9

Labeling

9.1 Introduction; Graceful Labelings

In general, a *labeling* (or *valuation*) of a graph is a map that carries some set of graph elements to numbers, most often to the positive or nonnegative integers. The most common choices of domain are the set of all vertices and edges (such labelings are called *total* labelings), the vertex-set alone (*vertex*-labelings), or the edge-set alone (*edge*-labelings). Other domains are possible.

We shall call two labelings of the same graph *automorphism-equivalent* if one can be transformed into the other by an automorphism of the graph.

Rosa ([102, 78]) introduced the idea of β-valuations, or as they are now called *graceful labelings*. A graceful labeling of a graph G is a one-to-one mapping γ from the set of all vertices to the integers $S_G = \{0, 1, \ldots, |E(G)|\}$ such that every non-zero member of S_G occurs as the difference between the labels on the endpoints of an edge. That is, if we extend γ to edges by defining $\gamma(xy) = |\gamma(x) - \gamma(y)|$, then every member of $1, 2, \ldots, |E(G)|\}$ arises (exactly) once among the edge-labels. A graph is called graceful if it has a graceful labeling.

Figure 9.1 shows a graceful labeling of K_4. In the right-hand diagram, the corresponding edge-labels are shown.

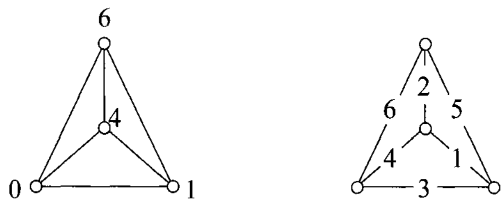

Fig. 9.1. Graceful labeling of K_4

If γ is a graceful labeling on G, define another labeling γ^* by

$$\gamma^*(x) = |E(G)| - \gamma(x)$$

for every vertex x. Then γ^* is also graceful; in fact, γ and γ^* induce the same edge-labeling. γ^* is called the *complementary labeling* or *dual* of γ. We formally define two graceful labelings γ and δ to be *equivalent* if δ is *automorphism-equivalent* to either γ or γ^*.

When testing a small graph with e edges to see whether it is graceful, first notice that there must be adjacent vertices labeled 0 and e in order for edge-label e to occur. Similarly there must be an edge from 0 to $e - 1$ or from 1 to e.

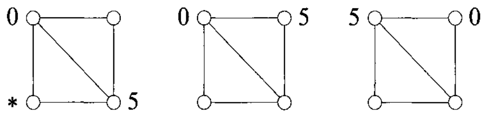

Fig. 9.2. Finding a graceful labeling of $K_4 - e$

Example. We shall find a graceful labeling of $K_4 - e$. This graph has two types of edge. Initially there are three ways of labeling an edge with 0 and 5, as shown in Figure 9.2. However the third is clearly dual to the second.

In order to achieve edge-label 1 in the first case, one of the remaining vertices must be labeled 1 or 4. Assume the vertex marked $*$ is labeled 1 (the 4 case is dual). Then labeling the other vertex 3 results in a graceful labeling.

The reader should check that the partial labeling in the second part of the diagram can also be completed.

Theorem 9.1. *There is no graceful labeling of K_v when $v > 4$.*

Proof. Suppose there is a graceful labeling of K_v, where $v \geq 5$. Let S be the set of vertex-labels used in the labeling. We write e for $\frac{1}{2}v(v-1)$, the number of edges in K_v. It causes no confusion if we use the symbol x for the vertex that receives label x.

The vertex labels must include 0 and e and (to achieve edge-label 1) either 1 or $e - 1$; the choices $\{0, 1, e\}$ and $\{0, e-1, e\}$ lead to dual labelings, so we can assume labels $\{0, 1, e\} \subseteq S$.

There can be no vertex labeled 2, for if there were, two edges would receive label 1: $2 \sim 1$ and $1 \sim 0$. Also label $e - 1$ is impossible: we would have $e \sim (e-1)$ and $1 \sim 0$. So $(e-2, e)$ is the only possible edge with label 2, and $\{0, 1, e-2, e\} \subseteq S$.

These vertex labels induce edges labeled $1, 2, e-3, e-2, e-1$ and e. Since $e > 6$, edge label $e - 4$ is still needed. An argument similar to those above shows that the only suitable new vertex label is 4, and the edge labels are

$$1,\ 2,\ 3,\ 4,\ e-6,\ e-4,\ e-3,\ e-2,\ e-1,\ e.$$

When $e \geq 10$, label $e - 5$ is not in this list. To obtain $e - 5$ we need vertex-label $3, 5, e-5, e-4$ or $e - 1$. But each of these possibilities leads to a duplicated edge-label: $3 - 0 = 3, 5 - 1 = 4, (e-5) - 1 = 4, e - (e-4) = 4, e - (e-1) = 1$. So no labeling is possible. □

Theorem 9.2. *Suppose G is an Eulerian graph with e edges. If G is graceful, then* $e \equiv 0$ *or* $3 \pmod 4$.

Proof. Suppose γ is a graceful labeling of G. Write $z_1, z_2, \ldots, z_e$ for the edges of G; and denote the endpoints of z_i by x_i and y_i, where $\gamma(x_i) > \gamma(y_i)$. Then

$$\begin{aligned}\sum_{i=1}^{e} \gamma(z_i) &= \sum_{i=1}^{e} \gamma(x_i) - \sum_{i=1}^{e} \gamma(y_i) \\ &= \sum_{i=1}^{e} \gamma(x_i) + \sum_{i=1}^{e} \gamma(y_i) - 2\sum_{i=1}^{e} \gamma(y_i).\end{aligned}$$

In the list $x_1, x_2, \ldots, x_e, y_1, y_2, \ldots, y_e$ the number of times each vertex occurs is equal to its degree. Since G is Eulerian, each of these degrees is even. So $\sum_{i=1}^{e} \gamma(x_i) + \sum_{i=1}^{e} \gamma(y_i)$ is even. Therefore $\sum_{i=1}^{e} \gamma(z_i)$ is even.

On the other hand, since γ is graceful, $\sum_{i=1}^{e} \gamma(z_i)$ is the sum of the first e positive integers, $\frac{1}{2}e(e-1)$. This is even only if $e \equiv 0$ or $3 \pmod 4$. □

There has been considerable interest in graceful labelings of trees; Kotzig (quoted in [102]) conjectured that all trees are graceful. However, this conjecture is far from settled. All stars and paths are graceful (see the exercises).

A *caterpillar* is a tree for which, if all *leaves* (vertices of degree 1 and their associated edges) were removed, the result is a path.

Theorem 9.3. [102] *All caterpillars are graceful.*

Proof. Suppose T is a caterpillar with v vertices and H is the path formed from T by deleting all the leaves. Select an endpoint of H (a vertex of degree 1 in the path) and name it x_0; the vertex adjacent to it in H is called x_1, and so on along H. Write X for the set of all vertices of T whose distance from x_0 is even (including x_0 itself), and Y for the set of vertices of odd distance. Every edge connects two vertices, one in X and the other in Y.

Assign label $v-1$ to x_0. Label the neighbors of x_0 with $0, 1, 2, \ldots,$ where the neighbor receiving the greatest label is x_1, the neighbor of x_0 in H. Assign labels $v-2, v-3, \ldots,$ to the neighbors of x_1 other than x_0; the largest label goes to x_2.

Continue as follows: after x_{2i} receives its label, assign increasing integer labels to its neighbors other than x_{2i-1} starting with the smallest unused label, assigning the largest label to x_{2i+1}; then assign labels to the neighbors of x_{2i+1} other than x_{2i} in decreasing order, starting with the largest unused integer smaller than v, ending by labeling x_{2i+2}.

The result will be a labeling in which members of X receive labels $v-1, v-2, \ldots, v-|X|$ and members of Y receive labels $0, 1, \ldots, |Y|$. It is easily checked to be graceful. □

An example is shown in Figure 9.3.

Graceful labelings of trees were first studied in an attempt to prove a conjecture of Ringel [100]. He conjectured that, given any tree T with n edges, it is possible to write the complete graph K_{2n+1} as a union of edge-disjoint copies of T. The connection is shown in the following theorem.

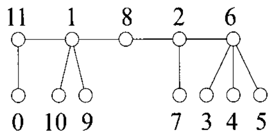

Fig. 9.3. Graceful labeling of a caterpillar

Theorem 9.4. *Suppose T is a graceful tree on $n + 1$ vertices. Then K_{2n+1} is a union of $2n + 1$ edge-disjoint copies of T.*

Proof. We use a K_{2n+1} whose vertices are the integers $\{0, 1, \ldots, 2n\}$ modulo $2n + 1$. Suppose γ is a graceful labeling of T. We identify the vertex x of T with the vertex $\gamma(x)$ of the K_{2n+1}. Then T is a subgraph of K_{2n+1} with vertices $\{0, 1, \ldots, n\}$.

For each integer $s \in \{0, 1, \ldots, 2n\}$ we construct a tree T_s as follows. Edge xy belongs to T_s if and only if $(x - s)(y - s)$ is an edge of T. Thus $T_0 = T$, and if the vertices of K_{2n+1} are written in order equally placed around a circle, T_s is obtained by rotating T_0 through $s/(2n + 1)$ of a revolution. Each T_s is isomorphic to T.

Now each edge xy of K_{2n+1} will belong to precisely one of the trees T_s, because there is precisely one edge of T that receives label $\pm(x - y)$ under γ. So we have the required decomposition.

If Kotzig's conjecture is true, that all trees are graceful, this will prove Ringel's conjecture. However, the Kotzig conjecture is stronger. A decomposition of K_{2n+1} into copies of an n-edge tree need not be of a cyclic nature.

Exercises 9.1

9.1.1 Find graceful labelings of P_3 and K_3.

9.1.2 Verify that the partial graceful labeling in Figure 9.2 can be completed.

A9.1.3 Find a graceful labeling of $P_3 \cup K_4$.

H9.1.4 Find a graceful labeling of $P_3 \cup K_3$.

9.1.5 Show that $P_2 \cup K_3$ and $P_3 \cup K_3$ have no graceful labelings.

A9.1.6 Show that no nontrivial forest (that is, a forest containing at least two trees) is graceful.

9.1.7 Show that the star $K_{1,n}$ is graceful for every n.

A9.1.8 Show that the path P_v is graceful for every v.

9.1.9 Suppose G is a graceful graph with e edges. Write X and Y for the sets of vertices with even and odd labels respectively. Show that the set $[X, Y]$ contains precisely $\frac{1}{2}(e + 1)$ edges.

9.2 Edge-Magic Total Labeling

A *magic square* of side n is an $n \times n$ array whose entries are an arrangement of the integers $\{1, 2, \ldots, n^2\}$, in which all elements in any row, any column, or either the main diagonal or main back-diagonal, add to the same sum. Small examples include

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

1	7	13	19	25
18	24	5	6	12
10	11	23	7	4
22	1	9	15	16
14	20	23	2	8

An excellent reference on magic squares is [2].

Various authors have introduced graph labelings that generalize the idea of a magic square; there has been a lot of duplication of terminology. To avoid confusion we define a labeling to be *edge-magic* if the sum of all labels associated with an edge equals a constant independent of the choice of edge, and *vertex-magic* if the same property holds for vertices. (This terminology could be extended to other substructures: face-magic, for example.) The domain of the labeling is specified by a modifier on the word "labeling." We shall always require that the labeling is a one-to-one map onto the appropriate set of consecutive integers starting from 1.

For example, Kotzig and Rosa [78] defined a magic labeling to be a total labeling in which the labels are the integers from 1 to $|V(G)|+|E(G)|$ and the sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [101] redefined this type of labeling and called the labelings *edge-magic*. We shall call them *edge-magic total labelings*. On the other hand, Stewart (see, for example, [111]), called such a labeling *supermagic*. Sedláček [110] originally proposed the study of edge-labelings with the magic property on vertices, but did not restrict the values of the labels in any way; they could be any reals.

To discuss these labelings, we define the *weight* of a graph element to be the sum of all labels associated with the element. For example, the weight of vertex x under an edge labeling α is

$$wt(x) = \sum_{y \sim x} \alpha(xy).$$

The weight of x under a total labeling λ is

$$wt(x) = \lambda(x) + \sum_{y \sim x} \lambda(xy),$$

while

$$wt(xy) = \lambda(x) + \lambda(xy) + \lambda(y).$$

If necessary, the labeling can be specified by a subscript, as in $wt_\lambda(x)$.

We formally define an *edge-magic total labeling* or EMTL on a graph G to be a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v + e$, where $v = |V(G)|$ and $e = |E(G)|$, with the property that, given any edge (xy),

$$\lambda(x) + \lambda(xy) + \lambda(y) = k$$

for some constant k. In other words, $wt(xy) = k$ for any choice of edge xy. k is called the *magic sum* of G. A graph is called *edge-magic* if it has an edge-magic total labeling.

As an example, Figure 9.4 shows an edge-magic total labeling of $K_4 - e$.

We shall frequently refer to the sum of consecutive integers, so we define

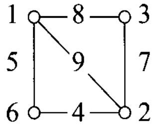

Fig. 9.4. An EMTL of $K_4 - e$ with $k = 12$.

$$\sigma_i^j = (i+1) + (i+2) + \cdots + j = i(j-i) + \binom{j+1}{2}. \tag{9.1}$$

Suppose the graph G has v vertices $\{x_1, x_2, \ldots, x_v\}$ and e edges. Vertex x_i has degree d_i and receives label a_i. Among the labels, write S for the set $\{a_i : 1 \le i \le v\}$ of vertex labels, and s for the sum of elements of S. Then S can consist of the v smallest labels, the v largest labels, or somewhere in between, so

$$\sigma_0^v \le s \le \sigma_e^{v+e},$$
$$\binom{v+1}{2} \le s \le ve + \binom{v+1}{2}. \tag{9.2}$$

Clearly, $\sum_{xy \in E}(\lambda(xy) + \lambda(x) + \lambda(y)) = ek$. This sum contains each label once, and each vertex label a_i an additional $d_i - 1$ times. So

$$ke = \sigma_0^{v+e} + \sum (d_i - 1)a_i. \tag{9.3}$$

If e is even, every d_i is odd and $v + e \equiv 2(\text{mod } 4)$, then (9.3) is impossible. We have

Theorem 9.5. [101] *If G has e even and $v + e \equiv 2(\text{mod } 4)$, and every vertex of G has odd degree, then G has no EMTL.*

Corollary 9.6. *The complete graph K_n is not edge-magic when $n \equiv 4(\text{mod } 8)$. The n-spoke wheel W_n is not edge-magic when $n \equiv 3(\text{mod } 4)$.*

(We shall see in Section 9.9.3 that K_n is never edge-magic for $n > 6$, so the first part of the Corollary really only eliminates K_4.)

Equation (9.3) may be used to provide bounds on k. Suppose G has v_j vertices of degree j, for each i up to Δ, the largest degree represented in G. Then the ke cannot be smaller than the sum obtained by applying the v_Δ smallest labels to the vertices of degree Δ, the next-smallest values to the vertices of degree $\Delta - 1$, and so on; in other words,

$$ke \ge (d_\Delta - 1)\sigma_0^{v_\Delta} + (d_{\Delta-1} - 1)\sigma_{v_\Delta}^{v_\Delta+(v_{\Delta-1})} + \sigma_{v_\Delta+(v_{\Delta-1})+\cdots+v_3}^{v_\Delta+(v_{\Delta-1})+\cdots+v_2} + \binom{v+e+1}{2}.$$

An upper bound is achieved by giving the *largest* labels to the vertices of highest degree, and so on.

In particular, suppose G is regular of degree d. Then (9.3) becomes

$$ke = (d-1)s + \sigma_0^{v+e} = (d-1)s + \tfrac{1}{2}(v+e)(v+e+1) \tag{9.4}$$

or, since $e = \frac{1}{2}dv$,

$$kdv = 2(d-1)s + (v+e)(v+e+1). \tag{9.5}$$

Given a labeling λ, its *dual* labeling λ' is defined by

$$\lambda'(x_i) = (v+e+1) - \lambda(x_i),$$

and for any edge xy,

$$\lambda'(xy) = (v+e+1) - \lambda(xy).$$

It is easy to see that if λ is an edge-magic labeling with magic sum k, then λ' is an edge-magic labeling with magic sum $k' = 3(v+e+1) - k$. The sum of vertex labels in the dual is $s' = v(v+e+1) - s$. Just as in the case of graceful labelings, we define two EMTLs λ and μ to be *equivalent* if λ is *automorphism-equivalent* to either μ or μ'.

Either s or s' will be less than or equal to $\frac{1}{2}v(v+e+1)$. This means that, in order to see whether a given graph has an EMTL, it suffices to check either all cases with $s \le \frac{1}{2}v(v+e+1)$ or all cases with $s \ge \frac{1}{2}v(v+e+1)$ (equivalently, either check all cases with $k \le \frac{3}{2}(v+e+1)$ or all with $k \ge \frac{3}{2}(v+e+1)$).

The cycle C_v is regular of degree 2 and has v edges. In that case, (9.2) becomes

$$v(v+1) \le 2s \le 2v^2 + v(v+1) = v(3v+1),$$

and (9.4) is

$$kv = s + v(2v+1),$$

whence v divides s; in fact $s = (k-2v-1)v$. When v is odd, s has $v+1$ possible values $\frac{1}{2}v(v+1), \frac{1}{2}v(v+3), \ldots, \frac{1}{2}v(v+2i-1), \ldots, \frac{1}{2}v(3v+1)$, with corresponding magic sums $\frac{1}{2}(5v+3), \frac{1}{2}(5v+5), \ldots, \frac{1}{2}(5v+2i+1), \ldots, \frac{1}{2}(7v+3)$. For even v, there are v values $s = \frac{1}{2}v^2 + v, \frac{1}{2}v^2 + 2v, \ldots, \frac{1}{2}v^2 + iv, \ldots, \frac{3}{2}v^2$, with corresponding magic sums $\frac{5}{2}v + 2, \frac{5}{2}v + 3, \ldots, \frac{5}{2}v + i + 1, \ldots, \frac{7}{2}v + 1$.

Kotzig and Rosa [78] proved that all cycles are edge-magic, producing examples with $k = 3v+1$ for v odd, $k = \frac{5}{2}v + 2$ for $v \equiv 2 \pmod 4$ and $k = 3v$ for $v \equiv 0 \pmod 4$. In [55], labelings are exhibited for the minimum values of k in all cases. We present two proofs here, and leave another as an exercise. In each case the proof consists of exhibiting a labeling.

Theorem 9.7. *If v is odd, then C_v has an edge-magic total labeling with $k = \frac{1}{2}(5v+3)$.*

Proof. Say $v = 2n+1$. Consider the cyclic vertex labeling $(1, n+1, 2n+1, n, \ldots, n+2)$, where each label is derived from the preceding one by adding $n \pmod{2n+1}$. The successive pairs of vertices have sums $n+2, 3n+2, 3n+1, \ldots, n+3$, which are all different. If $k = 5n+4$, the edge labels are $4n+2, 2n+2, 2n+3, \ldots, 4n+1$, as required. We have an edge-magic total labeling with $k = 5n+4 = \frac{1}{2}(5v+3)$ and $s = \frac{1}{2}v(v+1)$ (the smallest possible values).

By duality, we have:

Corollary 9.8. *Every odd cycle has an edge-magic total labeling with $k = \frac{1}{2}(7v+3)$.*

Theorem 9.9. *If v is even, then C_v has an edge-magic total labeling with* $k = \frac{1}{2}(5v + 4)$.

Proof. Write $v = 2n$. If n is even,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \ldots, n+1 \\ 3n & \text{for } i = 2 \\ (2n+i)/2 & \text{for } i = 4, 6, \ldots, n \\ (i+2)/2 & \text{for } i = n+2, n+4, \ldots, 2n \\ (2n+i-1)/2 & \text{for } i = n+3, n+5, \ldots, 2n-1, \end{cases}$$

while if n is odd,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \ldots, n \\ 3n & \text{for } i = 2 \\ (2n+i+2)/2 & \text{for } i = 4, 6, \ldots, n-1 \\ (n+3)/2 & \text{for } i = n+1 \\ (i+3)/2 & \text{for } i = n+2, n+4, \ldots, 2n-1 \\ (2n+i)/2 & \text{for } i = n+3, n+5, \ldots, 2n-2 \\ n+2 & \text{for } i = 2n. \end{cases}$$ □

Corollary 9.10. *Every even cycle has an edge-magic total labeling with* $k = \frac{1}{2}(7v + 2)$.

Figure 9.5 shows examples with $v = 7$ and $v = 8$ of the constructions in Theorems 9.7 and 9.9; they have $k = 19$ and 22 respectively. (Only the vertex labels are shown in the figure; the edge labels can be found by subtraction.)

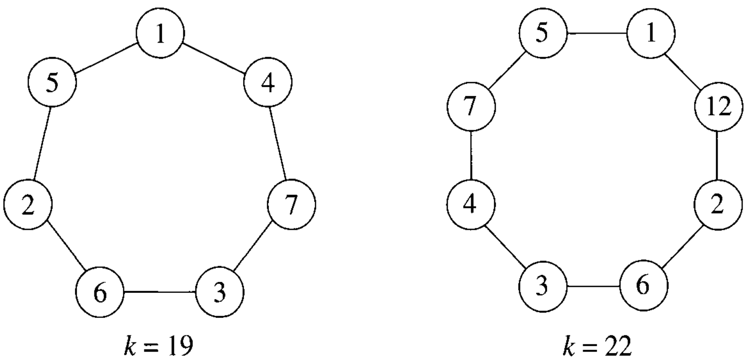

Fig. 9.5. Edge-magic total labelings of C_7 and C_8.

Theorem 9.11. *Every cycle of length divisible by 4 has an edge-magic total labeling with* $k = 3v$.

The path P_n can be viewed as a cycle C_n with an edge deleted. Say λ is an EMTL of C_n with the property that label $2n$ appears on an edge. If that edge is deleted, the result is a P_n with an EMTL.

For every n, there is a labeling of C_n in which $2n$ appears on an edge. Deleting this edge yields a path, on which the labeling is edge-magic. So:

Theorem 9.12. *All paths have EMTLs.*

Theorem 9.13. [78] *The complete bipartite graph $K_{m,n}$ is edge-magic for any m and n.*

Proof. The sets $S_1 = \{n+1, 2n+2, \ldots, m(n+1)\}$, $S_2 = \{1, 2, \ldots, n\}$, define an EMTL with $k = (m+2)(n+1)$. □

In particular, all EMTLs of stars $K_{1,n}$ are easily described.

Lemma 9.14. *In any EMTL of a star, the center receives label 1, $n+1$ or $2n+1$.*

Proof. Suppose the center receives label x. Then

$$kn = \binom{2n+2}{2} + (n-1)x. \tag{9.6}$$

Reducing (9.6) modulo n we find

$$x \equiv (n+1)(2n+1) \equiv 1$$

and the result follows. □

Theorem 9.15. *There are $3 \cdot 2^n$ EMTLs of $K_{1,n}$, up to equivalence.*

Proof. Denote the center of a $K_{1,n}$ by c, the peripheral vertices by $v_1, v_2, \ldots, v_n$ and edge (c, v_i) by e_i. From Lemma 9.14 and (9.6), the possible cases for an EMTL are $\lambda(c) = 1, k = 2n+4$, $\lambda(c) = n+1, k = 3n+3$ and $\lambda(c) = 2n+1, k = 4n+2$. As the labeling is edge-magic, the sums $\lambda(v_i) + \lambda(e_i)$ must all be equal to $M = k - \lambda(c)$ (so $M = 2n+3, 2n+2$ or $2n+1$). Then in each case there is exactly one way to partition the $2n+1$ integers $1, 2, \ldots, 2n+1$ into $n+1$ sets

$$\{\lambda(c)\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$$

where every $a_i + b_i = M$. For convenience, choose the labels so that $a_i < b_i$ for every i and $a_1 < a_2 < \cdots < a_n$. Then up to isomorphism, one can assume that $\{\lambda(v_i), \lambda(e_i)\} = \{a_i, b_i\}$. Each of these n equations provides two choices, according as $\lambda(v_i) = a_i$ or b_i, so each of the three values of $\lambda(c)$ gives 2^n EMTLs of $K_{1,n}$. □

It is conjectured ([78], also [101]), that all trees are edge-magic. Unfortunately, this has proven just as intractible as the corresponding conjecture for graceful labelings. Kotzig and Rosa [78] proved that all caterpillars are edge-magic. The proof is left as an exercise; as a hint, we give an example of an EMTL of a caterpillar in Figure 9.6.

Enomoto et al [38] carried out a computer search to show that all trees with fewer than 16 vertices are edge-magic.

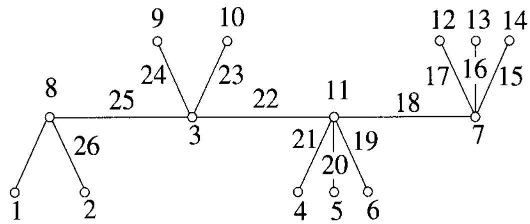

Fig. 9.6. An EMTL of a caterpillar

Vertex-magic total labelings, in which the sum of the labels of all edges adjacent to the vertex x, plus the label of x itself, is constant, have also been studied (see [85]). It is conceivable that the same labeling could be both vertex-magic and edge-magic for a given graph (not necessarily with the same constant). In that case the labeling, and the graph, are called *totally magic*. Totally magic graphs appear to be very rare.

Exercises 9.2

A9.2.1 Find all edge-magic total labelings of K_3.

A9.2.2 Show that every odd cycle has an edge-magic total labeling with $k = 3v + 1$.

9.2.3 Prove that the graph tK_4, consisting of t disjoint copies of K_4, has no edge-magic total labeling when t is odd.

9.2.4 Suppose a regular graph G of degree d is edge-magic. Prove

$$ke = (d-1)s + \sigma_0^{v+e} = (d-1)s + \tfrac{1}{2}(v+e)(v+e+1),$$
$$kdv = 2(d-1)s + (v+e)(v+e+1).$$

A9.2.5 A *triangular book* $B_{3,n}$ consists of n triangles with a common edge. Prove that all triangular books are edge-magic.

9.2.6 An n-sun is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex. Show that all suns are edge-magic. [7]

A9.2.7 What is the range of possible magic sums for an edge-magic total labeling of the *Petersen graph* P? Prove that P is edge-magic.

9.2.8 [7] An (n, t)-kite consists of a cycle of length n with a t-edge path (the *tail*) attached to one vertex. Show that an $(n, 1)$-kite (a kite with tail length 1) is edge-magic. [7]

9.2.9 Prove that all caterpillars are edge-magic.

9.2.10 Find a vertex-magic total labeling of $K_4 - e$.

9.2.11 Prove that C_5 has no totally magic labeling.

9.3 Edge-Magic Labelings of Complete Graphs

The discussion of edge-magic labelings of complete graphs is significantly harder than the "graceful" case, so we devote a section to it.

Suppose the graph G has an edge-magic total labeling λ, and suppose G contains a complete subgraph (or *clique*) H on n vertices. Let us write $x_1, x_2, \ldots, x_n$ for the vertices of H, and denote $\lambda(x_i)$ by a_i. Without loss of generality we can assume the names x_i to have been chosen so that $a_1 < a_2 < \cdots < a_n$.

If k is the magic sum, then $\lambda(x_i x_j) = k - a_i - a_j$, so the sums $a_i + a_j$ must all be distinct. This property is called being *well-spread*; this property will be used in discussing EMTLs of complete graphs. A *well-spread set* $A = \{a_1, a_2, \ldots, a_n)$ is a set of integers in which, if a_i, a_j, a_k, a_ℓ are all different, it never happens that $a_i + a_j = a_k + a_\ell$. If the elements are arranged in order, so that $0 < a_1 < a_2 < \cdots < a_n$, A is called a *well-spread sequence* or *Sidon sequence* of *length* n. Such sequences arise are related to the work of S. Sidon; their study was initiated by Erdős and Turan [40].

In discussing Sidon sequences (or, equivalently, cliques in edge-magic graphs), we write d_{ij} for $|a_j - a_i|$, the absolute difference between the i-th and j-th terms (the labels on the endpoints of the edge $x_i x_j$).

Lemma 9.16. *Suppose A is a Sidon sequence of length n. If $d_{ij} = d_{pq}$, then $\{a_i, a_j\}$ and $\{a_p, a_q\}$ have a common member. No three of the differences d_{ij} are equal.*

Proof. Suppose $d_{ij} = d_{pq}$. We can assume that $i > j$ and $p > q$. Without loss of generality we can also assume $p \geq i$. Then $a_i - a_j = a_p - a_q$, so $a_i + a_q = a_j + a_p$. Therefore $a_i = a_q$ and $i = q$ ($a_j = a_p$ is impossible), and $p > i > j$ — the common element is the middle one in order of magnitude.

Now suppose three pairs have the same difference. By the above reasoning there are two possibilities: the pairs must have a common element, or form a triangle. In the former case, suppose the differences are d_{ij}, d_{ik} and $d_{i\ell}$. From $d_{ij} = d_{ik}$ we must have either $k > i > j$ or $j > i > k$; let us assume the former. Then $d_{ij} = d_{i\ell}$ implies that $i > j > \ell$. So j is greater than both k and ℓ. But $d_{ik} = d_{i\ell}$ must mean that either $k > j > \ell$ or $\ell > j > k$, both of which are impossible. On the other hand, suppose the three pairs form a triangle, say $d_{ij} = d_{ik} = d_{jk}$. We can assume $i > j$. Then $d_{ij} = d_{jk}$ implies $i > j > k$, and $j > k$ and $d_{ik} = d_{jk}$ imply $k > i$, again a contradiction. □

Lemma 9.17. *Suppose A is a Sidon sequence of length n. If $d_{ij} = d_{ik}$, then $d_{ij} \leq \frac{1}{2} d_{1n}$.*

Proof. Suppose $d_{ij} = d_{ik}$, and assume $j < k$. Then $d_{jk} = a_k - a_j = a_i - a_j + a_k - a_i = d_{ij} = d_{ik} = 2d_{ij}$. But $a_1 \leq a - j$ and $a_k \leq a_n$, so $d_{jk} \leq d_{1n}$, giving the result. □

Theorem 9.18. *In any Sidon sequence of length n, $\binom{n}{2} \leq \lfloor \frac{3}{2} d_{1n} \rfloor$, or equivalently $d_{1n} \geq \lceil \frac{1}{3} n(n-1) \rceil$.*

Proof. There are $\binom{n}{2}$ unordered pairs of elements in the sequence, so there are $\binom{n}{2}$ differences. From Lemmas 9.16 and 9.17, the collection of values of these differences can contain the integers $1, 2, \ldots, \lfloor \frac{1}{2} r \rfloor$ at most twice each, and $\lfloor \frac{1}{2} r \rfloor + 1, \ldots, d_{1n}$ at most once each. The result follows. □

We now use Sidon sequences to show that only five complete graphs are edge-magic.

Theorem 9.19. *The complete graph K_v does not have an edge-magic total labeling if $v > 6$.*

Proof. One can show that K_7 has no EMTL by a complete search (or see Exercise 9.3.4 below). So we assume $v \geq 8$, and suppose there is an edge-magic total labeling of K_v. The vertex labels will form a Sidon sequence of length v, A say. Let us denote the edge labels by $b_1, b_2, \ldots, b_e$, where $b_1 < b_2 < \cdots < b_e$; of course, $e = \binom{v}{2}$. If the magic sum is k, then

$$\begin{aligned} k &= a_1 + a_2 + b_e && (9.7)\\ &= a_1 + a_3 + b_{e-1} && (9.8)\\ &= a_v + a_{v-1} + b_1 && (9.9)\\ &= a_v + a_{v-2} + b_2. && (9.10) \end{aligned}$$

Subtracting (9.7) from (9.8),

$$a_3 - a_2 = b_e - b_{e-1}, \tag{9.11}$$

while (9.9) and (9.10) yield

$$a_{v-1} - a_{v-2} = b_2 - b_1. \tag{9.12}$$

Suppose labels 1, 2, $v + e - 1$ and $v + e$ are all edge labels. Then $b_1 = 1, b_2 = 2, b_{e-1} = v+e-1$ and $b_e = v+e$. So, from (9.11) and (9.12), $a_3 - a_2 = a_{v-1} - a_{v-2} = 1$. But 2, 3, $v - 2$ and $v - 1$ are all distinct, so this contradicts Lemma 9.16. So one of $1, 2, v + e - 1, v + e$ is a vertex label. Without loss of generality we can assume either 1 or 2 is a vertex label (otherwise, the dual labeling will have this property). So $a_1 = 1$ or 2.

Equations (9.7) and (9.9) give

$$a_v = b_e - (a_{v-1} - a_2) - (b_1 - a_1). \tag{9.13}$$

Since $(a_2, a_3, \ldots, a_{v-1})$ is a Sidon sequence of length $v - 2$ (any subsequence of a Sidon sequence is also well-spread), Lemma 9.18 applies to it, and $(a_{v-1} - a_2) \geq \lceil \frac{1}{3}(v - 2)(v - 3) \rceil$, which is at least 10 because $v \geq 8$. Also $(b_1 - a_1) \geq -1$, $(b - 1$ is at least 1 and a_1 is at most 2), and $b_e \leq v + e$. So, from (9.13),

$$a_v \leq v + e - 9.$$

So the six largest labels are all edge labels:

$$b_{e-5} = v + e - 5, b_{e-4} = v + e - 4, \ldots, b_e = v + e.$$

From (9.7) and (9.8) we get

$$k = a_1 + a_2 + v + e = a_1 + a_3 + v + e - 1,$$

so $a_3 = a_2 + 1$. The next smallest sum of two vertex-labels, after $a_1 + a_2$ and $a_1 + a_3$, may be either $a_2 + a_3$ or $a_1 + a_4$.

If it is $a_2 + a_3$, then

$$k = a_2 + a_3 + v + e - 2$$

and by comparison with (9.8), $a_2 = a_1 + 1$. The next-smallest sum is $a_1 + a_4$, so

$$k = a_1 + a_4 + v + e - 3$$

and $a_4 = a_3 + 2$. Two cases arise. If $a_1 = 1$, then $a_2 = 2, a_3 = 3, a_4 = 5$. Also, a_5 cannot equal 6, because that would imply $a_1 + a_5 = 7 = a_2 + a_4$, contradicting the well-spread property. Every integer up to $v + e$ must occur as a label, so $b_1 = 4$ and $b_2 = 6$. So (9.12) is $a_{v-1} - a_{v-2} = b_2 - b_1 = 2$. But $a_4 - a_3 = 2$, so $d_{v-1,v-2} = d_{34}$, in contradiction of Lemma 9.16. In the other case, $a_1 = 2$, we obtain $a_2 = 3, a_3 = 4, a_4 = 6$, so $b_1 = 1, b_2 = 5$, and $a_{v-1} - a_{v-2} = 4 = a_4 - a_1$, again a contradiction.

If $a_1 + a_4$ is the next-smallest difference, we have

$$k = a_1 + a_4 + v + e - 2,$$

so $a_4 = a_3 + 1$. If $a_1 = 1$ and $a_2 = 3$, it is easy to see that $b_1 = 2, b_2 = 6$, and we get the contradiction $a_{v-1} - a_{v-2} = a_4 - a_1 = 4$. Otherwise $a_2 \geq 4$, so 3 is an edge-label. If $a_1 = 1$, then $b_1 = 2, b_2 = 3$, and $a_{v-1} - a_{v-2} = 1 = a_3 - a_2$. If $a_1 = 2$, then $b_1 = 1, b_2 = 3$, and $a_{v-1} - a_{v-2} = 2 = a_4 - a_2$. In every case, a contradiction is obtained. Therefore we have the result. □

This theorem was first proven in [79]; the above proof follows that in [32].

We know K_4 is not edge-magic; edge-magic total labelings of K_1, K_2 and K_3 are easy to find. One solution for K_5 is to use vertex labels $\{1, 2, 3, 5, 9\}$; one for K_6 is $\{1, 3, 4, 5, 9, 14\}$. (A complete list of solutions is given in [126].)

If $A = (a_1, a_2, \ldots, a_n)$ is any Sidon sequence of length n, we define the *size* of A as $\sigma(A) = a_n - a_1 + 1$. One usually assumes $a_1 = 1$ when constructing a sequence, and then the size equals the largest element. Another useful parameter is

$$\rho(A) = a_n + a_{n-1} - a_2 - a_1 + 1 = \sigma(A) + a_{n-1} - a_2.$$

We denote by $\sigma^*(n)$ and $\rho^*(n)$ the minimum values of $\sigma(A)$ and $\rho(A)$ respectively, taken over all Sidon sequences A of length n.

It is useful to know that there exist Sidon sequences of all positive lengths. The recursive construction $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2}$ gives a well-spread sequence. This is the well-known Fibonacci sequence, except that the standard notation for the Fibonacci numbers has $f_1 = f_2 = 1, f_3 = 2, \ldots$. So we have a well-spread sequence with its largest element equal to the $(n + 1)$-th term of the Fibonacci sequence: $a_n = f_{n+1}$. Therefore $\sigma^*(n) \leq f_{n+1}$, and

$$\rho^*(n) \leq f_{n+1} + f_n - 2 = f_{n+2} - 2.$$

It is well known (see texts such as [19]) that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

so we have

$$\sigma^*(n) \leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

and

$\sigma^*(3) = 3$	$f_4 = 3$	$\rho^*(3) = 3$	$f_5 - 2 = 3$
$\sigma^*(4) = 5$	$f_5 = 5$	$\rho^*(4) = 6$	$f_6 - 2 = 6$
$\sigma^*(5) = 8$	$f_6 = 8$	$\rho^*(5) = 11$	$f_7 - 2 = 11$
$\sigma^*(6) = 13$	$f_7 = 13$	$\rho^*(6) = 19$	$f_8 - 2 = 19$
$\sigma^*(7) = 19$	$f_8 = 21$	$\rho^*(7) = 30$	$f_9 - 2 = 32$
$\sigma^*(8) = 25$	$f_9 = 34$	$\rho^*(8) = 43$	$f_{10} - 2 = 53$
$\sigma^*(9) = 35$	$f_{10} = 55$	$\rho^*(9) = 62$	$f_{11} - 2 = 79$
$\sigma^*(10) = 46$	$f_{11} = 81$	$\rho^*(10) = 80$	$f_{12} - 2 = 134$

Table 9.1. Sidon sequence bounds compared to Fibonacci numbers

$$\rho^*(n) \le \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - 2.$$

However these are not the best-possible values. Table 9.1 shows the results of exhaustive computations.

Suppose G has an edge-magic total labeling λ and G contains a complete subgraph H on n vertices $x_1, x_2, \ldots, x_n$; denote $\lambda(x_i)$ by a_i. Assume $a_1 < a_2 < \cdots < a_n$, so $A = (a_1, a_2, \ldots, a_n)$ is a Sidon sequence of length n. Then

$$\lambda(x_n x_{n-1}) = k - a_n - a_{n-1},$$

and since $\lambda(x_n x_{n-1})$ is a label,

$$k - a_n - a_{n-1} \ge 1. \tag{9.14}$$

Similarly

$$\lambda(x_2 x_1) = k - a_2 - a_1,$$

and since $\lambda(x_2 x_1)$ is a label,

$$k - a_2 - a_1 \le v + e. \tag{9.15}$$

Combining (9.14) and (9.15) we have

$$v + e \ge a_n + a_{n-1} - a_2 - a_1 + 1 = \rho(A) \ge \rho^*(n).$$

Theorem 9.20. [79] *If the edge-magic graph G contains a complete subgraph with n vertices, then the number of vertices and edges in G is at least $\rho^*(n)$.*

Suppose $G = K_n + tK_1$. In other words, G consists of K_n together with t isolated vertices. The smallest t such that G is edge-magic is called the *magic number* $M(n)$. Theorem 9.20 enables us to find a lower bound

$$M(n) \ge \rho^*(n) - n - \binom{n}{2}.$$

(See Exercise 9.9.3.1, and also Exercise 9.9.3.2.)

An edge-magic injection is like an edge-magic total labeling, except that the labels can be any positive integers. We define an $[m]$-edge-magic injection of G to be an edge-magic injection of G in which the largest label is m, and call m the *size* of the injection. The *edge deficiency* $def_e(G)$ of G is the minimum value of $m - v(G) - e(G)$, such that an $[m]$-edge-magic injection of G exists.

Theorem 9.21. *Every graph has an edge-magic injection.*

Proof. Suppose G is a graph with v vertices and e edges. The empty graph is trivially edge-magic, so we assume that G has at least one edge. Let$(a_1, a_2, \ldots, a_v)$ be any Sidon sequence of length v with first element $a_1 = 1$. Define $k = a_{v-1} + 2a_v + 1$.

We now construct a labeling λ as follows. Select any edge of G and label its end-points with a_{v-1} and a_v, and label the remaining vertices with the other members of the Sidon sequence in any order. If xy is any edge, define $\lambda(xy) = k - \lambda(x) - \lambda(y)$. Every edge weight will be equal to k. The smallest edge label will be $k - a_{v-1} - a_v = a_v + 1$, which is greater than any vertex label. If two edge labels were equal, say $\lambda(xy) = \lambda(zt)$, then $\lambda(x) + \lambda(y) = \lambda(z) + \lambda(t)$, and since the labels of vertices are members of a Sidon sequence this implies that $xy = zt$. The vertex labels are distinct by definition. So λ is an edge-magic injection. □

The proof of Theorem 9.21 gives us an upper bound on the deficiency:

Corollary 9.22. *If G is a graph with v vertices and $(a_1, a_2, \ldots, a_v)$ is any Sidon sequence of length v with $a_1 = 1$, then*

$$def_e(G) \leq a_{v-1} + 2a_v - a_2 - v - e(G).$$

Proof. In the above construction, no label can be greater than $k - 1 - a_2$. □

This upper bound will not usually be very good. For example, consider the graph constructed from C_5 by joining two inadjacent vertices. Using the Sidon sequence $(1, 2, 3, 5, 8)$, a labeling with $k = 22$ is obtained, and the best assignment of the sequence to the vertices gives largest label 17, and deficiency 6. However, the graph is actually edge-magic. See Figure 9.7.

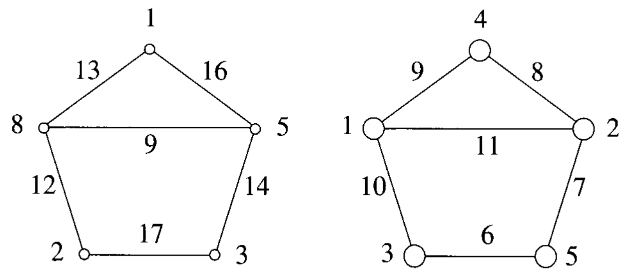

Fig. 9.7. Deficiency 6 on the left; magic on the right

Theorem 9.23. *The edge-magic deficiency of K_v equals the magic number $M(v)$.*

Proof. Consider a edge-magic total labeling λ of $K_v + M(v)K_1$. This graph has $v+M(v)$ vertices and $e(K_v)$ edges, so the largest label is $v+M(v)+e(K_v)$, and clearly this label occurs on a vertex or edge of K_v. The labeling constructed by restricting λ to

K_v is an $[v + M(v) + e(K_v)]$-edge-magic injection of K_v. Obviously any injection of size $v + m + e(K_v)$ gives rise to an edge-magic total labeling of $K_v + mK_1$ (apply the m unused labels to the extra vertices), so $v + M(v) + e(K_v)$ is the smallest possible size, and $def_e(K_v) = M(v)$. □

Exercises 9.3

A9.3.1 Suppose $G = K_n + tK_1$. Prove that if G is edge-magic, then

$$t \geq \rho^*(n) - n - \binom{n}{2}.$$

That is, $M(n) \geq \rho^*(n) - n - \binom{n}{2}$.

A9.3.2 Find an upper bound for $M(n)$. (It does not have to be a good upper bound. The point is to show that *some* upper bound exists.)

9.3.3 If G is an incomplete graph with v vertices and $(a_1, a_2, \ldots, a_v)$ is any Sidon sequence of length v with $a_1 = 1$, prove that

$$def_e(G) < a_{v-1} + 2a_v - a_2 - v - e(G).$$

H9.3.4 (i) Suppose K_v has an edge-magic total labeling with magic sum k. The number p of vertices that receive even labels satisfies the following conditions:

(i) If $v \equiv 0$ or $3 \pmod 4$ and k is even, then $p = \frac{1}{2}(v - 1 \pm \sqrt{v+1})$.

(ii) If $v \equiv 1$ or $2 \pmod 4$ and k is even, then $p = \frac{1}{2}(v - 1 \pm \sqrt{v-1})$.

(iii) If $v \equiv 0$ or $3 \pmod 4$ and k is odd, then $p = \frac{1}{2}(v + 1 \pm \sqrt{v+1})$.

(iv) If $v \equiv 1$ or $2 \pmod 4$ and k is odd, then $p = \frac{1}{2}(v + 1 \pm \sqrt{v+3})$.

(ii) Prove the following necessary conditions for K_v to have an edge-magic total labeling: if $v \equiv 0$ or $3 \pmod 4$, then $v + 1$ is a perfect square; if $v \equiv 1$ or $2 \pmod 4$, then either $v - 1$ is a perfect square and the magic sum of the labeling is even, or $v + 3$ is a perfect square and the magic sum of the labeling is odd.

(iii) Deduce that K_7 is not edge-magic. [118]

9.3.5 Suppose G is a graph with v vertices. Prove that

$$def_e(G) \leq M(v) + \binom{v}{2} - e(G).$$