Chapter 8

Higher Values of the Applied Field

The previous chapter dealt with minimizers of the Ginzburg–Landau functional when the applied field was $O(|\log \varepsilon|)$. The applied field behaving asymptotically like $\lambda|\log \varepsilon|$, letting $\lambda \to \infty$ in Theorem 7.2 indicates that for energy-minimizers for applied fields $h_{\text{ex}} \gg |\log \varepsilon|$, we must have $\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}} \to 1$, and $\frac{h_\varepsilon}{h_{\text{ex}}} \to 1$. But in this regime, $\frac{G_\varepsilon(u_\varepsilon, A_\varepsilon)}{{h_{\text{ex}}}^2} \to 0$ and the arguments of Chapter 7 do not give, even formally, the leading order term of the minimal energy. Moreover, the tools which were at the heart of the result, namely the vortex balls construction of Theorem 4.1 and the Jacobian estimate of Theorem 6.1 break down for higher values of h_{ex}.

On the other hand, we recall from Chapter 2 the prediction by Abrikosov that the transition from the mixed state, which we may as well call the vortex state, to the normal state, should occur for $h_{\text{ex}} \approx 1/\varepsilon^2$, i.e., much higher fields. We will show in this chapter how our techniques still allow us to find the minimum of the energy for applied fields satisfying $|\log \varepsilon| \ll h_{\text{ex}} \ll 1/\varepsilon^2$: in the scaling of Chapter 7 what we determine here is the first nonzero lower-order correction term. We find that minimizers have a uniform limiting density in the whole domain Ω, in agreement with Abrikosov lattices. In fact, the test-configurations we use below to obtain the upper bound on the minimal energy are constructed to be periodic.

Theorem 8.1. *Assume, as $\varepsilon \to 0$, that $|\log \varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$. Then, letting $(u_\varepsilon, A_\varepsilon)$ minimize G_ε, and letting $g_\varepsilon(u, A)$ denote the energy-density*

$\frac{1}{2}\left(|\nabla_A u|^2 + |h - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)$, *we have*

$$\frac{2g_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{ex}\log\frac{1}{\varepsilon\sqrt{h_{ex}}}} \rightharpoonup dx \quad \text{as } \varepsilon \to 0 \tag{8.1}$$

in the weak sense of measures, where dx denotes the 2-dimensional Lebesgue measure, and

$$\min_{(u,A)\in H^1\times H^1} G_\varepsilon(u,A) \sim \frac{|\Omega|}{2} h_{ex}\log\frac{1}{\varepsilon\sqrt{h_{ex}}} \quad \text{as } \varepsilon \to 0, \tag{8.2}$$

where $|\Omega|$ is the area of Ω.

Since in this regime $h_{\rm ex}\log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \ll {h_{\rm ex}}^2$, we deduce as an immediate corollary:

Corollary 8.1. *Assume that, as $\varepsilon \to 0$, $|\log\varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$ and $(u_\varepsilon, A_\varepsilon)$ minimize G_ε, letting $h_\varepsilon = \operatorname{curl} A_\varepsilon$ and $\mu(u_\varepsilon, A_\varepsilon) = \operatorname{curl}(iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + h_\varepsilon$, we have*

$$\frac{h_\varepsilon}{h_{ex}} \to 1 \quad \text{in } H^1(\Omega)$$
$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \to dx \quad \text{in } H^{-1}(\Omega).$$

Proof. Since $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε, it is a solution of (GL) and thus, using Lemma 3.3, we find

$$\|h_\varepsilon - h_{\rm ex}\|^2_{H^1(\Omega)} \le 2G_\varepsilon(u_\varepsilon, A_\varepsilon) \ll {h_{\rm ex}}^2$$

hence $h_\varepsilon/h_{\rm ex} \to 1$ in $H^1(\Omega)$. Since we have the relation $-\Delta h_\varepsilon + h_\varepsilon = \mu(u_\varepsilon, A_\varepsilon)$ obtained by taking the curl of the second Ginzburg–Landau equation, the convergence of $\mu(u_\varepsilon, A_\varepsilon)/h_{\rm ex}$ follows. □

The theorem is a direct consequence of Propositions 8.1 and 8.2 below, but let us briefly explain what problem occurs for high fields and how it is overcome. If $h_{\rm ex}$ is too high, say $h_{\rm ex} \gg 1/\varepsilon$, then a minimizer of G_ε is expected to have a number of vortices n of the order of $h_{\rm ex}$ and then the perimeter of the set where $|u| < 1/2$ should be of the order $n\varepsilon \gg 1$. This means that we can no longer hope that the a priori bound on the energy satisfied by a minimizer excludes, say, a line where $|u| = 0$. As we mentioned, the downside is that the vortex balls construction and the Jacobian estimate, which are based on covering the set $\{|u|=0\}$ by small balls, will not work anymore.

On the other hand, for such large fields, the problem of minimizing G_ε reduces to that of minimizing it on any subdomain, in other words the minimization problem becomes *local*. Thus we may perform blow-ups which yield the right lower bound. The effect of the blow-ups will be precisely to effectively reduce h_{ex} and allow our techniques to be applied on the smaller scale. On the other hand, that the upper bound that we need will demand a more rigid construction of a good test-configuration than in Proposition 7.4.

The rescaling formula is:

Lemma 8.1. *Given (u, A) and Ω, assuming $0 \in \Omega$, define u_λ, A_λ and Ω_λ by*

$$u_\lambda(\lambda x) = u(x), \quad \lambda A_\lambda(\lambda x) = A(x), \quad \Omega_\lambda = \lambda\Omega. \tag{8.3}$$

Then, for any h_{ex}, we have $G_\varepsilon(u, A, \Omega) = G_\varepsilon^\lambda(u_\lambda, A_\lambda, \Omega_\lambda)$, where

$$G_\varepsilon^\lambda(u_\lambda, A_\lambda, \Omega_\lambda) = \frac{1}{2}\int_{\Omega_\lambda} |\nabla_{A_\lambda} u_\lambda|^2 + \lambda^2\left(\operatorname{curl} A_\lambda - \frac{h_{ex}}{\lambda^2}\right)^2 + \frac{\left(1-|u_\lambda|^2\right)^2}{2(\lambda\varepsilon)^2}. \tag{8.4}$$

The proof is straightforward and we omit it.

8.1 Upper Bound

Proposition 8.1. *Assume, as $\varepsilon \to 0$, that $1 \ll h_{ex} \ll 1/\varepsilon^2$. Then for any ε small enough*

$$\min_{(u,A)\in H^1\times H^1} G_\varepsilon(u, A, \Omega) \le h_{ex}\frac{|\Omega|}{2}\left(\log\frac{1}{\varepsilon\sqrt{h_{ex}}} + C\right). \tag{8.5}$$

Proof. The proof is done by constructing a test configuration $(u_\varepsilon, A_\varepsilon)$ which is periodic, in the sense that gauge-invariant quantities are periodic. Let

$$\lambda = \sqrt{\frac{h_{\text{ex}}}{2\pi}}$$

and let $L_\varepsilon = \lambda\mathbb{Z} \times \lambda\mathbb{Z}$. We let h_ε be the solution in $\mathbb{R}^2$ of

$$-\Delta h_\varepsilon + h_\varepsilon = 2\pi \sum_{a \in L_\varepsilon} \delta_a. \tag{8.6}$$

It is thus periodic with respect to L_ε.

Then we define ρ_ε by

$$\rho_\varepsilon(x) = \begin{cases} 0 & \text{if } |x-a| \le \varepsilon \text{ for some } a \in L_\varepsilon, \\ \dfrac{|x-a|}{\varepsilon} - 1 & \text{if } \varepsilon < |x-a| < 2\varepsilon \text{ for some } a \in L_\varepsilon, \\ 1 & \text{otherwise.} \end{cases} \tag{8.7}$$

Finally, as in the proof of Proposition 7.3, we define A_ε to solve $\operatorname{curl} A_\varepsilon = h_\varepsilon$ and φ_ε, well defined modulo 2π, to solve $-\nabla^\perp h_\varepsilon = \nabla\varphi_\varepsilon - A_\varepsilon$ in $\mathbb{R}^2 \setminus L_\varepsilon$. Then we let $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$.

By construction, every gauge-invariant quantity is periodic with respect to the lattice L_ε, thus if we choose the origin carefully, the energy $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ will be estimated by computing the energy per unit cell. Indeed, let

$$K_\varepsilon = \left(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right) \times \left(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right)$$

be the unit cell of L_ε. For each $x \in K_\varepsilon$ we may define a translated lattice L_ε^x, and a corresponding test configuration $(u_\varepsilon^x, A_\varepsilon^x)$, with energy density $\mathrm{gl}_\varepsilon^x(y) = \mathrm{gl}_\varepsilon(y - x)$. Then, applying Fubini's theorem, we have

$$\int_{x \in K_\varepsilon} G_\varepsilon\left(u_\varepsilon^x, A_\varepsilon^x, \Omega\right) dx = \iint_{\substack{x \in K_\varepsilon \\ y \in \Omega}} \mathrm{gl}_\varepsilon^x(y)\, dx\, dy = |\Omega| G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon),$$

since gl_ε is periodic with respect to the lattice L_ε. It follows, using the mean value formula, that we may choose x such that

$$G_\varepsilon\left(u_\varepsilon^x, A_\varepsilon^x, \Omega\right) \le \frac{|\Omega|}{|K_\varepsilon|} G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon). \tag{8.8}$$

We estimate $G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon)$, arguing as in Proposition 7.3: we have $|\nabla_{A_\varepsilon} u_\varepsilon|^2 = |\nabla\rho_\varepsilon|^2 + {\rho_\varepsilon}^2 |\nabla\varphi_\varepsilon - A_\varepsilon|^2$ and ${\rho_\varepsilon}^2 |\nabla\varphi_\varepsilon - A_\varepsilon|^2 \ge |\nabla h_\varepsilon|^2$. Moreover, writing B_r for $B(0,r)$ and using (8.7)

$$\frac{1}{2} \int_{B_{2\varepsilon}} |\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2} \left(1 - {\rho_\varepsilon}^2\right)^2 \le C.$$

We deduce that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon) \le \frac{1}{2} \int_{K_\varepsilon \setminus B_\varepsilon} |\nabla h_\varepsilon|^2 + \frac{1}{2} \int_{K_\varepsilon} (h_\varepsilon - h_{\rm ex})^2\, dx + C. \tag{8.9}$$

To estimate the right-hand side, we perform a change of variables $y = \lambda x$. Then

$$\int_{K_\varepsilon \setminus B_\varepsilon} |\nabla h_\varepsilon|^2 + \int_{K_\varepsilon} (h_\varepsilon - h_{\rm ex})^2\, dx = \int_{K \setminus B_{\lambda\varepsilon}} |\nabla \tilde{h}_\varepsilon|^2 + \frac{2\pi}{h_{\rm ex}} \int_K \tilde{h}_\varepsilon^2\, dy \tag{8.10}$$

where $\tilde{h}_\varepsilon(y) = h_\varepsilon(x) - h_{\rm ex}$ and $K = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$. Now we decompose $\tilde{h}_\varepsilon - h_{\rm ex}$ as

$$\tilde{h}_\varepsilon(y) = g_\varepsilon(y) - \log|y|, \tag{8.11}$$

and we show that g_ε is bounded in $W^{1,q}(K)$ independently of ε for any $q > 0$.

First, by periodicity, the integral of h_ε in K_ε is 2π, thus the integral of $\tilde{h}_\varepsilon$ in K is $2\pi\lambda^2 - h_{\rm ex} = 0$. Therefore g_ε and $\log|\cdot|$ have the same mean value in K, and that value does not depend on ε. We deduce from Poincaré's inequality that

$$\|g_\varepsilon\|^2_{L^2(K)} \le C\left(1 + \|\nabla g_\varepsilon\|^2_{L^2(K)}\right). \tag{8.12}$$

Second note that h_ε, which is the solution to (8.6), is also the solution of $-\Delta h_\varepsilon + h_\varepsilon = 2\pi\delta_0$ in K_ε and $\partial_\nu h_\varepsilon = 0$ on ∂K_ε. Indeed, the problem (8.6) is symmetric with respect to each line containing a side of the square K_ε, hence h_ε is equal to its symmetrized and $\partial_\nu h_\varepsilon = 0$ on ∂K_ε. Therefore $g_\varepsilon(y) = h_\varepsilon(y/\lambda) - h_{\rm ex} + \log|y|$ solves

$$\begin{cases} -\Delta g_\varepsilon + \lambda^{-2}\left(g_\varepsilon + h_{\rm ex} - \log\right) = 0 & \text{in } K, \\ \partial_\nu g_\varepsilon = \partial_\nu \log & \text{on } \partial K. \end{cases}$$

Multiplying the equation by g_ε and integrating by parts in K yields

$$\int_K |\nabla g_\varepsilon|^2 + \frac{1}{\lambda^2}\left(g_\varepsilon^2 + g_\varepsilon h_{\rm ex} - g_\varepsilon \log\right) = \int_{\partial K} g_\varepsilon \partial_\nu g_\varepsilon.$$

We deduce, replacing λ by its value and using the facts that $\partial_\nu g_\varepsilon = \partial_\nu \log$ on ∂K and that the average of g_ε on K does not depend on ε,

$$\|\nabla g_\varepsilon\|^2_{L^2(K)} \leq C \left(1 + {h_{\mathrm{ex}}}^{-1} \|g_\varepsilon\|^2_{L^2(K)} + \|g_\varepsilon\|_{L^2(\partial K)}\right). \tag{8.13}$$

Since $1 \ll h_{\mathrm{ex}}$, if ε is small enough, then h_{ex} is large enough so that using (8.12) and bounding the L^2 norm of the trace of g_ε by the H^1 norm, the terms in the right-hand side of (8.13) are absorbed by $\|\nabla g_\varepsilon\|^2_{L^2(K)}$ yielding $\|g_\varepsilon\|_{H^1(K)} \leq C$. We deduce that g_ε is bounded independently of ε in $L^q(K)$ for every $q > 0$ and then, using the equation satisfied by g_ε, that for every $q > 0$

$$\|\nabla g_\varepsilon\|^2_{W^{1,q}(K)} \leq C.$$

Together with (8.11), this implies that

$$\int_{K \setminus B_{\lambda\varepsilon}} |\nabla \tilde{h}_\varepsilon|^2 \leq C + \int_{K \setminus B_{\lambda\varepsilon}} |\nabla \log|^2 \leq C + 2\pi \log \frac{1}{\lambda\varepsilon},$$

and also $\frac{2\pi}{h_{\mathrm{ex}}} \int_K \tilde{h}_\varepsilon^2 \leq C$. Together with (8.8), (8.9), and (8.10), this yields, since $|K_\varepsilon| = \lambda^{-2} = 2\pi/h_{\mathrm{ex}}$,

$$G_\varepsilon\left(u_\varepsilon^x, A_\varepsilon^x, \Omega\right) \leq \frac{|\Omega|}{|K_\varepsilon|}\left(\pi \log \frac{1}{\lambda\varepsilon} + C\right) \leq \frac{|\Omega|}{2} h_{\mathrm{ex}} \left(\log \frac{1}{\sqrt{h_{\mathrm{ex}}}\varepsilon} + C\right). \qquad \square$$

8.2 Lower Bound

We now wish to compute a lower bound for $G_\varepsilon(u, A)$ which matches the upper bound of the previous section. In the course of the proof we will see clearly that if (u, A) minimizes G_ε, then its energy is accounted for by the vortex-energy.

In what follows we denote $B_\lambda^x = B(x, \lambda^{-1})$ and we will often omit the subscript ε, where x is the center of the blow-up.

Proposition 8.2. *Assume $|\log \varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$ and $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε. Then there exists $1 \ll \lambda \ll \frac{1}{\varepsilon}$ such that for every $x \in \Omega$ such that $B_\lambda^x \subset \Omega$, we have*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, B_\lambda^x) \geq \frac{|B_\lambda^x|}{2} h_{ex} \log \frac{1}{\varepsilon\sqrt{h_{ex}}} (1 - o(1)). \tag{8.14}$$

Proof. As already mentioned, the proof is achieved by blowing up at the scale λ.

Define u_λ and A_λ as in (8.3), but taking the origin at x. From Lemma 8.1, (8.4), again with the origin at x, and dropping the ε subscripts, the left-hand side of (8.14) is equal to

$$\frac{1}{2}\int_{B_1} |\nabla_{A_\lambda} u_\lambda|^2 + \lambda^2 \left(\operatorname{curl} A_\lambda - \frac{h_{\rm ex}}{\lambda^2}\right)^2 + \frac{\left(1-|u_\lambda|^2\right)^2}{2(\lambda\varepsilon)^2}$$

thus, letting $u' = u_\lambda$, $A' = A_\lambda$, $\varepsilon' = \lambda\varepsilon$ and ${h_{\rm ex}}' = h_{\rm ex}/\lambda^2$, the inequality (8.14) that we wish to prove is equivalent to

$$\frac{1}{2}\int_{B_1} |\nabla_{A'} u'|^2 + \lambda^2 \left(\operatorname{curl} A' - {h_{\rm ex}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \geq \frac{|B_1|}{2} {h_{\rm ex}}' \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\left(1-o(1)\right). \quad (8.15)$$

Now we choose λ such that

$${h_{\rm ex}}' = |\log \varepsilon'|. \quad (8.16)$$

Let us check that this is possible and give the behavior of λ as $\varepsilon \to 0$. Condition (8.16) is equivalent to $\varepsilon^2 h_{\rm ex} = f(\varepsilon\lambda)$, where $f(x) = x^2\log(1/x)$. Since $\varepsilon^2 h_{\rm ex} \to 0$ as $\varepsilon \to 0$, it is easy to check that for ε small enough, there is a unique $x_\varepsilon \in (0, 1/2)$ satisfying $f(x_\varepsilon) = \varepsilon^2 h_{\rm ex}$. Moreover from $|\log\varepsilon| \ll h_{\rm ex} \ll 1/\varepsilon^2$ we deduce $\varepsilon \ll x_\varepsilon \ll 1$. Therefore (8.16) can indeed be verified, and the corresponding λ, ε' satisfy

$$1 \ll \lambda \ll \frac{1}{\varepsilon}, \quad \varepsilon' \ll 1, \quad \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \approx |\log\varepsilon'|,$$

the last identity being deduced from $\varepsilon^2 h_{\rm ex} = f(\varepsilon\lambda) = f(\varepsilon')$ by taking logarithms. Thus with this choice of λ, (8.15) becomes

$$\frac{1}{2}\int_{B_1} |\nabla_{A'} u'|^2 + \lambda^2 \left(\operatorname{curl} A' - {h_{\rm ex}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \geq \frac{|B_1|}{2} {h_{\rm ex}}' |\log\varepsilon'| \left(1-o(1)\right). \quad (8.17)$$

Two cases may now occur, depending on the blow-up origin x. Either

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - {h_{\rm ex}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \gg {h_{\rm ex}}'^2$$

as $\varepsilon \to 0$ and then, from (8.16), (8.17) is clearly satisfied, or

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - {h_{\rm ex}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \le C{h_{\rm ex}}'^2.$$

This way, we have reduced to the case of configurations with a relatively small energy, for which all the analysis of previous chapters apply.

In this case, since $\lambda \gg 1$ we find

$$\frac{\operatorname{curl} A' - {h_{\rm ex}}'}{{h_{\rm ex}}'} \to 0, \quad \text{in } L^2(B_1). \tag{8.18}$$

On the other hand, replacing ε by ε' and $h_{\rm ex}$ by ${h_{\rm ex}}'$, the hypotheses of Theorem 7.1, item 1) are satisfied and we deduce from (7.6), (7.8) that

$$\liminf_{\varepsilon'\to 0} \frac{1}{2{h_{\rm ex}}'^2}\int_{B_1} |\nabla_{A'}u'|^2 + \left(\operatorname{curl} A' - {h_{\rm ex}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \ge \frac{\|\mu'\|}{2},$$

where $\mu' = -\Delta h' + h'$ and h' is the limit of $\operatorname{curl} A'/{h_{\rm ex}}'$. From (8.18) we have $\mu' = 1$, hence

$$\liminf_{\varepsilon'\to 0} \frac{1}{2{h_{\rm ex}}'^2}\int_{B_1} |\nabla_{A'}u'|^2 + \left(\operatorname{curl} A' - {h_{\rm ex}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \ge \frac{\pi}{2},$$

and (8.17) is satisfied since for our choice of λ

$$\frac{\pi}{2}{h_{\rm ex}}'^2 = \frac{|B_1|}{2}{h_{\rm ex}}'\log\frac{1}{\varepsilon'}.$$

We have shown for our particular choice of λ that (8.17), hence (8.15) and then (8.14) are satisfied for every choice of blow-up origin x. $\square$

To conclude the proof of Theorem 8.1, we integrate (8.14) with respect to x. Letting U be any open subdomain of Ω, using Fubini's theorem, we have

$$\begin{aligned}\int_{x\in U} G_\varepsilon(u,A,B_\lambda^x\cap U) &= \iint_{\substack{x\in U\\ y\in B_\lambda^x\cap U}} g_\varepsilon(u,A)(y)\,dy\,dx\\ &= \iint_{\substack{x\in U\\ y\in B_\lambda^x\cap U}} g_\varepsilon(u,A)(y)\,dx\,dy\\ &= \int_{y\in U} |B_\lambda^y\cap U| g_\varepsilon(u,A)(y)\,dy \le \frac{\pi}{\lambda^2} G_\varepsilon(u,A,U).\end{aligned}$$

We deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{G_\varepsilon(u,A,U)}{h_{\mathrm{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}} &\ge \liminf_{\varepsilon\to 0}\int_{x\in U}\frac{\lambda^2 G_\varepsilon(u,A,B_\lambda^x\cap U)}{\pi h_{\mathrm{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}}\\ &\ge \liminf_{\varepsilon\to 0}\int_{x\in U, B_\lambda^x\subset U}\frac{\lambda^2 G_\varepsilon(u,A,B_\lambda^x\cap U)}{\pi h_{\mathrm{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}}\\ &\ge \int_{x\in U}\liminf_{\varepsilon\to 0}\left(\mathbf{1}_{B_\lambda^x\subset U}\frac{G_\varepsilon(u,A,B_\lambda^x)}{h_{\mathrm{ex}}|B_\lambda^x|\log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}}\right)\\ &\ge \frac{|U|}{2},\end{aligned} \tag{8.19}$$

where we have used Fatou's lemma and (8.14). In view of Proposition 8.1, we know that $\left(h_{\mathrm{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}\right)^{-1} g_\varepsilon(u_\varepsilon,A_\varepsilon)$ is bounded in $L^1(\Omega)$, hence has a weak limit g in the sense of measures. Since continuous functions on Ω can be uniformly approximated by characteristic functions, (8.19) allows to say that $g\ge\frac{dx}{2}$. But since (8.5) holds, there must be equality, which proves (8.1), and (8.2) immediately follows.

Bibliographic notes on Chapter 8: The result of this chapter was obtained in [170], but the proof is presented here under a much simpler form. The case of higher h_{ex}, of order b/ε^2 with $b<1$, was studied in [172].