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Afterword

Many of the results in previous chapters concerned bounded strongly pseudoconvex domains in complex Euclidean spaces. As it happens, almost all of these results can be extended in some form to more general situations. In particular, most of them apply in some suitable form to strongly pseudoconvex domains with compact closure in Stein manifolds. The restriction to the Euclidean space case earlier simplified the statements and made for a clearer exposition of the proof techniques. But it is of course important to realize that generalizations are possible when indeed they are possible. In this final chapter, we shall try to indicate these possibilities in enough detail that interested readers will be able to carry through the detailed statements and proofs for themselves in the more general situations which will be indicated.

If Ω is a connected open subset with a compact closure in a complex manifold M such that Ω has nonempty C^∞ strongly pseudoconvex boundary, then, according to [Grauert 1958], Ω is itself a Stein manifold provided that Ω contains no compact complex subvarieties of positive dimension. In particular, in this case, there is a slightly larger C^∞ strongly pseudoconvex domain $\widehat{\Omega}$ which contains the closure of Ω such that $\widehat{\Omega}$ is also a Stein manifold. Thus we shall lose no real generality if we assume that the C^∞ strongly pseudoconvex connected open sets to be considered lie with compact closure in some Stein manifold M of complex dimension n.

This assumption, which we make from now on, yields a number of important properties for Ω almost immediately. By the famous embedding theorem of Bishop, Narasimhan, and Remmert ([Bishop, E. 1961], [Narasimhan 1960], [Remmert 1956]), M can be properly embedded in some complex Euclidean space $\mathbb{C}^N$. In our case, applying this result to M, we find an embedding of Ω into $\mathbb{C}^N$ which is smooth on the closure of Ω, and indeed on an open neighborhood of the closure, which takes a suitable such neighborhood to a bounded set in $\mathbb{C}^N$.

It follows immediately that Ω admits an abundance of bounded holomorphic functions: every holomorphic function on $\mathbb{C}^N$, when restricted to (the image of) Ω, is bounded. This yields immediately that the Carathéodory

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metric of Ω is positive definite and hence that its Kobayashi metric is. (This latter can also be seen directly by Cauchy estimates.) Moreover, the pullback to Ω of a holomorphic $(n, 0)$-form on $\mathbb{C}^N$ will be a holomorphic $(n, 0)$-form on M which is necessarily L^2 on Ω. It is easy to check that there are enough such forms to guarantee that the intrinsic Bergman metric of Ω in terms of L^2 holomorphic $(n, 0)$-forms is defined and is a positive definite Kähler metric on Ω (see Section 3.2).

In particular, it follows that the automorphism group of Ω is a Lie group (see Section 7.2.3), that its isotropy subgroups I_p, $p \in \Omega$, are compact, and that the map of I_p into linear maps of the tangent space at p defined by $f \mapsto df|_p$, $f \in I_p$, is injective: the direct analogues of Theorem 1.3.1 and Corollary 1.3.3 are valid. Also, the action of $\mathrm{Aut}\,(\Omega)$ on Ω is proper. (cf. Theorem 7.2.10.)

To study Ω and in particular $\mathrm{Aut}\,(\Omega)$ further, it is useful to note that Ω (identified with its image as a submanifold in $\mathbb{C}^N$) can be exhibited as the intersection with M (similarly identified) of a C^∞ strongly pseudoconvex domain in $\mathbb{C}^N$ with certain special properties. We begin by noting from standard Stein manifold theory ([Docquier/Grauert 1960]) that there is a neighborhood U of M in $\mathbb{C}^N$ for which there is a *holomorphic retraction* onto M; i.e., there is a holomorphic map $F : U \to M$ such that $F(z) = z$ for every $z \in M$. (Here we identify M with its image in $\mathbb{C}^N$ as before.) Choose a C^∞ strictly plurisubharmonic function φ_1 defined in a neighborhood of the closure of Ω in M such that $\Omega = \{z\colon \varphi_1(z) < 1\}$ and $d\varphi_1$ is nowhere zero on the boundary of Ω. Set $\widehat{\varphi_1} = \varphi_1 \circ F$. Set $\varphi_{2,\epsilon}(z) = \epsilon^{-2}\ \mathrm{dis}^2(z, M)$. Then, for $\epsilon > 0$ sufficiently small, $\varphi_{2,\epsilon}$ is C^∞ for all z with $\varphi_{2,\epsilon}(z) < 2$ and z close enough to Ω. Now declare $\widehat{\Omega}$ to be the set of $z \in U$ such that $F(z)$ lies in the neighborhood of the closure of Ω on which φ_1 is defined and $\widehat{\varphi}_1(z) + \varphi_{2,\epsilon}(z) < 1$. It is straightforward to check that $\widehat{\varphi}_1 + \varphi_{2,\epsilon}$ is, again for $\epsilon > 0$ sufficiently small, C^∞ strictly plurisubharmonic in a neighborhood of the closure of Ω: the function $\widehat{\varphi}_1$ is strictly plurisubharmonic "parallel to M" and $\varphi_{2,\epsilon}$ is strictly plurisubharmonic "perpendicular to M" (cf. [Greene/Wu 1978] and [Elencwajg 1975]). Thus $\widehat{\Omega}$ is C^∞ strongly pseudoconvex—the nonvanishing of $d(\widehat{\varphi}_1 + \varphi_{2,\epsilon})$ at the boundary of $\widehat{\Omega}$ is also clear, for the C^∞ part. Moreover, $\widehat{\Omega} \cap M = \Omega$ and $F(\widehat{\Omega}) \subset \Omega$, since $\widehat{\varphi}_1 < 1$ on $\widehat{\Omega}$ by definition.

The utility of this somewhat intricate construction is that analysis of the $\overline{\partial}$ problem on Ω can be transferred to $\widehat{\Omega}$, a situation—C^∞ strongly pseudoconvex domains in $\mathbb{C}^N$—that is very familiar. Of course, $\overline{\partial}$ analysis can be carried out directly on domains in Stein manifolds. But the present approach will be advantageous when we wish to consider stability matters.

The construction just given yields immediately that, if p is a point of the boundary of Ω in M, then p is a "peak point" in the following (generalized) sense: there is a holomorphic function $f_p : \Omega \to \mathbb{C}$ such that $|f_p(z)| \to 1$ as $z \to p$ while $\limsup |f_p(z)| < 1$ as $z \to q$, $q \neq p$, $q \in \partial\Omega$. This follows since such "peaking functions" exist for each point of the boundary of a C^∞

bounded, strictly pseudoconvex domain in $\mathbb{C}^N$, so that peaking functions can be obtained for Ω by restricting a peaking function for $\widehat{\Omega}$.

The importance for our purposes of the existence of such peaking functions is that this means that the argument of [Rosay 1979] applies to yield the analogous theorem, not just for domains in $\mathbb{C}^n$ as in Section 9.2.4, but also for domains in Stein manifolds:

> *If Ω is a C^∞ strictly pseudoconvex domain in a Stein manifold and if* $\operatorname{Aut}(\Omega)$ *is noncompact, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} \Omega$.*

Actually, the existence of a global peaking function turns out to be unnecessary: the only hypothesis actually needed is strictly local. In particular, this optimal result is obtained in [Gaussier/Kim/Krantz 2002]:

> *If Ω is a domain in a complex manifold with C^2 boundary in a neighborhood of some boundary point p, if p is a strictly pseudoconvex boundary point, and if* $\operatorname{Aut}(\Omega)$ *has an orbit that accumulates at p, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} \Omega$.*

Returning now to the situation of a C^∞ strictly pseudoconvex domain in a Stein manifold, the embedding of Ω in $\widehat{\Omega}$ opens up, as mentioned earlier, the possibility of doing $\overline{\partial}$ analysis on Ω rather explicitly. first, suppose that ω is a $(0,1)$-form on Ω. By the construction of $\widehat{\Omega}$, there is a holomorphic retraction (projection) $F : \widehat{\Omega} \to \Omega$, which in fact is defined and holomorphic on a neighborhood of the closure of $\widehat{\Omega}$. Since holomorphic pullbacks commute with $\overline{\partial}$, we see that $F^*(\overline{\partial}_M \omega) = \overline{\partial}_{\mathbb{C}^N}(F^*\omega)$. In particular, $F^*\omega$ is $\overline{\partial}$ closed if ω is. Moreover, if $\overline{\partial} u = F^*\omega$, then $u|_\Omega$ satisfies $\overline{\partial}_M(u|_M) = \omega$. This setup means that the full power of the regularity theory for the Kohn solution of $\overline{\partial}$ on strongly pseudoconvex domains is available, even though in our setting there is no *a priori* canonical notion of a Kohn solution (orthogonal to holomorphic functions) on Ω, because Ω does not have a canonically specified metric.

In particular, $\overline{\partial}$ localization at boundary points holds in the form needed to make the *scaling method* apply in the form needed to establish the analogue of Theorem 3.4.3 and its stability under perturbation: Theorem 3.5.1 (and also 3.5.2).

Theorem 11.1 (Theorems 3.5.1 and 3.5.2 Extended). *If Ω_0 is a C^∞ strongly pseudoconvex domain with compact closure in a Stein manifold M, then the Bergman metric of Ω_0 is complete, and its holomorphic sectional curvature is asymptotically constant negative $-4/(n+1)$ in the sense that, given $\epsilon > 0$, there is a $\delta > 0$ such that, if $p \in \Omega_0$ with dis $(p, M \setminus \Omega_0) < \delta$, then $\left|K + \frac{4}{n+1}\right| < \epsilon$ for each holomorphic sectional curvature K at p of the Bergman metric of Ω_0. Moreover, this estimate is stable in the sense that there is a $\delta > 0$ and a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology of domains such that, if $\Omega \in \mathcal{U}$ and $p \in \Omega$ with dis $(p, M \setminus \Omega) < \delta$, then $\left|K + \frac{4}{n+1}\right| < \epsilon$ for each holomorphic sectional curvature K at p of the Bergman metric of Ω.*

Here dis means distance in a fixed Kähler metric on M.

It is actually the case that an asymptotic expansion of Fefferman type holds in a neighborhood of each boundary point. first $K(z,w)$, which is now a *double form* of type (n,n)—type $(n,0)$ in z and type $(0,n)$ in w—is again C^∞ on $\mathrm{cl}(\Omega) \times \mathrm{cl}(\Omega) \setminus \{(p,p) \colon p \in \partial\Omega\}$. And, given a boundary point p of Ω with holomorphic coordinates $(z_1, \ldots, z_n)$ defined in a neighborhood U of p in the Stein manifold M, we can write

$$K(z,w) = f(z,w)\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w_1} \wedge \cdots d\overline{w_n},$$

for z, w in the neighborhood U_p of p and in Ω. Then the function $f(z,w)$ has the same form of asymptotic expansion as does the Bergman kernel function in the Euclidean space case (cf. Section 3.4). This is established by using $\overline{\partial}$ localization of the Bergman kernel form, which implies that its asymptotic behavior near p is the same as that of the Bergman kernel of $U_p \cap \Omega$ (where we can take U_p to be itself strongly pseudoconvex). No essentially new ingredients arise here: after the localization argument, one is in the original Fefferman situation. This also holds in stable form, stable under C^∞ perturbation.

Thus, either from the full Fefferman expansion or from the less detailed but still sufficient information arising from the scaling method (see Section 10.1), one can consider boundary orbit accumulation from the curvature viewpoint. In particular, suppose that Ω is, as before, C^∞ strongly pseudoconvex (or even C^2, since the scaling method still applies in that case). Also, suppose that there is a sequence $\{\varphi_j\} \subset \mathrm{Aut}\,(\Omega)$ such that, for some $q \in \Omega$, the sequence $\{\varphi_j(q)\}$ converges to a point p_0 in the boundary of Ω. Then, as in Section 3.4, the (complete) Bergman metric of Ω has constant holomorphic sectional curvature. As in Corollary 3.4.4, one can then deduce that Ω is biholomorphic to the ball. As in the situation of Corollary 3.4.4, standard Kähler geometry gives that the universal cover of Ω is biholomorphic to the ball. To show that the covering map is injective, or equivalently that Ω is simply connected, any of the several methods used to deal with the question for 3.4.4 can be used here.

In particular, Lu Qi-Keng's theorem (Theorem 4.2.2) applies[1] in this case:

Theorem 11.2 (Lu Qi-Keng's Theorem for Stein Domains). *If Ω is a domain with compact closure in a Stein manifold M, and if the Bergman metric of Ω is complete and of constant (negative) holomorphic sectional curvature, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$.*

The proof of this result is obtained by the same method as for the case of domains in $\mathbb{C}^n$, with the one additional feature that a modified definition of

[1]If we are only concerned with establishing the simple connectivity of Ω, Lu's theorem is not really required, as noted in Section 3.4. But we exploit Lu's theorem here, because this generalization of Lu's theorem for such Stein domains is interesting by itself. Note: A domain is called *Stein* if it admits a strictly plurisubharmonic exhaustion function.

Bergman representative coordinates must be given. In the original definition as given in Section 4.2, the coordinates at $w_0 \in \Omega$ were obtained as $\overline{w}$-derivatives of $\log\bigl(K(z,w)/K(w,w)\bigr)$, with the derivatives evaluated at w_0. In the present Stein manifold case, the quotient $K(z,w)/K(w,w)$ is not as such defined, since now $K(z,w)$ and $K(w,w)$ are not functions, but are rather *double forms*, one at $(z,w) \in \Omega \times \Omega$ and the other at $(w,w) \in \Omega \times \Omega$ so that, if $z \neq w$, the quotient is not meaningful.

However, this apparent difficulty can be removed by choosing holomorphic coordinate systems $(z_1, \ldots, z_n)$ around the given $z \in \Omega$ and $(w_1, \cdots, w_n)$ around the given w and then writing

$$K(z,w) = f(z,w)\; dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w_1} \wedge \cdots d\overline{w_n}$$

and

$$K(w,w) = g(w,w)\; dw_1 \wedge \cdots \wedge dw_n \wedge d\overline{w_1} \wedge \cdots d\overline{w_n}.$$

Then the quotient f/g is well defined up to a product of two factors,[2] one a holomorphic function of z, the other a holomorphic function of w—these factors depending on the choice of z and w coordinate systems (the conjugate factors for the w-coordinates cancel since the same factor occurs in f and g). Thus $\overline{w}$-derivatives of $\log(f/g)$ are well defined even though f/g is not well defined itself. Once it is noted that Bergman representative coordinates can be thus defined, the remainder of the proof given in Section 4.2 (see Theorem 4.2.2) applies to establish Lu Qi-Keng's theorem in this Stein domain situation. □

We return now to the function-theoretic and geometric stability properties of compact-closure C^∞ strongly pseudoconvex domains Ω in a Stein manifold M (which, as before, we suppose to have a fixed proper embedding $E\colon M \to \mathbb{C}^N$). Using the construction for representing Ω as the intersection of $E(M) \subset \mathbb{C}^N$ with a C^∞ strongly pseudoconvex domain in $\mathbb{C}^N$ as already discussed, one obtains stable $\overline{\partial}$ estimates for variation of Ω in M from the stable $\overline{\partial}$ estimates for C^∞ strongly pseudoconvex domains in $\mathbb{C}^N$ ([Greene/Krantz 1982]). This stability is the needed ingredient to establish the extension of Theorems 3.5.1 and 3.5.2, as already stated. This theorem in particular gives stable bounds on the distance of orbits from the boundary, analogous to Theorem 3.5.2; this result comes directly from the stability part of the extension of Theorems 3.5.1 and 3.5.2.

Theorem 11.3 (Theorem 3.5.2 Extended). *Suppose that M is a Stein manifold with a fixed but arbitrary Kähler metric and suppose that Ω_0 is a C^∞ strictly pseudoconvex open subset of M with compact closure in M. If Ω_0 is not biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$, and if $p_o \in \Omega_0$,*

[2] One has to check what happens when we work with other holomorphic coordinate systems; that is what is discussed here.

then there is a $\delta > 0$ *and a neighborhood* $\mathcal{U}$ *of* Ω_0 *in the* C^∞ *topology on* C^∞ *compact-closure domains in* M *such that, if* $\Omega \in \mathcal{U}$*, then:*

(1) $p_0 \in \Omega$.
(2) *The domain* Ω *is real diffeomorphic to* Ω_0 *via a diffeomorphism that is* C^∞ *on the closure of* Ω *and with its inverse* C^∞ *on the closure of* Ω_0.
(3) *For every* $\varphi \in \operatorname{Aut}(\Omega)$*, the distance in the Kähler metric on* M *from* $\varphi(p_0)$ *to the boundary of* Ω *is* $\geq \delta$.

The proof here follows the pattern of the proof of Theorem 3.5.2.

This result together with the normal families results already noted make it possible to apply exactly the arguments used to prove Theorem 4.4.3 to prove a similar semicontinuity result for perturbation of a given Ω_0 in a Stein manifold, Ω_0 not biholomorphic to the ball.

Theorem 11.4 (Theorem 4.4.3 Extended). *If* Ω_0 *is a* C^∞ *strongly pseudoconvex domain in a Stein manifold* M *with* Ω_0 *not biholomorphic to the unit ball in* $\mathbb{C}^n$*,* $n = \dim_{\mathbb{C}} M$*, then there is a neighborhood* $\mathcal{U}$ *of* Ω_0 *in the* C^∞ *topology such that, if* $\Omega \in \mathcal{U}$*, then* $\operatorname{Aut}(\Omega)$ *is isomorphic to a subgroup of* $\operatorname{Aut}(\Omega_0)$ *via an isomorphism obtained by conjugation by a real diffeomorphism of* Ω *to* Ω_0*. first, there is a real diffeomorphism* $F\colon \Omega \to \Omega_0$ *such that the map* $\alpha \mapsto F \circ \alpha \circ F^{-1}$*,* $\alpha \in \operatorname{Aut}(\Omega)$*, is an injective homomorphism of* $\operatorname{Aut}(\Omega)$ *onto a subgroup of* $\operatorname{Aut}(\Omega_0)$.

A result for Stein domains analogous to Theorems 4.3.2 and 4.3.3 holds, and the same basic technique applies, but some additional technical considerations arise. The result itself is what one would perhaps expect.

Theorem 11.5 (Theorems 4.3.2 and 4.3.3 Extended). *Suppose that* Ω_0 *is a* C^∞ *compact-closure strictly pseudoconvex domain in a Stein manifold* M*. Then there is a* C^∞ *neighborhood* $\mathcal{O}$ *of the almost complex structure* J_M *of* M *restricted to the closure of* Ω_0 *within the space of all* C^∞ *almost complex structures on the closure of* Ω_0 *with the following property: for each* $J \in \mathcal{O}$ *with* J *integrable on* Ω_0*, there is a* C^∞ *compact-closure domain* Ω_J *in* M *such that* (Ω_0, J) *is biholomorphic to* $\left(\Omega_J, J_M\big|_{\Omega_J}\right)$*. Moreover, given any* C^∞ *neighborhood* $\mathcal{U}$ *of* Ω_0 *in the* C^∞ *topology on domains, the neighborhood* $\mathcal{O}$ *can be chosen so that, for each* $J \in \mathcal{O}$*, the domain* Ω_J *can be chosen to be in* $\mathcal{U}$.

The essential idea of the proof of this result is the same as that of the proof of Theorem 4.3.2, except that we *correct* not the coordinate functions of a domain in $\mathbb{C}^n$ but the embedding functions for M. Specifically, with $E\colon M \to \mathbb{C}^N$ a holomorphic proper embedding as before, write

$$E = (E_1, \ldots, E_N),$$

where each $E_i\colon M \to \mathbb{C}$ is a holomorphic function; holomorphic here means holomorphic in the J_M complex structure. The functions $E_i\big|_{\Omega_0}$ are of course

C^∞ on the closure of Ω_0. They need not be holomorphic relative to another integrable complex structure on (the closure of) Ω_0, but $\overline{\partial_J} E_i$, $\overline{\partial_J}$ relative to the J-structure, is C^∞ small on the closure of Ω_0. Suppose for the moment that $\overline{\partial_J}$ satisfies stable estimates in the same sense as for domains in the proof of Theorem 4.3.2. Then there are C^∞ functions u_j on Ω_0, which are C^∞ small up to the boundary of Ω_0, which satisfy $\overline{\partial_J} u_j = \overline{\partial_J} E_j$, $j = 1, \ldots, N$. Then the map of Ω_0 into $\mathbb{C}^N$ defined by setting (the j-th coordinate function of) $E_J \colon M \to \mathbb{C}^N = E_j - u_j$, $j = 1, \ldots, N$, is J-holomorphic, and C^∞ close to the map E.

Of course there is no guarantee that the image of E_J lies in $E(M)$. But by [Docquier/Grauert 1960] there is a *tubular neighborhood* of $E(M)$, first, an open set U in $\mathbb{C}^N$ that contains $E(M)$ and for which there is a holomorphic mapping $F \colon U \to E(M)$ with $F\big|_{E(M)} =$ identity. For short, there is a *holomorphic retraction* of U onto $E(M)$.

With F so chosen, it then follows from standard differential topology that, when E_J is sufficiently C^∞ (even C^1) close to E on the closure of Ω_0, the map $F \circ E_J$ is a holomorphic diffeomorphism of Ω_0 with the J-complex structure onto its image in $E(M)$, so that $E^{-1} \circ F \circ E_J$ is its desired biholomorphic realization of (Ω_0, J) as a compact-closure domain in M.

The required $\overline{\partial_J}$ estimates, stable in J, are obtained by working through the solution of the $\overline{\partial}$-Neumann problem for strictly pseudoconvex domains in Stein manifolds directly, and checking the stability of each step, as in [Greene/Krantz 1982]—a tedious and fairly difficult process.

If Ω_0 is a compact-closure domain in a Stein manifold M and if G is a compact subgroup of Aut (Ω_0), then a G-invariant Kähler metric on Ω_0 can be obtained as follows: Let φ be a C^∞ strictly plurisubharmonic function on M. Define $\psi : \Omega \to \mathbb{R}$ as the average of $\varphi\big|_{\Omega_0}$ with respect to the G-action. Then the Levi form of ψ is the desired G-invariant metric. If G on Ω_0 extends to act smoothly on the closure of Ω (as always happens, if Ω_0 is C^∞ strongly pseudoconvex in M), then this G-invariant metric will be C^∞ on the closure of Ω_0. In this case, the Kohn solution of the $\overline{\partial}$ problem (orthogonal to holomorphic functions with respect to the Kähler metric) will be G-invariant in the obvious sense. This in turn implies that a G-invariant abstract perturbation of the complex structure of Ω_0 can be realized G-equivariantly.

Equivalently, if G acting on Ω_0 arises as the restriction to Ω_0 of the action of the group G on all of M, the action preserving Ω_0, then every abstract G-invariant perturbation of the complex structure of Ω_0 that is sufficiently C^∞ close to the complex structure of Ω_0 can be realized in the sense of Theorem 11.5 as a G-invariant domain in M which is a C^∞ perturbation of Ω_0. In this sense, Theorem 11.5 holds equivariantly.

While some additional technical details can be expected to and indeed do arise in these developments, it is, from a certain viewpoint, almost to be expected that so much extends to the Stein manifold situation from the Euclidean space pseudoconvex situation. It is indeed one of the grand and recurrent themes of modern several complex variables, dating at least back to K. Oka, E. Cartan and H. Grauert, and in many aspects even back to E. E. Levi, that what happens for pseudoconvex domains in Euclidean space ought also to happen for Stein manifolds and pseudoconvex domains in these manifolds. In this sense, it is gratifying but not surprising that so many of the results developed in earlier chapters for domains in complex Euclidean space can be extended, and indeed extended by essentially the same arguments, to Stein manifolds.