Crystalline representations and F-crystals

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To Vladimir Drinfeld on his 50th birthday.

Summary. Following ideas of Berger and Breuil, we give a new classification of crystalline representations. The objects involved may be viewed as local, characteristic 0 analogues of the "shtukas" introduced by Drinfeld. We apply our results to give a classification of p-divisible groups and finite flat group schemes, conjectured by Breuil, and to show that a crystalline representation with Hodge–Tate weights 0, 1 arises from a p-divisible group, a result conjectured by Fontaine.

Subject Classifications: Primary 11S20. Secondary 14F30.

Introduction

Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ its ring of Witt vectors, $K_0 = W(k)[1/p]$, and K/K_0 a finite totally ramified extension. In [Br 4] Breuil proposed a new classification of p-divisible groups and finite flat group schemes over the ring of integers $\mathcal{O}_K$ of K. For p-divisible groups and $p > 2$, this classification was established in [Ki], where we also used a variant of Breuil's theory to describe flat deformation rings, and thereby establish a modularity lifting theorem for potentially Barsotti–Tate Galois representations.

In this paper we generalize Breuil's theory to describe crystalline representations of higher weight or, equivalently, their associated weakly admissible modules. To explain our main theorem, fix a uniformiser $\pi \in K$ with Eisenstein polynomial $E(u)$, and write $\mathfrak{S} = W[\![u]\!]$. We equip $\mathfrak{S}$ with the endomorphism φ, which acts via the Frobenius on W, and sends u to u^p. Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a map $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ whose cokernel is killed by some power of $E(u)$.

Theorem 0.1. *The category of crystalline representations with all Hodge–Tate weights $\leqslant 0$ admits a fully faithful embedding into the isogeny category* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ *of* $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$.

Unfortunately the embedding of the theorem is not essentially surjective. In this sense the situation is not as good as for p-divisible groups. However, we do give an explicit description of the image of the functor. To explain it, let $\mathcal{O}$ denote the ring of rigid analytic functions on the open unit u-disk. Then $\mathfrak{S}[1/p]$ corresponds to the bounded functions in $\mathcal{O}$. It turns out that the module $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ is equipped with a canonical connection which has poles at a sequence of points corresponding to the ideals $\varphi^n(E(u))\mathcal{O} \subset \mathcal{O}$. A module $\mathfrak{M}$ is in the image of our functor if and only if these poles are logarithmic (see Corollary 1.3.15 below).

In fact the theorem we prove is slightly more general than the above, and includes the case of semistable representations. We refer to the reader to the body of the text for the more general statement.

To prove the theorem we adapt the techniques of Berger [Be 1]. One can view his results as relating the weakly admissible module attached to a semistable representation and the (φ, Γ)-module attached to the same representation [Fo 1]. (φ, Γ)-modules are constructed using norm fields for the cyclotomic extension. We develop an analogue of Berger's theory in a setting where the cyclotomic extension has been replaced by the Kummer extension $K_\infty = \cup_{n\geq 1} K(\sqrt[p^n]{\pi})$ (cf. [Br 1]). As in Berger's case, a crucial role in the construction is played by Kedlaya's theory of slopes [Ke 1]. In particular, we again make use of Berger's beautiful observation that the notion of weak admissibility for filtered (φ, N)-modules is intimately related to that of a Frobenius module over the Robba ring being of slope 0 in the sense of [Ke 1]. For $K = K_0$, the analogue of the theorem in the setting of the cyclotomic extension is proved in [Be 2, Theorem 2].

Let us mention some applications of our results. Fix an algebraic closure $\bar{K}$ of K, and write $G_K = \mathrm{Gal}(\bar{K}/K)$ and $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$. The following result was conjectured by Breuil [Br 1], and proved by him for representations of G_K arising from p-divisible groups [Br 3, 3.4.3].

Theorem 0.2. *The functor from crystalline representations of G_K to p-adic G_{K_∞}-representations, obtained by restricting the action of G_K to G_{K_∞}, is fully faithful.*

We also obtain a proof of Fontaine's conjecture that weakly admissible modules are admissible (see Proposition 2.1.5). This is at least the fourth proof, following those of Colmez–Fontaine [CF], Colmez [Co], and Berger [Be 1]. Of course our proof is related to the one of Berger.

As alluded to above, for crystalline representations with all Hodge–Tate weights equal to 0 or -1, there is a refinement of Theorem 0.1. Namely the category of such representations is equivalent to $\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$, where $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ denotes the full subcategory of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of objects $\mathfrak{M}$ such that the cokernel of $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ is killed by $E(u)$. On the other hand there is a functor from $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ to the category of p-divisible groups. This functor was first constructed for $p > 2$ in [Br 4] using the theory of

[Br 2], and it was conjectured to exist and be an equivalence for all p [Br 4, 2.1.2]. Here we construct it for all p using Grothendieck–Messing theory. As a consequence, we establish the following two results.

Theorem 0.3. *Any crystalline representation with all Hodge–Tate weights equal to* 0 *or* 1 *arises from a p-divisible group.*

Theorem 0.4. *There is a functor from* $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ *to p-divisible groups. If $p > 2$ this functor is an equivalence. For $p = 2$ it induces an equivalence on the associated isogeny categories.*

Theorem 0.3 was conjectured by Fontaine [Fo 3, 5.2.5], and proved by Laffaille for ramification degree $e(K/K_0) \leqslant p-1$ [La, Section 2], and by Breuil for $p > 2$, and k finite [Br 2, Theorem 1.4].

For $p > 2$, Theorem 0.4 was proved in [Ki] by a completely different method. Finally, it was pointed out by Beilinson that, using Theorem 0.4, one can deduce a classification of finite flat group schemes over $\mathcal{O}_K$ when $p > 2$. A special case of this had been conjectured by Breuil [Br 4, 2.1.1]. To explain this result we denote by $(\mathrm{Mod}\,/\mathfrak{S})$ the category of finite $\mathfrak{S}$-modules $\mathfrak{M}$ which are killed by some power of p, have projective dimension 1 (i.e., $\mathfrak{M}$ has a two-term resolution by finite free $\mathfrak{S}$-modules) and are equipped with a map $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ whose cokernel is killed by $E(u)$. Then we have

Theorem 0.5. *For $p > 2$, the category* $(\mathrm{Mod}\,/\mathfrak{S})$ *is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$.*

During the writing of this paper, I learned from V. Lafforgue that, with Genestier, he had recently developed a theory remarkably parallel to ours in the function field case [GL]. The characteristic p analogues of modules in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ are a sort of local version of a "shtuka" in the sense of Drinfeld [Ka]. Drinfeld introduced these objects, with stunning success, in order to study the arithmetic of function fields. Lafforgue pointed out to us that the modules in our theory could be regarded as analogues of local shtukas in the case of mixed characteristic. The connection with shtukas gives a first hint that our theory, and related constructions using norm fields, which have no known geometric interpretation, may have some deeper meaning. The question of whether there is a global analogue of a shtuka for number fields is extremely tantalizing, and suggests that Drinfeld's ideas, which revolutionized the study of automorphic forms over function fields, may yet find an application in this case. It is a pleasure to dedicate this article to him.

1 F-crystals and weakly admissible modules

1.1 Preliminaries

Throughout the paper we will fix a uniformiser $\pi \in K$, and we denote by $E(u) \in K_0[u]$ the Eisenstein polynomial of π. We also fix an algebraic closure $\bar{K}$ of K, and a sequence of elements $\pi_n \in \bar{K}$, for n a nonnegative integer, such that $\pi_0 = \pi$, and $\pi_{n+1}^p = \pi_n$. We write $K_{n+1} = K(\pi_n)$.

1.1.1. Let $\mathfrak{S} = W[\![u]\!]$. We denote by $\widehat{\mathfrak{S}}_n$ the completion of $K_{n+1} \otimes_W \mathfrak{S}$ at the maximal ideal $(u - \pi_n)$. The ring $\widehat{\mathfrak{S}}_n$ is equipped with its $(u - \pi_n)$-adic filtration, and this extends to a filtration on the quotient field $\widehat{\mathfrak{S}}_n[1/(u - \pi_n)]$.

Denote by $D[0, 1)$ the open rigid analytic disk of radius 1 with co-ordinate u. Thus the $\bar{K}$-points of $D([0, 1))$ correspond to $x \in \bar{K}$ such that $|x| < 1$. Suppose that $I \subset [0, 1)$ is a subinterval. We denote by $D(I) \subset D[0, 1)$ the admissible open subspace whose $\bar{K}$-points correspond to $x \in \bar{K}$ with $|x| \in I$. We set $\mathcal{O}_I = \Gamma(D(I), \mathcal{O}_{D(I)})$, and $\mathcal{O} = \mathcal{O}_{[0,1)}$. If $I = (a, b)$ we will write $D(a, b)$ rather than $D((a, b))$, and similarly for half open and closed intervals.

Note that for any n we have natural maps $\mathfrak{S}[1/p] \to \mathcal{O} \to \widehat{\mathfrak{S}}_n$ given by sending u to u, where the first map has dense image. On $\mathfrak{S}$ we have the Frobenius φ which sends u to u^p, and acts as the natural Frobenius on W. We will write $\varphi_W : \mathfrak{S} \to \mathfrak{S}$ for the $\mathbb{Z}_p[\![u]\!]$-linear map which acts on W via the Frobenius, and by $\varphi_{\mathfrak{S}/W} : \mathfrak{S} \to \mathfrak{S}$ the W-linear map which sends u to u^p. For any $I \subset [0, 1)$, φ_W induces a map $\varphi_W : \mathcal{O}_I \to \mathcal{O}_I$, while $\varphi_{\mathfrak{S}/W}$ induces a map $\varphi_{\mathfrak{S}/W} : \mathcal{O}_I \to \mathcal{O}_{p^{-1}I}$, where $p^{-1}I = \{r : r^p \in I\}$. We will write $\varphi = \varphi_W \circ \varphi_{\mathfrak{S}/W} : \mathcal{O}_I \to \mathcal{O}_{p^{-1}I}$.

Let $c_0 = E(0) \in K_0$. Set

$$\lambda = \prod_{n=0}^{\infty} \varphi^n(E(u)/c_0) \in \mathcal{O}.$$

Thinking of functions in $\mathcal{O}$ as convergent power series in u, we define a derivation $N_\nabla := -u\lambda \frac{d}{du} : \mathcal{O} \to \mathcal{O}$. We denote by the same symbol the induced derivation $\mathcal{O}_I \to \mathcal{O}_I$, for each $I \subset [0, 1)$.

We adjoin a formal variable ℓ_u to $\mathcal{O}$ which acts formally like $\log u$. We extend the natural maps $\mathcal{O} \to \widehat{\mathfrak{S}}_n$ to $\mathcal{O}[\ell_u]$ by sending ℓ_u to

$$\log\left[\left(\frac{u - \pi_n}{\pi_n}\right) + 1\right] := \sum_{i=1}^{\infty} (-1)^{i-1} i^{-1} \left(\frac{u - \pi_n}{\pi_n}\right)^i \in \widehat{\mathfrak{S}}_n.$$

We extend φ to $\mathcal{O}[\ell_u]$ by setting $\varphi(\ell_u) = p\ell_u$, and we extend N_∇ to a derivation on $\mathcal{O}[\ell_u]$ by setting $N_\nabla(\ell_u) = -\lambda$. Finally, we write N for the derivation on $\mathcal{O}[\ell_u]$ which acts as differentiation of the formal variable ℓ_u. These satisfy the relations

$$N\varphi = p\varphi N \quad \text{and} \quad N_\nabla \varphi = (p/c_0)E(u)\varphi N_\nabla. \tag{1.1.2}$$

Finally, we remark that N and N_∇ commute.

1.1.3. Recall [Fo 2] that a φ-module is a finite-dimensional K_0-vector space D together with a bijective, Frobenius semilinear map $\varphi : D \to D$. A (φ, N)-module is a φ-module D, together with a linear (necessarily nilpotent) map $N : D \to D$ which satisfies $N\varphi = p\varphi N$. (φ, N)-modules (respectively, φ-modules) form a Tannakian category.

If D is a one-dimensional (φ, N)-module, and $v \in D$ is a basis vector, then $\varphi(v) = \alpha v$ for some $\alpha \in K_0$, and we write $t_N(D)$ for the p-adic valuation of α. If D has dimension $d \in \mathbb{N}^+$, then we write $t_N(D) = t_N(\bigwedge^d D)$.

A *filtered* (φ, N)-module (respectively, φ-module) is a (φ, N)-module (respectively, φ-module) D equipped with a decreasing, separated, exhaustive filtration on $D_K = D \otimes_{K_0} K$. These again form a Tannakian category. Given a one-dimensional filtered (φ, N)-module D, we denote by $t_H(D)$ the unique integer i such that $\mathrm{gr}^i D_K$ is nonzero. In general, if D has dimension d, we set $t_H(D) = t_H(\bigwedge^d D)$. A filtered (φ, N)-module D is called *weakly admissible* if $t_H(D) = t_N(D)$ and for any (φ, N)-submodule $D' \subset D$, $t_H(D') \leqslant t_N(D')$, where $D'_K \subset D_K$ is equipped with the induced filtration.

We will call a filtered (φ, N)-module *effective* if $\mathrm{Fil}^0 D = D$.

1.1.4. By a φ-module over $\mathcal{O}$ we mean a finite free $\mathcal{O}$-module $\mathcal{M}$, equipped with a φ-semilinear, injective map $\varphi : \mathcal{M} \to \mathcal{M}$. A (φ, N_∇)-module over $\mathcal{O}$ is a φ-module $\mathcal{M}$ over $\mathcal{O}$, together with a differential operator $N_\nabla^{\mathcal{M}}$ over N_∇. That is, for $f \in \mathcal{O}$, and $m \in \mathcal{M}$, we have

$$N_\nabla^{\mathcal{M}}(fm) = N_\nabla(f)m + fN_\nabla^{\mathcal{M}}(m).$$

φ and N_∇ are required to satisfy the relation $N_\nabla^{\mathcal{M}}\varphi = (p/c_0)E(u)\varphi N_\nabla^{\mathcal{M}}$. We will usually write N_∇ for $N_\nabla^{\mathcal{M}}$ since this will cause no confusion. The category of (φ, N_∇)-modules over $\mathcal{O}$ has a natural structure of a Tannakian category.

It will often be convenient to think of $\mathcal{M}$ as a coherent sheaf on $D[0, 1)$. Then we may speak of $\mathcal{M}$ or $1 \otimes \varphi : \varphi^*(\mathcal{M}) \to \mathcal{M}$ having some property (e.g., being an isomorphism) in the neighbourhood of a point of $D[0, 1)$, or over some admissible open subset. We will need the following.

Lemma 1.1.5. *Let $I \subset [0, 1)$ be an interval, $\mathcal{M}$ a finite free $\mathcal{O}_I$-module, and $\mathcal{N} \subset \mathcal{M}$ an $\mathcal{O}_I$-submodule. Then the following conditions are equivalent:*

(1) *$\mathcal{N} \subset \mathcal{M}$ is closed.*
(2) *$\mathcal{N}$ is finitely generated.*
(3) *$\mathcal{N}$ is finite free.*

Proof. We obviously have (3) $\implies$ (2). If $\mathcal{N}$ is finitely generated, then it is free of rank at most that of $\mathcal{M}$ by [Be 3, 4.13], so (2) $\implies$ (3). Moreover, in this case, $\mathcal{N}$ is the image of a map $\mathcal{M} \to \mathcal{M}$, hence by [Be 3, 4.12(5)], we may choose an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{O}_I^d$ under which $\mathcal{N}$ maps onto $\sum_{i=1}^d f_i\mathcal{O}_I$ for some $f_i \in \mathcal{O}_I$. Since $f_i\mathcal{O}_I \subset \mathcal{O}_I$ is a closed ideal by [Laz, 8.11], it follows that $\mathcal{N}$ is closed in $\mathcal{M}$.

Finally, suppose that $\mathcal{N} \subset \mathcal{M}$ is closed. We will show that $\mathcal{N}$ is free by induction on the $\mathcal{O}_I$-rank of $\mathcal{M}$. If $\mathcal{M}$ has rank 1, then this follows from [Laz, 7.3]. In general choose a nonzero element $n \in \mathcal{N}$. Let $\mathcal{M}' = (\mathcal{M} \cap n \cdot \mathcal{O}_I) \otimes_{\mathcal{O}_I} \mathrm{Fr}\,\mathcal{O}_I \subset \mathcal{M} \otimes_{\mathcal{O}_I} \mathrm{Fr}\,\mathcal{O}_I$, where $\mathrm{Fr}\,\mathcal{O}_I$ denotes the field of fractions of $\mathcal{O}_I$. Write $\mathcal{N}' = \mathcal{N} \cap \mathcal{M}'$. By Lazard's results and [Ke 1, Lemma 2.4], $\mathcal{M}' \subset \mathcal{M}$ is a direct summand and is free of rank 1 over $\mathcal{O}_I$. Since $\mathcal{N}'$ is closed in $\mathcal{M}'$, and $\mathcal{N}/\mathcal{N}'$ is closed in $\mathcal{M}/\mathcal{M}'$ by the open mapping theorem, we deduce by induction that both $\mathcal{N}'$ and $\mathcal{N}/\mathcal{N}'$ are finite free over $\mathcal{O}_I$, whence the same holds for $\mathcal{N}$. □

1.2 Filtered (φ, N)-modules and (φ, N_∇)-modules

Let D be an effective filtered (φ, N)-module. We define a (φ, N_∇)-module over $\mathcal{O}$, as follows: For each nonnegative integer n, write ι_n for the composite

$$\mathcal{O}[\ell_u] \otimes_{K_0} D \xrightarrow{\varphi_W^{-n} \otimes \varphi^{-n}} \mathcal{O}[\ell_u] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n \otimes_{K_0} D = \widehat{\mathfrak{S}}_n \otimes_K D_K,$$

where the second map is deduced from the map $\mathcal{O}[\ell_u] \to \widehat{\mathfrak{S}}_n$ defined in (1.1.2). We may extend this to a map

$$\iota_n : \mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n[1/(u - \pi_n)] \otimes_K D_K.$$

Set

$$\mathcal{M}(D) = \{x \in (\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D)^{N=0} : \iota_n(x) \in \mathrm{Fil}^0(\widehat{\mathfrak{S}}_n[1/(u - \pi_n)] \otimes_K D_K), \\ n \geq 0\}.$$

Note that $(\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D)^{N=0}$ is an $\mathcal{O}$-module with a φ-semilinear Frobenius given by those on D and $\mathcal{O}[\ell_u, 1/\lambda]$, where the latter ring is equipped with a Frobenius, because $\varphi(1/\lambda) = E(u)/(c_0\lambda)$. It is equipped with a differential operator N_∇, induced by the operator on $N_\nabla \otimes 1$ on $\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D$.

Lemma 1.2.1. *If we regard $\widehat{\mathfrak{S}}_n$ as an $\mathcal{O}$-module via φ_W^{-n}, then*

(1) *The map*

$$\widehat{\mathfrak{S}}_n \otimes_{\mathcal{O}} (\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0} \to \widehat{\mathfrak{S}}_n \otimes_K D_K$$

induced by ι_n is an isomorphism.

(2) *We have*

$$\widehat{\mathfrak{S}}_n \otimes_{\mathcal{O}} \mathcal{M}(D) \xrightarrow{\sim} \sum_{j \geq 0} (u - \pi_n)^{-j} \widehat{\mathfrak{S}}_n \otimes_K \mathrm{Fil}^j D_K \\ = \sum_{j \geq 0} \varphi^n_{\mathfrak{S}/W}(E(u))^{-j} \widehat{\mathfrak{S}}_n \otimes_K \mathrm{Fil}^j D_K.$$

Proof. Since both sides in (1) are easily seen to be free $\widehat{\mathfrak{S}}_n$-modules of the same rank, it suffices to show that the map obtained by reducing modulo $u - \pi_n$ is an isomorphism. The latter map is $(K_{n+1}[\ell_u] \otimes_{K_0} D)^{N=0} \overset{\ell_u \mapsto 0}{\to} K_{n+1} \otimes_K D_K$ (where N acts on $K_{n+1}[\ell_u]$ K_{n+1}-linearly), and this is easily seen to be an isomorphism. This establishes (1), and (2) follows easily. □

Lemma 1.2.2. *Suppose that D is effective. Then the operators φ and N_∇ on $(\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D)^{N=0}$ induce on $\mathcal{M}(D)$ the structure of a (φ, N_∇)-module over $\mathcal{O}$. Moreover, there is an isomorphism of $\mathcal{O}$-modules*

$$\mathrm{coker}(1 \otimes \varphi : \varphi^*\mathcal{M}(D) \to \mathcal{M}(D)) \xrightarrow{\sim} \oplus_{i \geq 0} (\mathcal{O}/E(u)^i)^{h_i}$$

where $h_i = \dim_K \mathrm{gr}^i D_K$.

Proof. First, we check that $\mathcal{M}(D)$ is finite free over $\mathcal{O}$. Let r be a nonnegative integer such that $\mathrm{Fil}^{r+1} D = 0$. Then $\mathcal{M}(D) \subset \lambda^{-r}(\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0}$, and the right-hand side is a finite free $\mathcal{O}$-module. Since the maps ι_n are continuous, and the filtration on $\widehat{\mathfrak{S}}_n[1/(u-\pi_n)]$ is by closed K-subspaces, this submodule is closed, and hence finite free by Lemma 1.1.5.

Now let $\mathcal{D}_0 = (\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0}$. To prove the rest of the lemma, we have to check that the natural map $\varphi^*(\lambda^{-r}\mathcal{D}_0) \to \lambda^{-r}\mathcal{D}_0$, induced by the isomorphism $1 \otimes \varphi_{\mathcal{D}_0} : \varphi^*\mathcal{D}_0 \xrightarrow{\sim} \mathcal{D}_0$, takes $\varphi^*(\mathcal{M}(\mathcal{D}))$ into $\mathcal{M}(D)$, and that the cokernel of $\varphi^*(\mathcal{M}(D)) \to \mathcal{M}(D)$ is as claimed. For this it will be convenient to think of finite $\mathcal{O}$-modules as coherent sheaves on $D[0, 1)$.

At any point of $D[0, 1)$ not corresponding to a maximal ideal of the form $\varphi^n(E(u))$ for some $n \geq 0$, $\mathcal{M}(D)$ is isomorphic to $\mathcal{D}_0$, and so $1 \otimes \varphi_{\mathcal{D}_0}$ induces an isomorphism $\varphi^*\mathcal{M}(D) \xrightarrow{\sim} \mathcal{M}(D)$ at such a point. Now for any $n \geq 1$, the map $\varphi_{\mathfrak{S}/W}$ on $\mathfrak{S}$ induces a map of K_{n+1}-algebras $\varphi_{\mathfrak{S}/W} : \widehat{\mathfrak{S}}_n \xrightarrow{u \mapsto u^p} \widehat{\mathfrak{S}}_{n+1}$, and we have a commutative diagram

$$\begin{array}{ccc} \lambda^{-r}\mathcal{D}_0 & \xrightarrow{\iota_n} & (u-\pi_n)^{-r}\widehat{\mathfrak{S}}_n \otimes_K D_K \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_{\mathfrak{S}/W}\otimes 1} \\ \lambda^{-r}\mathcal{D}_0 & \xrightarrow{\iota_{n+1}} & (u-\pi_{n+1})^{-r}\widehat{\mathfrak{S}}_{n+1} \otimes_K D_K. \end{array}$$

If we regard $\widehat{\mathfrak{S}}_n$ as an $\mathcal{O}$-module via φ_W^{-n}, then $\varphi_{\mathfrak{S}/W}$ becomes a φ-semilinear map, and the induced $\mathcal{O}$-linear map

$$1 \otimes \varphi_{\mathfrak{S}/W} : \varphi^*\widehat{\mathfrak{S}}_n[1/(u-\pi_n)] \to \widehat{\mathfrak{S}}_{n+1}[1/(u-\pi_{n+1})] \qquad (1.2.3)$$

is an isomorphism, which takes $\varphi^*(u-\pi_n)^s\widehat{\mathfrak{S}}_n$ onto $(u-\pi_{n+1})^s\widehat{\mathfrak{S}}_{n+1}$ for each integer s. Now let

$$\mathcal{M}_n(D) = \{x \in \mathcal{D}_0[1/\lambda] : \iota_n(x) \in \mathrm{Fil}^0(\widehat{\mathfrak{S}}_n[1/(u-\pi_n)] \otimes_K D_K)\}.$$

Then $\mathcal{M}(D) \subset \mathcal{M}_n(D)$ and this inclusion is an isomorphism at the point $x_n \in D[0, 1)$ corresponding to the ideal $(\varphi^n(E(u))) \subset \mathcal{O}$.

By Lemma 1.2.1, the map

$$\begin{aligned} \mathcal{D}_0/\varphi_W^n(E(u))\mathcal{D}_0 &= (\mathcal{O}/\varphi_W^n(E(u))\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0} \\ &\to \widehat{\mathfrak{S}}_n \otimes_K D_K/(u-\pi_n)\widehat{\mathfrak{S}}_n \otimes_K D_K \xrightarrow{\sim} K_{n+1} \otimes_K D_K \end{aligned}$$

induced by ι_n is a bijection. Hence we have an exact sequence

$$\begin{aligned} 0 &\to \mathcal{M}_n(D) \to \lambda^{-r}\mathcal{D}_0 \\ &\to ((u-\pi_n)^{-r}\widehat{\mathfrak{S}}_n \otimes_K D_K)/\mathrm{Fil}^0(\widehat{\mathfrak{S}}_n[1/(u-\pi_n)] \otimes_K D_K) \to 0 \end{aligned}$$

Denote by Q_n the term on the right of this exact sequence. Then its pullback by the flat map $\varphi : \mathcal{O} \to \mathcal{O}$ sits in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi^*\mathcal{M}_n(D) & \longrightarrow & \varphi^*(\lambda^{-r}\mathcal{D}_0) & \longrightarrow & \varphi^*(Q_n) & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_{n+1}(D) & \longrightarrow & \lambda^{-r}\mathcal{D}_0 & \longrightarrow & Q_{n+1} & \longrightarrow & 0. \end{array}$$

Here the map on the right is induced by the map $1 \otimes \varphi_{\mathfrak{S}/W}$ of (1.2.3), and the remarks above show that it is a bijection. The map in the middle has image $E(u)^r\lambda^{-r}\mathcal{D}_0$. In particular, we may fill in the left hand map $\varphi^*(\mathcal{M}_n(D)) \to \mathcal{M}_n(D)$, as shown, and we see that its cokernel is contained in $\lambda^{-r}\mathcal{D}_0/(E(u)^r\lambda^{-r}\mathcal{D}_0)$. Since the inclusions $\varphi^*(\mathcal{M}(D)) \subset \varphi^*(\mathcal{M}_n(D))$ and $\mathcal{M}(D) \subset \mathcal{M}_{n+1}(D)$ are isomorphisms at x_{n+1}, this shows that $1 \otimes \varphi_{\mathcal{D}_0}$ induces an isomorphism $\varphi^*(\mathcal{M}(D)) \xrightarrow{\sim} \mathcal{M}(D)$ at x_{n+1}.

Finally, since $\varphi(x_0) \neq x_n$ for any $n \geq 0$, the inclusion $\mathcal{D}_0 \subset \mathcal{M}(D)$ gives rise to an inclusion $\varphi^*\mathcal{D}_0 \subset \varphi^*(\mathcal{M}(D))$ which is an isomorphism at x_0. Since $1 \otimes \varphi_{\mathcal{D}_0}$ maps $\varphi^*(\mathcal{D}_0)$ isomorphically onto $\mathcal{D}_0 \subset \mathcal{M}(D)$, it induces a map $\varphi^*(\mathcal{M}(D)) \to \mathcal{M}(D)$ whose cokernel is supported on x_0. Moreover, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}_0 & \longrightarrow & \lambda^{-r}\mathcal{D}_0 & \longrightarrow & ((u-\pi)^{-r}\widehat{\mathfrak{S}}_0/\widehat{\mathfrak{S}}_0) \otimes_K D_K & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\| & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_0(D) & \longrightarrow & \lambda^{-r}\mathcal{D}_0 & \longrightarrow & Q_0 & \longrightarrow & 0. \end{array}$$

Hence

$$\begin{aligned} \operatorname{coker}(\varphi^*(\mathcal{M}(D)) \to \mathcal{M}(D)) &\xrightarrow{\sim} \mathcal{M}_0(D)/\mathcal{D}_0 \\ &\xrightarrow{\sim} \operatorname{Fil}^0(\widehat{\mathfrak{S}}_0[1/(u-\pi)] \otimes_K D_K)/(\widehat{\mathfrak{S}}_0 \otimes_K D_K) \end{aligned}$$

and the lemma follows. □

1.2.4. We will say that a φ-module $\mathcal{M}$ over $\mathcal{O}$ is of finite E-height if the cokernel of the $\mathcal{O}$-linear map $\varphi^*\mathcal{M} \to \mathcal{M}$ is killed by some power of $E(u)$, that is, if this cokernel is supported on $x_0 \in D[0, 1)$. A (φ, N_∇)-module over $\mathcal{O}$ is of finite E-height if it is of finite E-height as a φ-module. We denote by $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N_\nabla}$ (respectively, $\mathrm{Mod}_{/\mathcal{O}}^{\varphi}$) the category of (φ, N_∇)-modules (respectively, φ-modules) over $\mathcal{O}$ of finite E-height. Both these categories are stable under $\otimes$-products.

1.2.5. Suppose that $\mathcal{M}$ is in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi}$. We define a filtered φ-module $D(\mathcal{M})$ as follows: The underlying K_0-vector space of $D(\mathcal{M})$ is $\mathcal{M}/u\mathcal{M}$, and the operator φ is induced by φ on $\mathcal{M}$.

To construct the filtration on $D(\mathcal{M})_K$, it will be convenient to adopt the following notation: If $J \subset I \subset [0, 1)$ are intervals, and $\mathcal{M}$ is a finite $\mathcal{O}_I$-module, we will write $\mathcal{M}_J = \mathcal{M} \otimes_{\mathcal{O}_I} \mathcal{O}_J$. If we think of $\mathcal{M}$ as a coherent sheaf on $D(I)$, then $\mathcal{M}_J$ corresponds to the restriction of $\mathcal{M}$ to $D(J)$. Similarly, if $\xi : \mathcal{M} \to \mathcal{M}'$ is a map of finite $\mathcal{O}_I$-modules we denote by $\xi_J : \mathcal{M}_J \to \mathcal{M}'_J$ the induced map. We will need the following

Lemma 1.2.6. *Let $\mathcal{M}$ be a φ-module over $\mathcal{O}$. There is a unique $\mathcal{O}$-linear, φ-equivariant morphism*

$$\xi : D(\mathcal{M}) \otimes_{K_0} \mathcal{O} \to \mathcal{M}$$

whose reduction modulo u induces the identity on $D(\mathcal{M})$. ξ is injective, and its cokernel is killed by a finite power of λ. If $r \in (|\pi|, |\pi|^{1/p})$, then the image of the map $\xi_{[0,r)}$ induced by ξ coincides with the image of $1 \otimes \varphi : (\varphi^* \mathcal{M})_{[0,r)} \to \mathcal{M}_{[0,r)}$.

Proof. Recall that $\mathcal{O}$ is a Fréchet space, with its topology defined by the norms $|\cdot|_r$ for $r \in (0, 1)$, given by $|f|_r = \sup_{x \in D[0,r]} |f(x)|$. Since $\mathcal{M}$ is free we may identify it with $\mathcal{O}^d$, where $d = \mathrm{rk}_{\mathcal{O}} \mathcal{M}$, and we will again denote by $|\cdot|_r$ the norm on $\mathcal{M}$ obtained by taking the maximum of $|\cdot|_r$ applied to the co-ordinates of an element $m \in \mathcal{M} = \mathcal{O}^d$. For a subset $\Sigma \subset \mathcal{M}$ we set $|\Sigma|_r = \sup_{x \in \Sigma} |x|_r$.

Now choose any K_0-linear map $s_0 : D(\mathcal{M}) \to \mathcal{M}$ whose reduction modulo u is the identity. We define a new map $s : D(\mathcal{M}) \to \mathcal{M}$ by

$$s = s_0 + \sum_{i=1}^{\infty} (\varphi^i \circ s_0 \circ \varphi^{-i} - \varphi^{i-1} \circ s_0 \circ \varphi^{1-i})$$

To check that the right-hand side converges to a well defined map, fix an $r \in (0, 1)$, and let $L \subset D(\mathcal{M})$ be a $\mathcal{O}_{K_0}$-lattice. Then $\varphi^{-1}(L) \subset p^{-j} L$ for some nonnegative integer j. After increasing j, we may also assume that $|\varphi(m)|_r \leqslant |p^{-j} m|_r$ for all $m \in \mathcal{M}$. Since $\varphi \circ s_0 \circ \varphi^{-1} - s_0 \in u\mathcal{M}$, we have $\tilde{L} := u^{-1}(\varphi \circ s_0 \circ \varphi^{-1} - s_0)(L) \subset \mathcal{M}$ so that

$$|(\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i})(L)|_r \leqslant |p^{-ij} u^{p^i} \varphi^i(\tilde{L})|_r \leqslant p^{2ij} r^{p^i} |\tilde{L}|_r.$$

Since $|\tilde{L}|_r$ is finite, and $p^{2ij} r^{p^i} \to 0$ as $i \to \infty$, for any $j \geq 0$ and $r \in (0, 1)$, the map s is well defined. Once checks immediately that $\varphi \circ s = s \circ \varphi$.

Given any other such map s', the difference $s - s'$ sends $D(\mathcal{M})$ into $u\mathcal{M}$. But since φ is a bijection on $D(\mathcal{M})$, and $\varphi^j \circ (s - s') = (s - s') \circ \varphi^j$, for $j \geq 1$, we see that $(s - s')(D(\mathcal{M})) \subset u^{p^j} \mathcal{M}$, so that $s - s' = 0$. It follows that s is the unique such map. Extending s to $D(\mathcal{M}) \otimes_{K_0} \mathcal{O}$ by $\mathcal{O}$-linearity yields the required map ξ, and the uniqueness of s implies the that of ξ.

To establish the claim regarding the image of ξ, note that ξ is an isomorphism modulo u, so for some sufficiently large positive integer i, $\xi_{[0,r^{p^i})}$ is an isomorphism. Since ξ commutes with φ, we have a commutative diagram

$$\begin{array}{ccc} \varphi^*(D(\mathcal{M}) \otimes_{K_0} \mathcal{O}) & \xrightarrow{\varphi^*\xi} & \varphi^* \mathcal{M} \\ \downarrow \sim & & \downarrow 1 \otimes \varphi \\ D(\mathcal{M}) \otimes_{K_0} \mathcal{O} & \xrightarrow{\xi} & \mathcal{M}. \end{array}$$

If $i > 1$, then the restriction of the right vertical map to $[0, r^{p^{i-1}})$ is an isomorphism, so that $\xi_{[0,r^{p^{i-1}})}$ is also. Repeating this argument, we find that $\xi_{[0,r^p)}$ is an isomorphism, and making use of the above commutative diagram once more, we find that the image of $\xi_{[0,r)}$ coincides with $(1 \otimes \varphi)_{[0,r)}$.

Finally, we have seen that $\xi_{[0,r)}$ is injective with cokernel killed by a finite power $E(u)^s$ of $E(u)$. It follows from the same commutative diagram above that ξ is injective with cokernel killed by λ^s. □

1.2.7. Now define a decreasing filtration on $\varphi^*\mathcal{M}$ by

$$\mathrm{Fil}^i\, \varphi^*\mathcal{M} = \{x \in \varphi^*\mathcal{M} : 1 \otimes \varphi(x) \in E(u)^i\mathcal{M}\}.$$

This is a filtration on $\varphi^*\mathcal{M}$ by finite free $\mathcal{O}$-modules (for example, using Lemma 1.1.5), whose successive graded pieces are $E(u)$-torsion modules. By transport of structure, this defines a filtration on $(1 \otimes \varphi)(\varphi^*\mathcal{M})$, and hence on $(1 \otimes \varphi)(\varphi^*\mathcal{M})_{[0,r)}$, where r is as in Lemma 1.2.2. Using the map $\xi_{[0,r)}$ of Lemma 1.2.6, we obtain a filtration on $(D(\mathcal{M}) \otimes_{K_0} \mathcal{O})_{[0,r)}$. The required filtration on $D(\mathcal{M})_K$ is defined to be the image filtration under the composite

$$(D(\mathcal{M}) \otimes_{K_0} \mathcal{O})_{[0,r)} \to D(\mathcal{M}) \otimes_{K_0} \mathcal{O}/E(u)\mathcal{O} \xrightarrow{\sim} D(\mathcal{M}) \otimes_{K_0} K = D(\mathcal{M})_K.$$

Finally, if $\mathcal{M}$ is a (φ, N_∇)-module over $\mathcal{O}$ of finite E-height, then we equip $D(\mathcal{M})$ with a K_0-linear operator N, by reducing the operator N_∇ on $\mathcal{M}$ modulo u. This gives $D(\mathcal{M})$ the structure of a filtered (φ, N)-module.

We will show that the functors D and $\mathcal{M}$ induce quasi-inverse equivalences of categories.

Proposition 1.2.8. *Let D be an effective filtered (φ, N)-module. There is a natural isomorphism of filtered (φ, N)-modules $D(\mathcal{M}(D)) \xrightarrow{\sim} D$.*

Proof. As in Lemma 1.2.2, we set $\mathcal{D}_0 = (\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0}$. The natural inclusion $\mathcal{D}_0 \subset \mathcal{M}(D)$ is an isomorphism at $u = 0$, so that

$$D(\mathcal{M}(D)) = \mathcal{M}(D) \otimes_{\mathcal{O}} \mathcal{O}/u\mathcal{O} \xrightarrow{\sim} (K_0[\ell_u] \otimes_{K_0} D)^{N=0} \tag{1.2.9}$$

We claim that the composite map

$$\eta : (K_0[\ell_u] \otimes_{K_0} D)^{N=0} \subset K_0[\ell_u] \otimes_{K_0} D \overset{\ell_u \mapsto 0}{\to} D. \tag{1.2.10}$$

is an isomorphism of filtered (φ, N)-modules, where on the left-hand side N acts by $-N \otimes 1$. This is the operator induced by reducing the operator $N_\nabla \otimes 1$ on $\mathcal{O}[\ell_u] \otimes_{K_0} D$ modulo u. First, one checks easily that η is an injection, and that both sides have the same dimension. Hence η is a bijection. Since both maps in (1.2.10) are evidently compatible with φ, so is the composite. Finally, suppose that $d = \sum_{j \geq 0} d_j \ell_u^j \in (K_0[\ell_u] \otimes_{K_0} D)^{N=0}$, with $d_j \in D$. Since $N(d) = 0$, we see that $N(d_0) + d_1 = 0$. Hence

$$\eta(N_\nabla \otimes 1(d)) = -d_1 = N(d_0) = N(\eta(d)),$$

so η is compatible with N.

It remains to check that η is strictly compatible with filtrations. As remarked in the proof of Lemma 1.2.2, the submodule $\mathcal{D}_0 \subset \mathcal{M}(D)$, is contained in $(1 \otimes \varphi)(\varphi^* \mathcal{M})$, and this containment is an isomorphism at x_0. By definition of $\mathcal{M}$, an element $d \in \mathcal{D}_0$ is in $E(u)^i \mathcal{M}$ if and only if $\iota_0(d) \in \mathrm{Fil}^i(\widehat{\mathfrak{S}}_0 \otimes_K D_K)$. Hence, using Lemma 1.2.1, one sees that under the isomorphisms

$$\begin{aligned} D(\mathcal{M}(D))_K &= (K_0[\ell_u] \otimes_{K_0} D)^{N=0} \otimes_{K_0} \mathcal{O}/E(u)\mathcal{O} = \mathcal{D}_0/E(u)\mathcal{D}_0 \\ &\xrightarrow[\iota_0]{\sim} \widehat{\mathfrak{S}}_0 \otimes_K D_K/(u-\pi)\widehat{\mathfrak{S}}_0 \otimes_K D_K = D_K, \end{aligned} \tag{1.2.11}$$

the filtration on $D(\mathcal{M}(D))_K$ is identified with the given filtration on D_K.

Thus, to show that η is strictly compatible with filtrations, we have to check that the composite

$$D \stackrel{\eta^{-1}}{\to} D(\mathcal{M}(D)) \hookrightarrow D(\mathcal{M}(D))_K \stackrel{(1.2.11)}{\to} D_K.$$

is the natural inclusion. However, this is clear because both η and (1.2.11) send an element $\sum_{i \geq 0} \ell_u^i d_i \in (K_0[\ell_u] \otimes_{K_0} D)^{N=0}$ to d_0. □

Lemma 1.2.12. *Let* $\mathcal{M}$ *be in* $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$. *Then we have*

(1) *The* $\mathcal{O}$*-submodule* $(1 \otimes \varphi)\varphi^*\mathcal{M} \subset \mathcal{M}$ *is stable under* N_∇.
(2) *For* $i \geq 0$, $N_\nabla(E(u)^i \mathcal{M}) \subset E(u)^i \mathcal{M}$. *In particular, if we identify* $\varphi^*\mathcal{M}$ *with* $(1 \otimes \varphi)\varphi^*\mathcal{M}$ *via* $1 \otimes \varphi$, *then* N_∇ *respects the filtration on* $\varphi^*\mathcal{M}$ *defined in Section* 1.2.7.
(3) *The map*

$$\begin{aligned} (\mathcal{O}[\ell_u] \otimes_{K_0} D(\mathcal{M}))^{N=0} &= (K_0[\ell_u] \otimes_{K_0} D(\mathcal{M}))^{N=0} \otimes_{K_0} \mathcal{O} \\ &\stackrel{\eta \otimes 1}{\to} D(\mathcal{M}) \otimes_{K_0} \mathcal{O} \stackrel{\xi}{\to} \mathcal{M} \end{aligned}$$

is compatible with N_∇. *Here* η *is the isomorphism of* (1.2.10), *and* N_∇ *acts on the left via its action on* $\mathcal{O}[\ell_u]$.
(4) *For* $i \geq 1$, *applying* $\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}}$ *to the map of* (3) *and using the isomorphism of Lemma* 1.2.1(1) *induces an isomorphism*

$$\sum_{j \geq 0} E(u)^j \widehat{\mathfrak{S}}_0 \otimes_K \mathrm{Fil}^{i-j} D(\mathcal{M})_K \xrightarrow[\xi \circ (\eta \otimes 1)]{\sim} \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} (1 \otimes \varphi)(\mathrm{Fil}^i \varphi^* \mathcal{M}).$$

Proof. (1) follows from the relation $N_\nabla \varphi = E(u)\varphi N_\nabla$, while (2) follows from the Leibniz rule for N_∇, and the fact that $N_\nabla(E(u)) = -u\lambda E'(u) i E(u)^{i-1}$, since $E(u)$ divides λ in $\mathcal{O}$.

For (3) let $\sigma = N_\nabla \circ (\xi \circ \eta) - (\xi \circ \eta) \circ N_\nabla$, and write $D_0(\mathcal{M}) = (K_0[\ell_u] \otimes_{K_0} D(\mathcal{M}))^{N=0}$. Then σ is $\mathcal{O}$-linear, and it suffices to show that $\sigma(D_0(\mathcal{M})) = 0$. Since the map η of (1.2.10) is compatible with N, and ξ reduces to the identity modulo u, we have $\sigma(D_0(\mathcal{M})) \subset u\mathcal{M}$. On the other hand, $\xi \circ \eta$ is compatible with φ, so that $\sigma \circ \varphi = pE(u)/c_0 \varphi \circ \sigma$, and for $i \geq 1$

$$\begin{aligned}\sigma(D_0(\mathcal{M})) &= \sigma\circ\varphi^i(D_0(\mathcal{M}))\\ &= p^i E(u)/c_0\varphi(E(u)/c_0)\dots\varphi^{i-1}(E(u)/c_0)\varphi^i\circ\sigma(D_0(\mathcal{M}))\\ &\subset \mathcal{O}\cdot\varphi^i(u\mathcal{M})\subset u^{p^i}\mathcal{M}.\end{aligned}$$

It follows that $\sigma = 0$, which proves (3).

Finally, for (4) it will be convenient to again denote by N_∇ the operator $-u\lambda\frac{d}{du}$ on $\widehat{\mathfrak{S}}_0$, and to extend $N_\nabla^{\mathcal{M}}$ to a differential operator on $\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\mathcal{M}$, which we again denote by N_∇. By (2), N_∇ leaves $\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}(1\otimes\varphi)(\mathrm{Fil}^i\,\varphi^*\mathcal{M})$ stable.

Set $M_i = \widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}(E(u)(1\otimes\varphi)\varphi^*\mathcal{M}\cap E(u)^i\mathcal{M})$ for $i\geq 1$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \longrightarrow & M_{i-1} & \longrightarrow & M_{i-1}/M_i & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle N_\nabla|_{M_i}} & & \big\downarrow{\scriptstyle N_\nabla|_{M_{i-1}}} & & \big\downarrow & & \\ 0 & \longrightarrow & M_i & \longrightarrow & M_{i-1} & \longrightarrow & M_{i-1}/M_i & \longrightarrow & 0,\end{array}$$

where the vertical maps are induced by N_∇. We claim that $N_\nabla|_{M_i}$ is a bijection for $i\geq 0$.

By Lemmas 1.2.1(1) and 1.2.6 and (3) above, we have an N_∇-compatible isomorphism

$$\widehat{\mathfrak{S}}_0\otimes_{K_0}D(\mathcal{M})\xrightarrow{\sim}\widehat{\mathfrak{S}}_0\otimes_K D(\mathcal{M})_K\xrightarrow{\sim}\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\varphi^*\mathcal{M},$$

where N_∇ acts on the left via $N_\nabla\otimes 1$. Since N_∇ induces a bijection on $E(u)\widehat{\mathfrak{S}}_0$, our claim holds for $i = 0$. For $i\geq 1$, we may assume by induction that $N_\nabla|_{M_{i-1}}$ is a bijection. Hence $N_\nabla|_{M_{i-1}/M_i}$ is surjective, and it is therefore injective as M_{i-1}/M_i is a finite-dimensional K-vector space. Finally, it follows from the snake lemma that $N_\nabla|_{M_i}$ is surjective. In particular, we see that $N_\nabla|_{M_0/M_i}$ is bijective for all i.

To prove (4), we proceed by induction on i. For $i = 0$, this follows from Lemma 1.2.6. For $i\geq 1$, the induction hypothesis implies that

$$(u-\pi)\,\mathrm{Fil}^{i-1}(\widehat{\mathfrak{S}}_0\otimes_K D(\mathcal{M})_K) = (u-\pi)\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}(1\otimes\varphi)(\mathrm{Fil}^{i-1}\,\varphi^*\mathcal{M}).$$

Since the filtrations on $\varphi^*\mathcal{M}$ and on $\widehat{\mathfrak{S}}_0\otimes_K D(\mathcal{M})_K$ both induce the same filtration on their common quotient $D(\mathcal{M})_K$, it suffices to show that

$$\xi\circ(\eta\otimes 1)(\mathrm{Fil}^i\,D(\mathcal{M})_K)\subset\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}(1\otimes\varphi)(\mathrm{Fil}^i\,\varphi^*\mathcal{M}).$$

Let $d\in\xi\circ(\eta\otimes 1)(\mathrm{Fil}^i\,D(\mathcal{M})_K)$. We may write $d = d_0 + d_1$, with $d_0\in\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\mathrm{Fil}^i\,\varphi^*\mathcal{M}$, and $d_1\in E(u)\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\varphi^*\mathcal{M} = M_0$. Since $N_\nabla(d) = 0$,

$$N_\nabla(d_1) = -N_\nabla(d_0)\in\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\mathrm{Fil}^i\,\varphi^*\mathcal{M}\cap E(u)\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\varphi^*\mathcal{M} = M_i.$$

Hence, by what we saw above, we must have $d_1\in M_i\subset\widehat{\mathfrak{S}}_0\otimes_{\mathcal{O}}\mathrm{Fil}^i\,\varphi^*\mathcal{M}$. □

Proposition 1.2.13. *Let $\mathcal{M}$ be in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N_\nabla}$. There is a canonical isomorphism $\mathcal{M}(D(\mathcal{M}))\xrightarrow{\sim}\mathcal{M}$.*

Proof. Let $\mathcal{M}' = \mathcal{M}(D(\mathcal{M}))$. We will write $\mathcal{D}_0(\mathcal{M}) = (\mathcal{O}[\ell_u] \otimes_{K_0} D(\mathcal{M}))^{N=0}$. By construction $\mathcal{M}' \subset \mathcal{D}_0(\mathcal{M})[1/\lambda]$. On the other hand, if we identify $\mathcal{D}_0(\mathcal{M})$ with an $\mathcal{O}$-submodule of $\mathcal{M}$ via the map $\xi \circ (\eta \otimes 1)$ of Lemma 1.2.12(3), then $\mathcal{M} \subset \mathcal{D}_0(\mathcal{M})[1/\lambda]$, by Lemma 1.2.6. Since both these inclusions are compatible with N_∇ and φ, it suffices to check that $\mathcal{M}' = \mathcal{M}$.

It is enough to check that $\mathcal{M}'_{[0,r)} = \mathcal{M}_{[0,r)}$ where $r \in (|\pi|, |\pi|^{1/p})$, for then pulling back by $(\varphi^*)^i$, and using the fact that $\mathcal{M}$ and $\mathcal{M}'$ are both of finite E-height, we find that $\mathcal{M}'_{[0,r^{1/p^i})} = \mathcal{M}_{[0,r^{1/p^i})}$, and hence that $\mathcal{M} = \mathcal{M}'$.

Now at any point of $D[0, r)$ other than x_0, we have $\mathcal{M} = \mathcal{D}_0(\mathcal{M}) = \mathcal{M}'$, so we have to check that

$$\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{M} = \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{M}' \subset \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{D}_0(\mathcal{M})[1/\lambda] \underset{1.2.1}{\overset{\sim}{\longrightarrow}} \widehat{\mathfrak{S}}_0[1/(u-\pi)] \otimes_K D(\mathcal{M})_K$$

For this it suffices to check that an element $x \in \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{D}_0(\mathcal{M})$ is divisible by $E(u)^i$ in $\mathcal{M}$ for some $i \geq 0$ if and only if it is divisible by $E(u)^i$ in $\mathcal{M}'$. Now by Lemma 1.2.6, and the observations made in the proof of Lemma 1.2.2, we have

$$\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} (1 \otimes \varphi)\varphi^*\mathcal{M} = \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{D}_0(\mathcal{M}) = \widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} (1 \otimes \varphi)\varphi^*\mathcal{M}', \qquad (1.2.14)$$

so it is enough to show that the filtrations on the left- and right-hand sides of (1.2.14), defined in Section 1.2.7, coincide. This follows by comparing Lemma 1.2.1(2) with Lemma 1.2.12(4). □

Theorem 1.2.15. *The functors D and $\mathcal{M}$ induce exact, quasi-inverse equivalences of $\otimes$-categories between effective filtered (φ, N)-modules and the category* $\mathrm{Mod}_{/\mathcal{O}}^{\varphi, N_\nabla}$.

Proof. By Propositions 1.2.8 and 1.2.13 we know that D and $\mathcal{M}$ induce quasi-inverse equivalences of categories. It remains to check that they they are exact and compatible with tensor products.

Consider a sequence of filtered (φ, N)-modules

$$D^\bullet : 0 \to D'' \to D \to D' \to 0$$

and denote by $\mathcal{M}(D^\bullet)$ the corresponding sequence of (φ, N_∇)-modules over $\mathcal{O}$. If $D^\bullet$ is exact then, thinking of $\mathcal{M}(D^\bullet)$ as a sequence of coherent sheaves on $D[0, 1)$, we see that it is evidently exact outside the set of points $\{x_n\}_{n\geq 0}$, and the exactness at x_n follows from Lemma 1.2.1(2). Conversely, if $\mathcal{M}(D^\bullet)$ is exact then Lemma 1.2.1(2) implies that $D^\bullet$ is exact. Thus $\mathcal{M}$ and D are exact functors.

Suppose we are given filtered (φ, N)-modules D_1 and D_2. There is an obvious morphism of (φ, N_∇)-modules over $\mathcal{O}$, $\mathcal{M}(D_1) \otimes_{\mathcal{O}} \mathcal{M}(D_2) \to \mathcal{M}(D_1 \otimes_{K_0} D_2)$, which is an isomorphism outside the points $\{x_n\}_{n\geq 0}$. That it is an isomorphism at x_n follows from Lemma 1.2.1(2). Hence $\mathcal{M}$ commutes with tensor products.

Finally, suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are (φ, N_∇)-modules over $\mathcal{O}$. From the definitions, one sees that there is an isomorphism $D(\mathcal{M}_1) \otimes_{K_0} D(\mathcal{M}_2) \overset{\sim}{\longrightarrow} D(\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2)$ compatible with the action of φ, and that the map of K-vector spaces obtained by tensoring both sides by $\otimes_{K_0} K$ is compatible with filtrations. That it is strictly compatible with filtrations may be deduced from the strict compatibility with filtrations of the map

$$\varphi^*\mathcal{M}_1 \otimes_{\mathcal{O}} \varphi^*\mathcal{M}_2 \to \varphi^*(\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2). \qquad \square$$

1.3 Weakly admissible modules and F-crystals

In this section we show how to produce (φ, N_∇)-modules over $\mathcal{O}$ using φ-modules over $\mathfrak{S}$ of finite E-height.

1.3.1. We begin by reviewing the results of Kedlaya [Ke 1], [Ke 2]. Recall that the Robba ring $\mathcal{R}$ is defined by

$$\mathcal{R} = \lim_{r \to 1^-} \mathcal{O}_{(r,1)}$$

$\mathcal{R}$ is equipped with a Frobenius φ induced by the maps $\varphi : \mathcal{O}_{(r,1)} \to \mathcal{O}_{(r^{1/p},1)}$. We denote by $\mathrm{Mod}^{\varphi}_{/\mathcal{R}}$ the category of finite free $\mathcal{R}$-modules $\mathcal{M}$ equipped with an isomorphism $\varphi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$. This has a natural structure of a Tannakian category.

We also have the bounded Robba ring $\mathcal{R}^b$, defined by

$$\mathcal{R} = \lim_{r \to 1^-} \mathcal{O}^b_{(r,1)}$$

where $\mathcal{O}^b_{(r,1)} \subset \mathcal{O}_{(r,1)}$ denotes the functions on $D(r, 1)$ which are bounded. The ring $\mathcal{R}^b$ is a discrete valuation field, with a valuation $v_{\mathcal{R}^b}$ given by

$$v_{\mathcal{R}^b}(f) = -\log_p \lim_{r \to 1^-} \sup_{x \in D(r,1)} |f(x)|$$

The Frobenius φ on $\mathcal{R}$ induces a Frobenius φ on $\mathcal{R}^b$. We denote by $\mathrm{Mod}^{\varphi}_{/\mathcal{R}^b}$ the category of finite-dimensional $\mathcal{R}^b$-vector spaces $\mathcal{M}$ equipped with an isomorphism $\varphi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$.

Kedlaya defines an $\mathcal{R}$-algebra $\mathcal{R}^{\mathrm{alg}}$ (denoted by $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$ in [Ke 1]), which contains a copy of $W(\bar{k})$, where $\bar{k}$ denotes an algebraic closure of k, is equipped with a lifting φ of the Frobenius on $\mathcal{R}$, and such that for any $\mathcal{M}$ in $\mathrm{Mod}^{\varphi}_{/\mathcal{R}}$, there exists a finite extension E of $W(\bar{k})[1/p]$ such that $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{R}^{\mathrm{alg}} \otimes_{W(\bar{k})[1/p]} E$ admits a basis of φ-eigenvectors $\mathbf{v}_1, \dots \mathbf{v}_n$ such that $\varphi(\mathbf{v}_i) = \alpha_i \mathbf{v}_i$ for some $\alpha_i \in E$. The set of p-adic valuations of $\alpha_1, \dots, \alpha_n$ is uniquely determined by $\mathcal{M}$, and called the set of slopes of $\mathcal{M}$ [Ke 1, Theorem 4.16]. If these are all equal to some $s \in \mathbb{Q}$, then $\mathcal{M}$ is called pure of slope s. We denote by $\mathrm{Mod}^{\varphi,s}_{/\mathcal{R}}$ the full subcategory of $\mathrm{Mod}^{\varphi}_{/\mathcal{R}}$ consisting of modules which are pure of slope s. We write $\mathrm{Mod}^{\varphi,s}_{/\mathcal{R}^b}$ for the full subcategory of $\mathrm{Mod}^{\varphi}_{/\mathcal{R}^b}$ consisting of modules which are pure of slope s (as φ-modules over a discretely valued field).

Theorem 1.3.2. (1) *The functor $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{R}^b} \mathcal{R}$ induces an equivalence*

$$\mathrm{Mod}^{\varphi,s}_{/\mathcal{R}^b} \xrightarrow{\sim} \mathrm{Mod}^{\varphi,s}_{/\mathcal{R}}.$$

(2) *For any* $\mathcal{M}$ *in* $\mathrm{Mod}^{\varphi}_{/\mathcal{R}}$, *there exists a canonical filtration—called the slope filtration—*$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}$ *by* φ*-stable submodules such that* $\mathcal{M}_i/\mathcal{M}_{i-1}$ *is finite free over* $\mathcal{R}$ *and pure of slope* s_i, *and* $s_1 < s_2 < \cdots < s_r$.

Proof. The first part is [Ke 2, Theorem 6.3.3], while the second follows from [Ke 1, Theorem 6.10]. □

1.3.3. We want to show that if $\mathcal{M}$ in $\mathrm{Mod}^{\varphi}_{/\mathcal{R}}$ arises from a module $\mathcal{M}_{\mathcal{O}}$ in $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$, then the slope filtration of (4) is induced by a filtration on $\mathcal{M}_{\mathcal{O}}$.

We denote by N_∇ the operator $-u\lambda\frac{d}{du}$ on $\mathcal{R}$ and we write $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{R}}$ for the category whose objects consist of a module $\mathcal{M}$ in $\mathrm{Mod}^{\varphi}_{/\mathcal{R}}$ equipped with a differential operator $N_\nabla = N^{\mathcal{M}}_\nabla$ over the operator N_∇ on $\mathcal{R}$, such that $N_\nabla\varphi = (pE(u)/c_0)\varphi N_\nabla$.

For $\mathcal{M}$ a finite free $\mathcal{R}$-module (respectively, an $\mathcal{O}_I$-module for some interval $I \subset [0, 1)$,) we say that an $\mathcal{R}$ (respectively, $\mathcal{O}_I$) submodule $\mathcal{N} \subset \mathcal{M}$ is saturated if it is finitely generated and if $\mathcal{M}/\mathcal{N}$ is torsion-free or, equivalently, free over $\mathcal{R}$ (respectively, $\mathcal{O}_I$). If $\mathcal{N} \subset \mathcal{M}$ is any submodule, then there is a smallest submodule $\mathcal{N}' \subset \mathcal{M}$ containing $\mathcal{N}$ which is saturated, and we call this the saturation of $\mathcal{N}$.

Lemma 1.3.4. *Let* $\mathcal{M}$ *be a finite free* $\mathcal{O}$*-module equipped with a* φ*-semilinear map* $\varphi : \mathcal{M} \to \mathcal{M}$ *such that the induced map* $\varphi^*\mathcal{M} \to \mathcal{M}$ *is an injection. Let* $\mathcal{N}_{\mathcal{R}} \subset \mathcal{M}_{\mathcal{R}} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ *be a saturated submodule which is stable under* φ. *Then there is a unique saturated submodule* $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$ *such that* $\mathcal{N}_{(0,1)} \otimes_{\mathcal{O}_{(0,1)}} \mathcal{R} = \mathcal{N}_{\mathcal{R}}$. $\mathcal{N}_{(0,1)}$ *is* φ*-stable.*

Proof. Since $\mathcal{N}_{\mathcal{R}}$ is finitely generated, there exists $r \in (0, 1)$ and a saturated $\mathcal{O}_{(r,1)}$-submodule $\mathcal{N}_{(r,1)} \subset \mathcal{M}_{(r,1)}$ such that $\mathcal{N}_{(r,1)} \otimes_{\mathcal{O}_{(r,1)}} \mathcal{R} = \mathcal{N}$. Since $\mathcal{N}_{(r,1)}$ is clearly the unique such saturated submodule of $\mathcal{M}_{(r,1)}$, $1 \otimes \varphi$ induces a map $\varphi^*\mathcal{N}_{(r,1)} \to \mathcal{N}_{(r^{1/p},1)}$.

Set $\mathcal{N}_{(r^p,1)} = \mathcal{M}_{(r^p,1)} \cap \mathcal{N}_{(r,1)}$. Since $\mathcal{N}_{(r^p,1)}$ is clearly a closed $\mathcal{O}_{(r^p,1)}$-submodule, it is finitely generated, and one sees immediately that it is saturated. We claim that its rank is equal to $h = \mathrm{rk}_{\mathcal{O}_{(r,1)}} \mathcal{N}_{(r,1)}$. It suffices to show that $\varphi^*\mathcal{N}_{(r^p,1)}$ has $\mathcal{O}_{(r,1)}$-rank h. Since the map $\varphi : \mathcal{O}_{(r,1)} \to \mathcal{O}_{(r^p,1)}$ is finite flat, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \varphi^*\mathcal{N}_{(r^p,1)} & \longrightarrow & \varphi^*\mathcal{M}_{(r^p,1)} \oplus \varphi^*\mathcal{N}_{(r,1)} & \longrightarrow & \varphi^*\mathcal{M}_{(r,1)} \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathcal{N}_{(r,1)} & \longrightarrow & \mathcal{M}_{(r,1)} \oplus \mathcal{N}_{(r^{1/p},1)} & \longrightarrow & \mathcal{M}_{(r^{1/p},1)}.
\end{array}$$

Since the central and right vertical maps are injective, and the cokernel of the central vertical map is a torsion $\mathcal{O}_{(r,1)}$-module, we see that $\mathrm{rk}_{\mathcal{O}_{(r,1)}} \varphi^*\mathcal{N}_{(r^p,1)} = \mathrm{rk}_{\mathcal{O}_{(r,1)}} \mathcal{N}_{(r,1)} = h$.

Since $\mathcal{N}_{(r^p,1)} \otimes_{\mathcal{O}_{(r^p,1)}} \mathcal{O}_{(r,1)} \subset \mathcal{N}_{(r,1)}$ and both modules are saturated $\mathcal{O}_{(r,1)}$-submodules of $\mathcal{M}_{(r,1)}$ of the same rank, this inclusion must be an equality. Repeating the argument, we obtain for each $i \geq 0$ a saturated $\mathcal{O}_{(r^{p^i},1)}$-submodule $\mathcal{N}_{(r^{p^i},1)}$ of

$\mathcal{M}_{(r^{p^i},1)}$ such that the restriction of $\mathcal{N}_{(r^{p^i},1)}$ to $D(r^{p^{i-1}},1)$ is $\mathcal{N}_{(r^{p^{i-1}},1)}$. These modules glue to a coherent sheaf $\underline{\mathcal{N}}_{(0,1)}$ on $D(0,1)$. Write $\mathcal{N}_{(0,1)}$ for the global sections of $\underline{\mathcal{N}}_{(0,1)}$. Then $\mathcal{N}_{(0,1)}$ is a closed $\mathcal{O}_{(0,1)}$-submodule of $\mathcal{M}_{(0,1)}$, and hence finitely generated, and $\underline{\mathcal{N}}_{(0,1)}$ is the coherent sheaf corresponding to $\mathcal{N}_{(0,1)}$. In particular, we see that $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$ is saturated and that $\mathcal{N}_{(0,1)} \otimes_{\mathcal{O}_{(0,1)}} \mathcal{R} = \mathcal{N}_{\mathcal{R}}$. Since $\mathcal{N}_{(0,1)} = \mathcal{M}_{(0,1)} \cap \mathcal{N}_{\mathcal{R}}$ is the unique saturated submodule with this property, we see that $\mathcal{N}_{(0,1)}$ is stable under φ. □

Lemma 1.3.5. *Let $\mathcal{M}$ be a finite free $\mathcal{O}$-module equipped with a differential operator ∂ over $-u\frac{d}{du}$, and suppose that the operator $N : \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$ induced by ∂ is nilpotent. If $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$ is a saturated $\mathcal{O}_{(0,1)}$-submodule which is stable under ∂, then $\mathcal{N}_{(0,1)}$ extends uniquely to a saturated, ∂-stable $\mathcal{O}$-submodule $\mathcal{N} \subset \mathcal{M}$.*

Proof. This is part of the theory of connections with regular singular points. In fact one can even suppress the assumption on the nilpotence of N (cf. [De, Proposition 5.4]). Since we could not find a good reference, and for the convenience of the reader, we give a proof here. Closely related arguments may be found in the literature—see, for example, [Ba] and [An].

We equip $\mathcal{M}$ with the connection given by $\nabla(m) = -u^{-1}\partial(m)du$, and $\mathcal{M}' := \mathcal{M}/u\mathcal{M} \otimes_{K_0} \mathcal{O}$ with the logarithmic connection given by $\nabla(m \otimes f) = -N(m)/u \otimes f du + m \otimes df$. Then $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}', \mathcal{M})$ is naturally equipped with a logarithmic connection, given by $\nabla(f)(m') = -f(\nabla(m')) + \nabla(f(m'))$. Let $s_0 : \mathcal{M}/u\mathcal{M} \to \mathcal{M}$ be any K_0-linear map lifting the identity on $\mathcal{M}/u\mathcal{M}$. We define a new section by

$$s = \sum_{i=0}^{\infty} \nabla\left(\frac{d}{du}\right)^i (s_0)(-u)^i/i!.$$

Note that since s_0 lifts the identity section, $\nabla(\frac{d}{du})^i(s_0)(-u)^i/i!$ sends $\mathcal{M}/u\mathcal{M}$ into $u\mathcal{M}$ for each $i \geq 1$. Moreover, since N is nilpotent, this summand sends $\mathcal{M}/u\mathcal{M}$ into $u^{[i/d]}\mathcal{M}$, where d denotes the rank of $\mathcal{M}$. Using this one sees easily that there is a positive integer n such that the formula for s gives a well defined section $s : \mathcal{M}/u\mathcal{M} \to \mathcal{M}$ over $D[0, p^{-n})$. After replacing u by u/p^n, we may assume that s gives a well defined section over $D[0,1)$, so that $\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ as $\mathcal{O}$-modules with logarithmic connection. In particular, $\mathcal{M}^{\partial^d=0} \subset \mathcal{M}$ is a K_0-vector space of dimension d, which spans $\mathcal{M}$.

Now let $\mathcal{L}$ be any finite free $\mathcal{O}_{(0,1)}$-module equipped with a connection ∇, and define $\partial = \nabla(-u\frac{d}{du})$. We claim that the natural map $\mathcal{L}^{\partial^d=0} \otimes_{K_0} \mathcal{O}_{(0,1)} \to \mathcal{L}$ is injective. To see this we remark that we may replace $\mathcal{L}$ by the image of the above map, and assume that $\mathcal{L}^{\partial^d=0}$ spans $\mathcal{L}$. Note also that, if $\mathcal{L}'$ and $\mathcal{L}''$ are two finite free $\mathcal{O}_{(0,1)}$-modules with connection, and $\mathcal{L}$ is an extension of $\mathcal{L}'$ by $\mathcal{L}''$, then it suffices to prove the claim for $\mathcal{L}'$ and $\mathcal{L}''$. Furthermore if $\mathcal{L}' \subset \mathcal{L}$ is any $\mathcal{O}_{(0,1)}$-submodule which is stable by ∇, then $\mathcal{L}/\mathcal{L}'$ is equipped with a connection, and is hence $\mathcal{O}_{(0,1)}$-free [Kat 2, Proposition 8.9]. Applying these remarks with $\mathcal{L}' = a \cdot \mathcal{O}$ where $a \in \mathcal{L}^{\partial=0}$ is nonzero, and using induction on the rank of $\mathcal{L}$, it suffices to consider the case where $\mathcal{L} = \mathcal{O}_{(0,1)}$ with the trivial connection. In this case the result is clear since any

$f \in \mathcal{O}_{(0,1)}$ can be written as a convergent sum $f = \sum_{i \in \mathbb{Z}} a_i u^i$, so that $\partial^d(f) = 0$ if and only if f is constant.

Now the natural map $\mathcal{M}^{\partial^d=0} \otimes_{K_0} \mathcal{O} \to \mathcal{M}$ is evidently an isomorphism. Hence, applying the above remarks with $\mathcal{L} = \mathcal{N}_{(0,1)}$ and $\mathcal{M}_{(0,1)}/\mathcal{N}_{(0,1)}$, and using the snake lemma, we see that the map $\mathcal{N}_{(0,1)}^{\partial^d=0} \otimes_{K_0} \mathcal{O}_{(0,1)} \to \mathcal{N}_{(0,1)}$ is an isomorphism. In particular, $\mathcal{N}_{(0,1)}$ extends to $\mathcal{N} = \mathcal{N}_{(0,1)}^{\partial^d=0} \otimes_{K_0} \mathcal{O}$. □

Lemma 1.3.6. *Let $\mathcal{R}\xi$ denote a free $\mathcal{R}$-module of rank 1 with a generator ξ. We think of $\mathcal{L} := \mathcal{R} \oplus \mathcal{R}\xi$ as a right $\mathcal{R}$-module by setting $\xi \cdot a = a\xi + N_\nabla(a)$, and letting $\mathcal{R}$ act on itself in the natural way. This makes $\mathcal{L}$ into an $(\mathcal{R}, \mathcal{R})$-bimodule.*

Let $\mathcal{M}$ be in $\mathrm{Mod}_{/\mathcal{R}}^{\varphi,N_\nabla}$. Then $\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}$ has a natural structure of an object of $\mathrm{Mod}_{/\mathcal{R}}^{\varphi}$ given by

$$\varphi(a \otimes n + b\xi \otimes m) = \varphi(a)\varphi(n) + \varphi(b)(pE(u)/c_0)^{-1}\xi \otimes \varphi(m),$$

and the set of slopes of $\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}$ is equal to those of $\mathcal{M}$. More precisely, if s is a slope of $\mathcal{M}$ which appears with multiplicity h, then s appears with multiplicity $2h$ in $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{M}$.

Proof. First, observe that the formula giving φ defines a well defined Frobenius because

$$\begin{aligned}\varphi(\xi \otimes bm) &= (pE(u)/c_0)^{-1}\xi \otimes \varphi(bm)\\ &= (pE(u)/c_0)^{-1}\varphi(b)\xi \otimes \varphi(m) + (pE(u)/c_0)^{-1}N_\nabla(\varphi(b))\varphi(m)\\ &= \varphi(b\xi \otimes m + N_\nabla(b) \otimes m).\end{aligned}$$

To prove the second claim, we may reduce by dévissage to the case where $\mathcal{M}$ is irreducible and of pure slope $s \in \mathbb{Q}$. Then we have an exact sequence in $\mathrm{Mod}_{/\mathcal{R}}^{\varphi}$

$$0 \to \mathcal{M} \to \mathcal{L} \otimes_{\mathcal{R}} \mathcal{M} \to \mathcal{M}(1) \to 0.$$

Here $\mathcal{M}(1)$ denotes the object of $\mathrm{Mod}_{/\mathcal{R}}^{\varphi}$ whose underlying $\mathcal{R}$-module is equal to $\mathcal{M}$, but whose Frobenius is the Frobenius on $\mathcal{M}$ multiplied by $(pE(u)/c_0)^{-1}$. The first map is given by $m \mapsto m \oplus 0$, while the second sends $a + b\xi \otimes m$ to bm. It follows by [Ke 1, Proposition 4.5] that $\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}$ is pure of slope s. □

Proposition 1.3.7. *Let $\mathcal{M}$ be in $\mathrm{Mod}_{/\mathcal{O}}^{\varphi,N_\nabla}$ and set $\mathcal{M}_{\mathcal{R}} = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$. If*

$$0 = \mathcal{M}_{0,\mathcal{R}} \subset \mathcal{M}_{1,\mathcal{R}} \subset \cdots \subset \mathcal{M}_{r,\mathcal{R}} = \mathcal{M}_{\mathcal{R}}$$

denotes the slope filtration of $\mathcal{M}_{\mathcal{R}}$ then for $i = 0, 1, \dots, r$, $\mathcal{M}_{i,\mathcal{R}}$ extends uniquely to a saturated $\mathcal{O}$-submodule $\mathcal{M}_i \subset \mathcal{M}$ which is stable by φ and N_∇.

Proof. For any interval $I \subset [0, 1)$, $\mathcal{M}_I = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}_I$ is equipped with a differential operator induced by N_∇ on $\mathcal{M}$ and $-u\lambda \frac{d}{du}$ on $\mathcal{O}_I$. Passing to the limit we also get a differential operator on $\mathcal{M}_{\mathcal{R}}$. We again denote these differential operators by N_∇, By Lemma 1.3.4 $\mathcal{M}_{i,\mathcal{R}}$ extends to a saturated φ-stable submodule $\mathcal{M}_{i,(0,1)} \subset \mathcal{M}_{(0,1)}$. We claim that $\mathcal{M}_{i,(0,1)}$ is stable by N_∇. Since $\mathcal{M}_{i,(0,1)} = \mathcal{M}_{i,\mathcal{R}} \cap \mathcal{M}_{(0,1)}$, it suffices to show that $\mathcal{M}_{i,\mathcal{R}}$ is stable by N_∇. For this we use the notation of Lemma 1.3.6. Consider the map of $\mathcal{R}$-modules

$$\delta : \mathcal{L} \otimes_{\mathcal{R}} \mathcal{M} \to \mathcal{M}; \qquad (a + b\xi) \otimes m \mapsto am + bN_\nabla(m).$$

A simple calculation shows that δ respects the action of φ. Let $\mathcal{M}'_{i,\mathcal{R}} = \delta(\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}_{i,\mathcal{R}})$. We obviously have $\mathcal{M}_{i,\mathcal{R}} \subset \mathcal{M}'_{i,\mathcal{R}}$. To show this inclusion is an equality we proceed by induction on i. Let s_i denote the slope of $\mathcal{M}_i/\mathcal{M}_{i-1}$. When $i = 0$ there is nothing to prove. If $\mathcal{M}_{i-1,\mathcal{R}} = \mathcal{M}'_{i-1,\mathcal{R}}$, then we have surjections

$$\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}_{i,\mathcal{R}}/\mathcal{M}_{i-1,\mathcal{R}} \stackrel{\delta}{\to} \mathcal{M}'_{i,\mathcal{R}}/\mathcal{M}'_{i-1,\mathcal{R}} = \mathcal{M}'_{i,\mathcal{R}}/\mathcal{M}_{i-1,\mathcal{R}} \to \mathcal{M}'_{i.\mathcal{R}}/\mathcal{M}_{i,\mathcal{R}}.$$

Since the $\mathcal{R}$-submodule $\mathcal{M}'_{i,\mathcal{R}}/\mathcal{M}_{i,\mathcal{R}} \subset \mathcal{M}/\mathcal{M}_{i,\mathcal{R}}$ is finitely generated, it is finite free over $\mathcal{R}$ by Lemma 1.1.5. By Lemma 1.3.6, $\mathcal{L} \otimes_{\mathcal{R}} \mathcal{M}_{i,\mathcal{R}}/\mathcal{M}_{i-1,\mathcal{R}}$ is pure of slope s_i, so if $\mathcal{M}'_{i,\mathcal{R}}/\mathcal{M}_{i,\mathcal{R}}$ is nonzero, its smallest slope is $\leqslant s_i$ by [Ke 1, Lemma 4.1]. But then the smallest slope of $\mathcal{M}/\mathcal{M}_{i,\mathcal{R}}$ is $\leqslant s_i$, which is a contradiction as all the slopes of this module are $\geq s_{i+1} > s_i$.

Finally, N_∇ induces a differential operator $\partial = \lambda^{-1} N_\nabla$ over $-u\frac{d}{du}$ on $\mathcal{M}_{[0,p^{-2})}$. By Lemma 1.3.5, $\mathcal{M}_{i,(0,p^{-2})}$ extends to a unique ∂-stable saturated $\mathcal{O}_{[0,p^{-2})}$-submodule $\mathcal{M}_{i,[0,p^{-2})} \subset \mathcal{M}_{i,[0,p^{-2})}$. Hence $\mathcal{M}_{i,(0,1)}$ extends to a unique, N_∇-stable, saturated $\mathcal{O}$-submodule $\mathcal{M}_i \subset \mathcal{M}$. Since $\mathcal{M}_i = \mathcal{M} \cap \mathcal{M}_{i,\mathcal{R}}$ it is stable by φ. □

Theorem 1.3.8. *Let D be an effective filtered (φ, N)-module. Then D is weakly admissible if and only if $\mathcal{M}(D)$ is pure of slope* 0*.*

Proof. Suppose first that D has rank 1, and choose a basis $\boldsymbol{e} \in D$. Write $\varphi(e) = \alpha \boldsymbol{e}$ for some $\alpha \in K_0$. Set $\mathcal{D}_0 = (\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0}$. Then the definition of $\mathcal{M}$ shows that $\mathcal{M}(D) = \lambda^{-t_H(D)}\mathcal{D}_0$, so that

$$\varphi(\lambda^{-t_H(D)}\boldsymbol{e}) = (E(u)/c_0)^{t_H(D)}\alpha\lambda^{-t_H(D)}\boldsymbol{e}.$$

Hence $\mathcal{M}(D)$ has slope $t_N(D) - t_H(D)$. This proves the theorem for rank 1 (φ, N)-modules.

Suppose that D is weakly admissible. By Proposition 1.3.7, the slope filtration on $\mathcal{M}(D)_{\mathcal{R}}$ is induced by a filtration of $\mathcal{M}(D)$ by saturated $\mathcal{O}$-submodules, stable by φ and N_∇

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}.$$

Write s_i for the unique slope of $\mathcal{M}_{i,\mathcal{R}}/\mathcal{M}_{i-1,\mathcal{R}}$, and d_i for its $\mathcal{R}$-rank. By Theorem 1.2.15, $\mathcal{M}_1 = \mathcal{M}(D_1)$ for some filtered (φ, N) submodule $D_1 \subset D$. Then $\bigwedge^{d_1} \mathcal{M}_1$ has slope $d_1 s_1$ [Ke 1, Proposition 5.13], and the compatibility with tensor products in Theorem 1.2.15 and the rank 1 case considered above imply that this slope is $t_N(D_1) - t_H(D_1)$. Hence the weak admissibility of D implies that $s_1 \geq 0$. Since s_1 is the smallest slope this implies that $s_i \geq 0$ for all i. On the other hand, applying the rank 1 case as above, $\sum_{i=1}^r d_i s_i = t_N(D) - t_H(D) = 0$, so that $r = 1$ and $s_1 = 0$.

Conversely, suppose that $\mathcal{M}(D)$ is pure of slope 0. We have already seen that this implies $t_N(D) = t_H(D)$. If $D' \subset D$ is a (φ, N)-submodule, then $\mathcal{M}(D') \subset \mathcal{M}(D)$ has all slopes ≥ 0 by [Ke 1, Proposition 4.4]. In particular, the slope of the top exterior product of $\mathcal{M}(D')$ is ≥ 0, so we have $t_N(D') - t_H(D') \geq 0$. □

1.3.9. A (φ, N)-module over $\mathcal{O}$ is a φ-module $\mathcal{M}$ together with a K_0-linear map $N : \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$ which satisfies $N\varphi = p\varphi N$, where we have written φ for the endomorphism of $\mathcal{M}/u\mathcal{M}$ obtained by reducing $\varphi : \mathcal{M} \to \mathcal{M}$ modulo u. We say that $\mathcal{M}$ is pure of slope 0 if $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ is. As usual, $\mathcal{M}$ is said to be of finite E-height if it has this property as a φ-module over $\mathcal{O}$. We denote by $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$ the category of (φ, N)-modules over $\mathcal{O}$ of finite E-height, and by $\mathrm{Mod}^{\varphi,N,0}_{/\mathcal{O}}$ the full subcategory consisting of modules which are pure of slope 0. Each of these categories has a natural structure of a Tannakian category.

Given a module $\mathcal{M}$ in $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ we obtain a module $\tilde{\mathcal{M}}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$ by taking $\tilde{\mathcal{M}} = \mathcal{M}$ equipped with the operator φ, and taking N to be the reduction of N_∇ modulo u.

Lemma 1.3.10. *Let $\mathcal{M}$ be in $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$. Then*

(1) $\mathcal{M}[1/\lambda]$ *is canonically equipped with an operator N_∇ such that $N_\nabla\varphi = (p/c_0)E(u)\varphi N_\nabla$ and $N_\nabla|_{u=0} = N$.*
(2) *The functor $\mathcal{N} \mapsto \tilde{\mathcal{N}}$ is fully faithful, and a module $\mathcal{M}$ is in its image if and only if it is stable under the operator N_∇ on $\mathcal{M}[1/\lambda]$.*
(3) *Any $\mathcal{M}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$ which has $\mathcal{O}$-rank 1 is in the image of the functor in (2).*

Proof. The construction of Section 1.2.5 shows that given an $\mathcal{M}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$, we obtain a filtered (φ, N)-module $D(\mathcal{M})$, and that for $\mathcal{N}$ in $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ we have $D(\tilde{\mathcal{N}}) = D(\mathcal{N})$.

Now, given $\mathcal{M}$ we set $\mathcal{D}_0 = (\mathcal{O}[\ell_u] \otimes_{K_0} D(\mathcal{M}))^{N=0}$, equipped with an operator N_∇ induced by the corresponding operator on $\mathcal{O}[\ell_u]$. As in Lemma 1.2.12, we may consider the composite

$$\mathcal{D}_0 = (K_0[\ell_u] \otimes_{K_0} D(\mathcal{M}))^{N=0} \otimes_{K_0} \mathcal{O} \overset{\eta\otimes 1}{\to} D(\mathcal{M}) \otimes_{K_0} \mathcal{O} \overset{\xi}{\to} \mathcal{M} \qquad (1.3.11)$$

where η is a bijection, and ξ has cokernel killed by some power of λ by Lemma 1.2.6. Using (1.3.11), we obtain an isomorphism $\mathcal{D}_0[1/\lambda] \xrightarrow{\sim} \mathcal{M}[1/\lambda]$, which is compatible with the action of φ, and with N after applying $\otimes_{\mathcal{O}}\mathcal{O}/u\mathcal{O}$. From the definition of $\mathcal{M}(D)$, we have an isomorphism $\mathcal{D}_0[1/\lambda] \xrightarrow{\sim} \mathcal{M}(D(\mathcal{M}))[1/\lambda]$, compatible with φ and N_∇.

This proves (1). Moreover, by Theorem 1.2.15 $\mathcal{M}$ is in the image of $\mathcal{N} \mapsto \tilde{\mathcal{N}}$ if and only if $\mathcal{M} \xrightarrow{\sim} \mathcal{M}(D(\mathcal{M}))$ in $\mathcal{D}_0[1/\lambda]$, and this is equivalent to $\mathcal{M}$ being stable under N_∇. This also shows the claim regarding full faithfulness.

Finally, suppose $\mathcal{M}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$ has $\mathcal{O}$-rank 1. The above discussion shows that $\xi(D(\mathcal{M})) \subset \mathcal{M}$ is killed by N_∇. If $\boldsymbol{e}$ is a K_0-basis vector for $D(\mathcal{M})$, then there exists $f \in \mathcal{O}$ such that $\mathcal{M} = f^{-1}\mathcal{O}\boldsymbol{e}$, and

$$N_\nabla(f^{-1}\boldsymbol{e}) = -u\lambda\frac{df^{-1}}{du}\boldsymbol{e} = u\lambda\frac{df}{du}f^{-1}(f^{-1}\boldsymbol{e}).$$

So it suffices to show that $\lambda\frac{df}{du}f^{-1} \in \mathcal{O}$. Since $\mathcal{M} \subset \mathcal{D}_0[1/\lambda]$ the set of zeroes of f is contained in the set of zeroes of λ. Since $\frac{df}{du}f^{-1}$ has at most a simple pole at each such zero, this completes the proof of (3). □

1.3.12. A (φ, N)-module over $\mathfrak{S}$ is a finite free $\mathfrak{S}$-module $\mathfrak{M}$, equipped with a φ-semilinear Frobenius $\varphi : \mathfrak{M} \to \mathfrak{M}$, and a linear endomorphism $N : \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that $N\varphi = p\varphi N$ on $\mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We say that $\mathfrak{M}$ is of finite E-height if the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by some power of $E(u)$. We denote by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ the category (φ, N)-modules over $\mathfrak{S}$ of finite E-height, and by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ the associated isogeny category.

The reader may wonder why we do not insist that the operator N takes $\mathfrak{M}/u\mathfrak{M}$ to itself. The reason is that with this definition we could not prove Lemma 1.3.13 below. We do not know if the two definitions give rise to the same isogeny category.

Lemma 1.3.13. *The functor*

$$\Theta : \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Mod}^{\varphi,N,0}_{/\mathcal{O}}; \qquad \mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O} \tag{1.3.14}$$

is an equivalence of Tannakian categories.

Proof. Let $\mathcal{M}$ be in $\mathrm{Mod}^{\varphi,N,0}_{/\mathcal{O}}$. Then $\mathcal{M}_{\mathcal{R}} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ is in $\mathrm{Mod}^{\varphi,0}_{/\mathcal{R}}$, and hence Theorem 1.3.2 implies that $\mathcal{M}$ is of the form $\mathcal{M}_{\mathcal{R}^b} \otimes_{\mathcal{R}^b} \mathcal{R}$ for some $\mathcal{M}_{\mathcal{R}^b}$ in $\mathrm{Mod}^{\varphi,0}_{/\mathcal{R}^b}$, whose construction is functorial in $\mathcal{M}_{\mathcal{R}}$. Thus we have

$$\mathcal{M}_{\mathcal{R}^b} \otimes_{\mathcal{R}^b} \mathcal{R} \xrightarrow{\sim} \mathcal{M}_{\mathcal{R}} \xrightarrow{\sim} \mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}.$$

Choose an $\mathcal{R}^b$-basis for $\mathcal{M}_{\mathcal{R}^b}$, and an $\mathcal{O}$-basis for $\mathcal{M}$. The composite of the above isomorphisms is then given by a matrix with values in $\mathcal{R}$. By [Ke 1, Proposition 6.5], after modifying the chosen bases, we may assume that this matrix is the identity. In other words $\mathcal{M}_{\mathcal{R}^b}$ and $\mathcal{M}$ are spanned by a common basis. Let $\mathcal{M}^b$ denote the $\mathfrak{S}[1/p]$-span of this basis. Since $\mathfrak{S}[1/p] = \mathcal{O} \cap \mathcal{R}^b \subset \mathcal{R}$, we have $\mathcal{M}^b = \mathcal{M}_{\mathcal{R}^b} \cap \mathcal{M} \subset \mathcal{M}_{\mathcal{R}}$.

Hence $\mathcal{M}^b$ is stable by φ, and of finite E-height, since $\mathcal{M}$ is. This already shows that Θ is fully faithful, since given any $\mathcal{N}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$, $\mathcal{N} \otimes \mathbb{Q}_p$ can be recovered as $\Theta(\mathcal{N})_{\mathcal{R}^b} \cap \Theta(\mathcal{N})$. To show that it is essentially surjective, we have to check that $\mathcal{M}^b$ arises from an object of $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$.

Let $\mathcal{O}_{\mathcal{R}^b}$ denote the valuation ring of $\mathcal{R}^b$. Since $\mathcal{M}_{\mathcal{R}^b}$ has slope 0, there exists a φ-stable $\mathcal{O}_{\mathcal{R}^b}$-lattice $\mathcal{L}$ in $\mathcal{M}_{\mathcal{R}^b}$. Let $\mathfrak{M}' = \mathcal{M}^b \cap \mathcal{L}$, and set

$$\mathfrak{M} = \mathcal{O}_{\mathcal{R}^b} \otimes_{\mathfrak{S}} \mathfrak{M}' \cap \mathfrak{M}'[1/p] \subset \mathcal{M}_{\mathcal{R}^b}.$$

Then $\mathfrak{M} \subset \mathcal{M}_{\mathcal{R}^b}$ is a finite, φ-stable $\mathfrak{S}$-submodule. Moreover, the structure theory of finite $\mathfrak{S}$-modules shows that there exists an inclusion $\mathfrak{M}' \subset F$ of $\mathfrak{M}'$ into a finite free $\mathfrak{S}$-module F, such that $F/\mathfrak{M}'$ has finite length. This implies that $\mathfrak{M}$ may be identified with F. Thus $\mathfrak{M}$ is free over $\mathfrak{S}$.

To check that $\mathfrak{M}$ is in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$, we have to check that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by a power of $E(u)$. Let d be the $\mathfrak{S}$-rank of $\mathfrak{M}$. Then φ on $\bigwedge^d_{\mathfrak{S}} \mathfrak{M}$ with respect to some choice of basis vector is given by $p^r E(u)^s w$ where $r, s \geq 0$, and $w \in \mathfrak{S}^\times$. Since $\mathcal{M}_{\mathcal{R}^b} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{R}^b$, and $\wedge^d_{\mathcal{R}^b}\mathcal{M}_{\mathcal{R}^b}$ is pure of slope 0, we must have that $r = 0$. □

Corollary 1.3.15. *There exists a fully faithful $\otimes$-functor from the category of effective weakly admissible filtered (φ, N)-modules to* $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$.

If $\mathfrak{M}$ is in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$*, and $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$, then $\mathcal{M}[1/\lambda]$ is canonically equipped with a connection ∇ such that $\varphi \circ \nabla = \nabla \circ \varphi$. The module $\mathfrak{M}$ is in the image of the functor above if and only if ∇ induces a singular connection on $\mathcal{M}$ with only logarithmic singularities.*

Proof. By Theorems 1.3.8 and 1.2.15, $D \mapsto \mathcal{M}(D)$ is an equivalence between the category of effective weakly admissible filtered (φ, N)-modules, and $\mathrm{Mod}^{\varphi,N_\nabla,0}_{/\mathcal{O}}$. By Lemma 1.3.10, the latter category is a full subcategory of $\mathrm{Mod}^{\varphi,N,0}_{/\mathcal{O}}$ which is equivalent to $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ by Lemma 1.3.13. This proves the first claim, and the second follows from Lemma 1.3.10(2), the connection on $\mathcal{M}$ being given by $\nabla(m) = -\lambda^{-1} N_\nabla(m) \frac{du}{u}$. □

2 Galois representations and p-divisible groups

2.1 G_K-representations and G_{K_∞}-representations

In this section we will use the theory of the previous section to compare constructions of crystalline representations, and representations of finite E-height. We show that the functor from crystalline G_K-representations to G_{K_∞}-representations is fully faithful.

2.1.1. Let $\mathcal{O}_{\bar{K}}$ denote the ring of integers of $\bar{K}$. Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ where the transition maps are given by Frobenius. There is a unique surjective map $\theta : W(R) \to \widehat{\mathcal{O}_{\bar{K}}}$ to the p-adic completion $\widehat{\mathcal{O}_{\bar{K}}}$ of $\mathcal{O}_{\bar{K}}$, which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit.

Write $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$, where $\pi_n \in \bar{K}$ are the elements introduced in Section 1.1.1. Let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative. We embed the W-algebra $W[u]$ into $W(R)$ by $u \mapsto [\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$ this embedding extends to an embedding $\mathfrak{S} \hookrightarrow W(R)$, and $\theta|_{\mathfrak{S}}$ is the map $\mathfrak{S} \to \mathcal{O}_K$ sending u to π. This embedding is compatible with Frobenius endomorphisms.

We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\ker(\theta)$. As usual, we write $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$, we denote by B^+_{st} the ring obtained by formally adjoining the element "$\log[\underline{\pi}]$" to B^+_{cris}, and by B^+_{dR} the $\ker(\theta)$-adic completion of $W(R)[1/p]$.

Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with residue field the field of Laurent series $k((u))$. We write $\mathcal{E}$ for the field of fractions of $\mathcal{O}_{\mathcal{E}}$. If Fr R denotes the field of fractions of R, then the inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{E} \hookrightarrow W(\mathrm{Fr}\, R)$. Let $\mathcal{E}^{\mathrm{ur}} \subset W(\mathrm{Fr}\, R)[1/p]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\mathrm{Fr}\, R)[1/p]$, and $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ its ring of integers. Since Fr R is algebraically closed [Fo 1, A.3.1.6], the residue field $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ is a separable closure of $k((u))$. We denote by $\widehat{\mathcal{E}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$, and by $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ its ring of integers. $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is also equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}\, R)$. We write $\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \cap W(R) \subset W(\mathrm{Fr}\, R)$. We regard all these rings as subrings of $W(R)$.

Let $K_\infty = \cup_{n\geq 0} K_n$, and write $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$. Since G_{K_∞} fixes the subring $\mathfrak{S} \subset W(R)$, it acts on $\mathfrak{S}^{\mathrm{ur}}$ and $\mathcal{E}^{\mathrm{ur}}$.

Lemma 2.1.2. *Let $\mathfrak{M}$ be a finitely generated $\mathfrak{S}$-module equipped with an $\mathfrak{S}$-linear map $\varphi^*\mathfrak{M} \to \mathfrak{M}$. Suppose that $\mathfrak{M}$ is isomorphic as an $\mathfrak{S}$-module to a finite direct sum $\oplus_{i\in I}\mathfrak{S}/p^{n_i}\mathfrak{S}$ where $n_i \in \mathbb{N}^+$ and that* $\mathrm{coker}(1\otimes\varphi)$ *is killed by some power of $E(u)$. Then*

(1) *The association $\mathfrak{M} \mapsto \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}})$ is an exact functor in $\mathfrak{M}$.*
(2) *The natural map*

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{E}^{\mathrm{ur}}/\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})$$

is an isomorphism, and both sides are isomorphic to $\oplus_{i\in I}\mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$ as $\mathbb{Z}_p$-modules.

Proof. The first part of (2) follows from [Fo 1, B.1.8.4]. The rest of the lemma then follows from [Fo 1, Section A.1.2]. □

2.1.3. We denote by $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the category of finite free $\mathfrak{S}$-modules equipped with an $\mathfrak{S}$-linear map $1\otimes\varphi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ whose cokernel is killed by some power of $E(u)$. We may regard $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ as a full subcategory of $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ by taking the operator N to be 0 on an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$.

Corollary 2.1.4. *Let $\mathfrak{M}$ be in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. Then*

$$V_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$$

is a free $\mathbb{Z}_p$-module of rank $r = \mathrm{rk}_{\mathfrak{S}}\,\mathfrak{M}$, and the functor $\mathfrak{M} \mapsto V_{\mathfrak{S}}(\mathfrak{M})$ is exact in $\mathfrak{M}$. Moreover, the natural map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})$$

is a bijection.

Proof. This follows immediately from Lemma 2.1.2. □

Proposition 2.1.5. *Let D be an effective, weakly admissible filtered (φ, N)-module, and $\mathfrak{M}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ a module whose image in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ is equal to the image of D under the functor of Corollary* 1.3.15.

Then there exists a canonical bijection

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}),$$

which is compatible with the action of G_{K_∞} on the two sides. In particular, both sides have dimension $\dim_{K_0} D$*, and D is admissible.*

Proof. Set $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$. Using Proposition 1.2.8, we may identify D with $D(\mathcal{M})$. The inclusion $\mathfrak{S} \subset B^+_{\mathrm{cris}}$ admits a unique continuous extension to $\mathcal{O}$, and we will regard B^+_{cris} as an $\mathcal{O}$-algebra in this way. Since the inclusion of $\mathcal{O}$ in B^+_{cris} sends $E(u)$ to $E([\underline{\pi}]) \in \mathrm{Fil}^1 B^+_{\mathrm{dR}}$, it extends to an inclusion of $\widehat{\mathfrak{S}}_0$ into B^+_{dR}. Recall that the $\mathcal{O}$-module $\varphi^*\mathcal{M}$ is equipped with a decreasing filtration as in Section 1.2.7, while the ring $B^+_{\mathrm{cris}} \otimes_{K_0} K$ is equipped with a filtration via its inclusion into the discrete valuation ring B^+_{dR}.

Observe that we have natural maps

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathcal{O},\varphi}(\mathcal{M}, B^+_{\mathrm{cris}}) \to \mathrm{Hom}_{\mathcal{O},\mathrm{Fil},\varphi}(\varphi^*\mathcal{M}, B^+_{\mathrm{cris}}). \quad (2.1.6)$$

Here the term on the right means $\mathcal{O}$-linear, φ-compatible maps which induce a filtered map $\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \varphi^*\mathcal{M} \to B^+_{\mathrm{dR}}$, and the second map is obtained by composing morphisms with the inclusion $1 \otimes \varphi : \varphi^*\mathcal{M} \to \mathcal{M}$. It follows from the definition of the filtration on $\varphi^*\mathcal{M}$ (and the fact that $E([\underline{\pi}]) \in \mathrm{Fil}^1 B^+_{\mathrm{dR}})$) that any such composed morphism respects filtrations. Note that both maps in (2.1.6) are injective. This is clear for the first map, and for the second it follows from the fact that the cokernel of $1 \otimes \varphi$ is a killed by some power of λ, while B^+_{cris} is a domain.

Next, we set $\mathcal{D}_0 = (\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0}$. By Lemma 1.2.1 (and since we are identifying D and $D(\mathcal{M})$), we have an isomorphism $\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{D}_0 \xrightarrow{\sim} \widehat{\mathfrak{S}}_0 \otimes_K D_K$, and we regard the left-hand side of this isomorphism as equipped with the filtration induced by that on the right-hand side. By Lemma 1.2.12, $\xi \circ (\eta \otimes 1)$ induces a map

$$\mathrm{Hom}_{\mathcal{O},\mathrm{Fil},\varphi}(\varphi^*\mathcal{M}, B^+_{\mathrm{cris}}) \to \mathrm{Hom}_{\mathcal{O},\mathrm{Fil},\varphi}(\mathcal{D}_0, B^+_{\mathrm{cris}}), \quad (2.1.7)$$

where the term on the right means φ-compatible maps, which induce a map $\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{D}_0 \to B^+_{\mathrm{dR}}$ that is compatible with filtrations. Since the map of Lemma 1.2.12(3) is an isomorphism at the point x_0, (2.1.7) is an injection.

Finally, note that multiplication in the ring $\mathcal{O}[\ell_u]$ induces a natural isomorphism $\mathcal{O}[\ell_u] \otimes_{\mathcal{O}} \mathcal{D}_0 \underset{\mu\otimes 1}{\xrightarrow{\sim}} \mathcal{O}[\ell_u] \otimes_{K_0} D$ which is compatible with φ and N. Hence given any map f in the right-hand side of (2.1.7), we may form the composite

$$D \hookrightarrow \mathcal{O}[\ell_u] \otimes_{K_0} D \underset{(\mu\otimes 1)^{-1}}{\xrightarrow{\sim}} \mathcal{O}[\ell_u] \otimes_{\mathcal{O}} \mathcal{D}_0 \overset{1\otimes f}{\to} \mathcal{O}[\ell_u] \otimes_{\mathcal{O}} B^+_{\mathrm{cris}} \underset{\ell_u \mapsto \log[\underline{\pi}]}{\xrightarrow{\sim}} B^+_{\mathrm{st}}.$$

It follows from the definition of the filtration on $\widehat{\mathfrak{S}}_0 \otimes_{\mathcal{O}} \mathcal{D}_0$ that any such composed map respects filtrations after tensoring by $K \otimes_{K_0}$. Hence we obtain an injective map

$$\mathrm{Hom}_{\mathcal{O},\mathrm{Fil},\varphi}(\mathcal{D}_0, B^+_{\mathrm{cris}}) \to \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}). \tag{2.1.8}$$

Combining (2.1.6)–(2.1.8), we obtain a G_{K_∞}-equivariant inclusion

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Hom}_{\mathrm{Fil},\varphi,N}(D, B^+_{\mathrm{st}}),$$

Since the left-hand side has $\mathbb{Q}_p$-dimension $d = \dim_{K_0} D$ by Corollary 2.1.4, the dimension of the right-hand side is $\geq d$. But now an elementary argument [CF, Proposition 4.5] shows that the right-hand side has dimension d, and D is admissible. Hence our map is a bijection, as required. □

Lemma 2.1.9. *Let $h : \mathfrak{M} \to \mathfrak{M}'$ be a morphism in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ which becomes an isomorphism after tensoring by $\mathcal{O}_{\mathcal{E}}$. Then h is an isomorphism.*

Proof. Since h is a morphism of free $\mathfrak{S}$-modules of the same rank, it is an isomorphism if the induced map on determinants is. Hence we may assume that $\mathfrak{M}$ and $\mathfrak{M}'$ are free of rank 1 over $\mathfrak{S}$.

Let $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ and $\mathcal{M}' = \mathfrak{M}' \otimes_{\mathfrak{S}} \mathcal{O}$. By Lemmas 1.3.13 and 1.3.10(3), $\mathcal{M}$ and $\mathcal{M}'$ may be regarded as objects of $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$. Let $D = D(\mathcal{M})$ and $D' = D(\mathcal{M}')$. By Lemma 1.3.10(2) and Theorem 1.2.15 the map $D \to D'$ induced by h is nonzero, and hence is an isomorphism of filtered (φ, N)-modules. Hence h becomes an isomorphism after inverting p by Corollary 1.3.15. This means that in a suitable choice of bases h is given by multiplication by p^i for some nonnegative integer i. Since h becomes an isomorphism after tensoring by $\mathcal{O}_{\mathcal{E}}$, we must have $i = 0$. □

Lemma 2.1.10. *Let $\mathfrak{M}$ be in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, and let $V_{\mathfrak{S}}(\mathfrak{M})$ be as in Corollary 2.1.4. Then $\mathfrak{M}' = \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}), \mathfrak{S}^{\mathrm{ur}})$ is a free $\mathfrak{S}$-module of rank $d = \mathrm{rk}_{\mathfrak{S}}\, \mathfrak{M}$, and the natural map*

$$\mathfrak{M} \to \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}), \mathfrak{S}^{\mathrm{ur}}) = \mathfrak{M}'$$

is an injection.

Proof. Set $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. The natural map $\mathfrak{S}^{\mathrm{ur}}/p\mathfrak{S}^{\mathrm{ur}} \to \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ is an injection, so that we have an injection

$$\mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}), \mathfrak{S}^{\mathrm{ur}}/p\mathfrak{S}^{\mathrm{ur}}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}), \widehat{\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}).$$

By [Fo 1, A.1.2.7], the right-hand side is a $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} = k((u))$-vector space of dimension $d = \mathrm{rk}_{\mathbb{Z}_p} V_{\mathfrak{S}}(\mathfrak{M})$. The left-hand side is clearly a u-adically separated, torsion-free $k[\![u]\!]$-module. Hence it is a free $k[\![u]\!]$-module of rank at most d.

Now $\mathfrak{M}'$ is a p-adically separated torsion-free $\mathfrak{S}$-module. Moreover, we have an injection

$$\mathfrak{M}'/p\mathfrak{M}' \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}), \mathfrak{S}^{\mathrm{ur}}/p\mathfrak{S}^{\mathrm{ur}}).$$

Hence $\mathfrak{M}'$ is a quotient of $\mathfrak{S}^d$. On the other hand, the natural map $\mathfrak{M} \to \mathfrak{M}'$ is an injection because the map $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \to \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}), \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})$ is an isomorphism by [Fo 1, A.1.2.7]. Thus $\mathfrak{M}'$ must be a free $\mathfrak{S}$-module of rank $d = \mathrm{rk}_{\mathfrak{S}}\, \mathfrak{M}$ by Corollary 2.1.4. □

2.1.11. Denote by $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ the category of finite free $\mathcal{O}_{\mathcal{E}}$-modules $\mathcal{M}$ equipped with an isomorphism $\varphi^*\mathcal{M} \to \mathcal{M}$.

Proposition 2.1.12. *The functor*

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \to \mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}; \qquad \mathfrak{M} \mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$$

is fully faithful.

Proof. Let $\mathcal{M}$ be in $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$. If $\mathfrak{M} \subset \mathcal{M}$ is any finitely generated $\mathfrak{S}$-module which is stable under φ, and is such that $\mathfrak{M}/\varphi^*(\mathfrak{M})$ is killed by some power of $E(u)$, then we set $F(\mathfrak{M}) = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \cap \mathfrak{M}[1/p]$. As in the proof of Lemma 1.3.13, $F(\mathfrak{M})$ is a finite free $\mathfrak{S}$-module, and is naturally a submodule of $\mathcal{M}$, which contains $\mathfrak{M}$, is stable by φ, and such that $F(\mathfrak{M})/\varphi^*(F(\mathfrak{M}))$ is killed by some power of $E(u)$. In particular, $F(\mathfrak{M})$ is an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$.

Now suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, and write $\mathcal{M}_1 = \mathfrak{M}_1 \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_2 = \mathfrak{M}_2 \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Suppose we are given a morphism $h : \mathcal{M}_1 \to \mathcal{M}_2$ in $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$. We have to show this induces a map $\mathfrak{M}_1 \to \mathfrak{M}_2$.

Suppose first that h is the identity morphism. By Corollary 2.1.4, we have $V_{\mathfrak{S}}(\mathfrak{M}_1) = V_{\mathfrak{S}}(\mathfrak{M}_2)$, so both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are contained in $\mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V_{\mathfrak{S}}(\mathfrak{M}_1), \mathfrak{S}^{\mathrm{ur}})$, which is a finite $\mathfrak{S}$-module of rank $d = \mathrm{rk}_{\mathcal{O}_{\mathcal{E}}} \mathcal{M}_1$, by (2.1.10). In particular, $\mathfrak{M}_3 = \mathfrak{M}_1 + \mathfrak{M}_2 \subset \mathcal{M}_1$ is a finite $\mathfrak{S}$-module of rank d, which is stable under the action of φ, and $\mathfrak{M}_3/\varphi^*(\mathfrak{M}_3)$ is killed by a power of $E(u)$. Hence the morphism $\mathfrak{M}_1 \to F(\mathfrak{M}_3)$ is an isomorphism by Lemma 2.1.9, and similarly $\mathfrak{M}_2 = F(\mathfrak{M}_3) = \mathfrak{M}_1$.

Now consider the case of any map h. Let $\mathcal{M}_3 = h(\mathcal{M}_1)$, $\mathfrak{M}_3 = h(\mathfrak{M}_1)$, and $\mathfrak{M}'_3 = \mathcal{M}_3 \cap \mathfrak{M}_2$. Then $\mathcal{M}_3$ is in $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$, and $\mathfrak{M}_3$ and $\mathfrak{M}'_3$ are finitely generated, φ-stable $\mathfrak{S}$-modules, such that $\mathfrak{M}_3/\varphi^*(\mathfrak{M}_3)$ and $\mathfrak{M}'_3/\varphi^*(\mathfrak{M}'_3)$ are killed by some power of $E(u)$. To see this for $\mathfrak{M}'$ note that we have an exact sequence

$$0 \to \mathfrak{M}'_3 \to \mathcal{M}_3 \oplus \mathfrak{M}_2 \to \mathcal{M}_2$$

and that the map $1 \otimes \varphi$ is injective on all the terms of this sequence. Thus the cokernel of $1 \otimes \varphi$ on $\mathfrak{M}'_3$ may be identified with an $\mathfrak{S}$-submodule of the cokernel of $1 \otimes \varphi$ on $\mathfrak{M}_2$. By what we have seen above, we must have $F(\mathfrak{M}_3) = F(\mathfrak{M}'_3) \subset \mathcal{M}_3$, so h induces the composite map

$$\mathfrak{M}_1 \to F(\mathfrak{M}_3) = F(\mathfrak{M}'_3) \to F(\mathfrak{M}_2) = \mathfrak{M}_2. \qquad \square$$

2.1.13. Denote by $\mathrm{Rep}_{G_{K_\infty}}$ the category of continuous representations of G_{K_∞} on finite-dimensional $\mathbb{Q}_p$-vector spaces. Similarly, we denote by $\mathrm{Rep}^{\mathrm{cris}}_{G_K}$ the category of crystalline representations of $G_K = \mathrm{Gal}(\bar{K}/K)$. The following result had been conjectured by Breuil [Br 1, p. 202].

Corollary 2.1.14. *The functor* $\mathrm{Rep}^{\mathrm{cris}}_{G_K} \to \mathrm{Rep}_{G_{K_\infty}}$ *obtained by restricting the action of a G_K-representation to G_{K_∞} is fully faithful.*

Proof. It suffices to prove the corollary for the full subcategory $\mathrm{Rep}^{\mathrm{cris},+}_{G_K} \subset \mathrm{Rep}^{\mathrm{cris}}_{G_K}$ consisting of crystalline representations with nonnegative Hodge–Tate weights.

Consider the diagram of functors

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{cris},+}_{G_K} & \longrightarrow & \mathrm{Rep}_{G_{K_\infty}} \\ \downarrow & & \uparrow \\ \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p & \longrightarrow & \mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}} \otimes \mathbb{Q}_p. \end{array}$$

Here $\otimes\mathbb{Q}_p$ means that we have passed to the associated isogeny category. The map on the left is given by composing the (contravariant) functor from crystalline representations to weakly admissible modules with the fully faithful functor of Corollary 1.3.15. The map on the bottom is given by Proposition 2.1.12, and hence is fully faithful, while the map on the right is given by sending $\mathcal{M}$ in $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ to $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(\mathcal{M}, \widehat{\mathcal{E}^{\mathrm{ur}}})$, and this functor is an equivalence by [Fo 1, A.1.2.7]. That the square commutes (up to a natural equivalence) follows from Proposition 2.1.5. It follows that the top functor is also fully faithful. □

Lemma 2.1.15. *Let $\mathfrak{M}$ be in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ and set $V = V_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $\mathcal{M} = \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$. Then the map $\mathfrak{N} \mapsto \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathfrak{S}^{\mathrm{ur}})$ is a bijection between finite free, φ-stable $\mathfrak{S}$-submodules $\mathfrak{N} \subset \mathcal{M}$ such that $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{N} \xrightarrow{\sim} \mathcal{M}$ and $\mathfrak{N}/\varphi^*(\mathfrak{N})$ is killed by a power of $E(u)$, and G_{K_∞}-stable $\mathbb{Z}_p$-lattices $L \subset V$.*

Proof. By [Fo 1, A.1.2.7] the set of G_{K_∞}-stable lattices $L \subset V$ is in bijection with the set of finite free, φ-stable $\mathcal{O}_{\mathcal{E}}$-lattices $\mathcal{N} \subset \mathcal{M}$ such that the map $\varphi^*\mathcal{N} \to \mathcal{N}$ is an isomorphism.

Given $\mathfrak{N}$, $\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathfrak{S}^{\mathrm{ur}})$ is a G_{K_∞}-stable lattice in V by Corollary 2.1.4. Moreover, the above remarks together with Corollary 2.1.4 and Lemma 2.1.9 show that the map of the lemma is an injection. Suppose we are given a G_{K_∞}-stable lattice $L \subset V$, and let $\mathcal{N} = \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(L, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})$ be the corresponding finite free $\mathcal{O}_{\mathcal{E}}$-module. Let $\mathfrak{N} = \mathcal{N} \cap \mathfrak{M}[1/p] \subset \mathcal{M}$. As in the proof of Lemma 1.3.13 $\mathfrak{N}$ is a finite free $\mathfrak{S}$-module such that $\mathfrak{N}/\varphi^*(\mathfrak{N})$ is killed by some power of $E(u)$, and $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} = \mathcal{N}$. Hence $\mathfrak{N}$ maps to L by Corollary 2.1.4. □

2.2 Applications to p-divisible groups

In this section we apply the theory of Section 1 to the special case of p-divisible groups. We give a classification of p-divisible groups (up to isogeny when $p = 2$) using $\mathfrak{S}$-modules, and we show Fontaine's conjecture that a crystalline representation with Hodge–Tate weights 0 and 1 arises from a p-divisible group.

2.2.1. We will denote by $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ the full subcategory of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ consisting of objects $\mathfrak{M}$ such that $\mathfrak{M}/\varphi^*(\mathfrak{M})$ is killed by $E(u)$ (not just some power). Similarly we denote by $\mathrm{BT}^{\varphi,N_\nabla}_{/\mathcal{O}}$ (respectively, $\mathrm{BT}^{\varphi}_{/\mathcal{O}}$) the full subcategory of $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$ (respectively, $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$)

consisting of objects $\mathcal{M}$ such that $N_\nabla = 0$ modulo u, (respectively, $N = 0$) and $\mathcal{M}/\varphi^*(\mathcal{M})$ is killed by $E(u)$.

We say a weakly admissible module D is of Barsotti–Tate type if $\mathrm{gr}^i D_K = 0$ for $i \neq 0, 1$.

Proposition 2.2.2. *The functor of Corollary* 1.3.15 *induces an exact equivalence between the category of weakly admissible modules of Barsotti–Tate type and* $\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$.

Proof. Let $\mathfrak{M}$ be in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ and $\tilde{\mathcal{M}} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$ the corresponding object of $\mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$. Then $\tilde{\mathcal{M}}$ is evidently in $\mathrm{BT}^{\varphi}_{/\mathcal{O}}$. As in (1.3.11) we obtain a map $D(\tilde{\mathcal{M}}) \otimes_{K_0} \mathcal{O} \to \tilde{\mathcal{M}}$, which lifts the isomorphism $D(\tilde{\mathcal{M}}) \xrightarrow{\sim} \tilde{\mathcal{M}}/u\tilde{\mathcal{M}}$, and is compatible with φ and N_∇. Here N_∇ acts on $D(\tilde{\mathcal{M}}) \otimes_{K_0} \mathcal{O}$ as $1 \otimes -u\lambda\frac{d}{du}$. Now since $\tilde{\mathcal{M}}/\varphi^*(\tilde{\mathcal{M}})$ is killed by $E(u)$, one sees easily using Lemma 1.2.6 that $\tilde{\mathcal{M}}/(D(\tilde{\mathcal{M}}) \otimes_{K_0} \mathcal{O})$ is killed by λ. Let $m \in \tilde{\mathcal{M}}$, and write $m = \sum_{i=1}^r d_i \otimes \lambda^{-1} f_i$, where $d_i \in D(\tilde{\mathcal{M}})$ and $f_i \in \mathcal{O}$. Then

$$N_\nabla(m) = -u\lambda \sum_{i=1}^r d_i \otimes (-\lambda^{-2}\frac{d\lambda}{du} f_i + \lambda^{-1}\frac{df_i}{du}) = u\frac{d\lambda}{du}m - u\sum_{i=1}^r d_i \otimes \frac{df_i}{du} \in \tilde{\mathcal{M}}.$$

Hence, by Lemma 1.3.10, $\tilde{\mathcal{M}}$ arises from a module $\mathcal{M}$ in $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$, and $D(\tilde{\mathcal{M}}) = D(\mathcal{M})$ is weakly admissible by Theorems 1.3.8 and 1.2.15. By construction, the functor in Corollary 1.3.15 takes $D(\mathcal{M})$ to (an object isomorphic to) $\mathfrak{M}$.

It remains to remark that if D is an effective weakly admissible module, and $\mathfrak{M}$ in $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ is the image of D under the functor of Corollary 1.3.15, then $\mathfrak{M}$ is in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ if and only if D is of Barsotti–Tate type. This follows from Lemma 1.2.2. □

2.2.3. We will use the notation introduced in the appendix. Given a module $\mathfrak{M}$ in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, $\mathcal{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ has a natural structure of an object of $\mathrm{BT}^{\varphi}_{/S}$, where this is the category introduced in Section A.5. Here the tensor product is taken with respect to the map $\mathfrak{S} \to S$ sending u to u^p. Following [Br 4], we set

$$\mathrm{Fil}^1 \mathcal{M} = \{m \in \mathcal{M} : 1 \otimes \varphi(m) \in \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M}\},$$

and we define the map φ_1 as the composite

$$\varphi_1 : \mathrm{Fil}^1 \mathcal{M} \overset{1\otimes\varphi}{\to} \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\varphi_1\otimes 1}{\to} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}$$

By Lemma A.2, given a p-divisible group G over $\mathcal{O}_K$, $\mathcal{M}(G) := \mathbb{D}(G)(S)$ is naturally an object of $\mathrm{BT}^{\varphi}_{/S}$. By Proposition A.6 the functor $G \mapsto \mathcal{M}(G)$ is an equivalence between $\mathrm{BT}^{\varphi}_{/S}$ and the category of p-divisible groups if $p > 2$. If $p = 2$ it induces an equivalence between the corresponding isogeny categories.

Given a p-divisible group G over $\mathcal{O}_K$, we will denote by $T_p(G)$ its Tate module.

Lemma 2.2.4. *Let G be a p-divisible group over $\mathcal{O}_K$. If we regard the ring A_{cris} of Section 2.1.1 as an S-algebra via $u \mapsto [\underline{\pi}]$, then there is a canonical injection of G_{K_∞}-modules*

$$T_p(G) \hookrightarrow \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}(G), A_{\mathrm{cris}}).$$

This map is an isomorphism if $p > 2$, and has cokernel killed by p when $p = 2$.

Proof. An element of $T_p(G)$ is a map of p-divisible groups over $\mathcal{O}_{\bar{K}}$, $\mathbb{Q}_p/\mathbb{Z}_p \to G \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$. Since A_{cris} is a divided power thickening of $\widehat{\mathcal{O}_{\bar{K}}}$, we can pull this map back to $\widehat{\mathcal{O}_{\bar{K}}}$, and then evaluate the corresponding crystals on A_{cris} (see the appendix). This gives rise to a map $\mathcal{M} \otimes_S A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ compatible with filtrations and Frobenius. That the resulting map is injective, an isomorphism when $p > 2$, and has cokernel killed by p when $p = 2$, follows from [Fa, Theorem 7]. □

2.2.5. We remark that the fact that the map of Lemma 2.2.4 is an isomorphism when $p > 2$ also follows from the calculations of [Br 2, Section 5.3]; however, Faltings' argument is quite direct and does not rely on reduction to calculations with finite flat group schemes.

The following result had been conjectured by Fontaine [Fo 3, 5.2.5]

Corollary 2.2.6. *Let V be a crystalline representation of G_K with all Hodge–Tate weights equal to 0 or 1. Then there exists a p-divisible group G such that $V \xrightarrow{\sim} T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Proof. Let $D = \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(V, B^+_{\mathrm{cris}})$ denote the admissible filtered (φ, N)-module attached to V, and let $\mathfrak{M}$ in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ be the module associated to D by the functor of Proposition 2.2.2. We again denote by $\mathfrak{M}$ the object of $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ underlying $\mathfrak{M}$. Write $\mathcal{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ for the associated object of $\mathrm{BT}^{\varphi}_{/S}$. Then $\mathcal{M}$ is associated to a p-divisible group G as above, and by Lemma 2.2.4 we have an isomorphism of $\mathbb{Q}_p$-vector spaces with G_{K_∞}-action

$$T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{\mathrm{cris}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(D, B^+_{\mathrm{cris}}) = V.$$

Here the final isomorphism follows from the fact that, by [Br 2, 5.1.3], we have a canonical isomorphism $\mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} D \otimes_W S$, compatible with φ and filtrations. This fact is also easily deduced from Lemma 1.2.6. That this map is actually compatible with the action of G_K follows from Corollary 2.1.14. □

Theorem 2.2.7. *There exists an exact functor between $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ and the category of p-divisible groups over $\mathcal{O}_K$. When $p > 2$, this functor is an equivalence, and when $p = 2$ it induces an equivalence between the corresponding isogeny categories.*

Proof. Let $\mathfrak{M}$ be in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ and $\mathcal{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ the corresponding module in $\mathrm{BT}^{\varphi}_{/S}$. We have natural maps

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{\mathrm{cris}}) \qquad (2.2.8)$$

obtained by composing maps $\mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}$ with the inclusion $\mathfrak{S}^{\mathrm{ur}} \xrightarrow{\varphi} A_{\mathrm{cris}}$, and extending the resulting map to $\mathcal{M}$ by S-linearity. By Lemma 2.2.4 and Corollary 2.1.4, both sides of (2.2.8) are finite free $\mathbb{Z}_p$-modules of the same rank, and (2.2.8) is clearly injective. Hence it becomes an isomorphism after inverting p. In particular, any map in the right-hand side induces a map $\mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}[1/p]$. It follows that (2.2.8) is an isomorphism provided that any map $\mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}$ in the left-hand side whose composite with $\mathfrak{S}^{\mathrm{ur}} \xrightarrow{\varphi} A_{\mathrm{cris}}$ factors through pA_{cris} actually factors through $p\mathfrak{S}^{\mathrm{ur}}$. That this is the case for $p > 2$, was observed in the proof of [Br 3, 3.3.2].

Now given $\mathfrak{M}$ in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, the construction of Section 2.2.3 produces a p-divisible group $G(\mathfrak{M})$. Conversely, given a p-divisible group G, its Tate module $T_p(G)$ is a lattice in the Barsotti–Tate representation $V_p(G) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By Proposition 2.2.2 and Lemma 2.1.15, there is an $\mathfrak{M}$ in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, determined up to canonical isomorphism, such that $T_p(G) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$, and it follows from Proposition 2.1.12 that the assignment $G \mapsto \mathfrak{M} = \mathfrak{M}(G)$ is functorial.

Now suppose that $p > 2$. Then Lemma 2.2.4 and the fact that (2.2.8) is an isomorphism imply that for $\mathfrak{M}$ in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ there is a natural isomorphism $\mathfrak{M}(G(\mathfrak{M})) \xrightarrow{\sim} \mathfrak{M}$. On the other hand, if G is a p-divisible group over $\mathcal{O}_K$, then we have natural isomorphisms,

$$T_p(G(\mathfrak{M}(G))) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}(G), \mathfrak{S}^{\mathrm{ur}}) \xrightarrow{\sim} T_p(G),$$

and hence a natural isomorphism $G(\mathfrak{M}(G)) \xrightarrow{\sim} G$ by Tate's theorem.

For $p = 2$, the same arguments show that the functors G and $\mathfrak{M}$ induce equivalences on the associated isogeny categories. We could also have deduced the theorem in this case directly from Proposition 2.2.2 and Corollary 2.2.6. □

2.2.9. In [Ki, 2.2.22] we gave a different proof of the above theorem when $p > 2$, which, in particular, made no use of Tate's theorem. One can recover Tate's result from Theorem 2.2.7 by using the full faithfulness of Proposition 2.1.12 together with Lemma 2.2.4 and the isomorphism (2.2.8).

2.3 Classification of finite flat group schemes

In this final subsection of the paper, we use Theorem 2.2.7 to give a classification of finite flat group schemes over $\mathcal{O}_K$. The idea that one could do this is due to A. Beilinson, and we are grateful to him for allowing us to include his argument here. The final result was conjectured by Breuil [Br 4, 2.1.1]

2.3.1. Following the notation of [Ki] we denote by $'(\mathrm{Mod}\,/\mathfrak{S})$ the category consisting of $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semilinear map φ, such that the cokernel of $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ is killed by $E(u)$. We denote by $(\mathrm{Mod}\,/\mathfrak{S})$ the full subcategory of $'(\mathrm{Mod}\,/\mathfrak{S})$ consisting of modules $\mathfrak{M}$ such that $\mathfrak{M}$ has projective dimension 1 as an $\mathfrak{S}$-module and is killed by some power of p.

Later we will need the full subcategory $(\mathrm{ModFI}\,/\mathfrak{S})$ of $(\mathrm{Mod}\,/\mathfrak{S})$ consisting of modules $\mathfrak{M}$ which are of the form $\oplus_{i\in I}\mathfrak{S}/p^{n_i}\mathfrak{S}$, where I is a finite set and $n_i \in \mathbb{N}^+$.

Lemma 2.3.2. *A module* $\mathfrak{M}$ *in* $'(\mathrm{Mod}/\mathfrak{S})$ *is in* $(\mathrm{Mod}/\mathfrak{S})$ *if and only if* $\mathfrak{M}$ *is an extension in* $'(\mathrm{Mod}/\mathfrak{S})$ *of objects which are finite free* $\mathfrak{S}/p\mathfrak{S}$*-modules.*

Proof. We remark that since $\mathfrak{S}$ is a regular ring of dimension 2, the Auslander-Buchsbaum theorem implies that a finitely generated torsion $\mathfrak{S}$-module $\mathfrak{M}$ has projective dimension 1 if and only if it has depth 1. The latter condition holds if and only if the associated primes of $\mathfrak{M}$ are all of height 1 or, equivalently, if $\mathfrak{M}$ has no section supported on the closed point of Spec $\mathfrak{S}$.

Thus, if $\mathfrak{M}$ is in $(\mathrm{Mod}/\mathfrak{S})$ then the quotients $\mathfrak{M}[p^i]/\mathfrak{M}[p^{i-1}]$ for $i = 0, 1, 2\dots$ are easily seen to be free $\mathfrak{S}/p\mathfrak{S}$-modules, and one sees by descending induction on i that $\varphi^*(\mathfrak{M}[p^i]/\mathfrak{M}[p^{i-1}]) \to \mathfrak{M}[p^i]/\mathfrak{M}[p^{i-1}]$ has kernel killed by $E(u)$, and is therefore injective [Ki, 1.1.9]. Hence $\mathfrak{M}$ is an extension of objects which are free over $\mathfrak{S}/p\mathfrak{S}$.

Conversely, any such extension has projective dimension 1, and is killed by some power of p. □

2.3.3. Let $D^b(\mathrm{BT}^{\varphi}_{/\mathfrak{S}})$ denote the bounded derived category of the exact category $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$. We write $(\mathrm{Mod}/\mathfrak{S})^\bullet$ for the full subcategory of $D^b(\mathrm{BT}^{\varphi}_{/\mathfrak{S}})$ consisting of two-term complexes $\mathfrak{M}^\bullet = \mathfrak{M}_1 \to \mathfrak{M}_2$ concentrated in degrees 0 and -1, such that $H^{-1}(\mathfrak{M}^\bullet) = 0$, and $H^0(\mathfrak{M}^\bullet)$ is killed by a power of p. This is equivalent to asking that the map $\mathfrak{M}_1 \to \mathfrak{M}_2$ becomes an isomorphism in the isogeny category of $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$. Concretely, $(\mathrm{Mod}/\mathfrak{S})^\bullet$ is obtained by taking the category of two-term complexes $\mathfrak{M}^\bullet$, as above, dividing by homotopy equivalences—that is, by morphisms $\mathfrak{N}^\bullet \to \mathfrak{M}^\bullet$ of the form $(h \circ d, d \circ h)$, where $h : \mathfrak{N}_2 \to \mathfrak{M}_1$ is a morphism in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$—and inverting quasi-isomorphisms.

Lemma 2.3.4. *The functor* $\mathfrak{M}^\bullet \mapsto H^0(\mathfrak{M}^\bullet)$ *induces an equivalence between* $(\mathrm{Mod}/\mathfrak{S})^\bullet$ *and* $(\mathrm{Mod}/\mathfrak{S})$.

Proof. It is easy to check that the functor is fully faithful. To check essential surjectivity it suffices, given $\mathfrak{M}$ in $(\mathrm{Mod}/\mathfrak{S})$, to find $\tilde{\mathfrak{M}}$ in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ and a surjection $\tilde{\mathfrak{M}} \to \mathfrak{M}$ compatible with φ. Indeed, the kernel of any such surjection is automatically a finite free module, and since $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ is injective, this cokernel is in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$.

Let $L = \mathfrak{M}/(1 \otimes \varphi)(\varphi^*\mathfrak{M})$. Then L is a finite $\mathcal{O}_K$-module (via $\mathfrak{S} \overset{u\mapsto\pi}{\to} \mathcal{O}_K$). Let $\tilde{L}$ be a free $\mathcal{O}_K$-module, and $\tilde{L} \to L$ a surjection. Choose a free $\mathfrak{S}$-module $\tilde{\mathfrak{M}}$ and surjections of $\mathfrak{S}$-modules $\tilde{\mathfrak{M}} \to \tilde{L}$ and $\tilde{\mathfrak{M}} \to \mathfrak{M}$ compatible with the projections of $\tilde{L}$ and $\mathfrak{M}$ to L.

Since we may always replace $\tilde{\mathfrak{M}}$ with $\tilde{\mathfrak{M}} \oplus \mathfrak{S}^r$ for $r \in \mathbb{N}^+$, and map the second factor to 0 in $\tilde{L}$ and arbitrarily to $\mathfrak{N} = \ker(\mathfrak{M} \to L)$, we may assume that $\tilde{\mathfrak{N}} := \ker(\tilde{\mathfrak{M}} \to \tilde{L})$ surjects onto $\mathfrak{N}$. Finally, we may write $\tilde{\mathfrak{N}} = \tilde{\mathfrak{N}}_0 \oplus \tilde{\mathfrak{N}}_1$, where $\tilde{\mathfrak{N}}_1$ maps to 0 in $\mathfrak{N}$, and $\tilde{\mathfrak{N}}_0 \otimes_{\mathfrak{S}} k \overset{\sim}{\longrightarrow} \mathfrak{N} \otimes_{\mathfrak{S}} k$. Since $\varphi^*(\tilde{\mathfrak{M}})$ is a free $\mathfrak{S}$-module, the composite

$$\varphi^*(\tilde{\mathfrak{M}}) \to \varphi^*(\mathfrak{M}) \overset{\sim}{\longrightarrow} \mathfrak{N} \subset \mathfrak{M}$$

lifts to $\tilde{\mathfrak{N}}_0$, and any such lift is automatically a surjection. We may then lift this further to a surjection $\varphi^*(\tilde{\mathfrak{M}}) \to \tilde{\mathfrak{N}}$. Any such lift is an isomorphism, since both sides are free $\mathfrak{S}$-modules of the same rank.

The induced map $\varphi^*(\tilde{\mathfrak{M}}) \to \tilde{\mathfrak{M}}$ has cokernel $\tilde{L}$, and hence gives $\tilde{\mathfrak{M}}$ the structure of a module in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, which surjects onto $\mathfrak{M}$. □

Theorem 2.3.5. *If $p > 2$, there is an exact anti-equivalence between* $(\mathrm{Mod}\,/\mathfrak{S})$ *and the category* $(p\text{-}Gr\,/\mathcal{O}_K)$ *of finite flat group schemes over* $\mathcal{O}_K$.

Proof. Let $D^b(p\text{-div}\,/\mathcal{O}_K)$ denote the bounded derived category of the exact category of p-divisible groups over $\mathcal{O}_K$. We write $(p\text{-Gr}\,/\mathcal{O}_K)^\bullet$ for the full subcategory of $D^b(p\text{-div}\,/\mathcal{O}_K)$ consisting of isogenies of p-divisible groups $G_1 \to G_2$. This category has an explicit description analogous to the one given for $(\mathrm{Mod}\,/\mathfrak{S})^\bullet$ in Section 2.3.3.

The kernel of any isogeny is a finite flat group scheme, and conversely given any finite flat group scheme G there exists an embedding of G into a p-divisible group G_1 [BBM, 3.1.1]. The quotient G_1/G (taken, for example, in the category of fppf sheaves) is a p-divisible group. Hence one sees easily that the functor $(p\text{-Gr}\,/\mathcal{O}_K)^\bullet \to (p\text{-Gr}\,/\mathcal{O}_K)$ given by sending an isogeny to its kernel is an equivalence of categories.

On the other hand, $(p\text{-Gr}\,/\mathcal{O}_K)^\bullet$ is anti-equivalent to $(\mathrm{Mod}\,/\mathfrak{S})^\bullet$ by Theorem 2.2.7, and the theorem follows, since $(\mathrm{Mod}\,/\mathfrak{S})^\bullet$ is equivalent to $(\mathrm{Mod}\,/\mathfrak{S})$ by Lemma 2.3.4. □

Corollary 2.3.6. *If $p > 2$, the category* $(\mathrm{ModFI}\,/\mathfrak{S})$ *is anti-equivalent to the category of finite flat group schemes G over $\mathcal{O}_K$ such that $G[p^n]$ is finite flat for $n \geq 1$.*

Proof. This can be deduced by formal arguments from Theorem 2.3.5 in the same way that [Br 2, 4.2.2.5] is deduced from [Br 2, 4.2.1.6]. □

Appendix A: Crystals and p-divisible groups

A.1

Let T be a W-scheme on which p is locally nilpotent, and denote by $(T/W)_{\mathrm{cris}}$ the crystalline site of T over W, corresponding to embeddings of W-schemes $T \hookrightarrow T'$, defined by a sheaf of ideal J on T', which is equipped with divided powers, and such that the local sections of J are nilpotent.

Let G be a p-divisible group on T. Recall [MM, II Section 9] that there is a contravariant functor $G \mapsto \mathbb{D}(G)$ from the category of p-divisible groups over T to the category of crystals on $(T/W)_{\mathrm{cris}}$. The functor is defined using the Lie algebra of the universal vector extension of the dual p-divisible group G^*.

The formation of $\mathbb{D}(G)$ is compatible with arbitrary base change. In particular, if $p = 0$ on T, then we can pull G back by the Frobenius φ on T. The relative Frobenius on G, gives a map $G \to \varphi^*(G)$, and hence a map of crystals

$$\varphi^*(\mathbb{D}(G)) \xrightarrow{\sim} \mathbb{D}(\varphi^*(G)) \to \mathbb{D}(G).$$

Suppose now that T_0 is a W-scheme with $p = 0$ on T_0, and G_0 is a p-divisible group over T_0. Let $T_0 \hookrightarrow T$ be an object of $(T_0/W)_{\mathrm{cris}}$ on which p is locally nilpotent, and G a lifting of G_0 to T. By construction of $\mathbb{D}$, we have an isomorphism $\mathbb{D}(G_0)(T) \xrightarrow{\sim} \mathbb{D}(G)(T)$. Moreover, the $\mathcal{O}_T$-module $\mathbb{D}(G)(T)$ sits in an exact sequence

$$0 \to (\mathrm{Lie}\, G)^* \to \mathbb{D}(G)(T) \to \mathrm{Lie}\, G^* \to 0$$

where $(\mathrm{Lie}\, G)^*$ denotes the $\mathcal{O}_T$-dual of $\mathrm{Lie}\, G$. Hence specifying G equips $\mathbb{D}(G_0)(T)$ with an $\mathcal{O}_T$-submodule L such that $\mathbb{D}(G_0)(T)/L$ is a free $\mathcal{O}_T$-module.

The main result of [Me] asserts that if the divided powers on the ideal defining $T_0 \hookrightarrow T$ are nilpotent, then G is determined by L, and that, conversely, given a submodule $L \subset \mathbb{D}(G_0)(T)$ such that $\mathbb{D}(G_0)(T)/L$ is $\mathcal{O}_T$-free, and $L \otimes_{\mathcal{O}_T} \mathcal{O}_{T_0} \subset \mathbb{D}(G_0)(T_0)$ coincides with $(\mathrm{Lie}\, G_0)^*$, there is a p-divisible group G over T with $L = (\mathrm{Lie}\, G)^* \subset \mathbb{D}(G)(T) = \mathbb{D}(G_0)(T)$. (Strictly speaking, the result in [Me] applies when, locally on T_0, G_0 admits some lift to T, but this condition is always satisfied [MM, II Section 9]).

If $T = \mathrm{Spec}\, A$ is affine we will write $\mathbb{D}(G)(A)$ for $\mathbb{D}(G)(\mathrm{Spec}\, A)$.

Lemma A.2. *Let $A \to A_0$ be a surjection of p-adically complete and separated, local $\mathbb{Z}_p$-algebras with residue field k, whose kernel* $\mathrm{Fil}^1 A$ *is equipped with divided powers. Suppose that*

(1) *A is p-torsion-free, and equipped with an endomorphism $\varphi : A \to A$ lifting the Frobenius on A/pA.*
(2) *The induced map $\varphi^*(\mathrm{Fil}^1 A) \overset{1\otimes\varphi/p}{\to} A$ is surjective.*

If G is a p-divisible group over A_0, write $\mathrm{Fil}^1 \mathbb{D}(G)(A) \subset \mathbb{D}(G)(A)$ *for the preimage of* $(\mathrm{Lie}\, G)^* \subset \mathbb{D}(G)(A_0)$. *Then the restriction of $\varphi : \mathbb{D}(G)(A) \to \mathbb{D}(G)(A)$ to* $\mathrm{Fil}^1 \mathbb{D}(G)(A)$ *is divisible by p, and the induced map*

$$\varphi^* \mathrm{Fil}^1 \mathbb{D}(G)(A) \overset{1\otimes\varphi/p}{\longrightarrow} \mathbb{D}(G)(A)$$

is a surjection.

Proof. Let $\mathcal{M} = \mathbb{D}(G)(A)$. Let $\tilde{G}$ be a lifting of G to A, and set $\tilde{G}_0 = G \otimes_A A/pA$. Note that φ induces the zero endomorphism of $(\mathrm{Lie}\, \tilde{G}_0)^*$, and that φ restricted to $\mathrm{Fil}^1 A = \ker(A \to A_0)$ is divisible by p, since this ideal is equipped with divided powers. In particular, the map of (2) makes sense. Since

$$\mathrm{Fil}^1 \mathcal{M} = (\mathrm{Lie}\, \tilde{G})^* + \mathrm{Fil}^1 A \cdot \mathcal{M},$$

we see that $\varphi(\mathrm{Fil}^1 \mathcal{M}) \subset p\mathcal{M}$, so we may define a map

$$\varphi_1 = \varphi/p : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}.$$

We have to check that the image of this map generates the A-module $\mathcal{M}$. The hypothesis (2) implies that $\varphi(\mathcal{M}) = \varphi_1(\mathrm{Fil}^1 A)A\varphi(\mathcal{M}) \subset \varphi_1(\mathrm{Fil}^1 \mathcal{M})A$. Hence it suffices to show that the map

$$\varphi^*(\mathrm{Fil}^1 \mathcal{M} + p\mathcal{M}) \stackrel{1\otimes\varphi/p}{\longrightarrow} \mathcal{M} \tag{A.2.1}$$

is surjective.

There is a unique map $A \to W(k)$ which lifts the projection $A \to k$ and is compatible with the action of Frobenius. Write $H = \tilde{G} \otimes_A W(k)$ and $H_0 = H \otimes_{W(k)} k$. By [MM, II Section 15] $\mathbb{D}(H)(W(k))$ is naturally isomorphic to the Dieudonné module of H, and this isomorphism is compatible with the action of Frobenius. Hence if V denotes the Verschiebung, then we have

$$(\mathrm{Lie}\, H)^* = V(F/p)(\mathrm{Lie}\, H^*) \subset V\mathbb{D}(H)(W(k)).$$

Hence $(\mathrm{Lie}\, H_0)^* \subset V\mathbb{D}(H_0)(k)$, and this inclusion must be an equality since both sides have the same k-dimension. (They may both be identified with the quotient $\mathbb{D}(H_0)(k)/F\mathbb{D}(H_0)(k)$.) Hence $(\mathrm{Lie}\, H)^* + p\mathbb{D}(H)(W(k)) = V\mathbb{D}(H)(W(k))$, and since $(F/p)V = 1$, we see that F/p induces a surjection of $(\mathrm{Lie}\, H)^* + p\mathbb{D}(H)(W(k))$ onto $\mathbb{D}(H)(W(k))$. Hence (A.2.1) is also a surjection. □

A.3

By a *special* ring we shall mean a p-adically complete, separated, p-torsion-free, local $\mathbb{Z}_p$-algebra A with residue field k, equipped with an endomorphism φ lifting the Frobenius on A/pA.

For such an A, we denote by $\mathcal{C}_A$ the category of finite free A-modules $\mathcal{M}$, equipped with a Frobenius semilinear map $\varphi : \mathcal{M} \to \mathcal{M}$ and an A-submodule $\mathcal{M}_1 \subset \mathcal{M}$ such that $\varphi(\mathcal{M}_1) \subset p\mathcal{M}$ and the map $1 \otimes \varphi/p : \varphi^*(\mathcal{M}_1) \to \mathcal{M}$ is surjective.

Given a map of special rings $A \to B$, (that is a map of $\mathbb{Z}_p$-algebras compatible with φ) and $\mathcal{M}$ in $\mathcal{C}_A$, we give $\mathcal{M} \otimes_A B$ the structure of an object in $\mathcal{C}_B$, by giving it the induced Frobenius, and setting $(\mathcal{M} \otimes_A B)_1$ equal to the image of $\mathcal{M}_1 \otimes_A B$ in $\mathcal{M} \otimes_A B$.

Lemma A.4. *Let $h : A \to B$ be a surjection of special rings with kernel J. Suppose that for $i \geq 1$, $\varphi^i(J) \subset p^{i+j_i} J$, where $\{j_i\}_{i\geq 1}$ is a sequence of integers such that $\varinjlim_i j_i = \infty$.*

Let $\mathcal{M}$ and $\mathcal{M}'$ be in $\mathcal{C}_A$, and $\theta_B : \mathcal{M} \otimes_A B \xrightarrow{\sim} \mathcal{M}' \otimes_A B$ an isomorphism in $\mathcal{C}_B$ Then there exists a unique isomorphism of A-modules $\theta_A : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ lifting θ_B, and compatible with φ.

Proof. Let $\theta_0 : \mathcal{M} \to \mathcal{M}'$ be any map of A-modules lifting θ_B. Since $\varphi(J) \subset pA$ the truth of the proposition is unaffected if we replace $\mathcal{M}_1$ and $\mathcal{M}'_1$ by $\mathcal{M}_1 + J\mathcal{M}$ and $\mathcal{M}'_1 + J\mathcal{M}'$ respectively. In particular, we may assume that $\theta_0(\mathcal{M}_1) \subset \mathcal{M}'_1$.

We claim that the composite

$$\varphi^*(\mathcal{M}_1) \stackrel{\varphi^*(\theta_0|_{\mathcal{M}_1})}{\longrightarrow} \varphi^*(\mathcal{M}'_1) \stackrel{1\otimes\varphi/p}{\longrightarrow} \mathcal{M}' \tag{A.4.1}$$

factors through $\mathcal{M}$ via the map $1 \otimes \varphi/p$. To see this note that the map $\varphi^*\mathcal{M} \to \mathcal{M}$ is injective because, after inverting p, it becomes a surjection of finite free $A[1/p]$-modules of the same rank, and hence an isomorphism. Hence if $x \in \varphi^*(\mathcal{M}_1)$ is in the kernel of $1 \otimes \varphi/p$, then the image of x in $\varphi^*(\mathcal{M})$ is 0, and hence so is $\varphi^*(\theta_0)(x) \in \varphi^*(\mathcal{M}')$. It follows that (A.4.1) maps px and hence also x to 0.

Let $\theta_1 : \mathcal{M} \to \mathcal{M}'$ be the map induced by (A.4.1). Then for $x \in \mathcal{M}_1$ we have $\theta_1 \circ \varphi/p(x) = \varphi/p \circ \theta_0(x)$. Repeating the construction, we obtain a sequence of maps $\theta_0, \theta_1, \dots$ lifting θ_B, and such that $\theta_i \circ \varphi/p(x) = \varphi/p \circ \theta_{i-1}(x)$ for $x \in \mathcal{M}_1$. In particular, we have $(\theta_{i+1} - \theta_i) \circ \varphi/p = \varphi/p \circ (\theta_i - \theta_{i-1})$ on $\mathcal{M}_1$, and since $\varphi/p(\mathcal{M}_1)$ generates $\mathcal{M}$ as an A-module, we see that $(\theta_{i+1} - \theta_i)(\mathcal{M}) \subset (\varphi/p)^i(J)\mathcal{M}' \subset p^{j_i}\mathcal{M}'$. Hence the θ_i converge to a well defined map $\theta_A : \mathcal{M} \to \mathcal{M}'$, which commutes with φ and lifts θ_B.

If θ_A and θ'_A are two such maps, then as above, we obtain that $(\theta_A - \theta'_A)(\mathcal{M}) \subset (\varphi/p)^i(J)\mathcal{M}'$ for each i so that $\theta_A = \theta'_A$. □

A.5

We will apply Lemma A.4 in the following situation: J is equipped with divided powers, and there exist a finite set of elements $x_1, \dots x_n \in J$ such that J is topologically (for the p-adic topology) generated by the x_i and their divided powers, and $\varphi(x_i) = x_i^p$. The integers j_i may then be taken to be $v_p((p^i - 1)!) - i$.

Denote by S the p-adic completion of the divided power envelope of $W[u]$ with respect to the ideal $E(u)$. The ring S is equipped with an endomorphism φ given by the Frobenius on W, and $\varphi(u) = u^p$. We denote by $\mathrm{Fil}^1 S \subset S$ the closure of the ideal generated by $E(u)$ and its divided powers. Note that $\varphi(\mathrm{Fil}^1 S) \subset pS$. We set $\varphi_1 = \varphi/p|_{\mathrm{Fil}^1 S}$.

We will denote by $\mathrm{BT}^{\varphi}_{/S}$ the category of finite free S-modules $\mathcal{M}$ equipped with an S-submodule $\mathrm{Fil}^1 \mathcal{M}$ and a φ-semilinear map $\varphi_1 : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}$ such that

(1) $\mathrm{Fil}^1 S \cdot \mathcal{M} \subset \mathrm{Fil}^1 \mathcal{M}$, and the quotient $\mathcal{M}/\mathrm{Fil}^1 \mathcal{M}$ is a free $\mathcal{O}_K$-module.
(2) The map $\varphi^*(\mathrm{Fil}^1 \mathcal{M}) \stackrel{1\otimes\varphi_1}{\to} \mathcal{M}$ is surjective.

Any $\mathcal{M}$ in $\mathrm{BT}^{\varphi}_{/S}$ is equipped with a Frobenius semilinear map $\varphi : \mathcal{M} \to \mathcal{M}$ defined by $\varphi(x) = \varphi_1(E(u))^{-1}\varphi_1(E(u)x)$.

Proposition A.6. *There is an exact contravariant functor $G \mapsto \mathbb{D}(G)(S)$ from the category of p-divisible groups over $\mathcal{O}_K$ to $\mathrm{BT}^{\varphi}_{/S}$. If $p > 2$ this functor is an anti-equivalence, and if $p = 2$ it induces an anti-equivalence of the corresponding isogeny categories.*

Proof. Given a p-divisible group G over $\mathcal{O}_K$, the S-module $\mathcal{M}(G) := \mathbb{D}(G)(S)$ has a natural structure of an object of $\mathrm{BT}^{\varphi}_{/S}$ by (A.2). This gives a functor from p-divisible groups over $\mathcal{O}_K$ to $\mathrm{BT}^{\varphi}_{/S}$. We will construct a quasi-inverse (up to isogeny if $p = 2$).

Let $\mathcal{M}$ be in $\mathrm{BT}^{\varphi}_{/S}$. We begin by constructing from $\mathcal{M}$ a p-divisible group G_i over $\mathcal{O}_K/\pi^i$ for $i = 1, 2, \dots, e$. More precisely, for any such i let $R_i = W[u]/u^i$. It is equipped with a Frobenius endomorphism φ given by the usual Frobenius on W and $u \mapsto u^p$. We regard $\mathcal{O}_K/\pi^i$ as an R_i-algebra via $u \mapsto \pi$. This is a surjection with kernel pR_i, so R_i is a divided power thickening of $\mathcal{O}_K/\pi^i$ and given any p-divisible group G_i over $\mathcal{O}_K/\pi^i$ we may form $\mathbb{D}(G_i)(R_i)$. As in (A.2), we denote by $\mathrm{Fil}^1\,\mathbb{D}(G_i)(R_i)$ the preimage of $(\mathrm{Lie}\,G_i)^* \subset \mathbb{D}(G_i)(\mathcal{O}_K/\pi^i)$ in $\mathbb{D}(G_i)(R_i)$. On the other hand, we have a φ-compatible map $S \to R_i$, sending u to u, and $u^{ej}/j!$ to 0 for $j \geq 1$. Write I_i for the kernel of this map. We equip $\mathcal{M}_i = R_i \otimes_S \mathcal{M}$ with the induced Frobenius φ, and we set $\mathrm{Fil}^1\,\mathcal{M}_i \subset \mathcal{M}_i$ equal to the image of $\mathrm{Fil}^1\,\mathcal{M}$ in $\mathcal{M}_i$. Note that $1\otimes\varphi_1 : \varphi^*(\mathrm{Fil}^1\,\mathcal{M}) \to \mathcal{M}$ induces a surjective map $\varphi^*(\mathrm{Fil}^1\,\mathcal{M}_i) \to \mathcal{M}_i$. We will construct a p-divisible group G_i together with a canonical isomorphism $\mathbb{D}(G_i)(R_i) \xrightarrow{\sim} \mathcal{M}_i$ compatible with φ and filtrations.

Denote by $F : \mathcal{M}_1 \to \mathcal{M}_1$ the map induced by $\varphi : \mathcal{M} \to \mathcal{M}$. A simple computation shows that both sides of the surjective map $\varphi^*(\mathrm{Fil}^1\,\mathcal{M}_1) \to \mathcal{M}_1$, are free W-modules of the same rank, hence this map is an isomorphism. Composing the inverse of this isomorphism with the composite

$$\varphi^*(\mathrm{Fil}^1\,\mathcal{M}_1) \to \varphi^*(\mathcal{M}_1) \xrightarrow{\sim} \mathcal{M}_1,$$

where the first map is induced by the inclusion $\mathrm{Fil}^1\,\mathcal{M} \subset \mathcal{M}$, while the second is given by $a\otimes m \mapsto \varphi^{-1}(a)m$, gives a φ^{-1} semilinear map $V : \mathcal{M}_1 \to \mathcal{M}_1$, such that $FV = VF = p$. Denote by G_1 to be the p-divisible group associated (contravariantly) to this Dieudonné module. The tautological isomorphism $\mathbb{D}(G_1)(W) \xrightarrow{\sim} \mathcal{M}_1$ is compatible with Frobenius, and it is compatible with filtrations because $\mathrm{Fil}^1\,\mathbb{D}(G_1)$ may be identified with $V\mathbb{D}(G_1)$, as explained at the end of the proof of Lemma A.2.

Now suppose that $i \in [2, e]$ is an integer and that we have constructed G_{i-1} such that $\mathbb{D}(G_{i-1})(R_{i-1}) \xrightarrow{\sim} \mathcal{M}_{i-1}$ is compatible with Frobenius and filtrations. Note that the kernel of $R_i \to \mathcal{O}_K/\pi^{i-1}$ is equal to (u^{i-1}, p) which admits divided powers, so we may evaluate $\mathbb{D}(G_{i-1})$ on R_i. By Lemma A.2 and what we have already seen $\mathbb{D}(G_{i-1})(R_i)$, and $\mathcal{M}_i$ both have the structure of objects of $\mathcal{C}_{R_i}$, and the above isomorphism is an isomorphism in $\mathcal{C}_{R_{i-1}}$. Hence by Lemma A.4 applied to the surjection $R_i \to R_{i-1}$, it lifts to a unique φ-compatible isomorphism $\mathbb{D}(G_{i-1})(R_i) \xrightarrow{\sim} \mathcal{M}_i$. By the main result of [Me] there is a unique p-divisible group G_i over $\mathcal{O}_K/\pi^i$ which lifts G_{i-1}, and such that $(\mathrm{Lie}\,G_i)^* \subset \mathbb{D}(G_{i-1})(\mathcal{O}_K/\pi^i)$ is equal to the image of $\mathrm{Fil}^1\,\mathcal{M}_i$ under the composite

$$\mathrm{Fil}^1\,\mathcal{M}_i \subset \mathcal{M}_i \xrightarrow{\sim} \mathbb{D}(G_{i-1})(R_i) \to \mathbb{D}(G_{i-1})(\mathcal{O}_K/\pi^i).$$

By construction we have $\mathbb{D}(G_i)(R_i) \xrightarrow{\sim} \mathcal{M}_i$ compatible with φ and filtrations, which completes the induction.

We now apply Lemma A.4 to the surjection $S \to R_e$, and the modules $\mathcal{M}$ and $\mathbb{D}(G_e)(S)$ in $\mathcal{C}_S$. Note that the kernel of $S \to \mathcal{O}_K/\pi^e = \mathcal{O}_K/p$ admits divided powers, so we may evaluate $\mathbb{D}(G_e)$ on S, and the result is in $\mathcal{C}_S$ by Lemma A.2. Since $\mathcal{M}_e \xrightarrow{\sim} \mathbb{D}(G_e)(R_e)$ in $\mathcal{C}_{R_e}$, we have a canonical φ-compatible isomorphism $\mathcal{M} \xrightarrow{\sim} \mathbb{D}(G_e)(S)$ by Lemma A.4.

Suppose that $p > 2$. Then the divided powers on the kernel of $\mathcal{O}_K \to \mathcal{O}_K/p$ are nilpotent, and we may take $G = G(\mathcal{M})$ to be the unique lift of G_e to $\mathcal{O}_K$ such that

$(\mathrm{Lie}\, G)^* \subset \mathbb{D}(G_e)(\mathcal{O}_K)$ is equal to the image of $\mathrm{Fil}^1 \mathcal{M}$ under the composite of the above isomorphism and the projection $\mathbb{D}(G_e)(S) \to \mathbb{D}(G_e)(\mathcal{O}_K)$. Strictly speaking what Grothendieck–Messing theory produces is a sequence of p-divisible groups over $\mathcal{O}_K/p^i$ for $i = 1, 2, \ldots$ which are compatible under the maps $\mathcal{O}_K/p^i \to \mathcal{O}_K/p^{i-1}$. However, this data corresponds to a unique p-divisible group over $\mathcal{O}_K$ [deJ, 2.4.4].

From the construction we clearly have $\mathcal{M} \xrightarrow{\sim} \mathcal{M}(G(\mathcal{M}))$. On the other hand using the uniqueness at every stage of the construction, one sees by induction on i that for $i = 1, 2, \ldots, e$ and any p-divisible group G over $\mathcal{O}_K$, $G_i(\mathcal{M}(G))$ is isomorphic to $G \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^i$, and then that $G \xrightarrow{\sim} G(\mathcal{M}(G))$.

Now suppose that $p = 2$. We may regard the kernel of $\mathcal{O}_K/p^2 \to \mathcal{O}_K/p$ as being equipped with divided powers by taking the divided powers $p^{[i]}$ to be 0 for $i \geq 2$. We denote by G_{2e} the unique lift of G_e to $\mathcal{O}_K/p^2$, such that the image of the composite

$$\mathrm{Fil}^1 \mathcal{M} \to \mathcal{M} \xrightarrow{\sim} \mathbb{D}(G_e)(S) \to \mathbb{D}(G_e)(\mathcal{O}_K/p^2)$$

is equal to $(\mathrm{Lie}\, G_{2e})^*$. Finally, as for the case $p = 2$, we set G equal to the unique lift of G_{2e} to $\mathcal{O}_K$, such that the image of $\mathrm{Fil}^1 \mathcal{M}$ in $\mathbb{D}(G_{2e})(\mathcal{O}_K) = \mathbb{D}(G_e)(\mathcal{O}_K)$ is equal to $(\mathrm{Lie}\, G)^*$.

As for $p > 2$, we still have a natural isomorphism $\mathcal{M}(G(\mathcal{M})) \xrightarrow{\sim} \mathcal{M}$. Given a p-divisible group G over $\mathcal{O}_K$, we also obtain, as before, an isomorphism $G_e(\mathcal{M}(G)) \xrightarrow{\sim} G \otimes_{\mathcal{O}_K} \mathcal{O}_K/p$. In general, G_{2e} need not be isomorphic to $G'_{2e} := G \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^2$, because the divided power structure on $(p) \subset \mathcal{O}_K/p^2$, its not compatible with the divided powers on $(p) \subset S$. However, since both these p-divisible groups lift G_e, there exist maps $G_{2e} \leftrightarrows G'_{2e}$, lifting multiplication by p^2 on G_e [Kat, 1.1.3]. Since G and $G(\mathcal{M}(G))$ are obtained from G'_{2e} and G_{2e} as the unique lifts corresponding to the image of $\mathrm{Fil}^1 \mathcal{M}$ in

$$\mathbb{D}(G_{2e})(\mathcal{O}_K) \xrightarrow{\sim} \mathbb{D}(G_e)(\mathcal{O}_K) \xrightarrow{\sim} \mathbb{D}(G'_{2e})(\mathcal{O}_K),$$

these maps lift to maps $G(\mathcal{M}(G)) \leftrightarrows G$ whose composite in either order is multiplication by p^4. □

Acknowledgments. I would like to thank A. Beilinson, L. Berger, P. Berthelot, C. Breuil, J.-M. Fontaine, K. Kedlaya, and V. Lafforgue for extremely useful conversations and correspondence. I would especially like to thank Beilinson for allowing me to include the proof of Theorem 2.3.5.

The author is partially supported by NSF grant DMS-0400666 and a Sloan Research Fellowship.

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