8

Planarity

8.1 Representations and Crossings

The two diagrams in Figure 8.1 represent the same graph, a K_4 with vertices a, b, c and d. As diagrams they are quite different: in the version $K_4(1)$, the edges ac and bd cross; in $K_4(2)$ there are no crossings. We shall refer to the two diagrams as different *representations* of the graph in the plane. The *crossing number* of a representation is the number of different pairs of edges that cross; the crossing number $\nu(G)$ of a graph G is the minimum number of crossings in any representation of G. A representation is called *planar* if it contains no crossings, and a *planar graph* is a graph that has a planar representation. In other words, a *planar graph* G is one for which $\nu(G) = 0$. Figure 8.1 shows that $\nu(K_4) = 0$.

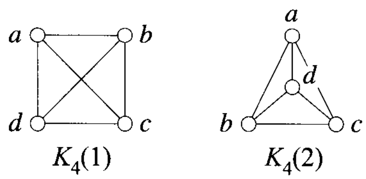


Fig. 8.1. Two representations of K_4

There are many applications of crossing numbers. An early use was in the design of railway yards, where it is inconvenient to have the different lines crossing, and it is better to have longer track rather than extra intersections. An obvious extension of this idea is freeway design. At a complex intersection, fewer crossings means fewer expensive flyover bridges. More recently, small crossing numbers have proven important in the design of VLSI chips; if two parts of a circuit are not to be connected electrically, but they cross, a costly insulation process is necessary.

In 1944, during the Second World War, Turán (see [60]) was forced to work in a brick factory, using hand-pulled carts that ran on tracks to move bricks from kilns to stores. When tracks crossed, several bricks fell from the carts and had to be replaced by hand. The tracks were modeled by a complete bipartite graph with one set of vertices representing kilns and the other representing stores, so to minimize the man-hours lost in replacing bricks, it was necessary to find $\nu(K_{m,n})$ (and find a representation of $K_{m,n}$ that realized the minimum number of crossings). This problem is called "Turán's brick factory problem," and is still unsolved. The best-known bound, which is conjectured to be the best possible, is given in Theorem 8.1.

Theorem 8.1. *The crossing number of $K_{m,n}$ satisfies*

$$\nu(K_{m,n}) \le \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor . \tag{8.1}$$

For a proof, see Exercise 8.1.8.

Kleitman [77] proved that equality holds in (8.1) when $m \le 6$. In particular,

$$\nu(K_{6,n}) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor . \tag{8.2}$$

The crossing numbers of complete graphs also present difficulties. A very simple (but very bad) upper bound is easily found. If the v vertices of K_v are arranged in a circle, and all the edges are drawn as straight lines, then every set of four vertices contributes exactly one crossing. So

$$\nu(K_v) \le \binom{v}{4}.$$

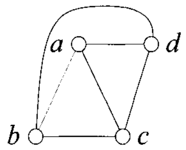
More sophisticated constructions (see, for example, [60]) can be used to prove

Theorem 8.2.

$$\nu(K_v) \le \frac{1}{4} \left\lfloor \frac{v}{2} \right\rfloor \left\lfloor \frac{v-1}{2} \right\rfloor \left\lfloor \frac{v-2}{2} \right\rfloor \left\lfloor \frac{v-3}{2} \right\rfloor .$$

It is easy to calculate the crossing numbers of some small graphs. For example, the crossing number of any tree is 0. To see this, we show how to draw a tree in the plane using ordinary Cartesian coordinates. Select any vertex x of the tree, and represent it by the point with coordinates $(0, 0)$. Suppose the vertices adjacent to x are $y_0, y_2, \ldots$. They are represented by $(0, 1)$, $(1, 1)$, $(2, 1)$, ... If there are k vertices other than x adjacent to y_i, they are represented by $(i, 2)$, $(i + \frac{1}{k}, 2)$, $(i + \frac{2}{k}, 2)$, ..., $(i + \frac{k-1}{k}, 2)$. This process of subdividing continues and provides a representation of the tree with no crossings.

To prove that $\nu(K_5) = 1$, we start by considering smaller complete graphs. The representation of K_3 as a triangle is essentially unique: one can introduce a crossing only by a fanciful, twisting representation of one or more edges. If the K_3 with vertices a, b, c is drawn as a triangle, we obtain a representation of K_4 by introducing a new vertex d and joining it to the other three. If it is inside the triangle, we get the representation shown as $K_4(2)$ in Figure 8.1; if it is outside, it is easy to draw the edges ad, bd, cd so that they do not cross the triangle: for example,



$K_4(3)$

This representation $K_4(3)$ is essentially equivalent to $K_4(2)$, with the vertices relabeled. (Remember, the shape of the edges is unimportant; only the connections matter.). So any planar representation of K_5 can be obtained by introducing another vertex into $K_4(2)$. If a new vertex e is introduced inside the triangle abd, then the representation of ce must cross one of ab, ad or bd. Similarly, the introduction of e inside any triangle causes a crossing involving the edge joining e to the vertex that is not on the triangle. (The "outer area" is considered to be the triangle abc.) So $\nu(K_5) \geq 1$. A representation with one crossing is easy to find, so $\nu(K_5) = 1$.

Another graph with crossing number 1 is $K_{3,3}$; proving this is left as an exercise.

Suppose G is planar. If a new graph were constructed by inserting a new vertex of degree 2 into the middle of an edge (*dividing an edge*), or by deleting a vertex of degree 2 and joining the two vertices adjacent to it (*eliding a vertex*), that new graph will also be planar. Graphs that can be obtained from each other in this way are called *homeomorphic*.

Since K_5 and $K_{3,3}$ are not planar, it follows that a graph having either as a subgraph could not be planar. Moreover, a graph that is homeomorphic to one with a *subgraph* homeomorphic to K_5 or $K_{3,3}$ cannot be planar. In fact, Kuratowski [82] proved that this necessary condition for planarity is also sufficient.

Theorem 8.3. *G is planar if and only if G is homeomorphic to a graph containing no subgraph homeomorphic to K_5 or $K_{3,3}$.*

The proof can be found, for example, in [17] or [66].

Exercises 8.1

H8.1.1 Prove that a graph G is planar if and only if every block of G is planar.

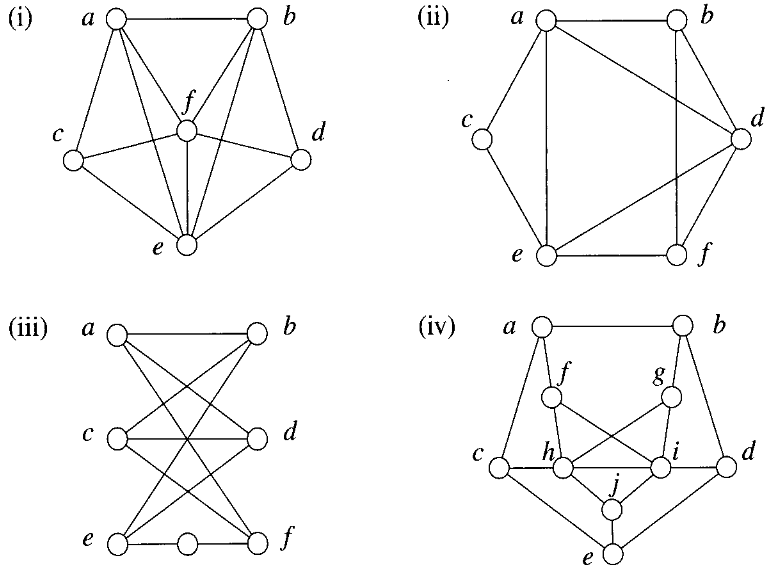
8.1.2 Prove that $\nu(K_{1,n}) = \nu(K_{2,n}) = 0$ for any n.

A8.1.3 Prove that $\nu(K_{3,3}) = 1$.

8.1.4 What is $\nu(K_{3,4})$?

A8.1.5 Prove that the Petersen graph is not planar. What is its crossing number?

8.1.6 Which of the following graphs are planar?



8.1.7 Suppose the graph G has seven or fewer vertices. Prove that either G or its complement $\overline{G}$ is planar.

8.1.8 Consider the complete bipartite graph $K_{m,n}$. In a Cartesian coordinate system, select $\lfloor \frac{m}{2} \rfloor$ points on the positive x-axis, $\lceil \frac{m}{2} \rceil$ points on the negative x-axis, $\lfloor \frac{n}{2} \rfloor$ points on the positive y-axis and $\lceil \frac{n}{2} \rceil$ points on the negative y-axis. The figure formed by joining all pairs with one point on the x-axis and one on the y-axis is a representation of $K_{m,n}$. Use this representation to prove Theorem 8.1.

8.2 Euler's Formula

In each plane representation of a connected graph, the plane is partitioned into regions called *faces*: a face is an area of the plane entirely surrounded by edges of the graph, that contains no edge. It is convenient to define one exterior face, corresponding to the plane outside the representation. For example, the exterior face is $abyz$ in Figure 8.2(a), and $abxyz$ in Figure 8.2(b). The cycle $abxyz$ is not a face in Figure 8.2(a), because it contains the edge bz.

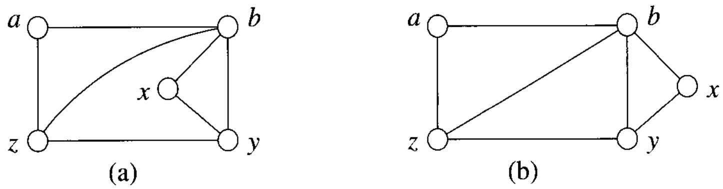


Fig. 8.2. Different faces in different representations

Special care must be taken with bridges. The representations shown in Figure 8.3 have exterior faces $abcx$, $abcxyz$ and abx respectively, and $abcx$ is a face in the third case. A tree has exactly one face.

The edges touching a face are said to *bound* that face, and are collectively called the *boundary*. It is common to think of the boundary as a cycle, but in the case of bridges this need not be true. In particular, any cycle must have at least three edges, but there are three connected graphs (K_1, K_2 and P_3) with faces having boundary smaller than three edges.

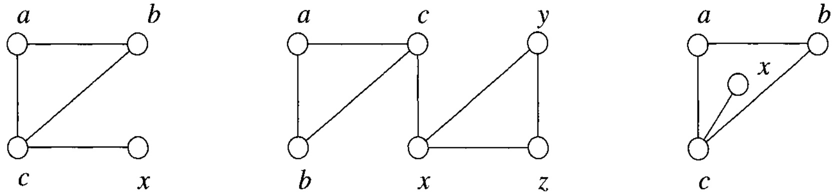


Fig. 8.3. Graphs with bridges

The following theorem was proved by Euler in the eighteenth century.

Theorem 8.4. *Suppose that a plane representation of the connected planar graph G has v vertices, e edges and f faces. Then*

$$v - e + f = 2. \tag{8.3}$$

Proof. By induction on e. It is easy to see that the theorem holds when e is small: if $e = 0$, then the graph must be K_1, which has $v = 1$ and $f = 1$; if $e = 1$ or $e = 2$, we have a path with $e + 1$ vertices and one face. Now assume the theorem is true for all graphs with E or less edges, and suppose $e = E + 1$.

If G is a tree, then $v = e + 1$ and $f = 1$, so

$$v - e + f = e + 1 - e + 1 = 2.$$

Otherwise G contains a cycle. Select an edge that lies in this cycle. That edge will lie separating two faces (one possibly being the exterior face). If it is deleted, one obtains a graph with one fewer edge and one fewer face than the original. It has E edges so, by induction, equation (8.3) is satisfied; in terms of the original graph,

$$v - (e - 1) + (f - 1) = 2,$$

and (8.3) follows. □

Corollary 8.5. *All plane representations of the same connected planar graph have the same number of faces.*

Proof. Suppose a graph has v vertices and e edges; suppose it has two plane representations, with f and f' faces respectively. Then

$$v - e + f = 2 = v - e + f',$$

so $f = f'$. □

Because of Corollary 8.5, one speaks of the number of faces of a connected planar graph, instead of the number of faces in a particular representation of it.

In a planar graph there are various restrictions on the values of v, e and f.

Theorem 8.6. *Suppose a connected planar graph G has v vertices, e edges and f faces, and no component has fewer than three vertices. Then*

$$3f \leq 2e. \tag{8.4}$$

Proof. The result is true for P_2 ($e = 2$, $f = 1$) so we can assume each face of G has at least three edges. The edge-face adjacency matrix A of G is the $e \times f$ matrix with entries a_{ij} defined by

$$a_{ij} = \begin{cases} 1 \text{ if the } i\text{-th edge lies in the boundary of the } j\text{-th face,} \\ 0 \text{ otherwise.} \end{cases}$$

Let σ be the sum of all the entries of A. Each edge bounds at most two faces, so the sum of each row is at most 2. There are e rows, so $\sigma \leq 2e$. As each face has at least three edges in its boundary, $3f \leq \sigma$. The result follows. □

Theorem 8.7. *If a connected planar graph has v vertices and e edges, where $v \geq 3$, then $e \leq 3v - 6$.*

Proof. Suppose the graph has f faces. By Theorem 8.6, $3f \leq 2e$, so $f \leq \frac{2e}{3}$. By Theorem 8.4, $v - e + f = 2$, so $v - e + \frac{2e}{3} \geq 2$. But this implies that $3v - e \geq 6$, and the theorem is proved. □

Example. The complete graph, K_v, is not planar for $v \geq 5$. If $v \geq 5$, then K_v contains K_5 as a subgraph. Hence it is sufficient to show that K_5 is not planar. We already saw this fact in the preceding section, but the following neat proof is now available: Suppose K_5 is planar. We have $v = 5$, $e = 10$, so that $e > 3v - 6$, contradicting Theorem 8.7.

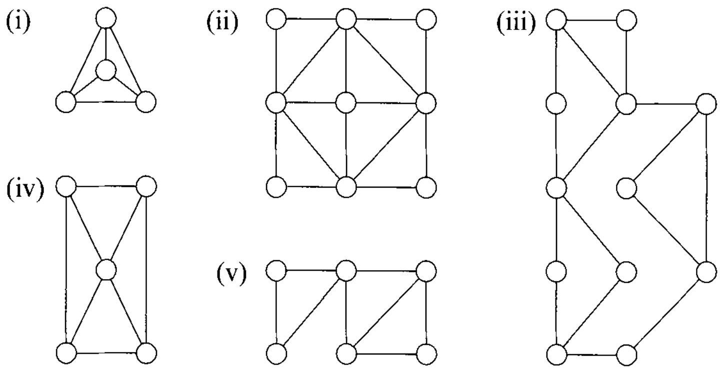
One can refine this example. Suppose G, a graph with v vertices and e edges, can be drawn with c crossings. Suppose you replace each crossing by a new vertex: in other words, if edges xy and zt cross, introduce a new vertex w and replace xy and zt by edges xw, wy, zw and wt. The new graph has $v + c$ vertices and $e + 2c$ edges and is planar, so $e + 2c \leq 3(v + c) - 6$, and $c \geq 6 + e - 3v$. For example, K_6 has 6 vertices and 15 edges, so $c \geq 3$. K_6 can be drawn with 3 crossings, so $\nu(K_6) = 3$.

The following useful consequence of Theorem 8.7 is left as an exercise (see Exercise 8.2.4).

Corollary 8.8. *Every planar graph has at least one vertex of degree smaller than 6.*

Exercises 8.2

8.2.1 Find the values of v, e and f for the following graphs, and verify Euler's formula for them.



H8.2.2 Prove that a connected planar graph with v vertices, $v \geq 3$, has at most $2v - 4$ faces.

8.2.3 Prove that if a connected planar graph has v vertices, $v \geq 3$, and every face in a certain plane representation has four edges, then the graph has $2v - 4$ edges. Hence prove:

(i) If a planar graph with v vertices contains no triangles, then it has at most $2v - 4$ edges;

(ii) The complete bipartite graph $K_{m,n}$ is not planar when $m \geq 3$ and $n \geq 3$.

A8.2.4 Suppose G is a graph with v vertices and e edges, and suppose every vertex of G has degree at least 6. Prove that

$$e \geq 3v.$$

Hence prove that every planar graph contains a vertex of degree at most 5.

8.2.5 Prove that in a planar bipartite graph with v vertices, e edges and f faces,

$$2f \leq e.$$

8.2.6 Verify that K_6 can be drawn with 3 crossings.

8.3 Maps, Graphs and Planarity

By a *map* we mean what is usually meant by a map of a continent (showing countries) or a country (showing states or provinces). However we make one restriction. Sometimes one state can consist of two disconnected parts (in the United States, Michigan consists of two separate land masses, unless we consider man-made constructions such as the Mackinaw Bridge). We exclude such cases from consideration.

Given a map, we can construct a graph as follows: the vertices are the countries or states on the map, and the two vertices are joined by an edge precisely when the corresponding countries have a common border. As an example, Figure 8.4 shows a map of the mainland of Australia, divided into states and territories, and its corresponding graph.

Suppose two states have only one point in common. This happens for example, in the United States, where Utah and New Mexico meet at exactly one point, as do Colorado and Arizona. We shall say that states with only one point in common have no border, and treat them as if they do not touch.

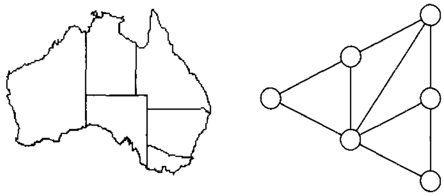


Fig. 8.4. The map and graph of mainland Australia

It is always possible to draw the graph corresponding to a map without crossings. To see this, draw the graph on top of the map by putting a vertex inside each state and joining vertices by edges that pass through common state borders so the graph of any map is planar. Conversely, any planar graph is easily represented by a map. Therefore the theory of maps (with the two stated restrictions) is precisely the theory of planar graphs.

In 1852, William Rowan Hamilton wrote to Augustus de Morgan concerning a problem that had been posed by a student, Frederick Guthrie. (Part of the correspondence is quoted in [14].) Guthrie said: cartographers know that any map (our definition) can be colored using four or less colors; is there a mathematical proof? (Guthrie [59] later pointed out that the question had come from his brother, Francis Guthrie.)

Kempe [75] published a purported proof in 1879. It was thought that the matter was over, but in 1890 Heawood pointed out a fallacy in Kempe's proof. (We explore the problem in Exercise 8.3.3.) Heawood did however repair the proof sufficiently to prove the following weaker result.

Theorem 8.9. [70] *Every planar graph can be colored in five colors.*

Clearly Theorem 8.9 is the five-color analog of Guthrie's four-color map problem.

Proof. We assume the theorem is false, so some planar graphs require six colors. From these, select one that has the minimum number of vertices and has no isolated vertices; call it G. By Corollary 8.8, there is a vertex x in G whose degree is at most 5. $G-x$ has fewer edges than G, so it is 5-colorable. Select a 5-coloring ξ of G. Observe that every color used in ξ must be represented among the neighbors of x: if color c were missing, one could set $\xi(x) = c$ and thus extend ξ to a 5-coloring of G, which is impossible. So $d(x) = 5$. We shall write x_1, x_2, x_3, x_4, x_5 for the five vertices adjacent to x in G, and assume $\xi(x_i) = c_i$. Without loss of generality we shall assume that the vertices x_1, x_2, x_3, x_4, x_5 occur in order around x in some plane representation of G, as shown in Figure 8.5(a).

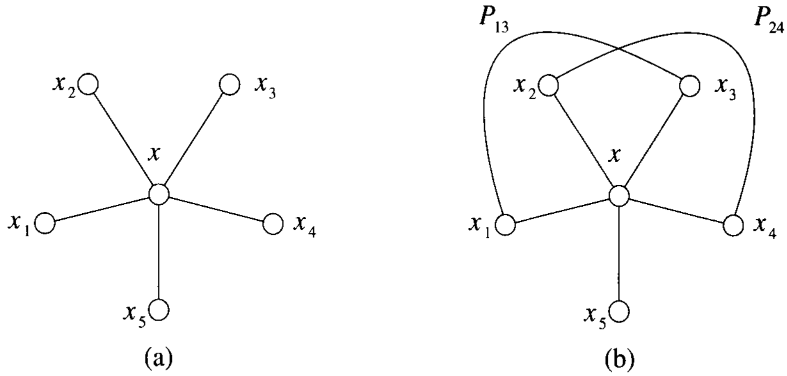


Fig. 8.5. A neighborhood used in the 5-color proof

Consider the induced subgraph G_{ij} of G whose vertex-set consists of all vertices that receive color c_i or color c_j under ξ. If x_i and x_j lie in different components of G_{ij}, then one could exchange colors among all the vertices in one component — say the component containing x_i — and the result would still be a 5-coloring of G. However there would be no vertex of color c_i adjacent to x, so the new coloring could be extended to a 5-coloring of G by allocating c_i to x. This is impossible. Therefore x_i and x_j must lie in the same component of G_{ij}, and there must exist a path P_{ij} from x_i to x_j in G, all of whose vertices receive either c_i or c_j under ξ.

Now consider the paths P_{13} and P_{24}. It is clear from Figure 8.5(b) that these two paths must cross. But the only way two paths in a plane representation can cross is at a common vertex, which is impossible because the vertices in P_{13} do not receive the same colors as those in P_{24}. □

After Heawood's paper appeared, there was renewed interest in the 4-color problem. Because it was easy to state and tantalizingly difficult to prove, it became one of the most celebrated unsolved problems in mathematics, second only to Fermat's Last Theorem. In 1976, Appel and Haken [3] finally proved that any planar graph — and therefore any planar map — can be colored in at most four colors. Their proof involved computer analysis of a large number of cases, so many that human analysis of all the cases is not feasible. We state their theorem as

Theorem 8.10. *Every planar graph can be colored using at most four colors.*

For details of the proof, see [4] or [106].

Exercises 8.3

8.3.1 Find the graph of the mainland provinces and territories of Canada. (Treat Labrador as a province.) How many colors does it need?

A8.3.2 Use the result of Exercise 8.2.4 to prove that any planar graph can be colored in at most six colors (without using any results from this section).

8.3.3 In Kempe's attempted proof of the four color theorem, it is assumed that G is a minimal planar graph requiring five colors, and that x is a vertex of degree 5. A 4-coloring ξ of $G - x$ is chosen. Kempe shows that if the coloring cannot be extended to a 4-coloring of G, it must happen that the neighbors of x receive all four colors under ξ, and are arranged as in Figure 8.6. Vertex x_i receives color c_i; both x_4 and y_4 receive c_4. G_i denotes the subgraph induced by vertices receiving colors c_i or c_j.

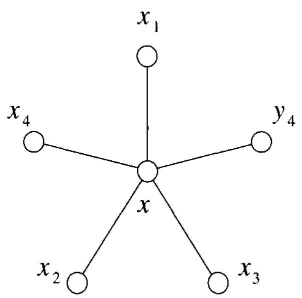


Fig. 8.6. A neighborhood in Kempe's proof

Kempe argues that there must be a path P_{12} from x_1 to x_2 in G_{12}, or else one could interchange colors c_1 and c_2 in the component of G_{12} that contains x_1, and then color x with c_1. Similarly there must be a path P_{13} from x_1 to x_3 in G_{13}.
By planarity, there can be no path from x_2 to y_4 in G_{24}, because such a path would need to cross P_{13}. So one can exchange colors c_2 and c_4 in the component of G_{24} that contains y_4. Similarly, one can exchange colors c_3 and c_4 in the component of G_{324} containing x_4. Now x can receive color c_4.
What is wrong with this argument?