1

Preliminaries

1.1 Automorphism Groups

A subset $\Omega \subseteq \mathbb{C}^n$ will be called a *domain* if it is connected and open. The *automorphism group* $\mathrm{Aut}\,(\Omega)$ of Ω is by definition the set of all holomorphic mappings $f : \Omega \to \Omega$ with inverse map f^{-1} existing and also holomorphic. The group operation is the composition of mappings, and it is easy to check that this binary operation makes $\mathrm{Aut}\,(\Omega)$ into a group. When $n = 1$, it is well known and easy to prove that f^{-1} will be automatically holomorphic when it is defined. This follows from the argument principle because a locally injective holomorphic function has nowhere zero first derivative. This result is also true in several complex variables, but requires more effort to prove. One must show that a locally injective, equi-dimensional holomorphic mapping has nowhere vanishing holomorphic Jacobian determinant; from this it follows immediately that f^{-1} is holomorphic. This result is conceptually fundamental, but plays little explicit role in what follows and will not be discussed further. [See, e.g., [Narasimhan 1971] for a proof.]

The definition of automorphism group can obviously be extended to the case where Ω is replaced by a complex manifold M. The same observation applies to the redundancy of the hypothesis that f^{-1} be holomorphic since the proof of that result can be performed in local coordinates. Much of the theory of automorphism groups of domains in space can be transferred, without any extra work, directly to the complex manifold case; we shall often treat the two situations simultaneously. Other results are quite different for manifolds than for domains in $\mathbb{C}^n$, and we shall indicate some of these distinctions later.

Just as, in one complex variable, the study of Riemann surfaces can clarify basic function-theoretic questions, the study of manifolds in higher dimensions can clarify the situation for domains in space. However, little detailed knowledge of complex manifold theory will be needed for the reading of this book.

The subject of the geometry of open sets in $\mathbb{C}^n$ and of the geometry of open complex manifolds in general divides itself rather naturally into two

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parts. It is really two subjects. In one of these, the domains and manifolds are such that their automorphism groups are finite dimensional and indeed are Lie groups. In the other, the automorphism groups involve infinitely many parameters. The one-variable, Riemann surface situation (for example) is deceptively simple. The group $\mathrm{Aut}\,(M)$ when M is a Riemann surface is *always* a Lie group, as we shall prove in Chapter 2. By contrast, if one takes $\Omega = \mathbb{C}^2$, then the group $\mathrm{Aut}\,(\Omega)$ is *not* a Lie group but rather is infinite dimensional in a certain sense. For example, if $f : \mathbb{C} \to \mathbb{C}$ is any entire function, then $(z_1, z_2) \mapsto (z_1 + f(z_2), z_2)$ is an automorphism of $\mathbb{C}^2$.

The present book is primarily about the situations in which $\mathrm{Aut}\,(\Omega)$ is a (finite-dimensional) Lie group and satisfies an additional condition that the action is *proper* in the following sense: the action map $A : \mathrm{Aut}\,(\Omega) \times \Omega \to \Omega \times \Omega$ defined by $(\varphi, z) \mapsto (\varphi(z), z)$ is proper. That is, $A^{-1}(C)$ is compact for each compact subset C of $\Omega \times \Omega$. In particular, the isotropy group $I_p \times \{p\} := \{\varphi \in \mathrm{Aut}\,(\Omega) : \varphi(p) = p\}$ is compact for any $p \in \Omega$ since $I_p = A^{-1}(p, p)$. For a statement like this to make sense, we need to define a topology on $\mathrm{Aut}\,(\Omega)$. The appropriate topology, which will be used throughout, is the compact-open topology, equivalently the topology of uniform convergence on compact sets. [It should be noted that all the complex manifolds that we shall consider in the sequel will be paracompact; thus no topological pathologies will arise. In particular, the compact-open topology is metrizable in this case.]

If Ω is a bounded domain in $\mathbb{C}^n$, then $\mathrm{Aut}\,(\Omega)$ is necessarily a Lie group. This was proved specifically by H. Cartan ([Cartan 1935]). Our approach to this will be via normal families and the Bochner–Montgomery theorem (Theorem 1.3.11 below), which characterizes the subgroups of the diffeomorphism group which are Lie groups. Our approach will also yield the properness of the action of $\mathrm{Aut}\,(\Omega)$ on Ω (Theorem 1.3.12).

Any covering-space quotient of a manifold M with $\mathrm{Aut}\,(M)$ acting properly, and in particular any covering-space quotient of a bounded domain, also has its automorphism group acting properly. Also, any Riemann surface except the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and $\mathbb{C}$ itself has this proper-action property.[1]

In addition to bounded domains in $\mathbb{C}^n$ and their quotients, there are other classes of complex manifolds for which the automorphism group action is proper. Some aspects of this phenomenon will be considered in Chapter 7.

The role of proper action can be made explicit even at this early stage of our development. This condition is necessary for the existence of a (smooth) Riemannian metric for which all the elements of the automorphism group are isometries. Actually, the condition of proper action is also sufficient for the

[1] That the property holds for tori and for $\mathbb{C}$ with one point removed is, in a sense, accidental: for these Riemann surfaces are both covered by $\mathbb{C}$, which itself does *not* have the desired property that the action of the automorphism group is proper. But all other Riemann surfaces (except the sphere and the cylinder) are quotients of the unit disc $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and for these the general principle applies.

existence of such an "invariant metric" [Palais 1961].[2] This will be discussed in more detail in Section 1.3.

Thus, for the domains and manifolds that we shall consider, the automorphism group, which is at first sight a function-theoretic object, will turn out to be also a geometric one via the existence of an invariant metric. These matters will usually be treated here by constructing explicitly an invariant metric rather than by appealing to the general results of Lie group theory.

In Riemann surface theory, this idea of relating function theory to geometry goes back at least to Poincaré and even Riemann. In higher dimensions, some aspects of the idea also have a long history, but many developments have occurred in recent times as well. It is this interaction between function theory and geometry that makes the whole subject so varied and interesting. And while we begin with the function theory, geometry soon takes center stage and plays a major role thereafter.

1.2 Some Fundamentals from Complex Analysis of Several Variables

We shall use systematically the standard notational conventions for coordinates in $\mathbb{C}^n$, first

$$z = (z_1, \ldots, z_n) \qquad \text{and} \qquad w = (w_1, \ldots, w_n).$$

We shall also write

$$|z| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}}.$$

Thus a mapping from an open subset of $\mathbb{C}^n$ into $\mathbb{C}^m$ is given by an m-tuple of complex-valued functions of n complex variables:

$$w = (w_1, \ldots, w_n) = f(z) = (f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n)).$$

Such a map is, by definition, holomorphic if each of the functions f_j, $j = 1, \ldots, m$, is holomorphic in one and hence any of the various equivalent senses of the word "holomorphic."

Here and elsewhere we take for granted basic elements of the theory of functions of several complex variables, for which see [Grauert/Fritzsche 1976], [Hörmander 1990], or [Krantz 2001] for instance. In particular, we assume that

[2] It is a familiar fact that the group of isometries of a (smooth) Riemannian manifold acts properly. But the partial converse, that a properly-acting subgroup of the group of diffeomorphisms acts as isometries for some smooth metric, is not obvious.

the reader is aware that, for $\mathbb{C}$-valued functions $f(z_1, \ldots, z_n)$ defined on an open subset of $\mathbb{C}^n$, the following ideas are equivalent:

- The function f is holomorphic in each variable separately; [3]
- The function f is real-continuously differentiable (C^1) and satisfies the Cauchy–Riemann equations in each variable separately;
- The function f has at each point $p = (p_1, p_2, \ldots, p_n)$ of its domain a power series expansion

$$f(z) = \sum_{i_1, i_2, \ldots, i_n \geq 0} a_{i_1 i_2 \cdots i_n} (z_1 - p_1)^{i_1} (z_2 - p_2)^{i_2} \cdots (z_n - p_n)^{i_n}$$

which converges absolutely to f for all $(z_1, z_2, \ldots, z_n)$ in some open neighborhood of p.

As will be taken for granted here, many of the ideas of one complex variable have more or less automatic extensions to several variables. These include the Cauchy integral formula in several variables: recall that the polydisc $D^n(p, r)$ of polyradius $r = (r_1, \ldots, r_n)$ with $r_j > 0$ for every j is defined to be

$$D^n(p, r) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - p_j| < r_j \text{ for every } j\}.$$

If the closure $\mathrm{cl}(D^n(p, r))$ of this polydisc is contained in the (open) domain of definition of a holomorphic function f then, for each $(z_1, \ldots, z_n)$ in the open polydisc,

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1 - p_1| = r_1} \cdots \oint_{|\zeta_n - p_n| = r_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_n \cdots d\zeta_1,$$

where the integral is an iterated line integral. This reconstructs the power series expansion of f around $(p_1, \ldots, p_n)$, by expansion of the integrand and integration term-by-term. Differentiation of this formula under the integral sign together with obvious estimates also yields the following, which we shall apply repeatedly: if a sequence $\{f_j\}$ of $\mathbb{C}$-valued holomorphic functions on an open subset U of $\mathbb{C}^n$ converges uniformly on each compact subset of U, then every derivative (of any order) of the sequence also converges uniformly on each compact subset, and the derivative of the limit is equal to the limit of the derivative.

This last result, which is a direct analogue of a familiar fact about one-variable theory, will be especially important to us since, in effect, it says that the compact-open topology for holomorphic functions is the same as the C^∞ topology. Thus sets or groups of holomorphic mappings have a natural, unique topology. This means that the subtle questions associated to the phrase

[3] In the background here is the famous theorem of Hartogs that a function holomorphic in each variable separately is automatically continuous, indeed real analytic.

"Hilbert's fifth problem" play no role here; such matters are automatically straightforward.

Hurwitz's theorem in one variable on limits of zero-free functions has a direct generalization to several variables: first, if $f_j\colon \Omega \to \mathbb{C}$, $j = 1, 2, 3, \ldots$, are holomorphic functions from a domain (i.e., a connected open set) in $\mathbb{C}^n$ with $0 \notin f_j(\Omega)$, and if the sequence $\{f_j\}$ converges uniformly on compact subsets of Ω to a (necessarily holomorphic) limit $f_0\colon \Omega \to \mathbb{C}$, then either $f_0(\Omega) = \{0\}$, i.e., $f_0 \equiv 0$, or $0 \notin f_0(\Omega)$, i.e., f_0 is nowhere zero. The proof is obtained by observing that, if $f_0(z_0) = 0$ for some $z_0 \in \Omega$, then, by the one-variable Hurwitz theorem, the function $\zeta \mapsto f_0(z_0 + a\zeta)$, for $\zeta \in \mathbb{C}$ with $|\zeta|$ small and for $a \in \mathbb{C}^n$ with $\|a\| = 1$, is defined and identically zero. Then that $f_0 \equiv 0$ follows by analytic continuation.

Since one of the main subjects of this book is self-mappings of domains in $\mathbb{C}^n$ or, on occasion, complex manifolds, we have some special interest in holomorphic mappings where domain and range have equal dimension; first, n-tuples $(f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ of holomorphic functions of n variables. Attached to this situation is the holomorphic Jacobian determinant $\mathcal{J}$, first, the ordinary determinant of the $n \times n$ complex matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{pmatrix}.$$

A linear algebra calculation shows that the Jacobian determinant of the mapping considered as a real mapping from an open subset of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ is $|\mathcal{J}|^2$. This is a generalization of the familiar fact from one variable that the real differential of a holomorphic function is a rotation followed by dilation by a factor of $|f'|$, so that its action on the area element is multiplication by $|f'|^2$.

Returning to the $\mathbb{C}^n$ situation in general, we see that the holomorphic mapping from an open subset into $\mathbb{C}^n$ again is nonsingular as a real mapping at a given point if and only if its holomorphic Jacobian determinant $\mathcal{J}$ is nonzero at that point. Combining this observation with Hurwitz's theorem, we see that the limit (uniformly on compact sets) of everywhere nonsingular mappings of a connected open set in $\mathbb{C}^n$ to $\mathbb{C}^n$ is either everywhere nonsingular or everywhere singular. In the latter case, the limit mapping has image with empty interior (by Sard's theorem (Theorem 5.3.2)). This line of thought is associated to the idea that the limit of biholomorphic mappings is either biholomorphic or in some sense "degenerate." This point will be explored in detail in later sections.

It is of interest to characterize holomorphic mappings in terms of their real differentials. This is done in effect by way of the Cauchy–Riemann equations. Let $(f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n))$ be a holomorphic mapping into $\mathbb{C}^m$ defined on an open subset of $\mathbb{C}^n$. Then we write $f_j = u_j + \sqrt{-1}v_j$, where u_j, v_j are real-valued. The Cauchy–Riemann equations are as usual

$$\frac{\partial u_j}{\partial x_\ell} = \frac{\partial v_j}{\partial y_\ell} \qquad \text{and} \qquad \frac{\partial u_j}{\partial y_\ell} = -\frac{\partial v_j}{\partial x_\ell}, \quad j = 1, \ldots, m, \quad \ell = 1, \ldots, n.$$

We write here, by convention, $z_\ell = x_\ell + \sqrt{-1}y_\ell$. This can be thought of in a less coordinate-dependent fashion as follows. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by sending $(z_1, \ldots, z_n)$ to $(x_1, y_1, \ldots, x_n, y_n)$. Define an $\mathbb{R}$-linear map J_{2n} of $\mathbb{R}^{2n}$ to itself by sending $(x_1, y_1, \ldots, x_n, y_n)$ to $(-y_1, x_1, \ldots, -y_n, x_n)$. Then the Cauchy–Riemann equations for a map $F : U \to \mathbb{C}^m$, with U open in $\mathbb{C}^n$, are equivalent to

$$J_{2m} \circ dF = dF \circ J_{2n},$$

where dF is the real differential of F considered as a C^∞ function from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$.

This characterization of holomorphicity has an immediate consequence that is important for the theory of complex manifolds. first, if two complex local coordinate systems $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ are holomorphically related, then the J operator determined from the z-coordinates is the same operator as the J operator determined from the w-coordinates. The meaning of this assertion is familiar in Riemann surface theory: J is rotation by 90° counterclockwise in the orientation determined by the Riemann surface structure. The meaning of this is the same in any holomorphic coordinate system because the real differential of the coordinate change is orientation-preserving and conformal. In higher dimensions, there is again a coordinate-invariant operator J on the real tangent space at each point of a complex manifold. This operator corresponds to the J operator in any coordinate system, and the observation in the previous paragraph shows that it is independent of coordinate choice.

The J operator thus obtained provides a way to connect real Riemannian geometry with complex behavior, since J is a real $(1,1)$ tensor but it completely determines which (locally defined) functions are holomorphic. This approach to the geometry of complex manifolds is presented systematically in, e.g., [Greene 1987], [Wells 1979]; see also [Kobayashi/Nomizu 1963].

1.3 Normal Families and Automorphisms

Let $D \subset \mathbb{C}$ denote the open unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Also $D(p,r) \subset \mathbb{C}$ denotes the open disc with radius r centered at p. For $r > 0$ we let

$$D^n(0,r) \equiv \underbrace{D(0,r) \times \cdots \times D(0,r)}_{n \text{ times}}.$$

Further, if $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$ and $r > 0$, then

$$D^n(p,r) \equiv D(p_1,r) \times \cdots \times D(p_n,r).$$

If $f : D \to D \subset \mathbb{C}$ is a holomorphic function with $f(0) = 0$ and $|f'(0)| = 1$, then f has the form $f(z) = f'(0)z$. In particular, if $f \in \mathrm{Aut}\,(D)$ and if such an f has $f'(0) = 1$, then $f(z) = z$. This is part of the classical Schwarz lemma. The following result is a direct generalization to several variables, and to arbitrary bounded domains. There are many possible generalizations of the Schwarz lemma, some of which will be discussed later on in this book, but this one is the one that will play the most direct role in our investigations. For example, it will enable us to see that, if Ω is a bounded domain, then $\operatorname{Aut}(\Omega)$ has compact isotropy group at each point.

Theorem 1.3.1 (H. Cartan). *Suppose that Ω is a bounded domain in $\mathbb{C}^n$. Let $\phi : \Omega \to \Omega$ be holomorphic and suppose that, for some $p \in \Omega$, $\phi(p) = p$ and $d\phi(p) = \mathrm{id}$. [Here $d\phi$ is the n-dimensional complex differential.] Then ϕ is the identity mapping from Ω to itself.*

Boundedness of Ω is an essential hypothesis: consider the automorphism of $\mathbb{C}^2$ given by $(z_1, z_2) \mapsto (z_1 + z_2^2, z_2)$.

Proof of Theorem 1.3.1. We may assume that $p = \mathbf{0}$ (the origin). For proof by contradiction, assume that ϕ does not coincide with the identity mapping. Expanding ϕ in a power series about $p = \mathbf{0}$ (and remembering that ϕ is vector-valued, hence so is the expansion) yields

$$\phi(z) = z + P_k(z) + O(|z|^{k+1}),$$

where P_k is the first nonvanishing homogeneous polynomial (of degree k) of order exceeding 1 in the Taylor expansion. Defining $\phi^j(z) = \phi \circ \cdots \circ \phi$ (j times); direct computation then gives that

$$\begin{aligned} \phi^2(z) &= z + 2P_k(z) + O(|z|^{k+1}) \\ \phi^3(z) &= z + 3P_k(z) + O(|z|^{k+1}) \\ &\vdots \\ \phi^j(z) &= z + jP_k(z) + O(|z|^{k+1}). \end{aligned}$$

Choose polydiscs $D^n(0,a) \subseteq \Omega \subseteq D^n(0,b)$. The Cauchy estimates imply then that, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n = k$,

$$j \cdot \left| \left(\frac{\partial}{\partial z}\right)^{\alpha} \phi\Big|_{\mathbf{0}} \right| = \left| \left(\frac{\partial}{\partial z}\right)^{\alpha} \phi^j\Big|_{\mathbf{0}} \right| \le n \cdot \frac{b \cdot \alpha!}{a^k},$$

where

$$\left(\frac{\partial}{\partial z}\right)^{\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}.$$

Note that the rightmost item in this estimate is independent of j. Hence, for each such multi-index α with $|\alpha| = k$, $(\partial/\partial z)^{\alpha}\phi\big|_{\mathbf{0}} = \mathbf{0}$. Thus $P_k = 0$, a contradiction. □

This argument in particular applies when the dimension $n = 1$ and the domain Ω is the unit disc. There it gives a conceptually direct proof of the corresponding part of the classical Schwarz lemma.

Cartan's result has some further immediate but surprising consequences.

Corollary 1.3.2. *Suppose that Ω is a bounded, circular domain in $\mathbb{C}^n$, that is $(e^{i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_n) \in \Omega$ whenever $(z_1, z_2, \ldots, z_n) \in \Omega$ for every $\theta \in \mathbb{R}$. If $\mathbf{0} \in \Omega$ and $f \in \mathrm{Aut}\,(\Omega)$ with $f(\mathbf{0}) = \mathbf{0}$, then f is a linear mapping.*

Proof. For $\theta \in \mathbb{R}$ and $z \in \Omega$, let $F(z) = e^{-i\theta} f(e^{i\theta} z)$. Then $F \in \mathrm{Aut}\,(\Omega)$, since Ω is circular. By the chain rule it follows that

$$d(f^{-1} \circ F)\big|_0 = \mathrm{id}.$$

Hence

$$f^{-1} \circ F = \mathrm{id}$$

on Ω, or equivalently $f = F$. If we write $f = (f_1, \ldots, f_n)$, $F = (F_1, \ldots, F_n)$, and

$$f_j(z) = \sum_{|N|=1}^{+\infty} a_N z^N$$

is the Taylor expansion of f_j, then the Taylor expansion of F_j is, by definition of F and by substitution,

$$F_j = \sum_{|N|=1}^{+\infty} e^{-i\theta} a_N e^{i|N|\theta} z^N.$$

But $F_j = f_j$. Therefore $e^{i(|N|-1)\theta} a_N = a_N$ for all multi-indices N and all $\theta \in \mathbb{R}$. This implies that $a_N = 0$ for $|N| \geq 2$.[4] Thus each f_j is linear. $\square$

It is easy to modify this argument to show that, if Ω_1, Ω_2 are two bounded, circular domains containing the origin $\mathbf{0}$ and if $F : \Omega_1 \to \Omega_2$ is biholomorphic with $F(\mathbf{0}) = \mathbf{0}$, then F is linear. This immediately implies that, when $n \geq 2$, the unit ball $\{(z_1, \ldots, z_n) \colon |z_1|^2 + \cdots + |z_n|^2 < 1\}$ and the unit polydisc $\{(z_1, \ldots, z_n) \colon |z_j| < 1, j = 1, \ldots, n\}$ are not biholomorphic: If there were a biholomorphic map between them, then applying suitable biholomorphic maps to each variable in the unit polydisc separately would produce a biholomorphic map that took $\mathbf{0}$ to $\mathbf{0}$. This would then have to be linear, which is not possible, since, e.g., the ball has smooth boundary and the polydisc does not (when $n \geq 2$). Thus the direct analogue of the Riemann mapping theorem fails in $\mathbb{C}^n$, $n \geq 2$: (bounded) domains can be homeomorphic to the ball without being biholomorphic to it. This failure, even for small perturbations of the ball, will be explained in much more detail in later chapters.

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[4] Here $N = (n_1, \ldots, n_n)$ and $|N| = n_1 + \cdots + n_n$.

The second corollary will play an important role in what follows.

Corollary 1.3.3. *If Ω is a bounded domain in $\mathbb{C}^n$ and $p \in \Omega$, then the mapping*

$$f \longmapsto df\big|_p$$

is an injective homomorphism of the group

$$I_p \equiv \{f \in \mathrm{Aut}\,(\Omega) : f(p) = p\}$$

into $GL(n, \mathbb{C})$.

Proof. If $df\big|_p = dg\big|_p$ for $f, g \in I_p$, then the chain rule gives that $d(f^{-1} \circ g)\big|_p =$ id, where the identity map id is given by the $n \times n$ identity matrix $I_n \in GL(n, \mathbb{C})$. By Theorem 1.3.1, $f^{-1} \circ g : \Omega \to \Omega$ is the identity mapping. Hence $f \equiv g$. We conclude that $f \mapsto df\big|_p$ is injective on I_p. The homomorphism property is a special case of the chain rule. □

If a group G acts on a space X through an action $G \times X \to X$, and if $x \in X$, then the *orbit* $\mathcal{O}_x$ of the point x is the set $\{gx : g \in G\}$. In a natural sense the orbit is the image of the group G. Indeed, $\mathcal{O}_x$ is naturally identified with the quotient G/I_x, where $I_x = \{g \in G\colon gx = x\}$. We shall be particularly interested in boundary points that are accumulation points of some orbit for the action of the automorphism group $\mathrm{Aut}\,(\Omega)$ on Ω. If the orbit $\mathcal{O}_x \subseteq \Omega$, considered as a point set, has a boundary point $p \in \partial\Omega$ as an accumulation point then we call p a *boundary orbit accumulation point.* These will be discussed in detail in Section 1.5.

Corollary 1.3.3 immediately yields the following observation. Fix $p_0 \in \Omega$. Then each $f \in \mathrm{Aut}\,(\Omega)$ is uniquely determined by $f(p_0)$ and $df|_{p_0}$. Now the possibilities for $f(p_0)$ range at most over Ω and for $df|_{p_0}$ over $\mathbb{C}^{n^2}$ (identifying $df|_{p_0}$ with its complex $n \times n$ matrix). So in a general sense $\mathrm{Aut}\,(\Omega)$ is parameterized by a subset of $\mathbb{C}^n \times \mathbb{C}^{n^2}$. Thus one might expect $\mathrm{Aut}\,(\Omega)$ to be a finite-dimensional group, and hence a Lie group. This expectation turns out to be justified. But of course this depends on adding the topology into the picture of $\mathrm{Aut}\,(\Omega)$: as it stands, this "parameterization" is only set-theoretic. We have already discussed the appropriate topology for $\mathrm{Aut}\,(\Omega)$, first the compact-open topology. Clearly the association $f \mapsto (f(p_0), df|_{p_0}) \in \mathbb{C}^n \times \mathbb{C}^{n^2}$ is continuous (for the second factor, by Cauchy estimates). To pursue this matter further, we shall need some results from normal families, to which we shall turn next.

Among results also associated to normal families and the closure properties of the group $\mathrm{Aut}\,(\Omega)$, when Ω is a bounded domain in $\mathbb{C}^n$, the following principle will in particular play an important role in our later considerations. While in a sense this is just an application of standard normal families ideas, the details are surprisingly subtle in this general, multi-variable situation.

Theorem 1.3.4 (Normal Families of Automorphisms). *Let Ω be a bounded domain in $\mathbb{C}^n$. If $\{f_j\}$ is a sequence in* $\mathrm{Aut}\,(\Omega)$ *which converges* *uniformly on compact subsets of* Ω *and if, for some* $p_0 \in \Omega$*, the limit* $\lim_{j\to\infty} f_j(p_0)$ *is a point of* Ω*, then the limit holomorphic mapping* $f_0 \equiv \lim f_j : \Omega \to cl(\Omega)$ *has image equal precisely to* Ω *and* $f_0 \in \mathrm{Aut}\,(\Omega)$*.*

Without the hypothesis about the point p_0, the conclusion can fail. For example, if $\Omega = D = \{z \in \mathbb{C} : |z| < 1\}$ and

$$f_j(z) = \frac{z - (1 - 1/j)}{1 - (1 - 1/j)z},$$

then $f_j \in \mathrm{Aut}\,(\Omega)$, but

$$\lim f_j = \text{the constant function } -1.$$

In one complex variable, such "degenerate limits," where $\lim f_j(p_0) \in \mathrm{cl}(\Omega)\setminus\Omega$ for some p_0 and hence (by the theorem) all $p_0 \in \Omega$, are necessarily constant functions. This is an easy consequence of Hurwitz's theorem on the limits of sequences of zero-free holomorphic functions. For, suppose to the contrary that $\lim f_j(p_0) = q \in \mathrm{cl}(\Omega) \setminus \Omega$. Then the limit of the zero-free functions $f_j(z) - q$ for $z \in \Omega$ has a zero at p_0 and is hence $\equiv 0$ on Ω.

This argument indeed shows that, under the hypotheses of the theorem, $\lim f_j$ is "interior," i.e., $(\lim f_j)(\Omega) \subset \Omega$, in the one-variable case. But the argument needed in general (i.e., higher dimensions) is much more intricate—even though Hurwitz's theorem on limits of sequences of zero-free holomorphic functions continues to play a role.

Proof of Theorem 1.3.4. Let $\mathcal{J}_{f_j}$ be the holomorphic Jacobian determinant of f_j as discussed earlier. Then $\mathcal{J}_{f_j}$ is zero-free on Ω. Write f_0 for the limit of the f_j. By Hurwitz's theorem, $\mathcal{J}_{f_0}$ is either identically 0 or is zero-free. To rule out the first possibility, we show that $\mathcal{J}_{f_0}(p_0) \neq 0$. For this, note that

$$\mathcal{J}_{f_0}(p_0) = \lim_{j\to\infty} \mathcal{J}_{f_j}(p_0) = \lim_{j\to\infty} \frac{1}{\mathcal{J}_{g_j}(f_j(p_0))},$$

where $g_j \equiv f_j^{-1}$.

Since $\lim f_j(p_0)$ exists by hypothesis and belongs to Ω, it follows that the set $\{f_j(p_0)\}$ belongs to a compact subset of Ω. Indeed it belongs to $\{\lim_j f_j(p_0)\} \cup \{f_j(p_0)\}$, which is surely compact. By Cauchy estimates, $\mathcal{J}_{g_j}$ is bounded on this compact set. Thus $\lim_j 1/\mathcal{J}_{g_j}(f_j(p_0)) \neq 0$, and that is what we wanted.

It would be pleasant if the fact that we just established, first that $\mathcal{J}_{f_0}$ is zero-free on Ω, implied immediately that $f_0(\Omega) \subset \Omega$. In the special case that Ω has a "nice boundary" (e.g., a regularly embedded C^2 hypersurface in $\mathbb{C}^n$), the result would actually follow. For in that case $\mathcal{J}_{f_0}$ being nowhere zero implies that $f_0(\Omega)$ is open in $\mathbb{C}^n$ and for a domain Ω with smooth boundary, every subset of the closure $\mathrm{cl}(\Omega)$ of Ω that is open in $\mathbb{C}^n$ is contained in Ω. But of course in a more general setting, wherein the boundary of Ω is not smooth,

$\mathrm{cl}(\Omega)$ can in fact contain points of $\mathrm{cl}(\Omega) \setminus \Omega$ in its interior (e.g., consider the case of Ω a punctured open ball). Thus a more refined argument is needed.

Fix a point $p \in \Omega$. Then $\mathcal{J}_{f_0}(p) \neq 0$ and of course the entire holomorphic Jacobian matrix of first derivatives of f_j at p converges to the matrix for f_0, which is nonsingular. Moreover, the second derivatives of the f_j on any fixed, closed ball $\mathrm{cl}(B^n(p, \epsilon)) \subset \Omega$, $\epsilon > 0$, are bounded uniformly in j by Cauchy estimates. Now it follows from the inverse function theorem (see, e.g., [Krantz/Parks 2002]) that there is a $\delta > 0$ such that $f_j(\Omega)$ contains an open ball of radius δ around $f_j(p)$. Here δ can be taken to be independent of j. In particular, since $f_j(\Omega) = \Omega$, the distance of $f_j(p)$ to $\mathbb{C}^n \setminus \Omega$ is at least δ for all j. It follows that $\lim_j f_j(p) = f_0(p)$ is in Ω, *not* in $\mathrm{cl}(\Omega) \setminus \Omega$. Thus, $f_0(\Omega) \subset \Omega$.

Now that we know that f_0 is "interior," i.e., *it maps the interior points to the interior points and hence no interior points are mapped to a boundary point*, we want to show that $f_0 \in \mathrm{Aut}\,(\Omega)$, i.e., that $f_0 : \Omega \to \Omega$ is one-to-one and onto. Passing to a subsequence if necessary, we can suppose that $\{g_j\} = \{f_j^{-1}\}$ converges uniformly on compact subsets to a limit $g_0 : \Omega \to \mathrm{cl}(\Omega)$. Our next goal is to show that g_0 is interior. By the argument used to show that f_0 was interior, it suffices to show that $g_0(f_0(p_0))$ belongs to Ω, not to $\mathrm{cl}(\Omega) \setminus \Omega$.

For this, choose $\lambda > 0$ such that the closed ball $\mathrm{cl}\big(B^n(f_0(p_0), 2\lambda)\big) \subset \Omega$. Notice that $f_j(p_0) \in \mathrm{cl}\big(B^n(f_0(p_0), \lambda)\big)$ whenever j is sufficiently large. Hence, by Cauchy estimates, there is a constant $M > 0$, independent of j, such that

$$\|g_j(f_j(p_0)) - g_j(f_0(p_0))\| \leq M\|f_j(p_0) - f_0(p_0)\|$$

for all j sufficiently large. But $g_j(f_j(p_0)) = p_0$. Hence

$$\|p_0 - g_j(f_0(p_0))\| \leq M\|f_j(p_0) - f_0(p_0)\|.$$

Since the righthand side goes to 0 as $j \to +\infty$, so does the lefthand side and hence

$$g_0(f_0(p_0)) = \lim_{j\to\infty} g_j(f_0(p_0)) = p_0.$$

We conclude that $g_0(f_0(p_0)) \in \Omega$ and therefore g_0 is interior.

We now must show that $f_0 \circ g_0 : \Omega \to \Omega$ and $g_0 \circ f_0 : \Omega \to \Omega$ are both identity maps of Ω to Ω. This of course will establish that $f_0 \in \mathrm{Aut}\,(\Omega)$. This final result is a consequence of the next lemma.

Lemma 1.3.5. *If $\{f_j : \Omega \to \Omega\}$ and $\{g_j : \Omega \to \Omega\}$ are sequences of holomorphic mappings which converge uniformly on compact subsets of Ω to interior limits $f_0 : \Omega \to \Omega$ and $g_0 : \Omega \to \Omega$, then the sequence $\{g_j \circ f_j : \Omega \to \Omega\}$ converges uniformly on compact subsets of Ω to $g_0 \circ f_0 : \Omega \to \Omega$.*

Assuming this lemma for the moment, we may apply it to f_j and g_j as before. Since $g_j \circ f_j$ is the identity map of Ω to Ω, for all j, it follows that $g_0 \circ f_0$ is also the identity map. Applying the lemma again with the roles of f and g interchanged gives that $f_0 \circ g_0$ is the identity. This completes the proof of the theorem. Thus, it remains to prove the lemma.

Proof of Lemma 1.3.5. Suppose that $K \subset \Omega$ is a compact subset. Then choose $\epsilon > 0$ such that

$$L_\epsilon \equiv \{z \in \Omega : \|z - w\| \leq \epsilon \text{ for some } w \in f_0(K)\}$$

is a compact subset of Ω. This choice is possible since $f_0(K)$ is a compact subset of Ω. For all j sufficiently large, $f_j(K) \subset L_\epsilon$. Furthermore, the members of $\{g_j\}$ are uniformly Lipschitz continuous on L_ϵ by Cauchy estimates. Thus, for $z \in K$ and j large, there is a j-independent constant M such that

$$\begin{aligned}\|g_j(f_j(z)) - g_0(f_0(z))\| &\leq \|g_j(f_j(z)) - g_j(f_0(z))\| + \|g_j(f_0(z)) - g_0(f_0(z))\| \\ &\leq M\|f_j(z) - f_0(z)\| + \|g_j(f_0(z)) - g_0(f_0(z))\|.\end{aligned}$$

Now $\|f_j(z) - f_0(z)\| \to 0$ uniformly for $z \in K$. Also, since $\{f_0(z) : z \in K\}$ is compact, $\|g_j(f_0(z)) - g_0(f_0(z))\| \to 0$ uniformly for $z \in K$. Thus $\lim_j g_j(f_j(z)) = g_0(f_0(z))$ uniformly for $z \in K$ as required. □

The proof of Theorem 1.3.4 is now complete. □

Corollary 1.3.6. *For each $p \in \Omega$, the orbit $\mathcal{O}_p := \{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ is closed in Ω.*

Proof. We need to show that, if $\{f_j(p)\}$ converges to $q \in \Omega$, then $q \in \mathcal{O}_p$, i.e., that $q = f(p)$ for some $f \in \mathrm{Aut}\,(\Omega)$. Choose a subsequence of $\{f_j\}$ which converges uniformly on compact subsets of Ω to $f : \Omega \to \mathrm{cl}(\Omega)$.[5] By Theorem 1.3.4, $f \in \mathrm{Aut}\,(\Omega)$ and clearly $f(p) = \lim_j f_j(p) = q$. □

Corollary 1.3.7. *The injective homomorphism $f \mapsto df|_p$ of I_p (the isotropy group $\{f \in \mathrm{Aut}\,(\Omega)\colon f(p) = p\}$) onto dI_p is a homeomorphism of I_p (in the compact-open topology) onto a compact subgroup of $GL(n, \mathbb{C})$.*

Proof. That $f \mapsto df|_p$ is an injective homomorphism of I_p onto dI_p has already been established (Corollary 1.3.3). The continuity is an immediate consequence of the Cauchy estimates for first derivatives. For the compactness, note that a sequence $\{df_j|_p : f_j \in I_p\}$ has a subsequence $\{df_{j_k}|_p : f_{j_k} \in I_p\}$ for which $\{f_{j_k}\}$ converges uniformly on compact subsets of Ω and, by Theorem 1.3.4, to an element $f_0 \in \mathrm{Aut}\,(\Omega)$ that fixes p. Again by the Cauchy estimates, $df_{j_k}|_p$ converges in $GL(n, \mathbb{C})$ to $df_0|_p \in dI_p$. □

The compactness part of Corollary 1.3.7 is a special case of a more general result which has essentially the same proof.

Corollary 1.3.8. *If K is a compact subset of Ω and $p \in \Omega$, then* $\{f \in \mathrm{Aut}\,(\Omega) : f(p) \in K\}$ *is a compact subset of* $\mathrm{Aut}\,(\Omega)$.

[5] We shall use the notation $\mathrm{cl}(\Omega)$ for the closure of Ω, instead of the more familiar $\overline{\Omega}$, to avoid confusion with the complex conjugate.

Proof. Let $\{f_j\}$ be a sequence in $\mathrm{Aut}\,(\Omega)$ with $f_j(p) \in K$ for all j. Since K is compact, we see by passing to a subsequence (still called f_j) that $\lim_j f_j(p)$ exists and lies in K. By normal families considerations, a further passage to a subsequence yields a sequence that converges uniformly on compact sets. By Theorem 1.3.4, this sequential limit is itself an automorphism. Obviously this limit takes p to some point in K. □

Corollary 1.3.9. *If, for some* $p \in \Omega$, $\{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ *is compact, then* $\mathrm{Aut}\,(\Omega)$ *is compact.*

Proof. In the corollary before this one, we simply take $K = \{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$. □

For all $p \in \Omega$, $\{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ is compact if $\mathrm{Aut}\,(\Omega)$ is compact, just because for a given p the mapping

$$\begin{aligned} F : \mathrm{Aut}\,(\Omega) &\to \Omega \\ f &\mapsto f(p) \end{aligned}$$

is continuous. Thus we have proved the following result.

Proposition 1.3.10. *If one orbit of* $\mathrm{Aut}\,(\Omega)$ *is compact, then* $\mathrm{Aut}\,(\Omega)$ *is compact and all of its orbits are compact.*

We know from Corollary 1.3.6 that any orbit of $\mathrm{Aut}\,(\Omega)$ is closed in Ω. Thus the only way that an orbit of $\mathrm{Aut}\,(\Omega)$ can be noncompact is to "run out to the boundary" of Ω, i.e., the closure must contain an element of $\mathrm{cl}(\Omega) \setminus \Omega$. One of the main points of the present book is to study what happens when $\mathrm{Aut}\,(\Omega)$ is noncompact. And one of the main approaches will be to study $\mathrm{cl}(\Omega) \setminus \Omega$ in a neighborhood of such a "boundary orbit accumulation point," that is, an element of $\mathrm{cl}(\Omega) \setminus \Omega$ that lies in the closure of some orbit of the automorphism group action.

We now see that the automorphism group of a bounded domain is a (finite-dimensional) Lie group. For this we shall use the following general theorem.

Theorem 1.3.11 ([Bochner/Montgomery 1946]). *Let G be a subgroup of the diffeomorphism group of a smooth manifold. If it is locally compact, then G is a Lie group.*

When the action of the automorphism group is proper, the group is necessarily locally compact. first, as before, we define the action map $A\colon \mathrm{Aut}\,(\Omega) \times \Omega \to \Omega \times \Omega$ by $A(\varphi, z) = (\varphi(z), z)$. Then A^{-1} of a compact-closure neighborhood of (z, z) for any $z \in \Omega$ has compact closure in $\mathrm{Aut}\,(\Omega) \times \Omega$, when A is a proper map. This gives a compact-closure neighborhood of the identity in $\mathrm{Aut}\,(\Omega)$, by projection to the first factor of $\mathrm{Aut}\,(\Omega) \times \Omega$. Thus to show that $\mathrm{Aut}\,(\Omega)$ is a Lie group when Ω is a bounded domain in $\mathbb{C}^n$, it suffices, in the presence of the Bochner–Montgomery theorem (Theorem 1.3.11), to show:

Theorem 1.3.12. *If Ω is a bounded domain in $\mathbb{C}^n$, then the action of* $\mathrm{Aut}(\Omega)$ *on Ω is proper, i.e., the map $(\varphi, z) \mapsto (\varphi(z), z) : \mathrm{Aut}(\Omega) \times \Omega \to \Omega \times \Omega$ is proper.*

Proof. Properness means explicitly that, if $C \subset \Omega \times \Omega$ is a compact set, then $\{(\varphi, z) : (\varphi(z), z) \in C\}$ is a compact set in $\mathrm{Aut}(\Omega) \times \Omega$. To check this property for $\mathrm{Aut}(\Omega)$, suppose that $\{(\varphi_j, z_j) : j = 1, 2, \ldots\}$ is a sequence with $(\varphi_j(z_j), z_j) \in C$ for all j. Passing to a subsequence if necessary, one can assume that $\{z_j\}$ converges to a point $z_0 \in \Omega$ and that the sequence $\{\varphi_j(z_j)\}$ converges to $w_0 \in \Omega$.

Since Ω is bounded, Cauchy estimates imply that $\varphi_j(z_0)$ converges to w_0: in more detail, this follows by noting from the Cauchy estimates that, for some $\epsilon > 0$, $B(z_0, 2\epsilon) \subset \Omega$, so that there is a constant $M > 0$ independent of j such that the norm of the (real) differential of φ_j is less than M at each point of $B(z_0, \epsilon)$. Thus the distance from $\varphi_j(z_j)$ to $\varphi_j(z_0)$ is bounded by $M\|z_j - z_0\|$, and hence goes to 0.

Since $\varphi_j(z_0)$ converges now to $w_0 \in \Omega$, it follows from Corollary 1.3.8 that $\{\varphi_j\}$ has a subsequence that converges to some $\varphi_0 \in \mathrm{Aut}(\Omega)$. The compactness of $\{(\varphi, z) : (\varphi(z), z) \in C\}$ has thus been established. □

Corollary 1.3.13. *If Ω is a bounded domain in $\mathbb{C}^n$, then* $\mathrm{Aut}(\Omega)$ *is a Lie group.*

Proof. Combine Theorem 1.3.12 with the Bochner–Montgomery theorem (Theorem 1.3.11). □

As already noted at the end of Section 1.1, this result implies, from the result of Palais [Palais 1961], the existence of a smooth Riemannian metric on Ω invariant under $\mathrm{Aut}(\Omega)$. Averaging this with respect to the almost complex structure produces a Hermitian metric on Ω invariant under $\mathrm{Aut}(\Omega)$. In Chapter 3, an explicit construction of such a metric will be presented, but it is worth noting that the existence of such an invariant metric is guaranteed by the general principles we have discussed.

The general situation just described gives at least a philosophical idea of why $\mathrm{Aut}(\Omega)$ is a Lie group when Ω is a bounded domain. The precise version of this idea is Theorem 1.3.11 by Bochner and Montgomery. The main point is to describe the elements of $G := \mathrm{Aut}(\Omega)$ locally, in a neighborhood of the identity element, by a finite number of parameters so as to make the group itself a manifold (of finite dimension). A way to think of this is to look for a point of minimal isotropy dimension. This idea makes sense because all the isotropy groups are closed subgroups of $GL(n, \mathbb{C})$ (actually $U(n)$), so the idea of dimension is just submanifold dimension. If p is such a point, and its orbit $\mathcal{O}_p := \{\gamma(p) : \gamma \in G\}$, then elements γ near the identity can be determined by specifying $\gamma(p)$, which is near p, and $d\gamma\big|_p$, which is near the "identity map," where the "identity map" is just the map from the tangent space at p to the tangent space at $\gamma(p)$ arising from the coordinates in $\mathbb{C}^n$. The set of such $d\gamma$ in Euclidean coordinates is a submanifold of $GL(n,\mathbb{C})$, although it is not in general a subgroup (if $\gamma(p) \neq p$). Using submanifold coordinates from that observation and submanifold-of-$\mathbb{C}^n$ coordinates of $\mathcal{O}_p$ near p gives a local parameterization of $G = \text{Aut}\,(\Omega)$ near the identity.

This picture will be clearer if one thinks of the case of Ω the unit disc and $p = 0$. Let γ be an element of $\text{Aut}\,(\Omega)$. Near the identity, we can parameterize $\text{Aut}\,(\Omega)$ by the image $\gamma(0)$ together with $d\gamma\big|_0$. The set of such $d\gamma\big|_0$ (when $\gamma(0)$ is near 0) is a submanifold of $GL(1,\mathbb{C}) = \mathbb{C} \setminus \{0\}$. It generally is not a subgroup:

$$\{d\gamma\big|_0 : \gamma(0) = a\} = \{\omega T_{-a}\big|_0 : |\omega| = 1\},$$

where $T_{-a} \in \text{Aut}\,(\Omega)$ is defined by $T_{-a}(z) = (z+a)/(1+\bar{a}z)$. But we still get a legitimate smooth parameterization of $\text{Aut}\,(\Omega)$ near the identity.

The reader is invited to consider the corresponding local parameterization of $\text{Aut}\,(\Omega)$ when Ω is the unit ball in $\mathbb{C}^2$—after this group is discussed in some detail in the next section.

Note that one obtains here a view of the general fact that, for $G = \text{Aut}\,(\Omega)$,

$$\dim \mathcal{O}_p + \dim\,(I_p) = \dim\,G,$$

when

$$\mathcal{O}_p = \text{orbit of } p = \{\gamma(p) : \gamma \in G\}.$$

[This holds in general: the restriction to minimal isotropy, maximal orbit dimensions we made was just for convenience of visualization purposes.]

A closed subgroup of $GL(n,\mathbb{C})$ which acts on $\mathbb{C}^n$ isometrically is necessarily a closed subgroup of $U(n)$ and is hence compact. Conversely, if a subgroup of $GL(n,\mathbb{C})$ is compact, then there is a Hermitian metric on $GL(n,\mathbb{C})$ for which the subgroup acts isometrically and hence belongs to the $U(n)$ associated to the Hermitian metric. This follows from a standard argument using averaging of the standard metric with respect to the group action of the given subgroup of $GL(n,\mathbb{C})$.

The fact that every compact subgroup of $GL(n,\mathbb{C})$ acts isometrically relative to some Hermitian metric combined with Corollary 1.3.7 implies that, at each point $p \in \Omega$, there is a Hermitian metric for which I_p acts isometrically on the tangent space at p. This strongly suggests that one ought to seek a Hermitian metric on Ω which is $\text{Aut}\,(\Omega)$-invariant. In other words, one ought to look for a C^∞ family h_p, $p \in \Omega$, of Hermitian metrics such that, for all $\gamma \in \text{Aut}\,(\Omega)$ and $p \in \Omega$, the map $d\gamma\big|_p$ from the tangent space at p with metric h_p is an isometry onto the tangent space at $\gamma(p)$ with metric $h_{\gamma(p)}$. Indeed, it even suggests a way to do this: for some selection of distinguished points p, one in each orbit, choose h_p more or less arbitrarily except that in some sense it varies nicely with the choices of orbit. Then, for q in the orbit of such a point p, determine h_q by the requirement that $d\gamma\big|_p$ must be isometric for a γ_q with $\gamma_q(p) = q$. This is well defined by Corollary 1.3.7, independently of which γ_q is chosen. Thus the only question is whether this can be done so that the resulting metric on all of Ω is C^∞. This involves finding smooth "slices" for orbits. This is the point addressed in [Palais 1961]. But since we shall construct such Aut(Ω)-invariant metrics directly later on, we leave Palais's general construction as a philosophical observation.

1.4 The Basic Examples

In this section we shall collect a number of examples for which the automorphism groups are obtained explicitly. Some of these are well known and elementary, and the derivations of their automorphism groups need be outlined only briefly. But it will be convenient to have them all in one place; and looking at them all at once will suggest various paths of exploration that we follow later.

(1) $\mathrm{Aut}\,(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$.

If $f : \mathbb{C} \to \mathbb{C}$ is injective, then the only possible singularity of f at ∞ is a simple pole. If instead ∞ were a removable singularity, then f would be constant by Liouville's theorem. If ∞ were an essential singularity, then f would not be injective in any neighborhood of ∞. Similarly, a pole at ∞ of higher order than 1 would preclude injectivity in a neighborhood of ∞. Thus the nonconstant injective function f is a polynomial of degree one. That any polynomial of degree one is an automorphism is clear. □

(2) $\mathrm{Aut}\,(D) = \{z \mapsto \omega \cdot (z - a)/(1 - \bar{a}z) : a, \omega \in \mathbb{C}, |\omega| = 1, |a| < 1\}$.

That

$$T_a : z \longmapsto \frac{z - a}{1 - \bar{a}z}$$

is defined and injective from D to D is easy algebra. Also $T_a(T_{-a}(z)) = z$; hence T_a is surjective.

Conversely, suppose that $f \in \mathrm{Aut}\,(D)$. Let $a = f^{-1}(0)$. Then $g := f/T_a$ is holomorphic and zero-free on D and

$$\lim_{|\zeta|\to 1} |g(\zeta)| = \lim_{|\zeta|\to 1} \left|\frac{f(\zeta)}{T_a(\zeta)}\right| = 1.$$

By the maximum principle applied to both g and $1/g$, we see that $|T_a/f| \equiv 1$ on D, hence $f = \omega T_a$ for some constant ω with $|\omega| = 1$.[6] □

[6] An alternative argument is to note that $T_a \circ f$ maps the disc to the disc and fixes 0. Then Schwarz's lemma implies that $|(T_a \circ f)(z)| \leq |z|$. Applying the same reasoning to the inverse of this mapping gives $|(T_a \circ f)(z)| \geq |z|$. Hence $|T_a \circ f(z)| \equiv |z|$ on D, and $T_a \circ f$ equals $w \cdot \mathrm{id}$ on D for some ω with $|\omega| = 1$.

(3) $\mathrm{Aut}\,(\mathbb{C} \setminus \{0\}) = \{z \mapsto az^\epsilon : \epsilon = \pm 1, a \in \mathbb{C}, a \neq 0\}$.

If $f \in \mathrm{Aut}\,(\mathbb{C}\setminus\{0\})$, then a connectivity argument shows that $\lim_{z\to 0} f(z) = 0$ or $\lim_{z\to 0} |f(z)| = +\infty$. Composing with an inversion, we may assume that the first alternative holds. But then f, considered as a holomorphic function, has a removable singularity at the origin. Thus the extension $f(0) = 0$ makes f an entire function that is an automorphism of the entire plane. From part (1), $f(z) = az$, for some $a \neq 0$. In case $\lim_{z\to 0} f(z) = \infty$, the same reasoning applied to $1/f$ gives $1/f(z) = az$. $\square$

(4) $\mathrm{Aut}\,(\{z \in \mathbb{C} : 0 < r_1 < |z| < r_2\}) = \{z \mapsto \omega z : \omega \in \mathbb{C}, |\omega| = 1\} \cup \{z \mapsto \omega r_1 r_2/z : \omega \in \mathbb{C}, |\omega| = 1\}$.

Denote the annulus by A. By a connectivity argument, for each $f \in \mathrm{Aut}\,(A)$, either

(a) $\lim_{|z|\to r_2} |f(z)| = r_2$ and $\lim_{|z|\to r_1} |f(z)| = r_1$;

or

(b) $\lim_{|z|\to r_2} |f(z)| = r_1$ and $\lim_{|z|\to r_1} |f(z)| = r_2$.

In either case, repeated application of Schwarz reflection to the boundary circles extends f to an automorphism $\widehat{f} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ of $\mathbb{C} \setminus \{0\}$. Thus, by Example (3), $f(z) = az$ or $f(z) = a/z$ for some nonzero $a \in \mathbb{C}$. The condition $f(A) = A$ tells us then that $a = \omega$ in the first instance and that $a = \omega r_1 r_2$ in the second instance. $\square$

(5) $\mathrm{Aut}\,(\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\})$.

The set

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

is of course the unit ball in $\mathbb{C}^2$. First notice that $I_{(0,0)} = U(2) \subset GL(2, \mathbb{C})$. Obviously $U(2) \subset I_{(0,0)}$. If $f \in I_{(0,0)}$, then f is $\mathbb{C}$-linear according to Corollary 1.3.2. Since f has to preserve the unit sphere (the boundary of B^2), it is immediate that $f \in U(2)$.

Now a direct calculation, analogous to that for the disc, shows that the mapping

$$T_{(a,0)}(z_1, z_2) \equiv \left(\frac{z_1 - a}{1 - \bar{a} z_1}, \frac{\sqrt{1 - |a|^2}\, z_2}{1 - \bar{a} z_1} \right)$$

sends the ball B^2 into itself. Furthermore, the inverse mapping to $T_{(a,0)}$ is $T_{(-a,0)}$. Thus $T_{(a,0)}$ is an automorphism.

If (z_1, z_2) is any point of B^2, then there is an element $\lambda \in U(2)$ that takes (z_1, z_2) to a point of the form $(a, 0)$. Also $T_{(a,0)}(a, 0) = (0, 0)$. These two pieces of information combined tell us that $\mathrm{Aut}\,(B^2)$ acts transitively on B^2: this means that any point of B^2 may be moved to any other by some element of the automorphism group. first, B^2 is *homogeneous.*

Let G denote the subgroup of $\operatorname{Aut}(B^2)$ generated by $U(2)$ together with $\{T_{(a,0)} : a \in \mathbb{C}, |a| < 1\}$. Then the isotropy subgroup of G at the origin obviously contains $U(2)$. Thus it equals $U(2)$. It follows that G is the full automorphism group, by Theorem 1.3.1.[7] For future reference, note that if $\varphi \in \operatorname{Aut}(B^2)$, then one can always express φ in the form $\mu_1 \circ T_{(b,0)} \circ \mu_2$, where μ_1, μ_2 are unitary rotations. first, let $\lambda_2 =$ a unitary rotation taking $\varphi^{-1}((0,0))$ to a point of the form $(b, 0)$. Then $T_{(b,0)} \circ \lambda_2$ takes $\varphi^{-1}((0,0))$ to $(0,0)$. Hence $T_{(b,0)} \circ \lambda_2 \circ \varphi$ takes $(0,0)$ to $(0,0)$. Thus, from our earlier observations, $T_{(b,0)} \circ \lambda_2 \circ \varphi$ is a unitary rotation, say λ_1. Hence $\varphi = \lambda_2^{-1} \circ T_{(b,0)} \circ \lambda_1$, which has the desired form. □

(6) $\operatorname{Aut}(\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\})$.

By Corollary 1.3.2, the elements of I_0 (the isotropy group at $0 = (0,0)$) are $\mathbb{C}$-linear. Such a map must take a point of the boundary of the form $(\alpha, 0)$ or $(0, \alpha)$ to another point with one coordinate 0. This is so because boundary points with one coordinate 0 are exactly those boundary points where $\partial\Omega$ makes higher than first-order contact with its complex tangent plane, a condition preserved by invertible complex linear maps. Thus

$$\begin{aligned} I_0 = &\big\{(z_1, z_2) \mapsto (\omega_1 z_1, \omega_2 z_2) : \omega_1, \omega_2 \in \mathbb{C}, |\omega_1| = |\omega_2| = 1\big\} \\ &\cup \big\{(z_1, z_2) \mapsto (\omega_1 z_2, \omega_2 z_1) : \omega_1, \omega_2 \in \mathbb{C}, |\omega_1| = |\omega_2| = 1\big\}. \end{aligned}$$

Next, we claim that any element of $\operatorname{Aut}(\Omega)$ must in fact fix the origin. Let ϕ be an automorphism. By standard results in several complex variables, ϕ and ϕ^{-1} are C^∞ up to the boundary of Ω (see [Bell 1981]). Weakly pseudoconvex boundary points must consequently be mapped only to weakly pseudoconvex boundary points. So ϕ must take the union of the two circles to itself. Thus ϕ must (after composition with the map permuting the coordinates if necessary) preserve the circle $\{(\alpha, 0) \in \partial\Omega\}$, and it must also preserve the circle $\{(0, \alpha) \in \partial\Omega\}$. By the Cauchy integral formula and continuity of ϕ at the boundary, it follows that ϕ preserves the entire discs $\{(\alpha, 0) : |\alpha| \le 1\}$ and $\{(0, \alpha) : |\alpha| \le 1\}$. We conclude that $\phi(0) = 0$. Hence ϕ is linear and in fact $\phi \in I_0$. So we have completely identified all elements of $\operatorname{Aut}(\Omega)$, and this verifies that $\operatorname{Aut}(\Omega) = I_0$. □

(7) $\operatorname{Aut}(\Omega)$ for $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < \alpha < |z_1|^2 + |z_2|^2 < 1\}$.

By the Hartogs extension phenomenon, each element $f \in \operatorname{Aut}(\Omega)$ extends uniquely to a holomorphic mapping $\widehat{f} : B^2 \to B^2$, where B^2 is the unit ball in $\mathbb{C}^2$ as usual. These extensions must all be invertible since

[7] Determining the automorphism group of B^2 as a recognizable Lie group requires additional work. It turns out that it is $PSL(2, \mathbb{C})$. See [Helgason 1962] for more on this matter.

clearly $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$ for all $f, g \in \mathrm{Aut}\,(\Omega)$ (and of course the extension of the identity map is the identity map). Each such $\widehat{f}$, $f \in \mathrm{Aut}\,(\Omega)$, is a unitary rotation. To see this, note that, by the remark at the end of Example (5), $\widehat{f} = \mu_1 \circ T_{(a,0)} \circ \mu_2$ for some unitary rotations μ_1, μ_2 with $T_{(a,0)}$, as in the discussion there. Both μ_1 and μ_2 preserve Ω, but $T_{(a,0)}$ definitely does not preserve Ω if $a \neq 0$. This point is simple to check algebraically by looking at points of the form $ta/|a|$ with $-1 < t < 1$. Thus $\widehat{f}$ can preserve Ω only if $a = 0$ and, hence, $\widehat{f}$ is a unitary rotation. Consequently, $\mathrm{Aut}\,(\Omega)$ consists of the restrictions to Ω of the set of unitary rotations around the origin $(0,0)$. □

(8) $\mathrm{Aut}\,(\Omega)$ for

$$\begin{aligned}\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : 1/100 < |z_1|^2 + |z_2|^2 < 1\} \\ \setminus \Big[\{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - 3/4|^2 + |z_2|^2 \leq r_1\} \\ \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2 - 7/8|^2 \leq r_2\}\Big],\end{aligned}$$

with some small positive numbers r_1 and r_2.

Notice first that each element of $\mathrm{Aut}\,(\Omega)$ again extends uniquely to an element of $\mathrm{Aut}\,(B^2)$, by the Hartogs extension theorem. Then each automorphism of Ω must either preserve the sphere $\Sigma = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1/100\}$ or map this sphere to one of the other deleted spheres, by topological considerations. Algebraic considerations show that the image of a Euclidean sphere around the origin under an automorphism of B^2 is a Euclidean sphere only if the automorphism fixes the origin and hence is a rotation.

The algebraic determination that the image of a sphere with a center at the origin is again a sphere only if the origin is fixed can be done conveniently as follows. Consider $T_{(a,0)}$, for $-1 < a < 1$, acting on $S(r) =$ the sphere of radius $0 < r < 1$ around the origin $(0,0)$. Then $T_{(a,0)}(r,0)$ and $T_{(a,0)}(-r,0)$ are diametrically opposite on the image sphere. Again, if the image is a sphere, it then follows that the vector from $T_{(a,0)}(0,r)$ to $T_{(a,0)}(-r,0)$ is perpendicular to the vector from $T_{(a,0)}(0,r)$ to $T_{(a,0)}(r,0)$. But direct calculation shows that the inner product of these two vectors is 0 if and only if $a = 0$.

As in the arguments for Example (7) above, f is now an automorphism of B^2 preserving the origin, that is the center of Σ. Consequently, any automorphisms of this Ω must be elements of $U(2)$. Since the elements of $U(2)$ are Euclidean isometries, and since the removed balls around $(3/4, 0)$ and $(0, 7/8)$ have centers that are at different distances from the origin, each of these balls must be mapped to itself. It follows that the automorphism which is an element of $U(2)$ must in fact be the identity mapping. Thus $\mathrm{Aut}\,(\Omega) = \{\mathrm{id}\}$: the automorphism group has just the single element, which is the identity. In this circumstance, we say that the domain Ω is *rigid*. □

(9) Aut (Ω) for

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2k} < 1\} , \quad k > 1 .$$

First we note that I_0 is linear from Corollary 1.3.3. Also this isotropy group clearly contains all linear maps of the form

$$(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2) , \quad \theta_1, \theta_2 \in \mathbb{R}.$$

By the same logic as in Example (6), the set $\{(\alpha, 0) \in \Omega\}$ must be mapped to itself by any element of this isotropy group. This and the compactness of I_0 imply that

$$I_0 = \{(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2) : \theta_1, \theta_2 \in \mathbb{R}\},$$

as follows. The invariance of the disc $\{(\alpha, 0) \in \Omega\}$ implies that the matrices in I_0 have the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}$$

with $\alpha_{11} \neq 0$ and $\alpha_{22} \neq 0$. If also α_{12} were not zero, then the powers of this matrix (which arise under multiple compositions of the mapping) would not be contained in a compact set in $GL(n, \mathbb{C})$. Thus in fact $\alpha_{12} = 0$.

For $a \in \mathbb{C}$, $|a| < 1$, consider the mapping

$$S_a : (z_1, z_2) \longmapsto \left(\frac{z_1 - a}{1 - \bar{a} z_1}, \frac{(1 - |a|^2)^{1/2k}}{(1 - \bar{a} z_1)^{1/2k}} z_2 \right) .$$

We see that S_a belongs to Aut (Ω). This assertion can be easily checked by direct calculation. Also S_{-a} is the inverse mapping of S_a. The orbit of $\mathbf{0}$ under Aut (Ω) consequently contains $\{(\alpha, 0) \in \Omega\}$. Again, by the logic of Example (6) using [Bell 1981] etc., it follows that the set $\{(\alpha, 0) \in \Omega\}$ is preserved by elements of Aut (Ω). Hence the Aut (Ω)-orbit of $\mathbf{0}$ is equal to $\{(\alpha, 0) \in \Omega\}$. This information then completely determines the automorphism group. □

(10) Aut (D^2), where $D^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$.

We write $\tau_a(z) = (z - a)/(1 - \bar{a}z)$ for $z \in D \subseteq \mathbb{C}$. The maps of the form $(z_1, z_2) \mapsto (\tau_{a_1}(z_1), \tau_{a_2}(z_2))$ act transitively on D^2. Also the isotropy subgroup I_0 at the origin $(0, 0)$ consists of linear maps only by Corollary 1.3.2. These linear maps must have the form $(z_1, z_2) \mapsto (\omega_1 z_1, \omega_2 z_2)$ or $(z_1, z_2) \mapsto (\omega_2 z_2, \omega_1 z_1)$ with $|\omega_1| = |\omega_2| = 1$, since they must preserve the *distinguished boundary* $\{(z_1, z_2) \colon |z_1| = 1, |z_2| = 1\}$: this set is exactly the points where ∂D^2 is not smooth, and the property of being not smooth is preserved by linear maps. It follows that Aut (D^2) is exactly the group generated by the maps $(z_1, z_2) \mapsto (\tau_{a_1}(z_1), \tau_{a_2}(z_2))$, $(z_1, z_2) \mapsto (\omega_1 z_1, \omega_2 z_2)$ with $|\omega_1| = |\omega_2| = 1$, and $(z_1, z_2) \mapsto (z_2, z_1)$.

Examples (5) and (10) yield the following historical theorem of Poincaré, which, as already discussed, shows that the Riemann mapping theorem does not hold in complex dimension higher than 1. The proof of this theorem by Poincaré (see below) demonstrated that automorphism groups could play an important role—especially in complex dimensions greater than 1. Of course we have already shown in the remarks after Corollary 1.3.2 that the ball and the polydisc are not biholomorphic, but Poincaré's proof is of historical interest.

Theorem 1.4.1 (Poincaré). *In complex dimension 2, the ball and the bidisc are not biholomorphic to each other.*

Proof. Suppose that there exists a biholomorphic map $f : B^2 \to D^2 = D \times D$. Composing with an automorphism of D^2, we may assume without loss of generality that f maps the origin to itself. Then the map $f_* : \mathrm{Aut}\,(B^2) \to \mathrm{Aut}\,(D^2)$ defined by $f_*(\gamma) \equiv f^{-1} \circ \gamma \circ f$ is a continuous group isomorphism. So, this map generates a group isomorphism between the identity components of the isotropy subgroups at the origin. Note that the identity component of the isotropy subgroup of $\mathrm{Aut}\,(B^2)$ at the origin contains $U(2)$, the group of 2×2 unitary matrices (and indeed $= U(2)$). On the other hand, the identity component of the isotropy subgroup of $\mathrm{Aut}\,(D^2)$ at the origin is the torus group consisting of rotations in each variable separately. But the torus group is commutative, while $U(2)$ is noncommutative. This is a contradiction. Therefore the desired conclusion follows immediately. □

1.5 Orbit Accumulation Boundary Points Are Pseudoconvex

In the preceding section, we have rather few examples in higher dimensions (i.e., $\mathbb{C}^n$, $n \geq 2$) of domains Ω with $\mathrm{Aut}\,(\Omega)$ noncompact. But the examples that we do have—numbers (5), (9), (10) in the last section—all have the notable property that they are convex and hence pseudoconvex. It turns out that if Ω is a bounded domain and p is a point of the boundary with the boundary smooth near p, then accumulation of an $\mathrm{Aut}\,(\Omega)$-orbit at p implies pseudoconvexity at p. More precisely:

Theorem 1.5.1 (Greene/Krantz [Greene/Krantz 1991]). *If $p_0 \in \partial\Omega$ is a boundary point of a bounded domain Ω in $\mathbb{C}^n$ whose boundary is C^2 smooth in a neighborhood of p_0, and if there exists a sequence $\varphi_j \in \mathrm{Aut}\,(\Omega)$ such that $\lim_{j\to\infty} \varphi_j(x_0) = p_0$ for some $x_0 \in \Omega$, then $\partial\Omega$ is Levi pseudoconvex at p_0.*

Proof. Assume the contrary, that $\partial\Omega$ is not pseudoconvex at p_0. Then there exists a compact set K contained in Ω such that the holomorphic hull $\widehat{K}$ of K contains a set of the form $\Omega \cap U$ where U is an open set in $\mathbb{C}^n$

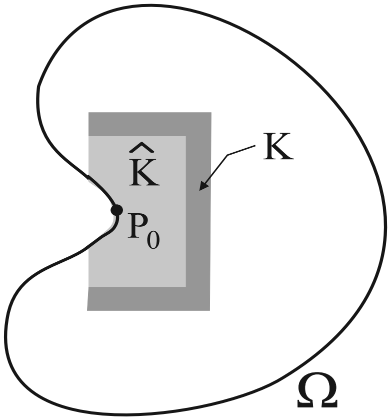

Fig. 1.1. The Hartogs figure and its holomorphic hull.

containing p_0.[8] [Recall that the holomorphic hull $\widehat{K}$ of a compact set K is by definition the set $\{p \in \Omega : |f(p)| \leq \max_K |f|, \forall f : \Omega \to \mathbb{C}$ holomorphic$\}$.] □

Now choose an $\epsilon > 0$ such that $B^n(x_0, 3\epsilon) \subset \Omega$. Let A_M be the set of $\varphi \in \mathrm{Aut}\,(\Omega)$ such that $\|d\varphi^{-1}|_{\varphi(x_0)}\| \leq M$, where $\|\cdot\|$ here represents the usual operator norm. Then we show:

Lemma 1.5.2. *There exists $\delta > 0$ such that $\varphi(B^n(x_0, \epsilon))$ contains $B^n(\varphi(x_0), \delta)$ for every $\varphi \in A_M$.*

Proof of the lemma. Since $d\varphi^{-1}|_{\varphi(x_0)} = (d\varphi|_{x_0})^{-1}$, we see that $\|(d\varphi|_{x_0})^{-1}\| \leq M$ whenever $\varphi \in A_M$. Consider the map

$$T(z) := (d\varphi|_{x_0})^{-1} \circ \varphi(z), \quad z \in B^n(x_0, \epsilon).$$

The differential at x_0 of this map is equal to the identity. And its second derivatives on $B^n(x_0, \epsilon)$ are bounded (Cauchy estimates on φ) by a constant depending only on M and the bound on $\|(d\varphi|_{x_0})^{-1}\|$ (and Ω and ϵ) but not on $\varphi \in A_M$. Hence, by standard information about the inverse function theorem, $T(B^n(x_0, \epsilon))$ contains a ball of radius $\alpha > 0$ centered at x_0, where α is independent of which φ is chosen from A_M: here α depends only on M (and ϵ and Ω). Thus the image of the map $\varphi = d\varphi|_{x_0} \circ T$ contains a ball of radius $\delta > 0$ centered at $\varphi(x_0)$, with δ independent of the choice of φ. [The radius δ depends only on M, ϵ, and Ω for the following reason: since $d\varphi|_{x_0}$ is a linear transformation with its inverse bounded above in operator norm, no such φ

[8] The usual construction of a compact set in Ω with holomorphic hull running out to a nonpseudoconvex boundary is casually called a "Hartogs tin can" in several complex variables (Figure 1.1). See [Grauert/Fritzsche 1976] for example. In case one "Hartogs tin can" does not provide a U of the sort we are after, one can perturb it and take the set K as the union of the perturbations to get the desired situation.

can take a given radius ball to a set not containing a definite radius ball. In fact, it cannot contract anything by more than a factor of $1/M$.] Thus the assertion of the lemma follows.

Altogether, one obtains that, if $\varphi_j(x_0) \to p_0 \in \partial\Omega$ as $j \to \infty$, then $\|d\varphi_j^{-1}|_{\varphi_j(x_0)}\| \to \infty$. Let $\psi_j = (\psi_j^1, \ldots, \psi_j^n)$ be the component representation of φ_j^{-1} for a moment. Passing to a subsequence, we may assume that

$$\left| \frac{\partial \psi_j^\ell}{\partial z_m}\bigg|_{\varphi_j(z_0)} \right| \to \infty$$

for some $\ell, m \in \{1, \ldots, m\}$. [Otherwise these φ_js would belong to A_M for some $M > 0$, and hence the image of φ_j contains a ball of radius δ, independent of j. A contradiction.] However, this is impossible, because $|\partial\psi_j^\ell/\partial z_m|$ is bounded near p_0 by its absolute value on the compact Hartogs figure K, and that is bounded by a constant independent of j, by Cauchy estimates. This completes the proof. □

We shall return to related considerations later in Chapter 7 (Proposition 7.6.2), using somewhat different, albeit related, methods.

1.6 Holomorphic Vector Fields and Their Flows

From the viewpoint of the Lie theory of transformation groups, it is natural to ask which (real) vector fields have the property that their flows consist of holomorphic mappings. We shall have explicit use for these ideas later (e.g., in Chapter 6), in addition to their general interest. To explore the matter in some detail, we recall first the general viewpoint.

Suppose that $\mathbf{V} : U \to \mathbb{R}^N$ is a "vector field" (at this state, it is just a vector-valued function) on an open set $U \subset \mathbb{R}^N$. If $\mathbf{V}$ has suitable regularity—even Lipschitz continuity will suffice—then, for each $p \in U$, there are an $\epsilon > 0$ and a neighborhood W of p, $p \in W \subset U$, such that, for each $q \in W$, there is a differentiable function $\gamma_q : (-\epsilon, \epsilon) \to U$ with

$$\frac{d\gamma_q}{dt}\bigg|_t = \mathbf{V}(\gamma_q(t))$$

for each $t \in (-\epsilon, \epsilon)$. Such a γ_q is called an *integral curve* of $\mathbf{V}$ with initial point q. Integral curves are unique up to the domain of definition in t if their initial point is given.

Such a vector field $\mathbf{V} : U \to \mathbb{R}^N$ thus defines a (local) flow $q \mapsto \gamma_q(t)$. We call this function φ_t so that $\varphi_t : W \to U$ is defined for all $t \in (-\epsilon, \epsilon)$. Also, $\varphi_0 =$ the identity map. Uniqueness of integral curves shows that

$$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1 + t_2}$$

for all t_1, t_2 with both $|t_1|$ and $|t_2|$ small enough that the φ-maps are defined.

This all makes sense for vector fields defined on an open subset of a manifold M of dimension n. In this case, the vector field $\mathbf{V}$ is a function from M into the tangent bundle $TM := \bigcup_{p \in M} T_pM$, where T_pM is the tangent space of M at p, and it is required that $\mathbf{V}(p) \in T_pM$ for every $p \in M$. The definitions of properties are the same as for the Euclidean space case, *mutatis mutandis*.

Now we are interested specifically in the question, either for $\mathbb{C}^n = \mathbb{R}^{2n}$, or on a complex manifold (locally the same as $\mathbb{C}^n$), of which vector fields $\mathbf{V}$ have the property that the associated local flows φ_t are holomorphic functions. Such a flow is called *holomorphic*, that is, a flow of a vector field is called holomorphic, if for each t, φ_t is holomorphic (where it is defined).

The answer to this question is straightforward, but it will be most easily explainable if we introduce some notation.

First we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by setting $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and then identifying $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $(x_1, y_1, \ldots, x_n, y_n)$. We set $\frac{\partial}{\partial x_j}$ = the $\mathbb{R}^{2n}$ vector with the $(2j-1)$-th component 1 and all other components 0, and then $\frac{\partial}{\partial y_j}$ = the $\mathbb{R}^{2n}$ vector with $(2j)$-th component 1 and all other components 0, for $j = 1, 2, \ldots, n$. [This notation makes sense because the directional derivative of a function along one such vector just considered is equal to the corresponding partial derivative, e.g., $\frac{\partial}{\partial x_1}$ of a function is its directional derivative along the vector $(1, 0, \ldots, 0) \in \mathbb{R}^{2n}$.] As usual, we set $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ as a differential operator.

If $\mathbf{V}$ is a real vector field on $U \subset \mathbb{C}^n = \mathbb{R}^{2n}$, then $\mathbf{V}$ has the form

$$\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y_j}$$

for some real-valued functions a_j and b_j and these are uniquely determined. We define

$$J\mathbf{V} = \sum_{j=1}^{n} a_j \frac{\partial}{\partial y_j} - \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}.$$

One can easily verify that

$$\mathbf{V} - iJ\mathbf{V} = 2\left(\sum_{j=1}^{n} (a_j + ib_j) \frac{\partial}{\partial z_j}\right).$$

We define the real vector field $\mathbf{V}$ to be *holomorphic* if, for each j, the function $a_j + ib_j$ is holomorphic. Thus a real vector field $\mathbf{V}$ is holomorphic if and only if $\mathbf{V}$ is the real part of a complex vector field of the form $\sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j}$ where the f_j are holomorphic functions. In these terms, we can answer the question about which real vector fields have (local) flows that are holomorphic.

Theorem 1.6.1 (Lie Theory Lemma). *A C^1 real vector field $\mathbf{V}$ has holomorphic local flows φ_t if and only if $\mathbf{V}$ is a holomorphic vector field in the sense just defined.*

If one is willing to use the standard methods of "Lie derivatives," then this assertion is easy to check. We shall present that proof first. Then we shall recast it in more concrete form in which the concept of Lie derivative is not used explicitly.

Proof of the lemma using Lie derivatives. The local flow φ_t for a fixed t value is holomorphic if and only if $d\varphi_t$ commutes with the J-mapping already defined. (This latter is just a restatement of the Cauchy–Riemann equations.) Here $d\varphi_t$ denotes the real differential of φ_t. Since φ_0 = the identity map, to check that $d\varphi_t \circ J = J \circ d\varphi_t$ for all t, we need only check that $L_{\mathbf{V}} J = 0$ where $L_{\mathbf{V}} J$ denotes the Lie derivative of the tensor J with respect to $\mathbf{V}$. Thus we need only check that, for each $j = 1, \dots, n$,

$$(L_{\mathbf{V}} J)\frac{\partial}{\partial x_j} = 0 \qquad \text{and} \qquad (L_{\mathbf{V}} J)\frac{\partial}{\partial y_j} = 0.$$

Now

$$(L_{\mathbf{V}} J)\frac{\partial}{\partial x_j} = L_{\mathbf{V}}\left(J\frac{\partial}{\partial x_j}\right) - J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right)\right)$$

by the Leibniz rule for Lie derivatives. But

$$\begin{aligned} L_{\mathbf{V}}\left(J\frac{\partial}{\partial x_j}\right) &= L_{\mathbf{V}}\left(\frac{\partial}{\partial y_j}\right) \\ &= -\sum_{\ell=1}^{n} \frac{\partial a_\ell}{\partial y_j}\frac{\partial}{\partial x_\ell} - \sum_{\ell=1}^{n} \frac{\partial b_\ell}{\partial y_j}\frac{\partial}{\partial y_\ell} \end{aligned}$$

while

$$\begin{aligned} J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right)\right) &= -J\left(\sum_{\ell=1}^{n} \frac{\partial a_\ell}{\partial x_j}\frac{\partial}{\partial x_\ell} + \sum_{\ell=1}^{n} \frac{\partial b_\ell}{\partial x_j}\frac{\partial}{\partial y_\ell}\right) \\ &= \sum_{\ell=1}^{n} \frac{\partial b_\ell}{\partial x_j}\frac{\partial}{\partial x_\ell} - \frac{\partial a_\ell}{\partial x_j}\frac{\partial}{\partial y_\ell}. \end{aligned}$$

Thus, $L_{\mathbf{V}}\big(J\frac{\partial}{\partial x_j}\big) = J\big(L_{\mathbf{V}}\big(\frac{\partial}{\partial x_j}\big)\big)$ if and only if

$$\frac{\partial a_\ell}{\partial y_j} = -\frac{\partial b_\ell}{\partial x_j} \qquad \text{and} \qquad \frac{\partial a_\ell}{\partial x_j} = \frac{\partial b_\ell}{\partial y_j}$$

for $\ell = 1, \dots, n$, in both cases. But these are precisely the Cauchy–Riemann equations for $a_\ell + ib_\ell$ to be holomorphic in the z_j variable. It is clear that if these hold, then $(L_{\mathbf{V}} J)\big(\frac{\partial}{\partial y_j}\big)$ is also 0 since

$$\begin{aligned} J\left(L_{\mathbf{V}}\left(J\frac{\partial}{\partial x_j}\right) - JL_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right)\right) &= L_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right) + J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial y_j}\right)\right) \\ &= -L_{\mathbf{V}}\left(J\left(\frac{\partial}{\partial y_j}\right)\right) + J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial y_j}\right)\right). \end{aligned}$$

For the converse direction, just trace the steps backwards. The conclusion follows.

To carry out essentially the same proof without introducing the Lie derivatives explicitly, we compute first, for each $j = 1, \ldots, n$,

$$\frac{\partial}{\partial t}\left(Jd\varphi_t|_{p,0}\left(\frac{\partial}{\partial x_j}\right) - d\varphi_t|_{p,0}\left(\frac{\partial}{\partial y_j}\right)\right).$$

For this, note that $p = \varphi_0(p)$ and write, for all q near p,

$$\varphi_t(q) = (x_{1,t}(q),\, y_{1,t}(q),\, \ldots,\, x_{n,t}(q),\, y_{n,t}(q)).$$

Then

$$d\varphi_t\left(\frac{\partial}{\partial x_j}\right) = \left(\frac{\partial x_{1,t}}{\partial x_j}, \frac{\partial y_{1,t}}{\partial x_j}, \ldots, \frac{\partial x_{n,t}}{\partial x_j}, \frac{\partial y_{n,t}}{\partial x_j}\right)$$

and

$$d\varphi_t\left(\frac{\partial}{\partial y_j}\right) = \left(\frac{\partial x_{1,t}}{\partial y_j}, \frac{\partial y_{1,t}}{\partial y_j}, \ldots, \frac{\partial x_{n,t}}{\partial y_j}, \frac{\partial y_{n,t}}{\partial y_j}\right),$$

while

$$J\left(d\varphi_t\left(\frac{\partial}{\partial x_j}\right)\right) = \left(\frac{\partial y_{1,t}}{\partial x_j},\ -\frac{\partial x_{1,t}}{\partial x_j}, \ldots\right).$$

So

$$\frac{\partial}{\partial t}J\left(d\varphi_t\left(\frac{\partial}{\partial x_j}\right)\right) = \left(\frac{\partial^2 y_{1,t}}{\partial t\partial x_j}, -\frac{\partial^2 x_{1,t}}{\partial t\partial x_j}, \ldots\right) = \left(\frac{\partial}{\partial x_j}\left(\frac{\partial y_{1,t}}{\partial t}\right), \ldots\right)$$

and

$$\frac{\partial}{\partial t}d\varphi_t\left(\frac{\partial}{\partial y_j}\right) = \left(\frac{\partial^2 x_{1,t}}{\partial t\partial y_j}, \ldots\right) = \left(\frac{\partial}{\partial y_j}\left(\frac{\partial x_{1,t}}{\partial t}\right), \ldots\right).$$

Note that

$$\left.\frac{\partial x_{\ell,t}}{\partial t}\right|_{t=0,p} = a_\ell(p) \qquad \text{and} \qquad \left.\frac{\partial y_{\ell,t}}{\partial t}\right|_{t=0,p} = b_\ell(p).$$

Translating the Cauchy–Riemann equations for the functions $a_\ell + ib_\ell$ back into the x, y notation gives

$$\frac{\partial}{\partial t}\left\{J\left(d\varphi_t\left(\frac{\partial}{\partial x_j}\right)\right) - d\varphi_t\left(J\left(\frac{\partial}{\partial x_j}\right)\right)\right\} = 0$$

when $t = 0$.

Working through the details of this calculation gives that this implication goes in both directions.

Now note that $d\varphi_{t+h}(\cdot) = d\varphi_t(d\varphi_h(\cdot))$ for small h and so $d\varphi_{t+h} - d\varphi_t = d\varphi_t(d\varphi_h - d\varphi_0)$. Hence $\lim_{h\to 0} \frac{1}{h}(d\varphi_{t+h} - d\varphi_t) = 0$, if $\lim_{h\to 0} \frac{1}{h}(d\varphi_h -$ identity$) = 0$. Thus, if $\mathbf{V}$ is holomorphic then $Jd\varphi_t(\frac{\partial}{\partial y_j}) - d\varphi_t(J(\frac{\partial}{\partial y_j})) = 0$. first, φ_t is holomorphic. These calculations also work in the opposite direction. □

This proof is essentially the same as the Lie derivative one: the Lie derivative concept has been replaced by equality of mixed partials, in effect.

Corollary 1.6.2. *If the local flow functions φ_t of a real vector field $\mathbf{V}$ are holomorphic, then so are the local flow functions of $J\mathbf{V}$.*

Proof. If $\mathbf{V} - iJ\mathbf{V}$ is a holomorphic linear combination of $\frac{\partial}{\partial z}$ vector fields, then so is $i(\mathbf{V} - iJ\mathbf{V})$. But Re $(i(\mathbf{V} - iJ\mathbf{V})) = J\mathbf{V}$. □

If $\mathbf{V}$ is a (real) vector field defined on an open set $U \subset \mathbb{R}^N$ (or on a manifold M), and if $q \in U$ (or, $q \in M$, respectively), then it may not be the case that the integral curve $\gamma_q(t)$ of $\mathbf{V}$ with $\gamma_q(0) = q$ is defined for all $t \in \mathbb{R}$. So the local flow functions φ_t of $\mathbf{V}$ may not be defined on all U for all t.

Note, however, that if there is an $\epsilon > 0$ such that $\varphi_t(q)$ is defined for all $t \in (-\epsilon, \epsilon)$ and all $q \in U$ (or $q \in M$), then φ_t is defined for all $t \in \mathbb{R}$: this result follows by "patching together" via uniqueness of integral curves the local flows for $|t| < \epsilon/2$. That is, one notes that φ_t should equal $\varphi_{t/k} \circ \cdots \circ \varphi_{t/k}$ (k-times) for any positive integer k and that, if k is large enough, then $|t/k| \le \epsilon/2$. Then one uses $\varphi_{t/k} \circ \cdots \circ \varphi_{t/k}$ as the definition of φ_t and verifies easily that this indeed has the defining property that $\frac{d}{dt}\varphi_t(q) = \mathbf{V}(\varphi_t(q))$.

Consequently, if M is a compact manifold and $\mathbf{V}$ a vector field on it, then the φ_t flows associated to $\mathbf{V}$ are defined for all $t \in \mathbb{R}$ since the existence of an ϵ uniform over M follows from the basic local existence result for ordinary differential equations and the compactness of M.

In noncompact complex instances, it can happen that a holomorphic vector field $\mathbf{V}$ has integral curves and flow functions φ_t defined for all $t \in \mathbb{R}$ but $J\mathbf{V}$, also a holomorphic vector field, does not. Consider, for instance, the vector field $\mathbf{V}(x, y) = (y, -x)$ on $U := \{z \in \mathbb{C} \mid |z| < 1\}$. The vector field $\mathbf{V}$ is the"infinitesimal generator" of rotations around the origin, and its flow φ_t, defined for all $t \in \mathbb{R}$, is the rotation clockwise around the origin through angle t. As guaranteed by the fact that V is holomorphic ($\mathbf{V} =$ Re $(-2iz \frac{\partial}{\partial z})$), these φ_t are indeed holomorphic. The vector field $J\mathbf{V}$ is (x, y). This too is holomorphic: $J\mathbf{V} =$ Re $2z \frac{\partial}{\partial z}$. Its local flow functions φ_t are given by $\varphi_t(x, y) = (e^t x, e^t y)$, as is easily verified. But of course these are not defined for all t: when t is large positive, $(e^t x, e^t y)$ no longer lies in U, unless $(x, y) = (0, 0)$, the origin $(0, 0)$ being a fixed point of the flow since $J\mathbf{V}(0, 0) = (0, 0)$.

But, when one passes to the compact case, things change. The vector field $\mathbf{V}$ extends to be a vector field on $\mathbb{C} \cup \{\infty\}$, the "Riemann sphere": it is again the infinitesimal generator of the one-parameter group of rotations around the origin (in the clockwise direction). Since this is a group of holomorphic mappings, it must be that $\mathbf{V}$ extended is holomorphic on $\mathbb{C} \cup \{\infty\}$. (One can of course check directly that V is holomorphic at ∞, using $w = 1/z$ as a local coordinate around ∞.) But now the flow of $J\mathbf{V}$ is defined for all t: the point $(e^t x, e^t y)$ is in $\mathbb{C}$ for all $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$, and the flow has $(0, 0)$ and ∞ as fixed points, with t going to $-\infty$ corresponding to motion towards 0. Thus one sees in action the important difference between the compact and noncompact cases. These themes will reappear in Chapter 6.