12) we have

$$A(\overrightarrow{T}) - \left\langle DA(\overrightarrow{S}),\ \overrightarrow{T} \right\rangle = \frac{1}{2}\left| \overrightarrow{T} - \overrightarrow{S} \right|^2 . \tag{9.98}$$

For integrands other than area, a Weierstrass condition would be used here instead of (9.12). Recalling from (9.7) that we may also treat $DA(\overrightarrow{S})$ as a differential M-form, we have

$$\int \left\langle DA(\overrightarrow{S}),\ \overrightarrow{T} \right\rangle d\|T_\sigma\| - \int \left\langle DA(\overrightarrow{S}),\ \overrightarrow{S} \right\rangle d\|S\| = [T_\sigma - S]\Big(DA(\overrightarrow{S}) \Big) . \tag{9.99}$$

Using (9.97), (9.98), and (9.99), we see that

$$\mathbf{A}[T_\sigma] - \mathbf{A}[S] = \frac{1}{2} \int \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T_\sigma\| + [T_\sigma - S]\Big(DA(\overrightarrow{S}) \Big) . \tag{9.100}$$

Use of the comparison surface and the second use of the minimality of $\boldsymbol{T}$. Since (9.94) tells us that $\partial(V + S - T_\sigma) = 0$, we have

$$V + S - T_\sigma = \partial R$$

for some $(M+1)$-dimensional current R, so (see (9.3) for notation)

$$(V + S - T_\sigma)\left(dx^M\right) = (\partial R)\left(dx^M\right) = R\left(d\,dx^M\right) = 0 .$$

Since (9.7) tells us that $DA(\mathbf{e}^M) = dx^M$, we conclude that

$$(V + S - T_\sigma)\left(DA(\mathbf{e}^M)\right) = 0 .$$

Thus we have

$$\begin{aligned} \mathbf{A}[T_\sigma] - \mathbf{A}[S] = {} & \frac{1}{2} \int \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T_\sigma\| \\ & + (T_\sigma - S)\left(DA(\overrightarrow{S}) - DA(\mathbf{e}^M) \right) \\ & + V\left(DA(\mathbf{e}^M) \right) . \end{aligned} \tag{9.101}$$

From(9.95), (9.100), and (9.101) we obtain

$$\begin{aligned} \mathbf{A}[V] &\geq \mathbf{A}[T_\sigma] - \mathbf{A}[S] \\ &\geq \frac{1}{2} \int \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T_\sigma\| \\ &\quad + (T_\sigma - S)\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M) \Big) \\ &\quad + V(DA(\mathbf{e}^M)) . \end{aligned} \tag{9.102}$$

By (9.90), we have $\mathbf{A}[V] = \|V\|(\mathbb{R}^{M+1}) \le c_{12}\, E^{1+\delta}$ and consequently also

$$\Big| V(\, DA(\mathbf{e}^M)\,) \Big| \le c_{12}\, E^{1+\delta} \, .$$

Thus we have

$$2c_{12}\, E^{1+\delta} \ge \frac{1}{2} \int \Big| \overrightarrow{T} - \overrightarrow{S} \Big|^2 \, d\|T_\sigma\| \\ + (T_\sigma - S)\Big(\, DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M)\,\Big) . \tag{9.103}$$

Estimating the second term on the right in (9.103). We wish to estimate the second term on the right in (9.103) by an expression similar to the first term on the right. The argument to obtain the desired estimate is sufficiently complicated that we state the result as a separate claim.

Claim. *There exist constants* c_{15} *and* c_{16} *such that*

$$\Big| (T_\sigma - S)\Big(\, DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M)\,\Big) \Big| \\ \le c_{15}\, E^{1+\delta} + 2\, c_{16}\, E^{\delta} \int \Big| \overrightarrow{S} - \overrightarrow{T} \Big|^2 \, d\|T_\sigma\| \, . \tag{9.104}$$

Proof of the Claim. We recall that h is as in Lemma 9.2.2 with $\gamma = 1$, and we introduce

$$T_\sigma^0 = G_\#^0(\mathbf{E}^M \llcorner \mathbb{B}^M(0,\sigma)) \, ,$$

where $G_0(x) = (x, h(x))$. By (9.24) of the Lipschitz approximation lemma, we have

$$\|T_\sigma^0 - T_\sigma\|\mathrm{C}(0,\sigma) \le c_4\, E \, , \tag{9.105}$$

because $\gamma = 1$, $\rho = 1$, and $\sigma < 1/8$.

The estimate (9.92) gives us the bound $|Du| \le c_{13}\, E^{\delta}$. Then, using (9.46), we obtain

$$\Big| DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M) \Big| \le 2\, c_{13}\, E^{\delta} \, . \tag{9.106}$$

By (9.105) and (9.106) we have

$$\Big| (T_\sigma - S)\Big(\, DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M)\,\Big) \Big| \\ \le \Big| (T_\sigma^0 - S)\Big(\, DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M)\,\Big) \Big| + \Big| (T_\sigma - T_\sigma^0)\Big(\, DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M)\,\Big) \Big| \\ \le \Big| (T_\sigma^0 - S)\Big(\, DA(\,\overrightarrow{S}\,) - DA(\mathbf{e}^M)\,\Big) \Big| + c_4\, E \cdot 2\, c_{13}\, E^{\delta} \, . \tag{9.107}$$

Because S is the current defined by integrating over the graph of u, we apply (9.39) with $f = u$ to obtain

$$DA(\overrightarrow{S}) - DA(\mathbf{e}^M)$$
$$= (1+|Du|^2)^{-1/2}\left(dx^M + \sum_{i=1}^{M}(D_{x_i}u)\,dx_{\widehat{\imath}}\right) - dx^M\,. \qquad (9.108)$$

Because T^0_σ is the current defined by integration over the graph of h, we may apply (9.36), (9.40), and (9.37), with $f = h$, and use (9.108) to find that

$$T^0_\sigma\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M)\Big)$$
$$= \int_{\mathbb{B}^M(0,\sigma)}\left[(1+|Du|^2)^{-1/2}\left(1+\sum_{i=1}^{M} D_{x_i}u\,D_{x_i}h\right) - 1\right] d\mathcal{L}^M\,. \quad (9.109)$$

Similarly, taking $f = u$, we obtain

$$S\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M)\Big)$$
$$= \int_{\mathbb{B}^M(0,\sigma)}\left[(1+|Du|^2)^{-1/2}\left(1+\sum_{i=1}^{M} D_{x_i}u\,D_{x_i}u\right) - 1\right] d\mathcal{L}^M\,. \quad (9.110)$$

Combining (9.109) and (9.110), we find that

$$(T^0_\sigma - S)\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M)\Big)$$
$$= \int_{\mathbb{B}^M(0,\sigma)}\left[(1+|Du|^2)^{-1/2}\sum_{i=1}^{M} D_{x_i}u\,D_{x_i}(h-u)\right] d\mathcal{L}^M\,. \quad (9.111)$$

We will simplify the integrand in (9.111) so that we can use the fact that u is a harmonic function. To this end we use (9.43) to bound

$$\left|\int_{\mathbb{B}^M(0,\sigma)}\left[(1+|Du|^2)^{-1/2}\sum_{i=1}^{M} D_{x_i}u\,D_{x_i}(h-u)\right] d\mathcal{L}^M\right.$$
$$\left. - \int_{\mathbb{B}^M(0,\sigma)}\left[\sum_{i=1}^{M} D_{x_i}u\,D_{x_i}(h-u)\right] d\mathcal{L}^M\right|$$

above by

$$\int_{\mathbb{B}^M(0,\sigma)} |Du| \left| \sum_{i=1}^{M} D_{x_i} u \, D_{x_i}(h-u) \right| d\mathcal{L}^M$$

$$\leq \int_{\mathbb{B}^M(0,\sigma)} |Du| \, |Du| \, |D(h-u)| \, d\mathcal{L}^M$$

$$\leq \int_{\mathbb{B}^M(0,\sigma)} |Du| \, |Du| \Big(|Dh| + |Du| \Big) \, d\mathcal{L}^M$$

$$\leq \int_{\mathbb{B}^M(0,\sigma)} |Du|^3 \, d\mathcal{L}^M + \int_{\mathbb{B}^M(0,\sigma)} |Du| \, |Du| \, |Dh| \, d\mathcal{L}^M$$

$$\leq \int_{\mathbb{B}^M(0,\sigma)} |Du|^3 \, d\mathcal{L}^M + \frac{1}{2} \int_{\mathbb{B}^M(0,\sigma)} |Du| \Big(|Du|^2 + |Dh|^2 \Big) \, d\mathcal{L}^M$$

$$\leq \frac{3}{2} \int_{\mathbb{B}^M(0,\sigma)} |Du| \Big(|Du|^2 + |Dh|^2 \Big) \, d\mathcal{L}^M \, .$$

So, using the bound $|Du| \leq c_{13} \, E^{\delta}$ from (9.92), we can write

$$(T_\sigma^0 - S)(DA(\overrightarrow{S}) - DA(\mathbf{e}^M)) = \int_{\mathbb{B}^M(0,\sigma)} \left[\sum_{i=1}^{M} D_{x_i} u \, D_{x_i}(h-u) \right] d\mathcal{L}^M + R \, , \tag{9.112}$$

where

$$|R| \leq (3/2) \, c_{13} \, E^{\delta} \int_{\mathbb{B}^M(0,\sigma)} \Big(|Du|^2 + |Dh|^2 \Big) \, d\mathcal{L}^M \, . \tag{9.113}$$

The fact that u is harmonic will allow us to express the integrand

$$\sum_{i=1}^{M} D_{x_i} u \, D_{x_i}(h-u)$$

in (9.112) as the divergence of a vector field, and thereby allow us to use the Gauss–Green theorem to replace the integral over the disk by an integral over the boundary of the disk.

Set

$$\boldsymbol{w} = (h-u) \sum_{i=1}^{M} D_{x_i} u \, \mathbf{e}_i \, .$$

We compute

$$\operatorname{div} \boldsymbol{w} = \sum_{i=1}^{M} \frac{\partial}{\partial x_i} [(h-u) D_{x_i} u]$$

$$= \sum_{i=1}^{M} D_{x_i} u \, D_{x_i}(h-u) + (h-u) \sum_{i=1}^{M} \frac{\partial^2 u}{\partial x_i^2}$$

$$= \sum_{i=1}^{M} D_{x_i} u \, D_{x_i}(h-u) \, .$$

Applying the Gauss–Green theorem (Theorem 6.2.6), we obtain

$$\int_{\mathbb{B}^M(0,\sigma)} \operatorname{div} \boldsymbol{w} \, d\mathcal{L}^M = \int_{\partial \mathbb{B}^M(0,\sigma)} \boldsymbol{w} \cdot \boldsymbol{\eta} \, d\mathcal{H}^{M-1} \, ,$$

where $\boldsymbol{\eta}$ is the outward unit normal to $\partial \mathbb{B}^M(0,\sigma)$. Hence we conclude that

$$\int_{\mathbb{B}^M(0,\sigma)} \left[\sum_{i=1}^{M} D_{x_i} u \, D_{x_i}(h-u) \right] d\mathcal{L}^M$$

$$= \int_{\partial \mathbb{B}^M(0,\sigma)} (h-u) \sum_{i=1}^{M} D_{x_i} u \, \boldsymbol{\eta}_i \, d\mathcal{H}^{M-1}$$

$$= \int_{\partial \mathbb{B}^M(0,\sigma)} (h-\widetilde{g}_\delta) \sum_{i=1}^{M} D_{x_i} u \, \boldsymbol{\eta}_i \, d\mathcal{H}^{M-1} \, ,$$

where we use the boundary condition in (9.91) to replace u by $\widetilde{g}_\delta$ in the last term. Thus we have

$$(T_\sigma^0 - S)\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M) \Big)$$

$$= \int_{\partial \mathbb{B}^M(0,\sigma)} (h-\widetilde{g}_\delta) \, \sum_{i=1}^{M} D_{x_i} u \, \boldsymbol{\eta}_i \, d\mathcal{H}^{M-1} + R \, .$$

Now, using (9.92) to estimate $|Du| \le c_{13} \, E^\delta$, (9.22) to estimate $|h - g_\delta| \le 2 \, c_2 \, E^{1/(2M)}$, (9.83) to estimate $|g_\delta - \widetilde{g}_\delta| \le E^{1+\delta}$, and (9.86) to estimate

$$\mathcal{H}^{M-1}\Big\{ x \in \partial \mathbb{B}^M(0,\sigma) : g_\delta(x) \ne h(x) \Big\} \le c_{10} \, E^{1-4M\delta} \, ,$$

and recalling that $\delta = 1/(9M^2)$, we obtain the estimate

$$\left| \int_{\partial \mathbb{B}^M(0,\sigma)} (h-\widetilde{g}_\delta) \sum_{i=1}^{M} D_{x_i} u \, \boldsymbol{\eta}_i \, d\mathcal{H}^{M-1} \right|$$

$$\le \left| \int_{\partial \mathbb{B}^M(0,\sigma)} (h-g_\delta) \sum_{i=1}^{M} D_{x_i} u \, \boldsymbol{\eta}_i \, d\mathcal{H}^{M-1} \right|$$

$$
\begin{aligned}
&+ \left| \int_{\partial \mathbb{B}^M(0,\sigma)} (g_\delta - \widetilde{g}_\delta) \sum_{i=1}^{M} D_{x_i} u \, \boldsymbol{\eta}_i \, d\mathcal{H}^{M-1} \right| \\
&\leq c_{13} \, E^\delta \, \left(\int_{\partial \mathbb{B}^M(0,\sigma)} |h - g_\delta| \, d\mathcal{H}^{M-1} \right. \\
&\qquad \left. + \int_{\partial \mathbb{B}^M(0,\sigma)} |g_\delta - \widetilde{g}_\delta| \, d\mathcal{H}^{M-1} \right) \\
&\leq c_{13} \, E^\delta \, \left(2 \, c_2 \, E^{1/(2M)} \ \ c_{10} \, E^{1-4M\delta} + E^{1+\delta} \, M \, \Omega_M \right) \\
&= \ c_{13} \left(2 \, c_2 \, c_{10} \, E^{6^{-1}\delta^{1/2}} + M \, \Omega_M \, E^\delta \right) \ E^{1+\delta} \, . \qquad (9.114)
\end{aligned}
$$

Combining equation (9.112) with the estimates (9.113) and (9.114), we obtain the estimate

$$
\begin{aligned}
&\left| (T_\sigma^0 - S) \Big(\, DA(\overrightarrow{S} \,) - DA(\mathbf{e}^M) \Big) \right| \\
&\qquad \leq c_{17} \, E^{1+\delta} + (3/2) \, c_{13} \, E^\delta \int_{\mathbb{B}^M((,0),\sigma)} (|Du|^2 + |Dh|^2) \, d\mathcal{L}^M \, ,
\end{aligned}
$$

where we set $c_{17} = c_{13} \, (\, 2 \, c_2 \, c_{10} + M \, \Omega_M \,)$, as we may since $E < 1$.

Next, noting that we have $\operatorname{Lip} u \leq 1$ and $\operatorname{Lip} h \leq 1$, we apply Proposition 9.3.6 to conclude that

$$
|Du|^2 + |Dh|^2 \leq 4 \left(| \, \overrightarrow{S} - \mathbf{e}^M |^2 + | \, \overrightarrow{T_\sigma^0} - \mathbf{e}^M |^2 \right) .
$$

Assume now that the function $\overrightarrow{T_\sigma^0}$ has been extended (as has $\overrightarrow{S}$) to all of $\mathrm{C}(0, \sigma)$ by defining $\overrightarrow{T_\sigma^0}(X) = \overrightarrow{T_\sigma^0}[\mathbf{p}\,(X), h(\mathbf{p}\,(X))]$ at points where the right-hand side is defined and $\overrightarrow{T_\sigma^0}(X) = \mathbf{e}^M$ otherwise. Using also the fact that the measure $\|T_\sigma\|$ is larger than the measure $\mathcal{L}^M$, we obtain

$$
\begin{aligned}
&\left| (T_\sigma^0 - S) \Big(\, DA(\overrightarrow{S} \,) - DA(\mathbf{e}^M) \Big) \right| \\
&\qquad \leq c_{17} \, E^{1+\delta} + c_{16} \, E^\delta \int \left(| \, \overrightarrow{S} - \mathbf{e}^M |^2 + | \, \overrightarrow{T_\sigma^0} - \mathbf{e}^M |^2 \right) d\|T_\sigma\| \, ,
\end{aligned}
$$

with $c_{16} = 4 \cdot (3/2) \, c_{13}$.

Since

$$
\left| \, \overrightarrow{S} - \mathbf{e}^M \right|^2 \leq \left(\left| \, \overrightarrow{S} - \overrightarrow{T} \, \right| + \left| \, \overrightarrow{T} - \mathbf{e}^M \right| \right)^2 \leq 2 \Big(\left| \, \overrightarrow{S} - \overrightarrow{T} \, \right|^2 + \left| \, \overrightarrow{T} - \mathbf{e}^M \right|^2 \Big) ,
$$

we deduce that

$$\left|(T^0_\sigma - S)\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M)\Big)\right|$$

$$\leq c_{17}\,E^{1+\delta}$$

$$+\,c_{16}\,E^{\delta}\int\Big(2\Big|\overrightarrow{S}-\overrightarrow{T}\Big|^2+2\Big|\overrightarrow{T}-\mathbf{e}^M\Big|^2+\Big|\overrightarrow{T^0_\sigma}-\mathbf{e}^M\Big|^2\Big)\,d\|T_\sigma\|$$

$$= c_{17}\,E^{1+\delta}+2\,c_{16}\,E^{\delta}\int\Big|\overrightarrow{S}-\overrightarrow{T}\Big|^2\,d\|T_\sigma\|$$

$$+\,2\,c_{16}\,E^{\delta}\int\Big|\overrightarrow{T}-\mathbf{e}^M\Big|^2\,d\|T_\sigma\|$$

$$+\,c_{16}\,E^{\delta}\int\Big|\overrightarrow{T^0_\sigma}-\mathbf{e}^M\Big|^2\,d\|T_\sigma\|$$

$$\leq c_{17}\,E^{1+\delta}+2\,c_{16}\,E^{\delta}\int\Big|\overrightarrow{S}-\overrightarrow{T}\Big|^2\,d\|T_\sigma\|$$

$$+\,4\,c_{16}\,E^{\delta}\cdot E+c_{16}\,E^{\delta}\int\Big|\overrightarrow{T^0_\sigma}-\mathbf{e}^M\Big|^2\,d\|T_\sigma\|\,. \tag{9.115}$$

Using the fact that $\overrightarrow{T^0_\sigma}$ and $\overrightarrow{T}$ are $\mathcal{H}^M$-almost always simple unit M-vectors, we note that

$$\int\Big|\overrightarrow{T^0_\sigma}-\mathbf{e}^M\Big|^2\,d\|T_\sigma\|$$

$$\leq\int\Big|\overrightarrow{T}-\mathbf{e}^M\Big|^2\,d\|T_\sigma\|+\int\Big|\,|\overrightarrow{T^0_\sigma}-\mathbf{e}^M|^2-\Big|\overrightarrow{T}-\mathbf{e}^M\Big|^2\Big|\,d\|T_\sigma\|$$

$$\leq\ 2\,E+\int\Big|\,\Big|\overrightarrow{T^0_\sigma}-\mathbf{e}^M\Big|^2-\Big|\overrightarrow{T}-\mathbf{e}^M\Big|^2\Big|\,d\|T_\sigma\|$$

$$\leq 2\,E+2\int\Big|(\overrightarrow{T^0_\sigma}-\overrightarrow{T})\cdot\mathbf{e}^M\Big|\,d\|T_\sigma\|$$

$$\leq 2\,E+2\int\Big|\overrightarrow{T^0_\sigma}-\overrightarrow{T}\Big|\,d\|T_\sigma\|\,.$$

By (9.24), we have

$$\|T^0_\sigma - T_\sigma\|\mathrm{C}(0,\sigma)\leq c_4\,E\,,$$

so

$$\int\Big|\overrightarrow{T^0_\sigma}-\overrightarrow{T}\Big|\,d\|T_\sigma\|\leq c_4\,E\,,$$

and we conclude that

$$\int \left| \overrightarrow{T_\sigma^0} - \mathbf{e}^M \right|^2 d\|T_\sigma\| \leq 2\,(1 + c_4)\, E\,. \tag{9.116}$$

Combining (9.107), (9.115), and (9.116), we obtain the estimate

$$\left| (T_\sigma - S)\Big(DA(\overrightarrow{S}) - DA(\mathbf{e}^M) \Big) \right|$$
$$\leq c_{15}\, E^{1+\delta} + 2\, c_{16}\, E^\delta \int \left| \overrightarrow{S} - \overrightarrow{T} \right|^2 d\|T_\sigma\|\,,$$

with

$$c_{15} = c_4 \cdot 2\, c_{13} + c_{17} + 4\, c_{16} + c_{16} \cdot 2\,(1 + c_4)\,.$$

Thus the claim has been proved.

Combining the estimates. Combining (9.101) and (9.104), we obtain the estimate

$$\Big(1/2 - 2\, c_{16}\, E^\delta\Big) \int \left| \overrightarrow{S} - \overrightarrow{T} \right|^2 d\|T_\sigma\| \leq 2\, c_{12}\, E^{1+\delta} + c_{15}\, E^{1+\delta}\,.$$

So we have

$$\int \left| \overrightarrow{S} - \overrightarrow{T} \right|^2 d\|T_\sigma\| \leq c_{18}\, E^{1+\delta}\,, \tag{9.117}$$

where $c_{18} = 4\,(2\, c_{12} + c_{15})$, provided that

$$c_{16}\, E^\delta \leq 1/8 \tag{9.118}$$

holds.

Considering candidates for θ. Consider an arbitrary $0 < \theta < \sigma/4$. We have

$$\int_{\mathrm{C}(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S}(0) \right|^2 d\|T\|$$
$$\leq 2 \int_{\mathrm{C}(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T\| + 2 \int_{\mathrm{C}(0,2\theta)} \left| \overrightarrow{S} - \overrightarrow{S}(0) \right|^2 d\|T\|$$
$$\leq 2 \int_{\mathrm{C}(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T\| + 2 \Big(\sup_{\mathrm{C}(0,2\theta)} \left| \overrightarrow{S} - \overrightarrow{S}(0) \right|^2 \Big) \cdot \|T\|\mathrm{C}(0, 2\theta)\,.$$

Now

$$\|T\|\mathrm{C}(0, 2\theta) - \Omega_M\, (2\theta)^M = \frac{1}{2} \int_{\mathrm{C}(0,2\theta)} \left| \overrightarrow{T} - \mathbf{e}^M \right|^2 d\|T\| \leq E$$

(see (9.16)), so that

$$\|T\|\mathrm{C}(0, 2\theta) \leq \Omega_M\, (2\theta)^M + E \leq (1 + \Omega_M 2^M)\, \theta^M\,, \tag{9.119}$$

provided that

$$E \le \theta^M \tag{9.120}$$

holds. Successively applying (9.41), (9.93), and Proposition 9.3.6, we see that

$$\begin{aligned}\sup_{C(0,2\theta)} \left| \overrightarrow{S} - \overrightarrow{S}(0) \right|^2 &\le \sup_{C(0,2\theta)} |Du - Du(0)|^2 \\ &\le c_{14}\,\theta^2 \int_{\mathbb{B}^M(0,\sigma)} |Du|^2\, d\mathcal{L}^M \\ &\le 4\,c_{14}\,\theta^2 \int \left| \overrightarrow{S} - \mathbf{e}^M \right|^2 d\|T_\sigma\| \,. \end{aligned} \tag{9.121}$$

Using (9.119) and (9.121), we then deduce, subject to (9.120), that

$$\begin{aligned} &\frac{1}{2}\int_{C(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S}(0) \right|^2 d\|T\| \\ &\quad\le \int_{C(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T\| \\ &\qquad + c_{19}\,\theta^{M+2} \int \left| \overrightarrow{S} - \mathbf{e}^M \right|^2 d\|T_\sigma\| \\ &\quad\le \int_{C(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T\| \\ &\qquad + 2\,c_{19}\,\theta^{M+2} \int \left(\left| \overrightarrow{S} - \overrightarrow{T} \right|^2 + \left| \overrightarrow{T} - \mathbf{e}^M \right|^2 \right) d\|T_\sigma\| \\ &\quad\le (1 + 2\,c_{19}) \int \left| \overrightarrow{T} - \overrightarrow{S} \right|^2 d\|T_\sigma\| + 4\,c_{19}\,\theta^{M+2}\, E \,, \end{aligned} \tag{9.122}$$

where $c_{19} = 4\,c_{14} \cdot (1 + \Omega_M 2^M)$. Combining (9.122) and (9.117), we deduce that

$$\frac{1}{2}\int_{C(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S}(0) \right|^2 d\|T\| \le (1 + 2\,c_{19}) \cdot 2\,c_{18}\, E^{1+\delta} + 4\,c_{19}\,\theta^{M+2}\, E \,,$$

so

$$\frac{1}{2}\,\theta^{-M}\int_{C(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S}(0) \right|^2 d\|T\| \le (1 + 4\,c_{19})\,\theta^2\, E \tag{9.123}$$

holds, provided that

$$c_{16}\, E^\delta \le 1/8, \qquad E \le \theta^M, \qquad (1 + 2\,c_{19})\, c_{18}\, E^\delta \le \theta^2 \,. \tag{9.124}$$

Note that (9.124) includes conditions (9.118) and (9.120).

Bounding the slope of the harmonic function at 0. By definition we have

$$\frac{1}{2}\,\theta^{-M}\int_{C(0,2\theta)}\left|\overrightarrow{T}-\mathbf{e}^M\right|^2 d\|T\| \le \theta^{-M}\,E\,. \tag{9.125}$$

Using $\Omega_M(2\,\theta)^M \le \|T\|[C(0,2\,\theta)]$, we can estimate

$$\begin{aligned}
&\left|\overrightarrow{S}(0)-\mathbf{e}^M\right|^2\\
&\quad=\frac{1}{\|T\|C(0,2\,\theta)}\int_{C(0,2\,\theta)}\left|\overrightarrow{S}(0)-\mathbf{e}^M\right|^2 d\|T\|\\
&\quad\le\frac{1}{\Omega_M\,(2\,\theta)^M}\int_{C(0,2\,\theta)}\left|\overrightarrow{S}(0)-\mathbf{e}^M\right|^2 d\|T\|\\
&\quad\le\frac{2}{\Omega_M\,(2\,\theta)^M}\int_{C(0,2\,\theta)}\left(\left|\overrightarrow{S}(0)-\overrightarrow{T}\right|^2+\left|\overrightarrow{T}-\mathbf{e}^M\right|^2\right)d\|T\|\\
&\quad\le\frac{1}{\Omega_M\,2^{M-2}}\,\frac{1}{2}\,\theta^{-M}\int_{C(0,2\,\theta)}\left|\overrightarrow{S}(0)-\overrightarrow{T}\right|^2 d\|T\|\\
&\qquad\quad+\frac{1}{\Omega_M\,2^{M-2}}\,\frac{1}{2}\,\theta^{-M}\int_{C(0,2\,\theta)}\left|\overrightarrow{T}-\mathbf{e}^M\right|^2 d\|T\|\,.
\end{aligned}$$

By (9.123) and (9.125), we have

$$\left|\overrightarrow{S}(0)-\mathbf{e}^M\right|^2 \le c_{20}\,\theta^{-M}\,E\,, \tag{9.126}$$

provided that (9.124) holds, where we may set $c_{20}=2^{3-M}\,\Omega_M^{-1}\,(1+2\,c_{19})$.

Defining the isometry. It is easy to see that there exists a constant c_{21} such that (9.126) implies the existence of a linear isometry $\mathbf{j}$ of $\mathbb{R}^{M+1}$ with

$$\left\langle \textstyle\bigwedge_M \mathbf{j},\ \overrightarrow{S}(0)\right\rangle = \mathbf{e}^M \quad \text{and} \quad \|\mathbf{j}-\mathbf{I}_{\mathbb{R}^{M+1}}\|^2 \le c_{21}\,\theta^{-M}\,E\,. \tag{9.127}$$

One way to construct such a $\mathbf{j}$ is to set $v_i=\langle\, Du(0),\ \mathbf{e}_i\,\rangle$ for $i=1,2,\ldots,M$. Then apply the Gram–Schmidt orthogonalization procedure to the set

$$\{v_1, v_2, \ldots, v_M, \mathbf{e}_{M+1}\}$$

to obtain the orthonormal basis $\{w_1, w_2, \ldots, w_{M+1}\}$. Finally, let $\mathbf{j}$ be the inverse of the isometry represented by the matrix having the vectors w_i as its columns.

Recall that $T_0 = T \llcorner C(0,1/2)$. By (H1) (see page 262), we have

$$\operatorname{spt}\partial T \subseteq \mathbb{R}^{M+1}\setminus C(0,1)\,.$$

So we see that

$$\operatorname{dist}(\,\operatorname{spt}\partial T_0,\ C(0,1/4)\,) = 1/4\,.$$

By Lemma 9.2.1 and the assumption that $0 \in \operatorname{spt} T$, we have

$$\sup_{X \in C(0,1/2) \cap \operatorname{spt} T} |\mathbf{q}(X)| \le c_4 E^{1/(2M)}, \tag{9.128}$$

so $\operatorname{spt} \partial T_0 \subseteq \overline{\mathbb{B}}(0, 1/2 + c_4 E^{1/(2M)})$. By (9.127), we have

$$|x - \mathbf{j}(x)| \le (c_{21} \theta^{-M} E)^{1/2} \cdot (1/2 + c_4 E^{1/(2M)})$$

for $x \in \operatorname{spt} \partial T_0$. Thus if

$$(c_{21} \theta^{-M} E)^{1/2} \cdot (1/2 + c_4 E^{1/(2M)}) < 1/4 \tag{9.129}$$

holds, then we have

$$\operatorname{spt} \partial \mathbf{j}_\# T_0 \subseteq \mathbb{R}^N \setminus C(0, 1/4).$$

A similar argument shows that if

$$(c_{21} \theta^{-M} E)^{1/2} \cdot (\theta + c_4 E^{1/(2M)}) < \theta \tag{9.130}$$

holds, then we have

$$\operatorname{spt} T_0 \bigcap \mathbf{j}^{-1} C(0, \theta) \subseteq C(0, 2\theta).$$

Selecting θ and ϵ_* to complete the proof of the lemma. If we satisfy the conditions (9.124), (9.129), and (9.130), then we obtain the estimates (9.123), (9.127), and (9.128). Those estimates are

$$\frac{1}{2} \theta^{-M} \int_{C(0,2\theta)} \left| \overrightarrow{T} - \overrightarrow{S}(0) \right|^2 d\|T\| \overset{(9.123)}{\le} (1 + 4 c_{19}) \theta^2 E,$$

$$\|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\|^2 \overset{(9.127)}{\le} c_{21} \theta^{-M} E,$$

$$\sup_{X \in C(0,1/2) \cap \operatorname{spt} T} |\mathbf{q}(X)| \overset{(9.128)}{\le} c_4 E^{1/(2M)}.$$

We must choose θ and ϵ_* so that the estimates (9.123), (9.127), and (9.128) will imply that (9.76), (9.78), and (9.79) hold. Finally, we need to meet the conditions (9.75) in the statement of the lemma and the condition (9.80) that allowed the use of Lemmas 9.2.1 and 9.2.2. Thus a full set of conditions that, if satisfied, complete the proof of the lemma is the following (of course, θ and ϵ_* must be positive):

$$\theta \overset{(9.75)}{<} 1/8, \tag{9.131}$$

$$\epsilon_* \overset{(9.75)}{\le} (\theta/4)^{2M},$$

$$\epsilon_* \overset{(9.80)}{<} \epsilon_0,$$

$$
\begin{aligned}
c_{16}\, E^{\delta} &\overset{(9.124)}{\leq} 1/8\,, \\
E &\overset{(9.124)}{\leq} \theta^{M}\,, \\
(1 + 2\, c_{19})\, c_{18}\, E^{\delta} &\overset{(9.124)}{\leq} \theta^{2}\,, \\
(c_{21}\, \theta^{-M}\, E)^{1/2} \cdot (1/2 + c_4\, E^{1/(2M)}) &\overset{(9.129)}{<} 1/4\,, \\
(c_{21}\, \theta^{-M}\, E)^{1/2} \cdot (\theta + c_4\, E^{1/(2M)}) &\overset{(9.130)}{<} \theta\,, \\
c_4\, E^{1/(2M)} &\overset{\text{so } (9.128)\Rightarrow(9.76)}{\leq} 1/8\,, \\
c_{21}\, \theta^{-M}\, E &\overset{\text{so } (9.127)\Rightarrow(9.78)}{\leq} \theta^{-2M}\, E\,, \qquad (9.132) \\
\theta^{-2M}\, E &\overset{(9.77)}{\leq} 1/64\,, \\
(1 + 4\, c_{19})\, \theta^{2}\, E &\overset{\text{so } (9.123)\Rightarrow(9.79)}{\leq} \theta\, E\,. \qquad (9.133)
\end{aligned}
$$

We first choose and fix $0 < \theta$ such that (9.131), (9.132), and (9.133) hold. This choice is clearly independent of the value of E and the choice of ϵ_*. Then we select $0 < \epsilon_*$ such that, assuming that $E < \epsilon_*$ holds, the remaining conditions are satisfied. □

9.6 The Regularity Theorem

The next theorem gives us a flexible tool that we can use in proving regularity; the proof of the theorem is based on iteratively applying Lemma 9.5.1.

Theorem 9.6.1. *Let θ and ϵ_* be as in Lemma* 9.5.1. *There exist constants c_{22} and c_{23}, depending only on M, with the following property:*

If $0 \in \operatorname{spt} T$, if $T_0 = T \llcorner \mathrm{C}(0, \rho/2)$, and if the hypotheses (H1–H5) (*see page* 262) *hold with*

$$y = 0\,, \quad \epsilon = \epsilon_*\,,$$

then

$$E(\,T,\ 0,\ r\,) \leq c_{22}\, E(\,T,\ 0,\ \rho\,)\,, \quad \textit{for} \quad 0 < r \leq \rho\,, \tag{9.134}$$

and there exists a linear isometry $\mathbf{j}$ of $\mathbb{R}^{M+1}$ such that

$$\operatorname{spt} \partial \mathbf{j}_{\#} T_0 \cap \mathrm{C}(0, \rho/4) = \emptyset\,,$$

$$\|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\| \leq 4\, \theta^{-2M}\, E(\,T,\ 0,\ \rho\,) \leq 4^{-2}\,, \tag{9.135}$$

$$E(\,\mathbf{j}_{\#} T_0,\ 0,\ r\,) \leq c_{23} \cdot \frac{r}{\rho} \cdot E(\,T,\ 0,\ \rho\,) \quad \textit{for} \quad 0 < r \leq \rho/4\,. \tag{9.136}$$

Proof. Set $\mathbf{j}_0 = \mathbf{I}_{\mathbb{R}^{M+1}}$. We will show inductively that, for $q = 1, 2, \ldots$, there are linear isometries $\mathbf{j}_q$ of $\mathbb{R}^{M+1}$ such that, writing

$$T_q = \mathbf{j}_{q\#} T_0 \,,$$

we have

$$\sup_{X \in \operatorname{spt} T_{q-1} \cap \mathrm{C}(0, \theta^{q-1}\rho/4)} |\mathbf{q}(X)| \le \theta^{q-1}\rho/2 \quad \text{for} \quad q \ge 2\,, \tag{9.137}$$

$$E(T_q,\, 0,\, \theta^q \rho) \le \theta\, E(T_{q-1},\, 0,\, \theta^{q-1}\rho) \quad \text{for} \quad q \ge 2\,, \tag{9.138}$$

$$\|\mathbf{j}_q - \mathbf{j}_{q-1}\| \le \theta^{-M}\, \theta^{(q-1)/2}\, E(T,\, 0,\, \rho)^{1/2}\,, \tag{9.139}$$

$$E(T_q,\, 0,\, \theta^q \rho) \le \theta^q\, E(T,\, 0,\, \rho)\,. \tag{9.140}$$

Note that for $q = 2, 3, \ldots$, (9.140) follows from (9.138) and from the instance of (9.140) in which q is replaced by $q - 1$. Thus we need only verify (9.140) for the specific value $q = 1$.

Start of induction on q to prove (9.137)–(9.140). For $q = 1$, conditions (9.137) and (9.138) are vacuous, so we need only verify (9.139) and (9.140). Let $\mathbf{j}_1$ be the isometry whose existence is guaranteed by Lemma 9.5.1. Then the inequality (9.78) gives us (9.139), and the inequality (9.79) gives us (9.140).

Inductive step. Now suppose that (9.137)–(9.140) hold for q. We apply Lemma 9.5.1 to T_q with ρ replaced by $\theta^q\rho$. We may do so because $T_q = \mathbf{j}_{q\#}T_0$ is mass-minimizing. Inequality (9.76) of Lemma 9.5.1 gives us (9.137) with q replaced by $q + 1$.

The isometry $\mathbf{j}$ whose existence is guaranteed by Lemma 9.5.1 satisfies

$$\|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\| \le \theta^M\, E(T_q, 0, \theta^q\rho)^{1/2}\,, \tag{9.141}$$

$$E\Big(\mathbf{j}_\#\Big(T_q \llcorner \mathrm{C}(0, \theta^q\rho/2)\Big),\, 0,\, \theta^{q+1}\rho\Big) \le \theta\, E(T_q,\, 0,\, \theta^q\rho)\,. \tag{9.142}$$

By (9.140) and (9.141), we have

$$\|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\| \le \theta^{-M}\, \theta^{q/2}\, E(T,\, 0,\, \rho)^{1/2}\,.$$

Setting $\mathbf{j}_{q+1} = \mathbf{j} \circ \mathbf{j}_q$, we obtain

$$\|\mathbf{j}_{q+1} - \mathbf{j}_q\| = \|(\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}) \circ \mathbf{j}_q\| = \|\mathbf{j} - \mathbf{I}_{\mathbb{R}^{M+1}}\| \le \theta^{-M}\, \theta^{q/2}\, E(T,\, 0,\, \rho)^{1/2}\,,$$

which gives us (9.139) with q replaced by $q + 1$.

Since

$$\mathbf{j}_\#\Big(T_q \llcorner \mathrm{C}(0, \theta^q\rho/2)\Big) \llcorner \mathrm{C}(0, \theta^{q+1}\rho) = (\mathbf{j}_\# T_q) \llcorner \mathrm{C}(0, \theta^{q+1}\rho)\,,$$

we have

$$E(T_{q+1},\, 0,\, \theta^{q+1}\rho) \\ = E\Big(\mathbf{j}_{\#}\Big(T_q \llcorner \mathrm{C}(0, \theta^q \rho/2)\Big),\, 0,\, \theta^{q+1}\rho\Big) \le \theta\; E(T_q,\, 0,\, \theta^q \rho)\,,$$

which gives us (9.138) with q replaced by $q+1$. The inductive step has been completed.

Next we show that $\mathbf{j}_q$ has a well-defined limit as $q \to \infty$. For $Q > q \ge 0$, we estimate

$$\|\mathbf{j}_Q - \mathbf{j}_q\| \le \sum_{s=q}^{Q+1} \|\mathbf{j}_{s+1} - \mathbf{j}_s\| \le \theta^{-M} \sum_{s=q}^{\infty} \theta^{s/2}\; E(T_0, 0, \rho)^{1/2}$$

$$= \theta^{(q/2)-M}\; E(T_0, 0, \rho)^{1/2} \cdot \frac{1}{1-\sqrt{\theta}} \le 2\,\theta^{(q/2)-M}\; E(T_0, 0, \rho)^{1/2}\,.$$

Thus the $\mathbf{j}_q$ form a Cauchy sequence in the mapping-norm topology. We set

$$\mathbf{j} = \lim_{q\to\infty} \mathbf{j}_q$$

and conclude that

$$\|\mathbf{j} - \mathbf{j}_q\|^2 \le 4\,\theta^{q-2M}\; E(T_0, 0, \rho) \le 1/16 \tag{9.143}$$

holds for $0 \le q$.

Recall Corollary 9.1.7, which tells us how the excess is affected by an isometry. Using (9.143) together with (9.137), (9.139), and (9.140), we see that with an appropriate choice of c_{24},

$$E(\mathbf{j}_{\#}T_0, 0, \theta^q \rho) \le c_{24}\, \theta^q\; E(T_0, 0, \rho) \tag{9.144}$$

holds for each $q \ge 1$. Using (9.144) together with (9.76) and (9.143) with $q = 0$, we see that, with an appropriate choice of c_{25},

$$E(\mathbf{j}_{\#}T_0, 0, r) \le c_{25}\; (r/\rho)\; E(T_0, 0, \rho)$$

holds for $0 < r < \rho/4$, proving (9.136). Finally, we see that (9.134) follows from (9.76), (9.136), (9.137), and (9.143), again with $q = 0$. □

We are now ready to state and prove the regularity theorem.

Theorem 9.6.2 (Regularity). *There exist constants*

$$0 < \epsilon_1\,, \qquad 0 < c_{26} < \infty\,,$$

depending only on M, with the following property:

If the hypotheses (H1–H5) (*see page* 262) *hold with*

$$\epsilon = \epsilon_1\,,$$

then spt $T \cap \mathrm{C}(y, \rho/4)$ *is the graph of a* C^1 *function* u. *Moreover,* u *satisfies the following Hölder condition with exponent* $1/2$*:*

$$\sup_{\mathbb{B}^M(y,\rho/4)} \|Du\| + \rho^{1/2} \sup_{x,z\in\mathbb{B}^M(y,\rho/4),x\neq z} |x-z|^{-1/2}\,\|Du(x)-Du(z)\|$$
$$\leq c_{26}\Big(E(T,\,y,\,\rho)\Big)^{1/2}. \tag{9.145}$$

Remark 9.6.3.

(1) Once (9.145) is established, the higher regularity theory applies to show that u is in fact real analytic. The treatise [Mor 66] is the standard reference for the higher regularity theory including the results for systems of equations needed when surfaces of higher codimension are considered.
(2) By the constancy theorem, the regularity theorem implies immediately that $T \llcorner \mathrm{C}(y,\rho/4) = G_{\#}\Big(\mathbf{E}^M \llcorner \mathbb{B}^M(y,\rho/4)\Big)$, where G is the mapping $x \longmapsto (x, u(x))$.

Proof. We set

$$\epsilon_1 = \min\{\theta^{2M}\,\epsilon_*,\ 2^{-M}\,c_6^{-2M}\,c_{22}^{-1}\},$$

where θ and ϵ_* are as in Lemma 9.5.1, c_{22} is as in (9.134) in Theorem 9.6.1, and c_6 is as in (9.32) in the proof of Lemma 9.2.2.

In (9.75) in the statement of Lemma 9.5.1, we required that $0 < \theta < 1/8$ and that $0 < \epsilon_* < (\theta/4)^{2M}$. Thus we have $\epsilon_1 < \epsilon_*/2^M$, so $E(T,\,y,\,\rho) < \epsilon_1$ implies that $E(T,\,z,\,\rho/2) < \epsilon_*$ for each $z \in \mathbb{B}^M(y,\rho/2)$. Therefore, after translating the origin and replacing ρ by $\rho/2$, we can apply Theorem 9.6.1 to conclude that

$$E(T,\,z,\,r) \leq c_{22}\,E(T,\,z,\,\rho/2) \leq 2^M\,c_{22}\,E(T,\,y,\,\rho) \tag{9.146}$$

holds for $0 < r \leq \rho/2$ and $z \in \mathbb{B}^M(y,\rho/2)$. Theorem 9.6.1 also tells us that

$$E(\mathbf{j}_{z\#}\,T_z,\,z,\,r) \leq c_{23}\cdot\frac{r}{\rho/2}\cdot E(T,\,z,\,\rho/2)$$
$$\leq 2^{M+1}\,c_{23}\,E(T,\,y,\,\rho) \tag{9.147}$$

holds for $0 < r \leq \rho/8$, where $T_z = T \llcorner \mathrm{C}(y,\rho/4)$. It also says that $\mathbf{j}_z$ is an isometry of $\mathbb{R}^{M+1}$ with spt $\partial\mathbf{j}_{z\#}T_z \cap \mathrm{C}(z,\rho/8) = \emptyset$, $\mathbf{j}_z(z,w) = (z,w)$ for some point $(z,w) \in$ spt T, and

$$\|D\mathbf{j}_z - \mathbf{I}_{\mathbb{R}^{M+1}}\| \leq 4\,\theta^{-2M}\,E(T,\,z,\,\rho/2) \leq 4^{-2}. \tag{9.148}$$

In (9.80) of the proof of Lemma 9.5.1 we required that $\epsilon_* < \epsilon_0$, where ϵ_0 is as in Lemma 9.2.1. Thus we also have $\epsilon_1 < \epsilon_0$. Now we look in detail at the construction in the proof of Lemma 9.2.2 with $\gamma = 1$. In particular, when the choice

$$\eta = c_6^{-2M}$$

is made in (9.34), we guarantee that $\eta = c_6^{-2M}$ is strictly less than ϵ_0. Since $\epsilon_1 \leq 2^{-M} c_6^{-2M}$ holds, (9.146) implies that

$$E(T, z, r) \leq c_6^{-2M} = \eta$$

holds for $0 < r \leq \rho/2$ and $z \in \mathbb{B}^M(y, \rho/2)$. Thus the set A defined in (9.28) contains all of $\mathbb{B}^M(y, \rho/2)$. We conclude that there exists a Lipschitz function $g : \mathbb{B}^M(y, \rho/4) \to \mathbb{R}$ such that

$$\text{Lip } g \leq 1, \tag{9.149}$$

$$T \llcorner \mathrm{C}(y, \rho/4) = G_\#\Big(\mathbf{E}^M \llcorner \mathbb{B}^M(y, \rho/4)\Big), \tag{9.150}$$

with $G : \mathbb{B}^M(y, \rho/4) \to \mathrm{C}(y, \rho/4)$ defined by $G(x) = (x, g(x))$.

If $L_z : \mathbb{R}^M \to \mathbb{R}$ denotes the linear map whose graph is mapped to $\mathbb{R}^M \times \{0\}$ by $D\mathbf{j}_z$, then estimates (9.147), (9.148), (9.149) and equation (9.150) imply that

$$r^{-M} \int_{\mathbb{B}^M(z,r)} \|Dg - L_z\|^2 \, d\mathcal{L}^M \leq c_{27} \, (r/\rho) \, E(T, y, \rho) \tag{9.151}$$

holds for $0 < r \leq \rho/8$ and $z \in \mathbb{B}^M(y, \rho/4)$, where c_{27} is an appropriate constant.

We will apply (9.151) with $z_1, z_2 \in \mathbb{B}^M(y, \rho/4)$ and with $r = |z_1 - z_2| < \rho/8$. Setting $z_* = (z_1 + z_2)/2$ and $B = \mathbb{B}^M(z_1, r) \bigcap \mathbb{B}^M(z_2, r)$, we estimate

$$\begin{aligned}
\Omega_M \, (r/2)^M \, \|L_{z_1} - L_{z_2}\|^2 &\leq \int_B \|L_{z_1} - L_{z_2}\|^2 \, d\mathcal{L}^M \\
&\leq 2 \int_B \Big(\|DL_{z_1} - Dg\|^2 + \|Dg - L_{z_2}\|^2\Big) \, d\mathcal{L}^M \\
&\leq 2 \int_{\mathbb{B}^M(z_1,r)} \|DL_{z_1} - Dg\|^2 \, d\mathcal{L}^M \\
&\quad + 2 \int_{\mathbb{B}^M(z_2,r)} \|Dg - L_{z_2}\|^2 \, d\mathcal{L}^M \\
&\leq 2 \, r^M \, c_{27} \, (r/\rho) \, E(T, y, \rho).
\end{aligned}$$

Thus we have

$$\|L_{z_1} - L_{z_2}\|^2 \leq 2^{M+1} \, \Omega_M^{-1} \, c_{27} \, (|z_1 - z_2|/\rho) \, E(T, y, \rho).$$

Since (9.151) also implies that

$$Dg(z) = L_z$$

holds for $\mathcal{L}^M$-almost all $z \in \mathbb{B}^M(y, \rho/4)$, we conclude that

$$\|Dg(z_1) - Dg(z_2)\| \leq c_{28}\,(|z_1 - z_2|/\rho)^{1/2}\,E(\,T,\ y,\ \rho\,)^{1/2} \tag{9.152}$$

holds for $\mathcal{L}^M$-almost all $z_1, z_2 \in \mathbb{B}^M(y, \rho/4)$, where we set

$$c_{28} = 2^{(M+1)/2}\,\Omega_M^{-1/2}\,c_{27}^{1/2}\,.$$

Since g is Lipschitz, we conclude that g is C^1 in $\mathbb{B}^M(y, \rho/4)$, that (9.152) holds for all $z_1, z_2 \in \mathbb{B}^M(y, \rho/4)$, and that (9.145) follows from (9.148) and (9.152) when we set $u = g$. □

9.7 Epilogue

In our exposition of the regularity results, we made the simplifying assumptions that the current being studied was of *codimension one* and that it minimized the integral of the *area integrand*. Relaxing these assumptions introduces notational and technical complexity and requires deeper results to obtain bounds for solutions of the appropriate partial differential equation or system of partial differential equations. Nonetheless the proof of the regularity theorem goes through—as Schoen and Simon showed.

What is affected fundamentally by relaxing the assumptions is the applicability of the regularity theorem and the further results that can be proved. It is the hypothesis (H3) that causes the most difficulty in applying Theorem 9.6.2.

Because we have limited our attention to the codimension-one case, we have Theorem 7.5.5 available to decompose a mass-minimizing current into a sum of mass-minimizing currents each of which is the boundary of the current associated with a set of locally finite perimeter. Thus we have proved the following theorem.

Theorem 9.7.1. *If T is a mass-minimizing, integer-multiplicity current of dimension M in $\mathbb{R}^{M+1}$, then, for $\mathcal{H}^M$-almost every $a \in \operatorname{spt} T \setminus \operatorname{spt} \partial T$, there is $r > 0$ such that $\mathbb{B}(a, r) \bigcap \operatorname{spt} T$ is the graph of a C^1 function.*

The more general form of the regularity theorem in [SS 82] extends Theorem 9.7.1 to currents minimizing the integral of smooth elliptic integrands and, in higher codimensions, yields a set of regular points that is dense, though not necessarily of full measure.

Suppose that T is an M-dimensional, integer-multiplicity current in $\mathbb{R}^N$, and suppose that T minimizes the integral of a smooth M-dimensional elliptic integrand F. Let us denote the set of regular points of the current T by $\operatorname{reg} T$ and the set of singular points of T by $\operatorname{sing} T$. More precisely, $\operatorname{reg} T$ is defined by

$$\begin{aligned}\operatorname{reg} T = {} & (\,\operatorname{spt} T \setminus \operatorname{spt} \partial T\,) \\ & \cap \left\{a : \exists r > 0 \text{ such that } \mathbb{B}(a, r) \textstyle\bigcap \operatorname{spt} T \text{ is the graph of a } C^1 \text{ function}\right\}\end{aligned}$$

and

$$\operatorname{sing} T = \operatorname{spt} T \setminus \left(\operatorname{spt} \partial T \textstyle\bigcup \operatorname{reg} T\right)\,.$$

Table 9.1. Interior regularity of minimizing currents.

	$F = A$	$F \neq A$
$N - M = 1$	$\dim_{\mathcal{H}}(\operatorname{sing} T) \leq M - 7$ [Fed 70]	$\mathcal{H}^{M-2}(\operatorname{sing} T) = 0$ [SSA 77]
$N - M \geq 2$	$\dim_{\mathcal{H}}(\operatorname{sing} T) \leq M - 2$ [Alm 00]	$\operatorname{reg} T$ is dense in $\operatorname{spt} T \setminus \operatorname{spt} \partial T$ [Alm 68]

Table 9.1 summarizes what is known about $\operatorname{reg} T$ and $\operatorname{sing} T$ (and gives a reference for each result). In the table, A denotes the M-dimensional area integrand.

One can also consider the question of what happens near points of $\operatorname{spt} \partial T$, that is, *boundary regularity* as opposed to the *interior regularity* considered above. The earliest results in the context of geometric measure theory are in William K. Allard's work [All 68], [All 75]. Allard's results focus on the area integrand. Robert M. Hardt considered more general integrands in [Har 77]. For area-minimizing hypersurfaces, the definitive result is that of Hardt and Simon [HS 79], which tells us that if ∂T is associated with a C^2 submanifold, then, near every point of $\operatorname{spt} \partial T$, the set $\operatorname{spt} T$ is a C^1 embedded submanifold-with-boundary. More recently, Frank Duzaar and Klaus Steffen (see [DS 02]) have given a unified argument applicable to the interior and boundary regularity of currents that "almost" locally minimize the integral of a general elliptic integrand.

Regularity theory is not a finished subject. The finer structure of the singular set is not generally known (2-dimensional area-minimizing currents are an important exception—see [Cha 88]), so understanding the singular set remains a challenge. Also, techniques created to answer questions about surfaces that minimize integrals of elliptic integrands have found applicability in other areas, for instance, to systems of partial differential equations (e.g., [Eva 86]), mean curvature flows (e.g., [Whe 05]), and harmonic maps (e.g., [Whe 97]). The future will surely see more progress.