6

Factorizations

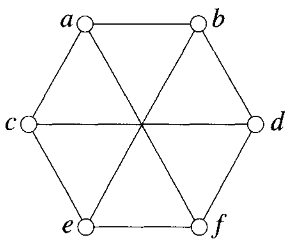
6.1 Definitions; One-Factorizations

If G is any graph, then a *factor* or *spanning subgraph* of G is a subgraph with vertex-set $V(G)$. A *factorization* of G is a set of factors of G that are pairwise *edge-disjoint* — no two have a common edge — and whose union is all of G.

Every graph has a factorization, quite trivially: since G is a factor of itself, $\{G\}$ is a factorization of G. However, it is more interesting to consider factorizations in which the factors satisfy certain conditions. In particular a *one-factor* is a factor that is a regular graph of degree 1. In other words, a one-factor is a set of pairwise disjoint edges of G that between them contain every vertex. Similarly, a *two-factor* in a graph G is a union of disjoint cycles that together contain all vertices of G.

A *one-factorization* of G is a decomposition of the edge-set of G into edge-disjoint one-factors.

Example. We find the one-factors of the graph



Each factor must contain one edge through a, which must be ab, ac or af. If it is ab, the rest of the factor consists of two edges covering $\{c, d, e, f\}$, and the only possibilities are cd, ef and ce, df. So there are two factors containing ab. In the same way we find two factors containing ac, and two with af. The factors are

$$\begin{array}{ccc} ab, cd, ef & ac, be, df & af, bd, de \\ ab, ce, df & ac, bd, cf & af, be, cd. \end{array} \tag{6.1}$$

To find one-factorizations that include ab, cd, ef, we check through this list to find all factors that are edge-disjoint from ab, cd, ef. In this small example it is easy to see that there are exactly two such factors, and together with ab, cd, ef they form a one-factorization that is shown in the first line of (6.1). There is one other one-factorization, shown in the second line.

This graph is one of the two nonisomorphic regular graphs of degree 3 on six vertices. The one-factors and one-factorizations of the other such graph are discussed in Exercise 6.1.3.

Another approach to the study of one-factors is through matchings. A *matching* between sets X and Y is a set of ordered pairs, one member from each of the two sets, such that no element is repeated. Such a matching is a set of disjoint edges of the $K_{m,n}$ with vertex-sets X and Y. The matching is called *perfect* if every member occurs exactly once, so a perfect matching is a one-factor in a complete bipartite graph. One can then define a matching in any graph to be a set of disjoint edges in that graph; in this terminology "perfect matching" is just another phrase for "one-factor."

It is sometimes useful to impose an ordering on the set of one-factors in a one-factorization, or a direction on the edges of the underlying graph. In those cases the one-factorization will be called *ordered* or *oriented* respectively.

Not every graph has a one-factor. In Section 6.3 we shall give a necessary and sufficient condition for the existence of a one-factor in a general graph. For the moment, we note the obvious necessary condition that a graph with a one-factor must have an even number of vertices. However, this is not sufficient; Figure 6.1 shows a 16-vertex graph without a one-factor (see Exercise 6.1.2).

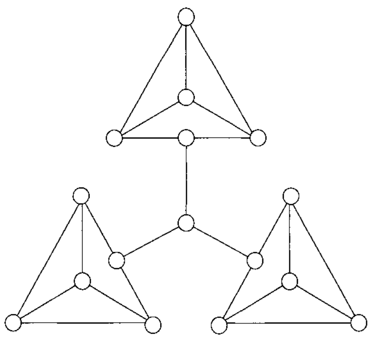


Fig. 6.1. N, the smallest cubic graph without a one-factor

In order to have a one-factorization, a graph not only needs an even number of vertices, but it must also be regular: if G decomposes into d disjoint one-factors, then every vertex of G must lie on precisely d edges. However, the following theorem shows that these conditions are not sufficient.

Theorem 6.1. *A regular graph with a bridge cannot have a one-factorization (except for the trivial case where the graph is itself a one-factor).*

Proof. Consider a regular graph G of degree d, $d > 1$, with a bridge $t = xy$; in $G - e$, label the component that contains x as E and label the component that contains y as F. The fact that t is a bridge implies that E and F are distinct. Suppose G is the edge-disjoint union of d one-factors, $G_1, G_2, \ldots, G_d$; and say G_1 is the factor that contains t. Now t is the only edge that joins a vertex of E to a vertex of F, so every edge of G_2 with one endpoint in E has its other endpoint in E. So G_2 contains an even number of vertices of E. Since G_2 contains every vertex of the original graph, it contains every vertex of E; so E must have an even number of vertices.

On the other hand, consider $G_1 \backslash E$. This contains a number of edges and the isolated vertex y, since t is in G_1. So $G_1 \backslash E$ has an odd number of vertices, and accordingly E has an odd number of vertices. We have a contradiction. □

This theorem can be used, for example, to show that the graph M of Figure 6.2 has no one-factorization, although it is regular and possesses the one-factor $\{ac, be, dg, fi, hj\}$. However, it clearly does not tell the whole story: the Petersen graph P (see Figure 2.4) has no one-factorization, but it also contains no bridge. The Petersen graph *does* contain a one-factor, however. In fact Petersen [94] showed that every bridgeless cubic graph contains a one-factor. We shall give a proof in Section 6.3.

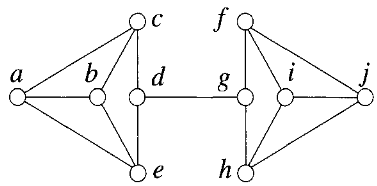


Fig. 6.2. M, the smallest cubic graph without a one-factorization

If the degree increases with the number of vertices, the situation is different. It has been conjectured that a regular graph with $2n$ vertices and degree greater than n will always have a one-factorization; this has only been proved in a very few cases, such as degree $2n - 4$, degree $2n - 5$, and degree at least $12n/7$ (for further details see [103, 25]). On the other hand, one can find regular graphs with degree near to half the number of vertices that do not have one-factorizations.

However, we can prove the existence of one-factorizations in many classes of graphs. Of basic importance are the complete graphs. There are many one-factorizations of K_{2n}. We present one that is usually called $\mathcal{GK}_{2n}$. To understand the construction, look at Figure 6.3. This represents a factor that we shall call F_0. To construct the factor F_1, rotate the diagram through a $(2n - 1)$-th part of a full revolution. Similar rotations provide $F_2, F_3, \ldots, F_{2n-2}$.

Theorem 6.2. *The complete graph K_{2n} has a one-factorization for all n.*

Proof. We label the vertices of K_{2n} as $x_\infty, x_0, x_1, x_2, \ldots, x_{2n-2}$. The spanning subgraph F_i is defined to consist of the edges

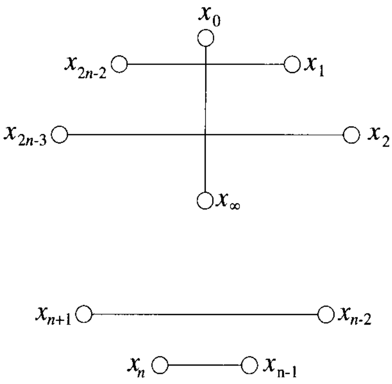


Fig. 6.3. The factor F_0 in $\mathcal{GK}_{2n}$

$$x_\infty x_i,\ x_{i+1}x_{i-1},\ \ldots,\ x_{i+j}x_{i-j},\ \ldots,\ x_{i+n-1}x_{i-n+1} \tag{6.2}$$

where the subscripts other than ∞ are treated as integers modulo $2n - 1$. Then F_i is a one-factor: every vertex appears in the list (6.2) exactly once. We prove that $\{F_0, F_1, \ldots, F_{2n-2}\}$ form a one-factorization. First, observe that every edge involving x_∞ arises precisely once: $x_\infty x_i$ is in F_i. If neither p nor q is ∞, then we can write $p + q = 2i$ in the arithmetic modulo $2n - 1$, because either $p + q$ is even or $p + q + 2n - 1$ is even. Then $q = i - (p - i)$, and $x_p x_q$ is $x_{i+j}x_{i-j}$ in the case $j = p - i$. Since i is uniquely determined by p and q, this means that $x_p x_q$ belongs to precisely one of the F_i. So $\{F_0, F_1, \ldots, F_{2n-2}\}$ is the required one-factorization. □

The complete bipartite graph $K_{n,n}$ is easily shown to have a one-factorization. If $K_{n,n}$ is defined to have vertex-set $\{1, 2, \ldots, 2n\}$ and edge-set $\{(x, y)\colon 1 \le x \le n,\ n + 1 \le y \le 2n\}$, then the factors $F_1, F_2, \ldots, F_n$, defined by

$$F_i = \{(x, x + n + i) : 1 \le x \le n\}$$

(where $x + n + i$ is reduced modulo n to lie between $n + 1$ and $2n$), form a one-factorization. This will be called the *standard factorization* of $K_{n,n}$.

Another important case is the family of cycles C_n: these have a one-factorization if and only if n is even. This fact will be useful — for example, one common way to find a one-factorization of a cubic graph is to find a spanning subgraph that is a union of disjoint even cycles: a Hamilton cycle will suffice. The complement of this subgraph is a one-factor, so the graph has a one-factorization. Reversing this reasoning, the union of two disjoint one-factors is always a union of disjoint even cycles; if the one-factors are not disjoint, the union consists of some even cycles and some isolated edges (the common edges of the two factors).

The use of unions in the preceding paragraph can be generalized. If G and H both have one-factorizations, then so does $G \oplus H$, the factorization being formed by listing all factors in the factorizations of each of the component graphs. At the other extreme, if G has a one-factorization, then so does nG. However, care must be exercised. If G

and H have some common edges, nothing can be deduced about the factorization of $G \cup H$ from factorizations of G and H.

A number of other types of factorization have been studied. One interesting problem is to decompose graphs into Hamilton cycles. Since each Hamilton cycle can be decomposed into two one-factors, such a Hamiltonian factorization gives rise to a special type of one-factorization. We give only the most basic result.

Theorem 6.3. *If v is odd, then K_v can be factored into $\frac{v-1}{2}$ Hamilton cycles. If v is even, then K_v can be factored into $\frac{v}{2} - 1$ Hamilton cycles and a one-factor.*

Proof. First, suppose v is odd: say $v = 2n + 1$. If K_v has vertices $0, 1, 2, \ldots, 2n$, then a suitable factorization is $Z_1, Z_2, \ldots, Z_n$, where

$$Z_i = (0, i, i+1, i-1, i+2, i-2, \ldots, i+j, i-j, \ldots, i+n, 0).$$

(If necessary, reduce integers modulo $2n$ to the range $\{1, 2, \ldots, 2n\}$.)

In the case where v is even, let us write $v = 2n + 2$. We construct an example for the K_v with vertices $\infty, 0, 1, 2, \ldots, 2n$. The factors are the cycles $Z_1, Z_2, \ldots, Z_n$, where

$$Z_i = (\infty, i, i-1, i+1, i-2, i+2, \ldots, i+n-1, \infty),$$

and the one-factor

$$(\infty, 0), (1, 2n), (2, 2n-1), \ldots, (n, n+1). \qquad \square$$

Exercises 6.1

6.1.1 Verify that every connected graph on four vertices, other than $K_{1,3}$, contains a one-factor.

A6.1.2 Prove that the graph N of Figure 6.1 contains no one-factor.

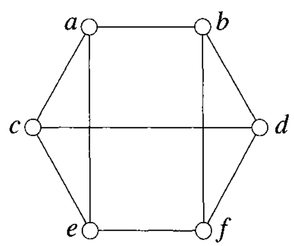
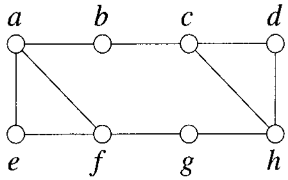


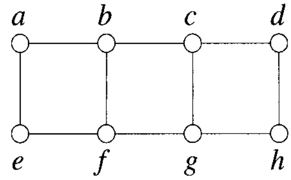
Fig. 6.4. Graph for Exercise 6.1.3

6.1.3 Find all one-factors and one-factorizations in the graph shown in Figure 6.4. Verify that it contains a one-factor that does not belong to any one-factorization.

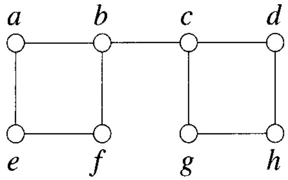
6.1.4 Verify that the Petersen graph has no one-factorization.

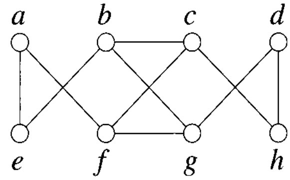
A6.1.5 Find all one-factors in the following graphs.

(i) 

(ii) 

6.1.6 Repeat the preceding exercise for the following graphs.

(i) 

(ii) 

6.1.7 The n-cube Q_n is defined as follows. Q_1 consists of two vertices and one edge; Q_2 is the cycle C_4; in general Q_n is formed by taking two copies of Q_{n-1} and joining each vertex in one copy to the corresponding vertex in the other copy.

(i) How many vertices does Q_n have?

(ii) Prove that Q_n is regular. What is the degree?

(iii) Prove that Q_n has a one-factorization.

(iv) Prove that Q_n has a Hamilton cycle, for $n > 1$.

H6.1.8 Prove that a tree contains at most one one-factor.

6.1.9 What is the number of distinct one-factors in K_{2n}?

HA6.1.10 Suppose G is a connected graph with an even number of vertices, and no *induced* subgraph of G is a star $K_{1,3}$. Prove that G has a one-factor. [112, 113]

6.1.11 In each case either prove the statement or find a counterexample.

(i) If G has a one-factor, then so does its complement $\overline{G}$.

(ii) If G has a one-factorization, then so does its complement $\overline{G}$.

6.1.12 G is a regular graph and every edge of G is contained in at least one one-factor of G. Does G necessarily have a one-factorization? Prove that it does, or provide a counterexample.

6.1.13 We say a two-factor of a graph is of *type* $[a, b, \ldots, c]$ if it consists of one cycle each of lengths $a, b, \ldots, c$. (Repetitions in the list correspond to several cycles of the same length.) For example, a graph on 6 vertices could perhaps have two-factors of types [6] or [3, 3], but no others; for 7 vertices, only [7] and [4, 3]

(i) What are the possible types of two-factors of graphs on 8 vertices?

H(ii) The graph G is formed from K_6 by deleting the edges of a one-factor. Show that G has two-factorizations of the following kinds:

- two factors of type [6];
- one factor of type [6] and one of type [3, 3];

but it has no two-factorization consisting of two factors of type [3, 3].

(iii) Show that K_7 has two-factorizations of the following kinds:

- three factors of type [7];
- three factors of type [4, 3];
- two factors of type [7] and one of type [4, 3];

but it has no two-factorization consisting of two factors of type [4, 3] and one of type [7].

6.2 Tournament Applications of One-Factorizations

Suppose several baseball teams play against each other in a league. The competition can be represented by a graph with the teams as vertices and with an edge xy representing a game between teams x and y. We shall refer to such a league — where two participants meet in each game — as a *tournament*. (The word "tournament" is also used for the directed graphs derived from this model by directing the edge from winner to loser. We shall consider these graphs in Section 11.2.)

Sometimes multiple edges will be necessary; sometimes two teams do not meet. The particular case where every pair of teams plays exactly once is called a *round robin tournament*, and the underlying graph is complete.

A very common situation is when several matches must be played simultaneously. In the extreme case, when every team must compete at once, the set of games held at one time is called a *round*. Clearly the games that form a round form a one-factor in the underlying graph. If a round robin tournament for $2n$ teams is to be played in the minimum number of sessions, we require a one-factorization of K_{2n}, together with an ordering of the factors (this ordering is sometimes irrelevant). If there are $2n-1$ teams, the relevant structure is a near-one-factorization of K_{2n-1}. In each case the (ordered) factorization is called the *schedule* of the tournament.

In many sports a team owns, or regularly plays in, one specific stadium or arena. We shall refer to this as the team's "home field." When the game is played at a team's home field, we refer to that team as the "home team" and the other as the "away team." Often the home team is at an advantage; and more importantly, the home team may receive a greater share of the admission fees. So it is usual for home and away teams to be designated in each match. We use the term *home-and-away schedule* (or just *schedule*) to refer to a round robin tournament schedule in which one team in each game is labeled the home team and one the away team. Since this could be represented by orienting the edges in the one-factors, a home-and-away schedule is equivalent to an oriented one-factorization. It is very common to conduct a double round robin, in which every team plays every other team twice. If the two matches for each pair of teams are arranged so that the home team in one is the away team in the other, we shall say the schedule and the corresponding oriented one-factorization of $2K_{2n}$ are *balanced*.

For various reasons one often prefers a schedule in which runs of successive away games and runs of successive home games do not occur (although there are exceptions: an east coast baseball team, for example, might want to make a tour of the west, and play several away games in succession). We shall define a *break* in a schedule to be a pair of successive rounds in which a given team is at home, or away, in both rounds. A schedule is *ideal* for a team if it contains no break for that team. Oriented factorizations are called ideal for a vertex if and only if the corresponding schedules are ideal for the corresponding team.

Theorem 6.4. [130] *Any schedule for* $2n$ *teams is ideal for at most two teams.*

Proof. For a given team x, define its *ground vector* v^x to have $v_j^x = 1$ if x is home in round j and $vv_j^x = 0$ if x is away in round j. If teams x and y play in round j, then $v_j^x \neq v_j^x$, so different teams have different ground vectors. But the ground vector corresponding to an ideal schedule must consist of alternating zeroes and ones. There are only two such vectors possible, so the schedule can be ideal for at most two teams. □

The following theorem shows that the theoretically best possible case can be attained.

Theorem 6.5. [130] *There is an oriented one-factorization of* K_{2n} *with exactly* $2n-2$ *breaks.*

Proof. We orient the one-factorization $\mathcal{P} = \{P_1, P_2, \ldots, P_{2n-1}\}$ based on the set $\{\infty\} \cup \mathbb{Z}_{2n-1}$, defined by

$$P_k = \{(\infty, k)\} \cup \{(k+i, k-i) : 1 \leq i \leq n-1\}. \tag{6.3}$$

Edge (∞, k) is oriented with ∞ at home when k is even and k at home when k is odd. Edge $(k+i, k-i)$ is oriented with $k-i$ at home when i is even and $k+i$ at home when i is odd.

It is clear that ∞ has no breaks. For team x, where x is in $\mathbb{Z}_{2n-1}$, we can write $x = k + (x-k) = k - (k-x)$. The way that x occurs in the representation (6.3) will be: as x when $k = x$, as $k+(x-k)$ when $1 \leq x-k \leq n-1$, and as $k-(k-x)$ otherwise. The rounds other than P_x where x is at home are the rounds k where $x-k$ is odd and $1 \leq x-k \leq n-1$, and the rounds k where $k-x$ is even and $1 \leq k-x \leq n-1$. It is easy to check that factors P_{2j-1} and P_{2j} form a break for symbols $2j-1$ and $2j$, and these are the only breaks. □

Exercises 6.2

A6.2.1 Two chess clubs, each of n members, wish to play a match over n nights. Each player will play one game per night against a different opponent from the other team. What mathematical structure is used? Give an example for $n = 4$.

6.2.2 $v = 3n$ card players wish to play for several nights. Each night, players sit three at a table, and play together for the full session. No two players are to play together (at the same table) on two nights. The players wish to play for as many nights as possible.

A(i) Describe the problem in terms of graph factorizations.

(ii) Prove that no more than $\lfloor \frac{v-1}{2} \rfloor$ nights of play are possible.

(iii) Show that the maximum can be achieved by nine players.

(iv) Show that twelve players cannot achieve five nights of play.

If $v \equiv 3 \bmod 6$, an optimal solution for v players is called a *Kirkman triple system*, and one exists for all such v. An optimal solution when $v \equiv 0 \bmod 6$ is called a *nearly Kirkman triple system*, and one exists for all v except 6 and 12. See, for example, [80, 123].

6.3 A General Existence Theorem

If W is any subset of the vertex-set $V(G)$ of a graph or multigraph G, we write $G - W$ to denote the graph constructed by deleting from G all vertices in W and all edges touching them. One can discuss the components of $G - W$; they are either *odd* (have an odd number of vertices) or *even*. Let $\Phi_G(W)$ denote the number of odd components of $G - W$.

Theorem 6.6. [115] *G contains a one-factor if and only if*

$$\Phi_G(W) \leq |W| \text{ whenever } W \subset V(G). \tag{6.4}$$

Proof (after [83].) First, suppose G contains a one-factor F. Select a subset W of $V(G)$, and suppose $\Phi_G(W) = k$; label the odd components of $G - W$ as $G_1, G_2, \ldots, G_k$. Since G_i has an odd number of vertices, $G_i \backslash F$ cannot consist of $\frac{1}{2}|G_i|$ edges; there must be at least one vertex, say x, of G_i that is joined by F to a vertex y_i that is not in G_i. Since components are connected, y_i must be in W. So W contains at least the k vertices $y_1, y_2, \ldots, y_k$, and

$$k = \Phi_G(W) \leq |W|.$$

So the condition is necessary.

To prove sufficiency, we assume the existence of a graph G that satisfies (6.4) but has no one-factor; a contradiction will be obtained. If such a G exists, we could continue to add edges until we reached a maximal graph G^* such that no further edge could be added without introducing a one-factor. (Such a maximum exists: it follows from the case $W = \emptyset$ that G has an even number of vertices; if we could add edges indefinitely, eventually an even-order complete graph would be reached.)

Moreover the graph G^* also satisfies (6.4) — adding edges may reduce the number of odd components, but it cannot increase them. So there is no loss of generality in assuming that G is already maximal. We write U for the set of all vertices of degree $|V(G)|-1$ in G. If $U = V(G)$, then G is complete, and has a one-factor. So $U \neq V(G)$. Every member of U is adjacent to every other vertex of G. We first show that every component of $G - U$ is a complete graph. Let G_1 be a component of $G - U$ that is not complete. Not all vertices of G_1 are joined, but G_1 is connected, so there must exist two vertices at distance 2 in G_1, say x and z; let y be a vertex adjacent to both. Since $d(y) \neq |V(G)| - 1$ there must be some vertex t of G that is not adjacent to y, and t is in $V(G - U)$ (since every vertex, y included, is adjacent to every member of U). So $G - U$ contains the configuration shown in Figure 6.5 (a dotted line means "no edge").

Since G is maximal, $G + xz$ has a one-factor — say F_1 — and $G + yt$ also has a one-factor — call it F_2. Clearly xz belongs to F_1 and yt belongs to F_2; and also xz is not in F_2.

Consider the graph $F_1 \cup F_2$; let H be the component that contains xz. As xz is not in F_2, H is a cycle of even length made up of alternate edges from F_1 and F_2. Either yt belongs to H or else yt belongs to another even cycle of alternate edges. These two cases are illustrated in Figure 6.6.

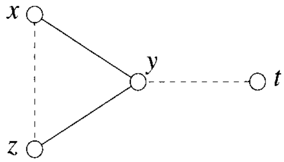


Fig. 6.5. A subgraph arising in the proof of Theorem 6.6

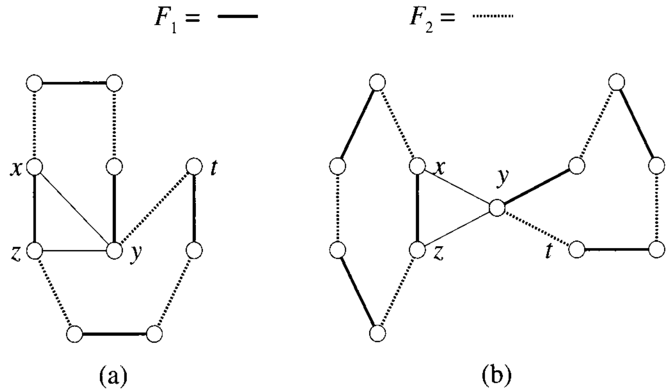


Fig. 6.6. Two cases needed in Theorem 6.6

In Case (a) we can assume that vertices x, y, t and z appear in that order along the cycle H, as shown in the figure (if not, interchange x and z). Then we can construct a one-factor of G as follows: take the edges of F_1 in the section $y, t, \ldots, z$ of H, the edges of F_2 in the rest of G, and yz. In Case (b) we can construct a one-factor in G by taking the edges of F_1 in H and the edges of F_2 in $G - H$. So in both cases we have a contradiction. So G_1 cannot exist — every component of $G - U$ must be complete.

Now $\Phi_G(U) \le |U|$, so $G - U$ has at most $|U|$ odd components. We associate with each odd component G_i of $G - U$ a different member u_i of U. We then construct a one-factor of G as follows. From every even component of $G - U$ select a one-factor (possible because the components are complete graphs); for each odd component G_i, select a one-factor of $G_i + u_i$ (again, $G_i + u_i$ is an even-order complete graph). Since G has an even number of vertices, there will be an even number of vertices left over, all in U; as they are all connected, a one-factor can be chosen from them. The totality is a one-factor in G, contradicting the hypothesis. □

Suppose G has v vertices and suppose the deletion of the w-set W of vertices results in a graph with k odd components; then $v - w \equiv k \pmod 2$. If v is even, then $w \equiv k \pmod 2$, and $k > w$ will imply that $k \ge w + 2$. So we have a slight improvement on Theorem 6.6.

Theorem 6.7. [121] *If the graph or multigraph G has an even number of vertices, then G has no one-factor if and only if there is some w-set W of vertices such that $G - W$ has at least $w + 2$ odd components.*

As an application of Theorem 6.6 we prove the following result.

Theorem 6.8. [31] *If n is even, then any regular graph of degree $n - 1$ on $2n$ vertices has a one-factor.*

Proof. If $n = 2$ or $n = 4$, then the result is easily checked by considering all cases. So we assume G is a regular graph of degree $n - 1$ on $2n$ vertices, where n is even and $n > 4$, and W is any set of w vertices of G, and we prove that the graph $G - W$ has at most w odd components.

If $w \geq n$, then $G - W$ has at most w vertices, so it has at most w components. If $w = 0$, then G has an odd component, which is impossible since w has odd degree. So we assume $1 \leq w \leq n - 1$.

The deletion of W cannot reduce the degree of a vertex of G by more than w, so every vertex of $G - W$ has degree at least $n - 1 - w$, and each component has at least $n - w$ vertices. If there are $w + 1$ or more components, then $G - W$ has at least $(w + 1)(n - w)$ vertices, and

$$(w + 1)(n - w) \leq 2n - w,$$

which simplifies to

$$w^2 - wn + n \geq 0.$$

For fixed n, this is a quadratic inequality which will have the solution

$$w \leq w_1 \text{ or } w \geq w_2$$

where w_1 and w_2 are the roots of $w^2 - wn + n = 0$: that is,

$$w_1 = \tfrac{1}{2}(n - \sqrt{n^2 - 4n}), \quad w_2 = \tfrac{1}{2}(n + \sqrt{n^2 - 4n}).$$

Now when $n > 4$,

$$n - 3 < \sqrt{n^2 - 4n} < n - 2,$$

so

$$1 < w_1 < 2, \quad n - 2 < w_2 < n - 1,$$

and the only integer values in the range $1 \leq w \leq n - 1$ that satisfy the inequality are $w = 1$ and $w = n - 1$.

If $w = 1$, then $G - W$ has every vertex of degree at least $n - 2$, so every component has at least $n - 1$ vertices. The only possible case is two components, one with $n - 1$ vertices and one with n. Only one is odd, so $G - W$ has at most w $(= 1)$ odd components.

If $w = n - 1$, then $G - W$ has $n + 1$ vertices. To get $w + 1$ odd components, the only possibility is $n + 1$ components, each of one vertex. So $G - W$ is empty: deletion of W has removed all $n(n - 1)$ edges of G. Since each vertex of W had degree $n - 1$, at most $(n - 1)^2$ edges can have been removed, which is a contradiction. □

Another application is Petersen's theorem that every bridgeless cubic graph contains a one-factor, which we promised in Section 6.1. In fact we prove the more general result, due to Schönberger [108], that there is a factor containing any specified edge. This obviously has Petersen's theorem as a corollary.

If W is any set of vertices of a graph G, define $z_G(W)$ to be the number of edges of G with precisely one endpoint in W.

Lemma 6.9. *If G is a regular graph of degree d and S is any set of vertices of G, then*

$$d|S| = 2e(\langle S\rangle) + z_G(S). \tag{6.5}$$

Proof. The sum of the degrees of vertices in S, and $2e(\langle S\rangle)$ is the contribution from edges with both endpoints in S; (6.5) follows. □

Corollary 6.10. *If G is a bridgeless cubic graph and S is a set of vertices of G whose order is odd, then $z_G(S) \geq 3$.*

Proof. Apply (6.5). Since d and $|S|$ are both odd, $z_G(S)$ must be odd also. If $z_G(S)$ $= 1$, then the unique edge joining S to $G - S$ would be a bridge, which is impossible in G. So $z_G(S) \geq 3$. □

Theorem 6.11. [108] *If G is a bridgeless cubic graph and e is any edge of G, then G has a one-factor that contains e.*

Proof. Suppose $e = xy$. We prove that $H = G - \{x, y\}$ has a one-factor. Then this factor together with xy is the required one-factor.

Suppose H has no one-factor. From Theorem 6.7, $\Phi_H(X) \geq |X| + 2$ for some subset X of $V(H)$. So

$$\Phi_G(W) \geq |W|$$

where W is the subset $X \cup \{x, y\}$ of $V(G)$. $\langle W\rangle$ contains at least one edge, xy, so from (6.5)

$$z_G(W) \leq 3|W| - 2. \tag{6.6}$$

Now if S is any odd component of $G - W$, then $z_G(S) \geq 3$, and the three edges coming from S must all have their other endpoints in W, so the number of edges into W from outside is at least 3 times the number of such subsets:

$$z_G(W) \geq 3\Phi_G(W) \geq 3|W|, \tag{6.7}$$

so (6.6) and (6.7) together give a contradiction. □

Exercises 6.3

A6.3.1 Does Theorem 6.8 apply to multigraphs?

6.3.2 Prove that there are exactly three regular graphs of degree 3 on eight vertices (up to isomorphism), and that each has a one-factor.

HA6.3.3 Prove that if a cubic graph has fewer than three bridges, then it has a one-factor. [108]

6.3.4 Suppose G is a regular graph of degree d that has a one-factorization. Prove that $z_G(W) \geq d-1$ for every odd-order subset W of $V(G)$. Is this necessary condition sufficient?

6.3.5 G is n-connected, regular of degree n, and has an even number of vertices. Prove that G has a one-factor.

6.3.6 Suppose G is a graph with $2n$ vertices and has minimum degree $\delta < n$. Prove that if

$$2n > \binom{\delta}{2} + \binom{2n - 2\delta - 1}{2} + \delta(2n - \delta),$$

then G has a one-factor. [17]

6.3.7 G is a cubic graph without a bridge. Prove that if xy is any edge of G, then G contains a one-factor that does *not* include xy.

6.3.8 Suppose G is a r-connected graph ($r \geq 1$) with an even number of vertices, and no *induced* subgraph of G is a star $K_{1,r+1}$. Prove that G has a one-factor. [113] (Compare with Exercise 6.1.10.)

6.4 Graphs Without One-Factors

Suppose G is a regular graph of degree d and suppose $G - W$ has a component with p vertices, where p is no greater than d. The number of edges within the component is at most $\frac{1}{2}p(p-1)$. This means that the sum of the degrees of these p vertices in $G - W$ is at most $p(p-1)$. But in G each vertex has degree d, so the sum of the degrees of the p vertices is pd, whence the number of edges joining the component to W must be at least $pd - p(p-1)$. For fixed d and for integer p satisfying $1 \leq p \leq d$, this function has minimum value d (achieved at $p = 1$ and $p = d$). So any odd component with d or fewer vertices is joined to W by d or more edges.

We now assume that G is a regular graph of degree d on v vertices, where v is even. If G has no one-factor, then by Theorem 6.7 there is a set W of w vertices whose deletion leaves at least $w + 2$ odd components. We call a component of $G - W$ *large* if it has more than d vertices, and *small* otherwise. The numbers of large and small components of $G - W$ are α_W and β_W, or simply α and β, respectively. Clearly

$$\alpha + \beta \geq w + 2. \tag{6.8}$$

There are at least d edges of G joining each small component of $G - W$ to W, and at least one per large component, so there are at least $\alpha + d\beta$ edges attached to the vertices of W; by regularity we have

$$\alpha + d\beta \leq wd. \tag{6.9}$$

Each large component has at least $d + 1$ vertices if d is even, and at least $d + 2$ if v is odd, so

$$v \geq w + (d+1)\alpha + \beta \text{ if } d \text{ is even,} \tag{6.10}$$

$$v \geq w + (d+2)\alpha + \beta \text{ if } d \text{ is odd.} \tag{6.11}$$

Since α is nonnegative, (6.9) yields $\beta \le w$, so from (6.8) we have $\alpha \ge 2$; but applying this to (6.9) again we get $\beta < w$, so from (6.8) we have $\alpha \ge 3$. So from (6.10) and (6.11) we see that if $w \ge 1$, then

$$v \ge 3d + 4 \text{ if } d \text{ is even,} \tag{6.12}$$

$$v \ge 3d + 7 \text{ if } d \text{ is odd.} \tag{6.13}$$

In the particular case $d = 4$, the bound in (6.12) cannot be attained. Suppose P is an odd component of $G - W$ with p vertices. Then the sum of the vertices in G of members of P is $4p$, which is even. On the other hand, if there are r edges joining W to P in G and s edges internal to W, the sum of the degrees is $r + 2s$. So r is even. So there are at least two edges from each large component to W, and (6.9) can be strengthened to

$$2\alpha + 4\beta \le 4t,$$

whence $2\beta \le 4k - (2\alpha + 2\beta)$; substituting from (6.8) we get

$$2\beta \le 2k - 4$$

and $\alpha \ge 4$. So (6.10) yields

$$v \ge 2 + 5{\cdot}4 = 22.$$

Summarizing this discussion, we have:

Theorem 6.12. [121] *If a regular graph G with an even number v of vertices and with degree d has no one-factor and no odd component, then*

$$\begin{array}{lll} v \ge 3d + 7 & \text{if} & d \text{ is odd, } d \ge 3; \\ v \ge 3d + 4 & \text{if} & d \text{ is even, } d \ge 6; \\ v \ge 22 & \text{if} & d = 4. \end{array} \tag{6.14}$$

The condition of "no odd component" is equivalent to the assumption that $w \ge 1$. The cases $d = 1$ and $d = 2$ are omitted, but in fact every graph with these degrees that satisfies the conditions has a one-factorization.

It follows from the next result that Theorem 6.12 is best possible.

Theorem 6.13. *If v is even and is at least as large as the bound of Theorem 6.12, then there is a regular graph of the relevant degree on v vertices that has no one-factor.*

Proof. We use two families of graphs. The graph $G_1(h, k, s)$ has $2s + 1$ vertices, and is formed from K_{2s+1} as follows. First factor K_{2s+1} into Hamilton cycles, as in Theorem 6.3. Then take the union of $h - 1$ of those cycles. Finally, take another of the cycles, delete k disjoint edges from it, and adjoin the remaining edges to the union. This construction is possible whenever $0 < h \le s$ and $0 \le k \le s$. If a graph has $2s + 1$ vertices, of which $2k$ have degree $2h - 1$ and the rest have degree $2h$, let us call it a 1-(h, k, s) graph; for our purposes, the essential property of $G_1(h, k, s)$ is that it is a 1-(h, k, s) graph.

The graph $G_2(h, k, s)$ has $2s + 1$ vertices. We construct it by taking a Hamilton cycle decomposition of K_{2s+1}, and taking the union of h of the cycles. Then another cycle is chosen from the factorization; from it are deleted paths with $2k+1$ vertices and

$s - k$ edges that contain each of the remaining $2s - 2k$ vertices once each. We define a 2-(h, k, s) graph to be a graph on $2s + 1$ vertices with $2k - 1$ vertices of degree $2h$ and the rest of degree $2h + 1$. Then $G_2(h, k, s)$ is a 2-(h, k, s) graph whenever $0 < h < s$ and $0 < k \le s$.

Finally, we define the composition $[G, H, J]$ of three graphs G, H and J, each of which have some vertices of degree d and all other vertices of degree $d - 1$, to be the graph formed by taking the disjoint union of G, H and J and adding to it a vertex x and edges joining all the vertices of degree $d - 1$ in G, H and J to x. It is clear that if d is even, say $d = 2h$, and $h \ge 3$, then

$$[G_1(h, 1, h), G_1(h, 1, h), G_1(h, h - 2, h + t)]$$

is a connected regular graph of degree d on $3d + 4 + 2t$ vertices that has no one-factor, since the deletion of x results in three odd components. Similarly, if $d = 2h + 1$, then

$$[G_2(h, 1, h + 1), G_2(h, 1, h + 1), G_2(h, h, h + t + 1)]$$

is a connected regular graph of degree d on $3d + 7 + 2t$ vertices that has no one-factor.

The first construction does not give a connected graph when $h = 2$, so another construction is needed for the case $d = 4$. Take three copies of $G_1(2, 1, 2)$, each of which has five vertices, two of degree 3 and three of degree 4, and one copy of $G_1(2, 1, 2 + t)$, which has two vertices of degree 3 and $3 + 2t$ of degree 4. Add two new vertices, x and y; join one vertex of degree 3 from each of the four graphs to x and the other to y. The result has $22 + 2t$ vertices, degree 4 and no one-factor. □

Exercises 6.4

6.4.1 Prove that $G_1(h, k, h)$ is uniquely defined up to isomorphism and that it is the only 1-(h, k, h) graph. Is the corresponding result true of 2-$(h, k, h + 1)$-graphs?

A6.4.2 What is the smallest value s such that there is a 1-$(h, 1, s)$ graph not of the type $G_1(h, 1, s)$?

A6.4.3 Consider the graph G constructed as follows. Take two copies of K_5. Select one edge from each; join the four endpoints of those edges to a new vertex. Then delete the two edges, and delete one further edge from one K_5. Prove that G is an 1-$(2, 1, 5)$ graph but is not of type $G_1(2, 1, 5)$.