Asymptotic behaviour of the Euler–Kronecker constant

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To Volodya Drinfeld with friendship and admiration.

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Summary. This appendix to the beautiful paper [1] of Ihara puts it in the context of infinite global fields of our papers [2] and [3]. We study the behaviour of Euler–Kronecker constant γ_K when the discriminant (genus in the function field case) tends to infinity. Results of [2] easily give us good lower bounds on the ratio $\gamma_K / \log \sqrt{|d_K|}$. In particular, for number fields, under the generalized Riemann hypothesis we prove

$$\liminf \frac{\gamma_K}{\log \sqrt{|d_K|}} \geq -0.26049 \ldots .$$

Then we produce examples of class-field towers, showing that

$$\liminf \frac{\gamma_K}{\log \sqrt{|d_K|}} \leq -0.17849 \ldots .$$

1 Introduction

Let K be a global field, i.e., a finite algebraic extension either of the field $\mathbb{Q}$ of rational numbers, or of the field of rational functions in one variable over a finite field of constants. Let $\zeta_K(s)$ be its zeta-function. Consider its Laurent expansion at $s = 1$,

$$\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + c_1(s-1) + \cdots .$$

In [1] Yasutaka Ihara introduces and studies the constant

$$\gamma_K = c_0/c_{-1}.$$

There are several reasons to study it:

- It generalizes the classical Euler constant $\gamma = \gamma_{\mathbb{Q}}$.
- For imaginary quadratic fields, it is expressed by a beautiful Kronecker limit formula.
- For fields with large discriminants, its absolute value is at most of the order of const $\times \log \sqrt{|d_K|}$, while the residue c_{-1} itself may happen to be exponential in $\log \sqrt{|d_K|}$; see [2].

In this appendix, we study the asymptotic behaviour of this constant when the discriminant (genus in the function field case) of the field tends to infinity. It is but natural to compare Ihara's results [1] with the methods of infinite zeta-functions developed in [2].

Let $\alpha_K = \log \sqrt{|d_K|}$ in the number field case and $\alpha_K = (g_K - 1)\log q$ in the function field case over $\mathbb{F}_q$. In the number field case, Ihara shows that

$$0 \geq \limsup_K \frac{\gamma_K}{\alpha_K} \geq \liminf_K \frac{\gamma_K}{\alpha_K} \geq -1.$$

We improve the lower bound to the following.

Theorem 1. *Assuming the generalized Riemann hypothesis, we have*

$$\begin{aligned}
&\liminf_K \frac{\gamma_K}{\alpha_K}\\
&\geq -\frac{\log 2 + \frac{1}{2}\log 3 + \frac{1}{4}\log 5 + \frac{1}{6}\log 7}{\frac{1}{\sqrt{2}-1}\log 2 + \frac{1}{\sqrt{3}-1}\log 3 + \frac{1}{\sqrt{5}-1}\log 5 + \frac{1}{\sqrt{7}-1}\log 7 + \frac{1}{2}(\gamma + \log 8\pi)}\\
&= -0.26049\ldots.
\end{aligned}$$

Remarks. Unconditionally we get $\liminf \gamma_K/\alpha_K \geq -0.52227\ldots$.

In the function field case, using the same method, we get $0 \geq \limsup \gamma_K/\alpha_K \geq \liminf \gamma_K/\alpha_K \geq -(\sqrt{q}+1)^{-1}$, which, of course, coincides with Ihara's result [1, Theorem 2].

Let us remark that the upper bound 0 is attained for any asymptotically bad family of global fields, and that the lower bound in the function field case is attained for any asymptotically *optimal* family (such that the ratio of the number of $\mathbb{F}_q$-points to the genus tends to $\sqrt{q} - 1$), which we know to exist whenever q is a square. Hence $\limsup \gamma_K/\alpha_K = 0$ and in the function field case with a square q, we have $\liminf \gamma_K/\alpha_K = -(\sqrt{q}+1)^{-1}$.

In Section 3 we construct examples of class-field towers proving (unconditionally) the following.

Theorem 2.

$$\liminf_K \gamma_K/\alpha_K \leq -\frac{2\log 2 + \log 3}{\log\sqrt{5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37}} = -0.17849\ldots.$$

This slightly improves the examples given by Ihara in [1].

In the number field case set $\beta_K = -(\frac{r_1}{2}(\gamma + \log 4\pi) + r_2(\gamma + \log 2\pi))$. If we complete γ_K by archimedean terms, we get the following.

Theorem 3. *Let $\tilde{\gamma}_K = \gamma_K + \beta_K$. Then under the generalized Riemann hypothesis, we have*

$$\liminf_K \frac{\tilde{\gamma}_K}{\alpha_K} \geq -\frac{\gamma + \log(2\pi)}{\gamma + \log(8\pi)} = -0.6353\ldots.$$

It is much easier to see that $\limsup \tilde{\gamma}_K/\alpha_K \leq 0$, and that 0 is attained for any *asymptotically bad* family (i.e., such that all ϕs vanish; see the definitions below).

The best example we know gives (unconditionally) the following.

Theorem 4.

$$\liminf_K \tilde{\gamma}_K/\alpha_K \leq -0.5478\ldots.$$

2 Bounds

Let us consider the asymptotic behaviour of γ_K. We treat the number field case. (The same argument in the function field case leads to [1, Theorem 2].) Let $|d_K|$ tend to infinity. By [2, Lemma 2.2], any family of fields contains an *asymptotically exact subfamily*, i.e., such that for any q there exists the limit ϕ_q of the ratio of the number $\Phi_q(K)$ of prime ideals of norm q to the "genus" α_K, and also the limits $\phi_{\mathbb{R}}$ and $\phi_{\mathbb{C}}$ of the ratios of r_1 and r_2 to α_K. To find $\liminf \gamma_K/\alpha_K$ and $\liminf \tilde{\gamma}_K/\alpha_K$, it is enough to find corresponding limits for a given asymptotically exact family, and then to look for their minimal values. In what follows, we consider only asymptotically exact families.

Theorem 5. *For an asymptotically exact family $\{K\}$, we have*

$$\lim_K \frac{\gamma_K}{\alpha_K} = -\sum \frac{\phi_q \log q}{q-1},$$

where q runs over all prime powers.

Proof. The right-hand side equals $\xi_\phi^0(1)$, where $\xi_\phi^0(s)$ is the log-derivative of the infinite zeta-function $\zeta_\phi(s)$ of [2]. The corresponding series converges for $\operatorname{Re} s \geq 1$ [2, Proposition 4.2]. We know [1, (1.3.3) and (1.3.4)] that

$$\gamma_K = -\lim_{s\to 1}\left(Z_K(s) - \frac{1}{s-1}\right),$$

where for $\mathrm{Re}(s) > 1$,

$$Z_K(s) = -\frac{\zeta_K'}{\zeta_K}(s) = \sum_{P,k\geq 1} \frac{\log N(P)}{N(P)^{ks}} = \sum_q \Phi_q(K) \frac{\log q}{q^s - 1}.$$

By the same [2, Proposition 4.2], $\frac{\zeta_K'}{\zeta_K}(s) \to \xi_\phi^0(s)$, and hence $\gamma_K/\alpha_K \to \xi_\phi^0(1)$. □

Proof of Theorem 1. We have to maximize $\sum \frac{\phi_q \log q}{q-1}$ under the following conditions:

- $\phi_q \geq 0$.
- For any prime p we have $\sum_{m=1}^{\infty} m\phi_{p^m} \leq \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}}$.
- $\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2}) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) \leq 1$ (the Basic Inequality, [2, GRH-Theorem 3.1]).

  If we put

$$a_0 = \log\sqrt{8\pi} + \frac{\pi}{4} + \frac{\gamma}{2}, \qquad a_1 = \log 8\pi + \gamma, \qquad a_q = \frac{\log q}{\sqrt{q}-1},$$

$$b_0 = b_1 = 0, \qquad b_q = \frac{\log q}{q-1},$$

we are under [2, Section 8, conditions (1)–(4) and (i)–(iv)].

Theorem 1 is now straightforward from [2, Proposition 8.3]. Indeed, the maximum is attained for $\phi_{p^m} = 0$ for $m > 1$, $\phi_{\mathbb{R}} = 0$, and $\phi_2 = \phi_3 = \phi_5 = \phi_7 = 2\phi_{\mathbb{C}}$. (Calculation shows that starting from $p' = 11$, the last inequality of [2, Proposition 8.3] is violated.) □

Proof of Theorem 3. This proof is much easier. Since in this case all coefficients are positive and the ratio of the coefficient of the function we maximize to the corresponding coefficient of the Basic Inequality is maximal for $\phi_{\mathbb{C}}$, the maximum is attained when all ϕs vanish except for $\phi_{\mathbb{C}}$. □

Remarks. If we want unconditional results, then instead of the Basic Inequality we have to use [2, Proposition 3.1]:

$$2\sum_q \phi_q \log q \sum_{m=1}^{\infty} \frac{1}{q^m+1} + \phi_{\mathbb{R}}\left(\frac{\gamma}{2} + \frac{1}{2} + \log 2\sqrt{\pi}\right) + \phi_{\mathbb{C}}(\gamma + \log 4\pi) \leq 1.$$

For $\tilde{\gamma}_K/\alpha_K$, one easily gets

$$\liminf \frac{\tilde{\gamma}_K}{\alpha_K} \geq -\frac{\gamma + \log(2\pi)}{\gamma + \log(4\pi)} = -0.7770\ldots.$$

The calculation for γ_K/α_K is trickier since the last condition of [2, Proposition 8.3] is not violated until very large primes. Changing the coefficients by the first term $(q+1)^{-1}$, Zykin [5] gets

$$\liminf \frac{\gamma_K}{\alpha_K} \geq -0.52227\ldots.$$

Note that (for an asymptotically exact family) $1 + \tilde{\gamma}_K/\alpha_K$ is just the value at 1 of the log-derivative $\xi(s)$ of the *completed infinite zeta-function* $\tilde{\zeta}(s)$ of [2].

3 Examples

Let us bound lim inf γ_K/α_K from above. To do this, one should provide some examples of families. The easiest is, just as in [2, Section 9], to produce quadratic fields having infinite class-field towers with prescribed splitting. The proof of Theorem 1 suggests that we should look for towers of totally complex fields, where 2, 3, 5, and 7 are totally split. This is, however, imprecise, because the sum of [2, Proposition 8.3] varies only slightly when we change p_0. Therefore, I also look at the cases when 2, 3, 5, 7, and 11 are split, and when only 2, 3, and 5 are split, or even only 2 and 3. This leads to a slight improvement on [1, (1.6.30)].

Each of the following fields has an infinite 2-class-field tower with prescribed splitting (just apply [2, Theorem 9.1]), and Theorem 5 gives the following list.

- For $\mathbb{Q}(\sqrt{11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 67})$ (the example of [2, Theorem 9.4]) $\mathbb{R}$, 2, 3, 5, 7 totally split, we get $\liminf \gamma_K/\alpha_K \leq -0.1515\ldots$.
- For $\mathbb{Q}(\sqrt{-13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\cdot 59\cdot 61\cdot 73\cdot 79})$ (the example of [2, Theorem 9.5]) with 2, 3, 5, 7, and 11 split, we get $-0.1635\ldots$.
- For $\mathbb{Q}(\sqrt{-7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 79})$ with 2, 3, 5 split, we get $-0.1727\ldots$.
- For $\mathbb{Q}(\sqrt{-7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 47\cdot 59})$ with 2, 3, 5 split, we get $-0.1737\ldots$.
- An even better example is found by Zykin [5]:

$$\mathbb{Q}(\sqrt{-5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37})$$

with 2 and 3 split gives us $-0.17849\ldots$. This proves Theorem 2.

For $\liminf \tilde{\gamma}_K/\alpha_K$, the Martinet field $\mathbb{Q}(\cos\frac{2\pi}{11}, \sqrt{2}, \sqrt{-23})$ (see [2, Theorem 9.2]) gives $-0.5336\ldots$ The best Hajir–Maire example (see [4, Section 3.2]) gives $\liminf \tilde{\gamma}_K/\alpha_K \leq -0.5478\ldots$ This proves Theorem 4.

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References

[1] Y. Ihara, On the Euler–Kronecker constants of global fields and primes with small norms, in V. Ginzburg, ed., *Algebraic Geometry and Number Theory: In Honor of Vladimir Drinfeld's* 50*th Birthday*, Progress in Mathematics, Vol. 850, Birkhäuser Boston, Cambridge, MA, 2006, 407–452 (this volume).

[2] M. A. Tsfasman and S. G. Vlăduţ, Infinite global fields and the generalized Brauer–Siegel theorem, *Moscow Math. J.*, **2**-2 (2002), 329–402.

[3] M. A. Tsfasman and S. G. Vlăduţ, Asymptotic properties of zeta-functions, *J. Math. Sci.* (*N.Y.*), **84**-5 (1997), 1445–1467.
[4] F. Hajir and C. Maire, Tamely ramified towers and discriminant bounds for number fields II, *J. Symbol. Comput.*, **33**-4 (2002), 415–423.
[5] A. Zykin, Private communication.