10

Ramsey Theory

10.1 The Graphical Case of Ramsey's Theorem

Suppose the edges of a graph G are painted in k colors. We say a subgraph H of G is *monochromatic* if all its edges receive the same color. We say a k-painting of G is *proper* with respect to H if G contains no monochromatic subgraph isomorphic to H in that painting. If no subgraph is specified, "proper" will mean proper with respect to triangles — graphs isomorphic to K_3.

For example, suppose G is a complete graph and its vertices represent people at a party. An edge xy is colored red if x and y are acquaintances, and blue if they are strangers. An old puzzle asks: given any six people at a party, prove that they contain either a set of three mutual acquaintances or a set of three mutual strangers. In graph-theoretic terms, the puzzle asks for a proof that there is no proper 2-painting of K_6.

To observe that the result is *not* true for fewer than six people, consider the complete graph K_5. It is easy to see that K_5 has a proper 2-painting: take all edges of a copy of C_5 in red and all other edges (they will form another copy of C_5) in blue. (See Figure 10.1.)

On the other hand, there is no proper 2-painting of K_6. To see this, select a vertex x in any 2-painting of K_6. There are five edges touching x, so there must be at least three of them that receive the same color, say red. Suppose xa, xb and xc are red edges. Now consider the triangle abc. If ab is red, then xab is a red triangle. Similarly, if ac or bc is red, there will be a red triangle. But if none are red, then all are blue, and abc is a blue triangle.

This proves that any 2-painting of K_v must contain a monochromatic triangle whenever $v \geq 6$: if $v > 6$, simply delete all but six vertices. The resulting 2-painted K_6 must contain a monochromatic triangle, and that triangle will also be a monochromatic triangle in K_v.

The same argument can be used when there are more than two colors, and applies to general graphs, not only to triangles. The general result is the graphical version of Ramsey's theorem. We first prove a particular case.

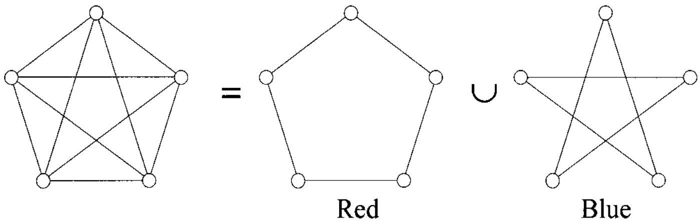

Fig. 10.1. Proper 2-painting of K_5

Lemma 10.1. *There exists a number $R(p, q)$ such that any painting of $K_{R(p,q)}$ in two colors c_1 and c_2 must contain either a K_p with all its edges in color c_1 or a K_q with all its edges in c_2.*

Proof. We proceed by induction on $p+q$. The lemma is clearly true when $p+q = 2$, since the only possible case is $p = q = 1$ and obviously $R(1, 1) = 1$. Suppose it is true whenever $p + q < N$, for some integer N. Consider any two positive integers P and Q that add to N. Then $P + Q - 1 < N$, so both $R(P - 1, Q)$ and $R(P, Q - 1)$ exist.

Consider any painting of the edges of K_v in two colors c_1 and c_2, where $v \geq R(P-1, Q)+R(P, Q-1)$, and select any vertex x of K_v. Then x must either lie on $R(P - 1, Q)$ edges of color c_1 or on $R(P, Q - 1)$ edges of color c_2. In the former case, consider the $K_{R(P-1,Q)}$ whose vertices are the vertices joined to x by edges of color c_1. Either this graph contains a K_{P-1} with all edges of color c_1, in which case this K_{P-1} together with x forms a K_p with all edges in c_1, or it contains a K_Q with all edges in c_2. In the latter case, the K_v again contains one of the required monochromatic complete graphs. So $R(P, Q)$ exists, and in fact $R(P, Q) \leq R(P, Q-1)+R(P-1, Q)$. □

Theorem 10.2. *Suppose $H_1, H_2, \ldots, H_k$ are any k graphs. Then there exists an integer $R(H_1, H_2, \ldots, H_k)$ such that whenever $v \geq R(H_1, H_2, \ldots, H_k)$, any k-painting of K_v must contain a subgraph isomorphic to H_i that is monochromatic in color i, for some i, $1 \leq i \leq k$.*

The numbers $R(H_1, H_2, \ldots, H_k)$ are called *Ramsey numbers*. In particular, if all the H_i are complete graphs, say $H_1 = K_{p_1}, H_2 = K_{p_2}, \ldots$, then the Ramsey number $R(K_{p_1}, K_{p_2}, \ldots, K_{p_k})$ is written $R(p_1, p_2, \ldots, p_k)$. If the p_i are all equal, with common value p, the notation $R_k(p)$ is used.

Proof of Theorem 10.2. It is sufficient to prove the theorem in the case where all the H_i are complete. Then, if v is sufficiently large that a k-painted K_v must contain a monochromatic $K_{v(H_i)}$ in color c_i, for some i, it must certainly contain a monochromatic copy of H_i in color c_i, so

$$R(H_1, H_2, \ldots, H_k) \leq R(v(H_1), v(H_2), \ldots, v(H_k)).$$

We proceed by induction on k to prove that $R(p_1, p_2, \ldots, p_k)$ exists for all parameters. In the case $k = 2$, the result follows from Lemma 10.1. Now suppose it is true

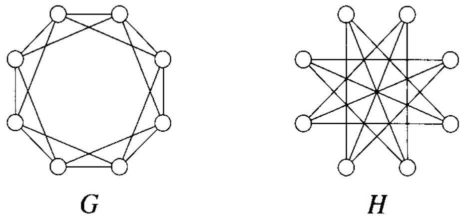

Fig. 10.2. Decomposition of K_8 proving $R(3,4) \geq 9$

for $k < K$, and suppose integers $p_1, p_2, \ldots, p_K$ are given. Then $R(p_1, p_2, \ldots, p_{K-1})$ exists.

Suppose

$$v \geq R(R(p_1, p_2, \ldots, p_{K-1}), p_K).$$

Select any k-painting of K_v. Then recolor by assigning a new color c_0 to all edges that received colors other than c_k. The resulting graph must contain either a monochromatic $K_{R(p_1,p_2,\ldots,p_{K-1})}$ in color c_0 or a monochromatic K_{p_K} in color c_K. In the former case, the corresponding $K_{R(p_1,p_2,\ldots,p_{K-1})}$ in the original painting has edges in the $K-1$ colors $c_1, c_2, \ldots, c_{K-1}$ only, so by induction it contains a monochromatic K_{p_i} in color c_i for some i. □

In discussing individual small Ramsey numbers, it is often useful to consider the graphs whose edges are precisely those that receive a given coloring in a painting of a complete graph. These are called the *monochromatic subgraphs*.

As an example, consider $R(3,4)$. Suppose K_v has been colored in red and blue so that neither a red K_3 nor a blue K_4 exists. Select any vertex x. Define R_x to be the set of vertices connected to x by red edges — that is, R_x is the neighborhood of x in the red monochromatic subgraph, and similarly define B_x in the blue monochromatic subgraph.

If $|R_x| \geq 4$, then either $\langle R_x \rangle$ contains a red edge yz, whence xyz is a red triangle, or else all of its edges are blue, and there is a blue K_4. So $|R_x| \leq 3$ for all x.

Next suppose $|B_x| \geq 6$. Then $\langle B_x \rangle$ is a complete graph on six or more vertices, so it contains a monochromatic triangle. If this triangle is red, it is a red triangle in K_9. If it is blue, then it and x form a blue K_4 in K_9.

It follows that every vertex x has $|R_x| \leq 3$ and $|B_x| \leq 5$, so $v \leq 9$. But $v = 9$ is impossible. If $v = 9$, then $|R_x| = 3$ for every x, and the red monochromatic subgraph has nine vertices each of (odd) degree 3, in contradiction of Corollary 1.3.

On the other hand, K_8 can be colored with no red K_3 or blue K_4. The graph G of Figure 10.2 has no triangle, and can be taken as the red monochromatic subgraph, while its complement $\overline{G}$ is the blue graph. (The construction of this graph will be discussed in Section 10.3, below.) So we have

Theorem 10.3. $R(3,4) = 9$.

The case where all the forbidden subgraphs are complete graphs is called *classical Ramsey theory*; if more general graphs are considered, the study is called *generalized Ramsey theory*. A great number of Ramsey numbers involving small graphs have been investigated; in particular, Burr [21] found the value of $R(G, G)$ whenever G is a graph with six or fewer edges and no isolated vertices.

Many results of generalized Ramsey theory have been obtained by *ad hoc* methods. We illustrate by finding $R(K_3, C_4)$. Clearly $R(K_3, C_4) \leq R(3, 4) = 9$. However, we can do rather better. Suppose K_v has been colored with no red K_3 and no blue C_4. As in the discussion of $R(3, 4)$, we see that no vertex can belong to more than three red edges. Suppose some vertex x was on four blue edges (if $R(K_3, C_4) = 9$, then every vertex must have this property). The graph generated by the other four endpoints of those edges can contain no blue path of length 2 and no red triangle. It is easy to see that the graph is the union of a red C_4 and two independent blue edges, as is shown in Figure 10.3(a) (blue edges are solid, red edges broken). Now suppose another vertex, y, is added. Since xy must be red, y can be joined to at most two other vertices by red edges, and those vertices cannot be adjacent in the red cycle. So y must lie on at least two blue edges of the type shown in Figure 10.3(b). But that graph contains a blue C_4. It follows that if any vertex lies on four blue edges, the graph has at most five vertices. If there is a solution for $v = 7$, then every vertex lies on three red and three blue edges, and both monochromatic subgraphs have an odd number of vertices and are regular of odd degree, which is impossible. So the maximum is $v = 6$. This can be attained: take the red subgraph to be $K_{3,3}$ and the blue one to be $2K_3$. So $R(K_3, C_4) = 7$.

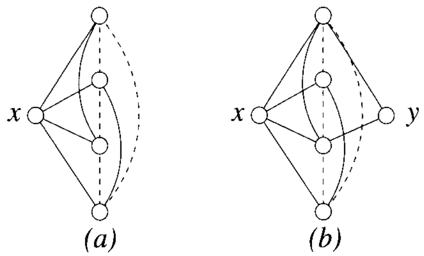

Fig. 10.3. Proving $R(K_3, C_4) < 8$

A good many families of Ramsey numbers have been found, but many more remain to be discussed. We give one example below. Further examples are given in the exercises, and in surveys of generalized Ramsey theory such as [20], [92] and [65].

Theorem 10.4. [28] . *If T is a tree with m vertices, then*

$$R(T, K_n) = (m - 1)(n - 1) + 1.$$

Proof. To see that $R(T, K_n) > (m-1)(n-1)$, consider a graph consisting of $m-1$ disjoint copies of K_{n-1}, with all edges colored red. Complete this graph to a $K_{(m-1)(n-1)}$ by coloring all remaining edges blue. Since the red subgraph contains no m-vertex component, it contains no copy of T. The blue graph is $(n-1)$-partite, so it can contain no K_n.

Equality is proved using induction on n. The case $n = 1$ is trivial. Suppose $n > 1$ and suppose the theorem is true of $R(T, K_s)$ whenever $s < n$. Suppose there is a coloring of the edges of $K_{(m-1)(n-1)+1}$ in red and blue that contains neither a red T nor a blue K_n, and examine some vertex x. If x lies on more than $(m-1)(n-2)$ blue edges, then the subgraph of G induced by the "blue" neighbors of x contains either a red copy of T or a blue K_{n-1}, by the induction hypothesis. In the former case $K_{(m-1)(n-1)+1}$ contains a red T; in the latter the blue K_{n-1} together with x forms a blue K_n. Therefore x lies on at most $(m-1)(n-2)$ blue edges, so it lies on at least $m-1$ red edges. Since x could be any vertex of the $K_{(m-1)(n-1)+1}$, the red subgraph has minimum degree at least $m-1$. Since T has $m-1$ edges, this red subgraph will contain a subgraph isomorphic to T, by Theorem 4.4. So the $K_{(m-1)(n-1)+1}$ contains a red copy of T, a contradiction. □

Exercises 10.1

A10.1.1 Consider $R(P_3, K_3)$.
- (i) Show that $R(P_3, K_3) \le 6$.
- (ii) Prove that any graph containing no P_3 consists of some disjoint edges together with some isolated vertices.
- (iii) Prove that if the graph described in part (ii) has at least four vertices and contains an isolated vertex, then its complement contains a triangle.
- (iv) Find $R(P_3, K_3)$.

10.1.2 Find $R(P_3, K_4)$.

10.1.3 Find $R(P_4, P_4)$, $R(P_4, C_4)$, $R(P_4, K_4)$ and $R(C_4, C_4)$.

10.1.4 Prove that $R(3, 5) \le 14$.

A10.1.5 Prove that $R(4, 4) \le 18$.

10.1.6 Suppose K_v can be colored in red and blue so that there is no red K_3 or blue $K_{1,s}$.
- (i) Prove that the red monochromatic subgraph has maximum degree s.
- (ii) Prove that the blue monochromatic subgraph has maximum degree $s-1$.
- (iii) Prove that $R(K_3, K_{1,s}) = 2s + 1$.

A10.1.7 Prove that if m or n is odd, then $R(K_{1,m}, K_{1,n}) = m + n$, and that if both m and n are even, then $R(K_{1,m}, K_{1,n}) = m + n - 1$. [65]

10.1.8 Chvátal and Harary [29] conjectured that if G and H are graphs with no isolated vertices, then

$$R(G, H) \ge \min\{R(G, G), R(H, H)\}.$$

Disprove this conjecture by using $G = K_{1,3}$ and $H = P_5$. (This result is due to Galvin; see [65].)

H10.1.9 If M is any matrix, its *principal $k \times k$ submatrices* are the submatrices formed by the intersection of rows $i_1, i_2, \ldots, i_k$ with columns $i_1, i_2, \ldots, i_k$ for some selection of k indices. (The principal submatrices are also called the *symmetrically placed* submatrices.) Prove that if k is an integer ≥ 2, and s is sufficiently large, then any $s \times s$ matrix M with entries from $\{0, 1\}$ contains a $k \times k$ principal submatrix with one of the following forms:

- all entries off the main diagonal are 0;
- all entries above the main diagonal are 0, all entries below the main diagonal are 1;
- all entries above the main diagonal are 1, all entries below the main diagonal are 0;
- all entries off the main diagonal are 1.

10.2 Ramsey Multiplicity

We know that any 2-painting of K_6 must contain a monochromatic triangle. However, it is not possible to find a painting with exactly one triangle. (This is easily checked by exhaustion, and follows from Theorem 10.5 below.) More generally, one can ask: what is the minimum number of monochromatic triangles in a k-painting of K_v?

Such questions are called *Ramsey multiplicity problems*. The k-Ramsey multiplicity $N_{k,v}(H)$ of H in K_v is defined to be the minimum number of monochromatic subgraphs isomorphic to H in any k-painting of K_v. Clearly $N_{k,v}(H) = 0$ if and only if $v < R_k(H)$.

The 2-Ramsey multiplicity of K_3 was investigated by Goodman, who proved the following Theorem in [56]. Our proof follows that given by Schwenk [109].

Theorem 10.5.

$$N_{2,n}(K_3) = \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor.$$

Proof. Suppose K_v is colored in two colors, red and blue. Write R for the number of red triangles, B for the number of blue triangles, and P for the number of *partial* triangles — triangles with at least one edge of each color. There are $\binom{n}{3}$ triangles in K_v, so

$$R + B + P = \binom{v}{3}.$$

Since $N_{2,v}(K_3)$ equals the minimum possible value of $R + B$,

$$N_{2,v}(K_3) = \binom{v}{3} - \max(P).$$

Suppose the vertices of K_v are $x_1, x_2, \ldots, x_v$, and x_i is incident with r_i red edges. Then it is adjacent to $v - 1 - r_i$ blue edges. Therefore the K_v contains $r_i(v - 1 - r_i)$ paths of length 2 in which one edge is red and the other blue. Let us call these mixed paths. The total number of mixed paths in the K_v is

$$\sum_{i=1}^{v} r_i(v - 1 - r_i).$$

The triangle xyz can be considered as the union of the three paths xyz, yzx and zxy. Moreover, the paths corresponding to different triangles will all be different. If the triangle is monochromatic, no path is mixed, but a partial triangle gives rise to two mixed paths. So there are $2P$ mixed paths in the K_v, and

$$P = \frac{1}{2}\sum_{i=1}^{v} r_i(v-1-r_i).$$

If v is odd, the maximum value of $r_i(v-1-r_i)$ is $(v-1)^2/4$, attained when $r_i = (v-1)/2$. If v is even, the maximum of $v(v-2)/4$ is given by $r_i = v/2$ or $(v-2)/2$. In either case, the maximum is

$$\left\lfloor \left(\frac{v-1}{2}\right)^2 \right\rfloor,$$

so

$$\begin{aligned} P &\leq \tfrac{1}{2}\textstyle\sum_{L=1}^{v} \left\lfloor \left(\frac{v-1}{2}\right)^2 \right\rfloor \\ &\leq \tfrac{v}{2}\left\lfloor \left(\frac{v-1}{2}\right)1^2 \right\rfloor, \end{aligned}$$

and since P is an integer,

$$P \leq \left\lfloor \frac{v}{2}\left\lfloor \left(\frac{v-1}{2}\right)^2 \right\rfloor \right\rfloor.$$

So

$$N_{2,v}(K_3) \geq \binom{v}{3} - \left\lfloor \frac{v}{2}\left\lfloor \left(\frac{v-1}{2}\right)^2 \right\rfloor \right\rfloor.$$

It remains to show that equality can be attained.

If v is even, say $v = 2t$, then partition the vertices of K_v into the two sets $\{x_1, x_2, \ldots, x_t\}$ and $\{x_{t+1}, x_{t+2}, \ldots, x_{2t}\}$ of size t, and color an edge red if it has one endpoint in each set, blue if it joins two members of the same set. (The red edges form a copy of $K_{t,t}$.) Each r_i equals $v/2$. If $v = 2t+1$, carry out the same construction for $2t$ vertices, except color edge $x_i x_{t+i}$ blue for $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$. Then add a final vertex x_{2t+1}. The edges $x_i x_{2t+1}$ and $x_{i+t} x_{2t+1}$ are red when $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$ and blue otherwise. In both cases it is easy to check that the number of triangles equals the required minimum. □

Substituting into the formula gives $N_{2,v}(K_3) = 0$ when $v \leq 5$, $N_{2,6}(K_3) = 2$, $N_{2,7}(K_3) = 4$, and so on.

The 3-Ramsey multiplicity of K_3 has not been fully investigated. We know that $R_3(3) = 17$, so the number $N_{3,17}(K_3)$ is of special interest. It is shown in [107] that $N_{3,17}(K_3) = 5$; the argument involves discussion of many special cases. A sketch of a proof that $N_{3,17}(K_3) \geq 3$ appears in Exercise 10.2.1.

The following theorem, which appears in a simplified form in [120], provides a recursive bound on $N_{k,v}(K_3)$.

Theorem 10.6.

$$N_{k,v+1}(K_3) \leq \left\lfloor \frac{v-1}{k} \right\rfloor + \left\lfloor \left(1 + \frac{3}{v}\right) N_{k,v}(K_3) \right\rfloor.$$

Proof. Suppose F is a k-painting of K_v that contains $N_{k,v}(K_3)$ monochromatic triangles. Select a vertex x of F that lies in the minimum number of monochromatic triangles. Since there are v vertices, one can assume that x lies on at most $\lfloor \frac{3}{v} N_{k,v}(K_3) \rfloor$ monochromatic triangles. Since x has degree $v-1$ in K_v, there will be a color — say R — such that x lies on at most $\lfloor \frac{v-1}{k} \rfloor$ edges of color R.

Construct a k-painting of K_{v+1} from F by adjoining a new vertex y. If z is any vertex other than x, then yz receives the same color as xz, and xy receives color R. Then xy lies in $\lfloor \frac{v-1}{k} \rfloor$ or fewer monochromatic triangles, all in color R. The original K_v contained $N_{k,v}(K_3)$ monochromatic triangles, so this is the number not containing x. Finally, the number of monochromatic triangles with y as a vertex but not x is at most $\lfloor \frac{3}{v} N_{k,v}(K_3) \rfloor$. So the maximum number of monochromatic triangles in the K_{v+1} is

$$\left\lfloor \frac{n-1}{k} \right\rfloor + \left\lfloor \left(1 + \frac{3}{v}\right) N_{k,v}(K_3) \right\rfloor. \qquad \square$$

This theorem provides the upper bounds 2 and 5 for $N_{2,6}(K_3)$ and $N_{3,17}(K_3)$, both of which can be met.

Exercises 10.2

10.2.1 Prove that $N_{2,3}(P_3) = 1$ and $N_{2,4}(P_3) = 4$. Suppose there exists a painting of K_{17} in the three colors red, blue and green that contains two or less monochromatic triangles. If v is any vertex, write $R(v)$, $B(v)$ and $G(v)$ for the sets of vertices joined to v by red, blue and green edges respectively, and write $r(v) = |R(v)|$, and so on.

(i) Select a vertex x that lies in no monochromatic triangle. Prove that one of $\{r(x), b(x), g(x)\}$ equals 6 and the other two each equal 5.

(ii) Without loss of generality, say $r(x) = 6$. Let S be the set of all vertices of K_{17} that lie in monochromatic triangles. Prove that $S \subseteq R(x)$.

(iii) If y is any member of $B(x)$, it is clear that S lies completely within $R(y)$, $B(y)$ or $G(y)$. Prove that, in fact, $S \subseteq R(y)$.

(iv) Prove that there must exist two vertices y_1 and y_2 in $B(x)$ such that $y_1 y_2$ is red.

(v) Use the fact that $S \subseteq R(y_1) \cap R(y_2)$ to prove that K_{17} contains more than two red triangles. So $N_{3,17}(K_3) \geq 3$. [120]

A10.2.2 It follows from Exercise 10.1.7 that $R(K_{1,n}, K_{1,n}) = 2n$ when n is odd and that $R(K_{1,n}, K_{1,n}) = 2n-1$ when n is even. Prove that

$$N_{2,2n}(K_{1,n}) = 2n-1, n \text{ odd},$$
$$N_{2,2n-1}(K_{1,n}) = 1, n \text{ even}.$$

10.3 Application of Sum-Free Sets

To introduce this section we derive the construction of the red graph of Figure 10.2. The vertices of the graph are labeled with the elements of the cyclic group $\mathbb{Z}_8$. The set $\mathbb{Z}_8^*$ of nonzero elements of $\mathbb{Z}_8$ is partitioned into two sets:

$$\mathbb{Z}_8^* = \{3, 4, 5\} \cup \{1, 2, 6, 7\}.$$

Call the two sets R and B respectively. Then $x \sim y$ in G if and only if $x - y \in R$. It follows that two vertices are joined in $\overline{G}$ if and only if their difference is in B. Observe that both R and B contain the additive inverses of all their elements; this is important because the differences $x - y$ and $y - x$ both correspond to the same edge xy. (This property might be relaxed for some applications to directed graphs.)

Notice that R contains no solution to the equation

$$a + b = c;$$

no element of R equals the sum of two elements of R. We say R is a *sum-free set*. By contrast, B is not sum-free; not only is $1 + 6 = 7$, but also $1 + 1 = 2$ (a, b and c need not be distinct). If xyz were a triangle in G, then $x - y$, $y - z$ and $x - z$ would all be members of R; but

$$x - y + y - z = x - z,$$

so R would not be sum-free.

In general, suppose G is any group, written additively. A nonempty subset S of G is a *sum-free set* if there never exist elements a, b of S such that $a + b \in S$. (This means that 0 cannot belong to S, since $0 + 0 = 0$.) S is *symmetric* if $-x \in S$ whenever x is in S. A *symmetric sum-free partition* of G is a partition of G^* into symmetric sum-free sets. As examples,

$$\mathbb{Z}_5^* = \{1, 4\} \cup \{2, 3\}$$
$$\mathbb{Z}_8^* = \{3, 4, 5\} \cup \{1, 7\} \cup \{2, 6\}$$

are symmetric sum-free partitions.

If S is a symmetric sum-free set in G, the *graph of* S is the graph with vertex-set G, where x and y are adjacent if and only if $x - y \in S$. From our earlier discussion, it follows that

Theorem 10.7. *If S is a symmetric sum-free set of order s in a group G of order g, then the graph of S is a triangle-free regular graph of degree s on g vertices.*

If there is a symmetric sum-free partition $S_1 \cup S_2 \cup \cdots \cup S_k$ of G^*, then one obtains a k-painting of $K_{|G|}$ that contains no monochromatic triangle, by applying c_i to all the edges of the graph of S_i for $i = 1, 2, \ldots, k$. So

Corollary 10.8. *If there exists a sum-free partition of a g-element group into k parts, then $R_k(3) > g$.*

For example, the partition

$$\mathbb{Z}_5^* = \{1, 4\} \cup \{2, 3\}$$

provides the well-known partition of $\mathbb{Z}_5$ into two 5-cycles that is used in proving that $R_2(3) = 6$. The partition

$$\mathbb{Z}_8^* = \{3, 4, 5\} \cup \{1, 7\} \cup \{2, 6\}$$

yields a (not very interesting) triangle-free 3-painting of K_8.

There are two abelian groups of order 16 that have symmetric sum-free partitions. The group $\mathbb{Z}_4 \times \mathbb{Z}_4$ can be written as the set of all ordered pairs xy where both x and y come from $\{0, 1, 2, 3\}$ and

$$xy + zt = (x+z)(y+t)$$

(additions modulo 4). Then

$$(\mathbb{Z}_4 \times \mathbb{Z}_4)^* = R \cup B \cup G$$

where

$$\begin{aligned} R &= \{02, 10, 30, 11, 33\}, \\ B &= \{20, 01, 03, 13, 31\}, \\ G &= \{22, 21, 23, 12, 32\}. \end{aligned} \tag{10.1}$$

$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)^*$ has a similar partition

$$\begin{aligned} R &= \{1000, 1100, 1010, 1111, 0001\}, \\ B &= \{0010, 0011, 1011, 0111, 1101\}, \\ G &= \{0100, 0110, 0101, 1110, 1001\}. \end{aligned} \tag{10.2}$$

The existence of these partitions proves of course that $R_3(3) \geq 17$. To see that $R_3(3) = 17$, we use the following argument. Suppose K_{17} could be colored in three colors. Select any vertex x. Since there are sixteen edges incident with x, there must be at least six in one color, red say. Consider the subgraph generated by the other endpoints of those edges. If it has a red edge, then there is a red triangle; if not, the subgraph is a K_6 colored in the two remaining colors, and it must contain a monochromatic triangle.

The above argument can be used to show that $R_4(3) \leq 66$, but there is no sum-free partition of a 65-element group into four parts. In fact, we know that $51 \leq R_4(3) \leq 65$ ([27, 43, 132]). The lower bound was proven by exhibiting a triangle-free coloring of K_{50}, while the upper bound comes from a lengthy argument proving that if a triangle-free 4-painting of K_{65} existed, then the adjacency matrices of the monochromatic subgraphs would have eigenvalues of irrational multiplicities.

The method of sum-free sets can be generalized to avoid larger complete subgraphs. For example, consider the subset $B = \{1, 2, 6, 7\}$ of K_8 that arose in discussing $R(3, 4)$. This set is not sum-free, and its graph will contain triangles. However, suppose there were a K_4 in the graph, with vertices a, b, c and d. Then B would contain a solution to the following system of three simultaneous equations in six unknowns:

$$\begin{array}{llll} x_{ab} & + x_{bc} & & = x_{ac} \\ & x_{ac} & + x_{cd} & = x_{ad} \\ & x_{bc} & + x_{cd} & = x_{bd} \end{array}$$

(in each case, x_{ij} will be either $i - j$ or $j - i$). But a complete search shows that B contains no solution to these equations. So the graph contains no K_4. (The graph is $\overline{G}$ in Figure 10.2.)

Exercises 10.3

10.3.1 Prove that $R(3,5) > 13$ by choosing $R = \{4, 6, 7, 9\}$, $B = \{1, 2, 3, 5, 8, 10, 11, 12\}$ in Z_{13}.

10.3.2 Show that $R(4,4) > 17$, by choosing $R = \{1, 2, 4, 8, 9, 13, 15, 16\}$, $B = \{3, 5, 6, 7, 10, 11, 12, 14\}$ in Z_{17}.

10.3.3 Verify that the partitions in (10.1) and (10.2) are in fact symmetric sum-free partitions.

10.4 Bounds on Classical Ramsey Numbers

Very few Ramsey numbers are known. Consequently much effort has gone into proving upper and lower bounds.

Lemma 10.9. *If p and q are integers greater than 2, then*

$$R(p,q) \le R(p-1,q) + R(p,q-1).$$

Proof. Write $m = R(p-1,q) + R(p,q-1)$. Suppose the edges of K_m are colored in red and blue. We shall prove that K_m contains either a red K_p or a blue K_q. Two cases arise.

(i) Suppose that one of the vertices x of K_m has at least $s = R(p-1,q)$ red edges incident with it, connecting it to vertices $x_1, x_2, \dots, x_s$. Consider the K_s on these vertices. Since its edges are colored red or blue, it contains either a blue K_q, in which case the lemma is proved, or a red K_{p-1}. Let the set of vertices of the red K_{p-1} be $\{y_1, y_2, \dots, y_{p-1}\}$. Then the vertices $x, y_1, \dots, y_{p-1}$ are those of a red K_p and again the lemma holds.

(ii) Suppose that no vertex of K_m has $R(p-1,q)$ red edges incident with it. Then every vertex must be incident with at least $m - 1 - [R(p-1,q) - 1] = R(p,q-1)$ blue edges. The argument is then analogous to that of part (i). □

Theorem 10.10. *For all integers, $p, q \ge 2$,*

$$R(p,q) \le \binom{p+q-2}{p-1}.$$

Proof. Write $n = p + q$. The proof proceeds by induction on n. Clearly $R(2,2) = 2 = \binom{2+2-2}{2-1}$. Since $p, q \ge 2$, we can have $n = 4$ only if $p = q = 2$. Hence the given bound is valid for $n = 4$. Also for any value of q, $R(2,q) = q = \binom{2+q-2}{2-1}$, and similarly for any value of p, $R(p,2) = p = \binom{p+2-2}{p-1}$, so the bound is valid if $p = 2$ or $q = 2$.

Without loss of generality assume that $p \ge 3, q \ge 3$ and that

$$R(p',q') \le \binom{p'+q'-2}{p'-1}$$

for all integers p', q' and n satisfying $p' \geq 2, q' \geq 2, p' + q' < n$ and $n > 4$. Suppose the integers p and q satisfy $p + q = n$.

We apply the induction hypothesis to the case $p' = p - 1, q' = q$, obtaining

$$R(p-1, q) \leq \binom{p+q-3}{p-2},$$

and to $p' = p, q' = q - 1$, obtaining

$$R(p, q-1) \leq \binom{p+q-3}{p-1}.$$

But by the properties of binomial coefficients,

$$\binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1},$$

and from Lemma 10.9

$$R(p, q) \leq R(p-1, q) + R(p, q-1),$$

so

$$R(p, q) \leq \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}. \qquad \square$$

If $p = 2$ or $q = 2$ or if $p = q = 3$, this bound is exact. But suppose $p = 3$, $q = 4$. Then $\binom{p+q-2}{p-1} = \binom{5}{2} = 10$, and the exact value of $R(3, 4)$ is 9. Again if $p = 3$, $q = 5$, then $\binom{p+q-2}{p-1} = \binom{6}{2} = 15$, whereas the exact value of $R(3, 5)$ is 14. In general, Theorem 10.10 shows that

$$R(3, q) \leq \binom{q+1}{2} = \frac{q(q+1)}{2} = \frac{q^2+q}{2}.$$

But for the case $p = 3$, this result can be improved [9]. It is shown there that for every integer $q \geq 2$,

$$R(3, q) \leq \frac{q^2+3}{2}.$$

The following lower bound for $R_n(k)$ was proved by Abbott [1].

Theorem 10.11. *For integers $s, t \geq 2$,*

$$R_n(st - s - t + 2) \geq (R_n(s) - 1)(R_n(t) - 1) + 1.$$

Proof. Write $p = R_n(s) - 1$ and $q = R_n(t) - 1$. Consider a K_p with vertices $x_1, x_2, \ldots, x_p$ and a K_q with vertices $y_1, y_2, \ldots, y_q$. Color the edges of K_p and K_q in n colors $c_1, c_2, \ldots, c_n$ in such a way that K_p contains no monochromatic K_s and K_q contains no monochromatic K_t (such colorings must be possible by the definitions of p and q).

Now let K_{pq} be the complete graph on the vertices z_{ij}, where $i \in 1, 2, \ldots, p$ and $j \in 1, 2, \ldots, q$. Color the edges of K_{pq} as follows:

(i) Edge $w_{gj}w_{gh}$ is given the color $y_j y_h$ received in K_q.

(ii) If $i \neq g$, $w_{ij}w_{gh}$ is given the color $y_j y_h$ received in K_p.

Now write $r = st - s - t + 2$ and let G be any copy of K_r contained in K_{pq}. Suppose G is monochromatic, with all its edges colored c_1. Two cases arise:

(i) There are s distinct values of i for which the vertex w_{ij} belongs to G. Then from the coloring scheme K_p contains a monochromatic K_s, which is a contradiction;
(ii) There are at most $s - 1$ distinct values of i for which w_{ij} belongs to G. Suppose there are at most $t - 1$ distinct values of j such that w_{ij} belongs to G. Then G has at most $(s-1)(t-1) = st - s - t + 1 = r - 1$ vertices, which is a contradiction. So there is at least one value of i such that at least t of the vertices w_{ij} belong to G. Applying the argument of Case (i), K_q contains a monochromatic K_t, which is again a contradiction.

Thus K_{pq} contains no monochromatic K_r. □

In order to develop the ideas of sum-free sets and obtain some bounds for $R_n(3)$, we define the *Schur function*, $f(n)$, to be the largest integer such that the set

$$1, 2, \dots, f(n)$$

can be partitioned into n mutually disjoint nonempty sets $S_1, S_2, \dots, S_n$, each of which is sum-free. Obviously $f(1) = 1$, and $f(2) = 4$ where $\{1, 2, 3, 4\} = \{1, 4\} \cup \{2, 3\}$ is the appropriate partition. Computations have shown that $f(3) = 13$ with

$$\{1, 2, \dots, 13\} = \{3, 2, 12, 11\} \cup \{6, 5, 9, 8\} \cup \{1, 4, 7, 10, 13\}$$

as one possible partition, that $f(4) = 44$ and $f(5) \geq 138$.

Lemma 10.12. *For any positive integer n*

$$f(n+1) \geq 3f(n) + 1,$$

and since $f(1) = 1$,

$$f(n) \geq \frac{3^n - 1}{2}.$$

Proof. Suppose that the set $S = \{1, 2, \dots, f(n)\}$ can be partitioned into the n sum-free sets $S_1 = \{x_{11}, x_{12}, \dots, x_{1\ell_1}\}, \dots, S_n = \{x_{n1}, x_{n2}, \dots, x_{n\ell_n}\}$. Then the sets

$$\begin{aligned}
T_1 &= \{3x_{11}, 3x_{11} - 1, 3x_{12}, 3x_{12} - 1, \dots, 3x_{1\ell_1}, 3x_{1\ell_1} - 1\},\\
T_2 &= \{3x_{21}, 3x_{21} - 1, 3x_{22}, 3x_{22} - 1, \dots, 3x_{2\ell_1}, 3x_{2\ell_1} - 1\},\\
&\ \vdots\\
T_n &= \{3x_{n1}, 3x_{n1} - 1, 3x_{n2}, 3x_{n2} - 1, \dots, 3x_{n\ell_n}, 3x_{n\ell_n} - 1\},\\
T_{n+1} &= \{1, 4, 7, \dots, 3f(n) + 1\}
\end{aligned}$$

form a partition of $\{1, 2, \dots, 3f(n) + 1\}$ into $n + 1$ sum-free sets. So

$$f(n+1) \geq 3f(n) + 1.$$

Now $f(1) = 1$, so equation 10.4 implies that

$$f(n) \geq 1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}.$$ □

Theorem 10.13. *For any positive integer*

$$\frac{3^n+3}{2} \le R_n(3) \le n(R_{n-1}(3)-1)+2.$$

Proof. (i) The proof of the upper bound is a generalization of the method used to establish $R_3(3) \le 17$.

(ii) Let $K_{f(n)+1}$ be the complete graph on the $f(n)+1$ vertices $x_0, x_1, \ldots, x_{f(n)}$. Color the edges of $K_{f(n)+1}$ in n colors by coloring $x_i x_j$ in the k-th color if and only if $|i-j| \in S_k$.

Suppose the graph contains a monochromatic triangle. This must have vertices x_a, x_b, x_c with $a.b.c$, such that $a-b, b-c, a-c \in S_k$. But now $(a-b)+(b-c) = a-c$, contradicting the fact that S_k is sum-free. Hence

$$f(n)+1 \le R_n(3)-1,$$

so that

$$\frac{3^n-1}{2}+2 = \frac{3^n-3}{2} \le R_n(3),$$

which proves the lower bound. □

Exercises 10.4

A10.4.1 Verify that, for any $p \ge 2$ and any $q \ge 2$,

$$R(2,q) = q, \qquad R(p,2) = p.$$

10.4.2 Is it possible to 2-color the edges of K_{35} so that no red K_4 or blue K_5 occurs?

A10.4.3 Is it possible to 2-color the edges of K_{25} so that no monochromatic K_5 occurs?

10.4.4 Verify that the sets $T_1, T_2, \ldots, T_{n+1}$ of Lemma 10.12 form a sum-free partition.

10.4.5 Verify the upper bound in Theorem 10.13.

10.5 The General Case of Ramsey's Theorem

In its general (finite) form, Ramsey's Theorem deals with the partition of the collection of all r-sets on a set. The graphical case is the case $r = 2$. The case $r = 1$ is the well-known pigeonhole principle: *if n objects are distributed among more than n sets, some set will contain at least two objects.* This obvious statement has some less-than-obvious applications.

We state Ramsey's Theorem in its general form as Theorem 10.14. The proof is left to the exercises.

Theorem 10.14. (Ramsey's Theorem) *Suppose S is an s-element set. Write $\Pi_r(S)$ for the collection of all r-element subsets of S, $r \ge 1$. Suppose the partition*

$$\Pi_r(S) = A_1 \cup A_2 \cup \cdots \cup A_n$$

is such that each r-subset of S belongs to exactly one of the A_i, and no A_i is empty. If the integers $p_1, p_2, \ldots, p_n$ satisfy $r \le p_i \le s$, for $i = 1, 2, \ldots, n$, then there exists an integer

$$R(p_1, p_2, \ldots, p_n; r),$$

depending only on n, $p_1, p_2, \ldots, p_n$ *and* r, *such that if* $s \geq R(p_1, p_2, \ldots, p_n; r)$, *then for at least one* i, $1 \leq i \leq n$, *there exists a* p_i*-element subset* T *of* S, *all of whose* r*-subsets belong to* A_i.

Exercises 10.5

10.5.1 Suppose p, q and r are integers satisfying $1 \leq r \leq p, q$. Prove:

(i) $R(p, q; 1) = p + q - 1$;

(ii) $R(r, q; r) = q$;

(iii) $R(p, r; r) = p$.

10.5.2 Prove Ramsey's Theorem for $n = 2$.

10.5.3 Prove Ramsey's Theorem.