1

Graphs

1.1 Sets, Binary Relations and Graphs

We shall use the standard concepts and notations of set theory. We write $x \in S$ and $y \notin S$ to indicate that x is a member of S, and that y is not a member of S. $|S|$ denotes the number of elements of S, also called the *order* of S. If all elements of S are also elements of T, then S is a subset of T, written $S \subseteq T$. The notation $S \subset T$ means that S is a subset of T but is not identical to T, so that T has at least one element that is not in S.

A set can be specified by writing a description ("the set of positive perfect squares less than 20") or by listing its members ("$\{1, 4, 9, 16\}$"). One can also state a membership law: for example, $\{x^2 \mid x \text{ is an integer}, x^2 < 20\}$. This is called *setbuilder notation*. The $|$ is read as "such that"; it is equally common to use a colon instead of $|$. Other mathematical abbreviations are often employed in setbuilder notation — for example, $\mathbb{Z}$ usually represents the set of integers, so one might replace "x is an integer" by "$x \in \mathbb{Z}$."

If S and T are any two sets, then $S \cup T$ means the *union* of S and T, the set of everything that is either a member of S or a member of T (or both), and $S \cap T$ is the *intersection*, the set common elements. The *set-theoretic difference* $S \backslash T$, also called the *relative complement* of T in S, consists of all elements of S that are *not* members of T. The cartesian product $S \times T$ is the set of all ordered pairs $\{x, y\}$, where x is a member of S and y is a member of T.

Various identities between sets can be proved. For example, both union and intersection satisfy the commutative laws

$$S \cup T = T \cup S \tag{1.1}$$

$$S \cap T = T \cap S \tag{1.2}$$

for any sets S and T, and the associative laws

$$R \cup (S \cup T) = (R \cup S) \cup T \tag{1.3}$$

$$R \cap (S \cap T) = (R \cap S) \cap T \tag{1.4}$$

for any sets R, S and T.

To prove that two sets are equal, one often proves that every member of one set is an element of the other, and conversely. In other words, to show that $A = B$, first prove $A \subseteq B$ and then prove $B \subseteq A$.

Example. To prove $R \cap (S \cap T) = (R \cap S) \cap T$, first observe that any member x of $R \cap (S \cap T)$ is both a member of R and a member of $S \cap T$, and the latter means that x belongs to both S and T. So all of $x \in R$, $x \in S$ and $x \in T$ are true. From these we see that both $x \in R$ and $x \in S$ are true, so $x \in R \cap S$, and also $x \in T$; therefore $x \in (R \cap S) \cap T$. We have actually shown that

$$R \cap (S \cap T) \subseteq (R \cap S) \cap T.$$

One proves $R \cap (S \cap T) \subseteq (R \cap S) \cap T$ in the same way, and equality has been established.

Example. Prove $R \backslash (S \cup T) = (R \backslash S) \cap (R \backslash T)$for any sets R, S and T.

$R \backslash (S \cup T)$ consists of precisely those members of R that are not members of $S \cup T$, in other words those elements of R that do not belong to S or to T. That is,

$$R \backslash (S \cup T) = \{x \mid x \in R \text{ and } x \notin S \text{ and } x \notin T\}.$$

On the other hand, $(R \backslash S)$ consists of all the things in R that are not in S, and $(R \backslash S) \cap (R \backslash T)$ consists of all the things in R that are not in T; the common elements of these sets are all the things in R but not in S and not in T, which is the same as the description of $R \backslash (S \cup T)$.

Binary relations occur frequently in mathematics and in everyday life. For example, the ordinary mathematical relations $<$, $=$, $>$, $\leq$ and $\geq$ are binary relations on number sets, $\subset$ and $\subseteq$ are binary relations on collections of sets, and so on. If S is the set of all living people, "is the child of" is a typical binary relation on S.

Formally, a *binary relation* $\sim$ on a set S is a rule that stipulates, given any elements x and y of S, whether x bears a certain relationship to y (written $x \sim y$) or not (written $x \not\sim y$). Alternatively, one can define a binary relation $\sim$ on the set S to consist of a set $\sim(S)$ of elements from $S \times S$ (the set of ordered pairs of elements of S), with the notation $x \sim y$ meaning that (x, y) belongs to $\sim(S)$.

One important class of binary relations is arithmetical *congruence*. Two integers a and b are *congruent modulo n* (written $a \equiv b (\text{mod}\, n)$) if $a - b$ is divisible by n. The *congruence class* of $a (\text{mod}\, n)$ is the set of all integers congruent to a modulo n. There are n different congruence classes modulo n.

One can represent any binary relation by a diagram. The elements of the set S are shown as points (*vertices*), and if $x \sim y$ is true, then a line (*edge*) is shown from x to y, with its direction indicated by an arrow. Provided the set S is finite, all information about any binary relation on S can be shown in the diagram. The diagram is a *directed graph* or *digraph*; if $x \sim x$ is ever true, the diagram is a *looped* digraph.

The binary relation $\sim$ on S is called *reflexive* if $x \sim x$ is true for all x in S, and *antireflexive* if $x \sim x$ is never true (or, equivalently, if $x \not\sim x$ is true for all x). If $y \sim x$ is true whenever $x \sim y$ is true, then $\sim$ is called *symmetric*. If the relation is symmetric, the arrows can be omitted from its diagram. The diagram of a symmetric, antireflexive binary relation on a finite set is called a *graph*.

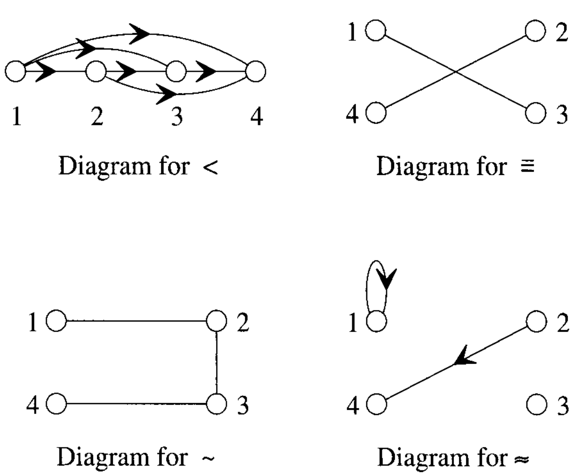


Fig. 1.1. Diagrams of binary relations

Example. Suppose the binary relations $<$, $\equiv$, $\sim$ and $\approx$ are defined on the set $S = \{1, 2, 3, 4\}$ as follows:

$$x < y \text{ means } x \text{ is less than } y;$$
$$x \equiv y \text{ means } x \text{ is congruent to } y \pmod 2 \text{ and } x \neq y;$$
$$x \sim y \text{ means } x = y \pm 1;$$
$$x \approx y \text{ means } y = x^2.$$

Then the corresponding subsets of $S \times S$ are

$$< (S) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\};$$
$$\equiv (S) = \{(1, 3), (3, 1), (2, 4), (4, 2)\};$$
$$\sim (S) = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3)\};$$
$$\approx (S) = \{(1, 1), (2, 4)\}.$$

The diagrams are shown in Figure 1.1. Relations $\equiv$ and $\sim$ yield graphs, $<$ gives a digraph, and $\approx$ a looped digraph.

In more general situations, it might make sense to use two or more lines to join the same pair of points. For example, suppose we want to describe the roads joining various townships. For many purposes we do not need to know the topography of the region, or whether different roads cross, or various other things. The important information is whether or not there is a road joining two towns. In these cases we could use a complete road map with the exact shapes of the roads and various other details shown, but it would be less confusing to make a diagram, as shown in Figure 1.2, that indicates two roads joining B to C, one road from A to each of B and C and one road from C to D with no direct roads joining A to D or B to D. We say that there is a *multiple edge* (of *multiplicity* 2) joining B to C. If any of the roads were one-way, an arrow could be employed.

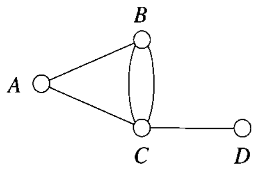


Fig. 1.2. Graphical representation of a road network

Example. Consider a football competition in which every team plays every other team once. At any point of the tournament we can represent the games that have been played by a graph. The vertices represent the teams; edge xy is included if and only if the teams x and y have already played each other. Figure 1.3 is the representation of a 6-team league after matches A v. B, A v. C, A v. D, B v. E and E v. F have been completed.

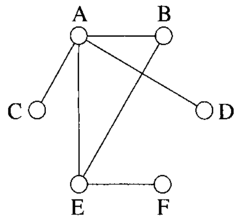


Fig. 1.3. Graphical representation of a football competition

Example. Suppose there are four jobs vacant and five men apply for them. Each man is capable of performing one or more of the jobs. The usual question is whether or not one can allocate jobs to four of the men so that all four jobs are allocated.

This situation can conveniently be represented by a graph. All the applicants and all the jobs are represented by vertices; two vertices are joined if and only if one represents an applicant, the other represents a job, and the applicant is capable of doing the job. Figure 1.4 shows the situation where A, B and C can all handle jobs 1 and 2, D can do 1 and 3 and E can do 2 and 4.

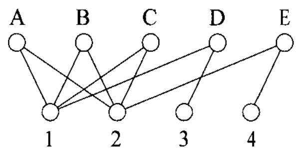


Fig. 1.4. Who is qualified for which job?

Exercises 1.1

1.1.1 Prove that the following identities hold for any sets R, S and T.
  (i) $R \cup (S \cup T) = (R \cup S) \cup T$.
  (ii) $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$.
  (iii) $R \backslash (S \cap T) = (R \backslash S) \cup (R \backslash T)$.

A1.1.2 In each part below, a binary relation $\sim$ is defined on
$$\{-3, -2, -1, 0, 1, 2, 3\}.$$
In each case, is the relation reflexive? Antireflexive? Symmetric?
  (i) $x \sim y$ means $x + y \leq 4$.
  (ii) $x \sim y$ means $x + y \leq 6$.
  (iii) $x \sim y$ means $x = y + 1$.
  (iv) $x \sim y$ means $x = \pm y$.

1.1.3 Repeat Exercise 1.1.2 for the following binary relations defined on
$$\{-3, -2, -1, 0, 1, 2, 3\}:$$
  (i) $x \sim y$ means $x \leq y^2$.
  (ii) $x \sim y$ means $x + y \geq 0$.
  (iii) $x \sim y$ means $x + y$ is odd.
  (iv) $x \sim y$ means xy is odd.

1.1.4 Draw graphical representations of the relations in Exercise 1.1.2.

1.1.5 Draw graphical representations of the relations in Exercise 1.1.3.

A1.1.6 Repeat Exercise 1.1.2 for the following relations defined on the positive integers.
  (i) $x \sim y$ means $x + y \leq 4$.
  (ii) $x \sim y$ means x divides y.
  (iii) $x \sim y$ means x and y have greatest common divisor 1.
  (iv) $x \sim y$ means $x + y$ is odd.

1.1.7 Repeat Exercise 1.1.2 for the following relations defined on the positive integers.
  (i) $x \sim y$ means x and y are both prime numbers.
  (ii) $x \sim y$ means $x = \pm y$.
  (iii) $x \sim y$ means xy is odd.

1.1.8 A relation is called *transitive* if every time (x, y) and (y, z) are in the relation, then (x, z) is also.
  (i) Describe the graph of a symmetric, transitive relation.
  (ii) Which, if any, of the relations in Exercises 1.1.2, 1.1.3, 1.1.6 and 1.1.7 are transitive?

1.1.9 Suppose the binary relation $\sim$ is defined on the set of real numbers as follows: $x \sim y$ means $x \equiv y \pmod 7$. Is $\sim$ reflexive? Antireflexive? Symmetric? Transitive?

1.1.10 A basketball league contains seven teams, denoted by A, B, C, D, E, F, G. Team A has played against each other team once; team B has played against C, E and G; and teams D, E and F have all played. Draw a graph to illustrate this situation.

1.2 Some Definitions

We start by formalizing some of the discussion and definitions from the preceding section. A *graph* G consists of a finite set $V(G)$ of objects called *vertices* together with a set $E(G)$ of unordered pairs of vertices; the elements of $E(G)$ are called *edges*. We write $v(G)$ and $e(G)$ for the orders of $V(G)$ and $E(G)$, respectively; these are often called the *order* and *size* of G. In terms of the more general definitions sometimes used, we can say that "our graphs are finite and contain neither loops nor multiple edges."

Graphs are usually represented by diagrams in which the vertices are points. An edge xy is shown as a line from (the point representing) x to (the point representing) y. To distinguish the vertices from other points in the plane, they are often drawn as small circles or large dots.

Example. A graph has five vertices, a, b, c, d, e, and edges ab, ac, ad, be, de. So its representation is

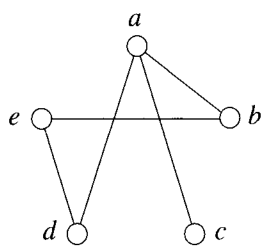
Often the same graph can give rise to several drawings that look quite dissimilar. For example, the three diagrams in Figure 1.5 all represent the same graph. Although the two diagonal lines cross in the first picture, their point of intersection does not represent a vertex of the graph.

The edge containing x and y is written xy or (x, y); x and y are called its *endpoints*. We say this edge *joins* x to y. If A and B are subsets of $V(G)$, then $[A, B]$ denotes the set of all edges of G with one endpoint in A and the other in B:

$$[A, B] = \{xy: x \in A, y \in B, xy \in E(G)\}. \tag{1.5}$$

If A consists of the single vertex a, it is usual to write $[a, B]$ instead of $[\{a\}, B]$.

An *isomorphism* of a graph G onto a graph H is a one-to-one map ϕ from $V(G)$ onto $V(H)$ with the property that a and b are adjacent vertices in G if and only if $a\phi$ and $b\phi$ are adjacent vertices in H; G is *isomorphic* to H if and only if there is an isomorphism of G onto H. An isomorphism from a graph G to itself is called an *automorphism* of G.



Because graphs are finite, one can prove that a map ϕ is an isomorphism of G onto H by showing that the two graphs have the same numbers of vertices and edges, that $a\phi$ is a vertex of H whenever a is a vertex of G, and that $a\phi$ and $b\phi$ are adjacent vertices in H whenever a and b are adjacent in G.

Given a set S of v vertices, the graph formed by joining each pair of vertices in S is called the *complete* graph on S and denoted by K_S. We also write K_v to mean any complete graph with v vertices. From the definition of isomorphism it follows that all complete graphs on v vertices are isomorphic. The notation K_v can be interpreted as being a generic name for the typical representative of the isomorphism class of all v-vertex complete graphs. The three drawings in Figure 1.5 are all representations of K_4.

A *multigraph* is defined in the same way as a graph except that there may be more than one edge corresponding to the same unordered pair of vertices. The *underlying graph* of a multigraph is formed by replacing all edges corresponding to the unordered pair $\{x, y\}$ by a single edge xy. Unless otherwise mentioned, all definitions and concepts pertaining to graphs will be applied to multigraphs in the obvious way.

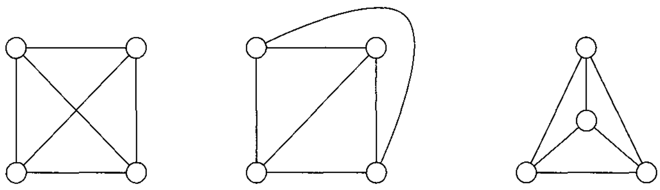


Fig. 1.5. Three representations of K_4

In some cases a direction is imposed on each edge. In this case we call the graph a *directed graph* or *digraph*. Directed edges are usually called *arcs*. An arc is an ordered pair of vertices, the first vertex is the *start* (or *tail* or *origin*) of the arc, and the second is the *finish* (or *head* or *terminus*). Directed graphs can have two arcs with the same endpoints, provided they have opposite directions. The *underlying graph* of a digraph is constructed by ignoring all directions and replacing any resulting multiple edges by single edges.

$G - xy$ denotes the graph produced by deleting edge xy from G. If xy is not an edge of G, then $G + xy$ is the graph constructed from G by adding an edge xy (one often refers to this process as *joining x to y in G*). Figure 1.6 illustrates these ideas. Similarly, $G - x$ means the graph derived from G by deleting one vertex x (and all the edges on which x lies). More generally, $G - S$ is the graph resulting from deleting some set S of vertices.

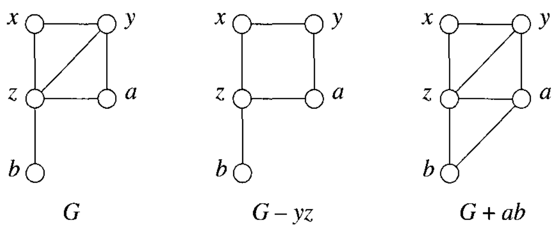


Fig. 1.6. Adding and deleting edges

If vertices x and y are endpoints of one edge in a graph or multigraph, then x and y are said to be *adjacent* to each other, and it is often convenient to write $x \sim y$. Vertices adjacent to x are called *neighbors* of x, and the set of all vertices adjacent to x is called the *neighborhood* of x, and denoted by $N(x)$. If G has v vertices, so that its vertex set is,

$$V(G) = \{x_1, x_2, \ldots, x_v\},$$

then its *adjacency matrix* M_G is the $v \times v$ matrix with entries m_{ij}, such that

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \sim x_j, \\ 0 & \text{otherwise.} \end{cases}$$

The particular matrix will depend on the order in which the vertices are listed.

Example. Consider the graph G shown in Figure 1.6. If its vertices are taken in the order x, y, z, a, b, then its adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Some authors define the adjacency matrix of a multigraph to be the adjacency matrix of the underlying graph; others set m_{ij} equal to the number of edges joining x_i to x_j. We shall not need to use adjacency matrices of multigraphs in this book.

A vertex and an edge are called *incident* if the vertex is an endpoint of the edge, and two edges are called *adjacent* if they have a common endpoint. A set of edges is called *independent* if no two of its members are adjacent, and a set of vertices is independent if no two of its members are adjacent. The *independence number* $\beta(G)$ of a graph G is the number of elements in the largest independent set in G. For example, the graph G of Figure 1.6 has independence number 3; its largest independent set (which happens to be unique) is $\{a, b, x\}$.

If the edge set of G is

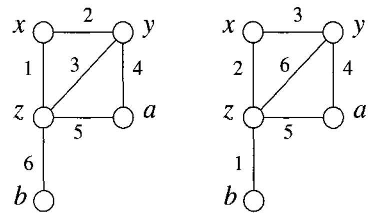
$$E(G) = \{a_1, a_2, \ldots, a_e\},$$

then the *incidence matrix* N_G of G is the $v \times e$ matrix with entries n_{ij}, such that

$$n_{ij} = \begin{cases} 1 & \text{if vertex } x_i \text{ is incident with edge } a_j, \\ 0 & \text{otherwise.} \end{cases}$$

(The adjacency and incidence matrices depend on the orderings chosen for $V(G)$ and $E(G)$; they are not unique, but vary only by row and/or column permutation.)

Example. Here are two copies of the graph of Figure 1.6, with the edges labeled 1, 2, 3, 4, 5, 6 in two different ways.



G_1 G_2

The incidence matrix for the labeling G_1, with edges taken in numerical order and vertices in the order x, y, z, a, b, is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

while the corresponding incidence matrix for the labeling G_2 is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If G is a graph, it is possible to choose some of the vertices and some of the edges of G in such a way that these vertices and edges again form a graph, say H. H is then called a *subgraph* of G; one writes $H \leq G$. Clearly every graph G has itself as a subgraph; we say a subgraph H is a *proper* subgraph of G, and write $H < G$, if it does not equal G. The 1-vertex graph (which we shall denote by K_1) is also a subgraph of every graph. If U is any set of vertices of G, then the subgraph consisting of U and all the edges of G that join two vertices of U is called an *induced* subgraph, the *subgraph induced by* U, and is denoted by $\langle U \rangle$ or $G[U]$. A subgraph G of a graph H is called a *spanning* subgraph if $V(G) = V(H)$. Clearly any graph G is a spanning subgraph of $K_{V(G)}$.

In particular, a *clique* in a graph G is a complete subgraph. In other words, it is a subgraph in which every vertex is adjacent to every other. A clique H in G is called *maximal* if no vertex of G outside of H is adjacent to all members of H. The clique structure of G can be illustrated by forming a new graph $C(G)$ called the *clique graph* of G. The vertices of this graph are in one-to-one correspondence with the maximal cliques of the original, and two vertices are adjacent if and only if the corresponding cliques have a common vertex. The size of the largest clique in G is called the *clique number* of G and denoted by $\omega(G)$.

Example. Figure 1.7 shows a graph G and its clique graph $C(G)$. The maximal cliques of G have vertex sets $\{0, 1, 3, 4\}$, $\{1, 2, 4, 5\}$, $\{5, 8\}$, $\{7, 8, 10\}$, $\{7, 9, 10\}$ and $\{6, 7, 9\}$, and are represented in $C(G)$ by a, b, c, d, e and f respectively.

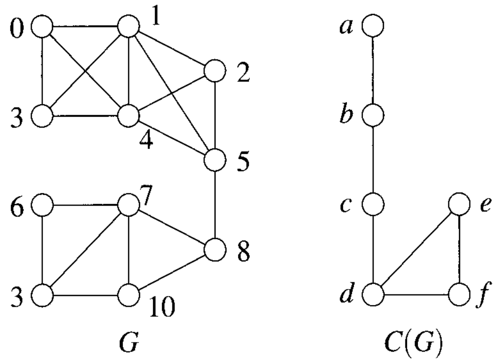


Fig. 1.7. A graph and its clique graph

Given any graph G, the set of all edges of $K_{V(G)}$ that are *not* edges of G will form a graph with $V(G)$ as vertex set; this new graph is called the *complement* of G, and written $\overline{G}$. More generally, if G is a subgraph of H, then the graph formed by deleting all edges of G from H is called the *complement of G in H*, denoted by $H - G$. The complement $\overline{K}_S$ of the complete graph K_S on vertex set S is called a *null graph*; we also write $\overline{K}_v$ for a null graph with v vertices.

A graph is called *disconnected* if its vertex set can be partitioned into two subsets, V_1 and V_2, that have no common element, in such a way that there is no edge with one endpoint in V_1 and the other in V_2; if a graph is not disconnected, then it is *connected*. A disconnected graph consists of a number of disjoint subgraphs; a maximal connected subgraph is called a *component*. As an example, instead of three representations of the same graph, Figure 1.5 might show one 12-vertex graph with three 4-vertex components. In a way, connectedness generalizes adjacency. In a connected graph, not all vertices are adjacent, but if x and y are not adjacent, then there must exist vertices $x_1, x_2, \ldots, x_n$ such that x is adjacent to x_1, x_1 is adjacent to x_2, ... and x_n is adjacent to y; such a sequence is called an *xy-walk*. Conversely, if every pair of nonadjacent vertices is joined by such a walk, the graph is connected. These ideas will be further explored and generalized in Chapters 2 and 3.

The *complete bipartite graph* on V_1 and V_2 has two disjoint sets of vertices, V_1 and V_2; two vertices are adjacent if and only if they lie in different sets. We write $K_{m,n}$ to mean a complete bipartite graph with m vertices in one set and n in the other. Figure 1.8 shows $K_{4,3}$; $K_{1,n}$ in particular is called an *n-star*. Any subgraph of a complete bipartite graph is called *bipartite*. More generally, the *complete r-partite graph* $K_{n_1,n_2,\ldots,n_r}$ is a graph with vertex set $V_1 \cup V_2 \cup \ldots \cup V_r$, where the V_i are disjoint sets and V_i has order n_i, in which xy is an edge if and only if x and y are in different sets. Any subgraph of this graph is called an *r-partite* graph. If $n_1 = n_2 = \ldots = n_r = n$, we use the abbreviation $K_n^{(r)}$.

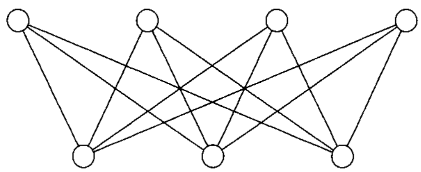
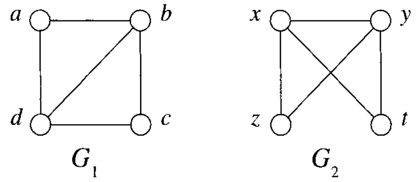


Fig. 1.8. $K_{4,3}$

Example. Prove that the graph G_1, formed by deleting an edge from K_4, and the graph G_2, constructed by adding to $K_{2,2}$ an edge joining two inadjacent vertices, are isomorphic.



Suppose the graphs are as shown. G_1 is formed by deleting edge ac from the complete graph with vertices a, b, c, d. (If instead edge ab were deleted, for example, the graph formed would be isomorphic to the one shown under the mapping $a \to a$, $b \to c$, $c \to b$, $d \to d$.) G_2 was formed from the complete bipartite graph with vertex-sets $\{x, y\}$ and $\{z, t\}$ by joining one pair — we have joined x to y, but the result of joining z to t is isomorphic. The two have the same numbers of vertices and of edges. On inspection it is seen that $a \to z, b \to x, c \to t, d \to y$ maps the vertices of G_1 to the vertices of G_2 and maps the edges of G_1 to the edges of G_2, so it is an isomorphism.

Several ways of combining two graphs have been studied. The *union* $G \cup H$ of graphs G and H has as vertex set and edge set the unions of the vertex sets and edge sets, respectively, of G and H. The *intersection* $G \cap H$ is defined similarly, using the intersections (but $G \cap H$ is defined only when G and H have a common vertex). If G and H are edge-disjoint graphs on the same vertex set, then their union is often also called their *sum* and written $G \oplus H$. ($A \oplus B$ is often written for the union of disjoint sets A and B; in this notation, the graph $G \oplus H$ has edge set $E(G) \oplus E(H)$.) At the other extreme, disjoint unions can be discussed, and the union of n disjoint graphs all isomorphic to G is denoted by nG.

The notation $G + H$ denotes the *join* of G and H, a graph obtained from G and H by joining every vertex of G to every vertex of H. (This notation is not consistent with the earlier use of the + symbol; $G + xy$ is not the join of G and the K_2 with vertex set $\{x, y\}$. Unfortunately, these two uses of + are common in the graph theory literature.) $G + H$ is also used when G and H represent isomorphism-classes of graphs, with the assumption that G and H are disjoint, so that for example

$$K_{m,n} = \overline{K}_m + \overline{K}_n.$$

The *cartesian product* $G \times H$ of graphs G and H is defined as follows:

(i) label the vertices of H in some way;
(ii) in a copy of G, replace each vertex of G by a copy of H;
(iii) add an edge joining vertices in two adjacent copies of H if and only if they have the same label.

In other words, if G has vertex set $V(G) = \{a_1, a_2, \ldots, a_g\}$ and H has vertex set $V(H) = \{b_1, b_2, \ldots, b_h\}$, then $G \times H$ has vertex set $V(G) \times V(H)$, and (a_i, b_j) is adjacent to (a_k, b_ℓ) if and only if *either* $i = k$ and b_j is adjacent in H to b_ℓ *or* $j = \ell$ and a_i is adjacent in G to a_k. It is clear that $G \times H$ and $H \times G$ are isomorphic. Similarly $(G \times H) \times J$ and $G \times (H \times J)$ are isomorphic, so one can omit parentheses and define cartesian products of three or more graphs in a natural way. An example is shown in Figure 1.9.

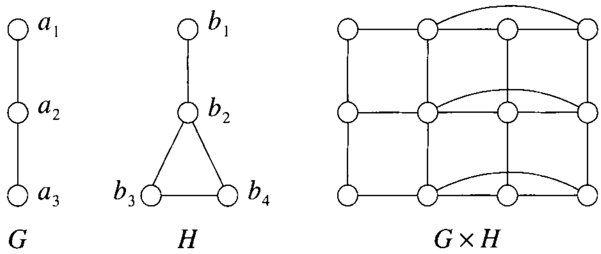


Fig. 1.9. The cartesian product of two graphs

Exercises 1.2

A1.2.1 Write down the incidence and adjacency matrices of the graphs G and H of Figure 1.9.

1.2.2 For each antireflexive symmetric relation in Exercise 1.1.2, write down the incidence and adjacency matrices of the corresponding graph.

1.2.3 Up to isomorphism, there are exactly six connected graphs, and exactly eleven graphs in total, on four vertices. Prove this.

1.2.4 If G is any graph, show that $\overline{\overline{G}} = G$ (that is, the complement of a complement equals the original graph).

A1.2.5 Prove that if G is not a connected graph, then $\overline{G}$ is connected.

1.2.6 Suppose G and H are any two graphs. Show that $\overline{G + H}$ (the complement of the join of G and H) is not connected.

1.2.7 A graph G is *self-complementary* if G and $\overline{G}$ are isomorphic. Prove that the number of vertices in a self-complementary graph must be congruent to 0 or 1 (mod 4).

1.2.8 For any graph G, prove that $\overline{G - x} = \overline{G} - x$, for any vertex x of G.

A1.2.9 G is a bipartite graph with v vertices. Prove that G has at most $\frac{v^2}{4}$ edges.

1.2.10 G is the graph shown in Figure 1.9 and H is the triangle K_3 ; G and H have disjoint vertex sets. Sketch the following graphs:

(i) $3G$
(ii) $2H$
(iii) $G \cup H$
(iv) $G \oplus H$
(v) $G + H$
(vi) $G \times H$
(vii) $G \times G$

1.2.11 How many edges does the star $K_{1,n}$ have? Write down the adjacency and incidence matrices of $K_{1,5}$.

A1.2.12 The graph W_n, called an *n-wheel*, has $n + 1$ vertices $\{x_0, x_1, \ldots, x_n\}$; x_0 is joined to every other vertex and the other edges are

$$x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1.$$

How many edges does W_n have? Write down the adjacency and incidence matrices of W_5.

1.2.13 The *n-ladder* L_n has $2n$ vertices $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$; x_i is joined to y_i for every i; x_i is joined to x_{i+1}, and y_i is joined to y_{i+1} for $i = 1, 2, \ldots, n-1$.
(i) How many edges does L_n have?
(ii) Write down the adjacency and incidence matrices of L_4.

1.2.14 Suppose G is the graph with vertices x, y, z, t and edges xy, yz, zt; H is the graph with vertices x, z and edge xz; K is a graph K_2 with vertex-set disjoint from G. Show that no two of the graphs $G + xz$, $G + H$ and $G + K$ are isomorphic.

1.2.15 A *directed multigraph* is a structure similar to a directed graph, in which multiple arcs in the same direction are allowed. Write down a formal definition of a directed multigraph in terms of a vertex set and an edge set. Give two practical examples of situations that are best modeled by directed multigraphs.

1.3 Degree

We define the *degree* or *valency* $d(x)$ of the vertex x to be the number of edges that have x as an endpoint. If $d(x) = 0$, then x is called an *isolated* vertex while a vertex of degree 1 is called *pendant*. The edge incident with a pendant vertex is called a *pendant edge*. A graph is called *regular* if all its vertices have the same degree. If the common degree is r, it is called *r-regular*. In particular, a 3-regular graph is called *cubic*. We write $\delta(G)$ for the smallest of all degrees of vertices of G, and $\Delta(G)$ for the largest. (One also writes either $\Delta(G)$ or $\delta(G)$ for the common degree of a regular graph G.) The degree $d(x)$ of x will equal the sum of the entries in the row of M_G or of N_G corresponding to x.

Theorem 1.1. *In any graph or multigraph, the sum of the degrees of the vertices equals twice the number of edges.*

Proof. It is convenient to work with the incidence matrix: we sum its entries. The sum of the entries in row i is just $d(x_i)$; the sum of the degrees is $\sum_{i=1}^{v} d(x_i)$, which equals the sum of the entries in N. The sum of the entries in column j is 2, since each edge is incident with two vertices; the sum over all columns is thus $2e$, so that

$$\sum_{i=1}^{v} d(x_i) = 2e,$$

giving the result. □

Corollary 1.1.1 *In any graph or multigraph, the number of vertices of odd degree is even. In particular, a regular graph of odd degree has an even number of vertices.*

Example. A graph with the degrees of vertices marked is shown in Figure 1.10. Observe that there are six odd vertices. The isolated vertex has degree 0.

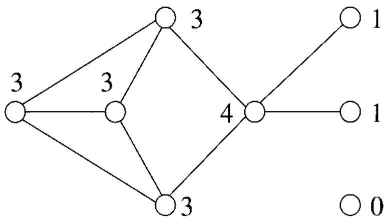


Fig. 1.10. Degrees of vertices

A collection of v nonnegative integers is called *graphical* if and only if there is a graph on v vertices whose degrees are the members of the collection. A graphical collection is called *valid* if and only if there is a *connected* graph with those degrees. (This is one situation where the distinction between graphs and multigraphs is very important: see Exercise 1.3.16.)

Theorem 1.2. [68, 61] *A collection*

$$S = \{d_0, d_1, \ldots, d_{v-1}\}$$

of v integers with $d_0 \geq d_1 \geq \ldots \geq d_{v-1}$, where $d_0 \geq 1$ and $v \geq 2$, is graphical if and only if the collection

$$S' = \{d_1 - 1, \ldots, d_{d_0} - 1, d_{d_0+1}, \ldots, d_{v-1}\}$$

is graphical.

Proof. (i) Suppose S' is graphical. Let H be a graph with vertices $u_1, u_2, \ldots, u_{v-1}$, where

$$\begin{aligned} d(u_i) &= d_i - 1, & 1 \leq i \leq d_0, \\ d(u_i) &= d_i, & d_0 + 1 \leq i \leq v - 1. \end{aligned}$$

Append a new vertex u_0, and join it to $u_1, u_2, \ldots, u_{d_0}$. The resulting graph has degree sequence S.

(ii) Suppose the collection S is graphical. Let G be a graph with vertex set

$$V(G) = \{x_0, x_1, \ldots, x_{v-1}\}$$

such that $d(x_i) = d_i$ for $0 \leq i \leq v - 1$. Two cases arise:

Case 1. Suppose G contains a vertex y of degree d_0, such that y is adjacent to vertices having degrees d_1, d_2, ..., d_{d_0}. In this case, we remove y and all the edges incident with it. The resulting graph has degree sequence S', whence S' is graphical.

Case 2. Suppose there is no such vertex y. We have the vertex set

$$x_0, x_1, x_2, x_3, x_4, \ldots, x_{d_0}, x_{d_0+1}, \ldots, x_k, \ldots, x_{v-1}$$

with degrees

$$d_0 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq \ldots \geq d_{d_0} \geq d_{d_0+1} \geq \ldots \geq d_k \geq \ldots \geq d_{v-1}.$$

Let $X = \{x_{j_1}, x_{j_2}, \ldots, x_{j_n}\}$ be the set of all vertices among $x_1, \ldots, x_{d_0}$ to which x_0 is *not* adjacent. Then $n \geq 1$. Because x_0 is adjacent to d_0 vertices altogether, there must be exactly n vertices in the set

$$Y = \{x_{k_1}, x_{k_2}, \ldots, x_{k_n}\}$$

among $x_{d_0+1}, \ldots, x_{v-1}$ to which x_0 *is* adjacent. We show that there is a vertex x_j in X and a vertex x_k in Y such that $d(x_j) > d(x_k)$. Suppose otherwise. Then all the vertices in X and all the vertices in Y have the same degree. Then interchanging x_{j_i} and x_{k_i} in the sequence of vertices for each i, $i = 1, \ldots, n$, produces a reordering of S, satisfying the conditions of the theorem, in which x_0 is adjacent to vertices having degrees d_1, d_2, ..., d_{d_0}. So G falls into Case 1, which we assumed was not true.

So there exist vertices x_j and x_k such that x_0 is not adjacent to x_j, x_0 is adjacent to x_k and $d(x_j) > d(x_k)$. Since the degree of x_j is greater than that of x_k, there must be a vertex x_m that is adjacent to x_j but not to x_k.

We delete the edges x_0x_k and x_jx_m from G and add edges x_0x_j and x_kx_m. The result is a graph G' having the same degree sequence s as G. However, the sum of the degrees of the vertices adjacent to x_0 in G' is larger than that in G. If G' falls into Case 1, then S' is graphical. If not, apply the argument to G', obtaining a new graph G'' with the same degree sequence as G, but such that the sum of the degrees of the vertices adjacent to x_0 is larger than the corresponding sum in G'. If G'' falls into Case 1, then S' is graphical; otherwise repeat again. Continuing this procedure must eventually result in a graph satisfying the hypothesis of Case 1, because the total sum of all the degrees remains the same for each new graph, while the sum of the degrees of vertices adjacent to x_0 increases. □

Example. Consider the sequence

$$S = \{6, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1\}.$$

This sequence is graphical if and only if

$$\begin{aligned} S' &= \{2, 2, 2, 2, 1, 1, 2, 2, 1, 1\} \\ &= \{2, 2, 2, 2, 2, 2, 1, 1, 1, 1\} \end{aligned}$$

is graphical; equivalently

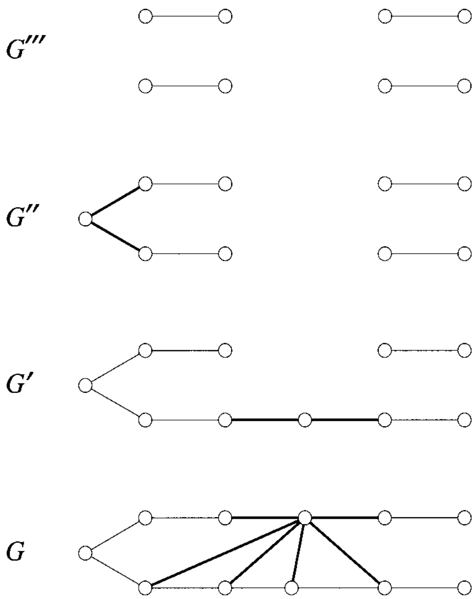


Fig. 1.11. Constructing a graph with given degrees

$$S'' = \{1, 1, 2, 2, 2, 1, 1, 1, 1\}$$
$$= \{2, 2, 2, 1, 1, 1, 1, 1, 1\}$$

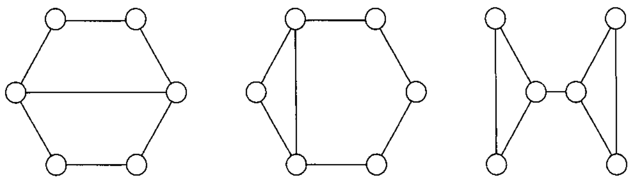
must be graphical, as must

$$S''' = \{1, 1, 1, 1, 1, 1, 1, 1\}.$$

Now S''' is easily seen to be graphical: the corresponding graph consists of four independent edges. So S is graphical. The method of constructing a suitable graph is illustrated in Figure 1.11, where G corresponds to S, G' to S', and so on. For example, S' was derived from S by subtracting 1 from six of the degrees, and the six resulting degrees are $\{2, 2, 2, 2, 1, 1\}$, so we select any six vertices in G' whose degrees are $\{2, 2, 2, 2, 1, 1\}$, and join a new vertex to them. In each case the new edges are shown with heavy lines.

As was shown by the above example, application of Theorem 1.2 not only finds whether a sequence is graphical, it also enables us to find a graph with the given degree sequence, if one exists. However, there is no guarantee of uniqueness. In fact, one can often find two graphs with the same degree sequence.

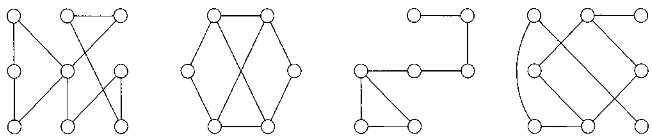
Example. Find three nonisomorphic graphs with degree sequence 3, 3, 2, 2, 2, 2.



The three graphs are obviously not isomorphic: they have different numbers of subgraphs K_3.

Exercises 1.3

A1.3.1 In each case, find the degrees of the vertices:



(i) (ii) (iii) (iv)

HA1.3.2 Suppose G has v vertices and $\delta(G) \geq (v-1)/2$. Prove that G is connected.

1.3.3 Prove that a regular graph of odd degree can have no component with an odd number of vertices.

A1.3.4 Prove that the collection $\{3, 2, 2, 2, 1\}$ is valid. Find a graph with this collection of degrees.

1.3.5 Prove that the collection $\{3, 3, 2, 1, 1\}$ is valid. Find a graph with this collection of degrees.

H1.3.6 Find two nonisomorphic graphs with degree collection 2, 2, 2, 1, 1.

1.3.7 Find two nonisomorphic graphs with degree collection 3, 3, 3, 3, 3, 3.

H1.3.8 Prove that there do not exist nonisomorphic graphs on four vertices with the same degree collection.

1.3.9 Prove that (up to isomorphism) there is exactly one graph with degree collection $\{5, 5, 4, 4, 3, 3\}$.

A1.3.10 Which of the following are graphical?

(i) $\{5, 5, 4, 4, 2, 2\}$;
(ii) $\{5, 4, 4, 3, 3, 3\}$;
(iii) $\{5, 4, 4, 4, 3, 3\}$;
(iv) $\{2, 2, 1, 1, 1, 1\}$?

1.3.11 Which of the following are graphical?

(i) $\{4, 3, 2, 2, 2, 1\}$;
(ii) $\{3, 3, 2, 2, 2, 1\}$;
(iii) $\{5, 4, 4, 4, 2, 1\}$;
(iv) $\{2, 2, 2, 2, 1, 1\}$?

A1.3.12 Find a graph on six vertices that has at least one vertex of each degree 1, 2, 3, 4, 5.

1.3.13 For each v, show that there exists a graph on v vertices that has at least one vertex of each degree $1, 2, \ldots, v-2, v-1$.

1.3.14 Prove that no graph has all its vertices of different degrees.

A1.3.15 Find a multigraph on four vertices that has all its vertices of different degrees.

1.3.16 Prove that there is a multigraph with degree sequence, $\{4, 3, 1, 1, 1\}$, but there is no multigraph with degree sequence $\{2, 0, 0, 0\}$. Deduce that Theorem 1.2 does not apply to multigraphs.

A1.3.17 Prove: if d and v are natural numbers, not both odd, with $v > d$, then there there is a regular graph of degree d with exactly v vertices.

A1.3.18 In a looped multigraph, each loop is defined to add 2 to the degree of its vertex.

(i) Do Theorem 1.1 and Corollary 1.3 hold for looped multigraphs?

(ii) Suppose D is any finite sequence of nonnegative integers such that the sum of all its members is even. Show that there is a looped multigraph with degree sequence D.