5

Lie Groups Realized as Automorphism Groups

5.1 Introduction

If Ω is a bounded domain in a complex Euclidean space, then the group Aut (Ω) of its holomorphic automorphisms is a finite-dimensional Lie group, as already discussed (Theorems 1.3.11, 1.3.12). It is natural to ask:

> **Question.** *Which Lie groups occur as the automorphism group of a bounded domain?*

Quite satisfactory answers are known. Bounded domains with noncompact automorphism group are in a sense unusual (cf. Corollary 3.4.4). Therefore it is natural to focus upon the compact Lie groups in asking which groups appear. In fact, every compact Lie group occurs as the automorphism group of a bounded domain in some complex Euclidean space, indeed a strictly pseudoconvex domain with real analytic boundary. This fact was proved independently and by different methods in [Bedford/Dadok 1987] and [Saerens/Zame 1987]. These proofs are the subject of this chapter.

In more detail:

Theorem 5.1.1 (Bedford–Dadok, Saerens–Zame). *Let G be a compact Lie group. Then there exist a positive integer N and a bounded strongly pseudoconvex domain Ω in $\mathbb{C}^N$ with a smooth (C^∞) boundary such that* Aut (Ω) *is Lie isomorphic to G.*

The semicontinuity theorem of Greene–Krantz (Theorem 4.4.3) makes it possible to choose the boundary of the domain in the theorem to be real analytic, as already stated. This will be discussed after the proof of the C^∞ result as stated.

5.2 General Philosophy

Before introducing the proofs, let us discuss the general philosophy underlying this theorem. Let G be a compact Lie group. It is a basic fact of Lie group

R. E. Greene et al., *The Geometry of Complex Domains,* Progress in Mathematics,
DOI 10.1007/978-0-8176-4622-6_5,

theory that G can be Lie isomorphically embedded into a unitary group $U(n)$, for some $n > 0$. (This fact is an aspect of the famous Peter–Weyl theorem. cf. [Chevalley 1946].) Therefore it is automatic to construct a bounded strongly pseudoconvex domain whose automorphism group contains a subgroup that is isomorphic to the given group G: the unit ball B^n suffices, since $U(n)$ is a subgroup of its automorphism group. On the other hand, it is a general principle that perturbation of the boundary of the domain in the smooth category will lose some of the automorphisms. [This was discussed earlier, in Chapter 4, in connection with the semicontinuity theorem (Theorem 4.4.3).] Hence the key issue here is how to perturb the ball—or some other domain with G contained in its automorphism group—so that G is kept while the other unwanted automorphisms are eliminated.

We first present the proof by Saerens and Zame and then the proof by Bedford and Dadok. The techniques are so different that both proofs are worth considering carefully.

5.3 The Saerens/Zame Proof

5.3.1 Unitary Representation

Start with the injective Lie group homomorphism $\iota : G \to U(n)$ of G into some unitary group $U(n)$ already mentioned. In order for such a *faithful unitary representation* to exist, n of course needs to be sufficiently large.

5.3.2 G-action by Left Multiplication

Consider the group $GL(n, \mathbb{C})$ of nonsingular $n \times n$ matrices with complex entries. Let G act on $GL(n, \mathbb{C}) \times \mathbb{C}^m$ as follows.

$$\begin{aligned} G \times (GL(n, \mathbb{C}) \times \mathbb{C}^m) &\longrightarrow GL(n, \mathbb{C}) \times \mathbb{C}^m \\ (g, (z, w)) &\mapsto g(z, w) := (g \cdot z, w), \end{aligned}$$

where:

- the action of g on $(z, w) \in GL(n, \mathbb{C}) \times \mathbb{C}^m$ is only on the first component $z \in GL(n, \mathbb{C})$ by left multiplication.
- the positive integer m will be determined later, and the role of $\mathbb{C}^m$ will also be clarified at the same time.

5.3.3 Averaging a Plurisubharmonic Exhaustion

Now consider the following real-valued function $\varphi : GL(n, \mathbb{C}) \times \mathbb{C}^m \to \mathbb{R}$ defined by

$$\varphi(z, w) = |\det z|^{-2} + \sum_{i,j=1}^{n} |z_{ij}|^2 + \sum_{k=1}^{m} |w_k|^2.$$

This function is a smooth (in fact real analytic), strictly plurisubharmonic (psh for shorthand) exhaustion function for $GL(n, \mathbb{C}) \times \mathbb{C}^m$, which is an open connected subset of $\mathbb{C}^{n^2+m}$.

Take a bi-invariant measure ν of total mass 1 on the compact Lie group G (the Haar measure), and consider the averaged function

$$\varphi^G(z, w) = \int_G \varphi(g \cdot z, w)\ d\nu(g).$$

This new function is also a real analytic, strictly psh exhaustion function for $GL(n, \mathbb{C}) \times \mathbb{C}^m$ and is obviously G-invariant.

5.3.4 A G-Invariant Strongly Pseudoconvex Domain

Now take a regular value $T \in \mathbb{R}$, that is, a real number T such that $d\varphi^G$ is nowhere singular on $(\varphi^G)^{-1}(T)$. [Such T are dense in $\mathbb{R}$, by the Morse-Sard theorem (Theorem 5.3.2); see Section 5.3.7 for more details on this matter.] One can take T to be sufficiently large that $(\varphi^G)^{-1}(-\infty, T)$ contains the set $U(n) \times \{0\}$. Denote by D^G the connected component of $(\varphi^G)^{-1}(-\infty, T)$ that contains the set $U(n) \times \{0\}$. By its construction, D^G is a G-invariant, bounded domain in $\mathbb{C}^{n^2+m}$ with a C^∞ smooth boundary. It has in fact real analytic boundary, by construction.

5.3.5 Preparation for Perturbation of the Boundary

Since $d\varphi^G$ is nonsingular at each point of ∂D^G, there exists an open neighborhood W of ∂D^G on which $d\varphi^G$ is nonsingular. Choose $r > 0$ such that $(\varphi^G)^{-1}(-r + T, T + r) \subset W$; such an $r > 0$ exists because ∂D^G is compact. Replacing W by $(\varphi^G)^{-1}(-r + T, T + r)$, we may assume that W itself is a G-invariant open neighborhood of ∂D^G, consisting of only regular points of φ^G.

Now consider the quotient by the G-action. By construction, the G-action is a fixed-point-free, properly discontinuous action. Therefore the quotient spaces W/G and $\partial D^G/G$ are smooth manifolds.

5.3.6 Scalar Invariants

Finding a suitable perturbation of the boundary of D^G uses an idea from the theory of curvature invariants in the sense of Tanaka–Chern–Moser. Here is a brief summary.

This concerns the local CR-invariants of the real hypersurfaces that will play an important role in the perturbation step. Consider a smooth real-valued function $\phi : \mathbb{C}^{n+1} \to \mathbb{R}$ that defines a smooth hypersurface $M = \{\phi = 0\}$ passing through the origin 0. In case M is strongly pseudoconvex, the function ϕ can be written, after a suitable change of coordinate system, say $(z_1, \ldots, z_n, \zeta)$ with $\zeta = u + iv$ about 0, in what is called the Chern–Moser normal form (see pp. 241–243 of [Burns/Shnider/Wells 1978] for further details and precise terminology). In this "normal form,"

$$\phi(z_1, \ldots, z_n, \zeta) = v - \sum_{\alpha=1}^{n} |z_\alpha|^2 - \sum_{p,q \geq 2} N_{p,q}$$

where each $N_{p,q}$ is a polynomial in the multi-variables $z, \bar{z}$ of type (p, q), first p of zs and q of $\bar{z}$s, with coefficients that are formal power series in the variable u as follows.

$$N_{p,q} = \sum N_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q}(u)\; z_1^{a_1} \cdots z_p^{a_p} \bar{z}_1^{b_1} \cdots \bar{z}_q^{b_q}$$

and

$$N_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q}(u) = \sum_{j=0}^{\infty} N^{(j)}_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q} u^j .$$

The origin 0 in M is called *spherical* (or *umbilical* in [Burns/Shnider/Wells 1978]; for the original introduction and developments, see [Chern/Moser 1974]) if the coordinates can be chosen so that $N^{(0)}_{a_1 a_2 \bar{b}_1 \bar{b}_2} = 0$ for any $a_1 a_2 \bar{b}_1 \bar{b}_2$. Otherwise, 0 is called *nonspherical*. This notion is independent of the choice of the normal form and is in fact preserved by biholomorphic transformations.

At a nonspherical point, further normalization, called the *restricted normal form*, is available (see Lemma 3.1 of [Burns/Shnider/Wells 1978]). In [Burns/Shnider/Wells 1978], "curvature invariants" for $j \geq 0, p \geq q \geq 2, p \geq 3$ are given by

$$K^j_{p,q} := \sum |N^{(j)}_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q}|^2$$

at the origin. (The curvature invariants make sense only at nonspherical points.) These are local CR invariants, meaning that the CR equivalences preserve the value of these terms.

5.3.7 Jets and Multi-Jets

The proof also involves the concept of *jets*. Again a brief summary.

Jets

Let X, Y be smooth manifolds and let $f, g : X \to Y$ be smooth maps with $f(x) = y = g(x)$ for some $x \in X$ and $y \in Y$. Then f and g are said to have first-order contact at x if every first-order partial derivative of f coincides with the corresponding derivative of g at x in some local coordinates around x and y in X and Y respectively. Notice that this concept does not depend upon the choices for local coordinate systems for X at x and for Y at y.

Likewise, f and g are said to have k-th order contact if they have the same partial derivatives at p of order up to and including k. Again, for every k, this concept does not depend upon the choices for local coordinate systems for X at x and for Y at y. For each k, it is obvious that this defines an equivalence relation; denote it by $\cong_k$, for the germs of smooth mappings. For a smooth map $f : X \to Y$ satisfying $f(x) = y$, denote by $j^k f|_{x,y}$ the equivalence class of the germ of f at x with respect to the relation $\cong_k$.

Denote by $J^k(X,Y)_{x,y}$ the collection of all the equivalence classes just defined. This is not in general a vector space as it does not have any obvious addition or scalar multiplication. However, in case Y is a Euclidean space, it is a vector space in an obvious way. In particular, $J^1(X,\mathbb{R})_{x,y}$ is naturally isomorphic to the cotangent space of X at x.

It is customary to call $J^k(X,Y)_{x,y}$ the space of *k-th order jets* (or simply *k-th jets*) of maps from X to Y at (x,y) and to consider the space

$$J^k(X,Y) = \bigcup_{(x,y)\in X\times Y} J^k(X,Y)_{x,y} \qquad \text{(disjoint union)}.$$

This union is usually called the *jet bundle* for smooth maps from X to Y. Notice that the space of k-th jets and the k-jet bundle are finite-dimensional smooth manifolds for each $k = 1, 2, 3, \ldots$.

Likewise, it makes sense to consider the map

$$j^k f : X \to J^k(X,Y) : x \mapsto j^k f|_{x,f(x)}$$

which is usually called the *k-jet* of the smooth map $f : X \to Y$. It is a smooth map with respect to the obvious smooth structure on $J^k(X,Y)$.

Multi-Jets

Now we shall introduce the concept of "multi-jets" (although, for our exposition here we only need double-jets).

First, we define

$$X^{(s)} := \left\{ (x_1, \ldots, x_s) \in \prod^s X \mid x_j \neq x_k \text{ if } j \neq k \right\}$$

and let $\alpha : J^k(X,Y) \to X$ be the projection defined by $\alpha(\sigma) = x$ if and only if $\sigma = j^k f|_{x,y}$ for some $y \in Y$ and some germ of a smooth $f : X \to Y$ with $f(x) = y$. Then let $\alpha^s := \prod^s \alpha : \prod_{\ell=1}^s J^k(X,Y) \to \prod^s X$ be the product map. Then one can consider the *space of s-fold k-th jets* defined by

$$J^k_{(s)}(X,Y) := (\alpha^s)^{-1}(X^{(s)}).$$

This is what is called in [Saerens/Zame 1987] a *multi-jet*. One can easily generalize this formalism to define the concept of the s-fold multi-jet bundle

$J^k_{(s)}(X,Y)$ and the map $j^k_{(s)}f : X^{(s)} \to J^k_{(s)}(X,Y)$, where the last is nothing but

$$j^k_{(s)}f(x_1,\dots,x_s) = (j^kf(x_1),\dots,j^kf(x_s))$$

for every $(x_1,\dots,x_s) \in X^{(s)}$.

Transversality

The *transversality* concept in differential topology is also needed for the proof. The idea of transversality grew out of the idea of regular value, already used in Section 5.3.4. For completeness and motivation, we discuss this first. Let $f : M \to N$ be a smooth map from a smooth manifold M to another smooth manifold N. Then one would like to know when the pre-image $f^{-1}(y)$ is necessarily a smooth submanifold of M for $y \in N$. A satisfactory answer comes of course from the implicit function theorem:

A point $y \in N$ is called a *regular value* of the smooth map $f : M \to N$ if, for any $x \in f^{-1}(y)$, the differential $df_x : T_xM \to T_yN$ is surjective. The implicit function theorem then implies:

Theorem 5.3.1. *Let M and N be smooth manifolds and let $f : M \to N$ be a smooth mapping. If $y \in N$ is a regular value for f, then the pre-image $f^{-1}(y)$ is an embedded submanifold of M.*

One notices that, due to the logic involving the empty set, any point $y \in N \setminus f(M)$ becomes a regular value. Of course in such a case $f^{-1}(y)$ coincides with the empty set, and that is surely a submanifold. (The dimension of empty submanifold is usually understood to be -1.) One might like to disregard such a “pathological” case, but in fact there is no particular reason to do so; in fact it will play an important role in many cases, including our current discussion.

Do regular values exist? The following familiar theorem guarantees their abundance.

Theorem 5.3.2 (Sard’s Theorem; cf. e.g., [Munkres 1966]). *The set of regular values for a smooth map $f : M \to N$ is dense in N.*

In fact, if we denote the set of regular values by R, then $N \setminus R$ is of measure zero. Note that the concept “measure zero” in differential topology does not have to involve any specific choice of a measure. A set is *measure zero* if and only if it has measure zero in every local coordinate system in the sense that its intersection with each coordinate domain has measure zero in $\mathbb{R}^n$ when mapped to $\mathbb{R}^n$ by the local coordinate map.

The following notion of transversality grew out of the concept of regular values.

Definition 5.3.3. Let M, N be smooth manifolds and let Z be a submanifold of N. Let $f : M \to N$ be a smooth mapping. Then we say that f is *transversal* to Z, if the equality

$$df_x(T_xM) + T_{f(x)}Z = T_{f(x)}N$$

for any $x \in f^{-1}(Z)$. It is customary to denote transversality by $f \pitchfork Z$.

The following is a well-known result in differential topology (cf. e.g., [Hirsch 1976]).

Theorem 5.3.4 (Transversality). *Let M, N be smooth manifolds and Z a submanifold of N. Let $f : M \to N$ be a smooth mapping. If f is transversal to Z, then $f^{-1}(Z)$ is an embedded submanifold of M.*

The reader must have noticed, by the logic involving the empty set, that f is transversal to Z whenever $f(M) \cap Z = \emptyset$. On the other hand, if it happens to be the case that $\dim N > \dim M + \dim Z$, then f can be transversal to Z if and only if $f(M) \cap Z = \emptyset$. Again, this seemingly somewhat pathological logic is going to play an important role in what follows.

Now what about the generalization of Sard's theorem (Theorem 5.3.2)? The Saerens/Zame proof uses the following standard theorems on this subject (cf. e.g., [Golubitsky/Guillemin 1973]):

Theorem 5.3.5. *Let X and Y be smooth manifolds.*

(1) [Thom transversality theorem] *Let W be a submanifold of $J^k(X, Y)$ and let $T_W := \{f \in C^\infty(X, Y) \mid j^k f \pitchfork W\}$. Then T_W is a dense G_δ-subset of $C^\infty(X, Y)$ in the C^∞ topology.*
(2) [Multi-jet transversality theorem] *Let W be a submanifold of $J^k_{(s)}(X, Y)$ and let $T_W := \{f \in C^\infty(X, Y) \mid j^k_{(s)} f \pitchfork W\}$. Then T_W is a dense G_δ-subset of $C^\infty(X, Y)$ in the C^∞ topology.*

5.3.8 Application of Transversality to ∂D^G

We now return to the actual proof of Theorem 5.1.1 at the point where we had a G-invariant domain D^G with a smooth strongly pseudoconvex boundary, and the regular G-invariant neighborhood W of ∂D^G (end of Subsection 5.3.5).

Consider Ψ the set of all smooth, strictly psh, G-invariant, proper functions defined on $GL(n, \mathbb{C}) \times \mathbb{C}^m$ that are nonsingular at every point of W, with W as in Subsection 5.3.5. This set is nonempty as we constructed such a function φ^G by an averaging method. However, unlike what is claimed in [Saerens/Zame 1987] by Saerens and Zame, it is actually *not* true that Ψ is an open subset of $C^\infty(GL(n, \mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$, since G-invariance is not an open condition.

Fortunately, this incorrect claim is not essential for the rest of the arguments. Here is a way to fix the situation. Consider the subset

$$\mathcal{D} := \{h \in C^\infty(GL(n, \mathbb{C}) \times \mathbb{C}^m, \mathbb{R}) \mid h(g \cdot x) = h(x), \quad \forall x \in GL(n, \mathbb{C}) \times \mathbb{C}^m \text{ and } \forall g \in G\}.$$

This is a closed linear subspace of the Fréchet (i.e., complete, semi-normed) space $C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$. Consider now the set

$$\Psi_G = \{\phi \in C^\infty(GL(n,\mathbb{C})/G \times \mathbb{C}^m, \mathbb{R}) \mid \phi \circ \pi \in \Psi\}.$$

Here, $\pi : GL(n,\mathbb{C}) \times \mathbb{C}^m \to GL(n,\mathbb{C})/G \times \mathbb{C}^m$ is the standard quotient map. Since G is compact, the map π is proper. It then follows by the chain rule that the map $\pi^* : C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R}) \to \mathcal{D}$ defined by $\pi^*(\psi) := \psi \circ \pi$ is a continuous mapping. Since the function space Ψ is an open subset of $\mathcal{D}$ in the inherited topology from $C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$, and since $\Psi_G = [\pi^*]^{-1}(\Psi)$, we see immediately that $\Psi_{/G}$ is an open subset of $C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$. This is what we need for the rest of the argument.

Let the correspondence $\phi \mapsto \phi_G : \Psi \to \Psi_G$ be defined by $\phi_{/G}(G \cdot x) = \phi(x)$. This gives rise to the natural map

$$\pi_k^* : J^k\Psi \to J^k\Psi_G,$$

defined by $\pi_k^*\big(j^k\phi_G\big|_{Gx}\big) = j^k\phi|_x$ for every $x \in GL(n,\mathbb{C}) \times \mathbb{C}^m$.

5.3.9 Elimination of Spherical Jets by Perturbation

Recall the definition of spherical (boundary) point in Section 5.3.6. The concept of spherical point depends only upon the jet of order at most 4. Therefore it makes sense to define the concept of *spherical jets* (of normalized defining functions) following the obvious method, instead of the concept of spherical point associated with the (normalized) defining function. Denote by S^k the set of spherical jets in $J^k\Psi$ and let $\Sigma^k := \pi_k^*(S^k)$. Furthermore, for p, q with $p > q \geq 3$ and $p + q \leq k$, the scalar curvature invariant functions $\widetilde{K}^0_{p,q}$ are also defined on $J^k\Psi_G \setminus \Sigma^k$, analogously to the curvature functions for Ψ. Also, let

$$S^k_{p,q} = \{\psi \in J^k\Psi \mid K^0_{p,q}(\psi) = 0\}$$

and

$$\Sigma^k_{p,q} = \{\psi \in J^k\Psi_G \mid K^0_{p,q}(\psi) = 0\}.$$

Lemma 5.3.6. *There exists $\ell > 0$ such that, for every $m \geq \ell$, the following estimates hold:*

$$\text{codim } (\Sigma^4 \text{ in } J^4\Psi_G) \geq 2(n^2 + m)$$

and

$$\text{codim } (\Sigma^4_{p,q} \text{ in } J^4\Psi_G) \geq 2(n^2 + m)$$

whenever the positive integers p, q satisfy the conditions $p > q \geq 3$ and $p + q \leq m$.

Notice that $2n^2+2m = \dim_{\mathbb{R}} W$. The proof of this lemma uses only general facts on the jets and the curvature invariants introduced in [Burns/Shnider/Wells 1978]. The proof we sketch here is reorganized by B.-L. Min in his thesis ([Min, B.-L. 2009]; see also [Min, B.-L. 2009a]). We refer to this last paper for further details.

A sketch of the proof of Lemma 5.3.6. The proof is a direct computation. In [Burns/Shnider/Wells 1978], the codimension of the space S^4 of spherical jet in $J^4\Psi$ was computed to be $t^2(t-1)^2/4-(t-1)^2$ where $t = n^2+m = \dim_{\mathbb{C}} W$. On the other hand, $\dim_{\mathbb{R}} J^4\Psi = \dim_{\mathbb{R}} W + 1 + \dim_{\mathbb{R}} A^4_{2n^2+2m}$ where A^k_r is the vector space of polynomials of degree $\leq k$ in r variables without constant terms.

Note that $\dim_{\mathbb{R}} G \leq n^2$ as $G \in U(n)$. Consequently, $\dim_{\mathbb{R}} J^4\Psi_{/G} \geq n^2 + 2m + 1 + \dim A^4_{n^2+2m}$, and this eventually gives rise to

$$\text{Codim } (\Sigma^4 \text{ in } J^4\Psi_G) \geq \frac{1}{4}m^4 - \text{ lower order terms in } m.$$

As n is fixed, and m can be chosen sufficiently large, one can see (due to the remarks in the first paragraph of this proof) that the assertion of the lemma follows. □

On the other hand, let $m \geq \ell$ be an integer as in the preceding lemma, and let

$$\Sigma = S^4 \cup \left(\bigcup_{\substack{p>q\geq 3 \\ p+q\leq m}} \Sigma^4_{p,q} \right).$$

Now apply the transversality theorem (Theorem 5.3.5) on jets and multi-jets introduced above. Recall the special neighborhood W for the boundary of the domain D^G defined earlier. For such a W, there exists a dense $\mathcal{G}_\delta$-subset of $\Psi_{/G}$ such that ψ in the G_δ-subset has the following two properties:

(1) If a map $j^4\psi : W/G \to J^4\Psi_G$ is transversal to Σ^4 and, at the same time, to $\Sigma^4_{p,q}$, then $j^4\psi(y) \notin \Sigma^4 \cup \Sigma^4_{p,q}$ for any $y \in W/G$.
(2) If

$$J^4{\Psi_G}^\times := \{\text{nonspherical jets in } J^4\Psi_G\}$$

and

$$J^4\Psi^\times := \{\text{nonspherical jets in } J^4\Psi\},$$

then there exists a set Q of $4(n^2+m)+1$ distinct curvature functions $\widetilde{K}_1, \ldots, \widetilde{K}_Q$, where $\widetilde{K}_\ell = \widetilde{K}_{p_\ell,q_\ell}$ for p_ℓ and q_ℓ satisfying $p_\ell > q_\ell \geq 3$ and $p_\ell + q_\ell \leq m$, such that the map

$$\widetilde{K} := (\widetilde{K}_1, \ldots, \widetilde{K}_Q) : J^4{\Psi_G}^\times \to \mathbb{R}^Q$$

has maximal rank. Let Δ denote the diagonal of $\mathbb{R}^Q \times \mathbb{R}^Q$. Then the inverse image $(\widetilde{K}, \widetilde{K})^{-1}(\Delta)$ is a submanifold of $J^4\Psi_G{}^\times \times J^4\Psi_G{}^\times$. The function ψ has its double jet $j^4_{(2)}\psi : (W/G)^{(2)} \to J^4_{(2)}\Psi_{/G}$, transversal to $(\widetilde{K}, \widetilde{K})^{-1}(\Delta)$.

Property (1) holds on a dense G_δ by the codimension estimates in the previous lemma. Property (2) holds on a dense G_δ by the multijet transversality theorem, Theorem 5.3.5 (2). Thus properties (1) and (2) hold simultaneously on a dense G_δ-set.

5.3.10 Construction of Ω

It may be useful to summarize what has been done up to this point. We started with the embedding of the given compact Lie group G into the unitary group $U(n)$ of some sufficiently large n. Then we considered the real analytic strictly psh function

$$\varphi(z, w) = |\det z|^{-2} + \sum |z_{jk}|^2 + \sum |w_\ell|^2$$

defined on $GL(n, \mathbb{C}) \times \mathbb{C}^m$. Then, exploiting the compactness of the given Lie group G, we have used the averaging process

$$\varphi^G(z, w) := \int_G \varphi(g \cdot z, w)\ d\nu(g)$$

so that the new function φ^G is invariant under the G-action and is strictly psh and real analytic. Then we choose a regular value T so that $D^G := (\varphi^G)^{-1}(-\infty, T)$ is defined to be a G-invariant, bounded, strongly pseudoconvex domain with a real analytic boundary. Furthermore, we observed that there exists a special G-invariant open neighborhood W of ∂D^G such that $d\varphi^G$ is nonsingular at every point of W.

Then, using jets and transversality theorems, we were able to perturb φ^G as follows.

Construct first $\phi : GL(n, \mathbb{C})/G \times \mathbb{C}^m \to \mathbb{R}$ by $\phi(G \cdot x) = \varphi^G(x)$. Then perturb ϕ to obtain $\psi : GL(n, \mathbb{C})/G \times \mathbb{C}^m$ so that $\widetilde{\psi} := \psi \circ \pi$ is still arbitrarily close to ϕ on compact subsets (and hence in particular on W). Notice that here one needs to take m sufficiently large. Of course $\widetilde{\psi}$ is still strictly psh and smooth of class $\mathcal{C}^\infty$, and $d\widetilde{\psi}$ is nonsingular at any point of W. Furthermore, if we now let

$$\Omega = \widetilde{\psi}^{-1}(-\infty, T),$$

then Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^{n^2+m}$ that has the following properties:

(i) $G \subset \text{Aut}\ (\Omega)$.
(ii) $\partial\Omega$ has no point at which the jet of ψ is spherical.
(iii) If $x, y \in \partial\Omega$ such that $x \notin G \cdot y$, then $K(j^4\psi(x)) \neq K(j^4\psi(y))$.

Now to continue the proof, we wish to show that $G = \text{Aut}\ (\Omega)$. Let $h \in \text{Aut}\ (\Omega)$. Since the scalar curvature invariant function is a CR invariant, and since h extends to a diffeomorphism of $\text{cl}(\Omega)$ by Fefferman's extension theorem, $h(x) = y$ implies that $x \in G \cdot y$.

Thus $h(x) = g_x \cdot x$ for some $g_x \in G$ that is *a priori* depending on x. But recall that the elements x and $h(x)$ are in $GL(n, \mathbb{C}) \times \mathbb{C}^m$. Hence we may write $x = (z, w)$ and $h(x) = h(z, w) = (h_1(z, w), h_2(z, w))$. Now $g_x \cdot x = h(x)$ means

$$g_{(z,w)} = h_1(z, w)z^{-1} \qquad \text{and} \qquad h_2(z, w) = w.$$

Therefore the map $g : \partial\Omega \to G$, $g(z, w) = g_{(z,w)}$, defines a CR-function. However, $U(n)$ inside $GL(n, \mathbb{C})$ is totally real. Therefore the differential of this map has to vanish identically. This means that $g = g_{(z,w)}$ is independent of $x = (z, w) \in \partial\Omega$ and hence depends only on h. first, for every $h \in \text{Aut}\ (\Omega)$ there exists $g \in G$ such that $h(z, w) = (g \cdot z, w)$ for any $(z, w) \in \Omega$. Hence $G = \text{Aut}\ (\Omega)$ as desired. This completes the construction and the proof of Theorem 5.1.1.

5.4 The Bedford/Dadok Proof

An alternative approach to the realization of a given compact Lie group as the automorphism group of a bounded domain was given by E. Bedford and J. Dadok ([Bedford/Dadok 1987]). Their essential idea was to realize the given group as the isometry group of a perturbation of the unit ball in some real Euclidean space $\mathbb{R}^n$ and then pass to the complex setting by considering a suitable modification of the "tube domain" in $\mathbb{C}^n$ over the domain in $\mathbb{R}^n$. Their paper also considers the question of realizing a given compact Lie group as the automorphism group of a compact-closure (and strongly pseudoconvex) domain in a Stein manifold, rather than in a complex Euclidean space: the point here is that this realization is possible in rather lower dimensions than if one requires a domain in $\mathbb{C}^n$. We shall outline the approaches in the two cases, the $\mathbb{C}^n$ case first. Complete details are given in [Bedford/Dadok 1987] for both.

5.4.1 Structure of the Proof

Suppose that G is a compact Lie group and that (following the notation of [Bedford/Dadok 1987]) ω is a bounded domain in some Euclidean space $\mathbb{R}^n$ with the following properties:

(a) there is an injective homomorphism of G into $O(n)$, the image of which we again denote by G, such that ω is invariant under G;
(b) if $g : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation with $g(\omega) = \omega$, then $g \in G$.

(We shall see later that, for suitable n, such an ω can be obtained as a C^∞ small perturbation of the unit ball in $\mathbb{R}^n$.) Now let

$$\Omega = (\omega + i\mathbb{R}^n) \setminus V \subset \mathbb{C}^n$$

where $V = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + \cdots + z_n^2 = \frac{1}{2}\}$. The role of removing V from the "tube domain" $\omega + i\mathbb{R}^n$ will become apparent momentarily. Note that each $g \in G$ takes Ω to itself if G is taken to act on $\mathbb{C}^n$ by complex linear extension of its action on $\omega \subset \mathbb{R}^n$: this is clear since g takes $\omega + i\mathbb{R}^n$ to itself and g takes V to itself—because g on Ω maps the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1^2 + \cdots + x_n^2 = \frac{1}{2}\}$ to itself, since $g \in O(n)$.[1]

The domain Ω is of course unbounded, but it is biholomorphic to a bounded open set (since it is contained in a proper cone). This immediately implies, using the fact that bounded holomorphic functions extend across deleted subvarieties, that any automorphism of Ω extends to be an automorphism of the tube domain $\omega + i\mathbb{R}^n$. Now automorphisms of tube domains are completely understood. In particular it is shown in [Yang 1982] that every automorphism of $\omega + i\mathbb{R}^n$ has the form $z \mapsto Az + b + ic$ for some $b, c \in \mathbb{R}^n$ and some $A \in GL(n, \mathbb{R})$. Here $Az + b$ must map ω to itself, so from property (b) of ω above, $b = 0$ and $A \in G \subset O(n)$. Now, for $z \mapsto Az + ic$ to map V to itself, it must be that $c = 0$: this is so because A maps V to itself but $V \neq V + ic$ if $c \neq 0$. Hence the original automorphism $z \mapsto Az + b + ic$ is in fact an element of G.

5.4.2 How to Obtain a Bounded Domain

The domain Ω does not as such answer the question of realizing G as the automorphism group of a bounded domain with a smooth boundary, since Ω is neither bounded nor smooth (because of the removal of V, which has real codimension 2). However one can modify Ω as follows: the domain Ω is pseudoconvex so it admits a C^∞ strictly plurisubharmonic exhaustion function $\varphi : \Omega \to \mathbb{R}$. By averaging with respect to the action of the compact group G on Ω, one can obtain such a φ that is G-invariant, so that its c-sublevel sets $\Omega_{\varphi,c} := \{z \in \Omega \colon \varphi(z) < c\}$ are C^∞, bounded and G-invariant, for generic choice of c (by Sard's theorem (Theorem 5.3.2)), first for c regular values of φ. For each fixed c, there is an arbitrarily small (in the C^∞ sense) perturbation, to be denoted $\widehat{\Omega}_{\varphi,c}$, which guarantees that $\widehat{\Omega}_{\varphi,c}$ is still contained in Ω, G-invariant and strongly pseudoconvex, and has the further property that $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c})$ preserves the function $\sum_{j=1}^n z_j^2$. This follows from the arguments discussed earlier (Section 5.4.1; see also Sections 5.3.3, 5.3.4, and 5.3.8.) about introducing orbit-stabilizing perturbations. Since G itself preserves $\sum_{j=1}^n z_j^2$,

---

[1] The inclusion relation $g(V) \subset V$ follows by the "persistence of identities" upon passing from a totally real maximal dimension submanifold to a whole connected open set in $\mathbb{C}^n$.

the possibility of carrying this perturbation process in a G-equivariant way follows easily. By choosing the perturbations sufficiently small (for each c_j), the property can be retained that for some fixed increasing sequence $c_j \to +\infty$, the $\widehat{\Omega}_{\varphi,c_j}$ are increasing (i.e., $\widehat{\Omega}_{\varphi,c_j} \subset \widehat{\Omega}_{\varphi,c_{j+1}}$ and $\bigcup_{j=1}^{+\infty} \widehat{\Omega}_{\varphi,c_j} = \Omega$.

With these choices made, it follows that, for j sufficiently large, $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$ must be exactly G.

To see this, it suffices to show that if $c_j \to +\infty$, c_j a regular (i.e., noncritical) value for φ, and $\alpha_j \in \mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$, then there is a subsequence α_{j_k} of the α_js which converges uniformly on compact subsets of Ω to an automorphism of Ω, first to an element of G. For, if this is known, then $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$ restricted to some fixed (nonempty) $\widehat{\Omega}_{\varphi,c}$ lies, when j is large enough, in a small neighborhood of $G\big|_{\Omega_{\varphi,c}}$ and hence, by the results of Chapter 4, in fact $= G$ (since it contains G).

To check the indicated convergence result for a subsequence of the α_j, note first that some subsequence α_{j_k} of the α_js converges uniformly on compact subsets of Ω to some holomorphic function $\alpha_0 : \Omega \to \Omega \cup \partial\Omega$. This follows from standard normal families arguments since Ω is biholomorphic to a bounded domain. Note that we need not worry about possible "divergence to infinity" for this reason: $\mathrm{Re}\,(\sum z_j^2)$ is preserved by $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c})$ by construction. And, the real parts of the z_js are bounded on $\widehat{\Omega}_{\varphi,c}$. It follows that the imaginary parts of the coordinates of $\varphi_j(0,\ldots,0)$ are bounded for $\varphi_j \in \mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$, the bound being uniform in j. The limit α_0 is in $\mathrm{Aut}\,(\Omega) = G$, provided it does not "degenerate," i.e., provided that $\alpha_0(\Omega) \subset \Omega$, for which it suffices to show that $\alpha_0(\Omega) \not\subset \partial\Omega$.

Now $\alpha_0(\Omega)$ cannot contain points of $\partial\omega + i\mathbb{R}^n$ that are not in V since such points are strongly pseudoconvex, by the standard argument about strongly pseudoconvex boundary points of domains not biholomorphic to the ball (cf. [Rosay 1979]) and the "scaling version" of Rosay's argument presented in Chapter 9 (see Theorem 9.2.1). On the other hand, it cannot be that $\alpha_0(\Omega) \subset V$ since this would give a retraction of $\Omega \cup V$ onto V, which is impossible for homological reasons: $\Omega \cup V$ is contractible, but $V \cap (\omega + i\mathbb{R}^n)$ is homologically nontrivial in dimension n since $\{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = \frac{1}{2}\}$ is not homologically trivial in V.

5.4.3 Construction of ω

Turning now to the construction of a suitable ω as a C^∞ small perturbation of the unit ball in some $\mathbb{R}^n$, we note first a general idea of metric perturbations and group actions: Suppose that (M, g_0) is a Riemannian manifold with metric g_0 invariant under a faithful action on M of a compact Lie group G. (By *faithful*, we mean here that only the identity in G acts as the identity map of M.) Thus, in effect, G can be thought of as a subgroup of the isometry group $\mathrm{Isom}\,(M, g_0)$ of M with respect to the metric g_0. Now, in general, it is not necessarily the case that there is a metric g on M that is C^∞ close to g_0

such that Isom $(M, g) = G$. For example, if a metric of the k-dimensional sphere S^k is invariant under the standard $SO(k+1)$ action on S^k, then it is necessarily a multiple of the standard S^k metric, and hence its isometry group is $O(k+1)$, not just $SO(k+1)$. However, what is true is that there is always a metric g on M, C^∞ close to g_0, such that the metric g is invariant under the G-action and Isom (M, g) has the same orbits as the G-action. Such an orbit-stabilizing perturbation of g_0 is obtained by making G-invariant alterations of the g_0-metric in tubular neighborhoods of sufficiently many G-orbits of maximal dimension. Then the detailed argument is similar to but easier than the corresponding ideas in the Saerens–Zame argument already presented, so we omit the details at this time. In summary, one can stabilize a given G-orbit by making a high-order derivative of the metric g in normal directions to the orbit larger than for other (remote) orbits: this will stabilize a neighborhood of the orbit. This process can be successively adjusted to stabilize smaller neighborhoods and the limit orbit itself. Then a dense set of other orbits can be stabilized, by the Baire category theorem. Hence all orbits can be stabilized.

Thus the problem of finding a suitable ω as above can be solved if it can be converted to an orbit stabilization situation. As pointed out in [Bedford/Dadok 1987], this can be arranged by choosing first a *diagonal embedding.* Suppose that the group G has a faithful representation as a subgroup of $O(n)$ for some n. Then G has an action on $\mathbb{R}^{n^2} \cong \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$ (n summands) by letting G act on each summand by its $O(n)$ representation. The G can be considered as a subgroup of $O(n^2)$, and this faithful representation of G has the following property: *If a subgroup H of $O(n^2)$ has the same orbits as G, i.e., $Hx = Gx$ for all $x \in \mathbb{R}^{n^2}$, then $H = G$.*

The role of the diagonal embedding process can be made more vivid by constructing a concrete example. Consider the action of $SO(3)$ on $S^2 \subset \mathbb{R}^3$, $S^2 = \{(x, y, z) \colon x^2 + y^2 + z^2 = 1\}$ as usual, and the action of $O(3)$ on S^2. The 2-sphere is itself an orbit for both actions, and, moreover, any metric invariant under $SO(3)$ has to be a multiple of the standard metric on S^2 and hence must be invariant under $O(3)$. No process of *orbit stabilization*—indeed no process whatever—can produce a metric on S^2 which is $SO(3)$ invariant but not $O(3)$ invariant: it cannot be arranged that Isom $(S^2, g) = SO(3)$ exactly with $SO(3)$ acting in the standard way as indicated. In fact it cannot be arranged that Isom $(S^2, g) = SO(3)$, acting any way at all. The reason is that a faithful $SO(3)$ action must *a priori* have an orbit of dimension $=$ $\dim SO(3) -$ maximum isotropy dimension $= 3 - 1 = 2$. Thus every faithful $SO(3)$ action on S^2 must make S^2 homogeneous so that an invariant metric must have constant Gauss curvature; and then S^2 with that metric must be isometric to S^2 with a multiple of its standard metric. But such a metric has isometry group $O(3)$, not just $SO(3)$.

All this difficulty of distinguishing $SO(3)$ from $O(3)$ by orbits can be repaired, as it were, by considering the *diagonal action.* first we let $SO(3)$ act on $\mathbb{R}^9$ as follows. Consider $A \in SO(3)$, a 3×3 orthogonal matrix. Then associate to A a *diagonal-associate* $\widehat{A} \in O(9)$, first the 9×9 matrix with three 3×3 diagonal blocks being A, and all other matrix elements 0:

$$\widehat{A}_{ij} = \begin{cases} A_{ij} & \text{if } 1 \le i, j \le 3, \\ A_{i-3,j-3} & \text{if } 4 \le i, j \le 6, \\ A_{i-6,j-6} & \text{if } 7 \le i, j \le 9, \\ 0 & \text{otherwise.} \end{cases}$$

The transformation $\widehat{A}$, constructed from $A \in SO(3)$, gives an orthogonal action on $\mathbb{R}^9$.

The crucial point that makes this construction of interest is this: if H is a subgroup of $O(9)$ the action of which on S^8 (or, equivalently on $\mathbb{R}^9$) has each H-orbit contained in some orbit of the *diagonal action* (action by $\{\widehat{A}\colon A \in SO(3)\}$), then each element $h \in H$ has the form $\widehat{A}$ for some $A \in SO(3)$. This will be checked momentarily. Note that this means that if a Riemannian metric g on S^8 is invariant under the action of $\{\widehat{A}\colon A \in SO(3)\}$ and also has the *orbit stabilization* property that Isom (g) has the same orbits as the orbits of $\{\widehat{A}\colon A \in SO(3)\}$, then Isom $(g) = \{\widehat{A}\colon A \in SO(3)\} \cong SO(3)$. Since such orbit stabilization can always be induced by a small perturbation of S^8, by making $\{A\}$-invariant perturbations normal to enough $\{\widehat{A}\colon A \in SO(3)\}$-orbits, one finds then a metric on S^8 with its isometry group isomorphic to $SO(3)$. The $O(3)$ versus $SO(3)$ difficulty for the actions on S^2 is eliminated by moving up to S^8. [Here we use implicitly the *rigidity* of small perturbations of S^8: for such, isometries of the metric are always realized as the restriction of a rigid motion of $\mathbb{R}^9$, hence, changing the origin if need be, by $O(9)$ elements. See the end of Subsection 5.4.4 for details of this idea.]

It remains to see why a subgroup H of $O(9)$ which has orbits contained in $\{\widehat{A}\}$-orbits must itself consist of elements of $\widehat{A}$ form. For this consider a 9×9 matrix $h \in H \subset O(9)$. We write images as column vectors here, so the first column of the matrix h is the image under h of $e_1 = (1, 0, \dots, 0)$, this image written in column form. This image is of course in the H-orbit of $e_1 = (1, 0, \dots, 0)$ and hence by hypothesis is in the $\{\widehat{A}\}$ orbit of $e_1 = (1, 0, \dots, 0)$: it equals $\widehat{A}e_1$ for some $A \in SO(3)$. In particular, this column has its bottom six entries $= 0$. Similarly, the fourth column of the h-matrix has its top three and bottom three entries $= 0$. The seventh column has its top six entries $= 0$.

Now we wish to see that the top three entries of column 1 of the h matrix $=$ the middle three entries of column 4 $=$ the bottom three entries of column 7 (same order, top to bottom, in the three cases). For this, we consider the h-image of $e_1 + e_4 + e_7$ where $e_i =$ the vector with 1 in the i-th position, all other components $= 0$. This h-image is (written as a column) the sum of the first, fourth and seventh columns. And, noting the forms of these columns already shown, this is the top three entries of the first column followed by the middle three of the fourth column followed by the bottom three of the seventh column. On the other hand $h(e_1 + e_4 + e_7)$ belongs to the H-orbit of $e_1 + e_4 + e_7$, and hence by hypothesis to the $\{\widehat{A}\}$-orbit of $e_1 + e_4 + e_7$. So $h(e_1 + e_4 + e_7) = \widehat{A}(e_1 + e_4 + e_7)$ for some $A \in SO(3)$. But $\widehat{A}(e_1 + e_4 + e_7)$ (as a column vector) consists of its top three entries repeated in order two additional times. This shows that the h-matrix has the correct form to be an $\widehat{A}$-matrix as far as the first, fourth, and seventh columns are concerned.

Similar reasoning applied to e_2, e_5 and e_8 together with $e_2 + e_5 + e_8$ and e_3, e_6 and e_9 together with $e_3 + e_6 + e_9$ completes the proof that the H-matrix has repeated block-diagonal form. The block, call it B, must belong to $O(3)$, since $h \in O(9)$. To see that $B \in SO(3)$, consider $h(e_1 + e_5 + e_9)$. This (column) vector is, from top to bottom, first column of B, second column of B, third column of B. Therefore, in order for the element $\widehat{B}(e_1 + e_5 + e_9)$ to coincide with the element $\widehat{A}(e_1 + e_5 + e_9)$, for some $A \in SO(3)$, it must be that $B = A$. So $h = \widehat{B}$ for some $B \in SO(3)$.

Note that the map of $SO(3)$ onto the orbit of $e_1 + e_5 + e_9$ is injective: $\widehat{A_1}(e_1 + e_5 + e_9) = \widehat{A_2}(e_1 + e_5 + e_9)$ implies that $A_1 = A_2$. It follows from general considerations that this is true generically: $A \mapsto \widehat{A}v$ is injective for generic vectors $v \in \mathbb{R}^9$, i.e., the set of v for which this is true is dense and open in $\mathbb{R}^9$.

Thus one is indeed in the situation where orbit stabilization suffices. The orbit stabilization process is in fact simpler in this case than for a general Riemannian action. And one sees that there is a G-invariant C^∞-small perturbation of the unit sphere which lies in the unit sphere except for a set of small measure and which stabilizes G-orbits in the sense that the (abstract) isometry group for the perturbation ω has the same orbits as G acting on the perturbation ω. It follows then that any affine mapping of $\mathbb{R}^{n^2}$ that preserves this perturbed domain ω is in fact in $O(n^2)$ and hence in G: the reason is that, because of the coincidence of the perturbation ω with the unit sphere everywhere but on a set of small measure, such an affine mapping must carry some open subset of the unit sphere to itself and hence be in $O(n^2)$. Further details can be found in [Bedford/Dadok 1987].

5.4.4 Isometry Group of a Riemannian Manifold

Note that, with ω so chosen, G is in fact the full isometry group of $\partial\omega$, the boundary of ω. This follows from the fact that $\partial\omega$, being C^∞ close to the unit sphere, is thus rigid in the sense that all its intrinsic (abstract) isometries extend to be isometries of $\mathbb{R}^{n^2}$. This rigidity follows from E. Cartan's "type number" local rigidity theorem: the unit sphere has maximal type number and hence so does every hypersurface C^∞ close enough to it. (Refer to [Hermann 1968] for these matters. See also [Spivak 1975], Volume 5, Chapter 12, p. 244 ff and the discussion on *type numbers* and rigidity.) From another only slightly different viewpoint, $\partial\omega$, being C^∞ close to the unit sphere, has positive sectional curvature and thus is rigid, again by E. Cartan's result. Thus any isometry of $\partial\omega$ extends to an isometry of $\mathbb{R}^{n^2}$ so that $G =$ the isometry group of $\partial\omega$ considered as an abstract Riemannian manifold. Thus one obtains: if G is a compact Lie group, then there is a compact Riemannian manifold (M, g) such that Isom $(M, g) \cong G$.

Curiously, the natural question in geometry that this result answers was never considered successfully in the context of pure Riemannian geometry itself, prior to its arising in the present context of complex analysis in [Bedford/ Dadok 1987] and [Saerens/Zame 1987].

5.4.5 Stein Domains

The second major line of thought in [Bedford/Dadok 1987] concerns realization of compact Lie groups as automorphism groups of bounded domains (i.e., domains with compact closure) in Stein manifolds which are not necessarily biholomorphic to bounded domains in $\mathbb{C}^n$. This more general class of domains yields a possible realization in lower dimensions. In effect, one can go from complex dimension n^2 for the Euclidean space case if $G \subset O(n)$ to dimension equal to that of G itself, clearly much lower when n is large.

Theorem 5.4.1 (Bedford–Dadok). *If G is a connected compact Lie group the dimension of whose center is not* 1*, then there is a strongly pseudoconvex domain Ω with the real analytic boundary contained in the complexification $G_{\mathbb{C}}$ of G and with $G \subset \Omega$ such that* Aut $(\Omega) \cong G$ *and* Aut (Ω) *consists exactly of the action of G on itself by translation extended holomorphically to Ω.*

If the dimension of the center of G is 1, then a similar domain Ω exists in $G_{\mathbb{C}} \times \mathbb{C}$.

This result is established by using the decomposition of G into the product of its center and simply connected simple factors, up to a finite quotient. The essential point is then to use the result of H. Cartan showing that, under quite general circumstances, the automorphism group of a product is the product of the automorphism groups of the factors. (This will be discussed in more detail later.)

5.4.6 Decomposition of G into $T \times G_s$

The product decomposition result is a standard part of Lie group theory (cf. [Helgason 1962]): *Every connected compact Lie group G has the form $(T^k \times G_1 \times \ldots \times G_\ell)/H$ where T^k is a k-dimensional torus ($k = 0$ is allowed), the G_is are simply connected compact simple groups, and H is a finite subgroup.* While the result is usually considered only in a Lie-group-theoretic context, it actually has an illuminating differential-geometric interpretation (and, indeed, proof).

This arises as follows: any left-invariant metric on the compact Lie group can be averaged with respect to the Haar measure on right translations of G. This produces a bi-invariant metric $\langle\,,\rangle$ on G. For this bi-invariant metric, the covariant derivative $D_X Y$, where X and Y are left-invariant vector fields, is $\frac{1}{2}[X,Y]$. And, again for left-invariant vector fields, the Riemann curvature tensor $R(X,Y,Z,W)$ is $-\frac{1}{4}\langle [X,Y],[Z,W]\rangle$ (cf. [Milnor 1963]; note that the sign convention for R in that reference is opposite to ours). This curvature tensor is parallel. Moreover, the Riemann sectional curvatures attached to it are all nonnegative, as follows immediately from the formula: the sectional curvature of the 2-plane spanned by an orthonormal pair X, Y is $-R(X,Y,X,Y) = \langle [X,Y],[X,Y]\rangle \geq 0$.

Let $\mathcal{I}$ be the set of all left-invariant vector fields X such that $[X,Y]=0$ for all left-invariant vector fields Y and set $\mathcal{I}_p = \{X(p)\colon X \in \mathcal{I}\}, p \in G$. If $X \in \mathcal{I}$, then X is globally parallel, since $D_Y X = \frac{1}{2}[Y,X] = 0$ for every (left-invariant) Y so $DX \equiv 0$. Thus the family of subspaces $\mathcal{I}_p \subset T_pG$, $\forall p \in G$, is a parallel family (i.e., invariant under parallel translation). The parallel nature of the family $\mathcal{I}_p$ can be interpreted in terms of the curvature tensor R: $\mathcal{I}$ is exactly the set of all left-invariant vector fields X such that $R(X,Y,Z,W)=0$ for all left-invariant vector fields Y,Z,W. So the parallel nature of R implies that of the family $\mathcal{I}_p$.

The de Rham decomposition theorem (cf. [Kobayashi/Nomizu 1963], Theorem 6.2, p. 192, Vol. I) now implies that the universal cover $\widehat{G}$ of G splits as a product $T \times G_s$ where the tangent space of the torus T at each point is the lift of $\mathcal{I}$ at the image of the point under the covering projection. And thus, for the pullback to $\widehat{G}$ of the metric of G, the torus T is flat. Moreover G_s is necessarily compact. (In the notation G_s, "s" stands for semi-simple, for reasons that will appear later.)

The group G_s is compact because, if G_s were noncompact, then there would be a geodesic ray $\gamma : [0,+\infty) \to G_x$ emanating from a pre-image of the identity. [Recall that a *ray* is a curve γ on $[0,+\infty)$ with $\mathrm{dis}(\gamma(0),\gamma(t)) = t$ for all $t \geq 0$.] But if v is the tangent vector $\gamma'(0)$ and V the associated left-invariant vector field on G, then there is a left-invariant vector field on G with $[V,W] \neq 0$. This would mean that $-R(V,W,V,W)$ would be a positive constant along the ray, implying the existence of a conjugate point to the initial point of the ray, a contradiction. Alternatively, one could show that G_s is compact by noting that it is complete and has positive Ricci curvature bounded away from 0: this follows by noting that, at a pre-image of the identity, there is at least one 2-plane of positive sectional curvature containing a given vector $v \neq 0$, associated, as above, to W such that $[V,W] \neq 0$. So the Ricci curvature of v is positive. Since curvature is parallel, the Ricci curvature is positive and bounded away from 0 everywhere. The compactness of G_s of course implies that any covering-space quotient of it is finite-to-one.

5.4.7 Decomposition of G_s

There is potentially a further decomposition of G_s that arises as follows. Since the metric is bi-invariant, its Lie derivative as a tensor with respect to a left-invariant vector field Y must be 0. This gives

$$0 = Y\langle X, Z\rangle - \langle [Y, X], Z\rangle - \langle X, [Y, Z]\rangle$$

for X, Y, Z left-invariant vector fields, using the usual Leibniz property to compute the $\mathcal{L}_Y$ Lie derivative of $\langle \, , \rangle$ as a tensor. But $\langle X, Z\rangle$ is constant so that $Y\langle X, Z\rangle = 0$. It follows that $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$. This same formula holds if we consider the lifts of left-invariant vector fields on G to vector fields on $\widehat{G}$. Let $\mathcal{L}$ = the Lie algebra of such lifts. Then the relationship $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$ implies that the orthogonal complement of an ideal in $\mathcal{L}$ is again an ideal, as one sees immediately. From this viewpoint, the space of vector fields in $\mathcal{L}$ tangent to G_s is exactly the orthogonal complement of the ideal in $\mathcal{L}$ consisting of vector fields tangent to T. Now the fact that orthogonal complements of ideals are again ideals implies that the tangent ideal of G_s can be successively decomposed into, finally, an orthogonal direct sum of simple ideals. Since G_s is simply connected, this implies a corresponding product decomposition of G_s into a product: the ideal decomposition is parallel by bi-invariance, so the de Rham decomposition theorem again applies. Thus, in outline, one arrives at the Lie group decomposition result as stated. Of course, the argument just discussed can be considered exclusively in Lie group terms: the appeal to the de Rham decomposition theorem is used just to give a differential geometric perspective.

The irreducibility of the ideals arising in this final decomposition implies that the positive Ricci curvature on each irreducible factor is in fact constant: the bi-invariant metrics are Einstein. Thus the Ricci curvature tensor itself can be thought of as being the original bi-invariant metric up to a constant factor. The $R(X, Y, Z, W) = -\langle [X, Y], [Z, W]\rangle$ formula shows that this Ricci curvature is in fact, again up to a constant, equal to the traditional "Killing form" $K(X, Y) = -tr\,(ad(X)ad(Y))$, where $ad(X)$ is the map on the tangent space determined by Lie bracketing with X. Thus the original metric and the Killing form metric are themselves Einstein metrics. The uniqueness (up to constant factors) of bi-invariant metrics on the simple factors can of course be seen directly from the irreducibility of the tangent ideals.

The decomposition of $\widehat{G}$ into $T \times G_s$, and the associated information about G itself, can also be viewed in the context of the Toponogov splitting theorem for complete manifolds of nonnegative sectional curvature, at least as far as the $T \times G_s$ decomposition is concerned. (The further decomposition of G_s into simple factors does not fit into this picture, however.) The reader is invited to consult [Cheeger/Ebin 1975] or [Petersen 2006] for further details of this perspective on decomposition.

5.4.8 Torus Group Case

We now begin constructing domains in the complexification of a compact connected Lie group G with automorphism group $= G$.

As already noted, the product decomposition of a compact connected Lie group offers a natural approach to finding domains with automorphism group equal to the given compact Lie group. If such domains can be found for each factor in the product then, under quite general and rather easily arranged circumstances, the product of these domains will serve for the whole (product) group. We now turn to this situation in more detail.

The first case to consider is that of a k-dimensional torus $T = \{(\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k : |\alpha_i| = 1, \forall i\}$. Recall the classical concept of a Reinhardt domain: an open and connected set $\Omega \subset \mathbb{C}^k$ such that Ω is invariant under the mappings $(z_1, \ldots, z_k) \mapsto (\alpha_1 z_1, \ldots, \alpha_k z_k)$ where each α_i has modulus 1. The torus T acts on such a domain, by definition.

A Reinhardt domain, say Ω, is completely specified by its "log profile"

$$\text{Log}\ (\Omega) := \{(\log |z_1|, \ldots, \log |z_k|) \in \mathbb{R} \cup \{-\infty\} : (z_1, \ldots, z_k) \in \Omega\}.$$

We allow $-\infty$ values to accommodate the possibility that U contains points with some or all coordinates $= 0$. We write Log $(z_1, \ldots, z_k)$ for the k-tuple $(\log |z_1|, \ldots, \log |z_k|)$, including the possible $-\infty$ values.

Note that $\text{Log}^{-1}(1, \ldots, 1) = T \subset \mathbb{C}^k$. Thus $\text{Log}^{-1}(V)$, where V is some neighborhood of $(1, \ldots, 1)$ in $\mathbb{R}^k$, is a tubular neighborhood of the real n-dimensional submanifold T of $\mathbb{C}^k$. Note also that T is a totally real submanifold of $\mathbb{C}^k$ in the sense that the tangent space of T and the J-image of this tangent space intersect in the 0-vector only. (Here J is the standard almost complex structure on $\mathbb{R}^{2k} = \mathbb{C}^k$.) That T is totally real is clear at the point $(1, 1, \ldots, 1) \in \mathbb{C}^k$, since the tangent space in $\mathbb{R}^{2k}$ coordinates $(x_1, y_1, \ldots, x_k, y_k)$, $x_j + iy_j = z_j$, is the set of vectors of the form $(0, b_1, 0, b_2, \ldots, 0, b_k)$, each $b_j \in \mathbb{R}$. The same holds at other points of T since these arise from $(1, 1, \ldots, 1)$ by a complex linear map which preserves T. Thus we can identify, for each (sufficiently small) neighborhood V in $\mathbb{R}^k$ of $(1, 1, \ldots, 1)$, the set $\text{Log}^{-1}(V)$ with a tubular neighborhood of T in its own complexification: $T_{\mathbb{C}}$ is characterized in a neighborhood of T by being a complex k-dimensional manifold containing T as a totally real submanifold.

Suppose now that Ω is a Reinhardt domain and Log (Ω) is a bounded convex domain in $\mathbb{R}^k$. Then, by [Bedford 1980], the automorphisms of Ω must have the form:

$$(z_1, \ldots, z_k) \mapsto (c_1 z^{m_1}, \ldots, c_k z^{m_k}),$$

where we are using multi-index notation

$$z^{m_j} = z_1^{m_j^1} \cdots z_k^{m_j^k},$$

and where it is required that the matrix $(m_j^\ell) \in GL(k, \mathbb{Z})$. A mapping of this form maps Ω to Ω if and only if the affine mapping $z \mapsto Mz + \log |c|$ is an affine mapping of Log (Ω) to itself. Here $M =$ the matrix (m_j^ℓ), $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ and $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$.

Now, if $k \geq 2$, then, generically, domains in $\mathbb{R}^k$ have no nontrivial affine self-mappings. In particular, there are domains V in $\mathbb{R}^k$ that are small perturbations of a (small) ball around the origin in $\mathbb{R}^k$. For such V, as before, the domain $\mathrm{Log}^{-1}(V)$ is a tubular neighborhood of T in its complexification $T_{\mathbb{C}}$, where as earlier we identify T with a totally real submanifold of $\mathbb{C}^k$. And, for such V (which have no nonidentity affine self-mapping), the automorphism group of $\mathrm{Log}^{-1}(V)$ is exactly T.

In case $k = 1$, any connected bounded open neighborhood of 0 in $\mathbb{R}^k = \mathbb{R}^1$ has an affine self-mapping that is not the identity, first reflection at its midpoint, the neighborhood being of course an open interval. Thus, in this case, for any V, $\mathrm{Log}^{-1}(V)$ has an automorphism other than those in T. (One such automorphism which is associated to the affine "inversion" indicated is the automorphism $z \mapsto R_1R_2/z$ of $\{z\colon R_1 < |z| < R_2\}$, $0 < R_1 < R_2 < +\infty$ to itself.) So special consideration and indeed an extra dimension (as stated in the theorem) is needed in this case. Indeed no Riemann surface has automorphism group isomorphic to $\{z \in \mathbb{C}\colon |z| = 1\}$ (cf. Chapter 2): the extra dimension is definitely required.

The reader can find an explicit construction dealing with this special case in [Bedford/Dadok 1987].

In summary form: set $\Omega = \{(z, w) \in \omega \times \mathbb{C}\colon r_1(z) < |w| < r_2(z)\}$, where ω is a smoothly bounded, triply-connected domain in $\mathbb{C}$ with $\mathrm{Aut}\,(\omega)$ being the identity alone, and r_1, r_2 are continuous functions on the closure of ω, smooth on ω itself, with $0 < r_1 < r_2$ on the closure of ω. Then, if $r_1(z)r_2(z)$ is not the modulus of a holomorphic function on ω, then $\mathrm{Aut}\,(\Omega)$ is isomorphic to $\{\alpha \in \mathbb{C}\colon |\alpha| = 1\} = T$. The proof can be found in [Bedford/Dadok 1987]. Note that it is not hard to see that there are, for example, perturbations of the unit ball in $\mathbb{C}^2$ for which the automorphisms group is exactly the set of maps (isomorphic to T)

$$(z_1, z_2) \mapsto (\alpha z_1, \alpha z_2), \quad \alpha \in \mathbb{C}, \quad |\alpha| = 1.$$

The point of the more intricate construction of Bedford/Dadok is that the above Ω lies in $T_{\mathbb{C}} \times \mathbb{C}$.

5.4.9 The Case of Simple Lie Groups

The next stage in the application of the product decomposition to finding domains in $G_{\mathbb{C}}$ with specified automorphism group is to consider the case $G =$ a compact simple group. In this case the usual representation of G acting on its own Lie algebra is faithful up to a finite kernel. In more detail, if v is a vector in the tangent space of G at the identity and $\gamma(t)$ is the corresponding one-parameter subgroup, then we define $Ad\ g$, $g \in G$ acting on v, by

$$(Ad\ g)(v) = \frac{d}{dt} g^{-1}\gamma(t)g\Big|_{t=0},$$

this being again a tangent vector to G at the identity. This gives a representation

$$G \to \text{linear endomorphism of the tangent space of } G \text{ at the identity.}$$

The simplicity of G implies that the kernel of this representation is finite. Indeed, to check this one needs only check that the kernel contains no 1-parameter subgroup, since the kernel is a closed subgroup of G. This follows from the simplicity of G and the associated nondegeneracy of the Killing form.

Thus, up to a finite quotient, G can be considered to be a matrix group. The image of the Ad representation is in fact a subgroup of the orthogonal group of linear transformations of the tangent space at the identity, orthogonal relative to the bi-invariant metric (which is the Killing form, as already discussed).

This gives an explicit way to construct a neighborhood basis of G inside $G_{\mathbb{C}}$; first, if ω is a neighborhood of zero in the tangent space of G at the identity, then we can set $\Omega_\omega = G \cdot \exp(i\omega)$ (ignoring the quotienting, which is easily handled by "lifting"), where exp is the usual exponentiation of matrices. Of course one can handle this matter "intrinsically": since exp in the 1-parameter subgroup sense is defined on ω, and since G is totally real in $G_{\mathbb{C}}$ and exp is real analytic, there is a unique way to define exp holomorphically on a sufficiently small neighborhood of the identity in $G_{\mathbb{C}}$. In particular, if w is sufficiently small, then $\exp(i\omega)$ is defined in this way, simply from holomorphic function theory.

Note that such a tubular neighborhood is G-invariant (for left multiplication action of G), and that this G-action is holomorphic on this G-invariant neighborhood of G in $G_{\mathbb{C}}$. The final step in completing the construction is to show that, for some suitable choice of ω, these G-induced automorphisms are the only automorphisms of the tubular neighborhood.

To begin with, we restrict the neighborhood ω of 0 in the Lie algebra of G (which we identify as usual with the tangent space of G at the identity) to be a perturbation of a small ball around 0 in the Lie algebra in the bi-invariant metric. As far back as Grauert's proof of the existence of real analytic embedding of real analytic manifolds [Grauert 1958], it was noted that for such ω, the associated tubular neighborhood Ω_ω is C^∞ strongly pseudoconvex. This is a general phenomenon, not involving the fact that G is a Lie group: every compact real analytic manifold has a neighborhood basis of smooth strongly pseudoconvex domain inside its own complexification (again [Grauert 1958]). In particular, such tubular neighborhoods are Stein manifolds, by Grauert's solution of the Levi problem since they have no compact positive-dimensional subvarieties. Each of these Stein tubular neighborhoods has compact closure in a slightly larger tubular neighborhood which is also a Stein manifold. Then it follows that a given such tubular neighborhood has a defined, positive definite Bergman metric in the manifold sense. This Bergman metric is constructed from the Bergman kernel obtained from the space of L^2 holomorphic $(k, 0)$ forms, $k =$ the complex dimension of the complexification, as discussed in Section 3.2. This follows easily from embedding in complex Euclidean space the slightly larger Stein manifold in which the given tubular neighborhood has compact closure. The given tubular neighborhood thus inherits holomorphic L^2 forms from the ambient Euclidean space; these restrictions/pullbacks to the tubular neighborhood in the submanifold (of $\mathbb{C}^N$) are automatically L^2, and there are enough of them to guarantee a positive definite Bergman metric. This argument is a straightforward generalization of the argument showing that a bounded domain in $\mathbb{C}^N$ has a defined and positive definite Bergman metric.

Returning to the specific situation of an Ω_ω in $G_{\mathbb{C}}$ with ω so chosen as above, note that $\mathrm{Aut}(\Omega_\omega)$ contains G in the sense that (left) multiplication by elements of G acts as biholomorphic maps on Ω_ω. *A priori*, it could be that $\mathrm{Aut}(\Omega_\omega)$ is larger than G, or even that the connected component of the identity in $\mathrm{Aut}(\Omega_\omega)$ was larger than G. [Note that $\mathrm{Aut}(\Omega_\omega)$ is a Lie group here and indeed a Lie group with the isotropy of points of Ω_ω compact, since $\mathrm{Aut}(\Omega_\omega)$ is a closed subgroup of the isometry group of the Bergman metric of Ω_ω.]

Now the homology group $H_d(\Omega_\omega, \mathbb{Z})$ is isomorphic to $H_d(G, \mathbb{Z})$, since ω is convex; thus Ω_ω has a strong deformation retract onto $G \subset \Omega_\omega$ by linearly contracting ω to 0 in the Lie algebra. Since $H_d(G, \mathbb{Z}) = \mathbb{Z}$, $d = \dim_{\mathbb{R}} G$, it follows by topological considerations that there is an orbit of $\mathrm{Aut}(\Omega_\omega)$ in Ω_ω with dimension at most d ([Bedford 1983a]). Since $\mathrm{Aut}(\Omega_\omega)$ contains G in the sense mentioned, such an $\mathrm{Aut}(\Omega_\omega)$-orbit of dimension at most d must in fact be a finite union of G-orbits (of dimension exactly d). And any one of these must be stable under the identity component $\mathrm{Aut}^0(\Omega_\omega)$ of $\mathrm{Aut}(\Omega_\omega)$, by continuity.

Let Gx_0 (following the notation of [Bedford/Dadok 1987]) be such an $\mathrm{Aut}^0(\Omega_\omega)$-stable orbit. Then $\mathrm{Aut}^0(\Omega_\omega)$ acts as isometries on Gx_0, when Gx_0 is equipped with the restriction of the Bergman metric of Ω_ω. Identifying Gx_0 with G (since left multiplication by "elements of G" is a simply transitive action on Gx_0), one obtains that $\mathrm{Aut}^0(\Omega_\omega)$ is in effect a subgroup of the identity component of the isometry group of G with the left-invariant metric obtained by restricting the Bergman metric to Gx_0 (identified with G). Note that this need not be the bi-invariant metric of G itself (if $x_0 \notin G \subset G_{\mathbb{C}}$), but it is left invariant. The form of such isometries was determined in [Ochiai/Takahashi 1976]: for each $f \in \mathrm{Aut}^0(\Omega_\omega)$, there are elements $a, b \in G$ such that $f(g \cdot x_0) = agb \cdot x_0$, where $\cdot$ denotes the G-action operation.

Since G is transitive on Gx_0, an "extra" automorphism in $\mathrm{Aut}^0(\Omega_\omega)$, that is one that is not in G, can be obtained as an automorphism φ fixing x_0 followed by one in G. Such an automorphism φ fixing x_0, and stabilizing the orbit Gx_0 at x_0, acts on the tangent space $T_{x_0}(Gx_0)$ of Gx_0 at x_0. The Cauchy–Riemann equations then determine an action on $J(T_{x_0}(Gx_0))$. Thus, since φ is a Bergman metric isometry, this determines the action of φ on geodesics with tangent vectors in $J(T_{x_0}(Gx_0))$.

The domain ω determines the domain Ω_ω as far as its transversal-to-G nature is concerned. So, in this situation, it is natural to suppose that a suitable choice of ω will rule out the possibility of any nontrivial such action of $d\varphi$ on the $J(T_{x_0}(Gx_0))$. And then, again by Cauchy–Riemann equations, the action $d\varphi$ along G would also be necessarily trivial. Then no "extra" automorphisms in $\mathrm{Aut}^0(\Omega_\omega)$ would exist.

This intuitive expectation is in fact correct. In [Bedford/Dadok 1987], it is shown that for this it suffices to choose ω so that (i) $\omega = -\omega$ and (ii) the only $\sigma \in$ automorphisms of the Lie algebra of G with $\sigma(\omega) = \omega$ is the identity. (Note here that multiplication by -1 is not an automorphism of the Lie algebra so the conditions are consistent.) Of course, ω continues to be chosen so that $G \cdot \exp(i\omega)$ is strongly pseudoconvex and smoothly bounded in $G_{\mathbb{C}}$. For the sufficiency of this genericity condition, the reader is referred to [Bedford/Dadok 1987].

5.4.10 Connected Lie Group Case with Product Decomposition

Once the situation is in hand for the torus factor and the simple group factors in the product decomposition $G = T \times G_1 \times \cdots \times G_\ell / H$, H finite, the group G as a whole is treated as follows. A domain in the complexification $G_{\mathbb{C}}$ of G written as $G \cdot \exp(i\omega)$, some ω, can be obtained in particular with $\omega = \omega^0 \times \cdots \times \omega^\ell$ in obvious notation. By a result of H. Cartan

$$\mathrm{Aut}\,(\Omega_\omega) = T \times \mathrm{Aut}\,(\Omega_1) \times \cdots \times \mathrm{Aut}\,(\Omega_\ell),$$

(where $T = \mathrm{Aut}\,(\Omega_0)$ and $\Omega_j = \Omega_{\omega^j}$) provided that the ws are chosen so that no permutation-of-factors automorphisms arise: this choice of ws is always possible. A lifting argument disposes of the H-quotienting (see [Bedford/Dadok 1987] for details), and one obtains a pseudoconvex product domain in $G_{\mathbb{C}}$ with automorphism group G.

We replace this domain with a bounded strongly pseudoconvex domain with smooth boundary by considering sub-level sets of a C^∞ strictly plurisubharmonic exhaustion function φ, first $\{z : \varphi(z) < \lambda\}$, λ a noncritical value of φ. The normal families method of [Greene/Krantz 1985b] can be applied to obtain a bounded, strongly pseudoconvex domain with smooth boundary which is clearly G-invariant and has no "extra" automorphisms so that its automorphism group is G. By using a real analytic φ, one can in fact make this final domain have real analytic boundary.

5.4.11 Some Remarks

If one is not restricted to bounded strongly pseudoconvex domains, for instance if one is interested in constructing complex manifolds with prescribed automorphism group, there is more recent work, even when the given Lie group is noncompact. See for instance [Winkelmann 2004], [Kan, S.-J. 2007].

On the other hand, the following question was posed by Greene and Krantz some years ago:

Question ([Greene/Krantz 1982a]). Let Ω be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary, whose automorphism group is compact. Let H be a closed subgroup of the automorphism group. Then, for any open neighborhood $\mathcal{U}$ of Ω in the C^∞ topology, does there exist $\Omega' \in \mathcal{U}$ such that Aut (Ω) is Lie-group-isomorphic to H?

A significant partial answer is reported recently: see [Min, B.-L. 2009]. the result is as follows.

Theorem 5.4.2 ([Min, B.-L. 2009]). *Let Ω be a bounded, strongly pseudoconvex domain in $\mathbb{C}^N$ with C^∞ boundary, with its automorphism group G compact. If $N > 5 \dim_{\mathbb{R}} G + 4$, then, for any closed subgroup H of G and any open neighborhood $\mathcal{U}$ (in the C^∞ topology on domains) of Ω, there exists $\Omega' \in \mathcal{U}$ such that* Aut (Ω') *is Lie-group-isomorphic to H.*

Whether the codimension condition $N > 5 \dim_{\mathbb{R}} G + 4$ is sharp is not known at this writing. Of course some restriction on the dimension is clearly required; see for example the discussion on $O(3)$ and $SO(3)$ actions in Section 5.4.3.