6

The Calculus of Differential Forms and Stokes's Theorem

In this chapter, we give a brief treatment of the classical theory of differential forms and Stokes's theorem. These topics provide motivation for the more abstract theory of currents.

6.1 Differential Forms and Exterior Differentiation

Multilinear Functions and m-Covectors

The dual space of $\mathbb{R}^N$ is very useful in the formulation of line integrals (see Appendices A.2 and A.3), but to define surface integrals we need to go beyond the dual space to consider functions defined on ordered m-tuples of vectors.

Definition 6.1.1. Let $(\mathbb{R}^N)^m$ be the Cartesian product of m copies of $\mathbb{R}^N$.

(1) A function $\phi : (\mathbb{R}^N)^m \to \mathbb{R}$ is *m-linear* if it is linear as a function of each of its m arguments; that is, for each $1 \le \ell \le m$, it holds that

$$\begin{aligned} &\phi(u_1, \dots, u_{\ell-1},\ \alpha u + \beta v,\ u_{\ell+1}, \dots, u_m) \\ &\quad = \alpha\, \phi(u_1, \dots, u_{\ell-1},\ u,\ u_{\ell+1}, \dots, u_m) \\ &\qquad + \beta\, \phi(u_1, \dots, u_{\ell-1},\ v,\ u_{\ell+1}, \dots, u_m)\,, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ and $u, v, u_1, \dots, u_{\ell-1}, u_{\ell+1}, \dots, u_m \in \mathbb{R}^N$. The more inclusive term *multilinear* means m-linear for an appropriate m.

(2) A function $\phi : (\mathbb{R}^N)^m \to \mathbb{R}$ is *alternating* if interchanging two arguments results in a sign change for the value of the function; that is, for $1 \le i < \ell \le m$, it holds that

$$\begin{aligned} &\phi(u_1, \dots, u_{i-1},\ u_i,\ u_{i+1}, \dots, u_{\ell-1},\ u_\ell,\ u_{\ell+1}, \dots, u_m) \\ &\quad = -\,\phi(u_1, \dots, u_{i-1},\ u_\ell,\ u_{i+1}, \dots, u_{\ell-1},\ u_i,\ u_{\ell+1}, \dots, u_m)\,, \end{aligned}$$

where $u_1, \dots, u_m \in \mathbb{R}^N$.

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(3) We denote by $\bigwedge^m(\mathbb{R}^N)$ the set of m-linear, alternating functions from $(\mathbb{R}^N)^m$ to $\mathbb{R}$. We endow $\bigwedge^m(\mathbb{R}^N)$ with the usual vector space operations of addition and scalar multiplication, namely,

$$(\phi + \psi)(u_1, u_2, \ldots, u_m) = \phi(u_1, u_2, \ldots, u_m) + \psi(u_1, u_2, \ldots, u_m)$$

and

$$(\alpha\,\phi)(u_1, u_2, \ldots, u_m) = \alpha \cdot \phi(u_1, u_2, \ldots, u_m)\,,$$

so $\bigwedge^m(\mathbb{R}^N)$ is itself a vector space. The elements of $\bigwedge^m(\mathbb{R}^N)$ are called *m-covectors of $\mathbb{R}^N$*.

Remark 6.1.2.

(1) In case $m = 1$, requiring a map to be alternating imposes no restriction; also, 1-linear is the same as linear. Consequently, we see that $\bigwedge^1(\mathbb{R}^N)$ is the dual space of $\mathbb{R}^N$; that is, $\bigwedge^1(\mathbb{R}^N) = (\mathbb{R}^N)^*$.

(2) Recalling that the standard basis for $\mathbb{R}^N$ is written $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$, we let $\mathbf{e}_i^*$ denote the dual of $\mathbf{e}_i$ defined by

$$\langle\, \mathbf{e}_i^*,\, \mathbf{e}_j \,\rangle = \begin{cases} 1 \text{ if } j = i, \\ 0 \text{ if } j \neq i. \end{cases}$$

Then $\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots, \mathbf{e}_N^*$ form the *standard dual basis* for $(\mathbb{R}^N)^*$.

(3) If $x_1, x_2, \ldots, x_N$ are the coordinates in $\mathbb{R}^N$, then it is traditional to use the alternative notation dx_i to denote the dual of $\mathbf{e}_i$; that is,

$$dx_i = \mathbf{e}_i^*\,, \quad \text{for } i = 1, 2, \ldots, N\,.$$

Example 6.1.3. The archetypical multilinear, alternating function is the determinant. As a function of its columns (or rows), the determinant of an N-by-N matrix is N-linear and alternating. It is elementary to verify that every element of $\bigwedge^N(\mathbb{R}^N)$ is a real multiple of the determinant function. □

The next definition shows how we can extend the use of determinants to define examples of m-linear, alternating functions when m is strictly smaller than N.

Definition 6.1.4. Let $a_1, a_2, \ldots, a_m \in \bigwedge^1(\mathbb{R}^N)$ be given. Each a_i can be written

$$a_i = a_{i\,1}\, dx_1 + a_{i\,2}\, dx_2 + \cdots + a_{i\,N}\, dx_N\,.$$

We define $a_1 \wedge a_2 \wedge \cdots \wedge a_m \in \bigwedge^m(\mathbb{R}^N)$, called the *exterior product of* $a_1, a_2, \ldots, a_m$, by setting

$$\begin{aligned} &(a_1 \wedge a_2 \wedge \cdots \wedge a_m)(u_1, u_2, \ldots, u_m) \\ &\quad = \det\left[\begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,N} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,N} \\ \vdots & \vdots & & \vdots \\ a_{m\,1} & a_{m\,2} & \ldots & a_{m\,N} \end{pmatrix} \begin{pmatrix} u_{1\,1} & u_{1\,2} & \ldots & u_{1\,m} \\ u_{2\,1} & u_{2\,2} & \ldots & u_{2\,m} \\ \vdots & \vdots & & \vdots \\ u_{N\,1} & u_{N\,2} & \ldots & u_{N\,m} \end{pmatrix}\right], \end{aligned} \tag{6.1}$$

where the u_{ij} are the components of the vectors $u_1, u_2, \ldots, u_m \in \mathbb{R}^N$; that is, each u_j is given by

$$u_j = u_{1\,j}\,\mathbf{e}_1 + u_{2\,j}\,\mathbf{e}_2 + \cdots + u_{N\,j}\,\mathbf{e}_N\,.$$

To see that the function in (6.1) is m-linear and alternating, rewrite it in the form

$$\begin{aligned}&(a_1 \wedge a_2 \wedge \cdots \wedge a_m)(u_1, u_2, \ldots, u_m)\\ &\quad = \det \begin{pmatrix} \langle a_1,\, u_1\rangle & \langle a_1,\, u_2\rangle & \ldots & \langle a_1,\, u_m\rangle \\ \langle a_2,\, u_1\rangle & \langle a_2,\, u_2\rangle & \ldots & \langle a_2,\, u_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle a_m,\, u_1\rangle & \langle a_m,\, u_2\rangle & \ldots & \langle a_m,\, u_m\rangle \end{pmatrix}, \end{aligned} \tag{6.2}$$

where $\langle a_i,\, u_j\rangle$ is the dual pairing of a_i and u_j (see Section A.2).

Elements of $\bigwedge^m \mathbb{R}^N$ that can be written in the form $a_1 \wedge a_2 \wedge \cdots \wedge a_m$ are called *simple m-covectors*.

Recall that $\bigwedge_m (\mathbb{R}^N)$ is the space of m-vectors in $\mathbb{R}^N$ defined in Section 1.4. It is easy to see that any element of $\bigwedge^m (\mathbb{R}^N)$ is well-defined on $\bigwedge_m (\mathbb{R}^N)$ (just consider the equivalence relation in Definition 1.4.1). Thus $\bigwedge^m (\mathbb{R}^N)$ can be considered the dual space of $\bigwedge_m (\mathbb{R}^N)$. Evidently

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}\,, \quad 1 \le i_1 < i_2 < \cdots < i_m \le N\,, \tag{6.3}$$

is the dual basis to the basis

$$\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_m}\,, \quad 1 \le i_1 < i_2 < \cdots < i_m \le N\,,$$

for $\bigwedge_m (\mathbb{R}^N)$.

Differential Forms

Definition 6.1.5. Let $W \subset \mathbb{R}^N$ be open. A *differential m-form* on W is a function $\phi : W \to \bigwedge^m (\mathbb{R}^N)$. We call m the *degree* of the form. We say that the differential m-form ϕ is C^k if for each set of (constant) vectors $v_1, v_2, \ldots, v_m$, the real-valued function $\langle \phi(p), v_1 \wedge v_2 \wedge \cdots \wedge v_m\rangle$ is a C^k function of $p \in W$.

The differential form can be rewritten in terms of a basis and component functions as follows: For each m-tuple $1 \le i_1 < i_2 < \cdots < i_m \le N$, define the real-valued function

$$\phi_{i_1, i_2, \ldots, i_m}(p) = \langle \phi(p), \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_m}\rangle\,.$$

Then we have

$$\phi = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le N} \phi_{i_1, i_2, \ldots, i_m}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}\,.$$

The natural role for a differential m-form is to serve as the integrand in an integral over an m-dimensional surface. This is consistent with and generalizes integration of a 1-form along a curve.

Definition 6.1.6. Suppose

(1) the m-dimensional surface $S \subseteq \mathbb{R}^N$ is parametrized by the function $F : U \to \mathbb{R}^N$, where U is an open subset of $\mathbb{R}^m$; that is, F is a one-to-one C^k ($k \geq 1$) function, DF is of rank m, and $S = F(U)$,
(2) $W \subseteq \mathbb{R}^N$ is open with $F(U) \subseteq W$, and
(3) ϕ is a differential m-form on W.

Then the *integral* of ϕ over S is defined by

$$\int_S \phi = \int_U \left\langle \phi \circ F(t), \frac{\partial F}{\partial t_1} \wedge \frac{\partial F}{\partial t_2} \wedge \cdots \wedge \frac{\partial F}{\partial t_m} \right\rangle d\mathcal{L}^m(t) \tag{6.4}$$

whenever the right-hand side of (6.4) is defined.

The surface S in Definition 6.1.6 is an oriented surface for which the orientation is induced by the orientation on $\mathbb{R}^m$ and the parametrization F. The value of the integral is unaffected by a reparametrization as long as the reparametrization is orientation-preserving.

Exterior Differentiation

In Appendix A.3 one can see how the exterior derivative of a function allows the fundamental theorem of calculus to be applied to the integrals of 1-forms along curves. The exterior derivative of a differential form, which we discuss next, is the mechanism that allows the fundamental theorem of calculus to be extended to higher-dimensional settings.

Definition 6.1.7. Suppose that $U \subset \mathbb{R}^N$ is open and $f : U \to \mathbb{R}$ is a C^k function, $k \geq 1$.

(1) The *exterior derivative* of f is the 1-form df on U defined by setting

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_N} dx_N \,. \tag{6.5}$$

Note that (6.5) is equivalent to

$$\langle df(p), \ v \rangle = \langle Df(p), \ v \rangle \,, \tag{6.6}$$

for $p \in U$ and $v \in \mathbb{R}^N$.

(2) The *exterior derivative* of the m-form $\phi = f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$, $m \geq 1$, is the $(m+1)$-form $d\phi$ given by setting

$$d\phi = (df) \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \,.$$

(3) The definition of exterior differentiation in (2) is extended by linearity to all C^k m-forms, $m \geq 1$.

The rules analogous to those for ordinary derivatives of sums and products of functions are given in the next lemma.

Lemma 6.1.8. *Let ϕ and ψ be C^1 m-forms and let θ be a C^1 ℓ-form. It holds that*

(1) $d(\phi + \psi) = (d\phi) + (d\psi)$,
(2) $d(\phi \wedge \theta) = (d\phi) \wedge \theta + (-1)^m \phi \wedge (d\theta)$.

Proof.
(1) Equation (1) follows immediately from Definition 6.1.7(3).

(2) Note that in case $m = 0$, equation (2) reduces to Definition 6.1.7(2) and the usual product rule. Now suppose that $m \geq 1$, $\phi = f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$, and $\theta = g\, dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}$. Using Definition 6.1.7(2), we compute

$$\begin{aligned}
&d(\phi \wedge \theta) \\
&\quad = d(fg)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell} \\
&\quad = [(df)\, g + f\,(dg)]\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell} \\
&\quad = [(df) \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}] \wedge [g\, dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}] \\
&\qquad + (-1)^m [f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}] \wedge [(dg) \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}] \\
&\quad = (d\phi) \wedge \theta + (-1)^m \phi \wedge (d\theta)\,.
\end{aligned}$$

□

In contrast to the situation for ordinary derivatives of functions, repeated exterior differentiation results in a trivial form.

Theorem 6.1.9. *If the differential m-form $\phi : U \to \bigwedge^m (\mathbb{R}^N)$ is C^k, $k \geq 2$, then $d\, d\phi = 0$ holds.*

Proof. For $m = 0$, ϕ is a real-valued function, so we have

$$\begin{aligned}
d\, d\phi &= \sum_{j \neq i} \sum_i \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right) dx_j \wedge dx_i \\
&= \sum_{i<j} \left[\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right)\right] dx_i \wedge dx_j = 0\,.
\end{aligned}$$

For $m \geq 1$ and $\phi = f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$, we have

$$\begin{aligned}
d\, d\phi &= \sum_{\substack{j \neq i \\ j \notin \{i_1, i_2, \ldots, i_m\}}} \sum_{i \notin \{i_1, i_2, \ldots, i_m\}} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) dx_j \wedge dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m} \\
&= \sum_{\substack{i<j \\ i, j \notin \{i_1, i_2, \ldots, i_m\}}} \left[\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right] dx_i \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m} \\
&= 0\,.
\end{aligned}$$

The result now follows from the linearity of exterior differentiation. □

Definition 6.1.10.

(1) An m-form ϕ is said to be *closed* if $d\phi = 0$.
(2) An m-form ϕ is said to be *exact* if there exists an $(m-1)$-form ψ such that $d\psi = \phi$.

Remark 6.1.11. Theorem 6.1.9 tells us that every exact form is closed. It is *not* the case that every closed form is exact. In fact, the distinction between closed forms and exact forms underlies the celebrated theorem of Georges de Rham (1903–1990) relating the geometrically defined singular cohomology of a smooth manifold to the cohomology defined by differential forms (see [DRh 31] or Theorem 29A in Chapter IV of [Whn 57]).

6.2 Stokes's Theorem

Motivation

Stokes's theorem[1] expresses the equality of the integral of a differential form over the boundary of a surface and the integral of the exterior derivative of the form over the surface itself. The simplest instance of this equality is found in the part of the fundamental theorem of calculus that assures us that the difference between the values of a (continuously differentiable) function at the endpoints of an interval is equal to the integral of the derivative of the function over that interval—here the interval plays the role of the surface and the endpoints form the boundary of that surface. In fact, Stokes's theorem can be considered the higher-dimensional generalization of the fundamental theorem of calculus.

Oriented Rectangular Solids in $\mathbb{R}^N$

In order to state Stokes's theorem, one needs to define the oriented geometric boundary of an m-dimensional surface. In fact, the general definitions are designed so that the proof of Stokes's theorem can be reduced to the special case of a nicely bounded region in $\mathbb{R}^N$, indeed, to the even more special case of a rectangular solid that has its faces parallel to the coordinate hyperplanes.

The space $\mathbb{R}^N$ itself is oriented by the unit N-vector $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N$. The orientation of a Lebesgue measurable subset of $\mathbb{R}^N$ will be induced by the orientation of $\mathbb{R}^N$ as described in the next definition.

Definition 6.2.1. Let $U \subseteq \mathbb{R}^N$ be $\mathcal{L}^N$-measurable, and let ω be a continuous differential N-form defined on U.

(1) The integral of ω over U is defined by setting

$$\int_U \omega = \int_U \langle \omega(x), \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N \rangle \, d\mathcal{L}^N(x) \,. \tag{6.7}$$

Note that on the left-hand side of (6.7), U denotes the *oriented* set, while on the right-hand side, U denotes the set of points. On the left-hand side of (6.7), U is

[1] George Gabriel Stokes (1819–1903).

deemed to have the *positive orientation* given by the unit N-vector $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N$. One must recognize from the context which meaning of U is being used. In Chapter 7, we will introduce a notation that allows us to explicitly indicate when U is to be considered an oriented set.

(2) If U is to be given the opposite, or negative, orientation, the resulting oriented set will be denoted by $-U$. We define

$$\int_{-U} \omega = \int_U -\langle \omega(x), \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N \rangle \, d\mathcal{L}^N(x) . \tag{6.8}$$

Definition 6.2.1 gives us a broadly applicable definition of the integral for an oriented set of top dimension. The matter is much more difficult for lower-dimensional sets.

A lower-dimensional case that is straightforward is that of a singleton set consisting of the point $p \in \mathbb{R}^N$. The point itself will be considered to be positively oriented. A 0-form is simply a function, and the "integral" over p is evaluation at p. Traditionally, evaluation at a point is called a *Dirac delta function*,[2] so we will use the notation

$$\boldsymbol{\delta}_p(f) = f(p)$$

for any real-valued function whose domain includes p.

The next definition will specify a choice of orientation for an $(N-1)$-dimensional rectangular solid in $\mathbb{R}^N$ that is parallel to a coordinate hyperplane.

Definition 6.2.2. Suppose that $N \geq 2$.

(1) An $(N-1)$-dimensional rectangular solid, parallel to a coordinate hyperplane in $\mathbb{R}^N$, is a set of the form

$$\mathcal{F} = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{c\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_N, b_N] ,$$

where $a_i < b_i$ for $i = 1, \ldots, i-1, i+1, \ldots, N$.

(2) The $(N-1)$-dimensional rectangular solid $\mathcal{F} \subseteq \mathbb{R}^N$ will be oriented by the $(N-1)$-vector

$$\widehat{\mathbf{e}_i} = \bigwedge_{j \neq i} \mathbf{e}_j = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{i-1} \wedge \mathbf{e}_{i+1} \wedge \cdots \wedge \mathbf{e}_N .$$

(3) Let ω be a continuous $(N-1)$-form defined on $\mathcal{F}$. The *integral of ω over $\mathcal{F}$* is defined by

$$\int_{\mathcal{F}} \omega = \int_{\mathcal{F}} \langle \omega(x), \widehat{\mathbf{e}_i} \rangle \, d\mathcal{H}^{N-1}(x) .$$

Similarly, the *integral of ω over $-\mathcal{F}$* is defined by

$$\int_{-\mathcal{F}} \omega = \int_{\mathcal{F}} -\langle \omega, \widehat{\mathbf{e}_i} \rangle \, d\mathcal{H}^{N-1} .$$

Note that $\int_{-\mathcal{F}} \omega = -\int_{\mathcal{F}} \omega$ holds.

[2] Paul Adrien Maurice Dirac (1902–1984).

(4) For a formal linear combination of $(N-1)$-dimensional rectangular solids as described in (1),

$$\sum \alpha_\ell \mathcal{F}_\ell \,, \tag{6.9}$$

we define

$$\int_{\sum \alpha_\ell \mathcal{F}_\ell} \omega = \sum \alpha_\ell \int_{\mathcal{F}_\ell} \omega \,. \tag{6.10}$$

We can now define the oriented boundary of the rectangular solid in $\mathbb{R}^N$ that has its faces parallel to the coordinate hyperplanes.

Definition 6.2.3. Let

$$\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \,,$$

where $a_i < b_i$, for $i = 1, 2, \ldots, N$.

(1) If $N \geq 2$, then for $i = 1, 2, \ldots, N$, set

$$\begin{aligned}\mathcal{R}_i^+ &= [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{b_i\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_N, b_N] \,, \\ \mathcal{R}_i^- &= [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{a_i\} \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_N, b_N] \,.\end{aligned}$$

In case $N = 1$, set $\mathcal{R}_1^+ = \boldsymbol{\delta}_{b_1}$ and $\mathcal{R}_1^- = \boldsymbol{\delta}_{a_1}$.

(2) The oriented boundary of $\mathcal{R}$, denoted by $\partial_{\circ}\mathcal{R}$ to distinguish it from the topological boundary, is the formal sum

$$\partial_{\circ}\mathcal{R} = \begin{cases} \boldsymbol{\delta}_{b_1} - \boldsymbol{\delta}_{a_1} & \text{if } N \geq 1 \,, \\ \displaystyle\sum_{i=1}^{N} (-1)^{i-1} \left(\mathcal{R}_i^+ - \mathcal{R}_i^-\right) & \text{if } N \geq 2 \,. \end{cases}$$

Stokes's Theorem on a Rectangular Solid

We now state and prove the basic form of Stokes's theorem.

Theorem 6.2.4. *Let*

$$\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \,,$$

where $a_i < b_i$, for $i = 1, 2, \ldots, N$. If ϕ is a C^k, $k \geq 1$, $(N-1)$-form on an open set containing $\mathcal{R}$, then it holds that

$$\int_{\partial_{\circ}\mathcal{R}} \phi = \int_{\mathcal{R}} d\phi \,.$$

Proof. For $N = 1$, the result is simply the fundamental theorem of calculus, so we will suppose that $N \geq 2$.

Write

$$\phi = \sum_{i=1}^{N} \phi_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N \, .$$

It suffices to prove that

$$\int_{\mathcal{R}} d(\phi_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N) \\ = \int_{\partial_o \mathcal{R}} (\phi_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N)$$

holds for each $1 \leq i \leq N$.

Fix an i between 1 and N. We compute

$$\begin{aligned} d(\phi_i \, & dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N) \\ &= (d\phi_i) \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N \\ &= \sum_{j=1}^{N} \frac{\partial \phi_i}{\partial x_j} \, dx_j \wedge \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N \\ &= \frac{\partial \phi_i}{\partial x_i} \, dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N \\ &= \frac{\partial \phi_i}{\partial x_i} \, (-1)^{i-1} \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_i \wedge dx_{i+1} \wedge \cdots \wedge dx_N \, , \end{aligned}$$

so we have

$$\begin{aligned} \int_{\mathcal{R}} & d(\phi_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N) \\ &= \int_{\mathcal{R}} (-1)^{i-1} \frac{\partial \phi_i}{\partial x_i} \, \langle dx_1 \wedge dx_2 \wedge \cdots \wedge dx_N, \; \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \mathbf{e}_N \rangle \, d\mathcal{L}^N \\ &= (-1)^{i-1} \int_{\mathcal{R}} \frac{\partial \phi_i}{\partial x_i} \, d\mathcal{L}^N \, . \end{aligned}$$

By applying Fubini's theorem to evaluate $\int_{\mathcal{R}} (\partial \phi_i / \partial x_i) \, d\mathcal{L}^N$, we obtain

$$\begin{aligned} \int_{\mathcal{R}} & \frac{\partial \phi_i}{\partial x_i} \, d\mathcal{L}^N \\ &= \int_{[a_1,b_1] \times \cdots \times [a_{i-1},b_{i-1}] \times [a_{i+1},b_{i+1}] \times \cdots \times [a_N,b_N]} \left(\int_{a_i}^{b_i} \frac{\partial \phi_i}{\partial x_i} \, d\mathcal{L}^1(x_i) \right) d\mathcal{L}^{N-1} \\ &= \int_{[a_1,b_1] \times \cdots \times [a_{i-1},b_{i-1}] \times [a_{i+1},b_{i+1}] \times \cdots \times [a_N,b_N]} \phi_i |_{x_i = b_i} \, d\mathcal{L}^{N-1} \end{aligned}$$

$$- \int_{[a_1,b_1]\times\cdots\times[a_{i-1},b_{i-1}]\times[a_{i+1},b_{i+1}]\times\cdots\times[a_N,b_N]} \phi_i|_{x_i=a_i}\, d\mathcal{L}^{N-1}$$

$$= \int_{\mathcal{R}_i^+} \phi_i\, d\mathcal{H}^{N-1} - \int_{\mathcal{R}_i^-} \phi_i\, d\mathcal{H}^{N-1} .$$

We conclude that

$$\int_{\mathcal{R}} d(\phi_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N)$$

$$= (-1)^{i-1} \left(\int_{\mathcal{R}_i^+} \phi_i\, d\mathcal{H}^{N-1} - \int_{\mathcal{R}_i^-} \phi_i\, d\mathcal{H}^{N-1} \right) . \tag{6.11}$$

On the other hand, we compute

$$\int_{\partial_\circ \mathcal{R}} \phi_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N$$

$$= \sum_{j=1}^{N} (-1)^{j-1} \int_{\mathcal{R}_j^+} \phi_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N$$
$$- \sum_{j=1}^{N} (-1)^{j-1} \int_{\mathcal{R}_j^-} \phi_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N$$

$$= \sum_{j=1}^{N} (-1)^{j-1} \int_{\mathcal{R}_j^+} \phi_i\, \langle dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N, \widehat{\mathbf{e}}_j \rangle\, d\mathcal{H}^{N-1}$$
$$- \sum_{j=1}^{N} (-1)^{j-1} \int_{\mathcal{R}_j^-} \phi_i\, \langle dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N, \widehat{\mathbf{e}}_j \rangle\, d\mathcal{H}^{N-1}$$

$$= (-1)^{i-1} \left(\int_{\mathcal{R}_i^+} \phi_i\, d\mathcal{H}^{N-1} - \int_{\mathcal{R}_i^-} \phi_i\, d\mathcal{H}^{N-1} \right) . \tag{6.12}$$

Since (6.11) and (6.12) agree, we have the result. □

The Gauss–Green Theorem

A vector field on an open set $U \subseteq \mathbb{R}^N$ is a function $V : U \to \mathbb{R}^N$. The component functions V_i, $i = 1, 2, \ldots, N$, are defined by setting

$$V_i(x) = V(x) \cdot \mathbf{e}_i ,$$

so we have $V = \sum_{i=1}^{N} V_i\, \mathbf{e}_i$. We say that V is C^k if the component functions are C^k. The *divergence* of V, denoted by div V, is the real-valued function

$$\operatorname{div} V = \sum_{i=1}^{N} \frac{\partial V_i}{\partial x_i} \,.$$

Given an $(N-1)$-form ϕ in $\mathbb{R}^N$ we can associate with it a vector field V by the following means: if ϕ is written

$$\phi = \sum_{i=1}^{N} \phi_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N \,,$$

then set

$$V = \sum_{i=1}^{N} (-1)^{i-1} \, \phi_i \, \mathbf{e}_i \,.$$

Direct calculation shows that

$$d\phi = (\operatorname{div} V) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_N$$

holds. One can also verify that

$$\int_{\partial_{\text{o}} \mathcal{R}} \phi = \int_{\partial \mathcal{R}} V \cdot \mathbf{n} \, d\mathcal{H}^{N-1}$$

holds, where $\mathbf{n}$ is the outward-pointing unit vector orthogonal to the topological boundary $\partial \mathcal{R}$. We call $\mathbf{n}$ the *outward unit normal vector*.

By converting the statement of Theorem 6.2.4 about integrals of forms into the corresponding statement about vector fields, one obtains the following result, called the Gauss–Green theorem[3] or the divergence theorem:

Corollary 6.2.5. *If V is a C^1 vector field on an open set containing $\mathcal{R}$, then*

$$\int_{\mathcal{R}} \operatorname{div} V \, d\mathcal{L}^N = \int_{\partial \mathcal{R}} V \cdot \mathbf{n} \, d\mathcal{H}^{N-1} \,.$$

By piecing together rectangular solids and estimating the error at the boundary, one can prove a more general version of Theorem 6.2.4 or of Corollary 6.2.5. Thus we have the following result.

Theorem 6.2.6. *Let $A \subseteq \mathbb{R}^N$ be a bounded open set with C^1 boundary, and let $\mathbf{n}(x)$ denote the outward unit normal to ∂A at x. If V is a C^1 vector field defined on $\overline{A}$, then*

$$\int_{A} \operatorname{div} V \, d\mathcal{L}^N = \int_{\partial A} V \cdot \mathbf{n} \, d\mathcal{H}^{N-1} \,.$$

Theorem 6.2.6 is by no means the most general result available. The reader should see [Fed 69, 4.5.6] for an optimal version of the Gauss–Green theorem.

[3] Johann Carl Friedrich Gauss (1777–1855), George Green (1793–1841).

The Pullback of a Form

Definition 6.2.7. Suppose that $U \subseteq \mathbb{R}^N$ is open and $F : U \to \mathbb{R}^M$ is C^k, $k \geq 1$. Fix a point $p \in U$. If the differential m-form ϕ is defined at $F(p)$, then the *pullback* of ϕ is the m-form, defined at p, denoted by $F^{\#}\phi$ and evaluated on $v_1, v_2, \ldots, v_m$ by setting

$$\langle F^{\#}\phi(p), v_1 \wedge v_2 \wedge \cdots \wedge v_m \rangle = \langle \phi[F(p)], D_{v_1}F \wedge D_{v_2}F \wedge \cdots \wedge D_{v_m}F \rangle , \quad (6.13)$$

where we use the notation

$$D_{v_i}F = \langle DF,\ v_i \rangle ,$$

for $i = 1, 2, \ldots, m$. In case $m = 0$, (6.13) reduces to $F^{\#}\phi = \phi \circ F$.

Remark 6.2.8. We now have three similar notations in use: $D_{v_i}F$ as above; $D_\lambda(\mu, x)$ for differentiation of measures, which was introduced in Section 4.3; and $D_S f(x)$ for the differential of f relative to the surface S, which was introduced in Section 5.3 for smooth surfaces and extended to rectifiable sets in Section 5.4. The notation that is meant should always be clear from context.

The next theorem tells us that the operations of pullback and exterior differentiation commute. This seems like an insignificant observation, but in fact, it is key to generalizing Stokes's theorem, i.e., Theorem 6.2.4.

Theorem 6.2.9. *Suppose that $U \subseteq \mathbb{R}^N$ is open and $F : U \to \mathbb{R}^M$ is C^k, $k \geq 2$. Fix a point $p \in U$. If the differential m-form ϕ is defined and C^k, $k \geq 2$, in a neighborhood of $F(p)$, then*

$$d(F^{\#}\phi) = F^{\#}(d\phi) \quad (6.14)$$

holds at p.

Proof. First we consider the case $m = 0$ in which $F^{\#}\phi = \phi \circ F$. Fix $v \in \mathbb{R}^N$. Using the chain rule and (6.6), we compute

$$\begin{aligned} \langle dF^{\#}\phi, v \rangle &= \langle d[\phi \circ F], v \rangle = \langle D[\phi \circ F],\ v \rangle \\ &= \langle D\phi[F(p)],\ \langle DF,\ v \rangle \rangle = \langle d\phi[F(p)],\ \langle DF,\ v \rangle \rangle . \end{aligned}$$

The most efficient argument to deal with the case $m \geq 1$ is to first consider a 1-form ϕ that can be written as an exterior derivative; that is, $\phi = d\psi$ for a 0-form ψ. Then we have

$$d(F^{\#}\phi) = d(F^{\#}d\psi) = d(dF^{\#}\psi) = 0 = F^{\#}(d\,d\psi) = F^{\#}(d\phi) .$$

Lemma 6.1.8 allows us to see that the set of forms satisfying (6.14) is closed under addition and exterior multiplication. The general case then follows by addition and exterior multiplication of 0-forms and exterior derivatives of 0-forms. □

In Appendix A.4, the reader can see an alternative argument that is less elegant, but which reveals the inner workings of interchanging a pullback and an exterior differentiation.

Stokes's Theorem

Let $\mathcal{R}$ be a rectangular solid in $\mathbb{R}^N$. If U is open with $\mathcal{R} \subseteq U \subseteq \mathbb{R}^N$ and $F : U \to \mathbb{R}^M$ is one-to-one and C^k, $k \geq 1$, then the F-image of $\mathcal{R}$ is an N-dimensional C^k surface parametrized by F. We denote this surface by

$$F_{\#}\mathcal{R}\,.$$

This definition extends to formal sums by setting $F_{\#}\Big[\sum_\alpha \mathcal{R}_\alpha\Big] = \sum_\alpha F_{\#}\mathcal{R}_\alpha$.

In Definition 6.1.6, we gave a definition for the integral of a differential form over a surface. The next lemma gives us another way of looking at that definition.

Lemma 6.2.10. *If ω is a continuous N-form defined in a neighborhood of $F(\mathcal{R})$, then*

$$\int_{F_{\#}\mathcal{R}} \omega = \int_{\mathcal{R}} F^{\#}\omega\,.$$

Proof. By Definition 6.1.6, we have

$$\int_{F_{\#}\mathcal{R}} \omega = \int_{\mathcal{R}} \left\langle \omega \circ F(t),\, \frac{\partial F}{\partial t_1} \wedge \frac{\partial F}{\partial t_2} \wedge \cdots \wedge \frac{\partial F}{\partial t_N} \right\rangle d\mathcal{L}^N(t)\,.$$

Observing that

$$\frac{\partial F}{\partial t_i} = \langle DF,\, \mathbf{e}_i \rangle\,,$$

for $i = 1, 2, \ldots, N$, we see that

$$\begin{aligned}
&\left\langle \omega \circ F(t),\, \frac{\partial F}{\partial t_1} \wedge \frac{\partial F}{\partial t_2} \wedge \cdots \wedge \frac{\partial F}{\partial t_N} \right\rangle \\
&\qquad = \langle \omega \circ F(t), \langle DF,\, \mathbf{e}_1\rangle \wedge \langle DF,\, \mathbf{e}_2\rangle \wedge \cdots \wedge \langle DF,\, \mathbf{e}_N\rangle\rangle \\
&\qquad = \left\langle F^{\#}\omega,\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_N \right\rangle,
\end{aligned}$$

and the result follows. □

The boundary of a smooth surface is usually defined by referring back to the space of parameters. That is our motivation for the next definition.

Definition 6.2.11. The *oriented boundary* of $F_{\#}\mathcal{R}$ will be denoted by $\partial_{\circ} F_{\#}\mathcal{R}$ and is defined by

$$\partial_{\circ} F_{\#}\mathcal{R} = \sum_{i=1}^{N} (-1)^{i-1} \left(F_{\#}\mathcal{R}_i^{+} - F_{\#}\mathcal{R}_i^{-} \right) = F_{\#}\partial_{\circ}\mathcal{R}\,.$$

Some explanation of this definition is called for because $F_\# \mathcal{R}_i^+$ and $F_\# \mathcal{R}_i^-$ do not quite fit our earlier discussion. Recall that $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ lie in planes parallel to the coordinate hyperplanes, so F restricted to either $\mathcal{R}_i^+$ or $\mathcal{R}_i^-$ can be thought of as a function on $\mathbb{R}^{N-1}$. Note that both $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$ are oriented in a manner consistent with this interpretation.

We are now in a position to state and prove a general version of Stokes's theorem.

Theorem 6.2.12 (Stokes's Theorem). *Let $\mathcal{R}$ be a rectangular solid in $\mathbb{R}^N$. Suppose that U is open with $\mathcal{R} \subseteq U \subseteq \mathbb{R}^N$ and that $F : U \to \mathbb{R}^M$ is one-to-one and C^k, $k \geq 1$, with DF of rank N at every point of U. If ω is a C^k, $k \geq 2$, $(N-1)$-form defined on $F(\mathcal{R})$, then*

$$\int_{F_\# \mathcal{R}} d\omega = \int_{\partial_\circ F_\# \mathcal{R}} \omega \, .$$

Proof. We compute

$$\int_{F_\# \mathcal{R}} d\omega \overset{\text{(Lemma 6.2.10)}}{=} \int_{\mathcal{R}} F^\#(d\omega) \overset{\text{(Thm. 6.2.9)}}{=} \int_{\mathcal{R}} d(F^\# \omega)$$

$$\overset{\text{(Thm. 6.2.4)}}{=} \int_{\partial_\circ \mathcal{R}} F^\# \omega \overset{\text{(Lemma 6.2.10)}}{=} \int_{F_\# \partial_\circ \mathcal{R}} \omega \overset{\text{(Def. 6.2.11)}}{=} \int_{\partial_\circ F_\# \mathcal{R}} \omega \, . \qquad \square$$

As was true for the earlier version of Stokes's theorem (Theorem 6.2.4) and for the Gauss–Green theorem (Corollary 6.2.5), a more general version of Theorem 6.2.12 may be obtained by piecing together patches of surface. Since the theory of currents gives a still more general expression to Stokes's theorem, we will defer further discussion of Stokes's theorem until we have introduced the language of currents.