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Exterior Dirichlet and Neumann Problems for the Helmholtz Equation as Limits of Transmission Problems

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24.1 Statement of the Problems

In this chapter, we propose a new way of understanding the classical exterior Dirichlet and Neumann problems for the Helmholtz equation as limiting situations of transmission problems, and study the stability of this limiting process under discretization. This kind of problems appear in the study of the scattering of time-harmonic acoustic and thermal waves.

We assume that $\Omega_{\rm int} \subset \mathbb{R}^d$, $d = 2$ or 3, is a bounded, simply connected, open set with smooth boundary Γ. If the obstacle is impenetrable, then the scattering amplitude of a time-harmonic wave with wavenumber λ^2 solves an exterior Dirichlet or Neumann problem for the Helmholtz equation $\Delta u + \lambda^2 u = 0$ in $\Omega_{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}_{\rm int}$. It satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} (\partial_r u - \imath\lambda u) = 0,$$

uniformly in all directions $\mathbf{x}/|\mathbf{x}| \in \mathbb{R}^d$, $r := |\mathbf{x}|$ (see [CK83]). When waves can propagate through Γ, that is, when the obstacle is penetrable, and the physical properties in both media are different, the problem in $\Omega_{\rm int}$ is modeled by $\Delta u + \mu^2 u = 0$. Both Helmholtz equations are coupled through two continuity conditions of the form

$$\begin{aligned} u^{\rm int} - u^{\rm ext} &= f, && \text{on } \Gamma, \\ \alpha\,\partial_n u^{\rm int} - \beta\,\partial_n u^{\rm ext} &= \beta\, g, && \text{on } \Gamma. \end{aligned}$$

Typically, $f = u_{\rm inc}$ and $g = \partial_n u_{\rm inc}$ are the Cauchy data on Γ of an incident wave, a known solution to the exterior Helmholtz equation. In acoustics, μ^2 is proportional to ρ/α^2, where ρ is the density and α the velocity of transmission in $\Omega_{\rm int}$. For thermal waves, μ^2 is proportional to $\imath\rho/\alpha$, where ρ is the density multiplied by the specific heat capacity and α is the conductivity. General conditions on the parameters λ, μ, α, and β guaranteeing uniqueness can be found in [RS06a] and the references therein.

Dirichlet, Neumann, and transmission problems have been studied successfully from both the analytical and the numerical points of view in a wide number of works with a special emphasis on the study of acoustic waves (see for instance [CK83], [CS85], [KM88], [KR78], and [TW93]). More recently, Helmholtz transmission problems have also appeared in the study of scattering of thermal waves (see [Man01], [RS06a], and [TSS02]).

When studying the behavior of the solution to the transmission problem depending on the interior parameters, physical experiments as well as numerical simulations seem to point out that, for a fixed interior wave number, when the parameter α tends to zero, the solution to

$$(P_\alpha)\quad\left|\begin{array}{ll} \Delta u_\alpha + \lambda^2 u_\alpha = 0, & \text{in } \Omega_{\text{ext}}, \\ \Delta u_\alpha + \mu^2 u_\alpha = 0, & \text{in } \Omega_{\text{int}}, \\ u_\alpha^{\text{int}} - u_\alpha^{\text{ext}} = f, & \text{on } \Gamma, \\ \alpha\, \partial_n u_\alpha^{\text{int}} - \beta\, \partial_n u_\alpha^{\text{ext}} = \beta\, g, , & \text{on } \Gamma, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} (\partial_r u_\alpha - \imath\lambda u_\alpha) = 0, & \end{array}\right.$$

tends to the solution to the exterior Neumann problem

$$(P_N)\quad\left|\begin{array}{ll} \Delta u_N + \lambda^2 u_N = 0, & \text{in } \Omega_{\text{ext}}, \\ \partial_n u_N = -g, & \text{on } \Gamma, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} (\partial_r u_N - \imath\lambda u_N) = 0, & \end{array}\right.$$

whereas if α goes to infinity, the solution (P_α) converges to the solution of the exterior Dirichlet problem

$$(P_D)\quad\left|\begin{array}{ll} \Delta u_D + \lambda^2 u_D = 0, & \text{in } \Omega_{\text{ext}}, \\ u_D = -f, & \text{on } \Gamma, \\ \lim_{r\to\infty} r^{\frac{d-1}{2}} (\partial_r u_D - \imath\lambda u_D) = 0. & \end{array}\right.$$

This can also be seen by taking limits formally. The aim of this work is to give a rigorous proof of these facts, providing the corresponding convergence rates. We want to point out that we are restricting ourselves to a particular family of transmission problems where only one of the two interior parameters varies. In this case we will show linear convergence. To improve our estimates, both interior parameters would have to converge to zero in the Neumann case or to infinity in the Dirichlet one. In view of numerical experiments in the two-dimensional setting, we believe that for the case of the Dirichlet problem, the faster the modulus of the interior wavenumber increases, the higher the convergence rate is, although we cannot predict any rate in terms of it. On the other hand, for the Neumann problem, we have not observed any substantial improvement by making the interior wavenumber tend to zero. At the current stage of our research, we cannot prove the results when both parameters vary, since our study is based on the very simple fact that all the integral operators involved in the boundary formulation do not depend on α. Taking into account that the fundamental solution depends on the wavenumber, our study cannot be adapted easily to the case of a family of transmission problems depending on both interior parameters.

24.2 Boundary Integral Formulations

Since we are dealing with exterior problems, a suitable way of inspecting them is by using boundary integral formulations. We introduce the fundamental solution to the Helmholtz equation $\Delta u + \rho^2 u = 0$,

$$\phi_\rho(\mathbf{x}, \mathbf{y}) := \begin{cases} \imath\, H_0^{(1)}(\rho\,|\mathbf{x}-\mathbf{y}|)/4, & \text{if } d = 2, \\ \exp(\imath\,\rho|\mathbf{x}-\mathbf{y}|)/(4\pi|\mathbf{x}-\mathbf{y}|), & \text{if } d = 3, \end{cases}$$

and the single-layer potential

$$\mathcal{S}^\rho\varphi \;:=\; \int_\Gamma \phi_\rho(\,\cdot\,,\mathbf{y})\,\varphi(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \mathbb{R}^d \longrightarrow \mathbb{C}.$$

We also define the boundary integral operators

$$V^\rho\varphi := \int_\Gamma \phi_\rho(\,\cdot\,,\mathbf{y})\,\varphi(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \Gamma \longrightarrow \mathbb{C},$$

$$J^\rho\varphi := \int_\Gamma \partial_{n(\,\cdot\,)}\phi_\rho(\,\cdot\,,\mathbf{y})\,\varphi(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \Gamma \longrightarrow \mathbb{C}.$$

We recall some well-known properties of the integral operators above (see [McL00]): (i) the bounded operator $V^\rho : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible if and only if $-\rho^2$ is not a Dirichlet eigenvalue of the Laplace operator in Ω_{int}; (ii) the bounded operator $-\frac{1}{2}I + J^\rho : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is invertible; and (iii) the bounded operator $\frac{1}{2}I + J^\rho : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is invertible if and only if $-\rho^2$ is not a Neumann eigenvalue of the Laplace operator in Ω_{int}.

We will use indirect formulations in terms of single-layer potentials that can fail if either $-\mu^2$ or $-\lambda^2$ are Dirichlet eigenvalues of the Laplace operator in Ω_{int} and if $-\mu^2$ is a Neumann eigenvalue of the Laplacian in Ω_{int}. To avoid these particular cases, we can adapt our results to the indirect formulation proposed in [RS06b] and based on Brakhage–Werner potentials.

The solution to the Dirichlet problem (P_D) can be represented as $u_D = \mathcal{S}^\lambda\psi_D$, where ψ_D is the unique solution to

$$V^\lambda\psi_D = -f. \tag{24.1}$$

The solution to the Neumann problem (P_N) is $u_N = \mathcal{S}^\lambda\psi_N$, where ψ_N is the unique solution to

$$(\tfrac{1}{2}I - J^\lambda)\psi_N = g. \tag{24.2}$$

Finally, the solution to the transmission problem (P_α) can be obtained as $u_\alpha = \mathcal{S}^\lambda\psi_\alpha$ in $\Omega_{\rm ext}$ and $u_\alpha = \mathcal{S}^\mu\varphi_\alpha$ in $\Omega_{\rm int}$, with $(\psi_\alpha, \varphi_\alpha)$ solving

$$\mathcal{H}_\alpha \begin{bmatrix} \varphi_\alpha \\ \psi_\alpha \end{bmatrix} := \begin{bmatrix} V^\mu & -V^\lambda \\ \alpha(\frac{1}{2}I + J^\mu) & \beta(\frac{1}{2}I - J^\lambda) \end{bmatrix} \begin{bmatrix} \varphi_\alpha \\ \psi_\alpha \end{bmatrix} = \begin{bmatrix} f \\ \beta\, g \end{bmatrix}. \tag{24.3}$$

The proof of these results can be found in [CZ92, Chap. 7] and [RS06a].

24.3 Convergence Analysis

We start by noticing that if $\mathbf{x} \in \Omega_{\rm ext}$, then

$$|u_\alpha(\mathbf{x}) - u_*(\mathbf{x})| = |\mathcal{S}^\lambda(\psi_\alpha - \psi_*)(\mathbf{x})| = |\langle \psi_\alpha - \psi_*, \phi_\lambda(\mathbf{x}, \,\cdot\,)\rangle| \le C_{\mathbf{x}} \|\psi_\alpha - \psi_*\|_{-1/2},$$

where the subscript "$*$" stands for either D or N. Therefore, the study of pointwise convergence in $\Omega_{\rm ext}$ can be carried out by analyzing the convergence of the densities in $H^{-1/2}(\Gamma)$. Indeed, here we use the natural $H^{-1/2}(\Gamma)$-norm, but when using a weaker or stronger norm, one obtains the same convergence rate in terms of α. The only difference is the constant appearing in the estimate. In any case, it does not depend on α, but depends on $\mathbf{x}$. It only blows up when we are close to Γ and it is uniformly bounded in the exterior of any ball containing Γ when $\lambda \notin \mathbb{R}$, whereas for $\lambda \in \mathbb{R}$, uniform boundedness is only assured in compact sets.

Proposition 1. *Consider the operators*

$$A := (\tfrac{1}{2}I + J^\mu)(V^\mu)^{-1} \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$
$$D := \beta^{-1}(\tfrac{1}{2}I - J^\lambda)^{-1} A V^\lambda \,:\, H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$
$$H_\alpha := \beta(\tfrac{1}{2}I - J^\lambda) + \alpha A V^\lambda \,:\, H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

Then

(a) *If* $|\alpha| < \|D\|^{-1}$, *then* H_α *is invertible. Moreover,*

$$\|H_\alpha^{-1}\| \le C, \qquad \forall\, |\alpha| \le \alpha_0 < \|D\|^{-1}.$$

(b) *If* $|\alpha| > \|D^{-1}\|$, *then* H_α *is invertible. Moreover,*

$$\|H_\alpha^{-1}\| \le C\, |\alpha|^{-1}, \qquad \forall\, |\alpha| \ge \alpha_0 > \|D^{-1}\|.$$

(c) *If either* $|\alpha| < \|D\|^{-1}$ *or* $|\alpha| > \|D^{-1}\|$, *then*

$$\psi_\alpha = H_\alpha^{-1}\left(-\alpha A f + \beta\, g\right). \tag{24.4}$$

Proof. First, we assume that $|\alpha| < \|D\|^{-1}$ and decompose

$$H_\alpha \;=\; \beta(\tfrac{1}{2}I - J^\lambda)(I + \alpha D). \tag{24.5}$$

Applying the geometric series theorem (see [AH01, Theorem 2.3.1]), we deduce that H_α is invertible. Furthermore, for all $|\alpha| \le \alpha_0 < \|D\|^{-1}$,

$$\|H_\alpha^{-1}\| \;\le\; \|\beta^{-1}(\tfrac{1}{2}I - J^\lambda)^{-1}\| \|(I + \alpha D)^{-1}\| \;\le\; \frac{C}{1 - |\alpha|\|D\|} \;\le\; C'.$$

For $|\alpha| > \|D^{-1}\|$, the proof is completely analogous: We now decompose

$$H_\alpha \;=\; \alpha A V^\lambda (I + \alpha^{-1} D^{-1}), \tag{24.6}$$

to deduce the invertibility of H_α and the uniform bound

$$\|H_\alpha^{-1}\| \le |\alpha|^{-1} \|(V^\lambda)^{-1} A^{-1}\| \|(I + \alpha^{-1} D^{-1})^{-1}\| \le \frac{C\,|\alpha|^{-1}}{1 - |\alpha|^{-1}\|D^{-1}\|} \le C'|\alpha|^{-1},$$

for all $|\alpha| \ge \alpha_0 > \|D^{-1}\|$. Finally, to show (c), we remark that

$$\mathcal{H}_\alpha = \begin{bmatrix} I & 0 \\ \alpha A & I \end{bmatrix} \begin{bmatrix} V^\mu & -V^\lambda \\ 0 & H_\alpha \end{bmatrix},$$

with $\mathcal{H}_\alpha$ being the operator introduced in (24.3). By (a) and (b), the operator H_α is invertible for the considered values of α, and

$$\begin{aligned} \mathcal{H}_\alpha^{-1} &= \begin{bmatrix} (V^\mu)^{-1} & (V^\mu)^{-1} V^\lambda H_\alpha^{-1} \\ 0 & H_\alpha^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\alpha A & I \end{bmatrix} \\ &= \begin{bmatrix} (V^\mu)^{-1}(I - \alpha V^\lambda H_\alpha^{-1} A) & (V^\mu)^{-1} V^\lambda\, H_\alpha^{-1} \\ -\alpha\, H_\alpha^{-1} A & H_\alpha^{-1} \end{bmatrix}. \end{aligned}$$

Finally, the result follows readily from (24.3).

Proposition 2. (a) *For all* $|\alpha| \le \alpha_0 < \|D\|^{-1}$,

$$\|\psi_\alpha - \psi_N\|_{-1/2} \;\le\; C\,|\alpha|.$$

(b) *For all* $|\alpha| \ge \alpha_0 > \|D^{-1}\|$,

$$\|\psi_\alpha - \psi_D\|_{-1/2} \;\le\; C\,|\alpha|^{-1}.$$

Proof. (a) From (24.2) and (24.4) it follows that

$$\psi_\alpha - \psi_N = -\alpha\, H_\alpha^{-1} A f + \big(\beta\, H_\alpha^{-1} - (\tfrac{1}{2}I - J^\lambda)^{-1}\big)\, g,$$

and, by (24.5), we can write

$$\beta\, H_\alpha^{-1} - (\tfrac{1}{2}I - J^\lambda)^{-1} \;=\; \beta\, H_\alpha^{-1} - \beta(I+\alpha D)H_\alpha^{-1} \;=\; -\alpha\beta D H_\alpha^{-1}.$$

Applying Proposition1(a), we now easily deduce the result. To prove (b), we proceed likewise: by direct computation using (24.1) and (24.4), we see that

$$\psi_\alpha - \psi_D = \left(-\alpha H_\alpha^{-1}A + (V^\lambda)^{-1}\right) f + \beta H_\alpha^{-1} g,$$

and, by (24.6), we have

$$-\alpha H_\alpha^{-1}A + (V^\lambda)^{-1} \;=\; -\alpha H_\alpha^{-1}A + (\alpha\, I + D^{-1})H_\alpha^{-1}A \;=\; D^{-1}H_\alpha^{-1}A.$$

The result is now a consequence of Proposition1(b).

Corollary 1. (a) *The solution of* (P_α) *converges to the solution of* (P_N) *in* $\Omega_{\rm ext}$ *as* $\alpha \to 0$. *Moreover, for all* $|\alpha| \le \alpha_0 < 1/\|D\|$,

$$|u_\alpha(\mathbf{x}) - u_N(\mathbf{x})| \le C_{\mathbf{x}}|\alpha|, \qquad \mathbf{x} \in \Omega_{\rm ext}.$$

(b) *The solution of* (P_α) *converges to the solution of* (P_D) *in* $\Omega_{\rm ext}$ *as* $\alpha \to \infty$. *Moreover, for all* $|\alpha| \ge \alpha_0 > \|D^{-1}\|$,

$$|u_\alpha(\mathbf{x}) - u_D(\mathbf{x})| \le C_{\mathbf{x}}|\alpha|^{-1}, \qquad \mathbf{x} \in \Omega_{\rm ext}.$$

24.4 Convergence at the Discrete Level

In this section, we describe briefly how the previous study applies when dealing with numerical approximations to (P_D), (P_N), and (P_α) obtained by an abstract class of discretizations sharing some common features. The hypotheses we will specify shortly are satisfied by a wide number of numerical methods; in particular, all the abstract Petrov–Galerkin schemes analyzed in [RS06a] fall into that setting, along with the quadrature methods studied in [DRS06].

We will assume that all the densities involved in the numerical solution to the corresponding boundary integral equations are approximated in a discrete space X_m of dimension m. In principle, X_m could not be a subspace of $H^{-1/2}(\Gamma)$ as happens when using quadrature methods where the discrete space is formed by Dirac delta distributions. As in the continuous case, the considered norm does not add any difficulty as indicated at the beginning of Section 24.3. We also assume that in order to compute the coordinates of the approximate densities in a basis of X_m, one has to solve linear systems of equations of the form

$$V_m^\lambda \psi_D^m = -f_m, \tag{24.7}$$

$$(\tfrac{1}{2}I_m - J_m^\lambda)\psi_N^m = g_m, \tag{24.8}$$

$$\begin{bmatrix} V_m^\mu & -V_m^\lambda \\ \alpha(\tfrac{1}{2}I_m + J_m^\mu) & \beta(\tfrac{1}{2}I_m - J_m^\lambda) \end{bmatrix} \begin{bmatrix} \varphi_\alpha^m \\ \psi_\alpha^m \end{bmatrix} = \begin{bmatrix} f_m \\ \beta\, g_m \end{bmatrix}, \tag{24.9}$$

for (P_D), (P_N), and (P_α), respectively, where the matrices V_m^λ, I_m, J_m^λ, V_m^μ, and J_m^μ do not depend on α. Obviously, to have a unique solution in (24.7)–(24.9), the corresponding matrices have to be invertible. Then, with the same arguments as in Propositions 1 and 2, the following bounds can be proven:

$$\|\psi_\alpha - \psi_N\| \leq C\,|\alpha|, \qquad \forall\,|\alpha| \leq \alpha_0,$$
$$\|\psi_\alpha - \psi_D\| \leq C\,|\alpha|^{-1}, \qquad \forall\,|\alpha| \geq \alpha_1,$$

where $\|\cdot\|$ is any norm in $\mathbb{C}^m$. From here one deduces the same kind of bounds for the densities in the norm of X_m. If the approximate solutions to (P_D), (P_N), and (P_α) are defined by simply introducing the discrete densities obtained in (24.7)–(24.9) in the definition of the single-layer potentials, then results analogous to those in Corollary 1 can be derived straightforwardly.

24.5 Numerical Examples

This last section is devoted to numerical illustrations in the two-dimensional setting. The numerical method we use here is an easy-to-implement quadrature method proposed in [DRS06].

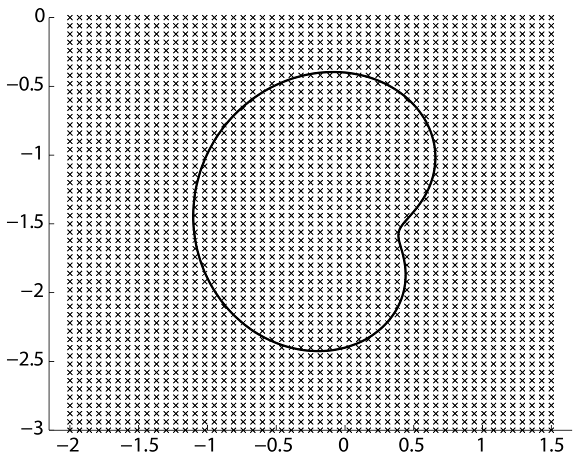

Fig. 24.1. Geometry of the problem.

We have considered the nonconvex domain represented in Figure 24.1, whose boundary is smooth. The physical parameters are $\lambda = \mu = 1 + \imath$ and $\beta = 1$, which correspond to a problem of scattering of thermal waves. We have taken $u_{\mathrm{inc}}(x_1, x_2) := \exp(-\imath\lambda x_2)$ as incident wave and have computed the total wave

$$u_{\mathrm{inc}} + u_\alpha \quad \text{in } \Omega_{\mathrm{ext}}, \qquad u_\alpha \quad \text{in } \Omega_{\mathrm{int}},$$

for some different values of α. In Figure 24.2, we represent the modulus of the total wave for five transmission problems with decreasing values of α as well as the modulus of the total wave that solves the exterior Neumann problem. Notice that the solution for $\alpha = 1$ is the planar incident wave.

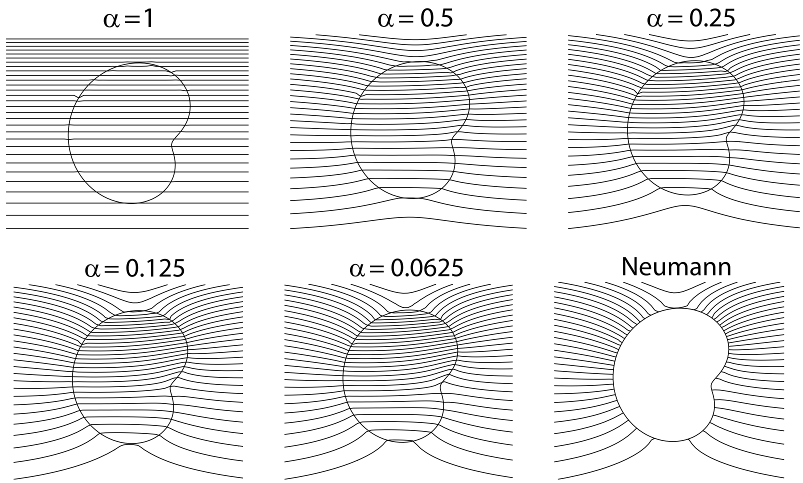

Fig. 24.2. $\alpha = 1,\ 1/2,\ 1/4,\ 1/8,\ 1/16$, and the Neumann exterior problem.

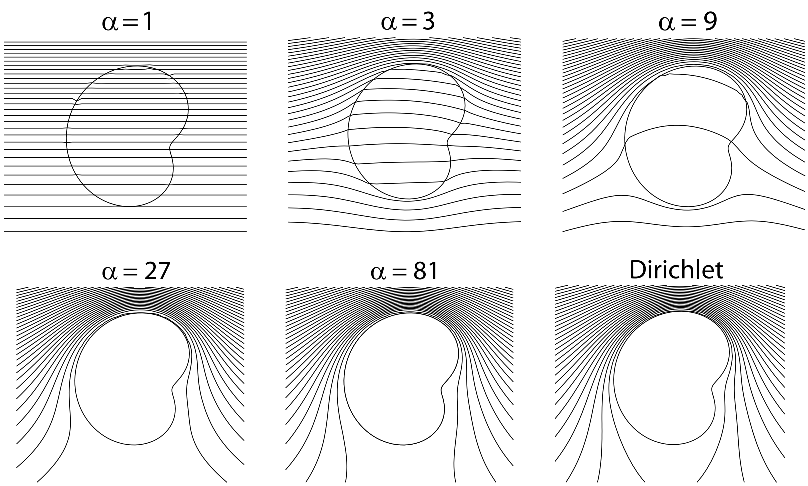

Fig. 24.3. $\alpha = 1,\ 3,\ 9,\ 27,\ 81$, and the Dirichlet exterior problem.

In Table 24.1, we write the errors $E^N_{\rm abs} := \max_i |u_\alpha(\mathbf{x}_i) - u_N(\mathbf{x}_i)|$ and $E^N_{\rm rel} := \max_i (|u_\alpha(\mathbf{x}_i) - u_N(\mathbf{x}_i)|/|u_N(\mathbf{x}_i)|)$, where $\mathbf{x}_i$ are the 50×50 points in the rectangle $[-2, 1.5] \times [-3, 0]$ represented in Figure 24.1. The corresponding estimated convergence rates (ecr) are computed by comparing errors for consecutive values of α in the usual way. It is clear that these numerical results fit with the theoretical ones.

We now solve the same problem for increasing values of α. In Figure 24.3, we represent the modulus of the total wave solution for the transmission and Dirichlet problems. Relative and absolute errors at the points $\mathbf{x}_i$ are written on the right of Table 24.1. Notice that in this case, although absolute errors have almost the same size as in the Neumann case, relative errors are now really large. This is not surprising, since the total wave in the Dirichlet problem is almost zero in the shadow of the obstacle.

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Table 24.1. Absolute and relative errors for the Neumann and Dirichlet problems.

α	$E^N_{\rm abs}$	ecr	$E^N_{\rm rel}$	ecr
10^{-1}	$5.95 \cdot 10^{-2}$		$1.24 \cdot 10^{-1}$	
10^{-2}	$6.33 \cdot 10^{-3}$	0.97	$1.34 \cdot 10^{-2}$	0.96
10^{-3}	$6.37 \cdot 10^{-4}$	0.99	$1.35 \cdot 10^{-3}$	0.99
10^{-4}	$6.37 \cdot 10^{-5}$	0.99	$1.35 \cdot 10^{-4}$	0.99
10^{-5}	$6.37 \cdot 10^{-6}$	0.99	$1.35 \cdot 10^{-5}$	0.99

α	$E^D_{\rm abs}$	ecr	$E^D_{\rm rel}$	ecr
10^{1}	$2.30 \cdot 10^{-1}$		$5.00 \cdot 10^{2}$	
10^{2}	$2.77 \cdot 10^{-2}$	-0.91	$7.01 \cdot 10$	-0.85
10^{3}	$2.82 \cdot 10^{-3}$	-0.99	7.25	-0.98
10^{4}	$2.82 \cdot 10^{-4}$	-0.99	$7.28 \cdot 10^{-1}$	-0.99
10^{5}	$2.82 \cdot 10^{-5}$	-0.99	$7.28 \cdot 10^{-2}$	-0.99

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