Chapter 14

A Guide to the Literature

Our goal here is to give a brief overview of results on Ginzburg–Landau, and point towards suitable references (in thematic, rather than chronological or hierarchical order). We apologize for not being able to be completely exhaustive.

There have been a few review-type papers on Ginzburg–Landau that one can also refer to, notably [40, 155, 85, 68].

14.1 Ginzburg–Landau without Magnetic Field

14.1.1 Static Dimension 2 Case in a Simply Connected Domain

The first studies of that model, i.e., of the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$

and its critical points, seem to date back to Elliott–Matano–Tang Qi [92] who proved that energy-minimizers have isolated zeroes, and to Fife and Peletier [96], who gave a formal justification of the "vanishing gradient property" for solutions.

The energy E_ε was then studied in details by Bethuel–Brezis–Hélein, in [42] for the case without vortices and in [43] for the case with vortices, both times with a fixed Dirichlet boundary data g of modulus one. They derived the "renormalized energy" (or the Γ-limit) of the problem:

$$W((a_1,d_1),\dots,(a_n,d_n)) = -\pi \sum_{i\neq j} d_i d_j \log|a_i - a_j| \\ - \pi \sum_i d_i R(a_i) + \frac{1}{2}\int_{\partial\Omega} \Phi_0 \left(ig, \frac{\partial g}{\partial \tau}\right).$$

where Φ_0 is the solution with zero average on the boundary of

$$\begin{cases} \Delta \Phi_0 = 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ \dfrac{\partial \Phi_0}{\partial \nu} = (ig, \dfrac{\partial g}{\partial \tau}) & \text{on } \partial\Omega \end{cases}$$

and $R(x) = \Phi_0(x) - \sum_i d_i \log|x - a_i|$. Convergence of minimizers and critical points under the assumption $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, and of their vortices, was established, with the derivation of the *renormalized energy* and of the "vanishing gradient property" presented here in Chapter 13. We sum up some of their results below:

Theorem 14.1. (Bethuel–Brezis–Hélein [43]). *Let Ω be a strictly star-shaped simply connected domain of $\mathbb{R}^2$ and $g : \partial\Omega \to \mathbb{S}^1$ a smooth map of degree $d > 0$.*

If u_ε minimizes E_ε among maps with values g on $\partial\Omega$. Then, as $\varepsilon \to 0$, up to extraction of a subsequence, there exist d distinct points $a_1,\dots,a_d \in \Omega$ such that $u_\varepsilon \to u_*$ in $C^k_{loc}(\Omega\backslash \cup_i \{a_i\})$ where

1. *u_* is an $\mathbb{S}^1$-valued harmonic map from $\Omega\backslash\{a_1,\dots,a_d\}$ to $\mathbb{S}^1$ with $u_* = g$ on $\partial\Omega$ and with degree $d_i = 1$ around each a_i.*

2. *$(a_1,\dots,a_d)$ is a minimizer of the renormalized energy W with $d_i = 1$.*

3. *$E_\varepsilon(u_\varepsilon) \ge \pi d|\log\varepsilon| + W(a_1,\dots,a_d) + d\gamma + o(1)$.*

If u_ε is a sequence of solutions with $u_\varepsilon = g$ on $\partial\Omega$ and $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, then, as $\varepsilon \to 0$ and up to extraction of a subsequence, there exist distinct points $a_1,\dots,a_n \in \Omega$, and degrees $d_1,\dots,d_n$ with $\sum_{i=1}^n d_i = d$, such that $u_\varepsilon \to u_*$ in $C^k_{loc}(\Omega\backslash \cup_i \{a_i\})$ where u_* is a harmonic map from $\Omega\backslash\{a_1,\dots,a_n\}$ to $\mathbb{S}^1$ with $u_* = g$ on $\partial\Omega$ and with degree d_i around each a_i. Moreover $((a_1,d_1),\dots,(a_n,d_n))$ is a critical point of W (the d_i's being fixed) and satisfies the "vanishing gradient property."

Their starshapedness assumption on the domain was removed and replaced for minimizers by simple-connectedness by Struwe [189].

A large literature followed, which we review in thematic rather than chronological order. Note that all the results we mention below in this section without magnetic field are under the assumption that $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$, i.e., concern bounded (as $\varepsilon \to 0$) numbers of vortices, and that this is one of the main limitations to adapting them to the case with magnetic field.

14.1.2 Vortex Solutions in the Plane

The existence of radial vortex solutions in the plane, i.e., solutions of the form $f_n(r)e^{in\theta}$ in polar coordinates, where f_n satisfies a certain ODE, was established by Hervé and Hervé [111] via the study of the ODE (note that these solutions have infinite energy for $n \neq 0$). As we saw in Theorem 3.2, it was established by Mironescu [142] that the only solution of degree ± 1 at infinity is the radial one (up to translation). For general solutions in the plane, the quantization result $\int_\Omega (1-|u|^2)^2 = 2\pi d^2$ where d is the total degree, was established by Brezis–Merle–Rivière [61], see Theorem 3.4; other qualitative results were obtained by Sandier and Shafrir [165, 186].

It is not yet fully known whether there can exist nonradial vortex solutions in the plane. These solutions would have a finite number of vortices of degree d_i which would have to satisfy the relation (related to the result of [61] and the Pohozaev identity)

$$\sum_i d_i^2 = (\sum_i d_i)^2.$$

Ovchinnikov and Sigal conjectured the existence of such solutions (having some rotational symmetry) and gave heuristic arguments to support this statement in [147] (see also Open Problem 4 in Chapter 15).

14.1.3 Other Boundary Conditions

More general Dirichlet data (of modulus not equal to one and even possibly vanishing) were studied by André–Shafrir [26]. Neumann boundary conditions were also considered, see for example Spirn [188] for a derivation of the renormalized energy in that case.

14.1.4 Weighted Versions

Versions of the energy with different potential terms, or weighted versions, meant to include possible pinning effects, such as

$$\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(a(x)-|u|)^2}{2\varepsilon^2}$$

or

$$\frac{1}{2}\int_\Omega p(x)|\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$

were studied by André–Shafrir [25], Hadiji–Beaulieu [33, 34], Du–Lin [86].

14.1.5 Construction of Solutions

Once the main result of [43] is known, namely that critical points/minimizers of E_ε have vortices which converge to critical points/minimizers of the renormalized energy, it is natural to examine the interesting inverse problem: given a critical point of the renormalized energy, can one find sequences of solutions of (1.3) whose vortices converge as $\varepsilon_n \to 0$ to these points? This has been solved under the restriction that vortices all be of degree ± 1; first for the case of local minimizers and min-max solutions by Lin [128] then more completely in the book by Pacard and Rivière [148] by a method of local inversion in weighted Hölder spaces, which also allowed them to establish a very nice uniqueness result, i.e., a one-to-one correspondance between solutions on the one hand, and critical points of the renormalized energy on the other hand, at least under this $d = \pm 1$ degree assumption. Another proof (via local inversion methods), which lifts the assumption of nondegeneracy of the renormalized energy, was recently given by Del Pino–Kowalczyk–Musso [82].

In the case of zero degree (or no vortices), a uniqueness result had been previously established by Ye and Zhou in [196].

Other unstable solutions were obtained by Almeida–Bethuel through topological methods [14].

14.1.6 Fine Behavior of the Solutions

The location and rate of convergence of the zeroes of solutions to the limiting vortices, was established by Comte–Mironescu [78] (results also follow from the study done in [148]). Also, the precise asymptotic expansion of the energy of (nonminimizing) solutions was established by Comte–Mironescu in [77, 79], through a minimality property of the solutions outside of their zero-set established in [79].

One may also mention a result of Bauman–Carlsson–Phillips [30] who proved that minimizing solutions with specific boundary data have a single zero.

14.1.7 Stability of the Solutions

In the case with Neumann boundary conditions, conditions on Ω for existence/nonexistence of nontrivial stable solutions (i.e., solutions with vortices) were given in [122, 123].

It was established in [183] that stable (resp. unstable) solutions of (1.3) have vortices which converge as $\varepsilon \to 0$ to stable (resp. unstable) critical points of the renormalized energy. A corollary of this result is that, for ε small enough, there does not exist a stable solution with vortices of (1.3) with Neumann boundary condition (in a simply connected domain), i.e., (1.3) with Neumann boundary condition cannot stabilize vortices. This had already been established but under the assumption that Ω is convex, and for every ε, by Jimbo and Sternberg in [125].

14.1.8 Jacobian Estimates

We saw in Chapter 6 that a crucial tool in the analysis of Ginzburg–Landau is the closeness between the Jacobian determinant $\mu = \mathrm{curl}(iu, \nabla u)$ and vortex densities $2\pi \sum_i d_i \delta_{a_i}$ measured in terms of the Ginzburg–Landau energy (see again Chapter 6 and [119]). A recent result of Jerrard and Spirn [120] gives improved estimates showing that the Jacobian can be made very close to some vortex density (where the vortices found this way are no longer the same ones as those given by the ball-construction method).

14.1.9 Dynamics

Heat-flow

Under the heat-flow for 2D Ginzburg–Landau, the limiting dynamical law of vortices, which is the gradient-flow of the renormalized energy

(up to collision time)

$$\frac{da_i}{dt} = -\frac{1}{\pi}\nabla_i W(a_1, \dots, a_n)$$

was proved, under a well-prepared data assumption, by Lin [129] and Jerrard–Soner [117], after slow motion had been observed by Rubinstein–Sternberg [161]. This result was retrieved through a more Γ-convergence or energy-based method in [174]. After the work of Bauman–Chen–Phillips–Sternberg [31], a few recent papers, by Bethuel–Orlandi–Smets [47, 48, 49] and by Serfaty [184], have extended the dynamical law passed collision and splitting times.

Schrödinger flow

This is also called the Gross–Pitaevskii equation, and is considered in superfluids, nonlinear optics and Bose–Einstein condensation. The limiting dynamical law of vortices

$$\frac{da_i}{dt} = -\frac{1}{\pi}\nabla_i^{\perp} W(a_1, \dots, a_n)$$

was established, still with well-prepared assumptions, by Colliander–Jerrard in [76] on a torus, and by Lin–Xin [134] in the whole plane. A recent result of Jerrard and Spirn [121] derives the same dynamical law for ε small but nonzero.

In the whole plane again, Bethuel and Saut [53] established the existence of some travelling wave solutions with vortices, as conjectured in the physics literature on the Gross–Pitaevskii equation, while Gravejat [104] proved the nonexistence of such solutions at supersonic speed.

Wave flow

In the case of the wave flow, the analogous limiting dynamical law was established by Lin in [130] and Jerrard in [114].

14.2 Higher Dimensions

14.2.1 Γ-Convergence Approach

In dimension 3, vortices become vortex-lines and in higher dimension, they become codimension 2 objects. The right way to capture them is to consider the analogue of the vorticity measure considered in this book (see Chapter 6.1), which is then a current, the Jacobian determinant of the function u. A result analogous to what is stated here in Theorem 6.1 was established by Jerrard–Soner in [119]. It served to prove similarly that these higher-dimensional vorticity-currents or weak Jacobians, $Ju = d(iu, du)$, are compact in the same weak norm, and that

$$\liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \geq \frac{1}{2}\|J\|$$

where $\|J\|$ is the total mass of the (rectifiable and integer-multiplicity) limiting Jacobian J; in other words, the Ginzburg–Landau functional is bounded below by $|\log \varepsilon|$ times half the mass of the limiting Jacobian, which is the mass (length, surface) of the limiting vortex lines or surfaces. A full Γ-convergence result (i.e., including the corresponding upper bound) was then established by Alberti–Baldo–Orlandi [12]. Some improvement of the lower bound, named "product-estimate", also used to estimate vortex velocities for vortex-dynamics, was established in [173].

14.2.2 Minimizers and Critical Points Approach

Even before the Γ-convergence approach, it was established that vortex-lines (in dimension 3 or higher) of minimizers should converge to minimal lines (or minimal connections): see Rivière [154], Sandier [167], Lin–Rivière [131]. It was also established that for critical points, they converge to stationary varifolds, see Lin–Rivière [132] and Bethuel–Brezis–Orlandi [44].

The case of the most general boundary data in 3D, i.e., boundary data in $H^{\frac{1}{2}}$ was examined in Bourgain–Brezis–Mironescu [57], in link with results on lifting of $\mathbb{S}^1$-valued maps in Sobolev spaces.

14.2.3 Inverse Problems

The inverse problem: given a curve which minimizes or is a critical point of length, construct solutions whose vortices converge to that curve, is beginning to be investigated. Montero–Sternberg–Ziemer [140] have proved that there exists such a locally minimizing solution (with Neumann boundary condition) if one starts from a straight line which is a local minimizer of length with endpoints on the boundary of the domain (hence the domain should be nonconvex), it was generalized to the case with magnetic field by Jerrard–Montero–Sternberg in [116]. By local inversion or Lyapounov–Schmidt type methods, Felmer–Kowalczyk–Del Pino [95] have established the existence of a critical point if one starts from a straight line whose endpoints are on the boundary, which is only a critical point of the length.

14.2.4 Dynamics

In dimension ≥ 3, the vortex-set of solutions of the Ginzburg–Landau heat-flow converges to a solution of mean curvature flow in the sense of Brakke (as for solutions to the Allen–Cahn equation). The first result in that direction was obtained in Lin–Rivière [133], and then a full proof was given by Bethuel–Orlandi–Smets [46].

As concerns the Schrödinger or Gross–Pitaevskii flow, of particular interest is the motion of a closed vortex loop. Such loops are expected to flow under binormal flow in the $\varepsilon \to 0$ limit of Gross–Pitaevskii. Results in that direction (but complete results only for the case of a travelling vortex circle) were obtained by Jerrard [115] and Bethuel–Orlandi–Smets [45]. Also, Chiron constructed travelling wave solutions, in particular helix-shaped ones [73, 74].

14.3 Ginzburg–Landau with Magnetic Field

14.3.1 Dependence on κ

As we saw in the phase diagram in Chapter 2, the qualitative behavior of the Ginzburg–Landau energy depends crucially on κ, the "Ginzburg–Landau parameter" which is a material constant.

The situation is most of the time divided into two cases: $\kappa < \frac{1}{\sqrt{2}}$ corresponding to type-I superconductivity, and $\kappa > \frac{1}{\sqrt{2}}$ corresponding to type-II superconductivity. The limiting situation $\kappa = \frac{1}{\sqrt{2}}$ is called the self-dual case. In that famous case, as observed by Bogomoln'yi, the functional can be rewritten into a sum of squares which can all be made equal to zero, and the Ginzburg–Landau equations decouple into a system of first order self-dual equations. For more on that case, refer to the book of Jaffe and Taubes [112].

The type of the superconductor is crucial for the behavior of vortices. Roughly speaking, when $\kappa < \frac{1}{\sqrt{2}}$, vortices (of same degree) would attract each other, hence they are not really observed but rather one observes interfaces (one-dimensional interfaces in 2D) between regions of superconducting phase $|u| \simeq 1$ and regions of normal phase $|u| \simeq 0$ (see for example [75] and references therein). In the self-dual case $\kappa = \frac{1}{\sqrt{2}}$, vortices do not interact and it was shown by Jaffe and Taubes in [112] that solutions with arbitrarily located vortices could be observed.

Then, for $\kappa > \frac{1}{\sqrt{2}}$ vortices of opposite sign attract and vortices of same sign repel, this is the regime where vortices and lattices of vortices are observed, as seen in this book. In this regime and in the context of the Yang–Mills–Higgs model on all $\mathbb{R}^2$, Rivière [156] showed that the unique (up to gauge-equivalence and reflection) minimizer is radially symmetric and of degree one.

However, the above classification is not completely accurate because it neglects size effects. The described classification with separation at the self-dual point $\kappa = \frac{1}{\sqrt{2}}$ corresponds rather to the situation for the whole plane (as in Abrikosov's study [1]) or large samples. In small samples, the scaling is such that the same behavior as for type-II superconductors (i.e., vortices) can be observed in superconductors with $\kappa < \frac{1}{\sqrt{2}}$, see for example Akkermans–Mallick [8] (and Schweigert–Peeters–Singha Deo [180] for corresponding numerical and experimental results) where branches of vortex-solutions such as in Chapter 11.1 are described. Another example of small size sample effect is described by Aftalion and Dancer in [3].

For a global picture, one may also refer to the paper by Aftalion and Du [4] which reviews the different regimes as a function of the parameters.

14.3.2 Vortex Solutions in the Plane

As we saw in Chapter 2, Section 2.5.1, the existence of the n-vortex, that is a finite-energy radial solution of the full Ginzburg–Landau equations (2.4) in $\mathbb{R}^2$, whose only zero is at the origin and of degree n, was first proved by Plohr [151, 152] and Berger–Chen [35]. Later on, their uniqueness (among radial solutions) was proved by Alama–Bronsard–Giorgi [10]. The stability of these vortex-solutions is crucially related to the type of the superconductor, as expected from the previous subsection. It was conjectured by Jaffe and Taubes and proved by Gustafson–Sigal [106] that

— for $|n| \leq 1$ the n-vortex is always stable
— for $|n| \geq 2$ the n-vortex is stable if $\kappa < \frac{1}{\sqrt{2}}$ and unstable if $\kappa > \frac{1}{\sqrt{2}}$.
The instability result had been previously established by Almeida–Bethuel–Guo [41] in the case of large enough κ. The stability of the degree 1 radial solution had also been established by Mironescu [141] (without magnetic field).

One can also search for possibly nonradial solutions in the plane, classifying them according to their homotopy class n, the homotopy class of a configuration being its topological degree at infinity, or its total degree. Jaffe and Taubes conjectured in [112] that for $\kappa > \frac{1}{\sqrt{2}}$, if $|n| > 1$ there are no finite action stable critical points in the n-homotopy class, and that for $n = 0, \pm 1$ the only stable critical point is the radial n-vortex solution described above. Rivière proved in [155] part of this in the strongly repulsive case of $\kappa \gg 1$. More precisely, he showed that for κ large enough, there is an energy-minimizer in the n-homotopy class if and only if $n = 0, \pm 1$, and that in that case it is the radial solution.

14.3.3 Static Two-Dimensional Model

Here we will restrict ourselves to the study of type-II superconductivity ($\kappa > \frac{1}{\sqrt{2}}$) and in particular, the London limit $\kappa \to +\infty$. There is abundant mathematical literature on 1-D solutions to the Ginzburg–Landau equations with studies of bifurcations, critical fields and asymptotics; we will not go into much detail, but refer to the works of Bolley–Helffer (for example [54]) and Aftalion–Troy [6].

Bethuel–Rivière [52] were the first to study vortices for the full Ginzburg–Landau model with magnetic-field, but with a Dirichlet type boundary condition (leading to a type of analysis similar to [43]). From now on, we restrict our attention to the standard full Ginzburg–Landau equations (GL), as studied in this book.

Critical fields and bifurcations

Here we will present the situation with decreasing applied fields.

Around H_{c_3}: As we already mentioned, above a third critical field H_{c_3}, the only solution is the (trivial) normal one $u \equiv 0$, $h \equiv h_{\mathrm{ex}}$. Giorgi and Phillips have proved in [102] that this is the case for $h_{\mathrm{ex}} \geq C\kappa^2$, which implies the upper bound $H_{c_3} \leq C\kappa^2$ for that constant C.

At H_{c_3}: Decreasing the applied field to H_{c_3}, a bifurcation from the normal solution of a branch of solutions with surface superconductivity occurs. The linear analysis of this bifurcation was first performed in the half-plane by De Gennes [80], then by Bauman–Phillips–Tang Qi [32] in the case of a disc (they thus analyze what is known as the "giant vortex" — a unique zero of u with very large degree); and for general domains, formally by Chapman [67], Bernoff–Sternberg [39], then rigorously by Lu and Pan [137], Del Pino–Felmer–Sternberg [81], Helffer–Morame [109], Helffer–Pan [108], see improved results in Fournais–Helffer [97, 98]. The nucleation of surface superconductivity takes place near the point of maximal curvature of the boundary, and the asymptotics for H_{c_3} is

Theorem 14.2.

$$H_{c_3} \sim \frac{\kappa^2}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} \max(curv(\partial\Omega))\kappa,$$

where β_0 is the smallest eigenvalue of a Schrödinger operator with magnetic field in the half-plane.

Between H_{c_2} and H_{c_3}: The behavior of energy minimizers for $H_{c_2} \leq h_{\text{ex}} \leq H_{c_3}$ has been studied by Pan [149], who showed that, as known by physicists, minimizers present surface superconductivity which spreads to the whole boundary, with exponential decay of $|u|$ from the boundary of the domain. More qualitative results of this type were obtained by Almog in [20, 17, 19].

Around H_{c_2}: At H_{c_2}, one goes from surface superconductivity to bulk-superconductivity. It was established by Pan [149] that

$$H_{c_2} = \kappa^2.$$

Qualitative results on bulk-superconductivity below H_{c_2} were obtained in [172], establishing, in particular, how bulk-superconductivity increases (average) as h_{ex} is lowered immediately below H_{c_2}. Results of successive bifurcations and of almost periodic behavior were obtained recently by Almog [19, 21].

Regime $\log \kappa \ll h_{ex} \ll H_{c_2}$*:* In this situation, a uniform density of vortices fills the domain, as presented in Chapter 8 (and first established in [170]). This is where the Abrikosov lattice is expected.

Around H_{c_1}: The value of H_{c_1} and the behavior of minimizers around H_{c_1} were presented in details in this book, and previously established in the references quoted in Chapters 7, 11, 12.

Special solutions

Meissner solution:
The existence and stability of the Meissner solution (solution without vortices) up to the "superheating field" was studied by Bonnet–Chapman–Monneau [55], its uniqueness was also studied in [182]. The superheating field is defined precisely as the value of the applied field for which the Meissner solution loses its stability, and it is of order κ.

Vortex-solutions below the subcooling field:
The existence of branches of vortex-solutions was presented in Chapter 11. Previously, the existence of vortex-solutions for small applied fields $h_{\rm ex} = O(1)$ had been established formally by Rubinstein [157, 158], and rigorously by Du and Lin [86]. The "subcooling field" is defined as the smallest applied field for which there exist stable vortex solutions. It is thus of order of a constant.

Radial solutions:
The radial degree-d (or d-vortex) solutions in a disc were studied by Sauvageot [177], for all values of κ. She established the existence and critical field for existence of these branches of solutions, as well as their stability and loss of stability through bifurcation of a branch of nonradial degree-d solutions.

Periodic solutions

We already mentioned the study of vortex solutions in the plane. In addition, periodic solutions naturally arise for the Ginzburg–Landau system, they are of critical importance to study the Abrikosov lattice. Since Abrikosov's original work [1], many periodic vortex solutions were exhibited, in general as bifurcating from the normal solution, in particular by Chapman [67] and Almog [16].

On the other hand, the study of the Ginzburg–Landau energy functional over periodic configurations (i.e., on a torus) was carried out by Dutour [89] and Aydi [28]. Dutour established a bifurcation diagram and studied in particular the bifurcation from the normal solution at $H_{c_2} = H_{c_3}$ (in the periodic case, there are no boundary effects). Aydi established that $H_{c_1} = \frac{1}{2}\log\kappa$ in the periodic setting, and studied the vorticity of minimizers for that order of applied fields, like in Chapter 7. He also constructed particular solutions which have vortices which concentrate on a finite number of lines.

14.3.4 Dimension Reduction

Chapman–Du–Gunzburger [70] have derived the two-dimensional limit of the 3D Ginzburg–Landau energy for thin films (when the thickness goes to 0). The limiting energy is like the 3D one but where the magnetic potential is prescribed, and the (possibly varying) thickness of the film results in a pinning term in the 2D model, see also Chapman–Héron [71] for a review of formal derivations. Jimbo and Morita [124] then proved that if there exists a nondegenerate solution of the two-dimensional problem, then the original 3D problem also has a local minimizer nearby.

Ginzburg–Landau in thin superconducting loops was also considered and Rubinstein and Schatzman (see [159] and references therein) derived the corresponding 1D model, with interpretation of the Little–Parks experiment. See also Rubinstein–Schatzman–Sternberg [160] for a model of thin loops including constrictions in order to model the Josephson effect.

14.3.5 Models with Pinning Terms

Various models containing weights were studied to take into account pinning effects: see Chapman–Héron [71] and the references therein, Aftalion–Sandier–Serfaty [5], Du–Ding [83], André–Bauman–Phillips [24] (who allowed zeroes of the pinning term). As mentioned just above, pinning terms arise naturally as a result of thin-film limits of the 3D Ginzburg–Landau model, they also serve to model impurities in the material. The analysis is also close to that done for the model without magnetic field and described above in Section 14.1.4.

14.3.6 Higher Dimensions

The full Ginzburg–Landau model in higher dimensions has not been studied as much as the two-dimensional one.

The main focus has been on the 3D analogue of the bifurcation study around H_{c_3}, on surface superconductivity and the influence of the geometry of the domain on its nucleation, see Pan [150], Almog [18], Helffer–Morame [110].

We already mentioned the inverse-type existence result of Jerrard–Montero–Sternberg [116]. More recently, Alama–Bronsard–Montero [11] derived a candidate for the first critical field in a ball in the presence of a uniform field, and constructed locally minimizing solutions with vortices. In the regime $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log \varepsilon|$, one may mention the result of Liu [136], which gives a curvature condition on the limiting vortex-lines of solutions, analogous to a result in Bethuel–Orlandi–Smets [45].

14.3.7 Dynamics

Here, again, the studies are quite similar in nature to the ones without magnetic field. For specific magnetic field results, see Du–Lin [86] and Spirn [187, 188] for the motion of a finite number of vortices in small applied fields, and Sandier–Serfaty [174] in large applied fields.

14.3.8 Mean-Field Models

A mean-field model describing the dynamics of a large number of vortices in the heat flow of Ginzburg–Landau was derived formally and through heuristic arguments by Chapman–Rubinstein–Schatzman [72] (see also similar work by E [90]). This model describes the evolution of vortices through an evolution-problem for the density-measure. Several mathematical papers were then interested in solving rigorously the evolution problem: see Schätzle–Styles [179], Lin–Zhang [135], Du–Zhang [87], Masmoudi–Zhang [139], Ambrosio–Serfaty [22].

The stationary case of the model is quite similar to the limiting conditions we obtained (rigorously) in Theorem 13.1. This stationary problem, in particular the regularity of the free-boundary (boundary of the support of the vorticity measure), was studied by Schätzle–Stoth [178], Bonnet–Monneau [56], Caffarelli–Salazar–Shagholian [66]. A higher dimensional-dynamical model was also proposed by Chapman [69], and later shown to be ill-posed by Richardson–Stoth [153].

14.4 Ginzburg–Landau in Nonsimply Connected Domains

In domain with holes, interesting phenomena of different qualitative nature occur, and many open problems remain. Due to the nontrivial topology, the order parameter can have a nonzero degree without vortices, in other words there can be vorticity (and permanent currents) without vortices.

For a review of such phenomena, we refer to the book edited by Berger and Rubinstein [36] completely devoted to the subject.

Let us mention that in the case with magnetic field, the existence and quantization of nontrivial solutions was studied by Rubinstein–Sternberg [162] and Almeida [13] (see also [15]). Also, Berger and Rubinstein [37] proved that in multiply-connected domains, the zero-set of the order parameter u can be of codimension 1, contrarily to the property of isolated zeroes for minimizers in simply-connected domains established by Elliott–Matano–Tang Qi. For a discussion on the Aharonov–Bohm effect see Helffer [107].

There is also some interesting dependence on the behavior of minimizers on the precise geometry of the domain, in particular on the conformal type in case of an annulus: see the results (obtained in the case without magnetic field) of Golovaty and Berlyand [103] (uniqueness of minimizer) and Berlyand and Mironescu [38].

Alama and Bronsard [9] have started to investigate the behavior of minimizers of the full energy G_ε under an applied magnetic field, i.e., the analogue of what is presented in this book but for nonsimply-connected domains. They establish, in particular, the existence and value of the first critical field for which vortices appear.