Integration in valued fields

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Summary. We develop a theory of integration over valued fields of residue characteristic zero. In particular, we obtain new and base-field independent foundations for integration over local fields of large residue characteristic, extending results of Denef, Loeser, and Cluckers. The method depends on an analysis of definable sets up to definable bijections. We obtain a precise description of the Grothendieck semigroup of such sets in terms of related groups over the residue field and value group. This yields new invariants of all definable bijections, as well as invariants of measure-preserving bijections.

Subject Classifications: Primary 03C60, 14C99, 11S80.

1 Introduction

Since Weil's *Foundations*, algebraic varieties have been understood independently of a particular base field; thus an algebraic group G exists prior to the abstract or topological groups of points $G(F)$, taken over various fields F. For Hecke algebras, or other geometric objects whose definition requires integration, no comparable viewpoint exists. One uses the topology and measure theory of each local field separately; since a field F has measure zero from the point of view of any nontrivial finite extension, at the foundational level there is no direct connection between the objects obtained over different fields. The main thrust of this paper is the development of a theory of integration over valued fields, which is geometric in the sense of Weil. At present the theory covers local fields of residue characteristic zero or, in applications, large positive residue characteristic.

Our approach to integration continues a line traced by Kontsevich, Denef–Loeser, and Loeser–Cluckers (cf. [7]). In integration over non-archimedean local fields there are two sources for the numerical values. The first is counting points of varieties over the residue field. Kontsevich explained that these numerical values can be replaced, with a gain of geometric information, by the isomorphism classes of the varieties themselves up to appropriate transformations, or more precisely by their classes in a certain Grothendieck ring. This makes it possible to understand geometrically the changes in integrals upon unramified base change. In this aspect our approach is very similar. The main difference is a slight generalization of the notion of variety over the residue field, which allows us to avoid what amounted to a choice of uniformizer in the previous theory.

The second source of numerical values is the piecewise linear geometry of the value group. We geometrize this ingredient, too, obtaining a theory of integration taking values in an entirely geometric ring, a tensor product of a Grothendieck ring of generalized varieties over the residue field, and a Grothendieck ring of piecewise linear varieties over the value group.

Viewed in this way, the integral is an invariant of measure-preserving definable bijections. We actually find all such invariants. In addition, we consider and determine all possible invariants of definable bijections; we obtain in particular two Euler characteristics on definable sets, with values in the Grothendieck group of generalized varieties over the residue field.

At the level of foundations, until an additive character is introduced, we are able to work with Grothendieck semigroups rather than with classes in Grothendieck groups.

1.1 The logical setting

Let L be a valued field, with valuation ring $\mathcal{O}_L$. $\mathcal{M}$ denotes the maximal ideal. We let $\mathrm{VF}^n(L) = L^n$. The notation VF^n is analogous to the symbol $\mathbb{A}^n$ of algebraic geometry, denoting affine n-space. Let $\mathrm{RV}^m(L) = L^*/(1+\mathcal{M})$, $\Gamma(L) = L^*/\mathcal{O}_L^*$, $\mathbf{k}(L) = \mathcal{O}_L/\mathcal{M}_L$. Let $\mathrm{rv} : \mathrm{VF} \to \mathrm{RV}$ and $\mathrm{val} : \mathrm{VF} \to \Gamma$ be the natural maps. The natural map $\mathrm{RV} \to \Gamma$ is denoted $\mathrm{val}_{\mathrm{rv}}$. The exact sequence

$$0 \to \mathbf{k}^* \to \mathrm{RV} \to \Gamma \to 0$$

shows that RV is, at first approximation, just a way to wrap together the residue field and value group.

We consider expressions of the form $h(x) = 0$ and $\mathrm{val}\, f(x) \geq \mathrm{val}\, g(x)$ where $f, g, h \in L[X]$, $X = (X_1, \ldots, X_n)$. A *semialgebraic formula* is a finite Boolean combination of such basic expressions. A semialgebraic formula ϕ clearly defines a subset $D(L)$ of $\mathrm{VF}^n(L)$. Moreover, if $f, g, h \in L_0[X]$, we obtain a functor $L \mapsto D(L)$ from valued field extensions of L_0 to sets. We will later describe more general definable sets; but for the time being take a *definable subset of* VF^n to be a functor $D = D_\phi$ of this form.

An intrinsic description of definable subsets of RV^m is given in Section 2.1. In particular, definable subsets of $(\mathbf{k}^*)^m$ coincide with constructible sets in the usual Zariski sense; while modulo $(\mathbf{k}^*)^m$, a definable set is a piecewise linear subset of Γ^m. The structure of arbitrary definable subsets of RV^m is analyzed in Section 3.3.

The advantages of this approach are identical to the benefits in algebraic geometry of working with arbitrary algebraically closed fields, over arbitrary base fields. One can use Galois theory to describe rational points over subfields. Since function fields are treated on the same footing, one has a mechanism to inductively reduce higher-dimensional geometry to questions in dimension one, and often, in fact, to dimension zero. (As in algebraic geometry, statements about fields, applied to generic points, can imply birational statements about varieties.)

1.2 Model theory

Since topological tools are no longer available, it is necessary to define notions such as dimension in a different way. The basic framework comes from [15]; we recall and develop it further in Sections 2 and 4. It is in many respects analogous to the o-minimal framework of [37], that has become well accepted in real algebraic geometry.

In addition, whereas in geometry all varieties are made as it were of the same material, here a number of rather different types of objects coexist, and the interaction between them must be clarified. In particular, the residue field and the value group are orthogonal in a sense that will be defined below; definable subsets of one can never be isomorphic to subsets of the other, unless both are finite. This orthogonality has an effect on definable subsets of VF^n in general; for example, closed disks behave very differently from open ones. Here we follow and further develop [16].

Note that the set of rational points of closed and open disks over discrete valuation rings, for instance, cannot be distinguished; as in rigid geometry, the geometric setting is required to make sense of the notions. Nevertheless, they have immediate consequences for local fields. As an example, we define the notion of a definable distribution; this is defined as a function on the space of polydisks with certain properties. Making use of model-theoretic properties of the space of polydisks, we show that any definable distribution agrees outside a proper subvariety with one obtained by integrating a function. This is valid over any valued field of sufficiently large residue characteristic. In particular, for large p, the p-adic Fourier transform of a rational polynomial is a locally constant function away from an exceptional subvariety, in the usual sense (Corollary 11.10). The analogue for $\mathbb{R}$ and $\mathbb{C}$ was proved by Bernstein using D-modules. For an individual $\mathbb{Q}_p$, the same result can be shown using Denef integration and a similar analysis of definable sets over $\mathbb{Q}_p$. These results were obtained independently by Cluckers and Loeser; cf. [8].

1.3 More general definable sets

Throughout the chapter, we discuss not semialgebraic sets, but definable subsets of a theory with the requisite geometric properties (called V-minimality). This includes also the rigid analytic structures of [23]. The adjective "geometrically" can be take to mean here "in the sense of the V-minimal theory."

While we work geometrically throughout the paper, the isomorphisms we obtain are canonical and so specialize to rational points over substructures. Thus a posteriori our results apply to definable sets over any Hensel field of large residue characteristic. See Section 12.

For model theorists, this systematic use of algebraically closed valued fields to apply to other Hensel fields is only beginning to be familar. As an illustration, see Proposition 12.9, where it is shown that after a little analysis of definable sets over algebraically closed valued fields, quantifier elimination for Henselian fields of residue characteristic zero becomes a consequence of Robinson's earlier quantifier elimination in the algebraically closed case.

A third kind of generalization is an a posteriori expansion of the language in the RV sort. Such an expansion involves loss of information in the integration theory, but is sometimes useful. For instance, one may want to use the Denef–Pas language, splitting the exact sequence into a product of residue field and value group. Another example occurs in Theorem 12.5, where it is explained, given a valued field whose residue field is also a valued field, what happens when one integrates twice. To discuss this, the residue field is expanded so as to itself become a valued field.

1.4 Generalized algebraic varieties

We now describe the basic ingredients in more detail. Let L_0 be a valued field with residue field $\mathbf{k}_0$ and value group A. For each point $\gamma \in \mathbb{Q} \otimes A$, we have one-dimensional $\mathbf{k}$-vector space

$$V_\gamma = \{0\} \cup \frac{\{x \in K : \mathrm{val}(x) = \gamma\}}{1 + \mathcal{M}}.$$

As discussed above, V_γ should be viewed as a functor $L \mapsto V_\gamma(L)$ on valued field extensions L of L_0, giving a vector space over the residue field functor. If $\gamma - \gamma' \in A$, then V_γ, V'_γ are definably isomorphic, so one essentially has V_γ for $\gamma \in (\mathbb{Q} \otimes A)/A$.

Fix $\bar{\gamma} = (\gamma_1, \ldots, \gamma_n)$, and $V_i = V_{\gamma_i}$, $V_{\bar{\gamma}} = \Pi_i V_{\gamma_i}$. A *$\bar{\gamma}$-polynomial* is a polynomial $H(X) = \sum a_\nu X^\nu$ with $\mathrm{val}_p(a_\nu) + \sum_i \nu(i)\gamma_i = 0$ for each nonzero term $a_\nu X^\nu$. The coefficients a_ν are described in Section 5.5; for the purposes of the introduction, and of Theorem 1.3 below, it suffices to think of integer coefficients. Such a polynomial clearly defines a function $H : V_{\bar{\gamma}} \to \mathbf{k}$. In particular, one has the set of zeroes $Z(H)$. The *generalized residue structure* RES_{L_0} is the residue field, together with the collection of one-dimensional vector spaces $V_\gamma (\gamma \in \mathbb{Q} \otimes A)$ over it, and the functions $H : V_{\bar{\gamma}} \to \mathbf{k}$ associated to each $\bar{\gamma}$-polynomial.

The intersection W of finitely many zero sets $Z(H)$ is called a *generalized algebraic variety over the residue field*. Given a valued field extensions L of L_0, we have the set of points $W(L) \subseteq V_{\bar{\gamma}}(L)$. When L is a local field, $W(L)$ is finite.

We will systematically use the Grothendieck group of generalized varieties over the residue field, rather than the usual Grothendieck group of varieties. They are fundamentally of a similar nature: base change to an algebraically closed value field makes them isomorphic. But the generalized residue field makes it possible to see canonically objects that are only visible after base change in the usual approach. One application is Theorem 1.3 below.

$K_+ \mathrm{RES}_{L_0}[n]$ denotes the Grothendieck group of generalized varieties of dimension $\leq n$; in the paper we will omit L_0 from the notation.

1.5 Rational polyhedra over ordered Abelian groups

Let A be an ordered Abelian group. A *rational polyhedron* Δ *over* A is given by an expression

$$\Delta = \{x : Mx \geq b\}$$

with $x = (x_1, \ldots, x_n)$, M a $k \times n$ matrix with rational coefficients, and $b \in A^k$. We view this as a functor $B \mapsto \Delta(B)$ on ordered Abelian group extensions B of A. This functor is already determined by its value at $B = \mathbb{Q} \otimes A$. In particular, when $A \leq \mathbb{Q}$, Δ is an ordinary rational polyhedron.

$K_+ \Gamma_A[n]$ is the semigroup generated by such polyhedra, up to piecewise $\mathrm{GL}_n(\mathbb{Z})$-transformations and A-translations; see Section 9. When A is fixed it is omitted from the notation.

In our applications, A will be the value group of a valued field L_0. If B is the value group of a valued field extension L, write $\Delta(L)$ for $\Delta(B)$.

1.6 The Grothendieck semiring of definable sets

Fix a base field L_0. The word "definable" will mean $\mathbf{T}_{L_0}$-definable, with $\mathbf{T}$ a fixed V-minimal theory. To have an example in mind one can read "semialgebraic over L_0" in place of "definable."

Let VF[n] be the category of definable subsets X of n-dimensional algebraic varieties over L_0; a morphism $X \to X'$ is a definable bijection $X \to X'$ (see Definition 3.65 for equivalent definitions). K_+ VF[n] denotes the Grothendieck semigroup, i.e., the set of isomorphism classes of VF[n] with the disjoint sum operation. $[X]$ denotes the class of X in the Grothendieck semigroup.

We explain how an isomorphism class of VF[n] is determined precisely by isomorphism classes of generalized algebraic varieties and rational polyhedra, whose dimensions add up to n.

If $X \subseteq \mathrm{RES}^m$ and $f : X \to \mathrm{RES}^n$ is a finite-to-one map, let

$$\mathbb{L}(X, f) = \mathrm{VF}^n \times_{\mathrm{rv}, f} X = \{(v_1, \ldots, v_n, x) : v_i \in \mathrm{VF}, x \in X, \mathrm{rv}(v_i) = f_i(x)\}.$$

The VF[n]-isomorphism class $[\mathbb{L}(X, f)]$ does not depend on f, and is also denoted $[\mathbb{L}X]$.

When S is a smooth scheme over $\mathcal{O}$, X a definable subset of $S(\mathbf{k})$, $\pi : S(\mathcal{O}) \to S(\mathbf{k})$ the natural reduction map, we have $[\mathbb{L}X] = [\pi^{-1}X]$.

We let RES[n] be the category of pairs (X, f) as above; a morphism $(X, f) \to (X', f')$ is just a definable bijection $X \to X'$. Let K_+ RES[$*$] be the direct sum of the Grothendieck semigroups K_+ RES[n].

On the other hand, we have already defined $K_+ \Gamma[n]$. Let $K_+ \Gamma[*]$ be the direct sum of the $K_+ \Gamma[n]$. An element of $K_+ \Gamma[n]$ is represented by a definable $X \subseteq \Gamma[n]$. Let $\mathbb{L}X = \mathrm{val}^{-1}(X)$, $\mathbb{L}[X] = [\mathbb{L}X]$.

It is shown in Proposition 10.2 that the Grothendieck semiring of RV is the tensor product $K_+ \mathrm{RES}[*] \otimes K_+ \Gamma[*]$ over the semiring $K_+\Gamma^{\mathrm{fin}}$ of classes of finite subsets of Γ; see Section 9.

Note that $\mathbb{L}([1]_1) = \mathbb{L}([1]_0) + \mathbb{L}([(0, \infty)]_1)$, where $[1]_1 \in K_+ \mathrm{RES}[1]$, $[1]_0 \in K_+ \mathrm{RES}[0]$ are the classes of the singleton set 1, and $[(0, \infty)]_1$ is the class in $K_+ \Gamma[1]$ of the semi-infinite segment $(0, \infty)$. Indeed, $\mathbb{L}([1]_1)$ is the unit open ball around 1, $\mathbb{L}([1]_0)$ is the point $\{1\}$, while $\mathbb{L}([(0, \infty)]_1)$ is the unit open ball around 0, isomorphic by a shift to the unit open ball around 1. This is the one relation that cannot be understood in terms of the Grothendieck semiring of RV; it will be seen to correspond to the analytic summation of geoemtric series in the Denef theory. Let I_{sp} be the congruence on the ring $K_+ \mathrm{RES}[*] \otimes K_+ \Gamma[*]$ generated by $[1]_1 \sim [1]_0 + [(0,\infty)]_1$.

The following theorem summarizes the relation between definable sets in VF and in RV; it follows from Theorem 8.4 together with Proposition 10.2 in the text.

Theorem 1.1. $\mathbb{L}$ *induces a surjective homomorphism of filtered semirings*

$$K_+ \mathrm{RES}[*] \otimes K_+ \Gamma[*] \to K_+(\mathrm{VF}).$$

The kernel is precisely the congruence I_{sp}.

The inverse isomorphism $K_+(\mathrm{VF}) \to K_+ \mathrm{RES}[*] \otimes K_+ \Gamma[*]/I_{sp}$ can be viewed as a kind of Euler characteristic, respecting products and disjoint sums, and can be functorial in various other ways.

The values of this Euler characteristic are themselves geometric objects, both on the algebraic-geometry side (RES) and the combinatorial-analytic side (Γ). This is valuable for some purposes; in particular, it becomes clear that the isomorphism is compatible with taking rational points over Henselian subfields (cf. Proposition 12.6).

For other applications, however, it would be useful to obtain more manageable numerical invariants; for this purpose one needs to analyze the structure of $K_+ \Gamma[*]$. We do not fully do this here, but using a number of homomorphisms on $K_+ \Gamma[*]$, we obtain a number of invariants. In particular, using the $\mathbb{Z}$-valued Euler characteristics on $K\,\Gamma[*]$ (cf. Section 9 and [26, 20]), we obtain two homomorphisms on $K_+ \mathrm{VF}[n]$ essentially to $K\,\mathrm{RES}[n]$. The reason there are two rather than one has to do with Poincaré duality; see Theorem 10.5.

For instance, when F is a field of characteristic 0, we obtain an invariant of rigid analytic varieties over $F((t))$, with values in the Grothendieck ring $K(\mathrm{Var}_F)$ of algebraic varieties over F; and another in $K(\mathrm{Var}_F)[[\mathbb{A}_1]^{-1}]$ (Proposition 10.8). It is instructive to compare this with the invariant of [25], with values in $K(\mathrm{RES}[n])/[G_m]$.[1] Since any two closed balls are isomorphic, via additive translation and multiplicative contractions, all closed balls must have the same invariant. Working with a discrete value group tends to force $[G_m] = 0$, since it appears that a closed ball B_0 of valuation radius 0 equals G_m times a closed ball B_1 of valuation radius 1. Since our technology is based on divisible value groups, the "equation" $[B_0] = [B_1][G_m]$ is replaced for us by $[B_0] = [B_0^o][G_m]$, where B_0^o is the open ball of valuation radius 0. Though B_1 and B_0^0 have the same $F((t))$-rational points, they are geometrically distinct (cf. Lemma 3.46) and so no collapse takes place. See also Sections 12.6 and 12.6 for two previously known cases.

By such Euler characteristic methods we can prove a statement purely concerning algebraic varieties, partially answering a question of Gromov and Kontsevich [13, p. 121]. In particular, two elliptic curves with isomorphic complements in projective

[1] The setting is somewhat different: Loeser–Sebag can handle positive characteristic, too, but assume smoothness.

space were previously known to be isogenous, by zeta function methods; we show that they are isomorphic. This also follows from [22]; the method there requires strong forms of resolution of singularities. See Theorem 13.1.

1.7 Integration of forms up to absolute value

Over local fields, data for integration consists of a triple (X, V, ω), with X a definable subset of a smooth variety V and ω a volume form on V. We are interested in an integral of the form $\int_X |\omega|$, so that multiplication of ω by a function with norm 1 does not count as a change, nor does removing a subvariety of V of smaller dimension. Using an equivalent description of $\mathrm{VF}[n]$, where the objects come with a distinguished finite-to-one map into affine space, we can represent an integrand as a pair (X, ω) with $X \in \mathrm{Ob}\,\mathrm{VF}[n]$ and ω a function from X into Γ. Isomorphisms are essential bijections, preserving the form up to a function of norm 1. See Definition 8.10 for a precise definition of this category, the category $\mu_\Gamma\mathrm{VF}[n]$.

Integration is intended to be an invariant of isomorphisms in this category. Thus we can find the integral if we determine all invariants. We do this in complete analogy with Theorem 1.1.

For $n \geq 0$ let $\Gamma[n]$ be the category whose objects are finite unions of rational polyhedra over the group A of definable points of Γ. A morphism $f : X \to Y$ of $\Gamma[n]$ is a bijection such that for some partition $X = \cup_{i=1}^k X_i$ into rational polyhedra, $f|X_i$ is given by an element of $\mathrm{GL}_n(\mathbb{Z}) \ltimes A^n$. Let $\mu\Gamma[n]$ be the category of pairs (X, ω), with X an object of $\Gamma[n]$, and $\omega : X \to \Gamma$ a piecewise affine map. A morphism $f : (X, \omega) \to (X', \omega')$ is a morphism $f : X \to X'$ of $\Gamma[n]$ such that $\sum_{i=1}^l x_i + \omega(x) = \sum_{i=1}^l x'_i + \omega'(x')$ whenever $(x'_1, \ldots, x'_n) = f(x_1, \ldots, x_n)$. Given $(X, \omega) \in \mathrm{Ob}\,\mu\Gamma[n]$, define $\mathbb{L}X$ as above, and adjoint the pullback of ω to obtain an object of $\mu_\Gamma\mathrm{VF}[n]$. This gives a homomorphism $K_+\,\mu\Gamma[n] \to K_+\,\mu_\Gamma\mathrm{VF}[n]$.

Theorem 1.2. $\mathbb{L}$ *induces a surjective homomorphism of filtered semirings*

$$K_+\,\mathrm{RES}[*] \otimes_{\mathbb{N}} K_+\,\mu\Gamma[*] \to K_+(\mu_\Gamma\mathrm{VF})[*].$$

The kernel is generated by the relations $p \otimes 1 = 1 \otimes [(\mathrm{val}_{\mathrm{rv}}(p), \infty)]$ *and* $1 \otimes a = {\mathrm{val}_{\mathrm{rv}}}^{-1}(a) \otimes 1$.

In the statement of the theorem, p ranges over definable points of RES (actually one value suffices), and a ranges over definable points of Γ.

This can also be written as

$$K_+\,\mathrm{RES}[*] \otimes_{K_+(\mu\Gamma^{\mathrm{fin}})} K_+\,\mu\Gamma[*]/\mathrm{I}_{\mathrm{sp}}^{\mu} \simeq K_+(\mu_\Gamma\mathrm{VF})[*],$$

where $K_+(\mu\Gamma^{\mathrm{fin}})$ is the subsemiringof subsets of $\mu\Gamma$ with finite support, and $\mathrm{I}_{\mathrm{sp}}^{\mu}$ is a semiring congruence defined similarly to I_{sp}. The base of the tensor leads to the identification of a point of Γ with with a coset of $\mathbf{k}^*$ in RES, while $\mathrm{I}_{\mathrm{sp}}^{\mu}$ identifies a point of RES with an infinite interval of Γ. The inverse isomorphism can be viewed as an integral.

We introduce neither additive nor multiplicative inverses in $K_+ \mathrm{RES}[*]$ formally, so that the target of integration is completely geometric.

We proceed to give an application of the first part of the theorem (the surjectivity) in terms of ordinary p-adic integration.

1.8 Integrals over local fields: Uniformity over ramified extensions

Let L be a local field, finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. We normalize the Haar measure μ in such a way that the maximal ideal has measure 1, the norm by $|a| = \mu\{x : |x| < |a|\}$. Let RES_L be the generalized residue field, and Γ_L be the value group. We assume $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ has value group $\mathbb{Z}$, and identify Γ_L with a subgroup of $\mathbb{Q}$.

Given $c = (c_1, \dots, c_k) \in L^k$ and $s = (s_1, \dots, s_k) \in \mathbb{R}^k$ with $s_i \geq 1$, let $|c|^s = \Pi_{i=1}^k |c_i|^{s_i}$.

Let λ be a multiplicative character $\mathbb{R}^n \to \mathbb{R}^*$. Define

$$\mathrm{ev}_\lambda(\Delta(B)) = \sum_{b \in \Delta(B)} \lambda(b),$$

provided this sum is absolutely convergent. Given linear functions $h_0, \dots, h_k$ on $\mathbb{R}^n$ and $s_1, \dots, s_k \in \mathbb{R}$, let $\mathrm{ev}_{h,s,Q} = \mathrm{ev}_\lambda$, where $\lambda(x) = Q^{h_0(x) + \sum s_i h_i(x)}$.

Theorem 1.3. *Fix $n, d, k \in \mathbb{N}$. Let p be a large prime compared to n, d, k, and let $f \in \mathbb{Q}_p[X_1, \dots, X_n]^k$ have degrees $\leq d$. Then there exist finitely many generalized varieties X_i over* $\mathrm{RES}(\mathbb{Q}_p)$, *rational polyhedra Δ_i, $\gamma(i) \in \mathbb{Q}^{\geq 0}$, $n_i \in \mathbb{N}$, and linear functions $h_0^i, \dots, h_k^i$ with rational coefficients, such that for any finite extension L of $\mathbb{Q}_p$ with residue field $GF(q)$ and* $\mathrm{val}(L^*) = (1/r)\mathbb{Z}$, $\mathrm{val}(p) = 1$, *and any $s \in \mathbb{R}^k_{\geq 1}$,*

$$\int_{\mathcal{O}_L^n} |f|^s = \sum_i q^{r\gamma(i)} (q-1)^{n_i} |X_i(L)| \, \mathrm{ev}_{h^i, s, q^r}(\Delta_i(L)).$$

Note the following:

(1) $\Delta_i(L)$ depends only on the ramification degree r of L over $\mathbb{Q}_p$.
(2) The formula is a sum of nonnegative terms.
(3) $\mathrm{ev}_{h,s,q^r}(\Delta_i((1/r)\mathbb{Z}))$ can be written in closed form as a rational function of q^{rs}. This follows from Denef, who shows it for more general sets Δ_i definable in Pressburger arithmetic; such analytic summation is an essential component of his integration theory. Since it plays no role in our approach we leave the statement in geometric form.
(4) The generalized varieties X_i and polyhedra Δ_i are simple functions of the coefficients f. Here we wish to emphasize not this, but the uniformity of the expression over ramified extensions of $\mathbb{Q}_p$.

The proof follows Proposition 10.10. (It uses only the easy surjectivity in this proposition and Proposition 4.5.)

1.9 Bounded and unbounded sets

The isomorphism of semirings of Theorem 1.2 obviously induces an isomorphism of rings. However, introducing additive inverses loses information on the Γ side; the class of the interval $[0, 1)$ becomes 0, since $[0, \infty)$ and $[1, \infty)$ are isomorphic. The classical remedy is to cut down to bounded sets before groupifying. This presents no difficulty, since the isomorphism respects boundedness.

In higher-dimensional local fields, stronger notions of boundedness may be useful, such as those introduced by Fesenko. Since these questions are not entangled with the theory of integration, and can be handled a posteriori, we will deal with them in a future work.

Here we mention only that even if one insists on integrating all definable integrands, with no boundedness condition, into a ring, some but not all information is lost. This is due to the existence of Euler characteristics on Γ, and thus again to the fact that we work geometrically, with divisible groups, even if the base field has a discrete group. We will see (Lemma 9.12) that $K_+(\mu\Gamma[n])$ can be identified with the group of definable functions $\Gamma \to K_+(\Gamma[n])$. Applying an appropriate Euler characteristic reduces to the group of piecewise constant functions on Γ into $\mathbb{Z}$. Recombining with RES we obtain a consistent definition of an integral on unbounded integrands, compatible with measure-preserving maps, sums, and products, with values in $K(\mathrm{RES})[A]/[\mathbb{A}_1]_1\, K(\mathrm{RES})[A]$, where A is the group of definable points of Γ, and $[\mathbb{A}_1]_1$ is the class of the affine line. See Theorem 10.11.

1.10 Finer volumes

We also consider a finer category of definable sets with RV-volume forms. This means that a volume form ω is identified with $g\omega$ only when $g - 1 \in \mathcal{M}$; $\mathrm{val}(g) = 0$ does not suffice. We obtain an integral whose values themselves are definable sets with volume forms; in particular, including algebraic varieties with volume forms over the residue field.

Theorem 1.4. $\mathbb{L}$ *induces a surjective homomorphism of graded semirings*

$$K_+\,\mu\mathrm{RV}[*] \to K_+(\mu\mathrm{VF})[*].$$

The kernel is precisely the congruence $\mathrm{I}^{\mu}_{\mathrm{sp}}$.

μRV is the category of definable subsets of $\mu\mathrm{RV}^*$ enriched with volume forms; see Definition 8.13. Again, an isomorphism is induced in the opposite direction, that can be viewed as a motivic integral

$$\int : K_+(\mu\mathrm{VF})[*] \to K_+\,\mu\mathrm{RV}[*]/\mathrm{I}^{\mu}_{\mathrm{sp}}.$$

This allows an iteration of the integration theory, either with an integral of the same nature if the residue field is a valued field, or with a different kind of integral if, for instance, the residue field is $\mathbb{R}$.

1.11 Hopes

We mention three. Until now, a deep obstacle existed to extending Denef's theory to positive characteristic; namely, the theory was based on quantifier elimination for Hensel fields of residue characteristic 0, or for finitely ramified extensions of $\mathbb{Q}_p$, and it is known that no similar quantifier elimination is possible for $\mathbb{F}_p((t))$, if any is. On the other hand, Robinson's quantifier elimination is perfectly valid in positive characteristic. This raises hopes of progress in this direction, although other obstacles remain.

It is natural to think that the theory can be applied to higher-dimensional local fields; we will consider this in a future work.

Another important target is asymptotic integration over $\mathbb{R}$. Nonstandard extensions of $\mathbb{R}$ admit natural valued field structures. This is the basis of Robinson's nonstandard analysis. These valued fields have divisible value groups, and so previous theories of definable integration do not apply. The theory of this paper applies, however, and we expect that it will yield connections between p-adic integration and asymptotics of real integrals.

1.12 Organization of the paper

After recalling some basic model theory in Section 2, we proceed in Section 3 to V-minimal theories.

In Section 4 we show that any definable subset of VF^n admits a constructible bijection with some $\mathbb{L}(X, f)$. In fact, only a very limited class of bijections is needed; a typical one has the form $(x_1, x_2) \mapsto (x_1, x_2 + f(x_1, x_2))$, so it is clearly measure preserving. The proof is simple and brief, and uses only a little of the preceding material. We note here that for many applications this statement is already sufficient; in particular, it suffices to give the surjectivity in Theorems 1.1 and 1.2, and hence the application Theorem 1.3.

In Section 5 we return to the geometry of V-minimal structures, developing a theory of differentiation. We show the compatibility between differentiation in RV and in VF. This is needed for Theorem 1.4. Differentiation in VF involves much finer scales than in RV; in effect RV can only see distances measured by valuation 0, while the derivative in VF involves distances of arbitrarily large valuation. The proof uses a continuity argument with respect to dependence on scales. It fails in positive characteristic, in its present form.

Section 6 is devoted to showing that $\mathbb{L}$ yields a well-defined map $K_+(\mathrm{RV}) \to K_+(\mathrm{VF})$; in other words, not only objects, but also isomorphisms can be lifted.

Sections 7 and 8 investigate the kernel of $\mathbb{L}$ in Theorem 1.1. This is the most technical part of the paper, and we have not been able to give a proof as functorial as we would have liked. See Question 7.9.

In Section 9 we study the piecewise linear Grothendieck group; see the introduction to this section.

Section 10 decomposes the Grothendieck group of RV into the components RES and Γ, used througout this introduction.

Section 11 introduces an additive character, and hence the Fourier transform. The isomorphism of volumes given by Theorem 1.4 suffices for this extension; it is not necessary to redo the theory from scratch, but merely to follow through the functoriality.

Section 12 contains the extension to definable sets over Hensel fields mentioned above, and Section 13 gives the application to the Grothendieck group of varieties.

2 First-order theories

The bulk of this paper uses no deep results from logic beyond Robinson's quantifier elimination for the theory of algebraically closed valued fields [33]. However, it is imbued with a model-theoretic viewpoint. We will not explain the most basic notions of logic: language, theory, model. Let us just mention that a language consists of basic relations and function symbols, and formulas are built out of these, using symbols for Boolean operations and quantifiers (cf., e.g., [11] or [19], or the first section of [9]); but we attempt in this section to bridge the gap between these and the model-theoretic language used in the paper.

A language L consists of a family of "sorts" S_i, a collection of variables ranging over each sort, a set of relation symbols R_j, each intended to denote a subset of a finite product of sorts, and a set of function symbols F_k intended to denote functions from a given finite product of sorts to a given sort. From these, and the logical symbols &, $\neg$, $\forall$, $\exists$ one forms *formulas*. A sentence is a formula with no free variables (cf. [11]). A theory T is a set of sentences of L. A theory is called *complete* if for every sentence ϕ of L, either ϕ or its negation $\neg\phi$ is in T.

A *universe* M for the language L consists, by definition, of a set $S(M)$ for each sort S of L. An L-structure consists of such a universe, together with an interpretation of each relation and a function symbol of L. One can define the truth value of a sentence in a structure M; more generally, if $\phi(x_1, \ldots, x_n)$ is a formula, with x_i a variable of sort S_i, then one defines the interpretation $\phi(M)$ of ϕ in M, as the set of all $d \in S_1(M) \times \cdots \times S_n(M)$ of which ϕ is true. If every sentence in T is true in M, one says that M is a model of T ($M \models T$). The fundamental theorem here is a consequence of Gödel's completeness theorem called the *compactness theorem*: a theory T has a model if every finite subset of T has a model.

The language L_{rings} of rings, for example, has one sort, three function symbols $+, \cdots, -$, two constants 0, 1; any ring is an L_{rings}-structure; one can obviously write down a theory T_{fields} in this language whose models are precisely the fields.

2.1 Basic examples of theories

We will work with a number of theories associated with valued fields:

(1) ACF, the theory of algebraically closed fields. The language is the language of rings $\{+, \cdot, -, 0, 1\}$, mentioned earlier. The theory states that the model is a field, and for each n, that every monic polynomial of degree n has a root. For instance, for $n = 2$,

$$(\forall u_1)(\forall u_0)(\exists x)(x^2 + u_1 x + u_0 = 0).$$

In addition, ACF(0) includes the sentence $1 + 1 \neq 0, 1 + 1 + 1 \neq 0, \dots$. This theory is complete (Tarski–Chevalley). It will arise as the theory of the residue field of our valued fields.

(2) Divisible ordered Abelian groups (DOAG). The language consists of a single sort, a binary relation symbol $<$, a binary function symbol $+$, a unary function symbol $-$, and a constant symbol 0. The theory states that a model is an ordered Abelian group. In addition, there are axioms asserting divisibility by n for each n, for instance, $(\forall x)(\exists y)(y + y = x)$.

This is the theory of the value group of a model of ACVF.

(3) The RV sort (extension of (2) by (1)). The language has one official sort, denoted RV, and includes Abelian group operations $\cdot, /$ on RV, a unary predicate $\mathbf{k}^*$ for a subgroup, and an operation $+ : \mathbf{k}^2 \to \mathbf{k}$, where $\mathbf{k}$ is $\mathbf{k}^*$ augmented by a constant 0. Finally, there is a partial ordering; the theory states that $\mathbf{k}^*$ is the equivalence class of 1; that $\leq$ is a total ordering on $\mathbf{k}^*$-cosets, making $\mathrm{RV}/\mathbf{k}^* =: \Gamma$ a divisible ordered Abelian group, and that $(\mathbf{k}, +, \cdot)$ is an algebraically closed field. (We thus have an exact sequence $0 \to \mathbf{k}^* \to \mathrm{RV} \to \Gamma \to 0$, but we treat Γ as an imaginary sort.) This theory TRV is complete, too.

We will sometimes view RV as an autonomous structure but it will arise from an algebraically closed valued field, as in (5) below.

(4) Let $M \models \mathrm{TRV}$, and let A be a subgroup of $\Gamma(M)$. Within TRV_A we see an interpretation of ACF, namely, the algebraically closed field $\mathbf{k}$. In addition, for each $a \in A$, we have a one-dimensional $\mathbf{k}$-space, the fiber of RV lying over Γ augmented by 0. Collectively, the field $\mathbf{k}$ with this collection of vector spaces will be denoted RES.

(5) ACVF, the theory of algebraically closed valued fields. According to Robinson, the completions, denoted $\mathrm{ACVF}(q, p)$, are obtained by specifying the characteristic q and residue characteristic p. We will be concerned with ACVF(0, 0) in this paper. However, since any sentence of ACVF(0, 0) lies in ACVF(0, p) for almost all primes p, the results will a posteriori apply also to valued fields of characteristic zero and large residue characteristic.

We will take ACVF(0, 0) to have two sorts, VF and $\mathrm{RV} = \mathrm{VF}^*/(1 + \mathcal{M})$. The language includes the language of rings (1) on the VF sort, the language (3) on the RV sort, and a function symbol rv for a function $\mathrm{VF}^* \to \mathrm{RV}$. Denote $\mathrm{rv}^{-1}(\mathrm{RV}^{\geq 0}) = \mathcal{O}$, $\mathrm{rv}^{-1}(0) = \mathcal{M}$.

The theory states that VF is a valued field, with valuation ring $\mathcal{O}$ and maximal ideal $\mathcal{M}$ such that $\mathrm{rv} : \mathrm{VF}^* \to \mathrm{RV}$ is a surjective group homomorphism, and the restriction to $\mathcal{O}$ (augmented by $0 \mapsto 0$) is a surjective ring homomorphism.

The structure that ACVF_A induces on Γ is of a uniquely divisible Abelian group, with constants for the elements of $\Gamma(A)$. Thus every definable subset of Γ is a finite union of points and open intervals (possibly infinite).

(6) Rigid analytic expansions (Lipshitz). The theory $\mathrm{ACVF}^{\mathrm{R}}$ of algebraically closed valued fields expanded by a family R of analytic functions. See [23] and [24].

Our theory of definable sets will be carried out axiomatically, and are thus also valid for these rigid analytic expansions.

A *definable set* D is not really a set, but a functor from the category of models of T to the category of sets of the form $M \mapsto \phi(M)$, where ϕ is a formula of L. Model theorists do not really distinguish between the definable set D and the formula ϕ defining it; we will usually refer to definable sets rather than to formulas. If $R \subseteq D \times D'$ and for any model $M \models T$, $R(M)$ is the graph of a function $D(M) \to D(M')$, we say R is a *definable function of* T. Similarly, we say D is *finite* if $D(M)$ is finite for any $M \models T$, etc. It follows from the compactness theorem that if D is finite, then for some integer m we have $|D(M)| \leq m$ for any $M \models T$. We sometimes write S^* to denote S^n for some unspecified n.

By a *map* between L-structures A, B we mean a family $f = (f_S)$ indexed by the sorts of L, with $f_S : S(A) \to S(B)$; one extends f to products of sorts by setting $f((x_1, \ldots, x_n)) = (f(x_1), \ldots, f(x_n))$. f is *an embedding of structures* if $f^{-1}R(B) = R(A)$ for any atomic formula R of L. Taking R to be the equality relation, this includes, in particular, the statement that each f_S is injective.

On occasion we will use ∞-definable sets. An ∞-definable set is a functor of the form $M \mapsto \cap\mathcal{D}$, where $\mathcal{D}$ is a given collection of definable sets. In a complete theory a definable set is determined by the value it has at a single model; this is, of course, false for ∞-definable sets.

We write $a \in D$ to mean $a \in D(M)$ for some $M \models T$. It is customary, since Shelah, to choose a single universal domain $\mathbb{U}$ embedding all "small" models, and let $a \in D$ mean $a \in D(\mathbb{U})$; we will not require this interpretation, but the reader is welcome to take it.

We will sometimes consider *imaginary sorts*. If D is a definable set, and E a definable equivalence relation on D, then D/E may be considered to be an *imaginary sort*; as a definable set it is just the functor $M \mapsto D(M)/E(M)$. A definable subset of a product $\Pi_{i=1}^n D_i/E_i$ of imaginary sorts (and ordinary sorts) is taken to be a subset whose preimage in $\Pi_{i=1}^n D_i$ is definable; the notion of a definable function is thus also defined. In this way, the imaginary sorts can be treated on the same footing as the others. The set of all elements of all imaginary sorts of a structure M is denoted M^{eq}. It is easy to construct a theory T^{eq} in a language L^{eq} whose category of models is (essentially) $\{M^{\mathrm{eq}} : M \models T\}$. See [35] and [31, Section 16d].

Given a definable set $D \subseteq S \times X$, where S, X are definable sets, and given $s \in S$, let $D(s) = \{x \in X : (s, x) \in D\}$. Thus D is viewed as a *family* of definable subsets of X, namely, $\{D(s) : s \in S\}$. If $s \neq s'$ implies $D(s) \neq D(s')$, we say that the parameters are *canonical*, or that s is a *code* for $D(s)$. In particular, if E is a definable equivalence relation, the imaginary elements a/E can be considered as codes for the classes of E.

T is said to *eliminate imaginaries* if every imaginary sort admits a definable injection into a product of some of the sorts of L. For instance, the theory of algebraically closed fields eliminates imaginaries. See [32] for an excellent exposition of these issues. We note that T admits elimination of imaginaries iff for any family $D \subseteq S \times X$

there exists a family $D' \subseteq S' \times X$ such that for any $t \in S$ there exists a *unique* $t' \in S'$ with $D(t) = D'(t')$.

(Recall that $t \in S$ means $t \in S(M)$ for some $M \models T$. The uniqueness of t' implies in this case that one can choose $t' \in S'(M)$, too.) In this case, we also say that t' is called a *canonical parameter* or *code* for $D(t)$.

Example 2.1. Let b be a nondegenerate closed ball in a model the theory ACVF of algebraically closed valued fields. Then $b = \{x : \mathrm{val}(x - c) \geq \mathrm{val}(c - c')\}$ for some elements $c \neq c'$ of the field. b is coded by $\bar{b} = (c, c')/E$, where $(c, c')E(d, d')$ iff $\mathrm{val}(c - c') = \mathrm{val}(d - d') \leq \mathrm{val}(c - d)$. However, we often fail to distinguish notationally between b and $\bar{b}$, and, in particular, we write $A(b) = A(\bar{b})$.

The only imaginary sorts that will really be essential for us are the sorts $\mathfrak{B}$ of closed and open balls. The closed balls around 0 can be identified with their radius, hence the valuation group $\Gamma(M) = \mathrm{VF}^*(M)/\mathcal{O}^*(M)$ of a valued field M is embedded as part of $\mathfrak{B}$.

Notation. Let $\mathfrak{B} = \mathfrak{B}^o \cup \mathfrak{B}^{\mathrm{cl}}$, the sorts of open and closed subballs of VF. Let $\Gamma^+ = \{\gamma \in \Gamma : \gamma \geq 0\}$.

$$\mathfrak{B}^{\mathrm{cl}} = \dot{\bigcup_{\gamma \in \Gamma}} \mathfrak{B}^{\mathrm{cl}}_\gamma, \qquad \mathfrak{B}^{\mathrm{cl}}_\gamma = \mathrm{VF}/\gamma\mathcal{O},$$

$$\mathfrak{B}^{o} = \dot{\bigcup_{\gamma \in \Gamma}} \mathfrak{B}^{o}_\gamma, \qquad \mathfrak{B}^{o}_\gamma = \mathrm{VF}/\gamma\mathcal{M}.$$

Here $\gamma\mathcal{M} = \{x \in \mathrm{VF} : \mathrm{val}(x) > \gamma\}$, $\gamma\mathcal{O} = \{x \in \mathrm{RES} : \mathrm{val}(x) \geq \gamma\}$. The elements of $\mathfrak{B}^{\mathrm{cl}}_\gamma$, $f B^o_\gamma$ will be referred to as *closed and open balls of valuative radius* γ; though this valuative definition of radius means that bigger balls have smaller radius. The word "distance" will be used similarly.

By a *thin annulus* we will mean a closed ball of valuative radius γ, with an open ball of valuative radius γ removed.

Fix a model M of T. A *substructure* A of M (written $A \leq M$) consists of a subset A_S of $S(M)$, for each sort S of L, closed under all definable functions of T. For example, the substructures of models of T_{fields} are the integral domains.

In general, the *definable closure* of a set $A_0 \subset M$ is the smallest substructure containing A_0; it is denoted $\mathrm{dcl}(A_0)$ or $\langle A_0 \rangle$. An element of $\langle A_0 \rangle$ can be written as $g(a_1, \ldots, a_n)$ with $a_i \in A_0$ and g a definable function; i.e., it is an element satisfying a formula $\phi(x, a_1, \ldots, a_n)$ of L_{A_0} in one variable that has exactly one solution in M. If A is a substructure, $\mathrm{dcl}(A \cup \{c\})$ is also denoted $A(c)$. These notions apply equally when A, c contain elements of the imaginary sorts. If B is contained in sorts $S_1, \ldots, S_n$, then $\mathrm{dcl}(B)$ is said to be an $S_1, \ldots, S_n$-generated substructure. In the special case of valued fields, where one of the sorts VF is the "main" valued field sort, a VF-generated structure will be said to be field-generated, or sometimes just "a field."

For any definable set D, we let $D(A)$ be the set of points of $D(M)$ with coordinates in A. If $S = D/E$ is an imaginary sort, $S(A)$ is the set of $a \in S$ whose preimage is defined over A. We have $D(A)/E(A) \subseteq S(A)$. $D(A)/E(A)$ is, of course, closed under definable functions $S^m \to S$ that lift to definable functions $D^m \to D$, but it is not necessarily closed under arbitrary definable functions, i.e., functions whose graph is the image of a definable subset of $D^m \times D$. For example $x \mapsto (1/n)x$ is a definable function on the value group of a model of ACVF, but if $A \leq M \models$ ACVF, $\Gamma(A)$ need not be divisible.

When $A \leq M, B \leq N$ with $M, N \models T$, a function $f : A \to B$ is called a *(partial) elementary embedding* $(A, M) \to (B, N)$ if for any definable set D of L, $f^{-1}D(B) = D(A)$. In particular, when $A = M$, $B = N$, one says that M is an *elementary submodel* of N.

By a *constructible set* over A, we mean the functor $L \mapsto \phi(L)$ on models $M \models T_A$, where $\phi = \phi(x_1, \dots, x_n, a_1, \dots, a_m)$ is a quantifier-free formula with parameters from A.

We say that T *admits quantifier elimination* if every definable set coincides with a constructible set. It follows in this case that for any A, any A-definable set is A-constructible. When T admits quantifier elimination, $f : A \to B$ is a partial elementary embedding iff it is an embedding of structures.

Theories (1)–(5) of Section 2.1 admit quantifier elimination in their natural algebraic languages (theorems of Tarski–Chevalley and Robinson; cf. [16]). The sixth admits quantifier elimination in a language that needs to be formulated with more care; see [23].

In all of this paper, except for Sections 12.1 and 12.3, we will only use structural properties of definable sets, and not explicit formulas. In this situation quantifier elimination can be assumed softly, by merely increasing the language by definition so that all definable sets become equivalent to quantifier-free ones. The above distinctions will only directly come into play in Sections 12.1 and 12.3.

If $A \leq M \models T$, L_A is the language L expanded by a constant c_a for each element a of A, so that an L_A-structure is the same as an L-structure M together with a function $A_S \to S(M)$ for each sort S. T_A is the set of L_A sentences true in M when the constant symbol c_a is interpreted as a; the models of T_A are models M of T, together with an isomorphic embedding of A as a substructure of M. In particular, M with the inclusion of A in M is an L_A-structure denoted M_A. For any subset $A_0 \subseteq M$, we write T_{A_0} for $T_{\langle A_0 \rangle}$, where $\langle A_0 \rangle$ is the substructure generated by A_0.

A definable set of T_A will also be referred to as A-definable; similarly for other notions such as those defined just below.

A *parametrically definable set* of T is by definition a T_A-definable set for some A.

An *almost definable* set is the union of classes of a definable equivalence relation with finitely many classes. An element e is called *algebraic* (respectively, definable) if the singleton set $\{e\}$ is almost definable (respectively, definable). When T is a complete theory, the set of algebraic (definable) elements of a model M of T forms a substructure that does not depend on M, up to (a unique) isomorphism.

Let $A_0 \subseteq M \models T$; the set of $e \in M$ almost definable over A_0 is called the *algebraic closure* of A_0, $\mathrm{acl}(A_0)$. If A_0 is contained in sorts $S_1, \dots, S_n$, any substructure of $\mathrm{acl}(A_0)$ containing $\mathrm{dcl}(A_0)$ is said to be *almost* $S_1, \dots, S_n$*-generated*.

Example 2.2. If a definable set D carries a definable linear ordering, then every algebraic element of D is definable. This is because the *least* element of a finite definable set F is clearly definable; the rest are contained in a smaller finite definable subset of D, so are definable by induction.

If, in addition, D has elimination of imaginaries, and Y is almost definable and definable with parameters from D, then Y is definable. Indeed, using elimination of imaginaries in D, the set Y can be defined using canonical parameters. These are algebraic elements of D, hence definable.

Two definable functions $f : X \to Y$, $f' : X \to Y'$ will be called *isogenous* if for all $x \in X$, $\mathrm{acl}(f(x)) = \mathrm{acl}(f'(x))$.

Compactness

Compactness often allows us to replace arguments in relative dimension one over a definable set, by arguments in dimension one over a different base structure. Here is an example.

Lemma 2.3. *Let $f_i : X_i \to Y$ be definable maps between definable sets of T $(i = 1, 2)$. Assume that for any $M \models T$ and $b \in Y(M)$, $X_1(b) := {f_1}^{-1}(b)$ is T_b-definably isomorphic to $X_2(b) = {f_2}^{-1}(b)$. Then X_1, X_2 are definably isomorphic.*

Proof. Let $\mathfrak{F}$ be the family of pairs (U, h), where U is a definable subset of Y, and $h : {f_1}^{-1}U \to {f_2}^{-1}U$ is a definable bijection.

Claim. For any $b \in Y(M)$, $M \models T$, there exists $(U, h) \in \mathfrak{F}$ with $b \in U$.

Proof. Let $b \in Y(M)$. There exists a T_b-definable bijection $X_1(b) \to X_2(b)$. This bijection can be written as $x \mapsto g(x, b)$, where g is a definable function. Let $U = \{y \in Y : (x \mapsto g(x, y))$ is a bijection $X_1(y) \to X_2(y)\}$. Then $(U, g(x, f_1(x))) \in \mathfrak{F}$, and $b \in U$. □

Now by compactness, there exist a finite number of definable subsets $U_1, \dots, U_k$ of Y, with $Y = \cup_i U_i$, and $(U_i, h_i) \in \mathfrak{F}$ for some h_i. We define $U_i' = U_i \setminus (U_1 \cup \dots \cup U_{i-1})$ and $h = \cup_i h_i | U_i'$. Then $h : X_1 \to X_2$ is the required bijection. □

Here is another example of the use of compactness.

Example 2.4. If D is a definable set, and for any $a, b \in D$, $a \in \mathrm{acl}(b)$, then D is finite. More generally, if $a \in \mathrm{acl}(b)$ for any $b \in D$, then $a \in \mathrm{acl}(\emptyset)$.

Proof. We prove the first statement, the second being similar. For any model M, pick $a \in M$; then $D(M) \subseteq \mathrm{acl}(a)$. For $b \in \mathrm{acl}(a)$. Let ϕ_b be the formula $x \neq b \& D(x)$. Thus the set of formulas $\mathrm{Th}(M)_M \cup \{\phi_b\}$ has no common solution. By compactness, some finite subset already has no solution; this is only possible if $D(M)$ is finite. □

Transitivity, orthogonality

A definable set D is *transitive* if it has no proper, nonempty definable subsets. (The usual word is "atomic." One also says that *D generates a complete type*.) It is *(finitely) primitive* if it admits no nontrivial definable equivalence relation (with finitely many classes).

Remark 2.5. Let A be a VF-generated substructure of a model of ACVF. When A is VF-generated, we will see that an ACVF_A-definable ball b is never transitive in ACVF_A; indeed, it always contains an A-definable finite set. But b is always $\mathrm{ACVF}_{A(b)}$-definable, and quite often it is transitive; cf. Lemma 3.8.

Two definable sets D, D' are said to be *orthogonal* if any definable subset of $D^m \times D^l$ is a finite union of rectangles $E \times F$, $E \subseteq D^m$, $F \subseteq D^l$. In this case, the rectangles E, F can be taken to be almost definable. If the rectangles can actually be taken definable, we say the D, D' are *strongly orthogonal*.

Types

Let S be a product of sorts, and let $M \models T$, $a \in S(M)$. We write $\mathrm{tp}(a) = \mathrm{tp}(a; M)$ (the type of a) for the set of definable sets D with $a \in D$; when $p = \mathrm{tp}(a)$ we write $a \models p$. A *complete type* is the type of some element in some model. If $q = \mathrm{tp}(a)$, we say that a is a *realization* of q. The set $\mathcal{T}p_S$ of complete types belonging to S can be topologized: a basic open set is the set of types including a given definable set D. The *compactness theorem* of model theory implies that this is a compact topological space: if $\{D_i\}$ is any collection of definable sets with nonempty finite intersections, the compactness theorem asserts the existence of $M \models T$ with $\cap_i D_i(M) \neq \emptyset$.

The compactness theorem is often used by way of a construction called *saturated models*; cf. [9]. These are models where all types over "small" sets are realized. They enjoy excellent Galois-theoretic properties: in particular, if M is saturated, then $\mathrm{dcl}(A_0) = \mathrm{Fix}\,\mathrm{Aut}(M/A_0)$ for any finite $A_0 \subseteq M$. If D is $\mathrm{acl}(A_0)$-definable, then there exists an A_0-definable D' which is a finite union of $\mathrm{Aut}(M/A_0)$-conjugates of D.

A type p can also be identified with the functor P from models of T (under elementary embeddings) into sets; $P(M) = \{a \in M : a \models p\}$. As with definable sets, we speak as if P is simply a set. Unlike definable sets, the value of $P(M)$ at a single model does not determine P. (It could be empty, but it does determine P if M is sufficiently saturated.)

Any definable map $f : S \to S'$ induces a map $f_* : \mathcal{T}p_S \to \mathcal{T}p_{S'}$; as another consequence of the compactness theorem, f_* is continuous. We also have a restriction map from types of T_A to types of T, $\mathrm{tp}_{T(A)}(a) \mapsto \mathrm{tp}_T(a)$.

If $L \subseteq L'$ and $T \subseteq T'$, we say that T' is an *expansion* of T. In this case any T'-type p' restricts to a T-type p. If p' is the *unique* type of T' extending p, we say that p implies p'.

The simplest kind of expansion is an expansion by constants, i.e., a theory T_A (where $A \leq M \models T$). If $c \in M^n$, or more generally if $c \in M^{\mathrm{eq}}$, the type of c for M_A

is denoted $\mathrm{tp}(c/A)$. It is rare for $\mathrm{tp}(c)$ to imply $\mathrm{tp}(c/A)$, but it is significant when it happens.

An instance of this is strong orthogonality: it is easy to see that strong orthogonality of two definable sets D, D' is equivalent to the following condition:

$$\text{If } A' \text{ is generated by elements of } D', \text{ then any type of elements of } D \text{ generates a complete type over } A'. \quad (*)$$

The asymmetry in $(*)$ is therefore only apparent.

Similarly, we have the following.

Lemma 2.6. *Let D, D' be definable sets. Then* (1) $\iff$ (2), (3) $\iff$ (4).

(1) *Every definable function $f : D \to D'$ is piecewise constant, i.e., there exists a partition $D = \cup_{i=1}^{n} D_i$ of D into definable sets, with f constant on D_i.*
(2) *If $d \in D$, $d' \in D'$, $d' \in \mathrm{dcl}(d)$, then $d' \in \mathrm{dcl}(\emptyset)$.*
(3) *If $f : E \to D$ is a definable finite-to-one map, and $g : E \to D'$ is definable, then $g(E)$ is finite.*
(4) *If $d \in D$, $d' \in D'$, $d' \in \mathrm{acl}(d)$, then $d' \in \mathrm{acl}(\emptyset)$.*

Proof. Let us show that (3) implies (4). Let $M \models T$, $d \in D(M)$, and $d' \in D'(M)$, $d' \in \mathrm{acl}(d)$. Then d' lies in some finite T_d-definable set $D'(d) \subseteq D'$. Since T_d is obtained from T by adding a constant symbol for d, there exists a formula $\phi(x, y)$ of the language of T and some m such that $M \models \phi(d, d')$ and $M \models (\exists^{\leq m} z)\phi(d, z)$. Let $X_0 = \{(x : (\exists^{\leq m} y)\phi(x, y)\}$, $E = \{(x, y) : x \in X_0, \phi(x, y)\}$, $f(x, y) = x$, $g(x, y) = y$. Then by (3), $g(E)$ is finite, but $d' \in g(E)$, so $d' \in \mathrm{acl}(\emptyset)$.

Next, (4) implies (3): let f, E, g be as in (3) , and suppose $g(E)$ is infinite. In particular, for any finite $F \subseteq \mathrm{acl}(\emptyset)$ there exists $d' \in g(E) \setminus F$. Thus the family consisting of $g(E)$ and the complement of all finite definable sets has nonempty intersections of finite subfamilies, so by the compactness theorem, in some $M \models T$, there exists $d' \in g(E) \setminus \mathrm{acl}(\emptyset)$.

Let $d \in E(M)$ be such that $d' = g(d)$. Then $d' \in \mathrm{acl}(f(d))$, but $f(d) \in D$, contradicting (4). Thus (4) implies (3).

The equivalence of (1)–(2) is similar. □

Example 2.7. Let P be a complete type, and f a definable function. Then $f(P)$ is a complete type P'. If f is injective on P, then there exist definable $D \supseteq P$, $D' \supseteq P'$ such that f restricts to a bijection of D with D'.

Proof. For any definable D', $f^{-1}D'$ is definable, so $P \subseteq f^{-1}D'$ or $P \cap f^{-1}D' = \emptyset$. Thus $P' \subseteq D'$ or $P' \cap D' = \emptyset$. Thus P' is complete.

Let $\{D_i\}$ be the family of definable sets containing P. Let $R_i = \{(x, y) \in D_i^2 : x \neq y, f(x) = f(y)\}$. Then $\cap_i R_i = \emptyset$. Since the family of $\{D_i\}$ is closed under finite intersections, it follows from the compactness theorem that for some i, $R_i = \emptyset$. Let $D = D_i$, $D' = f(D)$. □

Naming almost definable sets

As special case of an expansion by constants, we can move from a complete theory T to the theory T_A, where $A = \mathrm{acl}(\emptyset)$ is the set of all algebraic elements of a model M of T, including imaginaries. The effect is a theory where each class of any definable equivalence relation E with finitely many classes is definable. Since T is complete, the isomorphism type of $\mathrm{acl}(\emptyset)$ in a model M does not depend on the choice of model; so the theory T_A is determined. A definable set in this theory corresponds to an almost definable set in T.

When D is a constructible set, $T|D$ denotes the theory induced on D. If T eliminates quantifiers, the language is just the restriction to D of the relations and functions of L. If the language is countable, the countable models of D_A are of the form $D(M)$, where M is a countable model of T_A.

Stable embeddedness

A definable subset D of any product of sorts (possibly imaginary) is called *stably embedded* (in T) if for any A, any T_A-definable subset of D^m is T_B-definable for some $B \subset D$. For example, the set of open balls is not stably embedded in ACVF, since the set of open balls containing a point $a \in K$ cannot in general be defined using a finite number of balls.

Lemma 2.8. *Let D be a family of sorts of L; let $T|D$ be the theory induced on the sorts D. If D is stably embedded and $T|D$ admits elimination of imaginaries, then for any definable P and definable $S \subset P \times D^m$, viewed as a P-indexed family of subsets $S(a) \subseteq D^m$, $a \in P$, we have a definable function $f : P \to D^n$, with $f(a)$ a canonical parameter for $S(a)$.*

Proof. By stable embeddedness there exists a family $S' \subset P' \times D^m$ yielding the same family, i.e., $\{S(a) : a \in P\} = \{S'(a') : a' \in P'\}$, and with $P' \subseteq D^n$; using elimination of imaginaries we can take S' to be a canonical family; now a defines $f(a)$ to be the unique $a' \in P'$ with $S(a) = S'(a')$. □

Corollary 2.9. *If D is stably embedded and admits elimination of imaginaries, then for any substructure A,*

(1) $(T_A)|D = (T|D)_{A \cap D}$;
(2) *for $a \in A$,* $\mathrm{tp}(a/A \cap D)$ *implies* $\mathrm{tp}(a/D)$. □

Examples of definable sets of ACVF satisfying the hypotheses include the residue field $\mathbf{k}$, or the value group Γ, as well as $\mathrm{RV} \cup \Gamma$. The stable embeddedness in this case is an immediate consequence of quantifier elimination; cf. Lemma 3.30.

If M is saturated and D is stably embedded in T, then we have an exact sequence

$$1 \to \mathrm{Aut}(M/D(M)) \to \mathrm{Aut}(M) \to \mathrm{Aut}(D(M)) \to 1,$$

where $\mathrm{Aut}(M/D(M))$ is the group of automorphisms of M fixing $D(M)$ pointwise, and $\mathrm{Aut}(D(M))$ is the group of permutations of $D(M)$ preserving all definable relations. Moreover, $\mathrm{Aut}(M/D(M))$ has a good Galois theory; in particular, elements with a finite orbit are almost definable over some finite subset of D. This and some other characterizations can be found in [5, appendix].

Generic types

Let T be a complete theory with quantifier elimination. Let $\mathcal{C}$ be the category of substructures of models of T, with L-embeddings, and let $\mathcal{S}$ be the category of pairs (A, p) with $A \in \mathrm{Ob}\,\mathcal{C}$ and p a type over A. We define $\mathrm{Mor}((A, p), (B, q)) = \{f \in \mathrm{Mor}_{\mathcal{C}}(A, B) : f^*(q) = p\}$.

By a *generic type* we will mean a function p on $\mathrm{Ob}\,\mathcal{C}$, denoted $A \mapsto (p|A)$, such that $A \mapsto (A, p|A)$ is a functor $\mathcal{C} \to \mathcal{S}$. For example, when T, the theory of algebraically closed fields, is provided by any absolutely irreducible variety V: given a field F, let $p|F$ be the type of an F-generic point of V, i.e., the type of a point of $V(L)$ avoiding $U(L)$ for every proper F-subvariety U of V, where L is some extension field of F. Other examples will be given below, beginning with Example 3.3.

Lemma 2.10. *Let p be a generic type of T, and let $M \models T$, $a, b \in M$. Let $c \models p|M$.*

(1) *If $a \notin \mathrm{dcl}(\emptyset)$, then $a \notin \mathrm{dcl}(c)$.*
(2) *If $a \notin \mathrm{acl}(\emptyset)$, then $a \notin \mathrm{acl}(c)$.*
(3) *If $a \notin \mathrm{acl}(b)$, then $a \notin \mathrm{acl}(b, c)$.*

Proof.

(1) Since $a \notin \mathrm{dcl}(\emptyset)$, there exists $a' \neq a$ with $\mathrm{tp}(a) = \mathrm{tp}(a')$. Let $c' \models p|\langle\{a, a'\}\rangle$. Since $\mathrm{tp}(a) = \mathrm{tp}(a')$, there exists an isomorphism $\langle a\rangle \to \langle a'\rangle$; by functoriality of p, $\mathrm{tp}(a, c) = \mathrm{tp}(a', c)$. If $a \in \mathrm{dcl}(c)$, then a is the unique realization of $\mathrm{tp}(a/c)$, so $a = a'$; a contradiction.
(2) If $a \in \mathrm{acl}(c)$, then for some n there are at most n realizations of $\mathrm{tp}(a/c)$. Since $a \notin \mathrm{acl}(\emptyset)$, there exist distinct realizations $a_0, \dots, a_n$ of $\mathrm{tp}(a)$. Proceed as in (1) to get a contradiction.
(3) This follows from (2) for $T_{\langle b\rangle}$. □

2.2 Grothendieck rings

We define the Grothendieck group and associated objects of a theory T; cf. [10]. $\mathrm{Def}(T)$ is the category of definable sets and functions. Let $\mathcal{C}$ be a subcategory of $\mathrm{Def}(T)$. We assume $\mathrm{Mor}(X, Y)$ is a sheaf on X: if $X_1 = X_2 \cup X_3$ are subobjects of X, and $f_i \in \mathrm{Mor}(X_i, Y)$ with $f_1|(X_2 \cap X_3) = f_2|(X_2 \cap X_3)$, then there exists $f \in \mathrm{Mor}(X_1, Y)$ with $f|X_i = f_i$. Thus the disjoint union of two constructible sets in $\mathrm{Ob}\,\mathcal{C}$ is also the category theoretic disjoint sum.

If only the objects are given, we will assume $\mathrm{Mor}\,\mathcal{C}$ is the collection of all definable bijections between them.

The *Grothendieck semigroup* $K_+(\mathcal{C})$ is defined to be the semigroup generated by the isomorphism classes $[X]$ of elements $X \in \mathrm{Ob}\,\mathcal{C}$, subject to the relation

$$[X] + [Y] = [X \cup Y] + [X \cap Y].$$

In most cases, $\mathcal{C}$ has disjoint unions; then the elements of $K_+(\mathcal{C})$ are precisely the isomorphism classes of $\mathcal{C}$.

If $\mathcal{C}$ has Cartesian products, we have a semiring structure given by

$$[X][Y] = [X \times Y].$$

In all cases we will consider the cases when products are present, the symmetry isomorphism $X \times Y \to Y \times X$ will be in the category, as well as the associativity morphisms, so that $K_+(\mathcal{C})$ is a commutative semiring.

(The assumption on Cartesian products is taken to include the presence of an object $\{p\} = X^0$ such that the bijections $X \to \{p\} \times X$, $x \mapsto (p, x)$, and $X \to X \times \{p\}$, $x \mapsto (x, p)$, are in $\mathrm{Mor}_{\mathcal{C}}$ for all $X \in \mathrm{Ob}_{\mathcal{C}}$. All such p give the same element $1 = [\{p\}] \in K(\mathcal{C})$, which serves as the identity element of the semiring.)

Let $K(\mathcal{C})$ be the Grothendieck group, the formal groupification of $K_+(\mathcal{C})$. When $\mathcal{C}$ has products, $K(\mathcal{C})$ is a commutative ring.

We will often have dimension filtrations on our categories, and hence on the semiring.

By an *semiring ideal* we mean a congruence relation, i.e., an equivalence relation on the semiring R that is a subsemiring of $R \times R$. To show that an equivalence relation E is a congruence on a commutative semiring R, it suffices to check that if $(a, b) \in E$ then $(a + c, b + c) \in E$ and $(ac, bc) \in E$.

Remark. When T is incomplete, let S be the (compact, totally disconnected) space of completions of T. Then $\{K(t) : t \in S\}$ are the fibers of a sheaf of rings over S. $K(T)$ can be identified with the ring of continuous sections of this sheaf. In this sense, Grothendieck rings reduce to the case of complete theories.

This last remark is significant even when T is complete: if one adds a constant symbol c to the language, T becomes incomplete, and so the Grothendieck ring of T in $L(c)$ is the Boolean power of $K(T_a)$, where T_a ranges over all $L(c)$-completions of T. Say c is a constant for an element of a sort S. Then an $L(c)$-definable subset of a sort S' corresponds to an L-definable subset of $S \times S'$. This allows for an inductive analysis of the Grothendieck ring of a structure, given good information about definable sets in one variable (cf. Lemma 2.3).

Groups of functions into $\mathcal{R}$

Let $\mathcal{C}(T)$ be a subcategory of the category of definable sets and bijections, defined systematically for T and for expansions by constants T. Let $\mathcal{R}(T) = K_+(\mathcal{C}(T))$ be the Grothendieck semigroup of $\mathcal{C}(T)$. When V is a definable set, we let $\mathcal{C}_V$, $\mathcal{R}_V$ denote the corresponding objects over V; the objects of $\mathcal{C}_V$ are definable sets

$X \subseteq (V \times W)$ such that for any $a \in V$, $X_a \in \mathcal{C}_a$, and similarly the morphisms. In practice, $\mathcal{R}$ will be the Grothendieck semigroup of all definable sets and definable isomorphisms satisfying some definable conditions, such as a boundedness condition on the objects, or a "measure preservation" condition on the definable bijections.

To formalize the notion of "definable function into $\mathcal{R}$" we will need to look at classes X_a of parametrically definable sets. The class of X_a makes sense only in the Grothendieck groups associated with $\mathbf{T}_a$, not $\mathbf{T}$. Moreover, the equality of such classes, say, of X_a and of X_b, begins to make sense only in Grothendieck groups of $\mathbf{T}_{(a,b)}$. Expressions like

$$[X] \underset{a,b}{=} [Y]$$

will therefore mean that X, Y are both definable in $\mathbf{T}_{a,b}$, $[X], [Y]$ denote their classes in the Grothendieck group of $\mathbf{T}_{a,b}$, and these classes are equal.

If V is a definable set, we define the *semigroup of definable functions* $V \to \mathcal{R}$, denoted $\mathrm{Fn}(V, \mathcal{R})$. An element of $\mathrm{Fn}(V, \mathcal{R})$ is represented by a definable $X \in \mathcal{C}_V$, viewed as the function $a \mapsto [X_a]$, where $[X_a]$ is a class in $\mathcal{R}_a$. X, X' represent the same function if for all a, $[X_a], [X'_a]$ are the same element of $\mathcal{R}_a$. Note that despite the name, the elements of $\mathrm{Fn}(V, \mathcal{R})$ should actually be viewed as sections $V \to \Pi_{a \in V} \mathcal{R}_a$.

Addition is given by disjoint union in the image (i.e., disjoint union over X).

Usually $\mathcal{R}$ has a natural grading by dimension; in this case $\mathrm{Fn}(V, \mathcal{R})$ inherits the grading.

Assume that V is a definable group and $\mathcal{R} = K_+(T)$ is the Grothendieck semiring of all definable sets and functions of T, there is a natural convolution product on $\mathrm{Fn}(V, \mathbb{R})$. If $h_i(a) = [H_i(a)]$, $H_i \subset V \times B_i$, the convolution $h_1 * h_2$ is represented by

$$H = \{(a_1 + a_2, (a_1, a_2, y_1, y_2)) : (a_i, y_i) \in H_i\} \subseteq V \times (V^2 \times B_1 \times B_2)$$

so that $h_1 * h_2(a) = H(a) = \{(a_1, a_2, y_1, y_2) : (a_i, y_i) \in H_i, a_1 + a_2 = a\}$.

Grothendieck groups of orthogonal sets

Lemma 2.11. *Let T be a theory with two strongly orthogonal definable sets D_1, D_2, $D_{12} = D_1 \times D_2$. Let $K_+ D_i[n]$ be the Grothendieck semigroup of definable subsets of D_i^n. Then $K_+ D_{12}[n] \simeq K_+ D_1[n] \otimes K_+ D_2[n]$.*

Proof. This reduces to $n = 1$. Given definable sets $X_i \subseteq D_i^n$, it is clear that the class of $X_1 \times X_2$ in $K_+ D_{12}[n]$ depends only on the classes of X_i on $D_i[n]$. Define $[X_1] \otimes [X_2] = [X_1 \times X_2]$. This is clearly $\mathbb{Z}$-bilinear, and so extends to a homomorphism $\eta : K_+ D_1[1] \times K_+ D_2[1] \to K_+ D_{12}[1]$. By strong orthogonality, η is surjective.

To prove injectivity, note that any element of $K_+ D_1[n] \otimes K_+ D_2[n]$ can be written $\sum [X_1^i] \otimes [X_2^i]$, with $X_1^1, \dots, X_1^k$ pairwise disjoint. To see this, begin with some expression $\sum [X_1^i] \otimes [X_2^i]$; use the relation $[X' \dot\cup X''] \otimes [Y] = [X'] \otimes [Y] + [X''] \otimes [Y]$ to replace the X_1^i by the atoms of the Boolean algebra they generate, so that the new X_i^i are equal or disjoint; finally use the relation $[X' \otimes Y'] + [X' \otimes Y''] = [X'] \otimes [Y' \dot\cup Y'']$ to amalgamate the terms with equal first coordinate.

Hence it suffices to show that if $[\cup_i X_1^i \times X_2^i] = [\cup_j Y_1^i \times Y_2^i]$, with the X_1^i and the Y_1^i pairwise disjoint, then $\sum[X_1^i] \times [X_2^i] = \sum[Y_1^i] \times [Y_2^i]$. Let $F : \cup_i X_1^i \times X_2^i \to \cup_j Y_1^i \times Y_2^i$ be a definable bijection. By strong orthogonality, the graph of F is a disjoint union of rectangles. Since F is a bijection, it is easy to see that each of these rectangles has the form $f_1^k \times f_2^k$, where for $\nu = 1, 2$, $f_\nu^k : X_\nu(k) \to Y_\nu(k)$ is a bijection from a subset of $\cup_i X_\nu^i$ to a subset of $\cup_j Y_\nu^j$. The rest follows by an easy combinatorial argument; we omit the details, since a somewhat more complicated case will be needed and proved later; see Proposition 10.2. □

Integration by parts

The following will be used only in Section 9, to study the Grothendieck semiring of the valuation group.

Definition 2.12. Let us say that $Y \in \mathrm{Ob}\,\mathcal{C}$ is *treated as discrete* if for any $X \in \mathrm{Ob}\,\mathcal{C}$ and any definable $F \subset X \times Y$ such that $T \models F$ is the graph of a function, the projection map $F \to X$ is an invertible element of $\mathrm{Mor}_{\mathcal{C}}(F, X)$.

To explain the terminology, suppose each $X \in \mathrm{Ob}\,\mathcal{C}$ is endowed with a measure μ_X, and $\mathcal{C}$ is the category of measure-preserving maps. If μ_Y is the counting measure, and $\mu_{X\times Y}$ is the product measure, then for any function $f : X \to Y$, $x \mapsto (x, f(x))$ is measure preserving.

We will assume $\mathcal{C}$ is closed under products.

If Y_1, Y_2 are treated by $\mathcal{C}$ as discrete, so is $Y_1 \times Y_2$: if $F \subset X \times (Y_1 \times Y_2)$ is the graph of a function $X \to (Y_1 \times Y_2)$, then the projection to $F_1 \subset X \times Y_1$ is the graph of a function, hence the projection $F_1 \to X$ is in $\mathcal{C}$; now $F \subset (F_1 \times Y_2)$ is the graph of a function, and so $F \to F_1$ is invertibly represented, too; thus so is the composition. In particular, if Y is discretely treated, any bijection $U \to U'$ between subsets of Y^n is represented in $\mathcal{C}$.

If $\mathcal{R}$ is a Grothendieck group or semigroup, we write $[X] \underset{R}{=} [Y]$ to mean that X, Y have the same class in $\mathcal{R}$.

Lemma 2.13. *Let $f, f' \subset X \times L$ be objects of $\mathcal{C}$ such that $[f(c)] \underset{K(\mathcal{C}_c)}{=} [f'(c)]$ for any c in X. Then $[f] \underset{K(\mathcal{C})}{=} [f']$; similarly for K_+.*

Proof. By assumption, there exists $g(c)$ such that $f(c) + g(c)$, $f'(c) + g(c)$ are $\mathcal{C}_c$-isomorphic. By compactness (cf. the end of the proof of Lemma 2.3) this must be uniform (piecewise in L, and hence by glueing globally): there exists a definable $g \subset Z \times L$ and a definable isomorphism $f + g \simeq f' + g$, inducing the isomorphisms of each fiber. By the definition of $\mathcal{C}_c$, and since $\mathcal{C}$ is closed under finite glueing, $f + g$, $f' + g$ are in $\mathrm{Ob}\,\mathcal{C}$ and the isomorphism between them is in $\mathrm{Mor}\,\mathcal{C}$. □

Let L be an object of $\mathcal{C}$, treated as discrete in $\mathcal{C}$, and assume given a definable partial ordering on L.

Notation 2.14. Let $f \subset X \times L$. For $y \in L$, let $f(y) = \{x : (x, y) \in f\}$. Denote $\sum_{\gamma<\beta} f(\gamma) = [\{(x, y) : x \in f(y), y < \gamma\}]$.

Notation 2.15. Let $\phi : L \to K(X)$ be a constructible function, represented by $f \subset X \times L$; so that $\phi(y) = [f(y)]$, $f(y) = \{x : (x, y) \in f\}$. Denote $\sum_{\gamma<\beta} \phi(\gamma) = [\{(x, y) : x \in f(y), y < \gamma\}]$.

Note by Lemma 2.13 that this is well defined.

Below, we write fg for the pointwise product of two functions in $K(\mathcal{C})$; $[fg(y)] = [f(y) \times g(y)]$.

Lemma 2.16 (integration by parts). *Let* Γ *be an object of* $\mathcal{C}$*, treated as discrete in* $\mathcal{C}$*, and assume given a definable partial ordering of* Γ*. Let* $f \subset X \times \Gamma$*,* $F(\beta) = \sum_{\gamma<\beta} f(\gamma)$*,* $g \subset Y \times \Gamma$*,* $\mathbf{G}(\beta) = \sum_{\gamma\leq\beta} g(\gamma)$*.*

Then

$$F\mathbf{G}(\beta) = \sum_{\gamma<\beta} f\mathbf{G}(\gamma) + \sum_{\gamma\leq\beta} Fg(\gamma).$$

Proof. Clearly,

$$F\mathbf{G}(\beta) = \Sigma_{\gamma<\beta,\gamma'\leq\beta} f(\gamma)g(\gamma').$$

We split this into two sets, $\gamma < \gamma'$ and $\gamma' \leq \gamma$. Now

$$\Sigma_{\gamma<\gamma'\leq\beta} f(\gamma)g(\gamma') = \Sigma_{\gamma'\leq\beta} F(\gamma')g(\gamma'),$$
$$\Sigma_{\gamma'\leq\gamma<\beta} f(\gamma)g(\gamma') = \Sigma_{g<\beta} f(\gamma)\mathbf{G}(\gamma').$$

The lemma follows. □

This is particularly useful when L is treated as discrete in $\mathcal{C}$, since then, if the sets $f(\gamma)$ are disjoint, $[f] = [\cup_\gamma f_\gamma]$. Another version, with $G(\beta) = \sum_{\gamma<\beta} g(\gamma)$:

$$FG(\beta) = \sum_{\gamma<\beta} (fG + gF + fg)(\gamma).$$

3 Some C-minimal geometry

We will isolate the main properties of the theory ACVF, and work with an arbitrary theory **T** satisfying these properties. This includes the rigid analytic expansions $\mathrm{ACVF}^{\mathrm{R}}$ of [23].

The right general notion, C-minimality, has been introduced and studied in [15]. They obtain many of the results of the present section. Largely for expository reasons, we will describe a slightly less general version; it is essentially minimality with respect to an ultrametric structure in the sense of [31]. We will use notation suggestive of the case of valued fields; thus we denote the main sort by VF and a binary function by $\mathrm{val}(x - y)$. Some additional assumptions will be made explicit later on.

Let T be a theory in a language L, extending a theory T in a language L. T is said to be T-*minimal* if for any $M \models T$, any L_M-formula *in one variable* is T_M-equivalent to an L_M formula.

More generally, if D is a definable subset of T (i.e., a formula of L), we say that D is T-*minimal* if for any $M \models T$, any T_M-definable subset of D is T_M-equivalent to one defined by an L_M formula.

Strong minimality

Let $\mathsf{L} = \emptyset$. The only atomic formulas of L are thus equalities $x = y$ of two variables. T is the theory of infinite sets. T-minimality is known as *strong minimality*; see [1, 28, 29]. A theory T is strongly minimal iff for any $M \models T$, any T_M-definable subset of M is finite or cofinite. For us the primary example of a strongly minimal theory is ACF, the theory of algebraically closed fields.

Let $M \models T$. If D is strongly minimal, and X a definable subset of D^*, we define the D-*dimension* of X to be the least n such that X admits a T_M-definable map into D^n with finite fibers. In the situation we will work in, there will be more than one definable strongly minimal set up to isomorphism, and even up to definable isogeny; in particular, there will be the various sets of RES_M. However, between any of these, there exists an M-definable isogeny; so the $\mathbf{k}$ dimension agrees with the D dimension for any of them. We will call it the RES dimension. It agrees with *Morley rank*, a notion defined in greater generality, that we will not otherwise need here.

O-minimality

$\mathsf{L} = \{<\}$, $\mathsf{T} = DLO$ the theory of dense linear orders without endpoints (cf. [9]). DLO minimality is known as O-minimality, and can also be stated thusly: any T_M-definable subset of M is a finite union of points and intervals. This also forms the basis of an extensive theory; see [37].

Let D be O-minimal. Then the O-minimal dimension of a definable set $X \subseteq D^*$ is the least n such that X admits a T_M-definable map into D^n with bounded finite fibers.

The Steinitz exchange principle states that if $a \in \mathrm{acl}(B \cup \{b\})$ but $a \notin \mathrm{acl}(B)$, then $b \in \mathrm{acl}(B \cup \{a\})$.

This holds for both strongly minimal and O-minimal structures; cf. [37].

For us the relevant O-minimal theory is DOAG itself. We will occasionally use stronger facts valid for this theory. Quantifier elimination for DOAG implies the following.

Lemma 3.1.

(1) *Any parameterically definable function f of one variable is piecewise affine; there exists a finite partition of the universe into intervals and points, such that on each interval I in the partition, $f(x) = \alpha x + c$ for some rational α and some definable c.*

(2) DOAG *admits elimination of imaginaries.*

Proof.

(1) This follows from quantifier elimination for DOAG.
(2) This follows from (1) that any function definable with parameters in DOAG has a canonical code, consisting of the endpoints of the intervals of the coarsest such partition, together with a specification of the rationals α and the constants c. But from this it follows on general grounds that every definable set is coded (cf. [16, 3.2.2]). Thus DOAG admits elimination of imaginaries. □

C-minimality

Let $\mathsf{T} = \mathsf{T}_{um}$ be the theory of ultrametric spaces or, equivalently, chains of equivalence relations (cf. [31]).

In more detail, L has two sorts, VF and Γ_∞. The relations on Γ_∞ are a constant ∞ and a binary relation $<$. In addition, L has a function symbol $\mathrm{VF}^2 \to \Gamma_\infty$, written $\mathrm{val}(x-y)$. T states the following:

(1) Γ_∞ is a dense linear ordering with no least element, but with a greatest element ∞.
(2) $\mathrm{val}(x-y) = \infty$ iff $x = y$.
(3) $\mathrm{val}(x-y) \geq \alpha$ is an equivalence relation; the classes are called *closed α-balls*. Hence so is the relation $\mathrm{val}(x-y) > \alpha$, whose classes are called *open α-balls*.
(4) Let $\Gamma = \Gamma_\infty \setminus \{\infty\}$. For $\alpha \in \Gamma$, every closed α-ball contains infinitely many open α-balls.

A T_{um}-minimal theory will be said to be C-minimal. The notion considered in [15] is a little more general, but for theories T_{um} they coincide. Since we will be interested in fields, this level of generality will suffice.

A theory T extending ACVF is C-minimal iff for any $M \models T$, every T_M-definable subset of $\mathrm{VF}(M)$ is a Boolean combination of open balls, closed balls and points. If T is C-minimal, $A \leq M \models T$, and b is an A-definable ball, or an infinite intersection, let p_A^b be the collection of A-definable sets not contained in a finite union of proper subballs of b. Then by C-minimality, p_A^b is a complete type over A.

Let T be C-minimal. Then in T, Γ is O-minimal, and for any closed α-ball C, the set of open α-subballs of C is strongly minimal. Denote it $C/(1+\mathcal{M})$. (These facts are immediate from the definition.)

Assume T is C-minimal with a distinguished point 0. We define: $\mathrm{val}(x) = \mathrm{val}(x-0)$; $\mathcal{M} = \{x : \mathrm{val}(x) > 0\}$. Let $\mathfrak{B}_{\mathrm{cl}}$ be the family of all closed balls, including points. Among them are ${\mathfrak{B}_{\mathrm{cl}}}^c{}_\alpha(0) = \{x : \mathrm{val}(x) \geq \alpha\}$. Let $\mathrm{RV} = \dot{\bigcup}_{\gamma \in \Gamma} B_\gamma^c(0)/(1+\mathcal{M})$, and let $\mathrm{rv} : \mathrm{VF} \setminus \{0\} \to \mathrm{RV}$ and $\mathrm{val}_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$ be the natural map. By an rv-ball we mean an open ball of the form $\mathrm{rv}^{-1}(c)$.

The T-definable fibers of $\mathrm{val}_{\mathrm{rv}}$ are referred to, collectively, as RES_T. Later we will fix a theory $\mathbf{T}$, and write RES for $\mathrm{RES}_{\mathbf{T}}$; we will also write RES_A for $\mathrm{RES}_{\mathbf{T}_A}$. The unqualified notion "definable," as well as many derived notions, will implicity refer to $\mathbf{T}$.

A certain notion of genericity plays an essential role in these theories.

Example 3.2. Let T be a strongly minimal theory. For any $A \leq M \models T$, any A-definable set is finite or has finite complement. Therefore, the collection of cofinite sets forms a complete type. A realization of this type is called a *generic* element of M, over A.

Example 3.3. Let T be an O-minimal theory. For any $A \leq M \models T$, any A-definable set contains, or is disjoint from, an infinite interval (b, ∞) for some $b \in M$. The set of A-definable sets containing such an interval is thus a complete type, the generic type of large elements of Γ. Similarly, the set of A-definable sets containing an interval $(0, a)$ with $0 < a$ is *the generic type of small positive elements*. More generally, given a subset $S \subseteq A$ $S' = \{b \in A : (\forall s \in S)(s < b)\}$; then the definable sets $x > a (a \in S), x < b (b \in S')$ generate a complete type over A, called the type of elements *just bigger than* S.

Definition 3.4. Let T be C-minimal. Let b be a T_A-definable ball, or an infinite intersection of balls. The generic type p_b of b is defined by $p_b|A' = p^b_{A'}$, for any $A \leq A' \leq M \models T$.

The completeness follows from C-minimality, since for any A'-definable subset S of b, either S is contained in a finite union of proper subballs of b, or else the complement $b \setminus S$ is contained in such a finite union.

A realization of $p_b|A'$ is said to be a *generic point of* b *over* A'. An A'-definable set is said to be *b-generic* if it contains a generic point of b over A'.

See Section 3.2 for some generalities about generic types. For our purposes it will suffice to consider generic types in one VF variable. For more information see [16, Section 2.5].

Remark 3.5. If $A = \mathrm{acl}(A)$ then any type of a field element $\mathrm{tp}(c/A)$ coincides with $p_b|A$, where b is the intersection of all A-definable balls containing c.

This is intended to include the case of closed balls of valuative radius ∞, i.e., points; these are the algebraic types $x = c$. Note also the degenerate case that c is not in any A-definable ball; then $b = \mathrm{VF}$ and $\mathrm{tp}(c/A)$ is the generic type of VF over A.

Not every generic 1-type is of the form p_b for a ball b as above. For instance, let b be an open ball, $c \in b$; then the generic type $p_b((x - c)^{-1})$ is not of this form.

For V-minimal theories (defined below) it can be shown that every generic 1-type is of the form p_b or $p_b((x - c)^{-1})$.

Let $\mathbf{T}$ be a C-minimal theory. Let b be a definable ball, or an infinite intersection of definable balls. We say that b is centered if it contains a proper definable finite union of balls. If b is open, or a properly infinite intersection of balls, we have the following:

> If b contains a proper finite union of balls, then it contains a definable closed ball (the smallest closed ball containing the finite set). $(*)$

For C-minimal fields of residue characteristic 0, $(*)$ is true of closed balls: the set of maximal open subballs of b forms an affine space over the residue field $\mathbf{k}$, where the center of mass of a finite set is well defined.

Clearly, b is centered over $\mathrm{acl}(A)$ if and only if it is centered over A. The term "centered" will be justified to some extent by the assertion of Lemma 3.39, that when A is generated by elements of $\mathrm{VF} \cup \mathrm{RV} \cup \Gamma$, any A-definable closed ball contains an A-definable point, and thus a centered ball has a definable "center."

Lemma 3.6. *b is centered over A iff b is not transitive over A.*

This is immediate from the definition, and from C-minimality, since any proper definable subset would have to be a Boolean combination of balls.

An often useful corollary of C-minimality is the following.

Lemma 3.7. *Let T be C-minimal, X a definable subset of* VF*, and Y a definable set of disjoint balls. Then for all but finitely many $b \in Y$, either $b \subseteq X$ or $b \cap X = \emptyset$.*

Proof. X is a finite Boolean combination of balls, so it suffices to prove this when X is a ball; then X is contained in at most one ball $b \in Y$; for any other $b \in Y$, either $b \subseteq X$ or $b \cap X = \emptyset$. □

Lemma 3.8. *Let $(b_t : t \in Q)$ be a definable family of pairwise disjoint balls. Then for any nonalgebraic $t \in Q$, b_t is transitive over $\langle t \rangle$.*

Proof. Consider a definable $R' \subseteq Q \times \mathrm{VF}$ with $R'(t) \subseteq b_t$. Let $Y = \cup_{t \in Q} R'(t)$. Then Y is a definable subset of VF, hence a finite combination of a finite set H of balls. The b_t are pairwise disjoint, so at most finitely many can contain an element of H, and thus no nonalgebraic b_t contains an element of H. Thus each ball in H is disjoint from, or contains, any given b_t. It follows that Y is disjoint from, or contains, any given b_t. Thus $b_t \cap Y$ cannot be a nonempty proper subset of b_t. □

Internalizing finite sets

The following lemma will be generalized later to finite sets of balls. It is of such fundamental importance in this paper that we include it separately in its simplest form. The failure of this lemma in residue characteristic $p > 0$ is the main reason for the failure of the entire theory to generalize, in its present form. Recall the definition of RV (Section 2.1).

Lemma 3.9. *Let $\mathbf{T}$ be a C-minimal theory of fields of residue characteristic* 0 *(possibly with additional structure), $A \leq M \models \mathbf{T}$. Let F be a finite $\mathbf{T}_A$-definable subset of* VF^n*. Then there exists $F' \subseteq \mathrm{RV}^m$, and a $\mathbf{T}_A$-definable bijection $h : F \to F'$.*

Proof. First consider $F = \{c_1, \ldots, c_n\} \subseteq \mathrm{VF}$. Let $c = (\sum_{i=1}^n c_i)/n$ be the average; then F is $\mathbf{T}_A$-definably isomorphic to $\{c_1 - c, \ldots, c_n - c\}$. Thus we may assume the average is 0. If there is no nontrivial A-definable equivalence relation on F, then $\mathrm{val}(x - y) = \alpha$ is constant on $x \neq y \in F$. In this case rv is injective on F and one can take $h = \mathrm{rv}$. Otherwise, let E be a nontrivial A-definable equivalence relation on F. By an E-symmetric polynomial, we mean a polynomial $H(x_1, \ldots, x_n)$ with coefficients in A, invariant under the symmetric group on each E-class. For any such H, $H(F)$ is a $\mathbf{T}_A$-definable set with $< n$ elements. There exists H such that $H(F)$ has more than one element. By induction, there exists an injective A-definable function $h_0 : H(F) \to \mathrm{RV}^m$. Let $h_1 = h_0 \circ H$. For $d \in h_0(H(F))$, and $d' = {h_0}^{-1}d$, let $F_d = H^{-1}{h_0}^{-1}(d) = H^{-1}(d')$. By induction again, there exists an $A(d) = A(d')$-definable injective function $g_d : F_d \to \mathrm{RV}^{m'}$. (We can take the same m' for all d.) Define $h(x) = (h_1(x), g_{h_1(x)}(x))$. Then clearly h is A-definable and injective.

The case $F \subseteq \mathrm{VF}^n$ follows using a similar induction, or by finding a linear projection with $\mathbb{Q}$-coefficients $\mathrm{VF}^n \to \mathrm{VF}$ which is injective on F. □

3.1 Basic geography of C-minimal structures

Let $\mathbf{T}$ be a C-minimal theory. We begin with a rough study of the existence and nonexistence of definable maps between various regions of the structure: $\mathbf{k}$, Γ, RV, VF and $\mathrm{VF}/\mathcal{O}$.

We will occasionally refer to *stable* definable sets.

A definable set D of a theory T is called *stable* if there is no model $M \models T$ and M-definable relation $R \subseteq D^2$ and infinite subset $J \subseteq M(D)$ such that $R \cap J^2$ is a linear ordering. This is a model-theoretic finiteness condition, greatly generalizing finite Morley rank, and in turn strong minimality (cf. [28, 29]).

It is shown in [16] that a definable subset of $\mathrm{ACVF}_A^{\mathrm{eq}}$ is stable if and only if it has finite Morley rank, if and only if it admits no parametrically definable map onto an interval of Γ; and this is if and only if it embeds, definably over $\mathrm{acl}(A)$, into a finite-dimensional $\mathbf{k}$-vector space. These vector spaces have the general form $\Lambda/\mathcal{M}\Lambda$, with $\Lambda \leq \mathrm{VF}^n$ a lattice. Within the sorts we are using here, the relevant ones are the finite products of vector spaces of RES. More generally, in a C-minimal structure with sorts VF, RV, all stable sets are definably embeddable (with parameters) into RES. We will, however, make no use of these facts, beyond justifying the terminology. Thus "X is a stable definable set" can simply be read as " there exists a definable bijection between X and a subset of RES^*."

The first fact is the unrelatedness of $\mathbf{k}$ and Γ.

Lemma 3.10. *Let Y be a stable definable set. Then Y, Γ are strongly orthogonal. In particular, any definable map from Y to Γ has finite image.*

Proof. We prove the second statement first: let $M \models \mathbf{T}$. Let $f : Y \to \Gamma$ be an M-definable map. Then $f(Y)$ is stable, and linearly ordered by $<_\Gamma$; hence by the definition of stability, it is finite.

Let $\gamma = (\gamma_1, \ldots, \gamma_m) \in \Gamma$. We have to show that for a Y-generated structure A, $\mathrm{tp}(\gamma)$ implies $\mathrm{tp}(\gamma/A)$. It suffices to show that for any $a_, \ldots, a_n \in A$, $\mathrm{tp}(\gamma_i/\langle\gamma_1, \ldots, \gamma_{i-1}\rangle)$ implies $\mathrm{tp}(\gamma_i/\langle\gamma_1, \ldots, \gamma_{i-1}, a_1, \ldots, a_n\rangle)$, for each i. By passing to $T_{\langle\gamma_1,\ldots,\gamma_{i-1}\rangle}$ we may assume $m = 1$, $\gamma \in \Gamma$. Similarly, we may assume $n = 1$; let $a = a_1 \in Y$. To show that $\mathrm{tp}(\gamma)$ implies $\mathrm{tp}(\gamma/a)$, it suffices to show that any T_a-definable subset of Γ is definable. By O-minimality, any T_a-definable subset of Γ is a finite union of intervals, so (in view of the linear ordering) it suffices to show this for intervals (c_1, c_2). But if the interval is T_a-definable then so are the endpoints, so $c_i = c_i(a)$ is a value of a definable map $Y \to \Gamma$. But such maps have finite images, so c_i lies in a finite definable set. Using the linear ordering, we see that c_i itself is definable, and hence so is the interval. □

Lemma 3.11. *There are no definable sections of* $\mathrm{val}_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$ *over an infinite subset of* Γ. *In fact if* $Y \subset \mathrm{RV}^n$ *is definable and* $\mathrm{val}_{\mathrm{rv}}$ *is finite-to-one on* Y*, then* Y *is finite.*

Proof. Looking at the fibers of the projection of Y to RV^{n-1}, and using induction, we reduce the lemma to the case $n = 1$. In this case, by Lemma 3.7, every definable set is a Boolean combination of pullbacks by $\mathrm{val}_{\mathrm{rv}}$ of subsets of Γ and finite sets. □

Lemma 3.12. *Let* $M \models \mathbf{T}$ *and let* $Y \subset \mathfrak{B}_{\mathrm{cl}}{}^n$ *be an infinite definable set. Then there exists a surjective* M*-definable map of* Y *to a proper interval in* Γ.

Proof. Since Γ is O-minimal, any infinite M-definable subset contains a proper interval. Thus it suffices to find an M-definable map of Y onto an infinite subset of Γ.

If the projection of Y to $\mathfrak{B}_{\mathrm{cl}}{}^{n-1}$ as well as every fiber of this projection are finite, then Y is finite. Otherwise, replacing Y by one of the fibers or by the projection, we reduce inductively to the case $n = 1$.

Let $v(y) \in \Gamma$ be the valuative radius of the ball y. Then $v(Y)$ is an M-definable subset of Γ. If it is infinite, we are done; otherwise, we may assume all elements of Y have the same valuative radius γ.

Let $W = \cup Y$. By C-minimality, W is a Boolean combination of balls b_i (open, of valuative radius $< \gamma$, or closed, of valuative radii $\delta_i \leq \gamma$). If W contains some $W' = b_i \setminus (b_{j_1} \cup \cdots \cup b_{j_l})$, where b_{j_i} is a proper subball of b_i, and $\delta_i < \gamma$, pick a point c in W'; then for any δ with $\gamma > \delta > \delta_i$ there exists $c' \in W'$ with $\mathrm{val}(c - c') = \delta$. It follows that the balls $b_\gamma(c), b_\gamma(c')$ of radius γ around c, c' are both in Y; but infinitely many such δ exist; fixing c, we obtain a map $b_\gamma(c') \mapsto \mathrm{val}(c - c')$ into an infinite subset of Γ.

Otherwise, W can only be a finite set of balls of valuative radius γ. Thus Y is finite. □

Corollary 3.13. $\mathfrak{B}_{\mathrm{cl}}{}^n$ *contains no stable definable set. In particular,* VF *contains no strongly minimal set.* □

By contrast, we have the following.

Lemma 3.14. *Any infinite definable subset of* RV^n *contains a strongly minimal* M*-definable subset.*

Proof. By Lemma 3.11, the inverse image of some point in Γ^n must be infinite. □

Lemma 3.15. *Let* $M \models \mathbf{T}$. *Let* $Y \subseteq \mathfrak{B}_{\mathrm{cl}}$ *be a definable set. Let* $\mathrm{rad}(y)$ *be the valuative radius of the ball* y. *Then either* $\mathrm{rad} : Y \to \Gamma$ *is finite-to-one, or else there exists an* M*-definable map of an* M*-definable* $Y' \subseteq Y$ *onto a strongly minimal set.*

Proof. If rad is not finite-to-one, then Y contains an infinite set Y' of balls of the same radius α. Then $\cup Y'$ contains a closed ball b of valuative radius $\beta < \alpha$. The set S of open subballs b' of b of valuative radius β forms a strongly minimal set; the map sending $y \in Y'$ to the unique $b' \in S$ containing y is surjective. □

The following lemma regarding $\mathrm{VF}/\mathcal{O}$ will be needed for integration with an additive character (Section 11).

Lemma 3.16. *Let Y be a stable definable set, $Z \subset \mathrm{VF} \times Y$ a definable set such that for $y \in Y$, $Z(y) = \{x : (x, y) \in Z\}$ is additively $\mathcal{M}$ invariant. Then for all but finitely many $\mathcal{O}$-cosets C, $Z \cap (C \times Y)$ is a rectangle $C \times Y'$.*

Proof. For $y \in Y$, $Z(y)$ is a $\mathbf{T}_y$-definable subset of VF, hence a Boolean combination of a finite $\langle y \rangle$-definable set of balls $b_1(y), \ldots, b_k(y)$. Let $B_i(y)$ be the smallest closed ball containing $b_i(y)$. According to Lemma 3.13, since the set of closed balls occurring as $B_i(y)$ for some y is stable, it is finite:

$$\{B_i(y) : y \in Y\} = \{B_1, \ldots, B_l\}.$$

All the B_i are $\mathcal{O}$-invariant. Let R be the set of $\mathcal{O}$-cosets C that are equal to some B_i.

If $B_i(y)$ has valuative radius < 0 (i.e., it is bigger than an $\mathcal{O}$-coset), then so is $b_i(y)$, so the characteristic function of such a $b_i(y)$ is constant on any closed $\mathcal{O}$-coset C. If $C \notin R$, then it is disjoint from any B_i of valuative radius equal to (or greater than) 0, so the characteristic functions of the corresponding $b_i(y)$ are also constant on it. Thus with finitely many exceptional C, any such characteristic function is constant on C, and the claim follows. □

3.2 Generic types and orthogonality

Two generic types p, q are said to be *orthogonal* if for any base A', if $c \models p|A'$, $d \models q|A'$, then p generates a complete type over $A(d)$; equivalently, q generates a complete type over $A(c)$. We will see that generics of different kinds are orthogonal (cf. Lemma 3.19). This orthogonality of types is weaker than the orthogonality of definable sets mentioned in the introduction, and in the present case is only an indirect consequence of the orthogonality between the residue field and value group; these types do not have orthogonal definable neighborhoods.

If $\gamma \in \Gamma$ and $rk_{\mathbb{Q}}(\Gamma(C(a))/\Gamma(C)) = n$, we say that $\mathrm{tp}(\gamma/C)$ has Γ-dimension n.

Lemma 3.17. *Let p_Γ be a $\mathbf{T}_A$-type of elements of Γ^n of Γ dimension n. Let $P = \mathrm{val}^{-1}(p_\Gamma)$. Then we have the following:*

(1) $\mathrm{val}^{-1}(p_\Gamma)$ *is a complete type over A. In other words, for any A-definable set X, either* $\mathrm{val}^{-1}(p_\Gamma) \subseteq X$ *or* $\mathrm{val}^{-1}(p_\Gamma) \cap X = \emptyset$.
(2) *If D is a stable A-definable set and $d_1, \ldots, d_n \in D$, then P implies a complete type over $A(d_1, \ldots, d_n)$.*
(3) *If $c \in P$, then $D(A(c)) = D(A)$.*
(4) *P is complete over A.*

Proof.

(1) This reduces inductively to the case $n = 1$. Since $\mathrm{val}^{-1}(p_\Gamma)$ is a disjoint union of open balls, (1) for $n = 1$ follows from Lemma 3.7: an A-definable set X cannot intersect nontrivially each of an infinite family of open balls. Therefore, either X is disjoint from almost all, or X contains almost all open balls $\mathrm{val}^{-1}(c)$, $c \models p_\Gamma$; in the former case the complement of X contains $\mathrm{val}^{-1}(p_\Gamma)$, and in the latter X contains $\mathrm{val}^{-1}(p_\Gamma)$ since p_Γ is complete.

(2) By strong orthogonality, p_Γ generates a complete type q' over $A(d)$, of Γ-dimension n. By (1) over $A(d)$, $\mathrm{val}^{-1}(p_\Gamma)$ is complete over $A(d)$. But if $c \in P$ then $\mathrm{val}(c) \models p_\Gamma$ so $c \in \mathrm{val}^{-1}(p_\Gamma)$. Thus P implies a complete type over $A(d)$.

(3) follows from (2): if $d \in D(A(c))$ then there exists a formula ϕ such that $\models \phi(d, c)$ and such that $\phi(x, c)$ has a unique solution. By (2) ϕ is a consequence of $P(c) \cup \mathrm{tp}(d/A)$, and hence by compactness of a formula $\phi_1(x) \& \phi_2(c)$, where $\phi_2 \in \mathrm{tp}(d/A)$. Thus already $\phi_1(x)$ has the unique solution d, and thus $d \in D(A)$.

(4) This is immediate from (1). □

Lemma 3.18. *Let q be a $\mathbf{T}_A$-type of elements of RES^n_A of RES dimension n. Let $Q = \mathrm{rv}^{-1}(q)$. Then Q is complete over A. Moreover, if $\gamma_1, \dots, \gamma_m \in \Gamma$, then Q implies a complete type over $A(\gamma_1, \dots, \gamma_m)$.*

Proof. Again the lemma reduces inductively to the case $n = 1$, and for $n = 1$ follows from Lemma 3.7, since $\mathrm{val}^{-1}(q)$ is a union of disjoint annuli; the "moreover" also follows from orthogonality as in the proof of Lemma 3.17(2). □

Lemma 3.19 ([16, Section 2.5]).

(1) *If b is an open ball, or a properly infinite intersection of balls, and b' a closed ball, then p_b, $p_{b'}$ are orthogonal.*

(2) *Any b-definable map to $\mathbf{k}$ is constant on b away from a proper subball of b.*

Proof. We recall the proof from [16, Section 2.5]: The statement becomes stronger if the base set is enlarged. Thus we may assume that b and b' are centered; by translating we may assume both are centered at 0, and by a multiplicative renormalization that b' is the unit closed ball. Thus

$$c \models p_{b'}|A \quad \text{iff} \quad c \in \mathcal{O} \quad \text{and} \quad \mathrm{res}(c) \notin \mathrm{acl}(A). \tag{$*$}$$

On the other hand, let p_Γ be the type of elements of Γ that are just bigger than the valuative radius of b (cf. Example 3.3). Then $d \models p_b|A$ iff $\mathrm{val}(d) \models p_\Gamma$, i.e., p_b is now the type P described in Lemma 3.17. By Lemma 3.17, if $c' \in P$ then $\mathbf{k}(A(c')) = \mathbf{k}(A)$. It follows that if $c \models p_{b'}|A$, then $\mathrm{res}(c) \notin \mathrm{acl}(\mathbf{k}(A(c')))$. By $(*)$ $c \models p_{b'}|A(c')$.

For the second statement, let g be a definable map $b \to \mathbf{k}$; by Lemma 3.17(3), g is constant on the generic type of b; by compactness, g is constant on b away from some proper subball of b. □

Lemma 3.20. *Let* $a = (a_1, \dots, a_n) \in \mathrm{RV}^n$*, and assume* $a_i \notin \mathrm{acl}(A(a_1, \dots, a_{i-1}))$ *for* $1 \le i \le n$*. Then the formula* $D(x) = \bigwedge_{i=1}^n \mathrm{rv}(x_i) = a_i$ *generates a complete type over* $A(a)$*, and, indeed, over any* $\mathrm{RV} \cup \Gamma$*-generated structure* A'' *over* A*.*

In particular, if $q = \mathrm{tp}(a/A)$*, any* A*-definable function* $f : \mathrm{rv}^{-1}(q) \to \mathrm{RV} \cup \Gamma$ *factors through* $\mathrm{rv}(x) = (\mathrm{rv}(x_1), \dots, \mathrm{rv}(x_n))$.

Proof. This reduces inductively to the case $n = 1$. If we replace A by a bigger set M (such that $a_i \notin \mathrm{acl}(A(a_1, \dots, a_{i-1}))$ for $1 \le i \le n$), the assertion becomes stronger; so we may assume $A = M \models \mathbf{T}$. Let $\mathrm{rv}(c) = \mathrm{rv}(c') = a$. Either $\mathrm{val}(c) = \mathrm{val}(c') \notin M$, or else $\mathrm{val}(c) = \mathrm{val}(c') = \mathrm{val}(d)$ for some $d \in M$, and $\mathrm{res}(c/d) = \mathrm{res}(c'/d) \notin M$; in either case, by Lemma 3.17 or Lemma 3.18, we have $\mathrm{tp}(c/M) = \mathrm{tp}(c'/M)$. Thus $\mathrm{tp}(c, \mathrm{rv}(c)/M) = \mathrm{tp}(c', \mathrm{rv}(c')/M)$, i.e., $\mathrm{tp}(c/M(a)) = \mathrm{tp}(c'/M(a))$. This proves completeness over $A(a)$.

Let A' be a structure generated over A by finitely many elements of Γ. Then $A'(a) = A(\gamma_1, \dots, \gamma_k, a)$, where $\gamma_i \in \Gamma$, and $\gamma_i \notin A(\gamma_1, \dots, \gamma_{i-1}, \mathrm{val}(a))$. It follows that $\mathrm{rv}(a) \notin A(\gamma_1, \dots, \gamma_k)$, so $D(x)$ generates a complete type over $A(\gamma_1, \dots, \gamma_k)(a) = A'(a)$.

Let A'' be generated over $A'(a)$ by elements of stable A-definable sets. Since $D(x)$ is the (unique, and therefore) generic type of an open ball over $A'(a)$, by Lemma 3.17, it generates a complete type over A''.

Now if $A'' = A(\gamma_1, \dots, \gamma_k, r_1, \dots, r_n, d)$, where $\gamma_j \in \Gamma$, $r_i \in \mathrm{RV}$ and d lies in a stable set over A, let $A' = A(\gamma_1, \dots, \gamma_k, \mathrm{val}_{\mathrm{rv}}(r_1), \dots, \mathrm{val}_{\mathrm{rv}}(r_n))$; then A'/A is Γ-generated, and A''/A is generated by elements of stable sets (including $\mathrm{val}_{\mathrm{rv}}{}^{-1}(r_i)$). Thus the above applies.

The last statement follows by applying the first part of the lemma over $A'' = A(f(c))$: the formula $f(x) = f(c)$ must follow from the formula $D(x)$, since $D(x)$ generates a complete type over A''. □

3.3 Definable sets in group extensions

We will analyze the structure of RV in a slightly more abstract setting. In the following lemmas we assume R is a ring, and $0 \to A \to B \to C \to 0$ is a definable exact sequence of R-modules in T. This means that A, B, C are definable sets, and that one is also given definable maps $+_A : A^2 \to A$, $f_A^r : A \to A$ for each $r \in R$, and similarly for B, C; and definable maps $\iota : A \to B, \vartheta : B \to C$, such that in any $M \models T$, $A(M)$, $B(M)$, $C(M)$ are R-modules under the corresponding functions, and $0 \to A(M) \to_\iota B(M) \to_\vartheta C(M) \to 0$ is an exact sequence of homomorphisms of R-modules.

Lemma 3.21. *Consider a theory with a sequence* $0 \to A \to B \to_\vartheta C \to 0$ *of definable* R*-modules and homomorphisms* (*carrying additional structure*)*. Assume the following:*

(1) A, C *are stably embedded and orthogonal.*
(2) *Every almost definable subgroup of* A^n *is defined by finitely many* R*-linear equations.*

(3) *("No definable quasi-sections.") If P is a definable subset of B^n whose projection to C^n is finite-to-one, then P is finite.*

Then every almost definable subset Z of B^n is a finite union of sets of the form

$$\{b : \vartheta(b) \in W, Nb \in Y\},$$

where $N \in B_{n,k}(R)$ is an $n \times k$ matrix, Y is an almost definable subset of a single coset of A^k, W is an almost definable subset of C^n.

Note the following:

(1) To verify (3), it suffices to check it for $n = 1$ but for parametrically definable P.
(2) If C is definably linearly ordered, and Z is definable, then Y, W may be taken definable.

Proof. Using a base change as in Section 2.1, we may assume almost definable sets are definable. Replacing B by B^n and R by $M_n(R)$, we may assume $n = 1$. Let Z be a definable subset of B. Given $X \subset A$, let $[X]$ denote the class of X up to translation; so $[X] = [X']$ if $X = X' + a$ for some $a \in A$. Now a definable subset U of a coset $b + A$ of A has the form $b + X$, $X \subset A$; the class $[X]$ is well defined, and we will denote $[U] = [X]$. We obtain a map

$$c \mapsto [Z \cap \vartheta^{-1}(c)].$$

In more detail, for any $b \in (\vartheta^{-1}(c) \cap Z)$, we have $(\vartheta^{-1}(c) \cap Z) - b \subseteq A$, and so by stable embeddedness of A we can write $(\vartheta^{-1}(c) \cap Z) - b = X(a)$ for some $a \in A^m$. The tuple a is not well defined; but the class of a in the definable equivalence relation

$$x \sim x' \iff (\exists t \in A)(t + X(x)) = X(x')$$

is obviously a function of c alone. By the orthogonality assumption, this map is piecewise constant. Thus we may assume it is constant and fix C_0 with $[Z \cap \vartheta^{-1}(c)] = [C_0]$. Let S be the stabilizer $S = \{a \in A : a + C_0 = C_0\}$. Then for $a \in S$, $a + (Z \cap \vartheta^{-1}(c)) = (Z \cap \vartheta^{-1}(c))$ for any $c \in C$, so that also $S = \{a \in A : a + Z = Z\}$, and S is definable.

Now $Z \cap \vartheta^{-1}(c) = C_0 + f(c)$ for some $f(c) \in \vartheta^{-1}(c)$; $f(c) + S$ is well defined.

By assumption (2), $S = \mathrm{Ker}(r_1) \cap \ldots \cap \mathrm{Ker}(r_m)$ for some $r_i \in R$. Let $I = \{r_1, \ldots, r_m\}$. For $r \in I$, $f_r(c) := rf(c)$ is a well-defined element of B, and for all $c \in \vartheta(Z)$, $r(Z \cap \vartheta^{-1}(c)) = rC_0 + f_r(c)$.

We have $\vartheta f_r(c) = rc$. If $d \in \mathrm{Ker}(r : C \to C)$, then $f_r(d + c) = rc$ also, so $f_r(d + c) - f_r(c) \in A$. By orthogonality, for fixed r, $f_r(d + c) - f_r(c)$ takes finitely many values as c, d vary in C. In other words, $\{rf(c) : c \in \vartheta(Z)\}$ is a quasi-section above $r\vartheta(Z)$. By (3), $r\vartheta(Z)$ is finite, for each $r \in I$. Let $N = (r_1, \ldots, r_m)$, $Y' = NZ$. Then $\vartheta(Y')$ is finite. It follows that Y' is contained in a finite union of cosets of A, so C, Y' are orthogonal.

Thus $\{(\vartheta(z), Nz) : z \in Z\}$ is a finite union of rectangles; upon dividing Z further, we may assume this set is a rectangle $W \times Y$. Now if $\vartheta(b) \in W$ and $Nb \in Y$ then for some $z \in Z$, $\vartheta(b) = \vartheta(z)$ and $Nb = Nz$; it follows that $b - z \in A$ and $b - z \in S$; so $b \in S + Z = Z$. Thus Z is of the required form. □

Corollary 3.22. *Let T be a complete theory in a language L satisfying the assumptions of Lemma* 3.21. *Let $L \subseteq L'$, $T \subseteq T'$, and assume (1)–(3) persist to T'. If T, T' induce the same structure on A and on C, up to constants they induce the same structure on B, i.e., every T-definable subset of B^* is parameterically T'-definable.*

Proof. Apply Lemma 3.21 to T', and note that every definable set in the normal form obtained there is already parametrically definable in T. □

We will explicitly use imaginaries in RV only rarely; but our ability to work with RV, using Γ as an auxiliary, is partly explained by the following.

Corollary 3.23. *Let $0 \to A \to B \to_{\vartheta} C \to 0$ be as in Lemma* 3.21, *and assume C carries a definable linear ordering. Let $\bar{V}$ be the disjoint union of the definable cosets of A in B, with structure induced from T. Let e be an imaginary element of B. Then $\langle e \rangle = \langle (a', c') \rangle$ for some pair (a', c') consisting of an imaginary of $\bar{V}$ and an imaginary of C. Thus if $\bar{V}$, C eliminate imaginaries, so does $B \cup C \cup \bar{V}$.*

Proof. Let e be an imaginary element of B; let E_0 be the set of A, $\bar{V}$-imaginaries that are algebraic over e.

By Lemma 3.21, applied to a definable set with code e in the theory T_{E_0}, there exist almost definable subsets of $\bar{V}$, C^n from which e can be defined. These are coded by imaginaries permitted in the definition of E_0. Thus e is E_0-definable. Thus $e = g(d)$ for some definable function g and some tuple d from E_0. □

Let us now show that e is equidefinable with a finite set, i.e., an imaginary of the form $(f_1, \dots, f_n)/\operatorname{Sym}(n)$. Let W be the set of elements with the same type as d over e; W is finite by the definition of E_0, and is e-definable. But $e = g(w)$ for any element $w \in W$, so e is definable from $\{W\}$.

It remains to see that every finite set of elements of E_0 is coded by imaginaries of A and C and elements of B. Since C is linearly ordered, it suffices to consider finite sets whose image in C^m consists of one point. These are subsets of some definable coset of A^m, so again by elimination of imaginaries there they are coded. □

Corollary 3.24. *The structure induced on* $\mathrm{RV} \cup \Gamma$ *from* ACVF *eliminates imaginaries.*

Proof. Γ_{E_0} eliminates imaginaries, and so does ACF (cf. [31]). Note that $\bar{V}$ is essentially a family of one-dimensional **k**-vector spaces, closed under tensor products and roots and duals. Hence by [18], $\bar{V}_{E_0}$ eliminates imaginaries, too. Our only application of this lemma will be in a situation when parameters can be freely added; in this case, it suffices to quote elimination of imaginaries in ACF. □

Corollary 3.25. *Let T be a theory as in Lemma* 3.21, *with $R = \mathbb{Z}$, and C a linearly ordered group. Then every definable subset of B^n is a disjoint union of* $\mathrm{GL}_n(\mathbb{Z})$-*images of products $Y \times \vartheta^{-1}(Z)$, with $|\vartheta Y| = 1$. In particular, the Grothendieck semiring $K_+(B)$ (with respect to the category of all definable sets and functions of B) is generated by the classes of elements $Y \subset B^n$ with $|\vartheta Y| = 1$, and pullbacks $\vartheta^{-1}(Z)$, $Z \subset C^m$.*

Proof. By Lemma 3.21, the Grothendieck ring is generated by classes of sets X of the form $X = \{b \in B^n : \vartheta(b) \in W, Nb \in Y\}$. After performing row and column operations on the matrix N, we may assume it is the composition of a projection $p : R^n \to R^k$ with a diagonal $k \times k$ integer matrix with nonzero determinant. The composition $\vartheta p(X)$ is finite; since C is ordered, each element of $\vartheta p(X)$ is definable, and so we may assume $\vartheta p(X)$ has one element e. Thus $W = \{(e) \times W'\}$ for some W', and $X = pX \times \vartheta^{-1}(W')$. □

Lemma 3.26. *Let T be a theory, and let $0 \to A \to B \to_\vartheta C \to 0$ be an exact sequence of definable Abelian groups and homomorphisms. If $E \leq M \models T$, we will write $E_A = A(E)$, etc. Assume the following:*

(1) *A, C are orthogonal.*
(2) *Any parametrically definable subset of B is a Boolean combination of sets Y with $\vartheta(Y)$ finite, and of full pullbacks $\vartheta^{-1}(Z)$.*
(3) *C a uniquely divisible Abelian group, and for any $E \leq M \models T$, every divisible subgroup containing E_C is algebraically closed in C over E.*
(4) *For any prime $p > 0$, $T \models (\exists x \in A)(px = 0, x \neq 0)$.*

Let $Z \subset C^n$ and $f : Z \to C$ be definable, and suppose there exists E and E-definable $X \subset B^n$ and $F : X \to B$ lifting f: $\vartheta X = Z, \vartheta F(x) = f(\vartheta x)$. Then there exists a partition of Z into finitely many definable sets Z_ν, such that for each ν, for some $m \in \mathbb{Z}^n$, $f(x) - \sum_{i=1}^n m_i x_i$ is constant on Z_ν.

The main point is the integrality of the coefficients m_i.

Proof. It suffices to show that for any $M \models T$ and any $c = (c_1, \dots, c_n) \in Z(M)$, there exists $m = (m_1, \dots, m_n) \in \mathbb{Z}$ such that $f(c) - mc \in E^0$, where $E^0 = \mathrm{dcl}(\emptyset)$ is the smallest substructure of M. For if so, there exists a formula of one variable of sort C, such that $T \models (\exists^{\leq 1} z)\psi(z)$, $M \models \psi(f(c) - mc)$. By compactness there exists a finite set F of such pairs $\nu = (m, \psi)$, such that for any $M \models T$ and $c \in Z(M)$, for some $(m, \psi) \in F$, $M \models \psi(f(c) - mc)$; the required partition is given by $X_{m,\psi} = \{z \in Z : \psi(f(z) - mz)\}$.

Fix M and $c \in Z(M)$. Let $\langle c\rangle$ be the smallest divisible subgroup of $C(M)$ containing E^0_C and $c_1, \dots, c_n$. By (3), $\langle c\rangle$ is closed under f, so $f(c) \in \langle c\rangle$, i.e., $f(c) = \sum \alpha_i c_i + d$ for some $\alpha_i \in \mathbb{Q}$ and some $d \in E^0_C$. The only problem is to show that we can take $\alpha_i \in \mathbb{Z}$.

We will use induction on n. Let $K = \{\beta \in \mathbb{Q}^n : \beta \cdot c \in E^0_C\}$. K is a $\mathbb{Q}$-subspace of $\mathbb{Q}^n$. If $K \neq (0)$, there exists a primitive integral vector $\beta_1 \in K$. β_1 may be completed to a basis for a $\mathbb{Z}$-lattice in $\mathbb{Q}^n$. Applying a $\mathrm{GL}_n(\mathbb{Z})$ change of variables to B^n, we may assume $\beta_1 = (1, 0, \dots, 0)$, i.e., $c_1 \in E^0_C$. But then let $f'(z_2, \dots, z_n) = f(c_1, z_2, \dots, z_n)$. Then f' lifts to a definable function on B^n (with parameters, of the form $F(b_1, y_2, \dots, y_n)$) so by induction, $f(c_1, \dots, c_n) = f'(c_2, \dots, c_n) = \sum_{i \geq 2} m_i z_i + d'$ for some $m_2, \dots, m_n \in \mathbb{Z}$ and $d' \in E^0_C$, as required.

Thus we can assume $K = (0)$.

We can find $m, m_i \in \mathbb{Z}$, $e \in \mathrm{dcl}(\emptyset)$ with

$$mf(c) = \sum m_i c_i + e.$$

If $m|m_i$ we are done. We will now derive a contradiction from the contrary assumption that m does not divide each m_i in such an equation, with f a liftable function. We may assume that the greatest common divisor of $m, m_1, \dots, m_n$; so there exists a prime dividing m but not (say) m_1.

Let $g(x) = f(x, c_2, \dots, c_n) - e/m - \sum_{i=2}^{n} m_i c_i/m$; then $mg(c_1) = m_1 c_1$, m does not divide m_1, g is $E = \text{acl}(c_2, \dots, c_n)$-definable and liftable. Since $K = (0)$, by assumption (3), $c_1 \notin \text{acl}(E)$. Let $E' \supset E$ be such that g lifts to an E'-definable function G'. Enlarging the model if necessary, let c_1' realize $\text{tp}(c_1/E)$, with $c_1' \notin E'$ (cf. Example 2.4). Therefore, there exists E'' such that E'', c_1 and E', c_1' have the same type. In particular, g lifts to an E''-definable function G.

Consider any b_1 such that $\vartheta(b_1) = c_1$. Then $m\vartheta G(b_1) - m_1\vartheta(b_1) = 0$. Thus $mG(b_1) - m_1 b_1 \in A$.

Let p be prime, $p|m$ but $p \not| m_1$. Let $s, r \in \mathbb{Z}$ be such that $sp - rm_1 = 1$, and let $h(x) = sx - \frac{rm}{p} g(x)$. Then $ph(c_1) = psc_1 - rmg(c_1) = psc_1 - rm_1 c_1 = c_1$. Also h is liftable over E'': indeed, if G is E''-definable and lifts g, then $H(x) = sx - \frac{rm}{p} G(x)$ lifts h.

Thus $pH(b_1) = b_1 + d$, some $d \in A$. Let $b_2 = H(b_1)$; then $b_1 = pb_2 - d$, or

$$b_2 = H(pb_2 - d).$$

Now let $c_2 = h(c_1) = \vartheta(b_2)$. Then $pc_2 = c_1$, and so $c_2 \notin \text{acl}(E'')$, since by unique divisibility $c_1 \in \text{acl}(h(c_1))$. By (1), $c_2 \notin \text{acl}(E''(d))$. Let $C_2 = \vartheta^{-1} c_2$. By (2), any $E''(d)$-definable set either contains C_2 or is disjoint from C_2. Hence for any $y \in C_2$, $y = H(py - d)$.

By (4) there exists $0 \neq \omega_p \in A$ with $p\omega_p = 0$. Let $b_2' = b_2 + \omega_p$. Then $b_2 \in C_2$, so $b_2' = H(pb_2' - d)$. But $pb_2' = pb_2$, so $b_2 = b_2'$ and $\omega_p = 0$, a contradiction. □

Remark 3.27.

(1) It follows from Lemma 3.26 that a definable bijection between subsets of C^n that lifts to subsets of B^n is piecewise given by an element of $\text{GL}_n(\mathbb{Z}) \ltimes C^n$ (cf. Lemma 3.28).
(2) Assumption (4) on torsion does not hold in characteristic $p > 0$ for the sequence $\mathbf{k}^* \to \text{RV} \to \Gamma$. In this case there is l-torsion for $l \neq p$, but no p-torsion, and the corresponding group is $\text{GL}_n(\mathbb{Z}[1/p]) \ltimes C^n$.

Note as a corollary that there can be no definable sections of $B \to C$ over an infinite definable subset of C.

Lemma 3.28. *Let* $0 \to A \to B \to C \to 0$ *be as in Lemma* 3.26. *Let* $X \subset B^n$ *be definable, and let* $f : X \to B^l$ *be a definable function.* X *may be partitioned into finitely many pieces* X', *such that on each* X',

(1) $f(x) = Mx + b(x)$, *where* M *is a* $l \times n$*-integer matrix and* $\vartheta b(x)$ *is constant;*
(2) *there exists* $g \in \text{GL}_n(\mathbb{Z})$ *such that* $b \circ g$ *factors through a projection* $B^n \to_\pi B^k$, *where* $\vartheta\pi(X')$ *is one point of* C^k.

Proof. We first prove (1)–(2) for complete types.

(1) This reduces to $l = 1$. Let P be a complete type of elements of X. Then on P we have $\vartheta \circ f(x) = \sum m_i \vartheta(x_i) + d$ for some constant d (Lemma 3.26).

Thus $f(x) = \sum m_i x_i + b(x)$, where $b(x) = f(x) - \sum m_i x_i$, and $\vartheta b(x) = d$ is constant.

(2) Let $\pi : B^n \to B^k$ be a projection such that $\vartheta\pi(X)$ is one point of C^k, and with k maximal. Thus $P \subset P' \times P''$, $P' \subset B^{n-k}$, $P'' \subset B^k$, and $\vartheta(P'')$ is a single point of C^k, while $\vartheta(P')$ is not contained in any proper hypersurface $\sum n_i x_i =$ constant with $n_i \in \mathbb{Z}$. Pick $b'' \in P''$. Let $\gamma = (\gamma_1, \dots, \gamma_k) \in \vartheta(P')$, γ not in any such hypersurface. Let $a = (a_1, \dots, a_k)$, $\vartheta(a_i) = \gamma_i$, and let a' be another point with $\vartheta(a') = \gamma$. Let $e = f(a, b)$. Then $\mathrm{tp}(a/b, e) = \mathrm{tp}(a'/b, e)$, so $f(a', b) = e$. Thus $f(a, b)$ depends only on $b \in P''$ and not on a (with $(a, b) \in P$).

Since (1)–(2) hold on each complete type, there exists a definable partition such that they hold on each piece. □

3.4 V-minimality

We assume from now on that **T** is a theory of C-minimal valued fields, of residue characteristic 0. When using the many-sorted language, we will still say that **T** is *a theory of valued fields* when $\mathbf{T} = \mathrm{Th}(F, \mathrm{RV}(F))$ for some valued field F, possibly with additional structure. A C-minimal **T** satisfying assumption (3) below will be said to have *centered closed balls*. If, in addition, (1)–(2) hold, we will say **T** is V-*minimal*. Expansions by the definition of the language, i.e., the addition of a relation symbol $R(x)$ to the language along with a definition $(\forall x)(R(x) \iff \phi(x))$ to the theory, do not change any of our assumptions. Thus we can assume that **T** eliminates quantifiers.

(1) *Induced structure on* RV. **T** contains ACVF(0, 0), and every parametrically **T**-definable relation on RV* is parametrically definable in ACVF(0, 0).
(2) *Definable completeness.* Let $A \leq M \models T$, and let $W \subset \mathfrak{B}$ be a $\mathbf{T}_A$-definable family of closed balls linearly ordered by inclusion. Then $\cap W \neq \emptyset$.
(3) *Choosing points in closed balls.* Let $M \models \mathbf{T}$, $A \subseteq \mathrm{VF}(M)$, and let b be an almost A-definable closed ball. Then b contains an almost A-definable point.

T will be called *effective* if every definable finite disjoint union of balls contains a definable set, with exactly one point in each. A substructure A of a model of **T** will be called effective if $\mathbf{T}_A$ is effective.

If every definable finite disjoint union of rv-balls contains a definable set, with exactly one point in each, we can call **T** rv-effective. However, we have the following.

Lemma 3.29. *Let* **T** *be* V*-minimal. Then* **T** *is effective iff it is* rv*-effective.*

Proof. Assume **T** is rv-effective. Let b be an algebraic ball. If b is closed, it has an algebraic point by assumption (3) of Section 3.4. If b is open, let $\bar{b}$ be the closed ball surrounding it. Then $\bar{b}$ has an algebraic point a. Let $f(x) = x - a$. Then $f(b)$ is an rv-ball, so by rv-effectivity it has an algebraic point a'. Hence $a' + a$ is an algebraic point of b. □

In general, effectivity is needed for lifting morphisms from RV to VF, not for the "integration" direction.

If $\mathbf{T}$ is V-minimal and A is a $\mathrm{VF} \cup \mathrm{RV} \cup \Gamma$-generated structure, we will see that $\mathbf{T}_A$ is V-minimal, too. The analogue for points in open balls is true but only for $\mathrm{VF} \cup \Gamma$-generated substructures; for thin annuli it is true only for VF-generated structures. For this reason the condition on closed balls is more flexible; luckily we will be able to avoid the others.

Lemma 3.30. *Let* $\mathbf{T}$ *be a* C*-minimal theory of valued fields. Then* (1) $\implies$ (2) $\implies$ (3) $\implies$ (4):

(1) $\mathbf{T}$ *admits quantifier elimination in a three-sorted language* $(\mathrm{VF}, \mathbf{k}, \Gamma)$,*such that for any basic function symbol* F *with range* VF*, the domain is a power of* VF; *and no relations on* $\mathbf{k}$, Γ *beyond the field structure on* $\mathbf{k}$ *and the ordered Abelian group structure on* Γ.
(2) *Every parametrically definable relation on* $\mathbf{k}$ *is parameterically definable in* ACF(0), *and every parametrically definable relation on* Γ *is parameterically definable in* DOAG.
(3) *Assumption* (1) *of Section* 3.4.
(4) $\mathbf{k}$, Γ, *and* RV *are stably embedded.*

Proof.

(1) $\implies$ (2) Let $\phi(a, x)$ be an atomic formula with paramaters $a = (a_1, \dots, a_n)$ from VF and $x = (x_1, \dots, x_m)$ variables for the $\mathbf{k}, \Gamma$ sorts. Then ϕ must have the form $\psi(t(a), x)$, where t is a term (composition of function symbols) $\mathrm{VF}^* \to (\mathbf{k} \cup \Gamma)$. Thus $\phi(a, x)$ defines the same set as $\psi(b, x)$ where $b = t(a)$. Since every formula is a Boolean combination of atomic ones, (2) follows.
(2) $\implies$ (3) This follows from Corollary 3.22. The assumptions of Lemma 3.26 are satisfied: (1) is automatic since by C-minimality $\mathbf{k}$ is strongly minimal and Γ is O-minimal; (2) follows from C-minimality; (3)–(4) follow from the assumptions on $\mathbf{k}, \Gamma$.
(3) $\implies$ (4) This is immediate. □

Lemma 3.31. *Let* $\mathbf{T}$ *be a theory of valued fields satisfying assumption* (1) *of Section* 3.4, *such that* res *induces a surjective map on algebraic points. Then* (1) $\implies$ (2) $\implies$ (3) $\implies$ (4):

(1) *For any* VF*-generated substructure* A *of a model* M *of* $\mathbf{T}$*, if* $\Gamma(A) \neq (0)$*, then* $\mathrm{acl}(A) \models \mathbf{T}$.
(2) *For any* VF*-generated substructure* A *of a model of* $\mathbf{T}$, *any* $\mathbf{T}_A$*-definable nonempty finite union of balls contains a nonempty* $\mathbf{T}_A$*-definable finite set.*
(3) *Assumption* (3) *of Section* 3.4 *holds.*
(4) *Let* A *be* VF*-generated, and* Y *a finite* A*-definable set of disjoint closed balls. Then there exists an* A*-definable finite set* Z *such that* $|b \cap Z| = 1$ *for each* $b \in Y$.

Proof. We first show the following.

Claim. For any VF-generated A with $\Gamma(A) = (0)$, res : VF(acl(A)) $\to$ $\mathbf{k}$(acl(A)) is surjective.

Proof. It suffices to prove the claim for finitely generated A. For $A = \emptyset$ this is true by assumption. Using induction on the number of generators, it suffices to show that if the claim holds for A_0 and $c \in$ VF then it holds for $A = A_0(c)$).

Since $\Gamma(A) = (0)$, res is defined and injective on VF(A). If $c \in \mathrm{acl}(A_0)$ there is nothing to prove. Otherwise, by injectivity, $\mathrm{res}(c) \notin \mathrm{acl}(A_0)$. As a consequence of assumption (1) of Section 3.4, both dcl and acl agree with the corresponding field-theoretic notions on RV and, in particular, on the residue field.

By Lemma 3.20,

$$\mathbf{k}(A_0(c)) \subseteq \mathrm{dcl}(\mathrm{RV}(A_0), \mathrm{rv}(c)) = \mathrm{dcl}(\mathbf{k}(A_0), \mathrm{res}(c)) = \mathbf{k}(A_0)(\mathrm{res}(c)).$$

Now if $d \in \mathbf{k}(\mathrm{acl}(A))$ then $d \in \mathbf{k}$ and $d \in \mathrm{acl}(A)$, so by stable embeddedness of $\mathbf{k}$, we have $d \in \mathrm{acl}(\mathbf{k}(A))$; but $\mathrm{acl}(\mathbf{k}(A)) = \mathbf{k}(A)^{\mathrm{alg}}$ by assumption (1) of Section 3.4; so $d \in \mathbf{k}(A_0)(\mathrm{res}(c))^{\mathrm{alg}} \subseteq \mathrm{res}(A_0(c)^{\mathrm{alg}})$. □

Assume (1). If $\Gamma(\mathrm{acl}(A)) \neq (0)$, then by (1) $\mathrm{acl}(A) \models T$ and, in particular, every $\mathrm{acl}(A)$-definable ball has a point in $\mathrm{acl}(A)$, so (2) holds. Assume therefore that $\Gamma(\mathrm{acl}(A)) = 0$. Let b be an $\mathrm{acl}(A)$-definable ball. Then b must have valuative radius 0. If some element of b has valuation $\gamma < 0$ then all do, and $\gamma \in A$, a contradiction. Thus b is the (open or closed) ball of radius 0 around some $c \in \mathcal{O}$. If b is closed, then $b = \mathcal{O}$ and $0 \in b$. If b is open, then $b = \mathrm{res}^{-1}(b')$ for some element b' of the residue field $\mathbf{k}$; in this case b has an $\mathrm{acl}(A)$-definable point by the claim.

(3) is included in (2), being the case of closed balls.

Assume (3). In expansions of ACVF(0, 0), the *average* of a finite subset of a ball remains within the ball. Thus if Y is a finite A-definable set of disjoint balls, by (3), there exists a finite A-definable set Z_0 including a representative of each ball in Y. Let $Z = \{\mathrm{av}(b \cap Z_0) : b \in Y\}$, where $\mathrm{av}(u)$ denotes the average of a finite set u. □

Lemma 3.32. *When* **T** *is a complete theory, definable completeness is true as soon as* T *has a single* spherically complete *model* M *in the sense of Ribenboim and Kaplansky: every intersection of nested closed balls is nonempty.*

Proof. The proof is clear. □

Let $\mathrm{ACVF}^{\mathrm{an}}$ denote any of the rigid analytic theories of [23]. For definiteness, let us assume the power series have coefficients in $\mathbb{C}((X))$. See [14] for variants living over $\mathbb{Z}_p$.

Lemma 3.33. ACVF(0, 0) *is* V*-minimal and effective. Thus is* $\mathrm{ACVF}^{\mathrm{an}}$.

Proof. C-minimality is proved in [24]. Lemma 3.30(1) for ACVF is a version of Robinson's quantifier elimination; cf. [16].

$\mathrm{ACVF}^{\mathrm{an}}$ admits quantifier elimination in the sorts (VF, Γ) by [23, Theorem 3.8.2]. The residue field sort is not explicit in this language, but one can argue as follows. Let

$\mathbf{k}_1$ be a large algebraically closed field containing $\mathbb{C}$, and let $K = \cup_{n\geq 1}\mathbf{k}_1((X^{1/n}))$ be the Puiseux series ring. Then K admits a natural expansion to a model of the theory. K is not saturated, but by C-minimality the induced structure on the residue field is strongly minimal, so $\mathbf{k}_1$ is saturated. Now any automorphism of $\mathbf{k}_1$ as a field extends to an automorphism of K as a rigid analytic structure. Thus every K-definable relation on $\mathbf{k}_1$ is algebraic. (This could be repeated over a larger value group if necessary.) Lemma 3.30(2) thus holds in both cases; hence we have assumption (1) of Section 3.4.

Condition Lemma 3.31(1) is obviously true for ACVF. For ACVF^{an} it is proved in [24]. It is also evident that these theories have a spherically complete model. Thus by Lemmas 3.31 and 3.32, assumptions (3) and (2) of Section 3.4 hold, too. □

Remarks.

(1) Lemma 3.31(1)–(3) remain true for ACVF in positive residue characteristic, but (4) fails.
(2) ACVF(0, 0) also admits quantifier elimination in the two sorted language with sorts VF, RV; so assumption (1) of Section 3.4 can also be proved directly, without going through $\mathbf{k}$, Γ as in Lemma 3.30.
(3) Assumption (1) of Section 3.4 is needed for lifting definable bijections of RV to VF, Proposition 6.1, Lemma 6.3. Specifically, it implies the truth of assumptions (2) of Lemma 3.21 and (4) of Lemma 3.26. These lemmas are only needed for the injectivity of the Euler charactersitic and integration maps, not for their construction and main properties. It is also needed for the theory of differentiation and for comparing derivations in VF and RV; indeed, even for posing the question, since in general there is no notion of differentiation on RV. The theory of differentiation itself is needed neither for the Euler characteristic nor for integration of definable sets with a Γ-volume form. They are required only for the finer theory introduced here of integration of RV-volume forms.
(4) We know no examples of C-minimal fields where assumption (2) of Section 3.4 fails.
(5) Beyond effectivity of dcl(∅), assumption (3) of Section 3.4 imposes a condition on liftability of definable functions from VF to $\mathfrak{B}^{cl}$. Let $\mathbf{T}_1$ be the theory, intermediate between ACVF(0, 0) and a Lipshitz rigid analytic expansion, generated over ACVF(0, 0) by the relation

$$\mathrm{val}(f(t_0x) - y) \geq \mathrm{val}(t_1)$$

on $\mathcal{O}^2$, where t_0, t_1 are constants with $\mathrm{val}(t_1)\eth\,\mathrm{val}(t_0) > 0$ and f is an analytic function. It appears that balls do not necessarily remain pointed upon adding VF-points to $\mathbf{T}_1$; so assumption (3) of Section 3.4 is not redundant.

3.5 Definable completeness and functions on the value group

We assume $\mathbf{T}$ is C-minimal and definably complete. We show that the property of having centered closed balls is preserved under passage to $\mathbf{T}_A$ if A is RV, Γ, VF-generated; similarly for open balls if A is Γ, VF-generated. Also included is a lemma stating that every image of an RV-set in VF must be finite; from the point of view of content this belongs to the description of the "basic geography," but we need the lemmas on functions from Γ first.

Proposition 3.34. *Let $M \models \mathbf{T}$, $\gamma = (\gamma_1, \dots, \gamma_m)$ a tuple of elements of $\Gamma(M)$. Any almost $A(\gamma)$-definable ball b contains an almost A-definable ball b'.*

Proof. See [16, Proposition 2.4.4]. While the proposition is stated for ACVF there, the proof uses only C-minimality and definable completeness. We review the proof in the case that $b \in A(\gamma)$, i.e., $b = f(\gamma)$ for some definable function f with domain $D \subseteq \Gamma^M$.

Let $P = \mathrm{tp}(\gamma/A)$. Let $r(\gamma)$ be the valuative radius of $f(\gamma)$. By O-minimality, r is piecewise monotone; since P is a complete type, r is monotone, say, decreasing. For $a \in P$ let $P_a = \{b \in P : b < a\}$, and for $b \in P_a$ let $f_a(b)$ be the open ball of size $r(a)$ containing $f(b)$. By Lemma 3.15, the valuative radius map rad is finite-to-one on $f_a(P_a)$; but by definition it is constant, so $f_a(P_a)$ is finite. Using the linear ordering, $f_a(P_a)$ is constant on each complete type over a. Pick $b_1 \in P$, $\epsilon \in \Gamma$ with $\epsilon > 0$ but very small (over $A(b_1)$), and $\epsilon' \in \Gamma$ with $\epsilon' > 0$ but ϵ' very small (over $A(b_1, \epsilon)$). Let $b_2 = b_1 + \epsilon$, $a = b_2 + \epsilon'$. Then $\mathrm{tp}(b_1, a/A) = \mathrm{tp}(b_2, a/A)$, so $f_a(b_1) = f_a(b_2)$. Now if $f(b_1), f(b_2)$ are disjoint, let $\delta = \mathrm{val}(x_1 - x_2)$ for (some or any) $x_i \in f(b_i)$. Then $r(b_2) > \delta$. Since ϵ' is very small, $r(a) > \delta$ also. Thus $f_a(b_1)$, $f_a(b_2)$ are distinct, a contradiction. Thus $f(b_1) \subset f(b_2)$. Since $\mathrm{tp}(a/A) = \mathrm{tp}(b_2/A)$, we have $f(y) \subset f(a)$ for some $y \in P_a$. If $f(y) \subset f(a)$ for all $y \in P_a$, we are done; otherwise, let $c(a)$ be the unique smallest element such that f is monotone on $(c(a), a)$. We saw, however, that f is monotone on $(d, c(a))$ for some $d < c(a)$, hence also on (d, a), a contradiction. Thus f is monotone with respect to inclusion. By compactness, this is true on some A-definable interval, hence on some interval I containing P.

Let $U = \bigcap_{a \in I} f(a)$. By definable completeness (assumption (2) of Section 3.4), $U \neq \emptyset$. Clearly, U is a ball, and $U \subseteq b$. □

Lemma 3.35. *Let $M \models \mathbf{T}$, $\gamma = (\gamma_1, \dots, \gamma_m)$ a tuple of elements of $\Gamma(M)$. Then any $A(\gamma)$-definable ball contains an A-definable ball. If Y is a finite $A(\gamma)$-definable set of disjoint balls, then there exists a finite A-definable set Y' of balls, such that each ball of Y contains a unique ball of Y'.*

Proof. This reduces immediately to $m = 1$. For $m = 1$, by Proposition 3.34, any almost $A(\gamma)$-definable ball b contains an almost A-definable ball b'. Thus given a finite $A(\gamma)$-definable set Y of disjoint balls, there exists a finite A-definable set Z of balls, such that any ball of Y contains a ball of Z. Given $b \in Y$, let b' be the smallest ball containing every subball c of b with $c \in Z$. Then $Y' = \{b' : b \in Y\}$ is $A(\gamma)$-definable, finite, almost A-definable, and (since b'_1 is disjoint from b_2 if $b_1 \neq b_2 \in Y$) each ball of Y contains a unique ball of Y'. Using elimination of imaginaries in Γ, by Example 2.2, being $A(\gamma)$-definable and almost A-definable, Y' is A-definable. □

The following corollary of Lemma 3.35 concerning definable functions from Γ will be important for the theory of integration with an additive character in Section 11.

Corollary 3.36. *Let Y be a definable set admitting a finite-to-one map into Γ^n, and let into h be a definable map on Y into* VF *or* $\mathrm{VF}/\mathcal{O}$ *or* $\mathrm{VF}/\mathcal{M}$. *Then h has finite image.*

Proof. One can view h as a function from a subset of Γ^n into finite sets of balls. Since a ball whose radius is definable containing a definable ball is itself definable, Lemma 3.35 implies that $h(\gamma) \in \mathrm{acl}(\emptyset)$ for any $\gamma \in \Gamma^n$. By Lemma 2.6, the corollary follows. □

Corollary 3.37. *Let $Y \subseteq (\mathrm{RV} \cup \Gamma)^n$ and $Z \subseteq \mathrm{VF} \times Y$ be definable sets, with Z invariant for the action of $\mathcal{M}$ on* VF. *Then for all but finitely many $\mathcal{O}$-cosets C, $Z \cap (C \times Y)$ is a rectangle $C \times Y'$.*

Proof. Let $p : (\mathrm{RV} \cup \Gamma)^n \to \Gamma^n$ be the natural projection, and for $\gamma \in \Gamma^n$ let Z_γ be the fiber. For each γ, by Lemma 3.16, there exists a finite $F(\gamma) \subseteq \mathrm{VF}/\mathcal{O}$ such that for any $\mathcal{O}$-coset $C \notin F(\gamma)$, $Z_\gamma \cap (C \times Y)$ is $\mathcal{O}$-invariant. Now $\{(u, \gamma) : u \in F(\gamma)\}$ projects finite-to-one to Γ^n, so by Lemma 3.36, this set projects to a finite subset of $\mathrm{VF}/\mathcal{O}$. Thus there exists a finite $E \subset \mathrm{VF}/\mathcal{O}$ such that for any γ, and any $\mathcal{O}$-coset $C \notin E$, $Z_\gamma \cap (C \times Y)$ is $\mathcal{O}$-invariant. In other words, for any $C \notin E$, $Z \cap (C \times Y)$ is $\mathcal{O}$-invariant. □

Lemma 3.38. *Let $M \models \mathbf{T}$, A a substructure of M (all imaginary elements allowed), and let $r = (r_1, \ldots, r_m)$ be a tuple of elements of* $\mathrm{RV}(M) \cup \Gamma(M)$. *Then any closed ball almost defined over $A(r)$ contains a ball almost defined over A.*

Proof. This reduces to $m = 1$, $r = r_1$; moreover, using Lemma 3.35, to the case $r \in \mathrm{RV}(M)$, $\mathrm{val}_{\mathrm{rv}}(r) = \gamma \in A$. Let $E = \{y \in \mathrm{RV} : \mathrm{val}_{\mathrm{rv}}(y) = \gamma\}$. Then E is a k^*-torsor, and so is strongly minimal within M. If c is almost defined over $A(r)$, there exists an A-definable set $W \subset E \times \mathfrak{B}_{\mathrm{cl}}$, with $W(e) = \{y : (e, y) \in W\}$ finite, and $c \in W(r)$. But then W is a finite union of strongly minimals, and hence so is the projection P of W to $\mathfrak{B}_{\mathrm{cl}}$. But any strongly minimal subset of $\mathfrak{B}_{\mathrm{cl}}$ is finite. (Otherwise, it admits a definable map onto a segment in Γ; but Γ is linearly ordered and cannot have a strongly minimal segment.) Thus $c \in P$ is almost defined over A. □

Lemma 3.39. *Let $M \models \mathbf{T}$, $\mathbf{T}$ C-minimal with centered closed balls. Let B be substructure of* $\mathrm{VF}(M) \cup \mathrm{RV}(M) \cup \Gamma(M)$. *Then every B-definable closed ball has a B-definable point. If Y is a finite B-definable set of disjoint closed balls, there exists a finite B-definable set $Z \subset M$, meeting each ball of Y in a unique point.*

Proof. We may take B to contain a subfield K and be generated over K by finitely many points $r_1, \ldots, r_k \in RV$. Let Y be a finite B-definable set of disjoint closed balls, and let $b \in Y$. We may assume all elements of Y have the same type over B. By Lemma 3.38, there exists a closed ball b' defined almost over K and contained in b. By assumption (3) of Section 3.4, there exists a finite K-definable set Z' meeting b' in a unique point. Let $Y' = \{b'' \in Y : b'' \cap Z' \neq \emptyset\}$, and $Z = \{\mathrm{av}(Z' \cap b'') : b'' \in Y'\}$. Then Z meets each ball of Y' in a unique point, and Z, Y' are B-definable. As for $Y \setminus Y'$, it may be treated inductively. □

Corollary 3.40. *Let $M \models \mathbf{T}$, $\mathbf{T}$ C-minimal with centered closed balls, and effective. Let B be an almost Γ-generated substructure. Then $\mathbf{T}$ is effective.*

Proof. The proof is the same as the proof of Lemma 3.39, using Lemma 3.34 in place of Lemma 3.38. ☐

Lemma 3.41. *Let Y be a $\mathbf{T}$-definable set admitting a finite-to-one map into* RV^n. *Let $g : Y \to \mathrm{VF}^m$ be another definable map. Then $g(Y)$ is finite.*

Proof. It suffices to prove this for $\mathbf{T}_A$, where $A \models \mathbf{T}$. We may also assume $m = 1$. We will use the equivalence (3) $\iff$ (4) of Lemma 2.6. If $g(Y)$ is infinite, then by compactness there exists $a \in g(Y)$ $a \notin \mathrm{acl}(A)$. But for some b we have $a = g(b)$, so if $c = f(b)$, we have $c \in \mathrm{RV}^n$, $a \in \mathrm{acl}(c)$. Thus it suffices to show the following:

$$\text{If } a \in \mathrm{VF},\, c \in \mathrm{RV}^n \text{ and } a \in \mathrm{acl}(A(c)), \text{ then } a \in \mathrm{acl}(A). \qquad (*)$$

This clearly reduces to the case $n = 1$, $c \in \mathrm{RV}$. Let $d = \mathrm{val}_{\mathrm{rv}}(c)$, $A' = \mathrm{acl}(A(d))$. Then c lies in an A'-definable strongly minimal set S (namely, $S = \mathrm{val}_{\mathrm{rv}}{}^{-1}(d)$). Using Lemma 2.6 in the opposite direction, since $a \in \mathrm{acl}(A'(c))$ there exists a finite-to-one map $f : S' \to S$ and a definable map $g' : S' \to \mathrm{VF}$ with $a \in g'(f^{-1}(S'))$. By Corollary 3.13, $g'(f^{-1}(S'))$ is finite. Hence $a \in \mathrm{acl}(A(d))$. But then by Lemma 3.36, $a \in \mathrm{acl}(A)$. ☐

In particular, there can be no definable isomorphism between an infinite subset of RV^n and one of VF^m.

Lemma 3.42. *Let $M \models \mathbf{T}$, $\mathbf{T}$ C-minimal with centered closed balls, and let A be a substructure of M. Write A_{VF} for the field elements of A, A_{RV} for the* RV*-elements of A.*

Let $c \in \mathrm{RV}(M)$, and let $A(c) = dcl(A \cup \{c\})$. Then $A(c)_{\mathrm{VF}} \subset (A_{\mathrm{VF}})^{\mathrm{alg}}$, and $\mathrm{rv}(A(c)_{\mathrm{VF}}) \cap A_{\mathrm{RV}} = \mathrm{rv}(A_{\mathrm{VF}})$.

Proof. Let $e \in A(c)_{\mathrm{VF}}$. Then $e = f(c)$ for some A-definable function $f : W \to \mathrm{VF}$, $W \subseteq \mathrm{RV}$. By Lemma 3.41, the image of f is finite, $e \in \mathrm{acl}(A)$. This proves the first point. Now if $d \in \mathrm{RV}_A$ and $\mathrm{rv}^{-1}(d)$ has a point in $A(c)$, then it has a point in $(A_{\mathrm{VF}})^{\mathrm{alg}}$, by assumption (3) of Section 3.4. ☐

3.6 Transitive sets in dimension one

Let b be a closed ball in a valued field. Then the set Aff (b) of maximal open subballs of b has the structure of an affine space over the residue field. We will now begin using this structure. Without it, more general transitive annuli (missing more than one ball) could exist.

Lemma 3.43. *Let $X \subseteq \mathrm{VF}$ be a transitive $\mathbf{T}_B$-definable set, where B is some set of imaginaries. Then X is a finite union of open balls of equal size, or a finite union of closed balls of equal size, or a finite union of thin annuli.*

Proof. By C-minimality, X is a finite Boolean combination of balls. There are finitely many distinct balls $b_1, \dots, b_n$ that are almost contained in X (i.e., $b_i \setminus X$ is contained in a finite union of proper subballs of b_i) but such that no ball larger than b_i is almost contained in X. These b_i must be disjoint. If some of the b_i have different type than the others, their union (intersected with X) will be a proper B-definable subset of X. Thus they all have the same type over B; in particular, they have the same radius β.

Consider first the case where the balls b_i are open. Then $b_i \subseteq X$. Otherwise, $b_i \setminus X$ is contained in a unique smallest ball c_i. Say c_i has radius α; then $\alpha > \beta$. Let b_i' be the open ball of radius $(1/2)(\alpha + \beta)$ around c_i; then $\cup_i b_i'$ is a B-definable proper subset of X, a contradiction. Thus in the case of open balls, $X \supseteq \cup_i b_i$ and therefore $X = \cup_i b_i$.

If the balls b_i are closed, let c_{ij} be a minimal finite set of subballs of b_i needed to cover $b_i \setminus X$. The same argument shows that no c_{ij} has radius $< \beta$. Thus all c_{ij} are elements of the set V_i of open subballs of b_i of radius β. Now V_i is a $\mathbf{k}$-affine space, and if there is more than one c_{ij} then over $\operatorname{acl}(B)$, V_i admits a bijection with $\mathbf{k}$; so there is a finite B-definable set of bijections $V_i \to \mathbf{k}$; since any finite definable subset of $\mathbf{k}$ is contained in a strictly bigger one, the union of the pullbacks gives a B-definable subset of V_i properly containing the c_{ij}, leading to a proper B-definable subset of X. Thus either $b_i \subseteq X$ (and then $X = \cup_i b_i$), or else $b_i \setminus c_i \subseteq X$ for a unique maximal open subball c_i. Now $\cup c_i$ intersects X in a proper subset, which must be empty. Thus in this case $X = \cup_i (b_i \setminus c_i)$. □

Let X be a transitive B-definable set. Call $Y \subseteq X$ *potentially transitive* if there exists $B' \supset B$ such that Y is B'-definable and B'-transitive. Let $\mathcal{F}(X)$ be the collection of all proper potentially transitive subsets Y of X. Let $\mathcal{F}_{\max}(X)$ be the set of maximal elements of $\mathcal{F}(X)$.

Lemma 3.44.

(1) *If X is an open ball, $\mathcal{F}_{\max}(X) = \emptyset$.*
(2) *If X is a closed ball, $\mathcal{F}_{\max}(X) = \{X \setminus Y : Y \in \operatorname{Aff}(X)\}$.*
(3) *If X is a thin annulus $X' \setminus Y$ with X' closed, then $\mathcal{F}_{\max}(X) = \operatorname{Aff}(X) \setminus \{Y\}$.*

Proof. Any element of $\mathcal{F}(X)$ must be a ball or a thin annulus, so the lemma follows by inspection. □

Lemma 3.45. *Let b be a transitive closed ball (respectively, thin annulus). Let $Y = \operatorname{Aff}(b)$ be the set of maximal open subballs of b. Then the group of automorphisms of Y over $\mathbf{k}$ is definable, acts transitively on Y, and, in fact, contains $G_a(\mathbf{k})$ (respectively, $G_m(\mathbf{k})$).*

If b, b' are transitive definable closed balls, and $F : b \to b'$ a definable bijection, let $F_* : Y(b) \to Y(b')$ be the induced map. Then F_* is a homomorphism of affine spaces, i.e., there exists a vector space isomorphism $F_{**} : V(b) \to V(b')$ between the corresponding vector spaces, and $F_*(a + v) = F_*(a) + F_{**}(v)$. If $b = b'$ then $F_{**} = \operatorname{Id}$.

Proof. $Y = \mathrm{Aff}(b)$ is transitive, and there is a $\mathbf{k}$-affine space structure on Y (respectively, a $\mathbf{k}$-vector space structure on $V = Y' \dot\cup \{0\}$). Let $G = \mathrm{Aut}(Y/\mathbf{k})$ be the subgroup of the group $\mathrm{Aff} = (G_m \ltimes G_a)(\mathbf{k})$ of affine transformations of Y that preserve all definable relations. By definition, this is an intersection of definable subgroups of Aff. However, there is no infinite descending chain of definable subgroups of Aff, so G is definable.

If G is finite, then $Y \subseteq \mathrm{acl}(\mathbf{k})$, and it follows (cf. Section 2.1) that there are infinitely many algebraic points of Y, contradicting transitivity. Thus G is an infinite subgroup of $(G_m \ltimes G_a)(\mathbf{k})$ such that the set of fixed points Y^G is empty. Thus G must contain a translation, and by strong minimality it must contain $G_a(\mathbf{k})$. Similarly, in the case of the annulus, G is an infinite definable subgroup of $G_m(\mathbf{k})$, so it must equal $G_m(\mathbf{k})$.

As for the second statement, F induces a group isomorphism $\mathrm{Aut}(Y(b)/k) \to \mathrm{Aut}(Y(b')/k)$, and hence an isomorphism $G_a(k) \to G_a(k)$, which must be multiplication by some $\gamma \in \mathbf{k}^*$. Since $G_a(k)$ acts by automorphisms on $(Y(b), Y(b'))$, any definable function $Y(b) \to Y(b')$ commutes with this action and hence has the specified form. If $b = b'$ then $Y(b) = Y(b')$; now if $F_{**} \neq \mathrm{Id}$ then F_* would have a fixed point, contradicting transitivity. □

Lemma 3.46. *Let b be a transitive $\mathbf{T}_B$-definable closed (open) ball. Let F be a B-definable function, injective on b. Then $F(b)$ is a closed (open) ball.*

Proof. By Lemma 3.43, since $F(b)$ is also transitive, it is either a closed ball, or an open ball, or a thin annulus. We must rule out the possibility of a bijection between such sets of distinct types.

Consider the collection $\mathcal{F}_{\max}(b)$ defined above. Any definable bijection between b and b' clearly induces a bijection $\mathcal{F}_{\max}(b) \to \mathcal{F}_{\max}(b')$. By Lemma 3.44, the bijective image of an open ball is an open ball.

Let b be a closed ball, $b' = b'' \setminus b'''$ a closed ball minus an open ball, $A = \mathcal{F}_{\max}(b) \simeq \mathrm{Aff}(b)$, $A' = \mathcal{F}(b') \simeq \mathrm{Aff}(b'') \setminus \{b'''\}$, $G = \mathrm{Aut}(A/\mathbf{k})$, $G' = \mathrm{Aut}(A'/\mathbf{k})$. Then a definable bijection $A \to A'$ would give a definable group isomorphism $G \to G'$. But by Lemma 3.45, $G' = G_m(\mathbf{k})$ while G contains $G_a(\mathbf{k})$, so no such isomorphism is possible (say, because $G_m(\mathbf{k})$ has torsion points).

Thus the three types are distinct. □

We will see later that there can be no definable bijection between an open and a closed ball, whether transitive or not.

Lemma 3.47. *Let b be a transitive ball. Then every definable function on b into* $\mathrm{RV} \cup \Gamma$ *is constant. If b is a transitive thin annulus, every definable function on b into* $\mathbf{k} \cup \Gamma$ *is constant. More generally, this is true for definable functions into definable cosets C of* $\mathbf{k}^*$ *in* RV *that contain algebraic points.*

Proof. When a ball b is transitive, it is actually finitely primitive. For if E is a B-definable equivalence relation with finitely many classes, then exactly one of these classes is generic (i.e., is not contained in a finite union of proper subballs of b). This class is B-definable, hence must equal b.

Thus a definable function on b with finite image is constant.

Let F be a definable function on b into Γ. If F is not constant, then for some $\gamma \in \Gamma$, $F^{-1}(\gamma)$ is a proper subset of b; it follows that some finite union of proper subballs of b is γ-definable. By Lemma 3.35, it follows that some such finite union is already definable, a contradiction.

Thus it suffices to show that functions into a single coset $C = {\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma)$ of $\mathbf{k}^*$ are constant on b.

Assume first that b is open, or a properly infinite intersection of balls. By Lemma 3.19 definable functions on b into C are generically constant; but then by transitivity they are constant.

Now suppose b is closed, or a thin annulus. Let Y be the set of maximal open subballs b' of b. Each $b' \in Y$ is transitive over $\mathbf{T}_{b'}$, so $F|b'$ is constant. Thus F factors through Y.

In the case of the annulus, by Lemma 3.45, $G_m(\mathbf{k})$ acts transitively on Y by automorphisms over $\mathbf{k}$. This suffices to rule out nonconstant functions into $\mathbf{k}$. More generally, if a coset C of $\mathbf{k}^*$ has algebraic points, then $\mathrm{Aut}(C/\mathbf{k})$ is finite. Since $\mathrm{Aut}(Y/\mathbf{k})$ is transitive, it follows that if $f : Y \to C$ is definable then $f(Y)$ is finite. But Y is finitely primitive, so $f(Y)$ is a point.

Assume finally that b is a closed ball. Using Lemma 3.45, we can view $G_a(\mathbf{k})$ as a subgroup of $\mathrm{Aut}(Y/\mathbf{k})$. $\mathrm{Aut}(C/\mathbf{k})$ is contained in $G_m(\mathbf{k})$. Let $S = \mathrm{Aut}(Y \times C/\mathbf{k}) \cap (G_a(\mathbf{k}) \times G_m(\mathbf{k}))$. Then S projects onto $G_a(\mathbf{k})$. By strong minimality, $S \cap (G_a(\mathbf{k}) \times (0))$ is either $G_a(\mathbf{k})$ or a finite group. In the first case, $S = G_a \times T$ for some $T \leq G_m$. In the latter, S is the graph of an finite-to-one homomorphism $G_a \to T$; but this is impossible. Thus $G_a \times (0) \leq S$ and G_a acts transitively on Y by automorphisms fixing C; it follows that F is constant. □

3.7 Resolution and finite generation

Lemma 3.48. *Let $A \leq B$ be substructures of a model of* $\mathbf{T}$. *Assume B is finitely generated over A. Then* $\mathrm{RV}(B)$ *is finitely generated over* $\mathrm{RV}(A)$. *Also, if* $\mathrm{RV}(A) \leq C \leq \mathrm{RV}(B)$ *then C is finitely generated over* $\mathrm{RV}(A)$.

Proof. Suppose $\Gamma(B)$ has infinitely many $\mathbb{Q}$-linearly independent elements, modulo $\Gamma(A)$. By Lemma 3.1, they are algebraically independent. By Lemma 3.20, they lift to algebraically independent elements of B over A, contradicting the assumption of finite generation. Thus $rk_\Gamma \Gamma(B)/\Gamma(A) < \infty$. It is thus clear that any substructure of $\Gamma(B)$ containing $\Gamma(A)$ is finitely generated over $\Gamma(A)$. Thus it suffices to show that $\mathrm{RV}(B)$ is finitely generated over $A \cup \Gamma(B)$; replacing A by $A \cup \Gamma(B)$, we may assume $\Gamma(B) = \Gamma(A)$. In this case $\mathrm{RV}(B) \subset \mathrm{RES}$. See [17, Proposition 7.3] for a proof stated for ACVF_A, but valid in the present generality. Here is a sketch. One looks at $B = A(c)$ with $c \in \mathrm{VF}$. If $c \in \mathrm{acl}(A)$ then the Galois group $\mathrm{Aut}(\mathrm{acl}(A)/A(c))$ has finite index in $\mathrm{Aut}(\mathrm{acl}(A)/A)$. Hence the same is true of their images in $\mathrm{Aut}(\mathrm{acl}(A) \cap \mathrm{RV})$, and since RV is stably embedded (by clause (1) of the definition of V-minimality) it follows that there exists a finite subset C' of $A(c) \cap \mathrm{RV}$ such that any automorphism of $\mathrm{acl}(A)$ fixing $A(C')$ fixes $A(c) \cap \mathrm{RV}$. By Galois theory for saturated structures (Section 2.1) C' generates $A(c) \cap \mathrm{RV}$ over A.

On the other hand, if $c \notin \mathrm{acl}(A)$, then $\mathrm{tp}(c/\mathrm{acl}(A))$ agrees with the generic type over A of either a closed ball, an open ball, or an infinite intersection of balls. In the latter two cases, $\mathrm{RES}(A) = \mathrm{RES}(B)$ using Lemma 3.19. In the case of a closed ball b, let b' be the unique maximal open subball of b containing c. Then $b' \in A(c)$, and $\mathrm{tp}(c/A(b'))$ is generic in the open ball b'. Thus by Lemma 3.17, $\mathrm{RES}(B) = \mathrm{RES}(A(b'))$ so it is 1-generated. □

Recall $\mathfrak{B} = \mathfrak{B}^o \cup \mathfrak{B}^{\mathrm{cl}}$ is the sort of closed and open balls.

We require a variant of a result from [17] on canonical resolutions. We state it for $\mathfrak{B}$-generated structures, but it can be generalized to arbitrary ACVF-imaginaries [16].

The proposition and corollaries will have the effect of allowing free use of the technology constructed in this paper over arbitrary base (cf. Proposition 8.3).

For this proposition, we allow $\mathfrak{B}$ (and Γ) as sorts, in addition to VF and RV, so that a structure is a subset of $\mathfrak{B}$, Γ of a model of $\mathbf{T}$, closed under definable functions.

Assume for simplicity that $\mathbf{T}$ has quantifier elimination (cf. Section 3.4).

Let us call a structure A *resolved* if any ball and any thin annulus defined over $\mathrm{acl}(A)$ has a point over $\mathrm{acl}(A)$.

Lemma 3.49. *Let* $\mathbf{T}$ *be* V*-minimal. Let* $M \models \mathbf{T}$, *and let* A *be a substructure of* M. *Then* (1) *and* (2) *are equivalent; if* $\Gamma(A) \neq (0)$, *then* (3) *is equivalent to both.*

(1) A *is effective and* $\mathrm{VF}(\mathrm{acl}(A)) \to \Gamma(A)$ *is surjective.*
(2) A *is resolved.*
(3) $\mathrm{acl}(A)$ *is an elementary submodel of* M.

Proof. Clearly, (3) implies (1) and (2) implies (1). To prove that (1) implies (3) it suffices to show that every definable $\phi(x)$ of $\mathbf{T}_A$ in one variable, with a solution in M, has a solution in A. If x is an RV-variable it suffices to show that $\phi(\mathrm{rv}(y))$ has a solution; so we may assume x is a VF-variable, so ϕ defines $D \subseteq \mathrm{VF}$. By C-minimality D is a finite Boolean combination of balls. D can be written as a finite union of definable sets of the form $\cup_{j=1}^{m} D_j \setminus E_j$, where for each j, D_j is a closed ball, and E_j a finite union of maximal open subballs of D_j, or D_j is an open ball and E_j is a proper subball of D_j, or $E_j = \emptyset$, or $D_j = K$. In the third case, by effectivity there exists a finite set meeting each D_j in a point; since $A = \mathrm{acl}(A)$, this finite set is contained in A; so $D(A) \neq \emptyset$, as required. In the first and second cases, there exists similarly a finite set Y meeting each E_j. Since $A = \mathrm{acl}(A)$, $Y \subseteq A$. By picking a point and translating by it, we may assume $0 \in E_j$ for some j. Say E_j has valuative radius α; picking a point $d \in A$ with $\mathrm{val}(d) = \alpha$ and dividing, we may assume $\alpha = 0$. Now in the open case any element of valuation 0 will be in D_j. In the closed case, the image of E_j under res is a finite subset of the residue field; pick some element $\bar{a}$ of $\mathbf{k}(A)$ outside this finite set; by effectivity, pick $a \in A$ with $\mathrm{res}(a) = \bar{a}$; then $a \in D$. In the fourth case, we use the assumption that $\Gamma(A) \neq (0)$. This proves (3).

It remains to show that (1) implies (2). Let b be a thin annulus defined over $\mathrm{acl}(\emptyset)$; so $b = b' \setminus b''$ for a unique closed ball b' and maximal open subball b''. By effectivity, b'' has an algebraic point, so translation we may assume $0 \in b''$. In this case, the assumption that $\mathrm{VF}(\mathrm{acl}(A)) \to \Gamma(A)$ is surjective gives a point of $b' \setminus b''$.□

If $\mathbf{T}_0$ is V-minimal, A is a finitely generated structure (allowing $\mathfrak{B}$, or even ACVF-imaginareis), and $T = (\mathbf{T}_0)_A$, we will call T a finitely generated extension of a V-minimal theory.

Remark 3.50. If A is effective, then A is $\mathrm{VF} \cup \Gamma$-generated. If A is resolved, then A is VF-generated.

Proposition 3.51. *Let* $\mathbf{T}$ *be* V*-minimal.*

(1) *There exists an effective structure* E_{eff} *admitting an embedding into any effective structure* E*. We have* $\mathrm{RV}(E_{\mathrm{eff}}), \Gamma(E_{\mathrm{eff}}) \subseteq \mathrm{dcl}(\emptyset)$.
(2) *There exists a resolved* E_{rslv} *embedding into any resolved structure* E*. We have* $\mathbf{k}(E_{rslv}), \Gamma(E_{rslv}) \subseteq \mathrm{dcl}(\emptyset)$*. In fact,* $C(E_{rslv}) \subseteq \mathrm{dcl}(\emptyset)$ *for any cosets* C *of* $\mathbf{k}^*$ *in* RV *that contain algebraic points.*
(3) *Let* A *be a finitely generated substructure of a model of* $\mathbf{T}$*, in the sorts* $\mathrm{VF} \cup \mathfrak{B}$*. Then* (1)–(2) *hold for* $\mathbf{T}_A$.

Proof.

(1) Let $(b_i)_{i<\lambda}$ enumerate the definable balls. Define a tower of VF-generated structures A_i, and a sequence of balls b_i, as follows. Let $A_0 = \mathrm{dcl}(\emptyset)$; if κ is a limit ordinal, let $A_\kappa = \cup_{i<\kappa} A_i$. Assume A_i has been defined. If possible, let b_i be an A_i-definable, A_i-transitive ball, not a point; and let c_i be any point of b_i. If no such ball b_i exists, the construction ends, and we let $E_{\mathrm{eff}} = A_i$ for this i.

Suppose E is any effective substructure of a model of $\mathbf{T}$. We can inductively define a tower of embeddings $f_i : A_i \to E$. At limit stages κ let $f_\kappa = \cup_{i<\kappa} f_i$. Given f_i with $A_i \neq E$, let b_i' be the image under f of b_i. By effectivity, b_i' has a point $c_i' \in E$. Since b_i is transitive over A_i, the formula $x \in b_i$ generates a complete type; so $\mathrm{tp}(c_i/A_i)$ is carried by f to $\mathrm{tp}(c_i'/A_i')$. Thus there exists an embedding $f_{i+1} : A_{i+1} \to E$ extending f_i, and with $c_i \mapsto c_i'$.

Each A_i is VF-generated; by Lemma 3.31(3) $\implies$ (4), the process can only stop when $A_i = E_{\mathrm{eff}}$. This shows that E_{eff} embeds into E, and at the same time that the construction of E_{eff} itself must halt at some stage (of cardinality $\leq |\mathbf{T}|$).

By construction, E_{eff} is VF-generated; and hence $\mathbf{T}_{E_{\mathrm{eff}}}$ is V-minimal. Moreover, there are no E_{eff}-definable E_{eff}-transitive balls (except points). In other words all E_{eff}-definable balls are centered. By V-minimality (assumption (3) of Section 3.4) every closed ball has a definable point, so every centered ball has one. Thus E_{eff} is effective.

It remains only to show that $\mathrm{RV}(E_{\mathrm{eff}}), \Gamma(E_{\mathrm{eff}}) \subseteq \mathrm{dcl}(\emptyset)$. We show inductively that $\mathrm{RV}(A_i), \Gamma(A_i) \subseteq \mathrm{dcl}(\emptyset)$. At limit stages this is trivial, and at successor stages it follows from Lemma 3.47.

(2) The proof is identical to that of (1), but using thin annuli as well as balls. If a thin annulus is not transitive, it contains a proper nonempty finite union of balls, so by V-minimality it contains a proper nonempty finite set. Hence the construction of the A_i stops only when A_i is resolved.

(3) Let $A_0 = (A \cap (\mathrm{VF} \cup \Gamma))$. A is generated over A_0 by some $b_1, \ldots, b_n \in \mathfrak{B}$ with b_i of valuative radius $\gamma_i \in A_0$. Since $\mathbf{T}_{A_0}$ is V-minimal, we may assume $\mathbf{T} = \mathbf{T}_{A_0}$ and A is generated by $b_1, \ldots, b_n$, with γ_i definable.

Let J be a subset of $\{1, \ldots, n\}$ of smallest size such that $\mathrm{acl}(\{b_j : j \in J\}) = \mathrm{acl}(\{b_1, \ldots, b_n\})$. By minimality, no b_j is algebraic over $\{b_{j'} : j' \in J, j' \neq j\}$. Let $j \in J$, and let Y_j be the set of balls of radius γ_j; then Y_j is a definable family of disjoint balls. By Lemma 3.8 for $\mathbf{T}' = \mathbf{T}_{\langle\{b_{j'}:j' \in J, j' \neq j\}\rangle}$, b_j is transitive in $\mathbf{T}'_{b_j}$, i.e., in $\mathbf{T}_{\langle b_{j'}:j' \in J\rangle}$; hence b_j is transitive over $\mathrm{acl}(b_1, \ldots, b_n) = \mathrm{acl}(A)$. Let us now show, using induction on $|J|$, that $\Pi_{j \in J} b_j$ is transitive over A. Let $c_j \in b_j$. By Lemma 2.10 the $(\{b_{j'} : j' \in J, j' \neq j\})$ remain algebraically independent over $\langle c_j \rangle$. Thus by induction, $\Pi_{j \neq j'} b_{j'}$ is transitive over $A(c_j)$; since b_j is transitive over A, $\Pi_{j \in J} b_j$ is, too. Let $A' = A(c_j : j \in J)$.

Claim. If B is a $\mathrm{VF} \cup \Gamma$-generated structure containing A, then A' embeds into B over A.

Proof. Since B is $\mathrm{VF} \cup \Gamma$-generated, every ball of $\mathbf{T}_B$ is centered; in particular, b_j has a point c'_j defined over $\mathbf{T}_B$. Let $c' = (c'_j : j \in J)$. By transitivity of $\Pi_{j \in J} b_j$, we have $\mathrm{tp}(c/A) = \mathrm{tp}(c'/A)$. Thus A' embeds into B. □

Note that A' is almost $\mathrm{VF} \cup \Gamma$-generated; indeed, since γ_i is definable, $b_i \in \mathrm{dcl}(c_i)$ so $A' \subseteq \mathrm{acl}((c_j)_{j \in J})$. Thus $\mathbf{T}_{A'}$ is V-minimal. Thus (1)–(2) applies and prove (3).□

See Lemma 3.60 for a uniqueness statement.

Corollary 3.52. *Let* $f : \mathrm{VF} \to (\mathrm{RV} \cup \Gamma)^*$ *be a definable map.*

(1) *There exists a definable* $\tilde{f} : \mathrm{RV} \to (\mathrm{RV} \cup \Gamma)^*$ *such that for any* $\mathbf{x} \in \mathrm{RV}$, *for some* $x \in \mathrm{VF}$ *with* $\mathrm{rv}(x) = \mathbf{x}$, $\tilde{f}(\mathbf{x}) = f(x)$.
(2) *Let* $\Omega = \mathrm{VF}/\mathcal{M}$. *There exists a definable map* $\tilde{f} : \Omega \to (\mathrm{RV} \cup \Gamma)^*$ *such that for any* $\mathbf{x} \in \Omega$, *for some* $x \in \mathrm{VF}$ *with* $x + \mathcal{M} = \mathbf{x}$, $\tilde{f}(\mathbf{x}) = f(x)$.

Proof.
(1) In view of Lemma 2.3, it suffices to show that for a given complete type $P \subseteq \mathrm{RV}$, there exists such a function $\tilde{f}$ on P. We fix $\mathbf{a} \in P$, and show the existence of $\mathbf{c} \in \mathrm{dcl}(\mathbf{a})$ such that for some a with $\mathrm{rv}(a) = \mathbf{a}$, $f(a) = \mathbf{c}$.

By Proposition 3.51, there exists an effective substructure A with $\mathbf{a} \in A$ and $(\mathrm{RV} \cup \Gamma)(A) = (\mathrm{RV} \cup \Gamma)(\langle \mathbf{a} \rangle)$. Thus the open ball $\mathrm{rv}^{-1}(\mathbf{a})$ has an A-definable point a. Set $\mathbf{c} = f(a)$; since $f(a) \in \mathrm{RV}(A) = \mathrm{RV}(\langle \mathbf{a} \rangle)$ we have $\mathbf{c} = \tilde{f}_P(\mathbf{a})$ for some definable function $\tilde{f}_P$. Clearly, $\tilde{f}_P$ satisfies the lemma for the input $\mathbf{a}$, hence for any input from P.

(2) The proof is identical, using Lemma 3.51(3). □

Corollary 3.53. *Let* $\mathbf{T}$ *be* V*-minimal. Assume every definable point of* Γ *lifts to an algebraic point of* RV. *Then there exists a resolved structure* E_{rslv} *such that* E_{rslv} *can be embedded into any resolved structure* E, *and* $\mathrm{RV}(E_{rslv}), \Gamma(E_{rslv}) \subseteq \mathrm{dcl}(\emptyset)$. *If* A *is a finitely generated substructure of a model of* $\mathbf{T}$, *in the sorts* $\mathrm{VF} \cup \mathfrak{B}$, *the same is true for* $\mathbf{T}_A$.

Proof. Under the assumption of the corollary, the conclusion of Proposition 3.51 implies $\mathrm{RV}(E_{rslv}) \subseteq \mathrm{dcl}(\emptyset)$. □

Remark 3.54. It is easy to see using the description of imaginaries in [16] that in a resolved structure, any definable ACVF imaginary is resolved. In other words, if A is a resolved, and $\sim$ is a definable equivalence relation on a definable set D, then $D(A) \to (D/\sim)(A)$ is surjective.

If A is only effective, then there exists $\gamma \in \Gamma(A)^n$ such that for any t with $\mathrm{val}(t) = \gamma$, $(D/\sim)(A) \subseteq \mathrm{dcl}(D(A)/\sim, t)$; this can be seen by embedding $D/\sim$ into $B_n(K)/H$ for an appropriate $H \leq B_n(\mathcal{O})$, and splitting $B_n = T_n U_n$.

3.8 Dimensions

We define the VF-*dimension* of a T_M-definable set X to be the smallest n such that for some n, X admits a T_M-definable map with finite fibers into $\mathrm{VF}^n \times (\mathrm{RV} \cup \Gamma)^*$.

By *essential bijection* $Y \to Z$ we mean a bijection $Y_0 \to Z_0$, where $\dim_{\mathrm{VF}}(Y \setminus Y_0), \dim_{\mathrm{VF}}(Z \setminus Z_0) < \dim_{\mathrm{VF}}(Y) = \dim_{\mathrm{VF}}(Z)$; and where two such maps are identified if they agree away from a set of dimension $< \dim_{\mathrm{VF}}(Y)$.

We say that a map $f : X \to \mathrm{VF}^n$ has RV-fibers if there exists $g : X \to (\mathrm{RV} \cup \Gamma)^*$ with (f, g) injective.

Lemma 3.55. *Let* $X \subseteq \mathrm{VF}^n \times (\mathrm{RV} \cup \Gamma)^*$ *be a definable set. Then we have the following:*

(1) X *has* VF *dimension* $\leq n$ *iff there exists a definable map* $f : X \to \mathrm{VF}^n$ *with* RV*-fibers.*
(2) *If it exists, the map* f *is "unique up to isogeny": if* $f_1, f_2 : X \to \mathrm{VF}^n$ *have* RV*-fibers, then there exists a definable* $h : X \to Z \subseteq \mathrm{VF}^n \times (\mathrm{RV} \cup \Gamma)^*$ *and* $g_1, g_2 : Z \to \mathrm{VF}^n$ *with finite fibers, such that* $f_i = g_i h$.

Proof.

(1) If $f : X \to \mathrm{VF}^n$ has RV-fibers, let g be as in the definition of RV-fibers; then $(f, g) : X \to \mathrm{VF}^n \times (\mathrm{RV} \cup \Gamma)^*$ is injective, so certainly finite-to-one. If $\phi : X \to \mathrm{VF}^n \times \mathrm{RV}^*$ is finite-to-one, by Lemma 3.9, each fiber $\phi^{-1}(c)$ admits a c-definable injective map $\psi_c : {}^{-1}(c) \to \mathrm{RV}^*$. By Lemma 2.3 we can find $\theta : X \to \mathrm{VF}^n \to \mathrm{RV}^*$ that is injective on each ϕ-fiber. Let $f(x) = (\phi, \theta)$. This proves the equivalence.
(2) Now suppose $f_1, f_2 : X \to \mathrm{VF}^n$ both have RV-fibers. Let $h(x) = (f_1(x), f_2(x))$, $Z' = h(X)$, and define $g_i : Z' \to \mathrm{VF}^n$ by $g_1(x, y) = x$, $g_2(x, y) = y$. Then g_i has finite fibers. Otherwise, we can find $a \in X$ such that $f_1(a) \notin \mathrm{acl}(f_2(a))$ (or vice versa). But for any $a \in X$, we have $f_1(a) \in \mathrm{acl}(f_2(a), c)$ for some $c \in (\mathrm{RV} \cup \Gamma)^*$. By Lemma 3.41, $f_1(a) \in \mathrm{acl}(f_2(a))$, a contradiction. By Lemma 3.9 (cf. Lemma 2.3), there exists a definable bijection between Z' and a subset Z of $\mathrm{VF}^n \times \mathrm{RV}^*$. Replacing Z by Z' finishes the proof of the lemma. □

Corollary 3.56. *Let* $f : X \to \mathrm{RV} \cup \Gamma$, $X_a = f^{-1}(a)$. *Then* $\dim(X) = \max_a \dim X_a$.

Proof. Let $n = \max_a \dim X_a$. For each a there exist definable functions $g_a : X_a \to \mathrm{VF}^n$ and $h_a : X_a \to (\mathrm{RV} \cup \Gamma)^*$ with (g_a, f_a) injective on X_a. Thus by the compactness argument of Lemma 2.3, there exists definable functions $g : X \to \mathrm{VF}^n$ and $h : X_a \to (\mathrm{RV} \cup \Gamma)^*$ such that (g, h) is injective when restricted to each X_a. But then clearly (g, h, f) is injective, so $\dim(X) \leq n$. The other inequality is obvious.□

We continue to assume **T** is V-minimal.

Lemma 3.57. *Let $a, b \in \mathrm{VF}$. If $a \in \mathrm{acl}(b) \setminus \mathrm{acl}(\emptyset)$, then $b \in \mathrm{acl}(a)$.*

Proof. Suppose $b \notin \mathrm{acl}(a)$. Let $A_0 = \Gamma(\mathrm{acl}(a, b))$. Then by Lemma 3.36, $b \notin \mathrm{acl}(A_0(a))$.

Let C be the intersection of all $\mathrm{acl}(A_0)$-definable balls such that $b \in C$, and let C' be the union of all $\mathrm{acl}(A_0)$-definable proper subballs of C. Let $B = \cap_i \{B_i\}$ be the set of all balls defined over $\mathrm{acl}(A_0(a))$ with $b \in B_i$, and let $B' = \cup_j \{B'_j\}$ be the union of all $\mathrm{acl}(A_0(a))$-definable proper subballs of B.

Since $a \in \mathrm{acl}(b)$, we have $a \in \mathrm{acl}(b')$ for all $b' \in B \setminus B'$, outside some proper subball. It follows by compactness that for some $i, j, a \in \mathrm{acl}(b')$ for all $b' \in B_i \setminus B'_j$. Say $i = j = 1$, $B'_1 \subset B_1$. By Example 3.57, $a \in \mathrm{acl}(A_0(f_1))$, where $f_1 \in \mathfrak{B}$ codes the ball B_1.

If B_1 is a point, we are done. Otherwise, B_1 has valuative radius $\alpha_1 < \infty$ defined over A_0. It follows that if $B_1 \supseteq C$ then B_1 is $\mathrm{acl}(A_0)$-definable; but then $a \in \mathrm{acl}(A_0)$, contradicting the assumption. Since B_1 meets P nontrivially, we therefore have $B_1 \subset C$. Similarly, B_1 cannot contain any ball in C' since it is not $\mathrm{acl}(A_0)$-definable, but it cannot be contained in C' since $B_1 \cap P \neq \emptyset$. so $B_1 \cap C' = \emptyset$. Thus $B_1 \subset P$.

Let $\bar{B}_1$ be the closed ball of radius α_1 containing B_1, and let e_1 be the corresponding element of $\mathfrak{B}_{\mathrm{cl}}$. Since $\bar{B}_1$ is almost definable over $A_0(a)$, it follows from V-minimality that there exists an almost $A_0(a)$-definable point $c(a)$ in $\bar{B}_1$. Now if $a \in \mathrm{acl}(A_0(e_1))$, then $\bar{B}_1$ contains an $A_0(e_1)$-definable finite set $F_1 = F_1(e_1)$. But since B_1 is a proper subset of P, $e_1 \notin \mathrm{acl}(A_0)$, this contradicts Lemma 3.8. Thus $a \notin \mathrm{acl}(A_0(e_1))$.

Nevertheless, we have seen that $a \in \mathrm{acl}(A_0(f_1))$. Thus $B_1 \neq \bar{B}_1$, so B_1 is a maximal open subball of $\bar{B}_1$. Let b_1 be the point of $\mathrm{Aff}(\bar{B}_1)$ representing B_1. Then $a \in \mathrm{acl}(b_1)$. It follows that $\mathrm{tp}(a/\mathrm{acl}(A_0(e_1)))$ is strongly minimal, contradicting Lemma 3.13. We have obtained a contradiction in all cases; so $b \in \mathrm{acl}(a)$. □

Since the lemma continues to apply over any VF-generated structure, algebraic closure is a dependence relation in the sense of Steinitz (also called a prematroid or combinatorial geometry; cf. [34]). Define the VF-transcendence degree of a finitely generated structure B to be the maximal number of elements of $\mathrm{VF}(B)$ that are algebraically independent over $\mathrm{VF}(A)$. This is the size of any maximal independent set, and also the minimal size of a subset whose algebraic closure includes all VF-points. Hence we have the following.

Corollary 3.58. *The* VF *dimension of a definable set D is the maximal transcendence degree of $\langle b \rangle$.* □

We can now obtain a strengthening of Lemma 3.41, and a uniqueness statement in Proposition 3.51.

Corollary 3.59. *Let Y be a* $\mathbf{T}$*-definable set admitting a finite-to-one map f into $\mathfrak{B}^n$. Let $g : Y \to \mathrm{VF}^m$ be a definable map. Then $g(Y)$ is finite.*

Proof. We may assume $m = 1$. We will use the equivalence (3) $\iff$ (4) of Lemma 2.6. If $g(Y)$ is infinite, then by compactness there exists $a \in g(Y)$, $a \notin \mathrm{acl}(A)$. But for some b we have $a = g(b)$, so if $c = f(b)$, we have $c \in \mathfrak{B}^n$, $a \in \mathrm{acl}(c)$. Thus it suffices to show the following:

$$\text{If } a \in \mathrm{VF}, c \in \mathfrak{B}^n \text{ and } a \in \mathrm{acl}(A(c)), \text{ then } a \in \mathrm{acl}(A). \qquad (*)$$

This clearly reduces to the case $n = 1$, $c \in \mathfrak{B}$. Let γ be the valuative radius of c. As follows from Corollary 3.36, it suffices to show that $a \in \mathrm{acl}(A(\gamma))$. Thus in $(*)$ we may assume $\gamma \in A$.

Finally, to prove $(*)$ (using again the equivalence of Lemma 2.6), we may enlarge A, so we may assume $A \models \mathbf{T}$.

Since $\gamma \in A$, $c \in \mathrm{dcl}(A(e))$ for any element e of the ball c. Thus $a \in \mathrm{acl}(A(e))$. Suppose $a \notin \mathrm{acl}(A)$; then by exchange for algebraic closure in VF, $e \in \mathrm{acl}(A(a))$. Thus any two elements of the ball c are algebraic over each other. By Example 2.4, c has finitely many points; which is absurd. This contradiction shows that $a \in \mathrm{acl}(A)$. □

Lemma 3.60 (cf. Proposition 3.51). *Let* $\mathbf{T}$ *be a finitely generated extension of an effective* V*-minimal theory. Then if E_1, E_2 are effective and both embed into any effective E, then they are finitely generated, and $E_1 \simeq E_2$.*

Proof. The finite generation is clear. Since E_1, E_2 embed into each other, they have the same VF-transcendence degree We may assume $E_1 \leq E_2$. But then by Lemma 3.58, $E_2 \subseteq \mathrm{acl}(E_1)$. By Lemma 3.9, $E_2 \subseteq \mathrm{dcl}(E_1, F)$ for some finite $F \subseteq \mathrm{RV}^* \cap \mathrm{dcl}(E_2)$. But $\mathrm{RV}(E_1) = \mathrm{RV}(E_2)$, so $F \subseteq \mathrm{dcl}(E_1)$, and thus $E_2 = E_1$.□

Remark 3.61. The analogous statement is true for resolved structures. Note that if F is a finite definable subset of RV^n, then automatically the coordinates of the points of F lie in cosets of $\mathbf{k}^*$ that have algebraic points.

Remark. The hypothesis of Lemma 3.60 can be slightly weakened to the following: $\mathbf{T}$ is finitely generated over a V-minimal theory, and there exists a finitely generated effective E.

Example 3.62. In ACVF, when $X \subseteq \mathrm{VF}^n$, the VF dimension equals the dimension of the Zariski closure of X. This is proved in [36]. The idea of the proof: the VF dimension is clearly bounded by the Zariski dimension. For the opposite inequality, in the case of dimension 0, if X is a finite A-definable subset of VF, then using quantifier elimination there exists a nonzero polynomial f with coefficients in A, such that f vanishes on X. In general, if a definable $X \subseteq \mathrm{VF}^n$ has VF dimension $< n$, one can reduce to the case where all fibers of the projection $\mathrm{pr} : X \to \mathrm{pr}\, X \subset \mathrm{VF}^{n-1}$ are finite, then X is not Zariski dense in VF^n, using the zero-dimensional case.

The RV-*dimension* of a definable set $X \subseteq \mathrm{RV}^*$ is the smallest integer n (if any) such that X admits a parametrically definable finite-to-one map into RV^n. More generally for $X \subseteq (\mathrm{RV} \cup \Gamma)^*$, $\dim_{\mathrm{RV}}(X)$ is the smallest integer n (if any) such that X admits a parametrically definable finite-to-one map into $(\mathrm{RV} \cup \Gamma)^n$.

Note that RV is one dimensional, but Γ and every fiber of $\mathrm{val}_{\mathrm{rv}}$ are also one dimensional. In this sense $\mathrm{RV} \cup \Gamma$ dimension is not additive; model-theoretically it is closer to weight than to rank. We do have $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Dually, if a structure B is RV-generated over a substructure A, we can define the *weight* of B/A to be the least n such that $B \subseteq \mathrm{acl}(A, a_1, \dots, a_n)$, with $a_i \in \mathrm{RV}$.

For subsets of RV, RV dimension can be viewed as the size of a Steinitz basis with respect to algebraic closure. One needs to note that the exchange principle holds.

Lemma 3.63 (exchange). *Let* $a, b_1, \dots, b_n \in \mathrm{RV}$; *assume* $a \in \mathrm{acl}(A, b_1, \dots, b_n) \setminus \mathrm{acl}(A, b_1, \dots, b_{n-1})$. *Then* $b_n \in \mathrm{acl}(A, b_1, \dots, b_{n-1}, a)$.

Proof. We may take $n = 1$, $b_n = b$, and $A = \mathrm{acl}(A)$. Let $\alpha = \mathrm{val}_{\mathrm{rv}}(a) \in \Gamma$, $\beta = \mathrm{val}_{\mathrm{rv}}(b)$. If $\beta \in A$ then $\Gamma(A(a, b)) = \Gamma(A(b)) = \Gamma(A)$. The first equality is true since $a \in \mathrm{acl}(A(b))$ so $A(a, b) \subset \mathrm{acl}(A(b))$, and using the stable embeddedness of Γ (Section 2.1) and the linear ordering on Γ. The second equality follows from Lemma 3.10. Thus if $\beta \in A$, then a, b lie in A-definable strongly minimal sets, cosets of $\mathbf{k}^*$, and the lemma is clear.

Assume $\beta \notin A$. If $\alpha \in A$, then $\mathrm{tp}(a/A)$ is strongly minimal, and $\mathrm{tp}(a/A)$ implies $\mathrm{tp}(a/A(b))$ by Lemma 3.10; but then $a \in \mathrm{acl}(A)$, contradicting the assumption. Thus $\alpha, \beta \notin A$; from the exchange principle in Γ, it follows that $A' := \mathrm{acl}(A, \alpha) = \mathrm{acl}(A, \beta)$. Moreover, $a \notin \mathrm{acl}(\alpha)$ by Lemma 3.11 and Lemma 2.6. By the previous case, $b \in \mathrm{acl}(A', a)$, so $b \in \mathrm{acl}(A, a)$. □

Lemma 3.64. *A definable* $X \subseteq \mathrm{RV}^n$ *has* RV *dimension* n *iff it contains an* n*-dimensional definable subset of some coset of* $\mathbf{k}^{*n}$.

Proof. Assume X has RV dimension n. Then there exists $(a_1, \dots, a_n) \in X$ with $a_1, \dots, a_n$ algebraically independent. Let $c \in \Gamma$; then since $a_n \notin \mathrm{acl}(a_1, \dots, a_{n-1})$, it follows as in the proof of Lemma 3.63 that $a_n \notin \mathrm{acl}(a_1, \dots, a_{n-1}, c)$. This applies to any index, so $a_1, \dots, a_n$ remain algebraically independent over c; and inductively we may add to the base any finite number of elements of Γ. Let $c_i = \mathrm{val}_{\mathrm{rv}}(a_i)$, and let $A' = A(c_1, \dots, c_n)$. Then $a_1, \dots, a_n$ are algebraically independent over A', and they lie in $X' = X \cap \Pi_{i=1}^n \mathrm{rv}^{-1}(c_i)$; thus X' is an n-dimensional definable subset of a coset of $\mathbf{k}^{*n}$. □

Definition 3.65. $\mathrm{VF}[n, \cdot]$ be the category of definable subsets of $\mathrm{VF}^* \times \mathrm{RV}^*$ of dimension $\leq n$. Morphisms are definable maps.

Let $X \in \mathrm{Ob}\,\mathrm{VF}[n, \cdot]$. By Lemma 3.55, there exists a definable $f : X \to \mathrm{VF}^n$ with RV-fibers; and the maximal RV dimension of a fiber is a well-defined quantity, depending only on the isomorphism type of X (but not on the choice of f). In particular, the subcategory of definable sets of maximal fiber dimension 0 will be denoted $\mathrm{VF}[n]$.

Definition 3.66. We define $\mathrm{RV}[n,\cdot]$ to be the category of definable pairs (U, f), with $U \subseteq \mathrm{RV}^*$, $f : U \to \mathrm{RV}^n$. If $U, U' \in \mathrm{Ob}\,\mathrm{RV}[n,\cdot]$, a morphism $h : U \to U'$ is a definable map, such that $U'' = \{(f(u), f'(h(u)) : u \in U\}$ has finite-to-one first projection to RV^n. $\mathrm{RV}[n]$ is the full subcategory of pairs (U, f) with $f : U \to \mathrm{RV}^n$ finite-to-one.

$\mathrm{RES}[n]$ is the full subcategory of $RV[n]$ whose objects are pairs $(U, f) \in \mathrm{Ob}\,\mathrm{RV}[n]$ such that $\mathrm{val}_{\mathrm{rv}}(U)$ is finite, i.e., $U \subseteq \mathrm{RES}^*$.

Remark 3.67.

(1) For $X, Y \in \mathrm{Ob}\,\mathrm{RV}[n]$, any definable bijection $X \to Y$ is in $\mathrm{Mor}_{\mathrm{RV}[n]}(X, Y)$.
(2) The forgetful map $(X, f) \mapsto X$ is an equivalence of categories between $\mathrm{RV}[n]$ and the category of all definable subsets of RV^* of RV dimension $\leq m$, with all maps between them. The presentation with f is nonetheless useful for defining $\mathbb{L}$.

By Remark 3.67, $K_+(\mathrm{RV}[m])$ is isomorphic to the Grothendieck semigroup of definable subsets of RV^* of RV dimension $\leq m$. If $\dim(X) \leq m$, let $[X]_m$ denote the class $[X]_m = [(X, f)]_m \in \mathrm{RV}[m]$, where $f : X \to \mathrm{RV}^*$ is any finite-to-one definable map.

Unlike the case of $\mathrm{VF}[n,\cdot]$ or $\mathrm{RV}[n]$, for $(U, f) \in \mathrm{Ob}\,\mathrm{RV}[n,\cdot]$ the map f cannot be reconstructed from U alone, even up to isogeny, so it must be given as part of the data. We view (U, f) as a cover of $f(U)$ with "discrete" fibers.

We denote

$$\mathrm{RV}[\leq N,\cdot] := \oplus_{0\leq n\leq N}\mathrm{RV}[n,\cdot], \qquad \mathrm{RV}[\leq N] = \oplus_{0\leq n\leq N}\mathrm{RV}[n],$$
$$\mathrm{RV}[*,\cdot] := \oplus_{0\leq n}\mathrm{RV}[n,\cdot], \qquad \mathrm{RV}[*] := \oplus_{0\leq n}\mathrm{RV}[n],$$
$$\mathrm{RES}[*] := \oplus_{0\leq n}\mathrm{RES}[n].$$

We have natural multiplication maps $K_+\,\mathrm{RV}[k,\cdot] \times K_+\,\mathrm{RV}[l,\cdot] \to K_+[k+l,\cdot]$, $([(X, f)], [(Y, g)]) \mapsto [(X \times Y, f \times g)]$. This gives a semiring structure to $K_+(\mathrm{RV}[*])$. This differs from the Grothendieck ring $K_+(\mathrm{RV})$.

Alternative description of RV[$\leq N,\cdot$]

An object of $\mathrm{RV}[\leq N,\cdot]$ thus consists of a formal sum $\sum_{n=0}^{N}\mathbf{X}_n$ of objects $\mathbf{X}_n = (X_n, f_n)$ of $\mathrm{RV}[n,\cdot]$. This can be explained from another angle if one adds a formal element ∞ to RV, and extends rv to VF by $\mathrm{rv}(0) = \infty$. Define a function $f[k]$ by $f[k](x) = (f_n(x), \infty, \ldots, \infty)$ ($N-k$ times). If $\mathbf{X} = (X, f)$, let $\mathbf{X}[k] = (X, f[k])$. Then $\sum_{n=0}^{N}\mathbf{X}_n$ can be viewed as the disjoint union $\cup_{i=0}^{N}X_i \times \{\infty\}[N-i]$. The rv pullback is then a set of VF dimension N, invariant under multiplication by $1+\mathcal{M}$; the sum over dimensions $\leq N$ is necessary to ensure that any such invariant set is obtained (cf. Lemma 4.9). From this point of view, an isomorphism is a definable bijection preserving the function "number of finite coordinates." We will use $\mathrm{RV}[\leq N,\cdot]$ or $\mathrm{RV}_\infty[N,\cdot]$ interchangeably.

Lemma 3.68. *Let $X, X' \in \mathrm{Ob}\,\mathrm{RV}[n,\cdot]$, and assume a bijection $g : X' \to X$ lifts to $G : \mathbb{L}X' \to \mathbb{L}X$. Then $g \in \mathrm{Mor}_{\mathrm{RV}[n,\cdot]}(X', X)$.*

Proof. We only have to check the isogeny condition, i.e., that $f(g(a)) \in \mathrm{acl}(f'(a))$ for $a \in X'$ (and dually). By Lemma 3.42, for $x \in {\rho_{X'}}^{-1}(a)$, $G(x)_{\mathrm{VF}} \in \mathrm{acl}(x_{\mathrm{VF}})$, i.e., the VF-coordinates of $G(x)$ are algebraic over those of x. Thus $f(g(a)) \in \mathrm{acl}(x_{\mathrm{VF}})$. This is true for any $x \in {\rho_{X'}}^{-1}(a)$, so $f(g(a)) \in \mathrm{acl}(a)$. □

4 Descent to RV: Objects

We assume **T** is C-minimal with centered closed balls. We will find a very restricted set of maps that transform any definable set to a pullback from RV. This is related to Denef's cell decomposition theorem; since we work in C-minimal theories it takes a simpler form. Recall that this assumption is preserved under passage to $\mathbf{T}_A$, when A is a (VF, RV, Γ)-generated substructure of a model of **T** (Lemma 3.39).

Recall that $\mathrm{RV} = \mathrm{VF}^{\times}/(1+\mathcal{M})$, $\mathrm{rv} : \mathrm{VF}^{\times} \to \mathrm{RV}$ the quotient map. Let $\mathrm{RV}_{\infty} = \mathrm{RV} \cup \{\infty\}$, and define $\mathrm{rv}(0) = \infty$. We will also write rv for the induced map $\mathrm{rv}^n : (\mathrm{VF}^{\times})^n \to (\mathrm{RV})^n$.

Definition 4.1. Fix n. Let $\mathcal{C}^0$ be the category whose objects are the definable subsets of $\mathrm{VF}^n \times \mathrm{RV}_{\infty}^*$, and whose morphisms are generated by the inclusion maps together with functions of one of the following types:

(1) Maps

$$(x_1, \ldots, x_n, y_1, \ldots, y_l) \mapsto (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n, y_1, \ldots, y_l)$$

with $a = a(x_1, \ldots, x_{i-1}, y_1, \ldots, y_l) : \mathrm{VF}^{i-1} \times \mathrm{RV}_{\infty}^l \to \mathrm{VF}$ an A-definable function of the coordinates $y, x_1, \ldots, x_{i-1}$.

(2) Maps $(x_1, \ldots, x_n, y_1, \ldots, y_l) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_l, \mathrm{rv}(x_i))$.

The above functions are called *elementary admissible transformations over A*; a morphism in $\mathcal{C}_A^0$ generated by elementary admissible transformations over A will be called an *admissible transformation* over A. Taking $l = 0$, we see that all A-definable additive translations of VF^n are admissible.

Analogously, if Y is a given definable set, one defines the notion of a Y-family of admissible transformations.

If $e \in \mathrm{RV}$ and T_e is an $A(e)$-admissible transformation, then there exists an A-admissible T such that $\iota_e T_e = T \iota_e$, where $\iota_e(x_1, \ldots, x_n, y_1, \ldots, y_l) = (x_1, \ldots, x_n, e, y_1, \ldots, y_l)$. This is easy to see for each generator and follows inductively.

Informally, note that admissible maps preserve volume for any product satisfying Fubini's theorem of translation invariant measures on VF and counting measures on RV.

We will now see that any $X \subset \mathrm{VF}^n$ is a finite disjoint union of admissible transforms of pullbacks from RV. We begin with $n = 1$.

Lemma 4.2. *Let* **T** *be C-minimal with centered closed balls. Let X be a definable subset of* VF. *Then X is the disjoint union of finitely many definable sets Z_i, such that* *for some admissible transformations T_i, and definable subsets H_i of* $\mathrm{RV}_\infty^{l_i}$, $T_i Z_i = \{(x, y) : y \in H_i, \mathrm{rv}(x) = y_{l_i}\}$.

If X is bounded, H_i is bounded below; in fact, for any $h \in H_i$, $\mathrm{val}_{\mathrm{rv}}(h) \geq \mathrm{val}(x)$ *for some $x \in X$.*

Here VF will be considered a ball of valuative radius $-\infty$, and points as balls of valuative radius ∞.

Proof. We may assume X is a finite union of disjoint balls of the same valuative radius $\alpha \in \Gamma \cup \{\pm\infty\}$, each minus a finite union of proper subballs, since any definable set is a finite union of definable sets of that form.

Case 1: *X is a closed ball.* In this case, by the assumption of centered closed balls, X has a definable point a. Let $T(x) = x - a$. Then $TX \setminus \{0\}$ is the pullback of a subset of Γ, the semi-infinite interval $[\alpha, \infty)$ (where α is the valuative radius of X). Thus $TX = \mathrm{rv}^{-1}(H)$, where $H = {\mathrm{val}_{\mathrm{rv}}}^{-1}([\alpha, \infty)) \cup \{\infty\}$.

Case 2: *X is an open ball.* Let $\mathbf{X}$ be the surrounding closed ball of the same radius α, and as in Case 1 let $a \in \mathbf{X}$ be an definable point, $T(x) = x - a$. If $0 \in TX$ then $TX = \mathrm{rv}^{-1}(H)$, where $H = {\mathrm{val}_{\mathrm{rv}}}^{-1}((\alpha, \infty)) \cup \{\infty\}$. If $0 \notin TX$, then $TX = \mathrm{rv}^{-1}(H)$, where $H = \mathrm{rv}(TX)$ is a singleton of RV.

Case 3: *$X = C \setminus F$ is a ball with a single hole, the closed ball F.* Let β be the valuative radius of F. Let $a \in F$ be a definable point, $T(x) = x - a$. Then $TX = \mathrm{rv}^{-1}(H)$, $H = {\mathrm{val}_{\mathrm{rv}}}^{-1}(I)$, where I is the open interval (α, β) of Γ in case C is closed, the half-open interval $[\alpha, \beta)$ when C is open.

Case 4: *$X = C \setminus \cup_{j \in J} F_j$ is a closed ball, minus a finite union of maximal open subballs.* As in Case 1, find T_1 such that $0 \in T_1 X$. Then $T_1 X$ is the union of the maximal open subball S of radius α, with $\mathrm{rv}^{-1}(H)$, where $H = \mathrm{rv}(X \setminus S)$. S can be treated as in Case 2. Here H is a subset of ${\mathrm{val}_{\mathrm{rv}}}^{-1}(\alpha)$, consisting of ${\mathrm{val}_{\mathrm{rv}}}^{-1}(\alpha)$ minus finitely many points.

Cases 3a *and* 4a: *X is a union of m balls* (*perhaps with holes*) *of types* 1–4 *above*. Here we use induction on m; we have m balls C_j covering X. Let E be the smallest ball containing all C_j. As we may assume $m > 1$, E must be a closed ball; and each C_j is contained in some maximal open subball M_j of E. By the choice of E, not all C_j can be contained in the same maximal open ball of E. Let $a \in E$ be a definable point, $T_1(x) = x - a$. If $0 \in T_1 C_j$ for some j, the lemma is true by induction for this C_j and for the union of the others, hence also for X. Otherwise, $F = \mathrm{rv}(T_1(X))$ is a finite set, with more than one element. For $b \in F$, let $Y_b = T_1 X \cap \mathrm{rv}^{-1}(b)$. By Lemma 2.3, we can, in fact, find a definable Y whose fiber at b is Y_b. By induction again, there exists an admissible transformation T_b such that $T_b(Y)$ is a pullback of the required form. Let $T_2(x) = (x, \mathrm{rv}(x))$, $T_3((x, b)) = ((T_b(x), b))$. Then $T_3 T_2 T_1$ solves the problem.

General subsets of VF. Let $\beta \geq \alpha$ be the least size (i.e., greatest element of Γ) such that some ball of radius β contains more than one hole of X. Let $\{C_j : j \in J\}$ be the balls of radius β around the holes W of X, and let $C = \cup_{j \in J} C_j$. Then $X = (X \setminus C) \dot{\cup} (C \setminus W)$. Now $X \setminus C$ has fewer holes than X, so it can be dealt with inductively. Thus we may assume $X = C \setminus W$; and any proper subball of C of less than maximal size contains at most one hole of X. We may assume the $\{C_j\}$ form a single Galois orbit; so they each contain two or more holes of X. Since these holes are not contained in a proper subball of C_j, each C_j must be closed, and the maximal open subballs of C_j separate holes. Let $D_{j,k}$ be the maximal open subballs of C_j containing a hole $F_{j,k}$. Let $\bar{F}_{j,k}$ be the smallest closed ball containing $F_{j,k}$. Then $X = (C \setminus \cup_{j,k} D_{j,k}) \dot{\cup} \cup_{j,k} (D_{j,k} \setminus \bar{F}_{j,k}) \dot{\cup} \cup_{j,k} (\bar{F}_{j,k} \setminus F_{j,k})$. The second summand in this union falls into Case 3a, the first and third (when nonempty) into Case 4a. □

Remark. If we allow arbitrary Boolean combinations (rather than disjoint unions only), we can demand in Lemma 4.2 that the sets H_i be finite. More precisely, let X be a definable subset of VF. Then there exist definable sets Z_i, admissible transformations T_i, and finite definable subsets H_i of $\mathrm{RV}_\infty^{l_i}$ such that we have the following:

X is a Boolean combination of the sets Z_i, and $T_i Z_i$ is one of the following:

(1) VF;
(2) $(0) \times H_i$;
(4) $b_i \times H_i$, with b_i a definable ball containing 0;
(5) $\{(x, y) : y \in H_i, \mathrm{rv}(x) = f_i(y)\}$, for some definable function $f_i : H_i \to \mathrm{RV}_\infty$.

Corollary 4.3. *Let* $X \subseteq \mathrm{VF} \times \mathrm{RV}^*$ *be definable. Then there exists a definable* $\rho : X \to \mathrm{RV}^*$ *and* $c : \mathrm{RV}^* \to \mathrm{VF}$, $c' : \mathrm{RV}^* \to \mathrm{RV}_\infty$, $c'' : \mathrm{RV}^* \to \mathrm{RV}^*$ *such that every fiber* $\rho^{-1}(\alpha)$ *has the form* $(c(\alpha) + \mathrm{rv}^{-1}(c'(\alpha))) \times \{c''(a)\}$. *Moreover, c has finite image.*

Proof. The finiteness of the image of c is automatic, by Lemma 3.41. The corollary is obviously true for sets of the form $\mathbb{L}(H, h) = \{(x, u) \in \mathrm{VF} \times H : \mathrm{rv}(x) = h(u)\}$; take $\rho(x, u) = (\mathrm{rv}(x), u)$. If the statement holds for TX where T is an admissible transformation, then it holds for X. If true for two disjoint sets, it is also true for their union. (Add to ρ a map to $\{1, -1\} \subseteq \mathbf{k}^*$ whose fibers are the two sets.) Hence by Lemma 4.2 is true for all definable sets. □

Corollary 4.4. *Let* $\mathbf{T}$ *be* V*-minimal,* $X \subseteq \mathrm{VF}$ *and let* $f : X \to \mathrm{RV} \cup \Gamma$ *be a definable function. Then there exists a definable finite partition of* $X = \cup_{i=1}^m X_i$ *such that either* f *is constant on* X_i*, or else* X_i *is a finite union of balls of equal radius (possibly missing some subballs), there is a definable set* F_i *meeting each of the balls b in a single point, and for* $x \in X_i$*, letting* $n(x)$ *be the point of* F_i *nearest x, for some function* H, $f(x) = H(\mathrm{rv}(x - n(x)))$.

Proof. The conclusion is so stated that it suffices to prove it over $\mathrm{acl}(\emptyset)$, i.e., we may assume every almost definable set is definable; cf. Section 2.1. By compactness it suffices to show that for each complete type p, $f|p$ has the stated form. Let b be the intersection of all balls containing p. If b is transitive then by Lemma 3.47 $f|p$ is constant. Otherwise, by V-minimality b contains a definable point, and so we may assume $0 \in b$. It follows that $\mathrm{rv}(p)$ is infinite. Thus by Lemma 3.20, f factors through rv. □

Proposition 4.5. *Let* **T** *be C-minimal with centered closed balls, and let X be a definable subset of* $\mathrm{VF}^n \times \mathrm{RV}^l$. *Then X can be expressed as a finite disjoint union of A-definable sets Z, with each Z of the following form. For some A-admissible transformation T, A-definable subset H of* $\mathrm{RV}_\infty^{l^*}$, *and map of indices* $\nu \in \{1, \dots, n\} \mapsto \nu' \in \{1, \dots, l^*\}$,

$$TZ = \{(a, b) : b \in H, \mathrm{rv}(a_\nu) = b_{\nu'} (\nu = 1, \dots, n)\}.$$

If X projects finite-to-one to VF^n, *then the projection of H to the primed coordinates* $1', \dots, n'$ *is finite to one.*

If X is bounded, then H is bounded below in RV_∞.

Proof. By induction on n; the case $n = 0$ is trivial. Let $\mathrm{pr} : X \to \mathrm{pr}\, X$ be the projection of X to $\mathrm{VF}^{n-1} \times \mathrm{RV}^l$, so that $X \subset \mathrm{VF} \times \mathrm{pr}\, X$.

Let $\mathrm{pr}^*(Y) = \{v : (\exists y \in Y)(x, y) \in Y\}$. For any $c \in \mathrm{pr}\, X$, according to Lemma 4.2, we can write $\mathrm{pr}^*(c) = \dot\bigcup_{i=1}^{k} Z_i(c)$, where

$$T_i(c) Z_i(c) = \{(a, b) : b \in H_i(c), \mathrm{rv}(a) = b_{1'}\}$$

for some $A(c)$-admissible $T_i(c)$, $A(c)$-definable $Z_i(c)$, and $H_i(c) \subseteq \mathrm{RV} = \mathrm{RV}^{1'}$. We can write $Z_i(c) = \{x : (x, c) \in Z_i\}$, $H_i(c) = \{x : (x, c) \in H_i\}$ for some definable Z_i and $H_i \subset \mathrm{VF}^{n-1} \times \mathrm{RV}^{1'}$. By compactness, as in Lemma 2.3, one can assume that the $Z_i(c)$, $H_i(c)$, $T_i(c)$ are uniformly definable: there exists a partition of $\mathrm{pr}\, X$ into finitely many definable sets Y, and for each Y families Z_i, H_i, T_i over Y of definable sets and admissible transformations over Y, such that the integer k is the same for all $c \in Y$, and the $Z_i(c)$, $H_i(c)$, $T_i(c)$ are fibers over c of Z_i, H_i, T_i. In this case, $\mathrm{pr}^*(Y) = \dot\bigcup_{i=1}^{k} Z_i$. We can express X as a disjoint union of the various $\mathrm{pr}^*(Y)$; so we may as well assume $\mathrm{pr}\, X = Y$ and $X = Z_1$. Let T_1 be such that $\iota_c T_1(c) = T_1 \iota_c$. Then

$$T_1 X = \{(a, c, b) : (c, b) \in H_1, \mathrm{rv}(a) = b_{1'}\}.$$

Any admissible transformation is injective and so commutes with disjoint unions.

Now by induction, H_1 itself is a disjoint union $H_1 = \dot\bigcup_{j=1}^{k'} Z_j$, with

$$T_i' Z_i' = \{(d, b) : b \in H_i', \mathrm{rv}(d_\nu) = d_{\nu'} (\nu = 2, \dots, n)\}.$$

Notational remarks. Here $d = (d_2, \dots, d_n)$ are the VF-coordinates of c above. The $'$ depends on i but we will not represent this notationally.

Let $T_i^*(a, d, b) = (a, T_i'(d, b))$, i.e., T_i^* does not touch the first coordinate. Note that T_i^* also does not move the $1'$ coordinate, since in general admissible transformations can only add RV coordinates but not change existing ones. Let

$$Z_i = \{x : T_1(x) = (a, d, b), (d, b) \in Z_i', \mathrm{rv}(a) = b_{1'}\}.$$

Then (as one sees by applying T_1) $X = \dot{\bigcup}_{i=1}^{k} Z_i$, and if $T_i = T_i^* T_1$, we have

$$\begin{aligned} T_i Z_i &= \{(a, d', b') : (d', b') \in T_i' Z_i', \mathrm{rv}(a) = b_{1'}'\} \\ &= \{(a, d', b') : b \in H_i', \mathrm{rv}(a) = b_{1'}, \mathrm{rv}(d_\nu) = b_{\nu'}\}. \end{aligned}$$

As for the finiteness of the projection, if X admits a finite-to-one projection to VF^n, so does each Z in the statement of the proposition, and hence the isomorphic set TZ. We have $H \subset \mathrm{RV}^{n+l}$, $\pi : \mathrm{RV}^{n+l} \to \mathrm{RV}^n$, so $TZ = \{(a, b, b') : (b, b') \in H, \mathrm{rv}(a) = b'\}$. For fixed a, this yields an a-definable finite-to-one map $TZ'(a) = \{b' : (a, b, b') \in TZ\} \to \mathrm{VF}^n$. By Lemma 3.41, $TZ'(a)$ is finite. Now fix b and suppose $(b, b') \in H$ with b' not algebraic over b. Then for generic $a \in \mathrm{rv}^{-1}(b)$, b' is not algebraic over b, a. Yet $(a, b, b') \in TZ$ and so $b' \in TZ'(a)$, a contradiction.

The statement on boundedness is obvious from the proof; if $X \subseteq \{x : \mathrm{val}(x) \geq -\gamma\}^n \times \mathrm{RV}^m$, then H is bounded below by $-\gamma$ in each coordinate. □

A remark on more general base structures

Lemma 4.6. *Let* $\mathbf{T}$ *be* V*-minimal,* A *a* $\mathfrak{B}$*-generated substructure of a model of* $\mathbf{T}$. *Let* X *be a* $\mathbf{T}_A$*-definable subset of* $\mathrm{VF}^n \times \mathrm{RV}^l$. *Then there exist* $\mathbf{T}_A$*-definable subsets* $Y_i \subset \mathrm{RV}^{m_i}$ *and (projection) maps* $f_i : Y_i \to \mathrm{RV}^n$, *a disjoint union* Z *of*

$$Z_i = Y_i \times_{f_i, \mathrm{rv}} \mathrm{VF}^n$$

and a nonempty A*-definable family* $\mathfrak{F}$ *of admissible transformations* $X \to Z$. $\mathfrak{F}$ *will have an* A'*-point for any* $\mathrm{VF} \cup \mathrm{RV} \cup \Gamma$*-generated structure containing* A.

Proof. We may assume A is finitely generated. By Proposition 3.51 there exists an almost $\mathrm{VF} \cup \Gamma$-generated $A' \supset A$ embeddable over A into any $\mathrm{VF} \cup \Gamma$-generated structure containing A, and with $\mathrm{RV}(A') = \mathrm{RV}(A)$. By Proposition 4.5, the required objects Y_i, f_i exist over A'. But since RV is stably embedded, this data is defined over $\mathrm{RV}(A') \subseteq A$. The admissible transformations $X \to Z = \dot{\cup}(Y_i \times_{f_i, \mathrm{rv}} \mathrm{VF}^n)$ exist over A'; so one can find a definable set D with an A'-point, and such that any element of D codes an admissible transformation $X \to Z$. □

Remark. In fact, arbitrary ACVF-imaginaries may be allowed here.

Example 4.7. $\mathfrak{F}$ need not have an A-rational point. For instance, if A consists of an element of $\mathrm{VF}/\mathcal{M}$, i.e., an open ball c, then we can take $Y = Y_1$ to be the point $0 \in \mathrm{RV}$ (since c can be transformed to $\mathcal{M}$); but there is no A-definable bijection of c with $\mathcal{M}$.

A statement in terms of Grothendieck groups

Recall Definitions 3.65 and 3.66.

Definition 4.8. Define $\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[n, \cdot] \to \mathrm{Ob}\,\mathrm{VF}[n, \cdot]$ by

$$\mathbb{L}(X, f) = (\mathrm{VF}^{\times})^n \times_{\mathrm{rv}^n, f} X \subset \mathrm{VF}^n \times \mathrm{RV}^m,$$

where $\mathrm{VF}^{\times} = \mathrm{VF} \setminus \{0\}$.

For $\mathbf{X} = \sum_i \mathbf{X}_i \in \mathrm{RV}[*]$, we let $\mathbb{L}(\mathbf{X})$ be the disjoint sum $\sum_i \mathbb{L}(\mathbf{X}_i)$ over the various components in $\mathrm{RV}[i]$.

Let ρ denote the natural map $\mathbb{L}(X, f) \to X$.

Lemma 4.9. *The image of* $\mathbb{L} : \mathrm{Ob}\,\mathrm{RV}[\leq n, \cdot] \to \mathrm{Ob}\,\mathrm{VF}[n, \cdot]$ *meets every isomorphism class of* $\mathrm{VF}[n, \cdot]$.

Proof. For $X \subseteq \mathrm{RV}^*$ and $f : X \to \mathrm{RV}_{\infty}$, define $\mathrm{rv}(0) = \infty$ and

$$\mathbb{L}(X, f) = \mathrm{VF}^n \times_{\mathrm{rv}^n, f} X \subset \mathrm{VF}^n \times \mathrm{RV}^m.$$

Then in the statement of Proposition 4.5, we have $TZ = \mathbb{L}(H, h)$ where h is the projection to the primed coordinates. For $x \in H$, let $s(x) = \{i : h_i(x) = \infty\}$. For $w \subseteq \{1, \ldots, n\}$, let $H_w = \{x \in H : s(x) = w\}$. Let $\bar{H}_w = (H_w, h'_w)$ where $h'_w = (h_i)_{i \notin w}$. Then $\bar{H}_w \in \mathrm{RV}[|w|, \cdot]$, and $\mathbb{L}(H_w, h|H_w) \simeq \mathbb{L}(\bar{H}_w)$. Thus $\mathbb{L}(H, h) \simeq \mathbb{L}(\sum_w \bar{H}_w)$. □

A restatement in terms of VF alone

This restatement will not be used later in the paper.

Definition 4.10. Let A be a subfield of VF. Let $\mathcal{C}^1_A(n, l)$ be the category of definable subsets of $\mathrm{VF}^n \times (\mathrm{VF}^{\times})^l$, generated by composition and restriction to subsets by maps of one of the following types:

(1) Maps

$$(x_1, \ldots, x_n, y_1, \ldots, y_l) \mapsto (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n, y_1, \ldots, y_l)$$

with $a = a(x_1, \ldots, x_{i-1}, y_1, \ldots, y_l) : \mathrm{VF}^{i+l-1} \to \mathrm{VF}$ an A-definable function of the coordinates $y, x_1, \ldots, x_{i-1}$.

(2) Maps $(x_1, \ldots, x_n, y_1, \ldots, y_l) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_{i-1}, x_i y_i, y_{i+1}, \ldots, y_l) : X \to Y$ assuming $x_i \neq 0$ on X, and that this function takes X into Y.

Remark 4.11. The morphisms in this category are measure preserving with respect to Fubini products of invariant measures (additively for VF, multiplicatively for $\mathrm{VF}^{\times}$), viz. $dx_1 \wedge \cdots \wedge dx_n \wedge dy_1/y_1 \wedge \cdots \wedge dy_l/y_l$.

Lemma 4.12. *Let* $\mathbf{T}$ *be* C*-minimal with centered closed balls,* X *a definable subset of* VF^n*. Then* X *can be expressed as a disjoint union of* A*-definable sets* Z *with the following property. For some* $l \in \mathbb{N}$*, there exists an* $\mathcal{C}^1_A(n,l)$*-transformation* T *and a definable subset* H *of* $\mathrm{RV}^n_\infty \times \mathrm{RV}^l$*, such that*

$$T(Z \times ((1+\mathcal{M}))^l) = \mathrm{rv}^{-1}(H).$$

Moreover, the projection of H *to* RV^n_∞ *is finite-to-one.*

If $\mathrm{val}(x)$ *is bounded below, then* $\mathrm{val}(H)$ *may be taken to be bounded below in the* RV*-coordinates, and bounded in the* RV_∞*-coordinates.*

Proof. This follows from Proposition 4.5. □

5 V-minimal geometry: Continuity and differentiation

We work with a V-minimal theory.

5.1 Images of balls under definable functions

Proposition 5.1. *Let* X, Y *be definable subsets of* VF*, and let* $F : X \to Y$ *be a definable bijection. Then there exists a partition of* X *to finitely many definable equivalence classes, such that for any open ball* b *contained in one of the classes,* $F(b)$ *is an open ball; and dually, if* $F(b)$ *is an open ball, so is* b*.*

Proof. It suffices to show that such a partition exists over $\mathrm{acl}(\emptyset)$; for any finite almost definable partition has a finite definable refinement (cf. the discussion of Galois theory in Section 2.1). Thus as in Section 2.1 we may assume every almost definable set is named.

We will show that if p is a complete type, and b is an open subball of p, then $F(b)$ is an open ball; and that if b' is an open subball of $F(p)$, then b is an open ball. From this it follows by compactness that there exists a definable D_p containing p with the same property; by another use of compactness, finitely many D_p cover X; it then suffices to choose any partition, such that any class is contained in some D_p.

When p has a unique solution, the assertion is trivial. When p is the generic type of a closed ball, or of VF, or of a transitive open or ∞-definable ball, for any $\alpha \in \Gamma$, p remains complete over $\langle \alpha \rangle$. In the transitive cases, this follows from Lemma 3.47, while in the centered closed case it follows from Lemma 3.18.

Thus all open subballs b_t of p of any radius α have the same type over $\langle \alpha \rangle$; hence they are all transitive over $\langle t \rangle$, where $t \in K/\mathcal{M}_\alpha$, where $\mathcal{M}_\alpha = \{x : \mathrm{val}(x) > \alpha\}$ (Lemma 3.8, with $Q = p$). Thus by Lemma 3.46, $F(b_t)$ is an open ball.

The remaining case is that p is the generic type of a centered open or ∞-definable ball b_1. Thus b_1 contains a definable proper subball b_0. If b is an open subball of p, of radius α, then $b \cap b_0 = \emptyset$; let $\bar{b}$ be the smallest closed ball of containing b and b_0. Then b is contained in the generic type of $\bar{b}$, and so by the case of closed balls, $F(b)$ is an open ball. □

Remark 5.2. When $X \subseteq \mathrm{VF} \times \mathrm{RV}^n$, by a *ball contained in* X we will mean a subset of X of the form $b \times \{e\}$, where $b \in \mathfrak{B}$ and $e \in \mathrm{RV}^n$. With this understanding, the proposition extends immediately to such sets X. Indeed, for each $e \in \mathrm{RV}^n$, according to the proposition there is a finite partition of $X(e)$ with the required property; as in Lemma 2.3 these can be patched to form a single partition of X.

Remark 5.3. When $X \subseteq \mathrm{VF}$ there exists a finite set of points F (not necessarily A-definable) such that $F(b)$ is an open ball whenever b is an open ball disjoint from F. (This does not extend to $X \subseteq \mathrm{VF} \times \mathrm{RV}^*$.)

Indeed, by Proposition 5.1 there is a finite number of closed and open balls b_i and points, such that $F(b)$ is an open ball for any open ball b that is either contained in or is disjoint from each b_i. Now let c_i be a point of b_i. If b is an open ball and no $c_i \in b$, then b must be disjoint from, or contained in, each b_i; otherwise, b contains b_i and hence c_i.

5.2 Images of balls II

Lemma 5.4. *Let X, Y be balls, and $F : X \to Y$ a definable bijection taking open balls to open balls. Then for all $x, x' \in X$,*

$$\mathrm{val}(F(x) - F(x')) = \mathrm{val}(x - x') + v_0,$$

where v_0 is the difference of the valuative radii of X, Y.

Proof. Translating by some $a \in X$ and by $F(a) \in Y$, we may assume $0 \in X, 0 \in Y$, $F(0) = 0$; and by multiplying we may assume and both X, Y have valuative radius 0, i.e., $X = Y = \mathcal{O}$. Let $M(\alpha) = \{x : \mathrm{val}(x) < \alpha\}$. Then $F(M(\alpha)) = M(\beta)$ for some $\beta = \beta(\alpha)$. β is an increasing definable surjection from $\{\alpha \in \Gamma : \alpha > 0\}$ to itself; it must have the form $\beta(\alpha) = m\alpha$ for some rational $m > 0$. By Lemma 3.26, we have $m \in \mathbb{Z}$. Now reversing the roles of X, Y and using F^{-1} will transform m to m^{-1}, so $m^{-1} \in \mathbb{Z}$ also, i.e., $m = \pm 1$. Since $m > 0$, we have $m = 1$. □

Lemma 5.5. *Let X be a transitive open or closed ball* (*or infinite intersection of balls*), *and $F : X \to Y$ a definable bijection. Then there exists a definable $e_0 \in \mathrm{RV}$ such that for $x \neq x' \in X$, $\mathrm{rv}(F(x) - F(x')) = e_0 \,\mathrm{rv}(x - x')$.*

Proof. We first show a weaker statement.

Claim. For some definable $e_0 : \Gamma \to \mathrm{RV}$, $\mathrm{rv}(F(x)-F(x')) = e_0(\mathrm{val}(x-x'))\,\mathrm{rv}(x-x')$ for all $x \neq x' \in X$.

Proof. Fix $a \in X$. For $\delta \in \Gamma$, let $b_\delta = b_\delta(a)$, the closed ball around a of valuative radius δ. Consider those b_δ with $b_\delta \subseteq X$. As we saw in the proof of Lemma 5.1, as any $a \in X$ is generic, b_δ is transitive in $\mathbf{T}_{b_\delta}$. By Lemma 3.45, $\mathrm{rv}(F(x) - F(a)) = f_a(\delta)\,\mathrm{rv}(x - a)$, where $\mathrm{val}(x - a) = \delta$, and $f_a(\delta)$ is a function of a and δ. But then f_a is a function $\Gamma \to \mathrm{RV}$, so by Lemma 3.11 it takes finitely many values $v_1, \dots, v_n$.

Let $Y_i = f_a{}^{-1}(v_i)$. Y_i has a canonical code $e_i \in \Gamma^*$, consisting of the endpoints of the intervals making up Y_i. Using the linear ordering on Γ, each individual e_i is definable from the set $\{e_i\}_i$, and hence from a; thus $v_i = f_a(Y_i)$ is also definable from a. Thus f_a f_a is definable from $(e_i, v_i)_i$. (This last argument could have been avoided by quoting elimination of imaginaries in $\mathrm{RV} \cup \Gamma$.) However, as X is transitive, every definable function $X \to (\mathrm{RV} \cup \Gamma)$ is constant, and so $f_a = f_b$ for any $a, b \in X$. Let $e_0(\delta) = f_a(\delta)$. □

We now have to show that the function e_0 of the claim is constant. Using the O-minimality of Γ, it suffices to show for any definable $\delta \in \mathrm{dom}(e_0)$ that

(1) if $e_0(\delta) = e$, then $e_0(\gamma) = e$ for sufficiently small $\gamma > \delta$,

and if δ is not a minimal element of $\mathrm{dom}(e_0)$), then also

(2) if $e_0(\gamma) = e$ for sufficiently large $\gamma < \delta$, then $e_0(\delta) = e$.

To determine $e_0(\delta)$, it suffices to know $\mathrm{rv}(F(x) - F(x'))$ and $\mathrm{rv}(x - x')$ for one pair x, x' with $\mathrm{val}(x - x') = \delta$. Thus in (1) we may replace X by a closed subball Y of valuative radius δ, and in (2) by any closed subball Y of X of valuative radius $< \delta$. Since such closed balls Y are transitive (over their code), we may assume X is a closed ball.

Fix $a \in X$. Pick a generic c (over a) with $\mathrm{rv}(c) = e$.

To prove (1), note that type of such c is generic in an open ball, whereas the elements of X are generic in a closed ball; these generic types are orthogonal by Lemma 3.19; so X remains transitive in $\mathbf{T}_c$. Thus we may assume (by passing to $\mathbf{T}_c$) that c is definable.

Let q_a be the generic type of the closed ball $\{x : \mathrm{val}(a - x) \geq \delta\}$. For $x \models q_a$, let $v_0 = \mathrm{val}(F(a) - F(x) - c(a - x)) - \mathrm{val}(c)$.

By the definition of e, $\mathrm{val}(F(a) - F(x) - c(a - x)) > \mathrm{val}(F(a) - F(x))$, so we have

$$\begin{aligned} v_0 + \mathrm{val}(c) = \mathrm{val}(F(a) - F(x) - c(a - x)) &> \mathrm{val}(F(a) - F(x)) \\ = \mathrm{val}(c(a - x)) &= \delta + \mathrm{val}(c). \end{aligned} \tag{5.1}$$

If $\delta < \gamma < v_0$, find $x, x' \models q_a$ with $\mathrm{val}(x - x') = \gamma$. Then $\mathrm{val}(F(x) - F(x') - c(x - x')) \geq v_0 + \mathrm{val}_{\mathrm{rv}}(e) > \gamma + \mathrm{val}_{\mathrm{rv}}(e) = \mathrm{val}(c(x - x'))$, so $\mathrm{rv}(F(x) - F(x')) = \mathrm{rv}(c(x - x'))$ showing that $e_0(\gamma) = \mathrm{rv}(c) = e$. This proves (1).

For (2), let $Q_0 = \{\gamma : \gamma < \delta\}$, Q_0^{def} the set of definable elements of Q_0, and $Q = \{\gamma \in Q_0 : (\forall y \in Q_0^{\mathrm{def}})(\gamma > y)\}$. Thus Q is a complete type of elements of Γ. For $\gamma \in Q$, according to Lemma 3.17, the formula $\mathrm{val}(x - a) = \gamma$ generates a complete type $q_{\gamma;a}(x)$. By Lemma 3.47, X is transitive over γ, so the formula $x' \in X$ generates a complete $\mathbf{T}_\gamma$-type. Thus by transitivity a complete $\mathbf{T}_\gamma$-type $q_\gamma(x, x')$ is generated by $x, x' \in X$, $\mathrm{val}(x - x') = \gamma$; namely, $(a, b) \models q_\gamma$ iff $b \models q_{\gamma;a}$.

For some definable v_0, for $(a, x) \models q_\gamma$ we have, as in (1),

$$\mathrm{val}(F(a) - F(x) - c(a - x)) = v_0(\gamma) + \mathrm{val}(c) > \gamma + \mathrm{val}(c). \tag{5.2}$$

If we show that $v_0(\gamma) > \delta$ we can finish as in (1).

Now $v_0(\gamma) = m\gamma + \gamma_0$ for some definable $\gamma_0 \in \Gamma$, and some rational m. Letting $\gamma \to \delta$ in (5.2) gives $m\delta + \gamma_0 \geq \delta$. If $m < 0$, then $v_0(\gamma) = m\gamma + \gamma_0 > m\delta + \gamma_0 \geq \delta$ so we are done; hence we may take $m \geq 0$.

By Lemma 3.47, $\mathrm{RV}(\langle\emptyset\rangle) = \mathrm{RV}(\langle a\rangle)$; by Lemma 3.20, when $x \models q_{\gamma;a}$, $\mathrm{RV}(\langle a, x\rangle)$ is generated over $\mathrm{RV}(\langle a\rangle)$ by $\mathrm{rv}(a - x)$.

In particular, on $q_{\gamma,a}$, $x \mapsto \mathrm{rv}(F(a) - F(x) - c(a - x))$ is a function of $\mathrm{rv}(a - x)$. This function lifts v_0 to a function on RV; hence by Lemma 3.26, $m \in \mathbb{Z}$. (This and $m \geq 0$ are simplifications rather than essential points.) We have

$$\mathrm{val}((F(a) - F(x) - c(a - x))(a - x)^{-m}) = \gamma_0.$$

By Lemma 3.47, $(\mathrm{RV} \cup \Gamma)(\langle a\rangle) = (\mathrm{RV} \cup \Gamma)(\langle\emptyset\rangle)$. By Lemma 3.17, then ${\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_0) \cap \mathrm{dcl}(a, x) = {\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_0) \cap \mathrm{dcl}(a)$. Thus ${\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_0) \cap \mathrm{dcl}(a, x) = {\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_0) \cap \mathrm{dcl}(\emptyset)$. Thus $\mathrm{rv}((F(a) - F(x) - c(a - x))(a - x)^{-m}) \in \mathrm{dcl}(\emptyset)$; i.e.,

$$\mathrm{rv}((F(a) - F(x) - c(a - x))(a - x)^{-m}) = e_1$$

for some definable e_1. As in (1), we may assume there exists a definable c_1 with $\mathrm{rv}(c_1) = e_1$. Thus for $(a, x) \models q_\gamma$,

$$\begin{aligned}\mathrm{val}((F(a) - F(x) - c(a - x) - c_1(a - x)^m)) &> \mathrm{val}(F(a) - F(x) - c(a - x)) \\ &= v_0(\gamma) + \mathrm{val}(c). \end{aligned} \tag{5.3}$$

Let $x' \models q_{\gamma,a}$ be generic over $\{\gamma, a, x\}$, so in particular $\mathrm{val}(x - x') = \mathrm{val}(x - a) = \mathrm{val}(a - x') = \gamma$. We have

$$\begin{aligned}\mathrm{val}((F(a) - F(x') - c(a - x') - c_1(a - x')^m)) &> \mathrm{val}(F(a) - F(x) - c(a - x)) \\ &= v_0(\gamma) + \mathrm{val}(c).\end{aligned}$$

Subtracting from (5.3), we obtain

$$\begin{aligned}\mathrm{val}((F(x') - F(x) - c(x' - x) - c_1[(a - x)^m - (a - x')^m]) &> v_0(\gamma) + \mathrm{val}(c) \\ &= \mathrm{val}(c_1(a - x')^m).\end{aligned} \tag{5.4}$$

But since $(x, x') \models q_\gamma$, by (5.3) we have

$$\begin{aligned}\mathrm{val}((F(x) - F(x') - c(x - x') - c_1(x - x')^m)) &> v_0(\gamma) + \mathrm{val}(c) \\ &= \mathrm{val}(c_1(x - x')^m).\end{aligned} \tag{5.5}$$

Comparing (5.4) and (5.5) (and subtracting $\mathrm{val}(c_1)$), we see that

$$\begin{aligned}\mathrm{val}((a - x)^m - (a - x')^m - (x' - x)^m) &> \mathrm{val}((x - x')^m) \\ &= \mathrm{val}((a - x')^m) = \mathrm{val}((a - x)^m).\end{aligned}$$

Let $u = (a - x')/(x' - x)$; then $(a - x)/(x' - x) = u + 1$, $\mathrm{val}(u) = 0 = \mathrm{val}(u + 1)$, and $\mathrm{val}((u + 1)^m - u^m - 1) > 0$. If $U = \mathrm{res}(u)$, we get $(U + 1)^m = U^m + 1$.

Since the residue characteristic is 0 this forces $m = 1$. (Note that U is generic.) Thus $v_0(\gamma) = \gamma + \gamma_0$.

From (5.2), $\gamma + \gamma_0 + \mathrm{val}(c) > \gamma + \mathrm{val}(c)$, or $\gamma_0 > 0$. But $\delta - \gamma_0 \in Q_0^{\mathrm{def}}$, so since $\gamma \in Q$ we have $\gamma > \delta - \gamma_0$, or $v_0(\gamma) = \gamma + \gamma_0 > \delta$. As noted below (5.2) this proves the lemma. □

Remark 5.6. In ACVF(p, p), the claim following Lemma 5.5 remains true, but it is possible for e_0 to take more than one value; consider $x - cx^p$ on a closed ball of valuative radius 0, where $\mathrm{val}(c) < 0$.

Lemma 5.7. *Let X be a transitive open ball, and let $F : X \to X$ be a definable bijection. Then* $\mathrm{rv}(F(x) - F(y)) = \mathrm{rv}(x - y)$ *for all $x \neq y \in X$.*

Proof. This follows from the second assertion in Lemma 3.45 and from Lemma 5.5. □

At this point, Lemma 5.1 may be improved.

Definition 5.8. Call a function G on an open ball *nice* if for some e_0, for all $x \neq x' \in \mathrm{pr}\, X$, $\mathrm{rv}(G(x) - G(x')) = e_0\, \mathrm{rv}(x - x')$.

Proposition 5.9. *Let X, Y be definable subsets of* VF*, and let $F : X \to Y$ be a definable bijection. Then there exists a partition of X to finitely many definable classes, such that on any open ball b contained in one of the classes, $F(b)$ is an open ball, and $F|b$ is nice.*

Proof. The proof of Proposition 5.1 goes through verbatim, only quoting Lemma 5.5 along with Lemma 3.46. □

A definable translate of a ball $\mathrm{rv}^{-1}(\alpha)$ will be called a *basic* 1*-cell*. Thus Corollary 4.3 states that every fiber of ρ is a basic 1-cell. By a *basic* 2*-cell* we mean a set of the form

$$X = \{(x, y) : x \in \mathrm{pr}\, X, \mathrm{rv}(y - G(x)) = \alpha\},$$

where pr X is a basic 1-cell, and G is nice.

Corollary 5.10. *Let $X \subseteq \mathrm{VF}^2$ be definable. Then there exists a definable $\rho : X \to$ RV^* such that every fiber is a basic* 2*-cell.*

Proof. Let $X(a) = \{y : (a, y) \in X\}$. By Corollary 4.3 there exist an a-definable $\rho_a : X(a) \to \mathrm{RV}^*$ and functions c, c' such that every fiber ${\rho_a}^{-1}(\alpha)$ is a basic 1-cell $\mathrm{rv}^{-1}(c'(a, \alpha)) + c(a, \alpha)$. By Lemma 2.3 we can glue these together to a function $\rho_1 : X \to \mathrm{RV}^*$ with $\rho_a(y) = \rho_1(a, y)$. Let $\rho_2(x, y) = (\rho_1(x, y), c'(x, \rho(x, y)))$. Then any fiber D of ρ_2 has the form

$$\{(x, y) : x \in \mathrm{pr}_1 D, \mathrm{rv}(y - G_D(x)) = \alpha\},$$

where $G_D(x) = c(x, \alpha)$, α depending on the fiber D. Combining ρ_2 with a function whose fibers yield a partition as in Proposition 5.9, we may assume G takes open balls to open balls (cf. Remark 5.2). Now apply Corollary 4.3 to pr X to obtain a map $\rho' : \mathrm{pr}\, X \to \mathrm{RV}^*$ with nice fibers. □

5.3 Limits and continuity

We now assume **T** is a C-minimal theory of valued fields, satisfying assumption (1) of Section 3.4.

Let V be a VF-variety. By "almost all a" we will mean "all a away from a set of smaller VF dimension."

Lemma 5.11. *Let g be a definable function on a ball around 0. Then either $\operatorname{val} g(x) \to -\infty$ as $\operatorname{val}(x) \to \infty$ or there exists a unique $b \in$ VF such that $b = \lim_{x\to 0, x\neq 0} g(x)$; i.e.,*

$$(\forall \epsilon \in \Gamma)(\exists \delta \in \Gamma)(0 \neq x \,\&\, \operatorname{val}(x)) > \delta \implies \operatorname{val}(g(x) - b) > \epsilon.$$

Proof. Let p be the generic type of an element of large valuation; so $c \models p|A$ iff $\operatorname{val}(x) > \Gamma(A)$. and let $q = \operatorname{tp}(g(c)/A)$, where $c \models p|A$. By Remark 3.5, q coincides with the generic type of P over A where P is a closed ball, an open ball, or an infinite intersection of balls, or $P =$ VF. The last case means that $\operatorname{val} g(x) \to -\infty$. The existence of g shows that p, q are nonorthogonal, so it follows from Lemma 3.19 that the first case is impossible.

We begin by reducing to the case where P is centered. Assume therefore that P is transitive. For $b \in P$, let $q_b = \operatorname{tp}(g(c')/A(b))$, where $c' \models p|A(b)$. If q_b includes a proper b-definable subball P_b of P, or a finite union of such balls, we may take them all to have the same radius $\alpha(b)$; so $\alpha(b)$ is b-definable. By Lemma 3.47, α is constant. If as b varies there are only finitely many balls P_b, then P is after all centered. If not, then there are two disjoint $P_b, P_{b'}$; but this is absurd since if $c'' \models p|A(b, b')$ then $g(c'') \in P_b \cap P_{b'}$. Thus q_b cannot include a proper subball P_b of P; so q_b is just the generic type of P, over $A(b)$. Moving from A to $A(b)$ we may thus assume that P is centered.

Thus P is a centered open or infinitely-definable ball; therefore, it has a proper definable subball b. If $y \notin b$, write $\operatorname{val}(b - y)$ for the constant value of $\operatorname{val}(c - y)$, $c \in b$. By the definition of a generic type of P, $\operatorname{val}(b - g(c)) \notin \Gamma(A)$. Now $\operatorname{val}(b - g(c)) \in \Gamma(A(c)) = \Gamma(A) \oplus \mathbb{Q} \operatorname{val}(c)$ (by assumption (2) of the definition of V-minimality (Section 3.4)), and $\operatorname{val}(c) > \Gamma(A)$; it follows that $\operatorname{val}(b-g(c)) < \Gamma(A)$ or $\operatorname{val}(b-g(c)) > \Gamma(A)$. The first case is again the case of $P =$ VF, while the second implies that P is an infinite intersection of balls P_i, whose radius is not bounded by any element of $\Gamma(A)$. In other words, $P = \{b\}$. Unwinding the definitions shows that $b = \lim_{x\to 0, x\neq 0} g(x)$. □

Remark. In reality, the transitive case considered in the proof above cannot occur.

By an (open, closed) polydisc, we mean a product of (open, closed) balls. Let B be a closed polydisc. Let $M \models T$. Let $b \in B(M), a \in B(\operatorname{acl}(\emptyset))$. Write $b \to a$ if for any definable $\gamma \in \Gamma$, and each coordinate i, $\operatorname{val}(b_i - a_i) > \gamma$. Let p_0 be the type of elements of Γ greater than any given definable element. Then Lemma 5.11 can also be stated thusly: given a definable g on a ball B_0 around 0 into B, there exists $b \in \operatorname{dcl}(\emptyset)$ such that if $\operatorname{val}(t) \models p$, then $(t, g(t)) \to (0, b)$.

Stated this way, the lemma generalizes to functions defined on a finite cover of B_0.

Lemma 5.12. *Let B_0 be a ball around* 0*, and B a closed polydisc, both* 0*-definable. Let $t \in B_0$ have* $\mathrm{val}(t) \models p_0$*, and let $a \in \mathrm{acl}(t)$, $a \in B$. Then there exists $b \in B$, $b \in \mathrm{acl}(\emptyset)$ with $(t, a) \mapsto (0, b)$.*

Proof. The proof of Lemma 5.11 goes through. □

The following is an analogue of a result of Macintyre's for the p-adics. By the *boundary* of a set X, we mean the closure minus the interior of X.

Lemma 5.13.

(1) *Any definable $X \subseteq \mathrm{VF}^n$ of dimension n contains an open polydisc.*
(2) *Any definable function $\mathrm{VF}^n \to \mathrm{RV} \cup \Gamma$ is constant on some open polydisc.*
(3) *The boundary of any definable $X \subseteq \mathrm{VF}^n$ has dimension $< n$.*

Proof. Given (1) and (3) follows since the boundary is definable; so it suffices to prove (1)–(2). For a given n, (2) follows from (1): by Lemma 3.56, the fibers of the function cannot all have dimension $< n$.

For $n = 1$, (1) is immediate from C-minimality. Assume that (1)–(2) are true for n and let $X \subseteq \mathrm{VF} \times \mathrm{VF}^n$ be a definable set of dimension $n + 1$. For any $a \in \mathrm{VF}^n$ such that $X_a = \{b : (a, b) \in X\}$ contains an open ball, let $\gamma(a)$ be the infimum of all γ such that X_a contains an open γ-ball. By (2) for n, γ takes a constant value γ_0 on some polydisc U; pick $\gamma_1 > \gamma_0$. Let

$$X' = \{(u, z) \in X : u \in U \& (\forall z')(\mathrm{val}(z - z') > \gamma_0 \implies (u, z') \in X)\}.$$

Then $\dim(X') = n + 1$. Now consider the projection $(u, z) \mapsto z$. For some $c \in \mathrm{VF}$, the fiber $X'_c = \{u : (u, c) \in X'\}$ must have dimension n. By induction, X'_c contains a polydisc V. Now, clearly, $V \times B^o_{\gamma_1}(c) \subseteq X$. □

If $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_n)$, write $\mathrm{val}(x - x')$ for $\min \mathrm{val}(x_i - x'_i)$. Say a function F is δ-Lipschitz at x if whenever $\mathrm{val}(x - x')$ is sufficiently large, $\mathrm{val}(F(x) - F(x')) > \delta + \mathrm{val}(x - x')$. Say F is locally Lipschitz on X if for any $x \in X$, for some $\delta \in \Gamma$, F is δ-Lipschitz at x.

Lemma 5.14. *Let $F : X \subseteq \mathrm{VF}^n \to \mathrm{VF}$ be a definable function. Then F is continuous away from a subset X' of dimension $< n$. Moreover, F is locally Lipschitz on $X \setminus X'$.*

Proof. Let X' be the (definable) set of points x where F is not Lipschitz. We must show that X' has dimension $< n$. (In this case, by Lemma 5.13, the closure of X' has dimension $< n$, too.) Suppose otherwise. For $n = 1$ the lemma follows from Lemmas 5.1 and 5.4. Let $\pi_i : X' \subseteq \mathrm{VF}^n \to \mathrm{VF}^{n-1}$ be the projection along the ith coordinate axis. Let Y be the set of $b \in \mathrm{VF}^{n-1}$ such that ${\pi_i}^{-1}(b)$ is infinite or, equivalently, contains a ball; it is a definable set. For $b \in Y$, let

$$D_i(b) = \{x \in {\pi_i}^{-1}(b) : (\exists \delta \in \Gamma)(F|{\pi_i}^{-1}(b) \text{ is } \delta\text{-Lipschitz near } x)\}.$$

By the case $n = 1$, ${\pi_i}^{-1}(b) \setminus D_i(b)$ is finite. Thus if $D_i = \cup_{b \in Y} D_i(b)$, then π_i has finite fibers on $X \setminus D_i$, so $\dim(X \setminus D_i) < n$. Let $X^* = \cap_i D_i$, and for $x \in X^*$ let

$\delta(x)$ be the infinimum of all such Lipschitz constants δ (for all n projections). By Lemma 5.13, δ is constant on some open polydisc $U \subseteq X^*$. Let δ' be greater than this constant value. Then at any $x \in U$, the restriction of F to a line parallel to an axis is δ'-Lipschitz. It follows immediately (using the ultrametric inequality) that F is δ'-Lipschitz on U; but this contradicts the definition of X'. □

Remark 5.15. Via assumption (1) of Section 3.4, we used the existence of p-torsion points in the kernel of $\mathrm{RV} \to \Gamma$ for each p. In ACVF(p, p) this fails; one can still show that F is locally logarithmically Lipschitz, i.e., for some rational $\alpha > 0$, for any $x \in X \setminus X'$, for sufficiently close x', $\mathrm{val}(F(x) - F(x')) > \delta \,\mathrm{val}(x - x')$.

5.4 Differentiation in VF

Let $F : \mathrm{VF}^n \to \mathrm{VF}$ be a definable function, defined on a neighborhood of $a \in \mathrm{VF}^n$. We say that F is differentiable at a if there exists a linear map $L : \mathrm{VF}^n \to \mathrm{VF}$ such that for any $\gamma \in \Gamma$, for large enough $\delta \in \Gamma$, if $\mathrm{val}(x_i) > \delta$ for each i, $x = (x_1, \ldots, x_n)$, then $\mathrm{val}(F(a + x) - F(a) - Lx) > \delta + \gamma$. If such an L exists it is unique, and we denote it dF_a.

Lemma 5.16. *Let $F : X \subseteq \mathrm{VF}^n \to \mathrm{VF}^m$ be a definable function. Then each partial derivative is defined at almost every $a \in X$.*

Proof. We may assume $n = m = 1$. Let $g(x) = (F(a + x) - F(a))/x$. By Lemma 5.4, for almost every a, for some $\delta \in \Gamma$, for all x with $\mathrm{val}(x)$ sufficiently large, $\mathrm{val}(F(a+x) - F(a)) = \delta + \mathrm{val}(x)$; so $\mathrm{val}\, g(x)$ is bounded. By Lemma 5.11, and Proposition 5.1, $g(x)$ approaches a limit $b \in \mathrm{VF}$ as $x \to 0$ (with $x \neq 0$); the lemma follows. □

Corollary 5.17. *Let $F : \mathrm{VF}^n \to \mathrm{VF}$ be a definable function. Then F is continuously differentiable away from a subset of dimension $< n$.*

Proof. F has partial derivatives almost everywhere, and these are continuous almost everywhere, so the usual proof works. □

Lemma 5.18. *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be definable,* $\mathrm{pr} : X \to \mathrm{VF}^n$ *the projection. Then for almost every $p \in \mathrm{VF}^n$, there exists an open neighborhood U of p and $H \subseteq \mathrm{RV}^m$ such that* $\mathrm{pr}^{-1}(U) = U \times H$. *If $h : X \to \mathrm{VF}$, then for almost all $x \in X$, h is differentiable with respect to each* VF*-coordinate.*

Proof. For $x \in \mathrm{VF}^n$, let $H(x) = \{h \in \mathrm{RV}^m : (x, h) \in X\}$. By Corollary 3.24, Lemma 2.8, there exists $H' \subseteq \mathrm{RV}^m \times \mathrm{RV}^l \times \Gamma^k$ such that for any $x \in \mathrm{VF}^n$, there exists a unique $y = f(x) \in \mathrm{RV}^l \times \Gamma^k$ with $H(x) = H'(y)$. By Lemma 5.13, f is locally constant almost everywhere. Thus for almost all x, for some neighborhood U of x, for all $x' \in U$, $H(x) = H(x')$; so $\mathrm{pr}^{-1} U = U \times H(x)$. The last assertion is immediate. □

We can now define the partial derivatives of any definable map $F : X \to \mathrm{VF}$ (almost everywhere); we just take them with respect to the VF-coordinates, ignoring the RV-coordinates.

Given $h : X \to \mathrm{VF}^n$, $h' : X' \to \mathrm{VF}^n$ with RV-fibers, and a definable map $F : X \to X'$, we define the partials of F to be those of $h' \circ F$. Then the differential dF_x exists at almost every point $x \in X$ by Corollary 5.17, and we denote the determinant by Jcb, and refer to it as usual as the Jacobian.

Definition 5.19. Let $X, X' \in \mathrm{VF}[n, \cdot]$ and let $F : X \to X'$ be a definable bijection. F is *measure preserving* if $\mathrm{rv}\,\mathrm{Jcb}(x) = 1$ for almost all $x \in X$. $\mathrm{VF}_{\mathrm{vol}}[n, \cdot]$ is the subcategory of $\mathrm{VF}[n, \cdot]$ with the same objects, and whose morphisms are the measure-preserving morphisms of $\mathrm{VF}[n, \cdot]$.

Let $\mathrm{VF}_{\mathrm{vol}}$ be the category whose objects are those of $\mathrm{VF}[n, \cdot]$, and whose morphisms $X \to Y$ are the essential bijections $f : X \to Y$ that are measure preserving.

5.5 Differentiation and Jacobians in RV

Let X, Y be definable sets, together with finite-to-one definable maps $f_X : X \to \mathrm{RV}^n$, $f_Y : Y \to \mathrm{RV}^n$. Here X, Y can be subsets of RV^* or of $\mathrm{RV}^* \times \mathrm{VF}^*$, etc.; the notion of Jacobian will not depend on the particular realization of X, Y.

Let $h : X \to Y$ be a definable map.

The notion of Jacobian will depend not only on h, X, Y but also on f_X, f_Y; to emphasize this we will write $h : (X, f_X) \to (Y, f_Y)$.

We first define smoothness. When $A = f_X(X)$, $B = f_Y(Y)$ are definable subsets of $\mathbf{k}^n$, we say that h, X, Y are smooth if A, B are Zariski open, $\{(f_X(x), f_Y(h(x))) : x \in X\} \cap (A \times B) = Z$ for some nonsingular Zariski closed set $Z \subset A \times B$, and the differentials of the projections to A and to B are isomorphisms at any point $z \in Z$. In this case, composing the inverse of one of these differentials with the other, we obtain a linear isomorphism $T_a(A) \to T_b(B)$ for any $a = f_X(x), b = f_Y(h(x))$; since $T_a(A) = k^n = T_b(B)$, this linear isomorphism is given by an invertible matrix, whose determinant is the Jacobian J.

In general, to define smoothness of X, Y at $(x, y = h(x))$, we restrict to the cosets of $(\mathbf{k}^*)^n$ containing x and y, translate multiplicatively by x and y, respectively, and pose the same condition.

Any X, Y, h are smooth outside of a set E, where $E \cap C$ has dimension $< n$ for each coset C of $(\mathbf{k}^*)^n$. Equivalently (by Lemma 3.64), E has RV-dimension $< n$.

Assume now that X, Y, h are smooth. Define

$$\mathrm{Jcb}_{\mathrm{RV}}(h)(q) = \Pi(f_X(q))^{-1}\Pi(f_Y(q'))J(1, 1) \in \mathrm{RV},$$

where $\Pi(c_1, \dots, c_n) = c_1 \cdot \dots \cdot c_n$.

At times it is preferable not to use a different translation at each point of a coset of $(\mathbf{k}^*)^n$. The Jacobian $\mathrm{Jcb}_{\mathrm{RV}}(h)$ of h at $q \in X$ can also be defined as follows. Let $q' = h(q)$, $\gamma = \mathrm{val}_{\mathrm{rv}}(q)$, $\gamma' = \mathrm{val}_{\mathrm{rv}}(q') \in \Gamma^n$. Pick any $c, d \in \mathrm{RV}^n$ with $\mathrm{val}_{\mathrm{rv}}(c) = \gamma$, $\mathrm{val}_{\mathrm{rv}}(d) = \gamma'$ (one can take $c = f_X(q), d = f_Y(q')$). Let

$$W(\gamma,\gamma') = \{a : f_X(a) \in \mathrm{val_{rv}}^{-1}(\gamma),\ f_Y(h(a)) \in \mathrm{val_{rv}}^{-1}(\gamma')\},$$
$$H' = \{(c^{-1} f_X(a), d^{-1} f_Y(h(a))) : a \in W\}.$$

Since f_X, f_Y are finite-to-one, $H' \subset (\mathbf{k}^{*n})^2$ both projections of H' to $\mathbf{k}^{*n}$ are finite-to-one, and H' is nonsingluar by the smoothness of (X, Y, h). We can thus define the Jacobian J' of H' at any point. We have

$$\mathrm{Jcb_{RV}}(h)(q) = \Pi(c)^{-1}\Pi(d)J'(qc^{-1}, q'd^{-1}) \in \mathrm{RV}.$$

We also define $\mathrm{Jcb}_\Gamma(h)(q) = \sum \gamma' - \sum \gamma \in \Gamma$ (writing Γ additively). *Note that this depends only on the value of h at q*. We have

$$\mathrm{val_{rv}}\,\mathrm{Jcb_{RV}}(h)(q) = \mathrm{Jcb}_\Gamma(h)(q).$$

Example 5.20. Jacobian of maps on Γ. If $\bar{X}, \bar{Y} \subset \Gamma^n$, we saw that a definable map $\bar{f} : \bar{X} \to \bar{Y}$ lifts to RV iff it is piecewise given by an element of $\mathrm{GL}_n(\mathbb{Z})$ composed with a translation. Assume $\bar{f}$ is given by a matrix $M \in \mathrm{GL}_n(\mathbb{Z})$, let $X = \mathrm{val_{rv}}^{-1}(\bar{X})$, $Y = \mathrm{val_{rv}}^{-1}(\bar{Y})$, and let $f : X \to Y$ be given by the same matrix, but multiplicatively. Then X, Y, f are smooth, and

$$J(f)(x) = \Pi(y)\Pi(x)^{-1}\det M,$$

where $y = f(x)$, and $\det(M) = \pm 1$.

Alternative: Γ-weighted polynomials

We have seen that the geometry on $\mathrm{val_{rv}}^{-1}(\gamma)$ ($\gamma \in \Gamma^n$) translates to the geometry on $(\mathbf{k}^*)^n$, but this is true for the general notions and not for specific varieties; a definable subset of $C(\gamma) = \mathrm{val}^{-1}(\gamma)$ does not correspond canonically to any definable subset of $\mathrm{val}^{-1}(0)$. An invariant approach is therefore useful. Let $\Gamma_0 = \Gamma(\langle\emptyset\rangle)$. $X = (X_1, \ldots, X_n)$ be variables, $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_0^n$, and let $\nu = (\nu(1), \ldots, \nu(n)) \in \mathbb{N}^n$ denote a multi-index. By a γ-weighted monomial we mean an expression $a_\nu X^\nu$ with a_ν a definable element of RV, such that $\mathrm{val_{rv}}(a_\nu) + \sum \nu(i)\gamma_i = 0 \in \Gamma$. Let $\mathrm{Mon}(\gamma, \nu)$ be the set of γ-weighted monomials of exponent ν, together with 0. Then $\mathrm{Mon}(\gamma, \nu) \setminus \{0\}$ is a copy of $\mathrm{val_{rv}}^{-1}(-(a_\nu) + \sum \nu(i)\gamma_i)$; so $\mathrm{Mon}(\gamma, \nu)$ is a one-dimensional $\mathbf{k}$-space. In particular, addition is defined in $\mathrm{Mon}(\gamma, \nu)$. We also have a natural multiplication $\mathrm{Mon}(\gamma, \nu) \times \mathrm{Mon}(\gamma, \nu') \to \mathrm{Mon}(\gamma, \nu + \nu')$. Let $R[X; \gamma] = \oplus_{\nu \in \mathbb{N}^n} \mathrm{Mon}(\gamma, \nu)$. This is a finitely generated graded $\mathbf{k}$-algebra. It may be viewed as an affine coordinate ring of $C[\gamma]$; but the ring of the product $C[\gamma, \gamma']$ is $R[X, X'; (\gamma, \gamma')]$, in general a bigger ring than $R[X, \gamma] \otimes_{\mathbf{k}} R[X', \gamma']$. Nevertheless, a Zariski closed subset of $C(\gamma)$ corresponds to a radical ideal of $R[X'; \gamma]$. In this way, notions such as smoothness may be attributed to closed or constructible subsets of any $C(\gamma)$ in an invariant way.

Definition 5.21. Let $X, Y \in \mathrm{Ob}\,\mathrm{RV}[n, \cdot]$ and let $h : X \to Y$ be a definable bijection. h is *measure preserving* if $\mathrm{Jcb}_\Gamma\, h(x) = 0$ for all $x \in X$, and $\mathrm{Jcb_{RV}}\, h(x) = 1$ for all

$x \in X$ away from a set of RV dimension $< n$. If only the first condition holds, we say h is Γ-measure preserving.

For $X, Y \in \mathrm{RV}[\leq n, \cdot]$, we say that $h : X \to Y$ is measure preserving if this is true of the $\mathrm{RV}[n]$-component of h.

$\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$ (respectively, $\mathrm{RV}_{\Gamma\text{-vol}'}[n, \cdot]$) is the subcategory of $\mathrm{RV}[n, \cdot]$ with the same objects, and whose morphisms are the measure-preserving (respectively, Γ-measure-preserving) definable bijections.

$\mathrm{RV}_{\mathrm{vol}}[\leq n, \cdot] = \oplus_{k<n} \mathrm{RV}_{\Gamma\text{-vol}'}[k, \cdot] \oplus \mathrm{RV}_{\mathrm{vol}}[n, \cdot]$.

Note that when $X, Y \in \mathrm{Ob}\,\mathrm{RV}[n, \cdot]$, a bijection $h : X \to Y$ is Γ-measure preserving iff it leaves invariant the sets $S_\gamma = \{(a_1, \dots, a_n) : \sum_{i=1}^n \mathrm{val}_{\mathrm{rv}}(a_i) = \gamma\}$.

5.6 Comparing the derivatives

Consider a definable function $F : \mathrm{VF} \to \mathrm{VF}$ lying above $f : \mathrm{RV} \to \mathrm{RV}$, i.e., $\mathrm{rv}\, F = f\, \mathrm{rv}$. The fibers of the map $\mathrm{rv} : \mathrm{VF} \to \mathrm{RV}$ above $\mathbf{k}$, for instance, are open balls of valuative radius 0, whereas the derivative is defined on the scale of balls of radius r for $r \to +\infty$. Thus the comparison between the derivatives of F and f is not tautological. Nevertheless, one obtains the expected relation almost everywhere.

While this case of the affine line would suffice (using the usual technique of partial derivatives), it is easier to place oneself in the more general context of curves. More precisely, we consider definable sets C together with finite-to-one maps $f : C \to \mathrm{RV}$. Let $\mathbb{L}C$ and $\rho : \mathbb{L}C \to C$ be as above.

In the following lemma, H', h' denote, respectively, the VF-, RV derivatives of functions H, h defined on objects of $\mathrm{VF}[1]$, $\mathrm{RV}[1]$, respectively.

Proposition 5.22. *Let $C_i \subseteq \mathrm{RV}^*$ be definable sets, $f_i : C_i \to \mathrm{RV}$ finite-to-one definable maps $(i = 1, 2)$. Let $h : C_1 \to C_2$ be a definable bijection, and let $H : \mathbb{L}C_1 \to \mathbb{L}C_2$ be a lifting of h, i.e., $\rho H = h\rho$. Then we have the following:*

(1) *For all but finitely many $c \in C_1$, h is differentiable at c, H is differentiable at any $x \in \mathbb{L}c$, and* $\mathrm{rv}\, H'(x) = h'(\mathrm{rv}(x))$.
(2) *For all $c \in C_1$, H is differentiable at a generic $x \in \mathbb{L}c$, and* $\mathrm{val}\, H'(x) = (\mathrm{val}_{\mathrm{rv}} h')(x) = \mathrm{val}(f_2(h(x))) - \mathrm{val}(f_1(x))$.

Proof.

(1) Let Z' be the set of $x \in \mathbb{L}C_1$ such that H is not differentiable at x (a finite set) or that $\mathrm{rv}(H'(x)) \neq h'(\mathrm{rv}(x))$. We have to show that $\rho(Z') \subseteq C$ is finite or, equivalently, that $f_1 \circ \rho(Z')$ is finite. Otherwise, there exists $c \in \rho(Z')$ with $c \notin \mathrm{acl}(A)$. By Lemma 3.20, the formula $\mathrm{rv}(x) = f_1(c)$ generates a complete type q over $A(c)$; it defines a transitive open ball b_c over $A(c)$. Since $\rho \circ H = \rho \circ h$, we have $H(c, y) = (c, H_c(y))$ for some $A(c)$-definable bijection H_c of b_c. By Lemma 5.5, for some $e_0 \in \mathrm{RV}$, $\mathrm{rv}(H(u) - H(v)) = e_0\, \mathrm{rv}(u - v)$ for all $u, v \in b_c$; so $\mathrm{rv}((H(u) - H(v))/(u - v)) = e_0$. Since H is differentiable almost everywhere on b_c (Lemma 5.17) and b_c is transitive, it is differentiable at every point. Clearly, $\mathrm{rv}\, H'(u) = e_0$, contradicting the definition of Z'.

(2) This follows from Lemma 5.4. □

Corollary 5.23. *Let* $\mathbf{X} \in \mathrm{Ob}\,\mathrm{RV}[n]$, $F : \mathbb{L}\mathbf{X} \to \mathrm{VF}^n$ *a definable function,* $f : \mathbb{L}\mathbf{X} \to \mathrm{RV}^n$ *a definable function. Assume* $\mathrm{rv}\, F(x) = f(\mathrm{rv}(x))$. *Then Proposition* 5.22 *applies for each partial derivative of* F. *In particular,*

- *for all* $c \in X$ *away from a set of smaller dimension, for all* $x \in \mathbb{L}c$, F *is differentiable at* x, f *is differentiable at* c, *and* $\mathrm{rv}\,\mathrm{Jcb}(F)(x) = \mathrm{Jcb}_{\mathrm{RV}}(f)(x)$;
- *for all* $c \in X$, *for generic* $x \in \mathbb{L}c$, F *is differentiable at* x, *and* $\mathrm{val}\,\mathrm{Jcb}(F)(x) = (\mathrm{Jcb}_\Gamma\, f)(x)$. □

Corollary 5.24. *Let*

$$\mathbf{X}, \mathbf{Y} \in \mathrm{Ob}\,\mathrm{RV}[\leq n], \qquad f \in \mathrm{Mor}_{\mathrm{RV}[\leq n]}(\mathbf{X}, \mathbf{Y}), \qquad F \in \mathrm{Mor}_{\mathrm{VF}_{\mathrm{vol}}[n]}(\mathbb{L}\mathbf{X}, \mathbb{L}\mathbf{Y}).$$

Assume $\mathrm{rv}\, F(x) = f(\mathrm{rv}(x))$. *Then* $f \in \mathrm{Mor}_{\mathrm{RV}_{\mathrm{vol}}[n]}(\mathbf{X}, \mathbf{Y})$. □

Proof. The proof follows from Corollary 5.23. □

6 Lifting functions from RV to VF

Proposition 6.1. *Let* $\mathbf{T}$ *be an effective* V*-minimal theory. Let* $X \subset \mathrm{RV}^k$ *be definable and let* $\phi_1, \phi_2 : X \to \mathrm{RV}^n$ *be two definable maps with finite fibers. Then there exists a definable bijection* $F : X \times_{\phi_1,\mathrm{rv}} (\mathrm{VF}^\times)^n \to X \times_{\phi_2,\mathrm{rv}} (\mathrm{VF}^\times)^n$, *commuting with the natural projections to* X.

Proof. Let $A = \mathrm{dcl}(\emptyset) \cap (\mathrm{VF} \cup \Gamma)$. If $b \in \mathrm{dcl}(\emptyset) \cap \mathrm{RV}$, then viewed as a ball b has a point $a \in A$; since the valuative radius of b is also in A, we have $b \in \mathrm{dcl}(A)$. Thus ϕ_1, ϕ_2, X are ACVF_A-definable. Any ACVF_A-definable bijection F is a fortiori $\mathbf{T}$-definable; so the proposition for ACVF_A implies the proposition for $\mathbf{T}$. Moreover, ACVF_A is V-minimal and effective, since any algebaic ball of ACVF_A is $\mathbf{T}_A$-algebraic and hence has a point in $\mathrm{VF}(A)^{\mathrm{alg}}$. Thus we may assume $\mathbf{T} = \mathrm{ACVF}_A$.

The proof will be asymmetric, concentrating on $\phi_1 X$.

We may definably partition X, and prove the proposition on each piece.

Consider first the case where $\phi_1 : X \to U$ and $\phi_2 : X \to V$ are bijections to definable subsets $U, V \subseteq (\mathbf{k}^*)^k$. Our task is to lift the bijection $f = \phi_2 {\phi_1}^{-1}$ to VF^n. A definable subset of $\mathbf{k}^n$ (such as $\phi_i(X)$) is a disjoint union of smooth varieties. We thus consider a definable bijection $f : U \to V$ between $\mathbf{k}$ varieties $U \subset \mathbf{k}^n$ and $V \subset \mathbf{k}^n$. Induction on $\dim(U)$ will allow us to remove a subset of U of smaller dimension. Hence we may assume U is smooth, cut out by $h = (h_1, \ldots, h_l)$, $TU = \mathrm{Ker}(dh)$, $f = (f_1, \ldots, f_n)$, where f_i are regular on U (defined on an open subset of $\mathbf{k}^n$), and df is injective on TU at each point of U. Thus the common kernel of $dh_1, \ldots, dh_l, df_1, \ldots, df_n$ equals 0.

It follows that at a generic point of U (i.e., every point outside a proper subvariety), if Q is a sufficiently generic $n \times l$ matrix of elements of A (or integers) and we let $f_i' = f_i + Qh$, then the common kernel of $df_1', \ldots, df_n'$ vanishes. Note that

$f_i|U = f_i'|U$. Let W be a smooth variety contained in $f(U)$ and whose complement in $f(U)$ is a constructible set of dimension smaller than $\dim(U)$. Replacing U by $f^{-1}(W)$, we may assume $f(U)$ is also a smooth variety.

Let $\tilde{U} = \mathrm{res}^{-1}(U)$. Lift each f_i' to a polynomial F_i over $\mathcal{O}$, with definable coefficients. This is possible by effectiveness of ACVF_A. Obtain a regular map F, whose Jacobian is invertible at points of $\tilde{U}$. We have $\mathrm{res} \circ F = f \circ \mathrm{res}$. Since f is 1-1 on U, the invertibility of dF implies that F is 1-1 on $\tilde{U}$. Moreover, by Hensel's lemma, $F : \mathrm{rv}^{-1}(U) \to \mathrm{rv}^{-1}(W)$ is bijective.

Next consider the case where in place of a bijection $f : U \to V$ we have a finite-to-finite correspondence $\tilde{f} \subset U \times V$ (where $U = \phi_1(X)$, $V = \phi_2(X)$), $\tilde{f} = \{(\phi_1(x), \phi_2(x)) : x \in X\}$. We may take $\tilde{f} \subset U \times V$ to be a subvariety, unramified and quasi-finite over U and over V; and we can take U, V to be smooth varieties. As before we can lift $\tilde{f}$ to a correspondence $\widetilde{F} \subset \widetilde{U} \times \widetilde{V}$, such that $\widetilde{F} \cap \mathrm{rv}^{-1}(u) \times \mathrm{rv}^{-1}(v)$ is a bijection $\mathrm{rv}^{-1}(u) \to \mathrm{rv}^{-1}(v)$ whenever $(u, v) \in \tilde{f}$. It follows that a bijection $X \times_{\phi_1,\mathrm{rv}} (\mathrm{VF}^\times)^n \to X \times_{\phi_2,\mathrm{rv}} (\mathrm{VF}^\times)^n$ is given by $(x, y) \mapsto (x, y')$ iff $(y, y') \in \widetilde{F}$.

Let $\phi_1 : X \to U$ and $\phi_2 : X \to V$ be bijections to definable subsets U, V, each contained in a single coset of $(\mathbf{k}^*)^k$, say, $U \subseteq C(\gamma)$, $V \subseteq C(\gamma')$ for some $\gamma, \gamma' \in \Gamma^k$ (cf. Section 5.5). Let $Z = (Z_1, \ldots, Z_k)$ be variables, $R[Z; \gamma]$ be the subring of $\mathrm{VF}[Z]$ consisting of polynomials $\sum a_\nu Z^\nu$, with $\mathrm{val}(a_\nu) + \sum_{i=1}^k \nu(i)\gamma_i = 0$, and a_ν a definable element of VF. There is a natural homomorphism $R'[Z; \gamma] \to R[Z; \gamma]$, where $R[Z; \gamma]$ is the coordinate ring of $C(\gamma)$. By effectivity, this homomorphism is surjective. The proof now proceeds in exactly the same way as above.

This proves the proposition in case $\mathrm{val}_{\mathrm{rv}}\phi_i(X)$ consists of one point.

Next, assume $\mathrm{val}_{\mathrm{rv}}\phi_2$ consists of one point, and $\mathrm{val}_{\mathrm{rv}}\phi_1(X)$ is finite. Thus $\phi_1(X)$ lies in the union of finitely many cosets $(C(a) : a \in E)$, with E finite.

For $a \in E$, $A(a)$ remains almost VF, Γ-generated; since the proposition is true for ${\phi_1}^{-1}C(a)$ (definable in $\mathbf{T}_{A(a)}$), then by the one-coset case an appropriate isomorphism F exists; and the finitely many F obtained in this way can then be glued together, to yield a map defined over A.

The case of $\mathrm{val}_{\mathrm{rv}}\phi_1$, $\mathrm{val}_{\mathrm{rv}}\phi_2$ both finite, is treated similarly.

This proves the existence of a lifting in case $\mathrm{val}_{\mathrm{rv}}\phi_i(X)$ is finite. Now for the general case.

Claim. Let $P \subset X$ be a complete type. Then there exists a definable D with $P \subset D \subset X$, and definable functions θ on $\mathrm{val}_{\mathrm{rv}}(\phi_1(D))$ and θ' on $\mathrm{val}_{\mathrm{rv}}(\phi_2(D))$ such that for $x \in D$, $\theta(\mathrm{val}_{\mathrm{rv}}(\phi_1(x))) = \mathrm{val}_{\mathrm{rv}}\phi_2(x)$, $\theta'(\mathrm{val}_{\mathrm{rv}}(\phi_2(x))) = \mathrm{val}_{\mathrm{rv}}\phi_1(x)$.

Proof. Let $a \in P$, $\gamma_i = \mathrm{val}_{\mathrm{rv}}(\phi_i(a))$. Then γ_2 is definable over some points of $\phi_1^{-1}{\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_1)$. But ${\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_1)$ is a coset of $\mathbf{k}^*$, and ϕ_1 is finite-to-one, so $\phi_1^{-1}{\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma_1)$ is orthogonal to Γ. Thus γ_2 is algebraic over γ_1. Since Γ is linearly ordered, γ_2 is definable over γ_1; so $\gamma_2 = \theta(\gamma_1)$ for some definable θ. Similarly, $\gamma_1 = \theta'(\gamma_2)$. Clearly, θ restricts to a bijection $\mathrm{val}_{\mathrm{rv}}\phi_1 P \to \mathrm{val}_{\mathrm{rv}}\phi_2 P$, with inverse θ'. By Lemma 2.7 there exists a definable D with $\theta\phi_1 = \phi_2$, $\phi_1 = \theta'\phi_2$ on D. □

Now by compactness, there exist finitely many $(D_i, \theta_i, \theta_i')$ as in the claim with $\cup_i D_i = X$. We may cut down the D_i successively, so we may assume the union is disjoint. But in this case the proposition reduces to the case of each individual D_i, so we may assume $X = D$. Let $B_i = \mathrm{val}_{\mathrm{rv}}\phi_i(X)$. Given $b \in B_1$, let $X_b = (\mathrm{val}_{\mathrm{rv}}\phi_1)^{-1}(b)$. Then by the case already considered there exists an $A(b)$-definable $F_b : X_b \times_{\phi_1,\mathrm{rv}} (\mathrm{VF}^\times)^n \to X_b \times_{\phi_2,\mathrm{rv}} (\mathrm{VF}^\times)^n$. Let $F = \cup_{b \in B_1} F_b$. By Lemma 2.3, $F : X \times_{\phi_1,\mathrm{rv}} (\mathrm{VF}^\times)^n \to X \times_{\phi_2,\mathrm{rv}} (\mathrm{VF}^\times)^n$ is bijective (see the discussion in Section 2.1). □

We note a corollary.

Lemma 6.2. *Let* $\mathbf{T}$ *be* V*-minimal and effective, and let* A *be an almost* (VF, Γ)*-generated structure. Then* A *is effective.*

Proof. By Lemma 3.29 it suffices to show A is rv-effective. Note that if $A \subseteq \mathrm{acl}(\emptyset)$, then $\mathbf{T}$ is rv-effective iff $\mathbf{T}_A$ is rv-effective (see the proof of Lemma 3.31(2)–(3)). Thus it suffices to show that if $A_0 = \mathrm{acl}(A_0)$, $a \in \mathrm{VF} \cup \Gamma$, and $\mathbf{T}' = \mathbf{T}_{A_0}$ is effective, then so is $\mathbf{T}'(a)$. The case $a \in \Gamma$ is included in Corollary 3.40, so assume $a \in \mathrm{VF}$. Let P be the intersection of all A_0-definable balls containing a. If P is transitive over A_0, then by Lemma 3.47 we have $\mathrm{RV}(A_0(a)) = \mathrm{RV}(A_0)$, so rv-effectivity remains true trivially. Otherwise, P is centered over A_0, hence has an A_0-definable point, and by translation we may assume $0 \in P$. a is then a generic point of P over A_0. Let $c \in \mathrm{RV}(A_0(a))$; we must show that $\mathrm{rv}^{-1}(c)$ is centered over $A_0(a)$. By Lemma 3.20, if $c \in \mathrm{RV}(A_0(a))$ then $c = f(d)$ for some A_0-definable function $f : \mathrm{RV} \to \mathrm{RV}$, where $d = \mathrm{rv}(a)$. By Lemma 6.1 there exists an A_0-definable function $F : \mathrm{VF} \to \mathrm{VF}$ lifting f. Then $F(d) \in \mathrm{rv}^{-1}(c)$. □

Base change: Summary

Base change from $\mathbf{T}$ to $\mathbf{T}_A$ preserves V-minimality, effectiveness and being resolved, if A is VF-generated; V-minimality and effectiveness, if A is RV-generated; V-minimality, if A is Γ-generated. (Lemmas 6.2, 3.39, and 3.40; the resolved case follows using Lemma 3.49).

Though the notion of a morphism $g : (X_1, \phi_1) \to (X_2, \phi_2)$ does not depend on ϕ_1, ϕ_2, recall that the RV-Jacobian of g is defined with reference to these finite-to-one maps.

Lemma 6.3. *Let* $\mathbf{T}$ *be* V*-minimal and effective. Let* $X_i \subset \mathrm{RV}^{k_i}$ *be definable and let* $\phi_i : X \to \mathrm{RV}^n$ *be definable maps with finite fibers; let* $g : X_1 \to X_2$ *be a definable bijection. Assume given, in addition, a definable function* $\delta : X_1 \to \mathrm{RV}$*, such that*

(1) $\mathrm{val}_{\mathrm{rv}}\delta(x) = \mathrm{Jcb}_\Gamma\, g(x)$ *for all* $x \in X_1$;
(2) $\delta(x) = \mathrm{Jcb}_{\mathrm{RV}}\, g(x)$ *for almost all* $x \in X_1$ *(i.e., all* x *outside a set of dimension* $< n$*).*

Then there exists a definable bijection $G : X_1 \times_{\phi_1,\mathrm{rv}} (\mathrm{VF}^\times)^n \to X_2 \times_{\phi_2,\mathrm{rv}} (\mathrm{VF}^\times)^n$ *such that* $\rho_2 \circ G = g \circ \rho_1$*, where* ρ_i *are the natural projections to the* X_i*, and such that for any* $x \in X_1 \times_{\phi_1,\mathrm{rv}} (VF^*)^n$*,* G *is differentiable at* x*, and* $\mathrm{rv}(\mathrm{Jcb}(G)(x)) = \delta(x)$.

Proof. We follow closely the proof of Proposition 6.1. As there, we may assume $\mathbf{T} = \mathrm{ACVF}_A$, with A be an almost (VF, Γ)-generated substructure.

We first assume that $\mathrm{val}_{\mathrm{rv}}\phi_1(X_1)$ is a single point of Γ^n

Then X_1 can be definably embedded into $\mathbf{k}^N$ for some N, and it follows from the orthogonality of $\mathbf{k}$ and Γ that the image of X_1 in Γ under any definable map is finite. Thus $\phi_2 X_2$ is contained in finitely many cosets $(C(a) : a \in S)$ of $(\mathbf{k}^*)^n$; by partitioning X_1 working in $\mathbf{T}_{A(a)}$, we may assume $\phi_2 X_2$ is contained in a single coset (cf. Lemma 2.3).

As in Proposition 6.1, we may assume $\phi_i X \subseteq \mathbf{k}^n$, and, indeed, that $\phi_1 X = U, \phi_2 X = V$ are smooth varieties. If $\dim(U) = n$, then the lift constructed in Proposition 6.1 satisfies $\mathrm{rv}(\mathrm{Jcb}(G))(x) = \mathrm{Jcb}_{\mathrm{RV}}\, g(x)$ for $x \in X \times_{\phi_1,\mathrm{rv}} \mathrm{VF}^n$; thus by assumption (2), we have $\mathrm{rv}(\mathrm{Jcb}(G))(x) = \delta(x)$ for almost all x. The exceptional points have dimension $< n$, and may be partitioned into smooth varieties of dimension $< n$. Thus we are reduced to the case $\dim(U) < n$. We prove it by induction on $\dim(U)$. In this case choose any lifting G_0. We have an error term $e(x) = \mathrm{rv}(\mathrm{Jcb}(G_0))(x)^{-1}\delta(x)$. Now $A(x)$ is almost VF, Γ-generated, and so balls $\mathrm{rv}^{-1}(y)$ contain definable points; thus $e(x) = \mathrm{rv}\, E(x)$ for some definable $E : (X \times_{\phi_1,\mathrm{rv}} \mathrm{VF}^n) \to \mathrm{VF}$. Since U is a smooth subvariety of $\mathbf{k}^n$ of positive codimension, some regular h on $\mathbf{k}^n$ vanishes on V, while some partial derivative (say, h_1) vanishes only on a lower-dimensional subvariety. By induction, one may assume h_1 vanishes nowhere. Lift h to H; so H_1 lifts h_1. Compose G_0 with a map fixing all coordinates but the first, and multiplying the first coordinate by $E(x)H(y)/H_1(y)$. (Here $x = g^{-1}(y)$.) Where h vanishes, this has Jacobian $E(x)$; so the composition has RV-Jacobian $\delta(x)$ as required.

Now in general, for any $\gamma \in \Gamma^n$ let $X_1(\gamma) = \{x \in X_1 : \mathrm{val}_{\mathrm{rv}}\phi_1(x) = \gamma\}$, $X_2(\gamma) = g(X_1(\gamma))$. By the definitions of $\mathrm{Jcb}_{\mathrm{RV}}$ and Jcb_Γ, $\mathrm{Jcb}_{\mathrm{RV}}(g|X_2(\gamma)) = \mathrm{Jcb}_{\mathrm{RV}}(g)|X_2(\gamma)$ and likewise Jcb_Γ. By the case already analyzed (for the sets $X_1(\gamma), X_2(\gamma)$ defined in $\mathrm{ACVF}_{A(\gamma)}$) there exists an $A(\gamma)$-definable bijection $G_\gamma : X_1(\gamma) \times_{\phi_1,\mathrm{rv}} (VF^\times)^n \to X_2(\gamma) \times_{\phi_2,\mathrm{rv}} (VF^\times)^n$ with $\mathrm{rv}(\mathrm{Jcb}(G_\gamma)(x)) = \delta(x)$. As in Lemma 2.3 one can extend the G_γ by compactness to definable sets containing γ, cover X_1 by finitely many such definable sets, and glue together to obtain a single function G with the same property. □

Remark. Assume $\mathrm{Id}_X : (X, \phi_1) \to (X, \phi_2)$ has Jacobian 1 everywhere. Then it is possible to find F that is everywhere differentiable, of Jacobian precisely equal to 1. At the before the point where Hensel's lemma is quoted, it is possible to multiply the function by $J(F)^{-1}$ (not effecting the reduction, since $J(F) \in 1 + \mathcal{M}$). Then one obtains on each such coset a function of Jacobian 1 and therefore globally.

Example. Let $\phi_2(x) = \phi_1(x)^m$. A definable bijection

$$X \times_{\phi_1,\mathrm{rv}} (\mathrm{VF}^\times)^n \to X \times_{\phi_2,\mathrm{rv}} (\mathrm{VF}^\times)^n$$

is given by $(x, y) \mapsto (x, y^m)$. (If $\mathrm{rv}(u) = \phi(x)^m$, there exists a unique y with $\mathrm{rv}(y) = \phi(x)$ and $y^m = u$.)

Example 6.4. Proposition 6.1 need not remain valid over an RV-generated base set. Let $A = \mathrm{dcl}(c)$, c a transcendental point of $\mathbf{k}$. Let $f_1(y) = y$, $f_2(y) = 1$, $\mathbb{L}(Y, f_i) := \mathrm{VF} \times_{\mathrm{rv}, f_i} Y = \{(x, y) \in \mathrm{RV} \times Y : \mathrm{rv}(x) = y\}$. Then $\mathbb{L}(Y, f), \mathbb{L}(Y, f')$ are both open balls; over any field A' containing A, they are definably isomorphic, using a translation. But these balls are not definably isomorphic over A.

7 Special bijections and RV-blowups

We work with a V-minimal theory $\mathbf{T}$. Recall the lift $\mathbb{L} : \mathrm{RV}[\leq n, \cdot] \to \mathrm{VF}[n, \cdot]$, with $\rho_X : \mathbb{L}X \to X$. Our present goal is an intrinsic description in terms of RV of the congruence relation $\mathbb{L}X \simeq \mathbb{L}Y$.

A will denote a (VF, Γ, RV)-generated substructure of a model of $\mathbf{T}$. Note that $\mathbf{T}_A$ is also V-minimal (Corollary 3.39) so any lemma proved for $\mathbf{T}$ under our assumptions can be used for any $\mathbf{T}_A$.

The word "definable" below refers to $\mathbf{T}$. The categories VF, RV[$*$] defined below thus depend on $\mathbf{T}$; when necessary, we will denote them $\mathrm{VF}_{\mathbf{T}}$, etc. When $\mathbf{T}$ has the form $\mathbf{T} = \mathbf{T}^0_A$ for fixed $\mathbf{T}^0$ but varying A, we write VF_A, etc.

7.1 Special bijections

Let $X \subseteq \mathrm{VF}^{n+1} \times \mathrm{RV}^m$ be $\underset{\mathrm{rv}}{\sim}$-invariant. Say

$$X = \{(x, y, u) \in \mathrm{VF} \times \mathrm{VF}^n \times \mathrm{RV}^m : (\mathrm{rv}(x), \mathrm{rv}(y), u) \in \bar{X}\}.$$

(We allow x to be any of the $n + 1$ coordinates and y the others.)

Let $s(y, u)$ be a definable function into VF with $\underset{\mathrm{rv}}{\sim}$-invariant domain of definition

$$\mathrm{dom}(s) = \{(y, u) : (\mathrm{rv}(y), u) \in \bar{S}\}$$

and $\theta(u)$ a definable function on $\mathrm{pr}_u(\mathrm{dom}(s))$ into RV, such that $(s(y, u), y, u) \in X$ and $\mathrm{rv}(s(y, u)) = \theta(u)$ for $(y, u) \in \mathrm{dom}(s)$. Note that θ is uniquely defined (given s) if it exists. Let

$$X_1 = \{(x, y, u) \in X : (\mathrm{rv}(y), u) \in \bar{S}, \mathrm{rv}(x) = \theta(u)\}, \qquad X_2 = X \setminus X_1,$$
$$X_1' = \{(x, y, u) \in \mathrm{VF} \times \mathrm{dom}(s) : \mathrm{val}(x) > \mathrm{val}_{\mathrm{rv}}\theta(u)\}$$

and let $X' = X_1' \dot{\cup} X_2$. Also define $e_s : X' \to X$ to be the identity on X_2, and

$$e_s(x, y, u) = (x + s(y, u), y, u)$$

on X_1'.

Definition 7.1. $e_s : X' \to X$ is a definable bijection, called an *elementary bijection*. □

Lemma 7.2.

(1) *If* X *is* $\underset{\mathrm{rv}}{\sim}$*-invariant, so is* X'*. If* $X \to \mathrm{VF}^{n+1}$ *is finite-to-one, the same is true of* X'*.*
(2) *If* $X_i = \mathbb{L}\bar{X}_i$*,* $X'_1 = \mathbb{L}\bar{X}'_1$*, then* $\bar{X}'_1$ *is isomorphic to* $(\mathrm{RV}^{>0} \dot{\cup} \{1\}) \times \bar{S}$*, while* $\bar{X}_1$ *is isomorphic to* $\bar{S}$*.*
(3) *If the projection* $X \to \mathrm{VF}^{n+1}$ *has finite fibers, then so does the projection* $\mathrm{dom}(s) \to \mathrm{VF}^n$*, and also the projection* $\bar{S} \to \mathrm{RV}^n$*,* $(y', u) \mapsto y'$*.*
(4) e_s *has partial derivative matrix* I *everywhere, hence has Jacobian* 1*. Thus if* $F : X \to Y$ *is such that* $\mathrm{rv}\,\mathrm{Jcb}(F)$ *factors through* ρ_X*, then* $\mathrm{rv}\,\mathrm{Jcb}(F \circ e_s)$ *factors through* $\rho_{X'}$*.*

Proof. (1) and (4) are clear. The first isomorphism of (2) is obtained by dividing x by $\theta(u)$, the second is evident. For (3), note that if $(y, u) \in \mathrm{dom}(s)$ then $(s(y, u), y, u) \in X$ so by the assumption $u \in \mathrm{acl}(y, s(y, u))$. But for fixed y, $\{s(y, u) : u \in \mathrm{dom}(s)\}$ is finite, by Lemma 3.41. Thus, in fact, $(y, u) \in \mathrm{dom}(s)$ implies $u \in \mathrm{acl}(y)$. Hence $(y', u) \in \bar{S}$ implies $u \in \mathrm{acl}(y)$ for any y with $\mathrm{rv}(y) = y'$, so (fixing such a y) $\{u : (y', u) \in \bar{S}\}$ is finite for any given y'. □

A *special bijection* is a composition of elementary bijections and *auxiliary bijections* $(x_1, \ldots, x_n, u) \mapsto (x_1, \ldots, x_n, u, \mathrm{rv}(x_1), \ldots, \mathrm{rv}(x_n))$.

An elementary bijection depends on the data s of a partial section of $X \to \mathrm{VF}^n \times \mathrm{RV}^m$. Conversely, given s, if $\mathrm{rv}(s(y, u))$ depends only on u we can define $\theta(u) = \mathrm{rv}(s(y, u))$ and obtain a special bijection. If not, we can apply an auxiliary bijection to $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$, and obtain a set $X' \subseteq \mathrm{VF} \times \mathrm{RV}^{m+n}$, such that $\mathrm{rv}(x) = \mathrm{pr}_{m+1}(u)$ for $(x, u) \in X'$. For such a set X', the condition for existence of θ is automatic and we can define an elementary bijection $X'' \to X'$ based on s, and obtain a special bijection $X'' \to X$ as the composition.

The classes of auxiliary morphisms and elementary morphisms are all closed under disjoint union with any identity morphism, and it follows that the class special morphisms is closed under disjoint unions.

7.2 Special bijections in one variable and families of RV-valued functions

We consider here special bijections in dimension 1. An elementary bijection $X' \to X$ in dimension 1 involves a finite set B of rv-balls, and a set of "centers" of these balls (i.e., a set T containing a unique point $t(b)$ of each $b \in B$), and translates each ball so as to be centered at 0 (while fixing the RV coordinates). We say that $X' \to X$ blows up the balls in B, with centers T.

Given a special bijection $h' : X' \to X$, let $\mathrm{Fn}^{\mathrm{RV}}(X; h')$ be the set of definable functions $X \to \mathrm{RV}$ of the form $H(\rho_{X'}((h')^{-1}(x)))$, where H is a definable function. This is a finitely generated set of definable functions $X \to \mathrm{RV}$. There will usually be no ambiguity in writing $\mathrm{Fn}^{\mathrm{RV}}(X, X' \to X)$ instead.

Note that while a special bijection is an isomorphism in VF, an asymmetry exists: if $e : X' \to X$ is a special bijection, then $\mathrm{Fn}^{\mathrm{RV}}(X, X)$ is usually a proper subset of $\mathrm{Fn}^{\mathrm{RV}}(X, X' \to X)$.

What is the effect on $\mathrm{Fn}^{\mathrm{RV}}$ of passing from X' to X'', where $X'' \to X'$ is a special bijection? The auxiliary bijections have no effect. Assume rv is already a coordinate function of X'. Consider an elementary bijection $e_s : X'' \to X'$. Let $B = \{(x, u) \in X' : u \in \mathrm{dom}(s)\}$. Then the characteristic function 1_B lies in $\mathrm{Fn}^{\mathrm{RV}}(X', \mathrm{Id}_{X'})$; so $1_B \circ (h')^{-1}$ lies in $\mathrm{Fn}^{\mathrm{RV}}(X, h')$. Using this, we see that $\mathrm{Fn}^{\mathrm{RV}}(X', e_s)$ is generated over $\mathrm{Fn}^{\mathrm{RV}}(X', \mathrm{Id}_{X'})$ by the function $B \to \mathrm{RV}$, $(x, u) \mapsto \mathrm{rv}(x - s(u))$ (extended by 0 outside B). Thus if $h'' = h' \circ e_s : X'' \to X$, $\mathrm{Fn}^{\mathrm{RV}}(X, h'')$ is generated over $\mathrm{Fn}^{\mathrm{RV}}(X, h')$ by the composition of the function $(x, u) \mapsto \mathrm{rv}(x - s(u))$ with $(h')^{-1}$.

Conversely, if B is a finite union of open balls whose characteristic function lies in $\mathrm{Fn}^{\mathrm{RV}}(X, h')$, and if there exists a definable set T of representatives (one point $t(b)$ in each ball b of B), and a function $\phi = (\phi_1, \ldots, \phi_n)$, $\phi_i \in \mathrm{Fn}^{\mathrm{RV}}(X, h')$, with ϕ injective on T, then one can find a special bijection $X'' \to X'$ with composition $h'' : X'' \to X$, such that $\mathrm{Fn}^{\mathrm{RV}}(X, h'')$ is generated over $\mathrm{Fn}^{\mathrm{RV}}(X, h')$ by $y \mapsto \mathrm{rv}(y - t(y))$, $y \in b \in B$. Namely, let $\mathrm{dom}(s) = \phi(T)$, and for $u \in \mathrm{dom}(s)$ set $s(u) = h'^{-1}(t)$ if $t \in T$ and $\phi(t) = u$. In this situation, we will say that the balls in B are blown up by $X'' \to X'$, with centers T. Let $\theta(u) = \mathrm{rv}(s(u))$. Because $X' \to X$ may already have blown up some of the balls in B, $\mathrm{Fn}^{\mathrm{RV}}(X, h'')$ is generated over $\mathrm{Fn}^{\mathrm{RV}}(X, h')$ by the restriction of $y \mapsto \mathrm{rv}(y - t(y))$ to some subball of b, possibly proper. Nevertheless, we have the following.

Lemma 7.3. *The function* $y \mapsto \mathrm{rv}(y - t(y))$ *on* B *lies in* $\mathrm{Fn}^{\mathrm{RV}}(X, h'')$.

Proof. This follows from the following, more general claim. □

Claim. Let $c \in \mathrm{VF}$, $b \in \mathfrak{B}$ be definable, with $c \in b$. Let b' be an rv-ball with $c \in b'$. Then the function $\mathrm{rv}(x - c)$ on b is generated by its restriction to b', rv, and the characteristic function of b.

Proof. Let $x \in b \setminus b'$. From $\mathrm{rv}(x)$ compute $\mathrm{val}(x)$. If $\mathrm{val}(x) < \mathrm{val}(c)$, $\mathrm{rv}(x - c) = \mathrm{rv}(c)$. If $\mathrm{val}(x) > \mathrm{val}(c)$, $\mathrm{rv}(x - c) = \mathrm{rv}(x)$. When $\mathrm{val}(x) = \mathrm{val}(c)$, but $x \notin b'$, $\mathrm{rv}(x - c) = \mathrm{rv}(x) - \mathrm{rv}(c)$. Recall here that ${\mathrm{val}_{\mathrm{rv}}}^{-1}(\gamma)$ is the nonzero part of a $\mathbf{k}$-vector space; subtraction, for distinct elements u, v, can therefore be defined by $u - v = u(u^{-1}v - 1)$. □

Thus any special bijection can be understood as blowing up a certain finite number of balls (in a certain sequence and with certain centers). We will say that a special bijection $X'' \to X'$ is subordinate to a given partition of X if each ball blown up by $X'' \to X'$ is contained in some class of the partition.

It will sometimes be more convenient to work with the sets of functions $\mathrm{Fn}^{\mathrm{RV}}(X, h)$ than with the special bijections h themselves.

We observe that any finite set of definable functions $X \to \mathrm{RV}$ is contained in $\mathrm{Fn}^{\mathrm{RV}}(X; h)$ for some X', h.

Lemma 7.4. *Let* $X \subseteq \mathrm{VF} \times \mathrm{RV}^*$ *be* $\underset{\mathrm{rv}}{\sim}$*-invariant, and let* $f : X \to (\mathrm{RV} \cup \Gamma)$ *be a definable map. Then there exists an* $\underset{\mathrm{rv}}{\sim}$*-invariant* $X' \subseteq \mathrm{VF} \times \mathrm{RV}^*$ *a special bijection* $h : X' \to X$*, and a definable function* t *such that* $t \circ \rho_{X'} = f \circ h$*. Moreover, if* $X = \cup_{i=1}^m P_i$ *is a finite partition of* X *into sets whose characteristic functions factor through* ρ*, we can find* $X' \to X$ *subordinate to this partition.*

Proof. Say $X \subseteq \mathrm{VF} \times \mathrm{RV}^m$; let $\pi : X \to \mathrm{VF}$, $\pi' : X \to \mathrm{RV}^m$ be the projections. Applying an auxiliary bijection, we may assume $\mathrm{rv}(\pi(x)) = \mathrm{pr}_m \pi'(x)$, i.e., $\mathrm{rv}(\pi(x))$ agrees with one of the coordinates of $\pi'(x)$. We now claim that there exists a finite $F' \subseteq \mathrm{RV}^m$, such that away from $\pi'^{-1}(F')$, f factors through π'. To prove this, it suffices to show that if p is a complete type of X and $\pi'_* p$ is nonalgebraic (i.e., not contained in a finite definable set), then $f|p$ factors through π'; this follows from Lemma 3.20.

We can thus restrict attention to $\pi'^{-1}(F')$; our special bijections will be the identity away from this. Thus we may assume $\pi'(X)$ is finite. Recall that (since an auxiliary bijection has been applied) rv is constant on each fiber of π'. In this case there is no problem relativizing to each fiber of π', and then collecting them together (Lemma 2.3), we may assume, in fact, that $\pi'(X)$ consists of a single point $\{u\}$. In this case the partition (since it is defined via ρ) will automatically be respected.

The rest of the proof is similar to Lemma 4.2. We first consider functions f with finite support F (i.e., $f(x) = 0$ for $x \notin F$) and prove the analogue of the statement of the lemma for them. If $F = \{0\} \times \{u\}$ then $F = \rho^{-1}(\{(0)\} \times \{u\})$ so the claim is clear. If $F = \{(x_0, u)\}$, let $s : \{u\} \to \mathrm{VF}$, $s(u) = x_0$. Applying e_s returns us to the previous case. If $F = F_0 \times \{u\}$ has more than one point, we use induction on the number of points. Let $s(u)$ be the average of F_0. Apply the special bijection e_s. Then the result is a situation where rv is no longer constant on the fiber. Applying an auxiliary bijection to make it constant again, the fibers of $F \to \mathrm{RV}^{m+1}$ become smaller.

The case of the characteristic function of a finite union of balls is similar (following Lemma 4.2).

Now consider a general function f. Having disposed of the case of characteristic function, it suffices to treat f on each piece of any given partition. Thus we can assume f has the form of Corollary 4.4, $f(x) = H(\mathrm{rv}(x - n(x)))$. Translating by the $n(x)$ as in the previous cases, we may assume $n(x) = 0$. But then again f factors through ρ and rv, so one additional auxiliary bijection suffices. □

Corollary 7.5. *Let $X, Y \subseteq \mathrm{VF}^n \times \mathrm{RV}^*$, and let $f : X \to Y$ be a definable bijection. Then there exists a special bijection $h : X' \to X$, and t such that $\rho_Y \circ (f \circ h) = t \circ \rho_{X'}$.*

It can be found subordinate to a given finite partition, factoring through ρ_X. □

We wish to obtain a symmetric version of Corollary 7.5. We will say that bijections $f, g : X \to Y$ *differ by special bijections* if there exist special bijections h_1, h_2 with $h_2 g = f h_1$. We show that every definable bijection between $\underset{\mathrm{rv}}{\sim}$-invariant objects differs by special bijections from an $\underset{\mathrm{rv}}{\sim}$-invariant bijection.

Lemma 7.6. *Let $X \subseteq \mathrm{VF} \times \mathrm{RV}^m$, $Y \subseteq \mathrm{VF} \times \mathrm{RV}^{m'}$ be definable, $\underset{\mathrm{rv}}{\sim}$-invariant; let $F : X \to Y$ be a definable bijection. Then there exist special bijections $h_X : X' \to X$, $h_Y : Y' \to Y$, and an $\underset{\mathrm{rv}}{\sim}$-invariant bijection $F' : X' \to Y'$ with $F = h_Y F' h_X^{-1}$; i.e., F differs from an $\underset{\mathrm{rv}}{\sim}$-invariant morphism by special bijections.*

Proof. It suffices to find h_X, h_Y such that $\mathrm{Fn}^{\mathrm{RV}}(X, h_X) = F \circ \mathrm{Fn}^{\mathrm{RV}}(Y, h_Y)$; for then we can let $F' = h_Y^{-1} F h_X$.

Let $X = \cup_{i=1}^m P_i$ be a partition as in Proposition 5.1. By Lemma 7.4, there exist X_0, Y_1 and special bijections $X_0 \to X, Y_1 \to Y$, such that the characteristic functions of the sets P_i (respectively, the sets $F(P_i)$) are in $\mathrm{Fn}^{\mathrm{RV}}(X, X_0 \to X)$ (respectively, $\mathrm{Fn}^{\mathrm{RV}}(Y, Y_1 \to Y)$).

By Corollary 7.5, one can find a special $X_1 \to X_0$ such that $\mathrm{Fn}^{\mathrm{RV}}(X, X_1 \to X)$ contains $F \circ \mathrm{Fn}^{\mathrm{RV}}(Y, Y_1 \to Y)$. By another application of the same, one can find a special bijection $Y_* \to Y_1$ subordinate to $\{F(P_i)\}$ such that

$$\mathrm{Fn}^{\mathrm{RV}}(Y, Y_* \to Y) \supseteq F^{-1} \circ \mathrm{Fn}^{\mathrm{RV}}(X, X_1 \to X). \tag{7.1}$$

Now Y_* is obtained by composing a sequence $Y_* = Y_m \to \cdots \to Y_1$ of elementary bijections and auxiliary bijections. We define inductively $X_m \to \cdots \to X_2 \to X_1$, such that

$$\mathrm{Fn}^{\mathrm{RV}}(Y, Y_k \to Y) \circ F \subseteq \mathrm{Fn}^{\mathrm{RV}}(X, X_k \to X). \tag{7.2}$$

Let $k \geq 1$. Y_{k+1} is obtained by blowing up a finite union of balls B of Y, with a definable set T of representatives such that some $\phi \in \mathrm{Fn}^{\mathrm{RV}}(Y, Y_k \to Y)$ is injective on T; then $\mathrm{Fn}^{\mathrm{RV}}(Y, Y_{k+1} \to Y)$ is generated over $\mathrm{Fn}^{\mathrm{RV}}(Y, Y_k \to Y)$ by ψ, where for $y \in b \in B$ $\psi(y) = \mathrm{rv}(y - t(b))$ (Lemma 7.3). By the choice of the partition $\{P_i\}$, $F^{-1}(B)$ is also a finite union of balls.

Now $F^{-1}(B)$, with center set $F^{-1}(T)$, can serve as data for a special bijection: the requirement about the characteristic function of B and the injective function on T being in $\mathrm{Fn}^{\mathrm{RV}}$ are satisfied by virtue of Lemma 7.3. We can thus define $X_{k+1} \to X_k$ so as to blow up $F^{-1}(B)$ with center set $F^{-1}(T)$. By Lemma 5.4, $\mathrm{rv}(F(x) - F(x'))$ is a function of $\mathrm{rv}(x - x')$ (and conversely) on each of these balls, so $\mathrm{Fn}^{\mathrm{RV}}(X, X_{k+1} \to X)$ is generated over $\mathrm{Fn}^{\mathrm{RV}}(X, X_k)$ by $\psi \circ F$. Hence (7.2) remains valid for $k + 1$.

Now by (7.1), $\mathrm{Fn}^{\mathrm{RV}}(X, X_1 \to X) \subseteq \mathrm{Fn}^{\mathrm{RV}}(Y, Y_* \to Y) \circ F$; since the generators match at each stage, by induction on $k \leq m$,

$$\mathrm{Fn}^{\mathrm{RV}}(X, X_k \to X) \subseteq \mathrm{Fn}^{\mathrm{RV}}(Y, Y_m \to Y) \circ F. \tag{7.3}$$

By (7.2) and (7.3) for $k = m$, $\mathrm{Fn}^{\mathrm{RV}}(X, X_m \to X) = \mathrm{Fn}^{\mathrm{RV}}(Y, Y_* \to Y) \circ F$. □

For the sake of possible future refinements, we note that the proof of Lemma 7.6 also shows the following.

Lemma 7.7. *Let $X \subseteq \mathrm{VF} \times \mathrm{RV}^m$, $Y \subseteq \mathrm{VF} \times \mathrm{RV}^{m'}$ be definable, $\underset{\mathrm{rv}}{\sim}$-invariant; let $F : X \to Y$ be a definable bijection. If a Proposition* 5.1 *partition for F has characteristic functions factoring through ρ_X, ρ_Y, and if F is $\underset{\mathrm{rv}}{\sim}$-invariant, then for any special bijection $h'_X : X' \to X$, there exists a special bijection $h'_Y : Y' \to Y'$ such that $(h'_Y)^{-1} F h'_X$ is $\underset{\mathrm{rv}}{\sim}$-invariant.* □

7.3 Several variables

We will now show in general that any definable map from an $\underset{\mathrm{rv}}{\sim}$-invariant object to RV factors through the inverse of a special bijection, and the standard map ρ.

Lemma 7.8. *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be $\underset{\mathrm{rv}}{\sim}$-invariant, and let $\phi : X \to (\mathrm{RV} \cup \Gamma)$. Then there exists a special bijection $h : X' \to X$, and a definable function τ such that $\tau \circ \rho_{X'} = \phi \circ h$.*

Proof. By induction on n. For $n = 0$ we can take $X' = X$, since $\rho_X = \mathrm{Id}_X$.

For $n = 1$ and $X \subseteq \mathrm{VF}$, by Lemma 7.4, there exists $\mu = \mu(X, \phi) \in \mathbb{N}$ such that the lemma holds for some h that is a composition of μ elementary and auxiliary bijections. It is easy to verify the semicontinuity of μ with respect to the definable topology: if X_t is a definable family of definable sets, so that X_b is $A(b)$-definable, and $\mu(X_b, \phi|X_b) = m$, then there exists a definable set D with $b \in D$ such that if $b' \in D$, then $\mu(X_{b'}, \phi|X_{b'}) \leq m$.

Assume the lemma is known for n and suppose $X \subseteq \mathrm{VF} \times Y$, with $Y \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$. For any $b \in Y$, let $X_b = \{x : (x, b) \in X\} \subseteq \mathrm{VF}$; so X_b is $A(b)$-definable.

Let $\mu = \max_b \mu(X_b, \phi|X_b)$. Consider first the case $\mu = 0$. Then $\phi|X_b = \tau_b \circ \rho|X_b$, for some $A(b)$-definable function $\tau_b : \mathrm{RV}^m \to (\mathrm{RV} \cup \Gamma)$. By stable embeddedness and elimination of imaginaries in $\mathrm{RV} \cup \Gamma$, there exists (Section 2.1) a canonical parameter $d \in (\mathrm{RV} \cup \Gamma)^l$, and an A-definable function τ, such that $\tau_b(t) = \tau(d, t)$; and d itself is definable from τ_b, so we can write $d = \delta(b)$ for some definable $\delta : Y \to (\mathrm{RV} \cup \Gamma)^l$. Using the induction hypothesis for (Y, δ) in place of (X, ϕ), we find that there exists an $\underset{\mathrm{rv}}{\sim}$-invariant $Y' \subseteq \mathrm{VF}^n \times \mathrm{RV}^*$, a special $h_Y : Y' \to Y$, and a definable τ_Y, such that $\tau_Y \circ \rho_{Y'} = \delta \circ h_Y$. Let $X' = X \times_Y Y'$, $h(x, y') = (x, h_Y(y'))$. An elementary bijection to Y determines one to X, where the function s does not make use of the first coordinate; so $h : X' \to X$ is special. In this case, the lemma is proved: $\phi \circ h(x, y') = \phi(x, h_Y(y')) = \tau(\delta(h_Y(y')), \rho(x, y)) = \tau(\tau_Y(\rho_{Y'}(y')), \rho(x, y))$.

Next suppose $\mu > 0$. Applying an auxiliary bijection, we may assume that for some definable function (in fact, projection) p, $\mathrm{rv}(x) = p(u)$ for $(x, y, u) \in X$. For each $b \in Y(M)$ (with M any model of $\mathbf{T}_A$) there exists an elementary bijection $h_b : X'_b \to X_b$, such that $\mu(X'_b, \phi|X'_b) < \mu$; h_b is determined by s_b, θ_b,with $s_b \in \mathrm{rv}(s_b) = \theta_b$, and $(s_b, \theta_b) \in X$. (The u-variables have been absorbed into b.) By compactness, one can take $s_b = s(b)$ and $\theta_b = \theta'(b)$ for some definable functions s, θ'. By the inductive hypothesis applied to (Y, θ'), as in the previous paragraph, we can assume $\theta'(y, u) = \theta(u)$ for some definable θ. Applying the special bijection with data (s, θ) now amounts to blowing up (s_b, θ_b) uniformly over each b, and thus reduces the value of μ. □

Question 7.9. Is Proposition 7.6 true in higher dimensions?

Corollary 7.10. *Let $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ be definable. Then every definable function $\phi : X \to \Gamma$ factors through a definable function $X \to \mathrm{RV}^*$.*

Proof. By Lemma 4.5 we may assume X is $\underset{\text{rv}}{\sim}$-invariant; now the corollary follows from Lemma 7.8. □

(It is convenient to note this here, but it can also be proved with the methods of Section 3; the main point is that on the generic type of ball with center c, every function into $\mathrm{RV} \cup \Gamma$ factors through $\mathrm{rv}(x - c)$; while on a transitive ball, every function into $\mathrm{RV} \cup \Gamma$ is constant.)

Consider pairs (X', f') with $X', f' : X' \to \mathrm{VF}^n$ definable, such that f' has RV-fibers. A bijection $g : X' \to X''$ is said to be *relatively unary* (with respect to f', f'') if it commutes with $n - 1$ coordinate projections, i.e., $\mathrm{pr}_i\, f'' g = \mathrm{pr}_i\, f'$ for all but at most one value of i.

Given $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$, we view it as a pair (X, f) with f the projection to VF^n. Thus for $X, Y \subseteq \mathrm{VF}^n \times \mathrm{RV}^*$, the notion $F : X \to Y$ is relatively unary is defined.

Note that the elementary bijections are relatively unary, as are the auxiliary bijections.

Lemma 7.11. *Let $X, Y \subseteq \mathrm{VF}^n \times \mathrm{RV}^*$, and let $F : X \to Y$ be a definable bijection. Then F can be written as the composition of relatively unary morphisms of* $\mathrm{VF}[n, \cdot]$.

Proof. We have X with two finite-to-one maps $f, g : X \to \mathrm{VF}^n$ (the projection and the composition of F with the projection $Y \to \mathrm{VF}^n$). We must decompose the identity $X \to X$ into a composition of relatively unary maps $(X, f) \to (X, g)$.

Begin with the case $n = 2$; we are given (X, f_1, f_2) and (X, g_1, g_2).

Claim. There exists a definable partition of X into sets X_{ij} such that $(f_i, g_j) : X \to \mathrm{VF}^2$ is finite-to-one.

Proof. Let $a \in X$. We wish to show that for some i, j, $a \in \mathrm{acl}(f_i(a), g_j(a))$. This follows from the exchange principle for algebraic closure in VF: if $a \in \mathrm{acl}(\emptyset)$, there is nothing to show. Otherwise, $g_j(a) \notin \mathrm{acl}(\emptyset)$ for some j; in this case either $a \in \mathrm{acl}(f_1(a), g_j(a))$ or $f_1(a) \in \mathrm{acl}(g_j(a))$, and then $a \in \mathrm{acl}(f_2(a), g_j(a))$. The claim follows by compactness. □

Let $h : X' \to X$ be a special bijection such that the characteristic functions of X_{ij} are in $\mathrm{Fn}^{\mathrm{RV}}(X, X' \to X)$. (Lemma 7.8). Since h is composition of relatively unary bijections, we may replace X by X' (and f_i, g_i by $f_i \circ h, g_i \circ h$, respectively). Thus we may assume the characteristic function of X_{ij} is in $\mathrm{Fn}^{\mathrm{RV}}(X, X)$, i.e., $X_{ij} \in VFr[n]$. But then it suffices to treat each X_{ij} separately, say, X_{11}. In this case the identity map on X takes

$$\begin{aligned}(X, f_1, f_2) &\mapsto (X, f_1, g_1) \mapsto (X, g_2, g_1) \mapsto (X, g_2, g_1 - g_2)\\ &\mapsto (X, g_1, g_1 - g_2) \mapsto (X, g_1, g_2),\end{aligned}$$

where each step is relatively unary.

When $n > 2$, we move between $(X, f_1, \ldots, f_n)$ and $(X, g_1, \ldots, g_n)$, by partitioning, and on a given piece replacing each f_i by some g_j, one at a time. □

7.4 RV-blowups

We now define the RV-counterparts of the special bijections, which will be called RV-blowups. These will not be bijections; the kernel of the homomorphism $\mathbb{L} : K_+[\mathrm{RV}] \to K_+[\mathrm{VF}]$ will be seen to be obtained by formally inverting RV-blowups. Let $\mathrm{RV}_\infty^{>0} = \{x \in \mathrm{RV} : \mathrm{val}(x) > 0 \cup \{\infty\}\} \subseteq \mathrm{RV}_\infty$. In the RV[$\leq 1$]-presentation, $\mathrm{RV}_\infty^{>0} = [\mathrm{RV}^{>0}]_1 + [1]_0$ (cf. Section 3.8).

Definition 7.12.

(1) Let $\mathbf{Y} = (Y, f) \in \mathrm{Ob}\,\mathrm{RV}_\infty[n, \cdot]$ be such that $f_n(y) \in \mathrm{acl}(f_1(y), \ldots, f_{n-1}(y))$, and $f_n(y) \neq \infty$. Let $Y' = Y \times \mathrm{RV}_\infty^{>0}$. For $(y, t) \in Y'$, define $f' = (f_1', \ldots, f_n')$ by $f_i'(y, t) = f_i(y)$ for $i < n$, $f_n'(y, t) = tf_n(y)$. Then $\widetilde{\mathbf{Y}} = (Y', f')$ is an *elementary blowup* of $\mathbf{Y}$. It comes with the projection map $Y' \to Y$.

(2) Let $\mathbf{X} = (X, g) \in \mathrm{Ob}\,\mathrm{RV}_\infty[n, \cdot]$, $X = X' \dot{\cup} X''$, $g' = g|X'$, $g'' = g|X''$, and let $\phi : \mathbf{Y} \to (X', g')$ be an $\mathrm{RV}_{\mathrm{vol}}$-isomorphism. Then the RV-*blowup* $\widetilde{\mathbf{X}}_\phi$ is defined to be $\widetilde{\mathbf{Y}} + (X'', g'') = (Y' \dot{\cup} X'', f' \dot{\cup} g'')$. It comes with $b : Y' \dot{\cup} X'' \to X$, defined to be the identity on X'', and the projection on Y'. X' is called the *blowup locus* of $b : \widetilde{\mathbf{X}}_\phi \to \mathbf{X}$.

An *iterated* RV-*blowup* is obtained by finitely many iterations of RV-blowups.

Since blowups in the sense of algebraic geometry will not occur in this paper, we will say "blowup" for RV-blowup.

Remark 7.13. In the definition of an elementary blowup, $\dim_{\mathrm{RV}}(Y) < n$. For such Y, $\phi : \mathbf{Y} \to (X', g')$ is an $\mathrm{RV}_{\mathrm{vol}}[\leq n, \cdot]$-isomorphism iff it is an $\mathrm{RV}_{\Gamma\text{-vol}'}$-isomorphism (Definition 5.21).

Lemma 7.14.

(1) *Let* $\mathbf{Y}'$ *be an elementary blowup of* $\mathbf{Y}$. $\mathbf{Y}'$ *is* $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$-*isomorphic to* $\mathbf{Y}'' = (Y'', f'')$, *with* $Y'' = \{(y, t) \in Y \times \mathrm{RV}_\infty : \mathrm{val}_{\mathrm{rv}}(t) > f_n(y)\}$, $f''(y, t) = (f_1(y), \ldots, f_{n-1}(y), t)$.

(2) *An elementary blowup* $\mathbf{Y}'$ *of* $\mathbf{Y}$ *is* $\mathrm{RV}_\infty[n, \cdot]$-*isomorphic to* $(Y \times \mathrm{RV}_\infty, f')$ *for any* f' *isogenous to* $(f_1, \ldots, f_n, t)$.

(3) *Up to isomorphism, the blowup depends only on the blowup locus. In other words, if* X, X', g, g' *are as in Definition* 7.12, *and* $\phi_i : \mathbf{Y}_i \to (X', g')$ $(i = 1, 2)$ *are isomorphisms, then* $\widetilde{\mathbf{X}}_{\phi_1}, \widetilde{\mathbf{X}}_{\phi_2}$ *are* $\mathbf{X}$-*isomorphic in* $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$.

Proof.

(1) The isomorphism is given by $(y, t) \mapsto (y, tf_n(y))$.

(2) The identity map on $Y \times \mathrm{RV}$ is an $\mathrm{RV}_\infty[n, \cdot]$ isomorphism.

(3) Let $\psi_0 = {\phi_2}^{-1}\phi_1$, and define $\psi_1 : Y_1 \times \mathrm{RV}_\infty^{>0} \to Y_2 \times \mathrm{RV}_\infty^{>0}$ by $\psi(y, t) = (\psi_0(y), t)$. The sum of the values of the n coordinates of $\widetilde{\mathbf{Y}}_i$ is then $(\sum_{i<n} \mathrm{val}_{\mathrm{rv}} f_i) + (\mathrm{val}_{\mathrm{rv}}(t) + \mathrm{val}_{\mathrm{rv}} f_n)$ in both cases. Since by assumption $\psi_0 : Y_1 \to Y_2$ is an $\mathrm{RV}_{\mathrm{vol}}$-isomorphism, it preserves $\sum_{i \leq n} \mathrm{val}_{\mathrm{rv}} f_i$ and so ψ_1 too is an $\mathrm{RV}_{\Gamma\text{-vol}'}$-isomorphism; thus $\mathrm{Jcb}_{\mathrm{RV}}(\psi_1) \in \mathbf{k}^*$, i.e., let $\theta Y_1 \to \mathbf{k}^*$ be a definable map such

that $\theta = \mathrm{Jcb}_{\mathrm{RV}}(\psi_1)$ almost everywhere. Define $\psi : Y_1 \times \mathrm{RV}_\infty^{>0} \to Y_2 \times \mathrm{RV}_\infty^{>0}$ by $\psi(y,t) = (\psi_0(y), t/\theta(y))$. Then one computes immediately that $\mathrm{Jcb}_{\mathrm{RV}}(\psi) = 1$, so ψ is an $\mathrm{RV}_{\mathrm{vol}}[n,\cdot]$-isomorphism, and hence so is $\psi \dot\cup \mathrm{Id}_{X''} : \widetilde{\mathbf{X}}_{\phi_1} \to \widetilde{\mathbf{X}}_{\phi_2}$. □

Here is a coordinate-free description of RV-blowups; we will not really use it in the subsequent development.

Lemma 7.15.

(1) *Let* $\mathbf{Y} = (Y, g) \in \mathrm{Ob}\,\mathrm{RV}_\infty[n,\cdot]$, *with* $\dim(g(Y)) < n$; *let* $f : Y \to \mathrm{RV}^{n-1}$ *be isogenous to* g. *Let* $h : Y \to \mathrm{RV}$ *be definable, with* $h(y) \in \mathrm{acl}(g(y))$ *for* $y \in Y$, *and with* $\sum(g) = \sum(f) + \mathrm{val}_{\mathrm{rv}}(h)$. *Let* $Y' = Y \times \mathrm{RV}_\infty^{>0}$, *and* $f'(y,t) = (f(y), th(y))$. *Then* $\mathbf{Y}' = (Y', f')$ *with the projection map to* Y *is a blowup.*
(2) *Let* $\mathbf{Y}'' \to \mathbf{Y}$ *be a blowup with blowup locus* Y. *Then there exist* f, h *such that with* $\mathbf{Y}'$ *as in* (3), $\mathbf{Y}'', \mathbf{Y}'$ *are isomorphic over* $\mathbf{Y}$.

Proof.

(1) Since $\dim_{\mathrm{RV}}(g(Y)) < n$, $\mathrm{Id}_Y : (Y,(f,h)) \to (Y,g)$ is an $\mathrm{RV}_{\mathrm{vol}}$-isomorphism. Use this as ϕ in the definition of blowup.
(2) With notation as in Definition 7.12, let $h = g_n \circ \phi^{-1}$, $f = (g_1,\dots,g_{n-1}) \circ \phi^{-1}$. □

Definition 7.16. For $\mathcal{C} = \mathrm{RV}[\le n,\cdot]$ or $\mathcal{C} = \mathrm{RV}_{\mathrm{vol}}[\le n,\cdot]$, let $\mathrm{I_{sp}}[\le n]$ be the set of pairs $(\mathbf{X}_1, \mathbf{X}_2) \in \mathrm{Ob}\,\mathcal{C}$ such that there exist iterated blowups $b_i : \widetilde{\mathbf{X}}_i \to \mathbf{X}_i$ and an $\mathcal{C}$-isomorphism $F : \widetilde{\mathbf{X}}_1 \to \widetilde{\mathbf{X}}_2$.

When n is clear from the context, we will just write $\mathrm{I_{sp}}$.

Definition 7.17. Let 1_0 denote the one-element object of $\mathrm{RV}[0]$. Given a definable set $X \subseteq \mathrm{RV}^n$ let $\mathbf{X}_n$ denote $(X, \mathrm{Id}_X) \in \mathrm{RV}[n]$, and $[\mathbf{X}]_n$ the class in $K_+(\mathrm{RV}[n])$. Write $[1]_1$ for $[\{1\}]_1$ (where $\{1\}$ is the singleton set of the identity element of $\mathbf{k}$).

Lemma 7.18. *Let* $\mathcal{C} = \mathrm{RV}[\le n,\cdot]$ *or* $\mathcal{C} = \mathrm{RV}_{\mathrm{vol}}[\le n,\cdot]$.

(1) *Let* $f : \mathbf{X}_1 \to \mathbf{X}_2$ *be a* $\mathcal{C}$*-isomorphism, and let* $b_1 : \widetilde{\mathbf{X}}_1 \to \mathbf{X}_1$ *be a blowup. Then there exists a blowup* $b_2 : \widetilde{\mathbf{X}}_2 \to \mathbf{X}_2$ *and a* $\mathcal{C}$*-isomorphism* $F : \widetilde{\mathbf{X}}_1 \to \widetilde{\mathbf{X}}_2$ *with* $b_2 F = f b_1$.
(2) *If* $b : \widetilde{\mathbf{X}} \to \mathbf{X}$ *is a blowup, then so are* $b \dot\cup \mathrm{Id} : \widetilde{\mathbf{X}} \dot\cup \mathbf{Z} \to \mathbf{X} \dot\cup \mathbf{Z}$ *and* $(b \times \mathrm{Id}) : \widetilde{\mathbf{X}} \times \mathbf{Z} \to \mathbf{X} \times \mathbf{Z}$.
(3) *Let* $b_i : \widetilde{\mathbf{X}}_{\phi_i} \to \mathbf{X}$ *be a blowup* ($i = 1, 2$)*. Then there exist blowups* $b_i' : \mathbf{Z}_i \to \widetilde{\mathbf{X}}_{\phi_i}$ *and an isomorphism* $F : \mathbf{Z}_1 \to \mathbf{Z}_2$ *such that* $b_2 b_2' F = b_1 b_1'$.
(4) *Same as* (1)–(3) *for iterated blowups.*
(5) $\mathrm{I_{sp}}$ *is an equivalence relation. It induces a semiring congruence on* $K_+\,\mathrm{RV}[*,\cdot]$, *respectively,* $K_+\,\mathrm{RV}_{\mathrm{vol}}[*,\cdot]$.
(6) *As a semiring congruence on* $K_+\,\mathrm{RV}[*,\cdot]$, $\mathrm{I_{sp}}$ *is generated by* $([1]_1, [\mathrm{RV}^{>0}]_1 + 1_0)$.

Proof.

(1) This reduces to the case of elementary blowups. If $\mathcal{C} = \mathrm{RV}_{\mathrm{vol}}[n, \cdot]$, then the composition $f \circ b_1$ is already a blowup. If $\mathcal{C} = \mathrm{RV}[\leq n, \cdot]$, it is also clear using Lemma 7.14(2).

(2) This follows from the definition of blowup.

(3) If b_1 is the identity, let $b_1' = b_2, b_2' = \mathrm{Id}$, $F = \mathrm{Id}$; similarly if b_2 is the identity. If $X = X' \dot{\cup} X''$ and the statement is true above X' and above X'', then by glueing it is true also above X. We thus reduce to the case that b_1, b_2 both are blowups with blowup locus equal to X. But then by Lemma 7.14(3), there exists an isomorphism $F : \widetilde{\mathbf{X}}_{\phi_1} \to \widetilde{\mathbf{X}}_{\phi_2}$ over $\mathbf{X}$. Let $b_1' = b_2' = \mathrm{Id}$.

(4) For (1)–(2) the induction is immediate. For (3), write k-blowup as shorthand for "an iteration of k blowups." We show by induction on k_1, k' a more precise form.

Claim. If $\mathbf{X}_1 \to \mathbf{X}$ is a k_1-blowup, and $\mathbf{X}' \to \mathbf{X}$ is a k'-blowup, then there exists an k'-blowup $\mathbf{Z}_1' \to \mathbf{X}_1$ a k_1-blowup $\mathbf{Z}' \to \mathbf{X}$, and an $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$-isomorphism $\mathbf{Z}_1' \to \mathbf{Z}'$ over $\mathbf{X}$.

If $k_1 = k' = 1$, this is (3). Thus say $k' > 1$. The map $\mathbf{X}' \to \mathbf{X}$ is a composition $\mathbf{X}' \to \mathbf{X}_2 \to \mathbf{X}$, where $\mathbf{X}_2 \to \mathbf{X}$ is a $k' - 1$-blowup and $\mathbf{X}' \to \mathbf{X}_2$ is a blowup. By induction there is a $k' - 1$-blowup $\mathbf{Z}_1 \to \mathbf{X}_1$ and a k_1-blowup $\mathbf{Z}_2 \to \mathbf{X}_2$ and an $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$-isomorphism $\mathbf{Z}_1 \to \mathbf{Z}_2$ over $\mathbf{X}$.

By induction again there is a blowup and $\mathbf{Z}_2' \to \mathbf{Z}_2$, a k_1-blowup $\mathbf{Z}' \to \mathbf{X}'$ an $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$-isomorphism $\mathbf{Z}' \to \mathbf{Z}_2$ over $\mathbf{X}_2$. By (1) there exists a blowup $\mathbf{Z}_1' \to \mathbf{Z}_1$ and an $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$-isomorphism $\mathbf{Z}_1' \to \mathbf{Z}_2'$, making the $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_1', \mathbf{Z}_2$-square commute. Thus $\mathbf{Z}_1 \to \mathbf{X}_1$ is a k'-blowup, $\mathbf{Z}' \to \mathbf{X}'$ is a k_1-blowup, and we have a composed isomorphism $\mathbf{Z}_1' \to \mathbf{Z}_2' \to \mathbf{Z}'$ over $\mathbf{X}$.

(5) If $(\mathbf{X}_1, \mathbf{X}_2), (\mathbf{X}_2, \mathbf{X}_3) \in \mathrm{I}_{\mathrm{sp}}$, there are iterated blowups $\mathbf{X}_1' \to \mathbf{X}_1, \mathbf{X}_2' \to \mathbf{X}_2$ and an isomorphism $\mathbf{X}_1' \to \mathbf{X}_2'$; and also $\mathbf{X}_2'' \to \mathbf{X}_2, \mathbf{X}_3' \to \mathbf{X}_3$ and $\mathbf{X}_2'' \to \mathbf{X}_3$. Using (3) for iterated blowups, there exist iterated blowups $\widehat{\mathbf{X}_2}' \to \mathbf{X}_2', \widehat{\mathbf{X}_2}'' \to \mathbf{X}_2''$, and an isomorphism $\widehat{\mathbf{X}_2}' \to \widehat{\mathbf{X}_2}'$. By (1), for iterated blowups there are iterated blowups $\widehat{\mathbf{X}_1} \to \mathbf{X}_1', \widehat{\mathbf{X}_3} \to \mathbf{X}_3$ and isomorphisms $\widehat{\mathbf{X}_1} \to \widehat{\mathbf{X}_2}', \widehat{\mathbf{X}_2}'' \to \widehat{\mathbf{X}_3}$, with the natural diagrams commuting. Composing, we obtain $\widehat{\mathbf{X}_1} \to \widehat{\mathbf{X}_3}$, showing that $(\mathbf{X}_1, \mathbf{X}_3) \in \mathrm{I}_{\mathrm{sp}}$. Hence I_{sp} is an equivalence relation.

Isomorphic objects are I_{sp}-equivalent, so an equivalence relation on the semiring $K_+ \mathcal{C}$ is induced. If $(X_1, X_2) \in \mathrm{I}_{\mathrm{sp}}$, then by (2), $(X_1 \dot{\cup} Z, X_2 \dot{\cup} Z) \in \mathrm{I}_{\mathrm{sp}}$, and $(X_1 \times Z, X_2 \times Z) \in \mathrm{I}_{\mathrm{sp}}$. It follows that I_{sp} induces a congruence on the semiring $K_+ \mathcal{C}$.

(6) We can blow up 1_1 to $\mathrm{RV}_1^{>0} + 1_0$, so $([1]_1, [\mathrm{RV}^{>0}]_1 + 1_0) \in \mathrm{I}_{\mathrm{sp}}$. Conversely, under the conditions of Definition 7.12, let $\mathbf{Y}^- = [(Y, f_1, \ldots, f_{n-1})]$; then $[\mathbf{Y}] = [(Y, f_1, \ldots, f_{n-1}, 0)] = [\mathbf{Y}^-] \times [1]_1$ by Lemma 7.14, and we have

$$[\widetilde{\mathbf{X}}_{\mathbf{Y}}] = [\mathbf{Y}]_{n-1} + [\mathbf{Y}]_{n-1} \times [\mathrm{RV}^{>0}]_1 + [\mathbf{X}''] \cong_{\mathrm{I}_{\mathrm{sp}}} [\mathbf{Y}] \times [1]_1 + [\mathbf{X}''] = [\mathbf{X}]$$

modulo the congruence generated by $([1]_1, [\mathrm{RV}^{>0}]_1 + 1_0)$. □

We now relate special bijections to blowing ups. Given $\mathbf{X} = (X, f), \mathbf{X}' = (X', f') \in \mathrm{RV}[n, \cdot]$, say, $\mathbf{X}, \mathbf{X}'$ are *strongly isomorphic* if there exists a bijection $\phi : X \to X'$ with $f' = \phi f$. Strong isomorphisms are always in $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$.

Up to strong isomorphism, an elementary blowup of (Y, f) can be put in a different form: $(\widetilde{\mathbf{Y}}) \simeq (Y'', f'')$, $Y'' = \{(z, y) : y \in Y, \mathrm{val}_{\mathrm{rv}}(z) > \mathrm{val}_{\mathrm{rv}} f_n(y)\}$, $f_i(z, y) = f_i(y)$ for $i < n$, $f_n(z, y) = z$. The strong isomorphism $Y'' \to Y'$ is given by $(z, y) \mapsto (y, z/f_n(y))$. This matches precisely the definition of special bijection, and makes evident the following lemma.

Lemma 7.19. *Let* $\mathcal{C} = \mathrm{RV}_\infty[n, \cdot]$ *or* $\mathrm{RV}_{\mathrm{vol}}[\leq n, \cdot]$.

(1) $\mathbf{X}$, $\mathbf{Y}$ *are strongly isomorphic over* RV^n *iff* $\mathbb{L}\mathbf{X}$, $\mathbb{L}\mathbf{Y}$ *are isomorphic over the projection to* VF^n.
(2) *Let* $\mathbf{X}, \mathbf{X}' \in \mathrm{RV}[\leq n, \cdot]$, *and let* $G : \mathbb{L}\mathbf{X}' \to \mathbb{L}\mathbf{X}$ *be an auxiliary special bijection. Then* $\mathbf{X}'$ *is isomorphic to* $\mathbf{X}$ *over* RV^n.
(3) *Let* $\mathbf{X}, \mathbf{X}' \in \mathrm{RV}[\leq n, \cdot]$, *and let* $G : \mathbb{L}\mathbf{X}' \to \mathbb{L}\mathbf{X}$ *be an elementary bijection. Then* $\mathbf{X}'$ *is strongly isomorphic to a blowup of* $\mathbf{X}$.
(4) *Let* $\mathbf{X}, \mathbf{X}' \in \mathrm{RV}[\leq n, \cdot]$, *and let* $G : \mathbb{L}\mathbf{X}' \to \mathbb{L}\mathbf{X}$ *be a special bijection. Then* $\mathbf{X}'$ *is strongly isomorphic to an iterated blowup of* $\mathbf{X}$.
(5) *Assume* $\mathbf{T}$ *is effective. If* $\mathbf{Y} \to \mathbf{X}$ *is an* RV*-blowup, there exists* $\mathbf{Y}'$ *strongly isomorphic to* $\mathbf{Y}$ *over* $\mathbf{X}$ *and an elementary bijection* $c : \mathbb{L}\mathbf{Y}' \to \mathbb{L}\mathbf{Y}$ *lying over* $\mathbf{Y}' \to \mathbf{Y}$.

Proof.

(1) This is clear using Lemma 3.52.
(2) This is a special case of (1).
(3) This is clear from the definitions.
(4) This is clear from (1)–(3).
(5) It suffices to consider elementary blowups; we use the notation in the definition there. Thus $f_n(x) \in \mathrm{acl}(f_1(x), \ldots, f_{n-1}(x))$ for $x \in \phi(Y)$. By effectiveness and Lemma 6.2, there exists a definable function $s(x, y_1, \ldots, y_{n-1})$ such that if $\mathrm{rv}(y_i) = f_i(x)$ for $i = 1, \ldots, n-1$, then $\mathrm{rv}\, s(x, y) = f_n(x)$. This s is the additional data needed for an elementary bijection. □

Lemma 7.20. *Let* $\mathbf{X} = (X, f), \mathbf{X}' = (X', f') \in \mathrm{RV}[\leq n, \cdot]$, *and let* $h : X \to W \subseteq \mathrm{RV}^*$, $h' : X' \to W$ *be definable maps. Let* $X_c = h^{-1}(c)$, $\mathbf{X}_c = (X_c, f|X_c)$ *and similarly* $\mathbf{X}'_c$. *If* $(\mathbf{X}_c, \mathbf{X}'_c) \in \mathrm{I_{sp}}(\mathrm{RV}_c[n, \cdot])$; *then* $(\mathbf{X}, \mathbf{X}') \in \mathrm{I_{sp}}$.

Proof. Lemma 2.3 applies to $\mathrm{RV}_{\Gamma\text{-vol}'}$-isomorphisms, and hence using Remark 7.13 also to blowups. It also applies to $\mathrm{RV}[\leq n, \cdot]$-isomorphisms; hence to $\mathrm{I_{sp}}$-equivalence. □

Lemma 7.21. *If* $(\mathbf{X}, \mathbf{Y}) \in \mathrm{I_{sp}}$ *then* $\mathbb{L}\mathbf{X} \simeq \mathbb{L}\mathbf{Y}$.

Proof. Clear, since $\mathbb{L}[1]_1$ is the unit open ball around 1, $\mathbb{L}([\mathrm{RV}^{>0}]_1$ is the punctured unit open ball around 0, and $\mathbb{L}1_0 = \{0\}$. □

7.5 The kernel of $\mathbb{L}$

Definition 7.22. $\mathrm{VFR}[k,l,\cdot]$ is the set of pairs $\mathbf{X} = (X, f)$, with $X \subseteq \mathrm{VF}^k \times \mathrm{RV}^*$, $f : X \to \mathrm{RV}^l_\infty$, such that f factors through the projection $\mathrm{pr}_{\mathrm{RV}}(X)$ of X to the RV-coordinates. $\mathrm{I_{sp}}$ is the equivalence relation on $\mathrm{VFR}[k,l,\cdot]$:

$$(X, Y) \in \mathrm{I_{sp}} \iff (X_a, Y_a) \in \mathrm{I_{sp}}(\mathbf{T}_a) \quad \text{for each } a \in \mathrm{VF}^k.$$

$K_+\,\mathrm{VFR}$ is the set of equivalence classes.

By the usual compactness argument, if $(X, Y) \in \mathrm{I_{sp}}$ then there are uniform formulas demonstrating this. The relative versions of Lemmas 7.14 and 7.18 follow.

If $\mathbf{U} = (U, f) \in \mathrm{VFR}[k,l,\cdot]$, and for $u \in U$ we are uniformly given $\mathbf{V}_u = (V_u, g_u) \in \mathrm{VFR}[k',l',\cdot]$, we can define a sum $\sum_{u\in U} \mathbf{V}_u \in \mathrm{VFR}[k+k', l+l', \cdot]$: it is the set $\dot\bigcup_{u\in U} \mathrm{V}_u$, with the function $(u, v) \mapsto (f(u), g_u(v))$. When necessary, we denote this operation $\sum^{(k,l;k',l')}$. The special case $k = l = 0$ is understood as the default case.

By Proposition 7.6, the inverse of $\mathbb{L} : \mathrm{RV}[1,\cdot] \to \mathrm{VF}[1,\cdot]$ induces an isomorphism $I^1_1 : K_+\,\mathrm{VF}[1,\cdot] \to K_+\,\mathrm{RV}[1,\cdot]/\mathrm{I_{sp}}$:

$$I([X]) = [Y]/\mathrm{I_{sp}} \iff [\mathbb{L}Y] = [X].$$

Let J be a finite set of k elements. For $j \in J$, let $\pi^j : \mathrm{VF}^k \times \mathrm{RV}^* \to \mathrm{VF}^{J-\{j\}} \times \mathrm{RV}^*$ be the projection forgetting the jth VF coordinate. We will write VF^k, VF^{k-1} for VF^J, $\mathrm{VF}^{J-\{j\}}$, respectively, when the identity of the indices is not important.

Let $\mathbf{X} = (X, f) \in \mathrm{VFR}[k,l,\cdot]$. By assumption, f factors through π^j. We view the image $(\pi^j X, f)$ as an element of $\mathrm{VFR}[k-1,l,\cdot]$. Note that each fiber of π^j is in $\mathrm{VF}[1,\cdot]$.

Relativizing I^1_1 to π^j, we obtain a map

$$I^j = I^j_{k,l} : \mathrm{VFR}[k,l,\cdot] \to K_+\,\mathrm{VFR}[k-1, l+1, \cdot]/\mathrm{I_{sp}}.$$

Lemma 7.23. *Let* $\mathbf{X} = (X, f), \mathbf{X}' = (X', f') \in \mathrm{VFR}[k,l,\cdot]$.

(1) I^j *commutes with maps into* RV*: if* $h : \mathbf{X} \to W \subseteq \mathrm{RV}^*$ *is definable,* $\mathbf{X}_c = h^{-1}(c)$*, then* $I^j(\mathbf{X}) = \sum_{c\in W} I^j(\mathbf{X}_c)$.
(2) *If* $([\mathbf{X}], [\mathbf{X}']) \in \mathrm{I_{sp}}$ *then* $(I^j(\mathbf{X}), I^j(\mathbf{X}')) \in \mathrm{I_{sp}}$.
(3) I^j *induces a map* $K_+\,\mathrm{VFR}[k,l,\cdot]/\mathrm{I_{sp}} \to K_+\,\mathrm{VFR}[k-1,l+1,\cdot]/\mathrm{I_{sp}}$.

Proof.

(1) This reduces to the case of I^1_1, where it is an immediate consequence of the uniqueness, and the fact that $\mathbb{L}$ commutes with maps into RV in the same sense.
(2) All equivalences here are relative to the $k-1$ coordinates of VF other than j, so we may assume $k = 1$. For $a \in \mathrm{VF}$, $([\mathbf{X}_a], [\mathbf{X}'_a]) \in \mathrm{I_{sp}}(\mathbf{T}_a)$. By stable embeddedness of RV, there exists $\alpha = \alpha(a) \in \mathrm{RV}^*$ such that $\mathbf{X}_a, \mathbf{X}'_a$ are $\mathbf{T}_\alpha$-definable and $([\mathbf{X}]_a, [\mathbf{X}']_a) \in \mathrm{I_{sp}}(\mathbf{T}_\alpha)$. Fibering over the map α we may assume by (1) and Lemma 7.20 that α is constant; so for some $W \in \mathrm{VF}[1]$, $\mathbf{Y}, \mathbf{Y}' \in \mathrm{RV}[l,\cdot]$, we have $\mathbf{X} = W \times \mathbf{Y}$, $\mathbf{X}' = W \times \mathbf{Y}'$, and $([\mathbf{Y}], [\mathbf{Y}']) \in \mathrm{I_{sp}}$. Then $I^j(\mathbf{X}) = I^j(W) \times \mathbf{Y}$, $I^j(\mathbf{X}') = I^j(W) \times \mathbf{Y}'$, and the conclusion is clear.

(3) This follows from (2). □

Lemma 7.24. *Let* $\mathbf{X} = (X, f)$, $X \subseteq \mathrm{VF}^J \times \mathrm{RV}^\infty$, $f : X \to \mathrm{RV}^l$. *If* $j \neq j' \in J$, *then* $I^j I^{j'} = I^{j'} I^j : K_+ \mathrm{VFR}[k, l, \cdot]/\mathrm{I_{sp}} \to K_+ \mathrm{VFR}[k-2, l+2, \cdot]/\mathrm{I_{sp}}$.

Proof. We may assume $S = \{1, 2\}$, $j = 1$, $j' = 2$, since all is relative to $\mathrm{VF}^{S\setminus\{j,j'\}}$. By Lemma 7.23(1) it suffices to prove the statement for each fiber of a given definable map into RV.

Hence we may assume $X \subseteq \mathrm{VF}^2$ and f is constant; and by Lemma 5.10, we can assume X is a basic 2-cell:

$$X = \{(x, y) : x \in X_1, \mathrm{rv}(y - G(x)) = \alpha_1\}, \quad X_1 = \mathrm{rv}^{-1}(\delta_1) + c_1.$$

The case where G is constant is easy since then X is a finite union of rectangles. Otherwise, G is invertible, and by the niceness of G we can also write

$$X = \{(x, y) : y \in X_2, \mathrm{rv}(x - G^{-1}(y)) = \beta\}, \quad X_2 = \mathrm{rv}^{-1}(\delta_2) + c_2.$$

We immediately compute

$$I_2 I_1(X) = (\delta_1, \alpha_1), \qquad I_1 I_2(X) = (\alpha_2, \delta_2).$$

Clearly, $[(\delta_1, \alpha_1)]_2 = [(\alpha_2, \delta_2)]_2$. □

Proposition 7.25. *Let* $\mathbf{X}, \mathbf{Y} \in \mathrm{RV}[\leq n, \cdot]$. *If* $\mathbb{L}\mathbf{X}$, $\mathbb{L}\mathbf{Y}$ *are isomorphic, then* $([X], [Y]) \in \mathrm{I_{sp}}$.

Proof. Define $I = I_1 \dots I_n : \mathrm{VF}[n, \cdot] = \mathrm{VFR}[n, 0, \cdot] \to \mathrm{VFR}[0, n, \cdot] = \mathrm{RV}[\leq n, \cdot]$. Let $V \in \mathrm{VF}[n, \cdot]$.

Claim 1. If $\sigma \in \mathrm{Sym}(n)$ then $I = I_{\sigma(1)} \dots I_{\sigma(n)}$.

Proof. We may assume σ just permutes two adjacent coordinates, say, 2, 3 out of 1, 2, 3, 4. Then $I = I_1 I_2 I_3 I_4 = I_1 I_3 I_2 I_4$ by Lemma 7.24. □

Claim 2. When $F : V \to F(V)$ is a relatively unary bijection, we have $I(V) = I(F(V))$.

Proof. By Claim 1 we may assume F is relatively unary with respect to pr^n. Thus $F(V_a) = F(V)_a$, where V_a, $F(V)_a$ are the pr^n-fibers. By the definition of I_1^1, we have $I_1^1(V_a) = I_1^1(F(V)_a) \in \mathrm{RV}[1, \cdot](\mathbf{T}_a)$; but by the definition of I^n, $I_n(V)_a = I_1^1(V_a)$. Thus $I^n(V) = I^n(F(V))$ and thus $I(V) = I(F(V))$. □

Claim 3. When $F : V \to F(V)$ is any definable bijection, $I(V) = I(F(V))$.

Proof. The proof is immediate from Claim 2 and Lemma 7.11. □

Now turning to the statement of the proposition, assume $\mathbb{L}\mathbf{X}$, $\mathbb{L}\mathbf{Y}$ are isomorphic. We compute inductively that $\mathbb{L}(\mathbf{X}) = [\mathbf{X}]$. By Claim 3, $[\mathbf{X}] = I(\mathbb{L}\mathbf{X}) = I(\mathbb{L}\mathbf{Y}) = [\mathbf{Y}]$. □

Notation 7.26. Let $\mathbb{L}^* : K_+(\mathrm{VF}) \to K_+(\mathrm{RV}[*])/Isp$ be the inverse map to $\mathbb{L}$.

Remark 7.27. When **T** is rv-effective, one can restate the conclusion of Proposition 7.25 as follows: if $X, Y \in \mathrm{VF}[n, \cdot]$ are $\underset{\mathrm{rv}}{\sim}$-invariant and $F : X \to Y$ is a definable bijection, then there exist special bijections $X' \to X$ and $Y' \to Y$ and an $\underset{\mathrm{rv}}{\sim}$-invariant-definable bijection $G : X' \to Y'$. (This follows from Propositions 7.25 and 6.1 and Lemmas 7.18 and 7.19.) The effectiveness hypothesis is actually unnecessary here, as will be seen in the proof of Proposition 8.26. Perhaps Question 7.9 can be answered simply by tracing the connection between F and G through the proof.

8 Definable sets over VF and RV: The main theorems

In stating the theorems, we restrict attention to $\mathrm{VF}[n]$, i.e., to definable subsets of varieties, though the proof was given more generally for $\mathrm{VF}[n, \cdot]$ (definable subsets of $\mathrm{VF}^n \times \mathrm{RV}^*$).

8.1 Definable subsets of varieties

Let **T** be V-minimal. We will look at the category of definable subsets of varieties, and definable maps between them. The results will be stated for $\mathrm{VF}[n]$; analogous statements for $\mathrm{VF}[n, \cdot]$ are true with the same proofs.

We define three variants of the sets of objects. $\mathrm{VF}''[n]$ is the category of $\leq n$-dimensional definable sets over VF, i.e., of definable subsets of n-dimensional varieties. Let $\mathrm{VF}[n]$ be the category of definable subsets $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^*$ such that the projection $X \to \mathrm{VF}^n$ has finite fibers. $\mathrm{VF}'[n]$ is the category of definable subsets X of $V \times \mathrm{RV}^*$, where V ranges over all $\mathrm{VF}(A)$-definable sets of dimension $n, m \in \mathbb{N}$, such that the projection $X \to V$ is finite-to-one. VF, VF$'$, VF$''$ are the unions over all n. In all cases, the morphisms $\mathrm{Mor}(X, Y)$ are the definable functions $X \to Y$.

Lemma 8.1. *The natural inclusion of* $\mathrm{VF}[n]$ *in* $\mathrm{VF}'[n]$ *is an equivalence. If* **T** *is effective, so is the inclusion of* $\mathrm{VF}''[n]$ *in* $\mathrm{VF}'[n]$.

Proof. We will omit the index $\leq n$. The inclusion is fully faithful by definition, and we have to show that it hits every VF$'$-isomorphism type; in other words, that any definable $X \subseteq (V \times \mathrm{RV}^m)$ is definably isomorphic to some $X' \subseteq \mathrm{VF}^n \times \mathrm{RV}^{m+l}$ for some l (with $n = \dim(V)$). Definable isomorphisms can be glued on pieces, so we may assume V is affine, and admits a finite-to-one map $h : V \to \mathrm{VF}^m$. By Lemma 3.9, each fiber $h^{-1}(a)$ is $A(a)$-definably isomorphic to some $F(a) \subseteq \mathrm{RV}^l$. By compactness, F can be chosen uniformly definable, $F(a) = \{y \in \mathrm{RV}^l : (a, y) \in F\}$ for some definable $F \subseteq \mathrm{VF}^m \times \mathrm{RV}^l$; and there exists a definable isomorphism $\beta : V \to F$, over VF^m. Let $\alpha(v, t) = (\beta(v), t)$, $X' = \alpha(X)$.

Now assume **T** is effective. Let $X \in \mathrm{Ob}\,\mathrm{VF}'$; $X \subseteq V \times \mathrm{RV}^m$, $V \subseteq \mathrm{VF}^n$, such that the projection $X \to V$ has finite fibers. Then by effectivity, for any $v \in V$ (over any extension field), if $(v, c_1, \ldots, c_m) \in X$ then each c_i, viewed as a ball, has a point defined over $A(v)$. Hence the partial map $V \times \mathrm{VF}^m \to X$, $(v, x_1, \ldots, x_m) \mapsto (v, \mathrm{rv}(x_1), \ldots, \mathrm{rv}(x_m))$ has an A-definable section; the image of this section is a subset S of $V \times \mathrm{VF}^m$, definably isomorphic to X; and the Zariski closure V' of S in $V \times \mathrm{VF}^m$ has dimension $\leq \dim(V)$. □

The following definition and proposition apply both to the category of definable sets, and to the definable sets with volume forms.

Definition 8.2. X, Y are *effectively isomorphic* if

for any effective A, X, Y are definably isomorphic in $\mathbf{T}_A$. If $K_+^{\mathrm{eff}}(\mathrm{VF})$ is the semiring of effective isomorphic classes of definable sets. $K(\mathrm{VF})$ is the corresponding ring; similarly $K_+^{\mathrm{eff}}(\mathrm{VF}[n])$, etc.

Over an effective base, in particular, if $\mathbf{T}$ is effective over any field-generated base, effectively isomorphic is the same as isomorphic. But Example 4.7 shows that this is not so in general.

Proposition 8.3. *Let T be* V*-minimal, or a finitely generated extension of a* V*-minimal theory. The following conditions are equivalent* (*let $X, Y \in \mathrm{VF}[n]$*):

(1) $[\mathbb{L}^*X] = [\mathbb{L}^*Y]$ *in* $K_+(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}[\leq n]$.
(2) *There exists a definable family $\mathcal{F}$ of definable bijections $X \to Y$ such that for any effective structure A, $F(A) \neq \emptyset$.*
(3) *X, Y are effectively isomorphic.*
(4) *X, Y are definably isomorphic over any A such that* $\mathrm{VF}^*(A) \to \mathrm{RV}(A)$ *is surjective.*
(5) *For some finite* $A_0 \subseteq \mathrm{RV}(\langle\emptyset\rangle)$, *$X$, Y are definably isomorphic over any A such that* $A_0 \subseteq \mathrm{rv}(\mathrm{VF}^*(A))$.

Proof.

(1) implies (5): By Proposition 6.1 (Proposition 6.3 in the measured case), the given isomorphism $[\mathbb{L}^*X] \to [\mathbb{L}^*Y]$ lifts to an isomorphism $\mathbb{L}\mathbb{L}^*X \to \mathbb{L}\mathbb{L}^*Y$; since $\mathbf{T}_A \supseteq \mathrm{ACVF}_A$, this is also a $\mathbf{T}_A$ isomorphism; it can be composed with the isomorphisms $X \to \mathbb{L}\mathbb{L}^*X$, $Y \to \mathbb{L}\mathbb{L}^*Y$.

(2) implies (3), (5) implies (4) implies (3), trivially.

(3) implies (1)–(2): Let E_{eff} be as in Proposition 3.51. By (3), X, Y are E_{eff}-isomorphic. By Proposition 7.25, $[\mathbb{L}^*X] = [\mathbb{L}^*Y]$ in $K_+(\mathrm{RV}_{E_{\mathrm{eff}}}[*])/\mathrm{I_{sp}}$. But $\mathrm{RV}(E_{\mathrm{eff}}), \Gamma(E_{\mathrm{eff}}) \subseteq \mathrm{dcl}(\emptyset)$, so every E_{eff}-definable relation on RV is definable; i.e., $\mathrm{RV}_{E_{\mathrm{eff}}}$, RV are the same structure. Thus (1) holds.

Now by assumption, there exists an E_{eff}-definable bijection $f' : X \to Y$. f' is an E_{eff}-definable element of a definable family $\mathcal{F}$ of definable bijections $X \to Y$. Since this family has an E_{eff}-point, and E_{eff} embeds into any effective B, it has a B point, too. Thus (3) implies (2). □

8.2 Invariants of all definable maps

Let $[X]$ denote the class of X in $K_+^{\mathrm{eff}}(\mathrm{VF}[n])$.

Proposition 8.4. *Let* $\mathbf{T}$ *be* V*-minimal. There exists a canonical isomorphism of Grothendieck semigroups*

$$\oint : K_+^{\mathrm{eff}}(\mathrm{VF}[n]) \to K_+(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}[\leq n]$$

satisfying

$$\oint [X] = W/\mathrm{I_{sp}}[\leq n] \iff [X] = [\mathbb{L}W] \in K_+^{\mathrm{eff}}(\mathrm{VF}[n]).$$

Proof. Recall Definition 4.8. Given $\mathbf{X} = (X, f) \in \mathrm{Ob}\,\mathrm{RV}[k]$ we have $\mathbb{L}\mathbf{X} \in \mathrm{Ob}\,\mathrm{VF}[k] \subseteq \mathrm{Ob}\,\mathrm{VF}[n]$. If $\mathbf{X}, \mathbf{X}'$ are isomorphic,then by Proposition 6.1, $\mathbb{L}\mathbf{X}, \mathbb{L}\mathbf{X}'$ are effectively isomorphic. Direct sums are clearly respected, so we have a semigroup homomorphism $\mathbb{L} : K_+(\mathrm{RV}[\leq n]) \to K_+^{\mathrm{eff}}(\mathrm{VF}[n])$. It is surjective by Proposition 4.5. By Proposition 8.3, the kernel is precisely $\mathrm{I_{sp}}[\leq n]$. Inverting, we obtain $\oint$. □

Definition 8.5. Let $K_+\,\mathrm{VF}[n]/(\dim < n)$ be the Grothendieck ring of the category of definable subsets of n-dimensional varieties, and essential bijections between them. Let $\mathrm{I_{sp}}'[n]$ be the congruence on $RV[n]$ generated by pairs $(X, X \times \mathrm{RV}^{>0})$ (where $X \subseteq \mathrm{RV}^*$ is definable, of dimension $< n$).

Corollary 8.6. $\oint$ *induces an isomorphism*

$$K_+^{\mathrm{eff}}(\mathrm{VF}[n])/(\dim < n) \to \mathrm{RV}[n]/\mathrm{I_{sp}}'[n].$$ □

Corollary 8.7. *Let* $A, B \in \mathrm{RV}[\leq n]$*. Let* $n' > n$*, and let* A_N, B_N *be their images in* $\mathrm{RV}[\leq N]$*. If* $(A_N, B_N) \in \mathrm{I_{sp}}[\leq N]$*, then* $(A, B) \in \mathrm{I_{sp}}[\leq n]$*.*

Proof. By Proposition 8.4, $(A, B) \in \mathrm{I_{sp}}[\leq n]$ iff $\mathbb{L}A, \mathbb{L}B$ are definably isomorphic; this latter condition does not depend on n. □

Putting Proposition 8.4 together for all n, we obtain the following.

Theorem 8.8. *Let* $\mathbf{T}$ *be* V*-minimal. There exists a canonical isomorphism of filtered semirings*

$$\oint : K_+(\mathrm{VF}) \to K_+(\mathrm{RV}[*])/\mathrm{I_{sp}}.$$

Let $[X]$ *denote the class of* X *in* $K_+(\mathrm{VF})$*. Then*

$$\oint [X] = \frac{W}{\mathrm{I_{sp}}} \iff [X] = [\mathbb{L}W] \in K_+^{\mathrm{eff}}(\mathrm{VF}).$$ □

On the other hand, using the Grothendieck group isomorphisms of Proposition 8.4 and passing to the limit, we have the following.

Corollary 8.9. *Let* **T** *be* V*-minimal. The isomorphisms of Proposition* 8.4 *induce an isomorphism of Grothendieck groups:*

$$\int^K : K^{\mathrm{eff}}(\mathrm{VF}[n]) \to K(\mathrm{RV}[n]).$$

The isomorphism $\oint$ *of Theorem* 8.8 *induces an injective ring homomorphism*

$$\int^K : K^{\mathrm{eff}}(\mathrm{VF}) \to K(\mathrm{RV})[J^{-1}],$$

where $J = \{1\}_1 - [\mathrm{RV}^{>0}]_1 \in K(\mathrm{RV})$.

Proof. We may work over an effective base. With subtraction allowed, the generating relation of $\mathrm{I_{sp}}$ can be read as $[\{1\}]_0 = \{1\}_1 - [\mathrm{RV}^{>0}]_1 := J$, so that the groupification of $K_+(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}[\leq n]$ is isomorphic to $K(\mathrm{RV}[n])$, via the embedding of $K_+(\mathrm{RV}[n])$ as a direct factor in $K_+(\mathrm{RV}[\leq n])$. Thus the groupification of the homomorphism of Theorem 8.8 is a homomorphism

$$\int^K : K(\mathrm{VF}) \to \lim_{n\to\infty} K(\mathrm{RV}[n]),$$

where the direct limit system maps are given by $[X]_d \mapsto ([X]_{d+1} - ([X]_d \times (\mathrm{RV}^{>0}))) = [X_d]J$. This direct limit embeds into $K(\mathrm{RV})[J^{-1}]$ by mapping $X \in K(\mathrm{RV}[n])$ to XJ^{-n}. □

8.3 Definable volume forms: VF

We will now define the category $\mu\mathrm{VF}[n]$ of "n-dimensional $\mathbf{T}_A$-definable sets with definable volume forms, up to RV-equivalence" and the same up to Γ-equivalence. We will represent the forms as functions to RV, that transform in the way volume forms do.

By way of motivation, in a local field with an absolute value, a top differential form ω induces a measure $|d\omega|$. For a regular isomorphism $f : V \to V'$, we have $\omega = hf^*\omega'$ for a unique h, and f is measure preserving between $(V, |\omega|)$ and $(V', |\omega'|)$ iff $|h| = 1$.

We do not work with an absolute value into the reals, but instead define the analogue using the map rv or, a coarser version, the map val into Γ. When $\Gamma = \mathbb{Z}$, the latter is the usual practice in Denef-style motivic integration. Using rv leaves room for considering an absolute value on the residue field, and iterating the integration functorially when places are composed, for instance, $\mathbb{C}((x))((y)) \to \mathbb{C}((x)) \to \mathbb{C}$. This functoriality will be described in a future work.

In the definition below, the words "almost every $y \in Y$" will mean for all y outside a set of VF dimension $< \dim_{\mathrm{VF}}(Y)$.

Definition 8.10. Ob μVF$[n, \cdot]$ consists of pairs (Y, ω), where Y is a definable subset of $\mathrm{VF}^n \times \mathrm{RV}^*$, and $\omega : Y \to \mathrm{RV}$ is a definable map. A morphism $(Y, \omega) \to (Y', \omega')$ is a definable essential bijection F such that for almost every $y \in Y$,

$$\omega(y) = \omega'(F(y)) \cdot \mathrm{rv}(\mathrm{Jcb}\, F(y)).$$

(We will say "$F : (Y, \omega) \to (Y', \omega')$ is measure preserving.")

μ_ΓVF$[n, \cdot]$ is the category of pairs (Y, ω) with $\omega : Y \to \Gamma$ a definable function A morphism $(Y, \omega) \to (Y', \omega')$ is a definable essential bijection $F : Y \to Y'$ such that for almost every $y \in Y$,

$$\omega(y) = \omega'(F(y)) + \mathrm{val}(\mathrm{Jcb}\, F(y)).$$

("$F : (Y, \omega) \to (Y', \omega')$ is Γ-measure preserving.")

μVF$[n]$, μ_ΓVF$[n]$ are the full subcategories of μVF$[n, \cdot]$, μ_ΓVF$[n, \cdot]$ (respectively) whose objects admit a finite-to-one map to VF^n.

In this definition, let $t_1(y), \ldots, t_n(y)$ be the VF-coordinates of $y \in Y$. One can think of the form as $\omega(y) dt_1 \cdot \cdots \cdot dt_n$.

Note that $\mathrm{VF}_{\mathrm{vol}}$ of Definition 5.19 is isomorphic to the full subcategory of μVF whose objects are pairs $(Y, 1)$.

Remark 8.11. When **T** is V-minimal and effective, the data ω of an object (Y, ω) of μVF$[n]$ can be written as $\mathrm{rv} \circ \Phi$ for some $\Phi : Y \to \mathrm{VF}$. (Write $\omega = \bar{\omega} \circ \mathrm{rv} \circ F$ for some F, and use Proposition 6.1 to lift $\bar{\omega}$ to some G, so that $\omega = \mathrm{rv} \circ G \circ F$.) It is thus possible to view ω as the RV-image (respectively, Γ-image) of a definable volume form on Y. One could equivalently take ω to be a definable section of $\Lambda^n TY/(1+\mathcal{M})$, where TY is the (appropriately defined) tangent bundle, Λ^n the nth exterior power with $n = \dim(Y)$.

For VF_Γ the category we take is slightly more flexible than taking varieties with absolute values of volume forms, even if **T** is V-minimal and effective, in that expressions such as $\int |\sqrt{x}| dx$ are allowed.

In either of these categories, one could restrict the objects to bounded ones.

Definition 8.12. Let $\mu\mathrm{VF}_{\mathrm{bdd}}[n]$ be the full subcategory of μVF$[n]$ whose objects are bounded definable sets, with bounded definable forms ω. Similarly, one defines $\mu\mathrm{VF}_{\Gamma;\mathrm{bdd}}$.

Here *bounded* means that there is a lower bound on the valuation of any coordinate of any element of the set. A similar definition applies in RV and μRV.

Note that if an object of μVF$[n]$ is μVF$[n]$-isomorphic to an object of $\mu\mathrm{VF}_{\mathrm{bdd}}[n]$, it must lie in $\mu\mathrm{VF}_{\mathrm{bdd}}[n]$.

8.4 Definable volume forms: RV

We will define a category $\mu\mathrm{RV}[n]$ of definable subsets of $(\mathrm{RV})^m$, with additional data that can be viewed as a volume form. Unlike $\mu\mathrm{VF}[n]$, in $\mu\mathrm{RV}[n]$ subsets of dimension $< n$ are *not* ignored: a point of RV^n corresponds to an open polydisc of VF^n, with nonzero n-dimensional volume.

In particular, the Jacobian of a morphism needs to be defined at every point, not just away from a lower-dimensional set. However, in accord with Lemma 6.3, it may be modified by $\mathbf{k}^*$-multiplication on a lower-dimensional set.

Definition 8.13. The objects of $\mu\mathrm{RV}[n]$ are definable triples (X, f, ω), $X \subseteq \mathrm{RV}^{n+m}$, $f : X \to \mathrm{RV}^n$ finite-to-one, and $\omega : X \to \mathrm{RV}$.

We define a multiplication $\mu\mathrm{RV}[n] \times \mu\mathrm{RV}[n'] \to \mu\mathrm{RV}[n+n']$ by $(X, f, \omega) \times (X', f', \omega') = (X \times X', f \times f', \omega \cdot \omega')$. Here $\omega \cdot \omega'(x, x') = \omega(x)\omega'(x')$.

Given $\mathbf{X} = (X, f, \omega)$, we define an object $\mathbb{L}\mathbf{X}$ of $\mathrm{VF}[n]$; namely, $(\mathbb{L}X, \mathbb{L}f, \mathbb{L}\omega)$, where $\mathbb{L}X = X \times_{f,\mathrm{rv}} (\mathrm{VF}^\times)^n$, $\mathbb{L}f(a, b) = f(a, \mathrm{rv}(b))$, $\mathbb{L}\omega(a, b) = \omega(a, \mathrm{rv}(b))$. (Sometimes we will write f, ω for $\mathbb{L}f, \mathbb{L}\omega$.)

A morphism $\alpha : \mathbf{X} = (X, f, \omega) \to \mathbf{X}' = (X', f', \omega')$ is a definable bijection $\alpha : X \to X'$ such that

$$\omega(y) = \omega'(\alpha(y)) \cdot \mathrm{rv}(\mathrm{Jcb}_{\mathrm{RV}}(\alpha)(y)) \quad \text{for almost all } y,$$

where "almost all" means "away from a set Y with $\dim_{\mathrm{RV}}(f(Y)) < n$"; and

$$\mathrm{val}_{\mathrm{rv}}\omega(y) + \sum_{i=1}^{n} \mathrm{val}_{\mathrm{rv}} f_i(y) = \mathrm{val}_{\mathrm{rv}}\omega'(\alpha(y)) + \sum_{i=1}^{n} \mathrm{val}_{\mathrm{rv}} f_i'(\alpha y) \quad \text{for all } y.$$

The objects of $\mu_\Gamma\mathrm{RV}[n]$ are triples (X, f, ω), with $f : X \to \mathrm{RV}^n$, $\omega : X \to \Gamma$. A morphism $\alpha : (X, f, \omega) \to (X', f', \omega')$ is a definable bijection $\alpha : X \to X'$ such that $\mathrm{val}_{\mathrm{rv}}\omega(y) + \sum_{i=1}^{n} \mathrm{val}_{\mathrm{rv}} f_i(y) = \mathrm{val}_{\mathrm{rv}}\omega'(\alpha(y)) + \sum_{i=1}^{n} \mathrm{val}_{\mathrm{rv}} f_i'(\alpha y)$ for all y. Disjoint sums and products are defined as for $\mu\mathrm{RV}$.

$\mu_\Gamma\mathrm{RES}[n]$ is the full subcategory of $\mu_\Gamma\mathrm{RV}[n]$ with objects (X, f, ω), such that $\mathrm{val}_{\mathrm{rv}}(X)$ is finite. In this case, ω takes finitely many values, too.

$K_+^{\mathrm{eff}}\, \mu\mathrm{RV}[n]$ is the Grothendieck semigroup of $\mu\mathrm{RV}[n]$ with respect to effective isomorphism. $K_+^{\mathrm{eff}}\, \mu\mathrm{RV}$ is the direct sum $\oplus_n K_+^{\mathrm{eff}}\, \mu\mathrm{RV}[n]$; it clearly inherits a semiring structure from Cartesian multplication, $(X, f, \omega) \times (X', f', \omega') = (X \times X', (f, f'), \omega \cdot \omega')$.

The morphisms of $\mu_\Gamma\mathrm{RV}[n]$ are called Γ-measure preserving.

The category $\mathrm{RV}_{\mathrm{vol}}[n, \cdot]$ of Definition 5.21 is isomorphic to the full subcategory whose objects have $\omega = 1$.

Remark. The semiring $K_+^{\mathrm{eff}}\, \mathrm{RV}_{\mathrm{vol}}$ is naturally a subsemiring of $K_+^{\mathrm{eff}}\, \mu\mathrm{RV}$. The latter is obtained by inverting $[\{a\}]_1$ for $a \in \mathrm{RV}$ and taking the zeroth graded component. This process is needed in order to identify integrals of functions in n variables with volumes in $n + 1$ variables. Thus as semirings they are closely related. But if the dimension grading is taken into account, the subsemiring of RV-volumes contains finer information connected to integrability of forms.

8.5 The kernel of $\mathbb{L}$ in the measured case

The description of the kernel of $\mathbb{L}$ on the semigroups of definable sets with volume forms is essentially the same as for definable sets. We will now run through the proof, indicating the modficiations. The principal change is the introduction of a category with fewer morphisms, defined not only with reference to RV but also to VF. For effective bases, the category is identical to μRV, so it will be invisible in the statements of the main theorems; but during the induction in the proof, bases will not in general be effective and the mixed category introduced here has better properties.

Both the introduction of the various intermediate categories and the repetition of the proof would be unnecessary if we had a positive answer to Question 7.9. In this case the proof of Lemma 8.23 would immediately lift to higher dimensions. Indeed, the characterization of the kernel of the map $\mathbb{L}$ on Grothendieck groups would be uniformized not only for the categories we consider, but for a range of categories carrying more structure.

The integer n will be fixed in this subsection.

Lemma 8.14. *Let* $(X, \omega) \in \mathrm{Ob}\, \mu\mathrm{VF}[n, \cdot]$, $Y \in \mathrm{Ob}\, \mathrm{VF}[n, \cdot]$, *and let* $F : Y \to X$ *be a definable bijection.*

(1) *There exists* $\psi : Y \to \mathrm{RV}$ *such that* $F : (Y, \psi) \to (X, \omega)$ *is measure preserving.*
(2) ψ *is essentially unique in the sense that if* ψ' *meets the same condition, then* ψ, ψ' *are equal away from a subset of* X *of lower dimension.*
(3) *Dually, given* F, X, Y, ψ, *there exists an essentially unique* ω *such that* $F : (Y, \psi) \to (X, \omega)$ *is measure preserving.*
(4) *Lemma* 7.11 *applies to* $\mu\mathrm{VF}[n, \cdot]$ *and to* $\mu\mathrm{VF}[n]$.

Proof.

(1)–(2) Let $\psi(y) = \omega(\alpha(y)) \cdot \mathrm{rv}(\mathrm{Jcb}_{\mathrm{RV}}(\alpha)(y))$. By the definition of μVF this works, and is the only choice "almost everywhere."
(3) This follows from the case of F^{-1}.
(4) Now let $\mathbf{X}, \mathbf{Y} \in \mathrm{Ob}\, \mu\mathrm{VF}[n]$ and let $F \in \mathrm{Mor}_{\mu\mathrm{VF}[n]}(X, Y)$. We have $\mathbf{X} = (X, \omega_X)$, $\mathbf{Y} = (Y, \omega_Y)$ for some $X, Y \in \mathrm{Ob}\, \mathrm{VF}[n]$ and $\omega_X : X \to \mathrm{RV}, \omega_Y : Y \to \mathrm{RV}$. By Lemma 7.11 there exist $X = X_1, \ldots, X_n = Y \in \mathrm{Ob}\, \mathrm{VF}[n]$ and essentially unary $F_i : X_i \to X_{i+1}$ with $F = F_{n-1} \circ \cdots \circ F_1$. Let $\omega_1 = \omega_X$, and inductively let ω_{i+1} be such that $F_i \in \mathrm{Mor}_{\mu\mathrm{VF}[n]}((X_i, \omega_i), (X_{i+1}, \omega_{i+1}))$. Then $F \in \mathrm{Mor}_{\mu\mathrm{VF}[n]}((X, \omega), (Y, \omega_n))$. By uniqueness it follows that ω_Y, ω_n are essentially equal. □

Definition 8.15. Given $\mathbf{X}, \mathbf{Y} \in \mathrm{Ob}\, \mu\mathrm{RV}[n, \cdot]$ call a definable bijection $h : X \to Y$ *liftable* if there exists $F \in \mathrm{Mor}_{\mu\mathrm{VF}[n,\cdot]}(\mathbb{L}X, \mathbb{L}Y)$ with $\rho_Y F = h \rho_X$.

Let $\mathcal{C} = \mu_l\mathrm{RV}[n, \cdot]$ be the subcategory of $\mu\mathrm{RV}[n, \cdot]$ consisting of all objects and liftable morphisms.

By Proposition 5.22, liftable morphisms must preserve the volume forms, so $\mathcal{C}$ is a subcategory of $\mu\mathrm{RV}[n, \cdot]$.

Over an effective base, $\mathcal{C} = \mu\mathrm{RV}[n, \cdot]$ (Lemma 6.3), and the condition of existence of s in Definition 8.16(1) below is equivalent to $f_n(y) \in \mathrm{acl}(f_1(y), \dots, f_{n-1}(y))$.

Definition 8.16.

(1) Let $\mathbf{Y} = (Y, f, \omega) \in \mathrm{Ob}\,\mu\mathrm{RV}[n, \cdot]$ be such that there exists $s : \mathbf{Y} \times_{f_1,\dots,f_{n-1}} \mathrm{VF}^{n-1} \to \mathrm{VF}$ with $\mathrm{rv}(s(y, u_1, \dots, u_{n-1})) = f_n(y)$. Let $Y' = Y \times \mathrm{RV}^{>0}$. For $(y, t) \in Y'$, define $f' = (f'_1, \dots, f'_n)$ by $f'_i(y, t) = f_i(y)$ for $i < n$, $f'_n(y, t) = t f_n(y)$. Let $\omega'(y, t) = \omega(y)$. Then $\widetilde{\mathbf{Y}} = (Y', f', \omega')$ is an *elementary blowup* of $\mathbf{Y}$. It comes with the projection map $Y' \to Y$.

(2) Let $\mathbf{X} = (X, g, \omega) \in \mathrm{Ob}\,\mu\mathrm{RV}[n, \cdot]$, $X = X' \dot\cup X''$, $g' = g|X'$, $g'' = g|X''$, $\omega' = \omega|X', \omega'' = \omega|X''$, and let $\phi : \mathbf{Y} \to (X', g', \omega')$ be a $\mu_l\mathrm{RV}[n, \cdot]$-isomorphism. Then the RV-*blowup* $\widetilde{\mathbf{X}}_\phi$ is defined to be $\widetilde{\mathbf{Y}} + (X'', g'', \omega'') = (Y' \dot\cup X'', f' \dot\cup g'', \omega' \dot\cup \omega'')$. It comes with $b : Y' \dot\cup X'' \to X$, defined to be the identity on X'', and the projection on Y'. X' is called the *blowup locus* of $b : \widetilde{\mathbf{X}}_\phi \to \mathbf{X}$.

An *iterated* RV-*blowup* is obtained by finitely many iterations of RV-blowups.

Definition 8.17. Let $\mathrm{I}^{\mu}_{\mathrm{sp}}[n]$ be the set of pairs $(\mathbf{X}_1, \mathbf{X}_2) \in \mathrm{Ob}\,\mu\mathrm{RV}[n, \cdot]$ such that there exist iterated blowups $b_i : \widetilde{\mathbf{X}}_i \to \mathbf{X}_i$ and a $\mu_l\mathrm{RV}[n, \cdot]$-isomorphism $F : \widetilde{\mathbf{X}}_1 \to \widetilde{\mathbf{X}}_2$.

When n is fixed, we will simply write $\mathrm{I}^{\mu}_{\mathrm{sp}}$. On the other hand, we will need to make explicit the dependence on the theory; we write $\mathrm{I}^{\mu}_{\mathrm{sp}}(A)$ for the congruence $\mathrm{I}^{\mu}_{\mathrm{sp}}$ of the theory $\mathbf{T}_A$.

When $\mathbf{X} = (X, f, \omega) \in \mathrm{Ob}\,\mu\mathrm{RV}[n, \cdot]$, $h : X \to W$ is a definable map, and $c \in W$, define $\mathbf{X}_c = (h^{-1}(c), f|h^{-1}(c), \omega|h^{-1}(c))$.

Let $X_1, X_2 \in \mathrm{Ob}\,\mu\mathrm{RV}[n, \cdot]$, and let $f_i : X_i \to Y$ be a definable map, with $Y \subseteq \mathrm{RV}^*$. In this situation the existence of $\mu\mathrm{RV}[n, \cdot](\langle a\rangle)$-isomorphisms between each pair of fibers $X_1(a)$, $X_2(a)$ $(a \in Y)$ does not necessarily imply that $X_1 \simeq_{\mu\mathrm{RV}[\leq n, \cdot]} X_2$, because of the explicit reference to dimension in the definition of morphisms; the dimension of the allowed exceptional sets may accumulate over Y. The definition of morphisms for $\mu\mathrm{VF}[n]$ also allows a lower-dimensional exceptional set; but this does not create a problem when fibered over $W \subseteq \mathrm{RV}^*$, since by Lemma 3.56 $\max_{c \in W} \dim_{\mathrm{VF}}(Z_c) = \dim_{\mathrm{VF}}(Z)$. Thus an RV-disjoint union of $\mu\mathrm{VF}[n]$-isomorphisms is again a $\mu\mathrm{VF}[n]$-isomorphism, and it follows that the same is true for $\mu_l\mathrm{RV}[n, \cdot]$. We thus have the following.

Lemma 8.18. *Let* $\mathbf{X} = (X, f, \omega)$, $\mathbf{X}' = (X', f', \omega) \in \mu\mathrm{RV}[n, \cdot]$, *and let* $h : X \to W \subseteq \mathrm{RV}^*$, $h' : X' \to W$ *be definable maps. If for each* $c \in W$, $(\mathbf{X}_c, \mathbf{X}'_c) \in \mathrm{I}^{\mu}_{\mathrm{sp}}(\langle c\rangle)$, *then* $(\mathbf{X}, \mathbf{X}') \in \mathrm{I}^{\mu}_{\mathrm{sp}}$.

Proof. Lemma 2.3 applies to $\mathrm{RV}_{\mathrm{vol}}$-isomorphisms, and hence using Remark 7.13, also to blowups. It also applies to $\mu_l\mathrm{RV}[n, \cdot]$-isomorphisms by the discussion above, and hence to $\mathrm{I}^{\mu}_{\mathrm{sp}}$-equivalence. □

In other words, there exists a well-defined direct sum operation on $\mu\mathrm{RV}[n, \cdot]/\mathrm{I}^{\mu}_{\mathrm{sp}}$, with respect to RV-indexed systems.

Lemma 8.19.

(1) *Let* $\mathbf{Y}'$ *be an elementary blowup of* $\mathbf{Y}$. $\mathbf{Y}'$ *is* $\mathcal{C}$*-isomorphic to* $\mathbf{Y}'' = (Y'', f'', \omega')$, *with*

$$Y'' = \{(y,t) \in Y \times \mathrm{RV}_\infty : \mathrm{val}_{\mathrm{rv}}(t) > f_n(y)\},$$
$$f''(y,t) = (f_1(y), \dots, f_{n-1}(y), t), \quad \omega'(y,t) = \omega(y).$$

(2) *Up to isomorphism, the blowup depends only on the blowup locus. In other words, if* X, X', g, g', ω, ω' *are as in Definition* 8.16, *and* $\phi_i : \mathbf{Y}_i \to (X', g', \omega')$ $(i = 1, 2)$ *are* $\mu_l\mathrm{RV}[n,\cdot]$*-isomorphisms, then* $\widetilde{\mathbf{X}}_{\phi_1}$, $\widetilde{\mathbf{X}}_{\phi_2}$ *are* $\mathbf{X}$*-isomorphic in* $\mu_l\mathrm{RV}[n,\cdot]$.

Proof.

(1) The isomorphism is given by $h((y,t)) = (y, tf_n(y))$; since f_n always lifts to a function $F_n : \mathbb{L}Y \to \mathrm{VF}$ (a coordinate projection), h can be lifted to H defined by $H((y,t)) = (y, tF_n(y))$.
(2) By assumption, ϕ_1, ϕ_2 lift to measure-preserving maps $\Phi_i : \mathbb{L}\mathbf{Y}_i \to \mathbb{L}\mathbf{X}'$. On the other hand, by the assumption on existence of a section s of f_n, we have measure-preserving isomorphisms $\alpha_1 : \mathbb{L}\mathbf{Y}_1 \to \mathbb{L}\widetilde{\mathbf{Y}}_1$, $(y, u_1, \dots, u_n) \mapsto (y, u_1, \dots, u_{n-1}, (u_n - s)/s)$. Similarly, we have $\alpha_2 : \mathbb{L}\mathbf{Y}_2 \to \mathbb{L}\widetilde{\mathbf{Y}}_2$. Composing, we obtain $\alpha_2 {\Phi_2}^{-1} \Phi_1 {\alpha_1}^{-1} : \mathbb{L}\widetilde{\mathbf{Y}}_1 \to \mathbb{L}\widetilde{\mathbf{Y}}_2$; it is easy to check that this is $\underset{\mathrm{rv}}{\sim}$-invariant and shows that $\mathbb{L}\widetilde{\mathbf{Y}}_1, \mathbb{L}\widetilde{\mathbf{Y}}_2$ are $\mathbf{Y}$-isomorphic in $\mu_l\mathrm{RV}[n,\cdot]$. Taking the disjoint sum with the complement X'' of X', we obtain the result. □

Remark. There is also a parallel of Lemma 7.15: Let $\mathbf{Y} = (Y, g) \in \mathrm{Ob}\,\mathrm{RV}_\infty[n,\cdot]$, with $\dim(g(Y)) < n$; let $f : Y \to \mathrm{RV}^{n-1}$ be isogenous to g. Let $h : Y \to \mathrm{RV}$ be definable, with $h(y) \in \mathrm{acl}(g(y))$ for $y \in Y$, and with $\sum(g) = \sum(f) + \mathrm{val}_{\mathrm{rv}}(h)$. Let $Y' = Y \times \mathrm{RV}_\infty^{>0}$, and $f'(y,t) = (f(y), th(y))$. Then for appropriate ω', $\mathbf{Y}' = (Y', f', \omega')$ with the projection map to Y is a blowup. This follows from Lemma 7.15 and Lemma 8.14(3).

Notation. For $X \in \mathrm{RV}[n,\cdot]$, $[X] = [(X, 1)]$ denotes the corresponding object of $\mu\mathrm{RV}[n,\cdot]$ with form 1.

Lemma 8.20. *Lemma* 7.18(1)–(5) *holds for* $\mu_l\mathrm{RV}[n,\cdot]$. *We also have the following:*

(6) *As a semiring congruence on* $K_+\,\mu_l\mathrm{RV}[n,\cdot]$, $\mathrm{I}_{\mathrm{sp}}^{\mu}$ *is generated by* $([[1_{\mathbf{k}}]_1], [[\mathrm{RV}^{>0}]_1])$ (*with the forms* 1).

Proof. (1)–(5) go through with the same proof. For (6), Let $\sim$ be the congruence generated by this element. By blowing up a point one sees immediately that $([[1]_1], [[\mathrm{RV}^{>0}]_1]) \in \mathrm{I}_{\mathrm{sp}}^{\mu}$, so $\sim \leq \mathrm{I}_{\mathrm{sp}}^{\mu}$. For the converse direction we have to show that $(\widetilde{\mathbf{Y}}, \mathbf{Y}) \in \sim$ whenever $\widetilde{\mathbf{Y}}$ is a blowup of $\mathbf{Y}$; the elementary case suffices, since the $\mu_l\mathrm{RV}[n,\cdot]$-isomorphisms of Definition 8.16(2) are already accounted for in the semigroup $K_+\,\mu_l\mathrm{RV}[n,\cdot]$. Now $\mathbf{Y} = (Y, f, \omega)$ with $f_n(y) \in \mathrm{RV}$. Since $\dim(Y) < n$, we have $\mathbf{Y} \simeq (Y, f', \omega')$ where $f'_i = f_i$ for $i < n$, $f'_n = 1$, and $\omega' = f_n\omega$. Thus we may assume $f_n = 1$. In this case, as in the proof of Lemma 7.18(6), $(\widetilde{\mathbf{Y}}, \mathbf{Y}) \in \sim$. □

Definition 8.21. Let J be a k-element set of natural numbers. $\mathrm{VFR}_\mu[J, l, \cdot]$ is the set of triples $\mathbf{X} = (X, f, \omega)$, with $X \subseteq \mathrm{VF}^J \times \mathrm{RV}^*$, $f : X \to \mathrm{RV}^l_\infty$, $\omega : X \to \mathrm{RV}$, and such that f and ω factor through the projection $\mathrm{pr}_{\mathrm{RV}}(X)$ of X to the RV-coordinates. $\mathrm{I}^\mu_{\mathrm{sp}}$ is the equivalence relation on $\mathrm{VFR}_\mu[J, l, \cdot]$:

$$(X, Y) \in \mathrm{I}^\mu_{\mathrm{sp}} \iff (X_a, Y_a) \in \mathrm{I}^\mu_{\mathrm{sp}}(\langle a \rangle) \quad \text{for each } a \in \mathrm{VF}^J.$$

$K_+ \mathrm{VFR}_\mu$ is the set of equivalence classes.

For $j \in J$, let $\pi^j : \mathrm{VF}^k \times \mathrm{RV}^* \to \mathrm{VF}^{J-\{j\}} \times \mathrm{RV}^*$ be the projection forgetting the jth VF coordinate. We will write $\mathrm{VFR}_\mu[k, l, \cdot]$, VF^k, VF^{k-1} for $\mathrm{VFR}_\mu[J, l, \cdot]$ VF^J, $\mathrm{VF}^{J-\{j\}}$, respectively, when the identity of the indices is not important.

The map $\mathbb{L} : \mathrm{Ob}\, \mu\mathrm{RV}[n, \cdot] \to \mathrm{Ob}\, \mu\mathrm{VF}[n]$ induces, by Lemma 6.3, a homomorphism $\mathbb{L} : K_+\, \mu\mathrm{RV}[n, \cdot] \to K_+\, \mu\mathrm{VF}[n]$. By Proposition 4.5 it is surjective.

Lemma 8.22. *Let $\mathbf{X}, \mathbf{X}' \in \mu\mathrm{RV}[n, \cdot]$, and let $G : \mathbb{L}\mathbf{X}' \to \mathbb{L}\mathbf{X}$ be a special bijection. Then $\mathbf{X}'$ is isomorphic to an iterated blowup of $\mathbf{X}$.*

Proof. The proof is clear from Lemma 7.19 since strong isomorphisms are also $\mu_l\mathrm{RV}[n, \cdot]$-isomorphisms. □

Lemma 8.23. *The homomorphism $\mathbb{L} : K_+\, \mu\mathrm{RV}[1, \cdot] \to K_+\, \mu\mathrm{VF}[1, \cdot]$ is surjective, with kernel equal to $\mathrm{I}^\mu_{\mathrm{sp}}[1]$. The image of $K_+\, \mathrm{RV}_{\mathrm{vol}}[1, \cdot]$ is $K_+\, \mathrm{VF}_{\mathrm{vol}}[1, \cdot]$*

Proof. Let $\mathbf{X}, \mathbf{Y} \in \mu\mathrm{RV}[1, \cdot]$, and let $F : \mathbb{L}\mathbf{X} \to \mathbb{L}\mathbf{Y}$ be a definable measure-preserving bijection. We have $\mathbf{X} = (X, f, \omega)$, $\mathbf{Y} = (Y, g, \omega)$ with $(X, f), (Y, g) \in \mathrm{RV}[1, \cdot]$. By Lemma 7.6 there exist special bijections $b_X : \mathbb{L}\mathbf{X}' \to \mathbb{L}\mathbf{X}$, $b_Y : \mathbb{L}\mathbf{Y}' \to \mathbb{L}\mathbf{Y}$ and an $\underset{\mathrm{rv}}{\sim}$-invariant definable bijection $F' : \mathbb{L}\mathbf{X}' \to \mathbb{L}\mathbf{Y}'$ such that $b_Y F' = F b_X$. We used here that any $\underset{\mathrm{rv}}{\sim}$-invariant object can be written as $\mathbb{L}\mathbf{X}'$ for some $\mathbf{X}'$. Since F, b_X, b_Y are measure-preserving bijections, so is F'. By Lemma 8.22, $\mathbf{X}' \to \mathbf{X}$ and $\mathbf{Y}' \to \mathbf{Y}$ are blowups; and F' descends to a definable bijection between them. This bijection is measure preserving by Lemma 5.22. Hence by definition $(\mathbf{X}, \mathbf{Y}) \in \mathrm{I}^\mu_{\mathrm{sp}}$.□

By Proposition 8.23, the inverse of $\mathbb{L} : \mathrm{RV}[1, \cdot] \to \mathrm{VF}[1, \cdot]$ induces an isomorphism $I^{\mathrm{vol}}_1 : K_+\, \mathrm{VF}_{\mathrm{vol}}[1, \cdot] \to K_+\, \mathrm{RV}_{\mathrm{vol}}[1, \cdot]/\mathrm{I}^\mu_{\mathrm{sp}}$.

$$I^{\mathrm{vol}}_1([X]) = [Y]/\mathrm{I}^\mu_{\mathrm{sp}} \iff [\mathbb{L}Y] = [X].$$

Let $\mathbf{X} = (X, f, \omega) \in \mathrm{VFR}_\mu[k, l, \cdot]$. By assumption, f, ω factor through π^j, so that they can be viewed as functions on $\pi^j X$. We view the image $(\pi^j X, f, \omega)$ as an element of $\mathrm{VFR}_\mu[k-1, l, \cdot]$. Each fiber of π^j is a subset of VF; it can be viewed as an element of $\mathrm{VF}_{\mathrm{vol}}[1] \subseteq \mu\mathrm{VF}[1] \subseteq \mu\mathrm{VF}[1, \cdot]$.

Claim. Relative $\mathrm{I}^\mu_{\mathrm{sp}}$-equivalence implies $\mathrm{I}^\mu_{\mathrm{sp}}$-equivalence, in the following sense. Let $X_i \subseteq \mathrm{RV}^*$ $(i = 1, 2)$; $h_i : X_i \to W \subseteq \mathrm{RV}^*$; $f_W : W \to \mathrm{RV}^l$, $\omega : W \to \mathrm{RV}$, and $f_i : X \to \mathrm{RV}^k$ be definable sets and functions. Let $\mathbf{X}_i = (X_i, (f_W \circ h_i, f_i), \omega \circ h_i)$. Let $\mathbf{X}_i(w) = (X_i(w), f_i|X_i(w), \omega \circ h_i|X_i(w))$, where $X_i(w) = {h_i}^{-1}(w)$. If $\mathbf{X}_1(w), \mathbf{X}_2(w) \in \mathrm{I}_{\mathrm{sp}}(\langle w \rangle)$ for each $w \in W$, then $(\mathbf{X}_1, \mathbf{X}_2) \in \mathrm{I}^\mu_{\mathrm{sp}}$.

Proof. The proof is clear using Lemma 8.18. □

The claim allows us to relativize I_1^{vol} to π^j. We obtain a map

$$I^j = I^j_{k,l} : \mathrm{VFR}_\mu[k,l,\cdot] \to K_+ \mathrm{VFR}_\mu[k-1,l+1,\cdot]/\mathrm{I}^\mu_{\mathrm{sp}}.$$

Lemma 8.24. *Let* $\mathbf{X} = (X, f, \omega), \mathbf{X}' = (X', f', \omega') \in \mathrm{VFR}_\mu[k,l,\cdot]$.

(1) I^j *commutes with maps into* RV*: if* $h : \mathbf{X} \to W \subseteq \mathrm{RV}^*$ *is definable,* $\mathbf{X}_c = h^{-1}(c)$*, then* $I^j(\mathbf{X}) = \sum_{c\in W} I^j(\mathbf{X}_c)$.
(2) *If* $([\mathbf{X}], [\mathbf{X}']) \in \mathrm{I}^\mu_{\mathrm{sp}}$*, then* $(I^j(\mathbf{X}), I^j(\mathbf{X}')) \in \mathrm{I}^\mu_{\mathrm{sp}}$.
(3) I^j *induces a map* $K_+ \mathrm{VFR}_\mu[k,l,\cdot]/\mathrm{I}^\mu_{\mathrm{sp}} \to K_+ \mathrm{VFR}_\mu[k-1,l+1,\cdot]/\mathrm{I}^\mu_{\mathrm{sp}}$.

Proof.

(1) This reduces to the case of I_1^{vol}, where it is an immediate consequence of uniqueness, and the fact that $\mathbb{L}$ commutes with maps into RV in the same sense.
(2) All equivalences here are relative to the $k-1$ coordinates of VF other than j, so we may assume $k = 1$, and write I for I^j. For $a \in \mathrm{VF}$, $([\mathbf{X}_a], [\mathbf{X}'_a]) \in \mathrm{I}^\mu_{\mathrm{sp}}(\langle a\rangle)$. By stable embeddedness of RV, there exists $\alpha = \alpha(a) \in \mathrm{RV}^*$ such that $\mathbf{X}_a, \mathbf{X}'_a$ are $\langle\alpha\rangle$-definable there are $\langle\alpha\rangle$-definable blowups $\widetilde{\mathbf{X}}_a, \widetilde{\mathbf{X}'}_a$ and an $\langle\alpha\rangle$-definable isomorphism between them, lifting to an a-definable isomorphism. Using (1) and Lemma 8.18 we may assume that α is constant. Thus for some $W \in \mathrm{Ob}\,\mathrm{VF}[1]$, $\mathbf{Y}, \mathbf{Y}' \in \mu\mathrm{RV}[l+1,\cdot]$, we have $\mathbf{X} = W \times \mathbf{Y}$, $\mathbf{X}' = W \times \mathbf{Y}'$, $\widetilde{\mathbf{Y}}, \widetilde{\mathbf{Y}}'$ are blowups of $\mathbf{Y}, \mathbf{Y}'$, respectively, $\phi : \mathbf{Y} \to \mathbf{Y}'$ is a bijection, and for any $w \in W$ there exists a measure-preserving $F_w : \mathbb{L}\widetilde{\mathbf{Y}} \to \mathbb{L}\widetilde{\mathbf{Y}}$ lifting ϕ. Then $I(\mathbf{X}) = I(W) \times \mathbf{Y}$, $I(\mathbf{X}') = I(W) \times \mathbf{Y}'$ and the bijection $\mathrm{Id}_{I(W)} \times \phi$ is lifted by the measure-preserving bijection $(w, y) \mapsto (w, F_w(y))$.
(3) This follows by (2). □

Lemma 8.25. *Let* $\mathbf{X} = (X, f, \omega) \in \mathrm{Ob}\,\mathrm{VFR}_\mu[J,l,\cdot]$*. If* $j \neq j' \in J$*, then* $I^j I^{j'} = I^{j'} I^j : K_+ \mathrm{VFR}_\mu[J,l,\cdot]/\mathrm{I}^\mu_{\mathrm{sp}} \to K_+ \mathrm{VFR}_\mu[J \setminus \{j,j'\}, l+2,\cdot]/\mathrm{I}^\mu_{\mathrm{sp}}$.

Proof. We may assume $S = \{1,2\}$, $j = 1$, $j' = 2$, since all is relative to $\mathrm{VF}^{S\setminus\{j,j'\}}$. By Lemma 7.23(1) and Lemma 8.18 it suffices to prove the statement for each fiber of a given map into RV[l]. Hence we may assume $X \subseteq \mathrm{VF}^2$ so that f is constant; and by Lemma 5.10, we can assume X is a basic 2-cell:

$$X = \{(x,y) : x \in X_1, \mathrm{rv}(y - G(x)) = \alpha_1\}, \quad X_1 = \mathrm{rv}^{-1}(\delta_1) + c_1.$$

The case where G is constant is easy since then X is a finite union of rectangles. Otherwise, G is invertible, and by the niceness of G we can also write

$$X = \{(x,y) : y \in X_2, \mathrm{rv}(x - G^{-1}(y)) = \beta\}, \quad X_2 = \mathrm{rv}^{-1}(\delta_2) + c_2.$$

We immediately compute

$$I_2 I_1(X) = (\delta_1, \alpha_1), \qquad I_1 I_2(X) = (\alpha_2, \delta_2)$$

and necessarily $\mathrm{val}_{\mathrm{rv}}\delta_1 + \mathrm{val}_{\mathrm{rv}}\alpha_1 = \mathrm{val}_{\mathrm{rv}}\alpha_2 + \mathrm{val}_{\mathrm{rv}}\delta_2$ (Lemma 5.4). We have bijections $F_j : X \to \mathbb{L}I_j(X)$. The map $F_1 F_2 F_1^{-1} F_2^{-1} : \mathbb{L}I_2 I_1(X) \to \mathbb{L}I_1 I_2(X)$ lifts the unique bijection between the singleton sets $\{(\delta_1, \alpha_1)\}$, $\{(\alpha_2, \delta_2)\}$, and shows that $[(\delta_1, \alpha_1)]_2 = [(\alpha_2, \delta_2)]_2$. □

Proposition 8.26. *Let* $\mathbf{X}, \mathbf{Y} \in \mu\mathrm{RV}[\le n, \cdot]$. *If* $\mathbb{L}\mathbf{X}$, $\mathbb{L}\mathbf{Y}$ *are isomorphic, then* $([X], [Y]) \in \mathrm{I}_{\mathrm{sp}}^{\mu}$.

Proof. The proof is identical to the proof of Proposition 7.25, only quoting Lemma 8.25 in place of Lemma 7.24, and Lemma 8.14 to enable using Lemma 7.11. □

Proposition 8.27. *Proposition* 8.3 *is valid for* $\mu\mathrm{VF}[n]$, $\mu\mathrm{RV}[n]$, $\mathrm{I}_{\mathrm{sp}}^{\mu}[n]$.

Proof. The proof is the same as that of Proposition 8.3, but using Proposition 6.3 in place of 6.1 and Proposition 8.26 in place of Proposition 7.25. □

8.6 Invariants of measure-preserving maps, and some induced isomorphisms

Theorem 8.28. *Let* **T** *be* V*-minimal. There exists a canonical isomorphism of Grothendieck semigroups*

$$\not{\int} : K_+^{\mathrm{eff}}\, \mu\mathrm{VF}[n, \cdot] \to K_+(\mu\mathrm{RV}[n, \cdot])/\mathrm{I}_{\mathrm{sp}}^{\mu}[n].$$

Let $[X]$ *denote the class of* X *in* $K_+^{\mathrm{eff}}(\mu\mathrm{VF}[n])$. *Then*

$$\not{\int}[X] = W/\mathrm{I}_{\mathrm{sp}}^{\mu}[n] \iff [X] = [\mathbb{L}W] \in K_+^{\mathrm{eff}}(\mu\mathrm{VF}[n]).$$

Proof. Given $\mathbf{X} = (X, f, \omega) \in \mathrm{Ob}\, \mu\mathrm{RV}[n]$ we have $\mathbb{L}\mathbf{X} \in \mathrm{Ob}\, \mu\mathrm{VF}[n]$. If $\mathbf{X}, \mathbf{X}'$ are isomorphic, then by Lemma 6.3, $\mathbb{L}\mathbf{X}, \mathbb{L}\mathbf{X}'$ are effectively isomorphic. Direct sums are clearly respected, so we have a semigroup homomorphism $\mathbb{L} : K_+(\mu\mathrm{RV}[n]) \to K_+^{\mathrm{eff}}(\mu\mathrm{VF}[n])$. It is surjective by Proposition 4.5 and injective by Proposition 8.3. Inverting, we obtain I. □

Let ${\mathrm{I}_{\mathrm{sp}}^{\mu}}'$ be the semigroup congruence on $\mathrm{RV}_{\mathrm{vol}}[n]$ generated by $((Y, f), (Y \times \mathrm{RV}^{>0}, f'))$, where Y, f, f' are as in Definition 7.12. Let $\mu_\Gamma \mathrm{I}_{\mathrm{sp}}$ be the congruence on $K_+\, \mu_\Gamma\mathrm{RV}[n]$ generated by $([[1_{\mathbf{k}}]_1], [[\mathrm{RV}^{>0}]_1])$, with the constant Γ-form $0 \in \Gamma$.

Assume given a distinguished subgroup N_1 of the multiplicative group of the residue field $\mathbf{k}$. For example, N_1 may be the group of elements of norm one, with respect to some absolute value $|,|$ on $\mathbf{k}$. With this example in mind, write $|x| = 1$ for $x \in N_1$. Let $|\mu|\mathrm{VF}[n]$ be the subcategory of $\mathrm{VF}[n]$ with the same objects, and such that $F \in \mathrm{Mor}_{|\mu|\mathrm{VF}[n]}$ iff $F \in \mathrm{Mor}_{\mu_\Gamma \mathrm{VF}[n]}$ and $|JRV(F)| = 1$ almost everywhere. Define $|\dot{\mu}|\mathrm{RV}[n]$ similarly.

Theorem 8.29. *The isomorphism* $\not{\int}$ *of Theorem* 8.28 *induces isomorphisms:*

$$K_+^{\mathrm{eff}}\, \mathrm{VF}_{\mathrm{vol}}[n] \to K_+\, \mathrm{RV}_{\mathrm{vol}}[n]/{\mathrm{I}_{\mathrm{sp}}^{\mu}}'[n], \tag{8.1}$$

$$K_+^{\text{eff}}\ \mathrm{VF}^{\text{bdd}}_{\text{vol}}[n] \to K_+\ \mathrm{RV}^{\text{bdd}}_{\text{vol}}[n]/\mathrm{I}^{\mu}_{\text{sp}}[n], \tag{8.2}$$

$$K_+^{\text{eff}}\ \mu\mathrm{VF}^{\text{bdd}}[n] \to K_+\ \mu\mathrm{RV}^{\text{bdd}}[n]/\mathrm{I}^{\mu}_{\text{sp}}[n], \tag{8.3}$$

$$K_+^{\text{eff}}\ |\mu|\mathrm{VF}[n] \to K_+\ |\dot{\mu}|\mathrm{RV}[n]/\mathrm{I}^{\mu}_{\text{sp}}[n], \tag{8.4}$$

$$K_+^{\text{eff}}\ \mu_\Gamma\mathrm{VF}[n] \to K_+\ \mu_\Gamma\mathrm{RV}[n]/\mu_\Gamma\mathrm{I}_{\text{sp}}[n]. \tag{8.5}$$

Proof. Since Proposition 4.5 uses measure-preserving maps, Proposition 6.1 does not go out of the subcategory VF_{vol}, and $\mathrm{RV}_{\text{vol}}[n]$ is a full subcategory of $\mu\mathrm{RV}[n]$, we have (8.1). It is similarly easy to see that "dimension $< n$" and boundedness are preserved, hence (8.2)–(8.3).

We have $K_+^{\text{eff}}\ |\mu|\mathrm{VF} = K_+^{\text{eff}}\ \mu\mathrm{VF}/N_{\mathrm{VF}}$, where $N_{\mathrm{VF}} = \{([X,\omega],[X,g\omega]) : g : X \to \mathrm{RV}, |g| = 1\}$; similarly for $K_+^{\text{eff}}\ |\dot{\mu}|\mathrm{RV}$. Thus for (8.4) it suffices to show that $(\not\oint(\mathbf{X}), \not\oint(\mathbf{Y})) \in N_{\mathrm{RV}} \iff (\mathbf{X},\mathbf{Y}) \in N_{\mathrm{VF}}$. For $X \in \mathrm{Ob}\,\mu\mathrm{VF}[n]$ or $X \in \mathrm{Ob}\,\mu\mathrm{RV}[n]$ with RV-volume form ω, given $g : X \to \mathrm{RV}$, let ${}^gX$ denote the same object but with volume form $g\omega$. In one direction, we have to show that $(\mathbb{L}X, \mathbb{L}Y) \in N_{\mathrm{VF}}$ if $(X,Y) \in N_{\mathrm{RV}}$. This is clear since $\mathbb{L}({}^gX) ={}^g (\mathbb{L}X)$. Conversely we have to show that $(\not\oint[{}^gZ], \not\oint[Z]) \in N_{\mathrm{RV}}$. Since $\not\oint$ commutes with RV-sums, we may assume g is constant, with value a. But then $\mathbb{L}({}^aX) ={}^a (\mathbb{L}X)$ implies $\not\oint({}^aZ) ={}^a \not\oint Z$ as required. This gives (8.4); (8.5) is a special case. □

9 The Grothendieck semirings of Γ

Let $T = \mathrm{DOAG}_A$ be the theory of divisible ordered Abelian groups Γ, with distinguished constants for elements of a subgroup A. Let $\mathrm{DOAG}_A[*]$ be the category of all DOAG_A definable sets and bijections. Our primary concern is not with DOAG_A, but rather a proper subcategory $\Gamma[*]$, having the same objects but only piecewise integral morphisms (Definition 9.1). Our interest in $\Gamma[*]$ derives from this: the morphisms of $\Gamma[*]$ are precisely those that lift to morphisms of $\mathrm{RV}[*]$, and it is $K_+[\Gamma[*]]$ that forms a part of $K_+[\mathrm{RV}[*]]$ (cf. Section 3.3). This category depends on A, but will nevertheless be denoted $\Gamma[*]$ when A is fixed and understood.

We will first describe $K(\Gamma^{\text{fin}}[*])$, the subring of classes of finite definable sets. Next, we will analyze $K(\mathrm{DOAG}_A)$, obtaining two Euler characteristics. This repeats earlier work by Maříková. We retain our proofs as they give a rapid path to the Euler characteristics, but [26] includes a complete analysis of the semiring $K(\mathrm{DOAG}_A)$, that may well be useful in future applications.

At the level of Grothendieck rings, the categories $\Gamma[*]_A$ and DOAG_A may be rather close; see Lemma 9.8 and Question 9.9. But the semiring homomorphism $K_+(\Gamma[*]_A) \to K(\mathrm{DOAG}_A)$ is far from being an isomorphism, and it remains important to give a good description of $K_+(\Gamma[*]_A)$. We believe that further invariants can be found by mapping $K_+[\Gamma[*]]$ into the Grothendieck semirings of other completions of the universal theory of ordered Abelian groups over A, as well as DOAG, in the manner of Proposition 9.2; it is possible that all invariants appear in this way.

A description of $K_+(\Gamma[*]_A)$ would include information about the Grothendieck group of subcategories, such as the category of bounded definable sets. We will only sample one bit of the information available there, in the form of a "volume" map on bounded subsets of $K_+[\Gamma[*]]$ into the rationals, and a discrete analogue.

Definition 9.1. An object of $\Gamma[n]$ is a finite disjoint union of subsets of Γ^n defined by linear equalities and inequalities with $\mathbb{Z}$-coefficients and parameters in A. Given $X, Y \in \mathrm{Ob}\,\Gamma[n]$, $f \in \mathrm{Mor}_\Gamma(X, Y)$ iff f is a bijection, and there exists a partition $X = \cup_{i=1}^n X_i$, $M_i \in \mathrm{GL}_n(\mathbb{Z})$, $a_i \in A^n$, such that for $x \in X_i$,

$$f(x) = M_i x + a_i.$$

$\Gamma[*]$ is the category of definable subsets of Γ^n for any n, with the same morphisms. Since there are no morphisms between different dimensions, it is simply the direct sum of the categories $\Gamma[n]$, and the Grothendieck semiring $K_+[\Gamma]$ of $\Gamma[*]$ is the graded direct sum of the semigroups $K_+(\Gamma[n])$. We will write $K[\Gamma]$ for the corresponding group.

Let $\Gamma^{\mathrm{bdd}}[*]$ be the full subcategory of $\Gamma[*]$ consisting of bounded sets, i.e., an element of $\mathrm{Ob}\,\Gamma^{\mathrm{bdd}}[n]$ is a definable subset of $[-\gamma, \gamma]^n$ for some $\gamma \in \Gamma$.

Γ_A is a subcategory of $\Gamma_{\mathbb{Q}\otimes A}$ (a category with the same objects, but more morphisms, generated by additional translations) and this in turn is a subcategory of $\mathrm{DOAG}_{\mathbb{Q}\otimes A}$.

There is therefore always a natural morphism from $K_+(\Gamma_A[*])$ to the simpler semigroup $K_+(\mathrm{DOAG}_{\mathbb{Q}\otimes A})$. We will exhibit two independent Euler characteristics on $\mathrm{DOAG}_{\mathbb{Q}\otimes A}$ and show that they define an isomorphism $K(\mathrm{DOAG}_{\mathbb{Q}\otimes A}) \to \mathbb{Z}^2$. Taking the dimension grading into account, this will give rise to two families of Euler characteristics on $K(\Gamma_A)$, with $\mathbb{Z}[T]$-coefficients.

9.1 Finite sets

Let $\Gamma^{\mathrm{fin}}[n]$ be the full subcategory of $\Gamma_A[n]$ consisting of finite sets. The Grothendieck semiring of $\Gamma^{\mathrm{fin}}[*]$ embeds into the semirings of both Γ_A and RES, within the Grothendieck semiring of RV_A, and we will see that $K_+(\mathrm{RV}_A)$ is freely generated by them over $K_+(\Gamma^{\mathrm{fin}}[*])$. We proceed to analyze $K_+(\Gamma^{\mathrm{fin}}[*])$ in detail.

Let $\boldsymbol{\tau} = [0]_1 \in K_+(\Gamma^{\mathrm{fin}}[1])$ be the class of the singleton $\{0\}$.

The unit element of $K(\Gamma)$ is the class of Γ^0. Note that the bijection between $\boldsymbol{\tau}$ and Γ^0 is not a morphism in $\Gamma[*]$; in fact $1, \boldsymbol{\tau}, \boldsymbol{\tau}^2, \ldots$ are distinct and $\mathbb{Q}$-linearly independent in $K(\Gamma)$. The motivation for this choice of category becomes clear if one thinks of the lift to RV: the inverse image of $\boldsymbol{\tau}^n$ in RV (also denoted $\boldsymbol{\tau}^n$) has dimension n, and cannot be a union of isomorphic copies of $\boldsymbol{\tau}^m$ for smaller m.

Let $K(\Gamma^{\mathrm{fin}})[\boldsymbol{\tau}^{-1}]$ be the localization. This ring is a naturally $\mathbb{Z}$-graded ring; let H_{fin} be the zero-dimensional component.

Let Ξ_A be the space of subgroups of $(\mathbb{Q} \otimes A)/A$ or, equivalently, of subgroups of $\mathbb{Q} \otimes A$ containing A. View it as a closed subspace of the Tychonoff space $2^{(\mathbb{Q}\otimes A)/A}$, via the characteristic function 1_s of a subgroup $s \in \Xi_A$. Let $C(\Xi_A, \mathbb{Z})$ be the ring of continuous functions $\Xi_A \to \mathbb{Z}$ (where $\mathbb{Z}$ is discrete).

A *cancellation* semigroup is a semigroup where $a + b = a + c$ implies $b = c$; in other words, a subsemigroup of an Abelian group.

Proposition 9.2. *$K_+(\Gamma^{\text{fin}}[n])$ is a cancellation semigroup. As a semiring, $K_+(\Gamma^{\text{fin}}[*])$ is generated by $K_+(\Gamma^{\text{fin}}[1])$. We have*

$$K(\Gamma^{\text{fin}})[\boldsymbol{\tau}^{-1}] = H_{\text{fin}}[\boldsymbol{\tau}, \boldsymbol{\tau}^{-1}],$$
$$H_{\text{fin}} \simeq C(\Xi_A, \mathbb{Z}).$$

Proof. Since Γ is ordered, any finite definable subset of Γ^n is a union of definable singletons. Thus the semigroup $K_+(\Gamma^{\text{fin}}[n])$ is freely generated by the isomorphism classes of singletons $a \in \Gamma^n$ and, in particular, is a cancellation semigroup. The displayed equality is thus clear; we proceed to prove the isomorphism.

A definable singleton of Γ^n has the form $(a_1, \dots, a_n)$, where for some $N \in \mathbb{N}$, $Na_1, \dots, Na_n \in A$. Thus $[(a_1, \dots, a_n)] = [(a_1)] \cdots [(a_n)]$.

For any commutative ring R, let $\text{Idem}(R)$ be the Boolean algebra of idempotent elements of a commutative ring R with the operations $1, 0, xy, x + y - xy$. Note that the elements $[(a_1, \dots, a_n)]\boldsymbol{\tau}^{-n} \in H_{\text{fin}}$ belong to $\text{Idem}(H_{\text{fin}})$: in $K_+(\Gamma^{\text{fin}})$: for any $a \in \Gamma$ we have the relation $[a]^2 = [a]\boldsymbol{\tau}$. Let B be the Boolean subalgebra of $\text{Idem}(H_{\text{fin}})$ generated by the elements $[(a_1, \dots, a_n)]\boldsymbol{\tau}^{-n}$. For a maximal ideal M of B, let I_M be the ideal of H_{fin} generated by M. Note $H_{\text{fin}} = \mathbb{Z}B$. Hence we have to show the following:

(1) The Stone space of B is Ξ_A.
(2) For any maximal ideal M of B, $H_{\text{fin}}/I_M \simeq \mathbb{Z}$ naturally.

For any commutative ring R, a finitely generated Boolean ideal of $\text{Idem}(R)$ is generated by a single element b; if $b \neq 1$, then $bR \neq R$ since $b(1 - b) = 0$. Thus if M is a proper ideal of $\text{Idem}(R)$, then MR is a proper ideal of R. Applying this to B, viewed as a Boolean subalgebra of $\text{Idem}(\mathbb{Q} \otimes H_{\text{fin}})$, we see that $I_M \cap \mathbb{Z} = (0)$ for any maximal ideal M of B. Thus the composition $\mathbb{Z} \to H_{\text{fin}} \to H_{\text{fin}}/I_M$ is injective. On the other hand, H_{fin} is generated over $\mathbb{Z}$ by the elements $[a]/\boldsymbol{\tau}$, and each of them equals 0 or 1 modulo I_M, so the map is surjective, too. This proves the second point.

To prove the first, we define a map $\Phi : \Xi_A \to \text{Stone}(B)$.

Let $t = T/A$, $T \leq \mathbb{Q} \otimes A$. If $[(a_1, \dots, a_n)] = [(b_1, \dots, b_n)]$, then some element of $\text{GL}_n(\mathbb{Z}) \ltimes A^n$ takes $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$; in this case, if $a_i \in T$ for each i then $b_i \in T$ for each i; so $\Pi_{i=1}^n 1_t(a_i + A) = \Pi_{i=1}^n 1_t(b_i + A)$. Thus, given $t \in \Xi_A$, we can define a homomorphism $h_t : H_{\text{fin}} \to \mathbb{Z}$ by

$$[(a_1, \dots, a_n)]/\boldsymbol{\tau}^n \mapsto \Pi_{i=1}^n 1_t(a_i + A).$$

Let $M(t) = \ker(h_t) \cap B$.

The map $\Phi : t \mapsto M(t)$ is clearly continuous. If t, t' are distinct subgroups, let $a \in t, a \notin t'$ (say); then $[a]/\boldsymbol{\tau} \in M(t)$, $[a]/\boldsymbol{\tau} \notin M(t')$. Thus Φ is injective. If P is a maximal filter of B, let $t_P = \{a + A : [a]/\boldsymbol{\tau} \in P\}$.

Claim. t_P is a subgroup.

Proof. Suppose $a + A, b + A \in t_P$ and let $c = a + b$. Then we have the relation

$$[a][b]\boldsymbol{\tau} = [a][b][c]$$

in $K_+(\Gamma^{\text{fin}})$, arising from the map

$$(x, y, z) \mapsto (x, y, xyz).$$

Thus $([a]/\boldsymbol{\tau})([b]/\boldsymbol{\tau})(1-[c]/\boldsymbol{\tau}) = 0$. As $([a]/\boldsymbol{\tau}), ([b]/\boldsymbol{\tau}) \in P$ we have $(1-[c]/\boldsymbol{\tau}) \notin P$, so $[c]/\boldsymbol{\tau} \in P$. □

Clearly, $P = M(t_P)$. Thus Φ is surjective, and so a homeomorphism. □

Example. We always have a homomorphism $K(\Gamma^{\text{fin}}) \to \mathbb{Z}$ (by counting points of a finite set in the divisible hull); when A is divisible, this identifies $K(\Gamma^{\text{fin}})$ with $\mathbb{Z}[\tau]$. In general, we have the surjective morphism $K(\Gamma^{\text{fin}})) \to K(\Gamma^{\text{fin}}_{\mathbb{Q}\otimes A}) = \mathbb{Z}[\tau]$.

Lemma 9.3. *Let Y be an A-definable subset of Γ^n, of dimension $< n$. Then Y is a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugates of sets $Y_i \subseteq \{c_i\} \times \Gamma^{n-1}$, with $c_i \in \mathbb{Q} \otimes A$.*

Proof. Y can be divided into finitely many A-definable pieces, each contained in some A-definable hyperplane of Γ^n. Thus we may assume Y itself is contained in some such hyperplane, i.e., $\sum r_i y_i = c$ for some $c \in \mathbb{Q} \otimes \mathrm{val}_{\mathrm{rv}}(A)$. We may assume $r_i \in \mathbb{Z}$ and $(r_1, \ldots, r_n)$ have no common divisor. In this case $\mathbb{Z}^n/\mathbb{Z}(r_1, \ldots, r_n)$ is torsion free, hence free, so $\mathbb{Z}(r_1, \ldots, r_n)$ is a direct summand of $\mathbb{Z}^n$. Thus after effecting a transformation of $\mathrm{GL}_n(\mathbb{Z})$, we may assume $(r_1, \ldots, r_n) = (1, 0, \ldots, 0)$, i.e., Y lies in the hyperplane $y_1 = c$. Let Z be the projection of Y to the coordinates $(2, \ldots, n)$. Then $Y = \{c\} \times Z$. □

9.2 Euler characteristics of DOAG

We describe two independent Euler characteristics on A-definable subsets of Γ, i.e., additive, multiplicative $\mathbb{Z}[\boldsymbol{\tau}]$-valued functions invariant under all definable bijections. The values are in $\mathbb{Z}[\boldsymbol{\tau}]$ rather than $\mathbb{Z}$ because $\Gamma[*] = \oplus_n \Gamma[n]$ is graded by ambient dimension. Proposition 9.4–Lemma 9.6 were obtained earlier in [26], and independently in [20].

In fact, these two Euler characteristics come from Euler characteristics of $\mathrm{DOAG}_{\mathbb{Q}\otimes A}$. Ther they are the only ones.

Proposition 9.4. *Let A be a divisible ordered Abelian group. Then $K(\mathrm{DOAG}_A) \simeq \mathbb{Z}^2$.*

Proof. We begin by noting that there are at most two possibilities.

In DOAG, all definable singletons are isomorphic. The identity element of the ring $K(\mathrm{DOAG})$ is the class of any singleton. Thus the image of $K(\Gamma^{\text{fin}}[*])$ in $K(\mathrm{DOAG}_A)$ is isomorphic to $\mathbb{Z}$.

Claim. The image of $K(\Gamma^{\text{bdd}}_A)$ in $K(\mathrm{DOAG}_A)$ equals the image of $K(\Gamma^{\text{fin}}[*])$ there.

Translation by a gives an equality of classes in $K(\Gamma)$, $[(0,\infty)] = [(a,\infty)]$, so

$$[(0,a)] + [\{pt\}] = [(0,a] = 0.$$

Thus bounded segments are equivalent to linear combinations of points. This can be seen directly by induction on dimension and on ambient dimension: consider the class of a bounded set $Y \subset \Gamma^{n+1}$. Y is a Boolean combination of sets of the form $\{(x,y) : x \in X, f(x) < y < g(x)\}$. This is DOAG_A-isomorphic to $Y' = \{(x,y) : x \in X, 0 < y < h(x)\}$, where $h(x) = g(x) - f(x)$. Let $Z = \{(x,y) : x \in X, y > 0\}$, $Z' = \{(x,y) : x \in X, y > h(x)\}$. Then the map $(x,y) \mapsto (x, y + h(x))$ shows that $[Z] = [Z']$. On the other hand, Z' is the disjoint union of Z, Y and a lower-dimensional set W. Thus $[Z] = [Z'] = [Z'] + [Y] + [W]$ so $[Y] = -[W]$, and by induction $[Y]$ lies in the image of $K(\Gamma^{\mathrm{fin}}[*])$.

Now consider $t = [(0,\infty)] \in K(\Gamma_A)$. We have a homomorphism $K(\Gamma_A^{\mathrm{bdd}})[t] \to K(\Gamma)$. To see that it is surjective, again by induction it suffices to look at sets such as $\{(x,y) : x \in X, f(x) < y\}$ or $\{(x,y) : x \in X, f(x) < y < g(x)\}$. The latter is equivalent to a lower-dimensional set, by induction, as above. The former is equivalent to $\{(x,y) : x \in X, 0 < y\}$ so that it has the class $[X] \times t$ and thus is in the image of $K(\Gamma_A^{\mathrm{bdd}})[t]$.

Let $T = \{(x,y) : 0 < y \le x\}$. The map $(x,y) \mapsto (x, y + x)$ takes T to $\{(x,y) : 0 < x < y \le 2x\}$, so $2[T] = [\{(x,y) : 0 < y \le 2x\}]$. The same map shows that $t^2 - [T] = t^2 - 2[T]$ so $[T] = 0$. But then $[\{(x,y) : 0 < x \le y\}] = 0$, and adding we obtain $0 + 0 = t^2 + [\{(x,x) : 0 < x\}] = t^2 + t$. Thus $K(\mathrm{DOAG}_A)$ is a homomorphic image of $\mathbb{Z}[t]/(t^2+t) \simeq \mathbb{Z}^2$. To see that the homomorphism is bijective, it remains to exhibit a homomorphism $K(\mathrm{DOAG}_A) \to \mathbb{Z}$ with $t \mapsto 0$ and another with $t \mapsto -1$. The two lemmas below show this, in a form suitable also for a dimension-graded version. □

Lemma 9.5. *There exists a ring homomorphism* $\chi_O : K(\Gamma) \to \mathbb{Z}[\boldsymbol{\tau}]$, *such that* $\chi_O((0,\infty)) = \boldsymbol{\tau}$. *It is invariant under* $\mathrm{GL}_n(\mathbb{Q})$ *acting on* Γ^n.

Proof. Let RCF be the theory of real closed fields. See [37] for the existence and definability of an Euler characteristic map $\chi : K(\mathrm{RCF}) \to \mathbb{Z}$. For any definable $X, P, f : X \to P$ of RCF, there exists $m \in \mathbb{N}$ and a definable partition $P = \cup_{-m \le i \le m} P_i$, such that for any i, any $M \models \mathrm{RCF}$ and $b \in P_i(M)$, $\chi(X_b) = i$. Here $X_b = f^{-1}(b)$, and $\chi(X_b) = i$ iff there exists an M-definable partition of X_b into definable cells C_j, with $\sum_j (-1)^{\dim(C_j)} = i$.

The language of Γ (the language of ordered Abelian groups) is contained in the language of RCF. Thus if $X, P, f : X \to P$ are definable in the language of ordered Abelian groups, they are RCF-definable. Therefore, the above result specializes, and we obtain an Euler characteristic map $\chi : K(\Gamma_A[n]) \to \mathbb{Z}$, valid for any divisible group A. This χ is invariant under all definable bijections (not only the morphisms of $\Gamma[*]$), and is additive and multiplicative. We have $\chi_O(\{0\}) = 1$, $\chi_O((a,b)) = -1$ for $a < b$, and $\chi_O(0,\infty) = -1$, too (though $(0,1)$ and $(0,\infty)$ are not definably isomorphic in the linear structure). Now let $\chi_O(X) = \chi(X)\boldsymbol{\tau}^n$ for $X \subseteq \Gamma^n$, and extend to $\Gamma[*]$ by additivity. □

Remark. The Euler characteristic constructed in this proof appears to depend on an embedding of A into the additive group of a model of RCF. But by the uniqueness shown above, it does not. In fact, as pointed out to us by Van den Dries, Ealy and Maříková, an Euler characteristic with the requisite properties is defined in [37] directly for any O-minimal structure; moreover, the use of RCF in the lemma below can also be replaced by a direct inductive argument, and some simple facts about Fourier–Motzkin elimination.

Another Euler characteristic can be obtained as follows: given a definable set $Y \subset \Gamma^n$, let

$$\chi'(Y) = \lim_{r \to \infty} \chi(Y \cap C_r),$$

where C_r is the bounded closed cube $[-r, r]^n$. By O-minimality, the value of $\chi(Y \cap C_r)$ is eventually constant.

Note that χ' is not invariant under semialgebraic bijections, since the bounded and unbounded open intervals are given different measures. Still,

Lemma 9.6. *χ' induces a group homomorphism $K(\Gamma[n]) \to \mathbb{Z}$; and yields a ring homomorphism $K(\Gamma[*]) \to \mathbb{Z}[\boldsymbol{\tau}]$. Moreover, χ' is invariant under piecewise* $\mathrm{GL}_n(\mathbb{Q})$-*transformations.*

Proof. χ' is clearly additive and multiplicative. Isomorphism invariance can be checked as follows: First, we make the following claim.

Claim. If $X \neq \emptyset$ is defined by a finite number of weak ($\leq$) affine equalities and inequalities, then $\chi'(X) = 1$.

Proof. It suffices to show that this is true in $(\mathbb{R}, +)$; since then it is true in any model of the theory of divisible ordered Abelian groups. Now we may compute the Euler characteristic χ of the bounded sets $X \cap C_r$ in $(\mathbb{R}, +, \cdot)$. Let $p \in X$. For large enough r, $p \in X \cap C_r$ there is a definable retraction of the closed bounded set $X \cap C_r$ to p (along lines through p). Thus $X \cap C_r$ has the same homology groups as a point, and so Euler characteristic 1. □

To prove the lemma we must show that if $\phi : X \to Y$ is a definable bijection, $X, Y \subseteq \Gamma^n$, then $\chi'(X) = \chi'(Y)$. We use induction on $\dim(X)$. By additivity, if X is a Boolean combination of finitely many pieces, it suffices to prove the lemma for each piece. We may therefore assume that ϕ is linear (rather than only piecewise linear) on X. Let ϕ' be a linear automorphism extending ϕ. Expressing X as a union of basic pieces, we may assume X is defined by some inequalities $\sum \alpha_i x_i \leq c$, as well as some equalities and strict inequalities. Thus X is convex. We have to show that $\chi'(X) = \chi'(\phi' X)$. Let $\bar{X}$ be the closure of X (defined by the corresponding weak inequalities). Then $\bar{X} \setminus X$ has dimension $< \dim(X)$, so by induction $\chi'(\phi'(\bar{X} \setminus X) = \chi'(\bar{X} \setminus X)$. But $\bar{X}$ is closed and convex, so $\chi_{O'}(\bar{X}) = 1 = \chi_{O'}(\phi' \bar{X})$. Subtracting, $\chi'(\phi'(\bar{X})) = \chi'(\bar{X})$.

Once again, using the ambient dimension grading, we can define $\chi'_O : \Gamma[*] \to \mathbb{Z}[\boldsymbol{\tau}]$ with $\chi'_O(x) = \chi'(x)\boldsymbol{\tau}^n$ for $x \in \Gamma[n]$. □

In the following lemma, all classes are taken in $K(\Gamma_A)[*]$. Let e_a be the class in $K(\Gamma_A)[1]$ of the singleton $\{a\}$, and τ_a the class of the segment $(0, a)$.

Lemma 9.7. *Let $a \in \mathbb{Q} \otimes A$, $b \in A$.*

(1) $\tau_a = \tau_{a+b}$, $e_a = e_{a+b}$.
(2) *If* $b < c \in A$ *then* $[(b, c)] = -e_0$.
(3) $e_a e_0 = e_a^2$.
(4) $\tau_a(\tau_a + e_0) = 0$.
(5) *If* $2a \in A$ *then* $2\tau_a + e_a = -e_0$, *and* $e_0(e_a - e_0) = 0$.

Proof.

(1) $\tau_a = [(0, a)] = [(0, \infty)] - [(a, \infty)] - e_a$, and similarly τ_{a+b}. The map $x \mapsto x+b$ shows that $[(a, \infty)] = [(a + b, \infty)]$ and $e_a = e_{a+b}$, hence also $\tau_a = \tau_{a+b}$.
(2) $[(b, c)] = [(b, \infty)] - [(c, \infty)] - e_c = -e_0$ by (1), since $c - b \in A$.
(3) The map $(x, y) \mapsto (x, y+x)$ is an $SL_2(\mathbb{Z})$-bijection between $\{(a, 0)\}$ and (a, a).
(4) Let

$$D = \{(x, y) : 0 < x < a, 0 < y \leq x\},$$
$$D' = \{(x, y) : 0 < y < a, 0 < x \leq y\},$$
$$D_1 = \{(x, y) : 0 < x < a, y > 0\},$$
$$T(x, y) = (x, y + x).$$

Then $T(D_1) = D_1 \setminus D$. Since $[T(D_1)] = [D_1]$, $[D] = 0$. Similarly, $[D'] = 0$. Note also

$$T((0, a) \times \{0\}) = \{(x, x) : 0 < x < a\}.$$

Thus

$$0 = [D] + [D'] = [(0, a)^2] + [\{(x, x) : 0 < x < a\}] = \tau_a^2 + \tau_a e_0.$$

(5) Let $0 < 2a \in A$. Then $[(0, a)] = [(a, 2a)]$ using the map $x \mapsto 2a - x$. Thus $2\tau_a + e_a = [(0, a) \cup \{a\} \cup (a, 2a)] = [(0, 2a)] = -e_0$ (by (2)). Therefore, $(-e_0 - e_a)(e_0 - e_a) = (2\tau_a)(2\tau_a + 2e_0) = 0$ by (1). Thus $e_a e_0 = e_a^2 = e_0^2$. □

The next lemma will not be used, except as a partial indication towards the question that follows, regarding the difference at the level of Grothendieck groups between $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Q})$ transformations. Let $\mathrm{Ann}(e_0)$ be the annihilator ideal of e_0; it is a graded ideal. Let $R = K(\Gamma_A)[*]/\mathrm{Ann}(e_0)$, the image of $K(\Gamma_A)[*]$ in the localization $K(\Gamma_A)[*](e_0{}^{-1})$. In the next lemma, the classes of definable sets are taken in R, viewed as a subring of $K(\Gamma_A)[*](e_0{}^{-1})$. Let $\mathbf{e}_a = e_a/e_0$, $t_a = \tau_a/e_0$.

Lemma 9.8. *Let $A' = \{a \in \mathbb{Q} \otimes A : \mathbf{e}_a = 1\}$.*

(1) *If $X \subseteq \Gamma^n$ is definable by linear inequalities over A, and $T \in \mathrm{GL}_n(\mathbb{Z}) \ltimes (A')^n$, then $[TX] = [X] \in R$.*
(2) *A' is a subgroup of $\mathbb{Q} \otimes A$.*

(3) $\mathbf{e}_a^2 = \mathbf{e}_a$, $t_a(t_a + 1) = 0$.
(4) *A' is 2-divisible.*

Proof.

(1) It suffices to show this when T is a translation by an element $a \in (A')^n$. The map $(x, y) \mapsto (x + y, y)$ is in $SL_{2n}(\mathbb{Z})$, hence $[X \times \{a\}] = [TX \times \{a\}]$ in $K(\Gamma_A)[2n]$. Since $a \in (A')^n$, $[a] = e_0^n$. Thus $[X]e_0^n = [TX]e_0^n$, and upon dividing by e_0^n the statement follows.
(2) This is clear from (1). For the following clauses, note that by (1)–(2), Lemma 9.7 applies with A replaced by A'.
(3) This follows from Lemma 9.7(3)–(4) divided by e_0^2.
(4) By Lemma 9.7(5) applied to A', if $2a \in A'$ then $e_0(e_a - e_0) = 0$; so $\mathbf{e}_a - 1 = 0$, i.e., $a \in A'$. □

Question 9.9. Is it true that $K(\Gamma_A[*])/\operatorname{Ann}(e_0) = K(\mathrm{DOAG}_A[*])/\operatorname{Ann}(e_0)$?

A positive answer would follow from an extension of (4) to odd primes, over arbitrary A; by an inductive argument, or by integration by parts.

9.3 Bounded sets: Volume homomorphism

Let $\bar{A} = \mathbb{Q} \otimes A$. Recall that $\Gamma^{\mathrm{bdd}}[n]$ is the category of bounded A-definable subsets of Γ^n, with piecewise $\mathrm{GL}_n(\mathbb{Z}) \ltimes \bar{A}$-bijections for morphisms. Let $\mathrm{Sym}(\bar{A})$ be the symmetric algebra on A.

Proposition 9.10. *There exists a natural "volume" ring homomorphism* $K(\Gamma^{\mathrm{bdd}}[*]) \to \mathrm{Sym}(\bar{A})$.

Proof. We first work with DOAG without parameters, defining a polynomial associated with a family of definable sets.

Let $C(x, u) = C(x_1, \ldots, x_n; u_1, \ldots, u_m)$ be a formula of DOAG. Write $C_b = \{x : C(x, b)\}$; this is a definable family of definable sets. Assume the sets C_b are uniformly bounded: equivalently, as one easily sees, for some $q \in \mathbb{N}$, for each i, $C(x, u)$ implies $|x_i| \leq q \sum_j |u_j|$. For $b \in \mathbb{R}^m$, let $v(b) = \mathrm{vol} C_b(\mathbb{R}^n)$. Here vol is the Lebesgue measure.

By a *constructible function into* $\mathbb{Q}$, we mean a $\mathbb{Q}$-linear combination of characteristic functions of definable sets of DOAG. Let R be the $\mathbb{Q}$-algebra of constructible functions into $\mathbb{Q}$.

Claim 1. There exists a polynomial $P_C(u) \in R[u]$ such that for all $b \in \mathbb{R}^m$, $\mathrm{vol} C_b(\mathbb{R}^n) = P_C(b)$.

In other words, the volume of a rational polytope is piecewise polynomial in the parameters, with linear pieces. The proof of the claim is standard, using iterated integration. For each C, fix such a polynomial P_C.

At this point we reintroduce A. Any A-definable bounded subset of Γ^n has the form C_b for some C as above and some $b \in \bar{A}^m$.

Claim 2. If $C_b = C'_{b'}$ then $P_C(b) = P_{C'}(b')$.

Proof. (See also below for a more algebraic proof). Fix the formulas C, C'. Write $b = Ne, b' = N'e$ where $e \in \bar{A}^l$ is a vector of $\mathbb{Q}$-linearly independent elements of $\bar{A}$, and N, N' are rational matrices. Write $P_C = \sum a_\nu(u)u^\nu$ where a_ν is a constructible function into $\mathbb{Q}$; similarly for $P_{C'}$.

Now note that any formula $\psi(x_1, \dots, x_l)$ of DOAG of dimension l has a solution in $\mathbb{R}^l$ whose entries are algebraically independent. Use this to find algebraically independent $\tilde{e} \in \mathbb{R}^l$ such that $C_{N\tilde{e}} = C'_{N'\tilde{e}}$, and $a_\nu(N\tilde{e}) = a_\nu(b), a_\nu(N'\tilde{e}) = a'_\nu(b')$ for each multi-index ν of degree d.

By the definition of P_C we have $P_C(N\tilde{e}) = P_{C'}(N'\tilde{e})$. Thus $\sum a_\nu(b)(N\tilde{e})^\nu = \sum a'_\nu(b')(N'\tilde{e})^\nu$. By algebraic independence, $\sum a_\nu(b)(Nv)^\nu = \sum a'_\nu(b')(N'v)^\nu$ as $\mathbb{Q}$-polynomials. Therefore, $P_C(Ne) = P_{C'}(N'e)$. □

Thus we can define: $v(C_b) = P_C(b)$. Let us show that v defines a ring homomorphism.

Given C, C' one can find C'' such that $C''_{b,b'} = C_b \cup C_{b'}$, and similarly C''' with $C'''_{b,b'} = C_b \cap C_{b'}$. Then $P_C + P_{C'} = P_{C''} + P_{C'''}$. It follows that v is additive. Similarly, v is multiplicative, and translation invariant. Since $|\det(M)| = 1$ for $M \in \mathrm{GL}_n(\mathbb{Z})$, if $\phi^M(x, u) = \phi(Mx, u)$ then $P_{\phi^M} = P_\phi$. □

Van den Dries, Ealy, and Maříková pointed out that Claim 2 can also be reduced to the following statement: if $Q \in R[u]$, B is any 0-definable set of , and Q vanishes on $B(\mathbb{R})$, then Q vanishes on $B(\Gamma)$. They prove it as follows: let $\bar{B}$ be the Zariski closure of B; $\bar{B}$ is clearly a finite union of linear subspaces, and by intersecting B with each of these, we may assume $\bar{B}$ is linear, so it is cut out by homogeneous linear polynomials $Q_1, \dots, Q_m$. Each Q_i vanishes on $B(\mathbb{R})$ and hence on $B(\Gamma)$. Thus Q lies in the (radical) ideal generated by $Q_1, \dots, Q_m$, hence vanishes on $B(\Gamma)$.

The counting homomorphism in the discrete case

Suppose A has a least positive element 1, and assume given a homomorphism $h_p : A \to \mathbb{Z}_p$ for each p. Then A embeds into a $\mathbb{Z}$-group $\widetilde{A}$, i.e., an ordered Abelian group whose theory is the theory $\mathrm{Th}(\mathbb{Z})$ of $(\mathbb{Z}, <, +)$. (We have $\widetilde{A} \cap (\mathbb{Q} \otimes A) = \{a/n \in \mathbb{Q} \otimes A : (\forall p)(n | h_p(a)).\}$) We have a homomorphism $[X] \mapsto [X(\widetilde{A})]$ from $K_+(\Gamma[*])$ to $K_+(\mathrm{Th}(\mathbb{Z})_A)$. On the other hand, the polynomial formula for the number of integral points in a polytope defined by linear equations over $\mathbb{Z}$ yields a homomorphism $K(\mathrm{Th}(\mathbb{Z})^{\mathrm{bdd}}[*]) \to \mathbb{Q}[A]$. By composing we obtain a homomorphism $K(\Gamma^{\mathrm{bdd}}[*]) \to \mathbb{Q}[A]$.

Remark. Using integration by parts, one can see that the homomorphism

$$K(\mathrm{Th}(\mathbb{Z})^{\mathrm{bdd}}[*]) \to \mathbb{Q}[A]$$

above is actually an isomorphism.

9.4 The measured case

We repeat the definition of $\mu\Gamma$ from the introduction, along with two related categories. The category $\mathrm{vol}\,\Gamma$ corresponds to integrable volume forms, i.e., those that can be transformed by a definable change of variable to the standard form on a definable subsets of affine n-space. By Lemma 3.26, the liftability condition in (2) is equivalent to being piecewise in $\mathrm{GL}_n(\mathbb{Z}) \ltimes A^n$, A^n being the group of definable points.

Definition 9.11.

(1) For $c = (c_1, \ldots, c_n) \in \Gamma^n$, let $\sum(c) = \sum_{i=1}^n c_i$.
(2) For $n \geq 0$, let $\mu\Gamma[n]$ be the category whose objects are pairs (X, ω), with $X \in \mathrm{Ob}\,\Gamma[n]$ and $\omega : X \to \Gamma$ a definable map. A morphism $(X, \omega) \to (X', \omega')$ is a definable bijection $f : X \to X'$ liftable to a definable bijection ${\mathrm{val}_{\mathrm{rv}}}^{-1}X \to {\mathrm{val}_{\mathrm{rv}}}^{-1}X'$, such that $\sum(x) + \omega(x) = \sum(x') + \omega'(x')$ for $x \in X$, $x' = f(x)$.
(4) Let $\mu\Gamma^{\mathrm{bdd}}[n]$ be the full subcategory of $\mu\Gamma[n]$ with objects $X \subseteq [\gamma, \infty)^n$ for some $\gamma \in \Gamma$.
(3) Let $\mathrm{Ob}\,\mathrm{vol}\,\Gamma[n]$ be the set of finite disjoint unions of definable subsets of Γ^n. Given $X, Y \in \mathrm{Ob}\,\mathrm{vol}\,\Gamma[n]$, $f \in \mathrm{Mor}_{\mathrm{vol}\,\Gamma[n]}(X, Y)$ iff $f \in \mathrm{Mor}_{\Gamma[n]}$ and $\sum(x) = \sum(f(x))$ for $x \in X$.
(5) $\mu\Gamma[*]$ is the direct sum of the $\mu\Gamma[n]$, and similarly for the related categories.

Recall the Grothendieck rings of functions from Section 2.2. $\mathrm{Fn}(\Gamma, K_+(\Gamma))$ is a semigroup with pointwise addition. We also have a convolution product: if f is represented by a definable $F \subseteq \Gamma \times \Gamma^m$, in the sense that $f(\gamma) = [F(\gamma)]$, and g by a definable $G \subseteq \Gamma \times \Gamma^n$, let

$$f * g(\gamma) = [\{(\alpha, b, c) : \alpha \in \Gamma, b \in F(\alpha), c \in G(\gamma - \alpha)\}].$$

The coordinate α in the definition is needed in order to make the union disjoint. In general, it yields an element represented by a subset of $\Gamma \times \Gamma^{m+n+1}$ rather than $m + n$. But let $\mathrm{Fn}(\Gamma, K_+(\Gamma))[n]$ be the set of $[F] \in \mathrm{Fn}(\Gamma, K_+(\Gamma[n]))$ such that $\dim(F(a)) < n$ for all but finitely many $a \in \Gamma$. If $f \in \mathrm{Fn}(\Gamma, K_+(\Gamma))[m]$ and $g \in \mathrm{Fn}(\Gamma, K_+(\Gamma))[n]$, then $f * g \in \mathrm{Fn}(\Gamma, K_+(\Gamma))[m+n]$. Let $\mathrm{Fn}(\Gamma, K_+(\Gamma))[*] = \oplus_m \mathrm{Fn}(\Gamma, K_+(\Gamma))[m]$, a graded semiring.

Lemma 9.12.

(1) $K_+(\mu\Gamma)[n] \simeq \mathrm{Fn}(\Gamma, K_+(\Gamma))[n]$.
(2) $K_+\,\mu\Gamma^{\mathrm{bdd}}[n] \simeq \{f \in \mathrm{Fn}(\Gamma, K_+(\Gamma^{\mathrm{bdd}}))[n] : (\exists\gamma_0)(\forall\gamma < \gamma_0)(f(\gamma) = 0)\}$.
(3) $K_+\,\mathrm{vol}\,\Gamma[n] \simeq \mathrm{Fn}(\Gamma, K_+(\Gamma[n-1]))$.

Proof.

(1) Let $(X, \omega) \in \mathrm{Ob}\,\mu\Gamma[n]$, with $X \subseteq \Gamma^n$ and $\omega : X \to \Gamma$. Let $d(x) = \omega(x) + \sum(x)$. For $a \in \Gamma$, let $X_a = \{x \in X : d(x) = a\}$. This determines an element $F(X, \omega) \in \mathrm{Fn}(\Gamma, K_+(\Gamma[n]))$, namely, $a \mapsto [X_a]$. It is clear from additivity of dimension that $\dim(X_a) < n$ for all but finitely many a; so

$F(X,\omega) \in \mathrm{Fn}(\Gamma, K_+(\Gamma))[n]$. If $h \in \mathrm{Mor}_{\mu\Gamma[n]}(X,Y)$, then by the definition of $\mu\Gamma$ we have $h(X_a) = Y_a$; so $[X_a] = [Y_a]$ in $K_+(\Gamma)[n]$. Conversely if for all $a \in \Gamma$ we have $[X_a] = [Y_a]$ in $K_+(\Gamma)[n]$, then $\mathrm{val_{rv}}^{-1}(X_a)$, $\mathrm{val_{rv}}^{-1}(Y_a)$ are a-definably isomorphic. By Lemma 2.3 there exists a definable $H : \mathrm{val_{rv}}^{-1}(X) \to \mathrm{val_{rv}}^{-1}(Y)$ such that for any $x \in \mathrm{val_{rv}}^{-1}(X)$, $H(x) = h_a(x)$, where $a = \sum \mathrm{val_{rv}}(x)$. Clearly, H descends to $\bar{H} : X \to Y$; by construction $\bar{H}$ lifts to RV, and preserves $\sum + \omega$, so $\bar{H} \in \mathrm{Mor}_{\mu\Gamma[n]}(X,Y)$. We have thus shown that $[X] \mapsto [F(X)]$ is injective. It is clearly a semiring homomorphism.

For surjectivity, let $g \in \mathrm{Fn}(\Gamma, K_+(\Gamma))[n]$ be represented by $G \subseteq \Gamma \times \Gamma^n$. It suffices to consider either g with singleton support $\{\gamma_0\}$, or g such that $\dim(G(a)) < n$ for all $a \in \Gamma$. In the first case, $g = F(X,\omega)$ where $X = G(\gamma_0)$ and $\omega(x) = \gamma_0 - \sum(x)$. In the second: after effecting a partition and a permuation of the variables, we may assume $G(a) \subseteq \Gamma^{n-1} \times \{\psi(a)\}$ for some definable function $\psi(a)$. With another partition of Γ, we may assume g is supported on $S \subseteq \Gamma$, i.e., $g(x) = 0$ for $x \notin S$, and ψ is either injective or constant on S. In fact, we may assume ψ is injective on S: if ψ is constant on S, let $G' = \{(a,(b_1,\dots,b_{n-1},b_n + a)) : (a,(b_1,\dots,b_n)) \in G, a \in S\}$. Then G' also represents g, and for G' the function ψ is injective. Now let $X = \cup_{a \in S} G(a)$, and let $\omega(x) = -\sum(x) + \psi^{-1}(x_n)$. Then $F(X,\omega) = g$.

(2) This follows from (1) by restricting the isomorphism.

(3) This is proved in a similar manner to (1) though more simply and we omit the details. The key point is that $\mathrm{GL}_n(\mathbb{Z})$ acts transitively on $\mathbb{P}^n(\mathbb{Q})$; this can be seen as a consequence of the fact that finitely generated torsion free Abelian groups are free. More specifically, the covector $(1,\dots,1)$ is $\mathrm{GL}_n(\mathbb{Z})$-conjugate to $(1,0,\dots,0)$. Thus the catgegory $\mathrm{vol}\,\Gamma[n]$ is equivalent to the one defined using the weighting x_1 in place of $\sum(x_i)$. For this category the assertion is clear. □

This lemma reduces the study of $K_+(\mu\Gamma)$ to that of $K_+(\Gamma)$.

10 The Grothendieck semirings of RV

10.1 Decomposition to Γ, RES

Recall that RV is a structure with an exact sequence

$$0 \to \mathbf{k}^* \to RV \underset{\mathrm{val_{rv}}}{\to} \Gamma \to 0.$$

We study here the Grothendieck semiring of RV in a theory $\mathbf{T}_{\mathrm{RV}}$ satisfying the assumptions of Lemma 3.26. The intended case is the structure induced from ACVF_A for some RV, Γ-generated base structure A.

We show that the Grothendieck ring of RV decomposes into a tensor product of those of RES, and of Γ.

The category $\Gamma[*]$ was described in Section 9. We used $\mathrm{GL}_n(\mathbb{Z})$ rather than $\mathrm{GL}_n(\mathbb{Q})$ morphisms. The reason is given by the following.

Lemma 10.1. *The morphisms of* $\Gamma[n]$ *are precisely those definable maps that lift to morphisms of* $\mathrm{RV}[n]$. *The map* $X \mapsto \mathrm{val_{rv}}^{-1}(X)$ *therefore induces a functor* $\Gamma[n] \to \mathrm{RV}[n]$, *yielding an embedding of Grothendieck semirings* $K_+[\Gamma[n]] \to K_+[\mathrm{RV}[n]]$.

Proof. Any morphism of $\Gamma[*]$ obviously lifts to RV, since $\mathrm{GL}_n(\mathbb{Z})$ acts on C^n for any group C. The converse is a consequence of Lemma 3.28. □

We also have an inclusion morphism $K_+(\mathrm{RES}) \to K_+(\mathrm{RV})$.

Observe that $K_+(\Gamma^{\mathrm{fin}})$ forms a part of both $K_+(\mathrm{RES}[*])$ and $K_+(\Gamma[*])$: the embedding of $K_+(\Gamma[*])$ into $K_+(\mathrm{RV}[*])$ takes $K_+(\Gamma^{\mathrm{fin}})$ to a subring of $K_+(\mathrm{RES}[*])$, namely, the subring generated by the pullbacks $\mathrm{val_{rv}}(\gamma)$, $\gamma \in \Gamma$ a definable point.

Given two semirings R_1, R_2 and a homomorphism $f_i : S \to R_i$, define $R_1 \otimes_S R_2$ by the universal property for triples (R, h_1, h_2), with R a semiring and $h_i : R_i \to R$ a semiring homomorphism, satisfying $h_1 f_1 = h_2 f_2$.

We have a natural map $K_+(\mathrm{RES}) \otimes K_+(\Gamma[*]) \to K_+(\mathrm{RV})$, $[X] \otimes [Y] \mapsto [X \times \mathrm{val_{rv}}^{-1}(Y)]$. By the universal property it induces a map on $K_+(\mathrm{RES}) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*])$. A typical element of the image is represented by a definable set of the form $\dot\bigcup(X_i \times \mathrm{val_{rv}}^{-1}(Y_i))$, with $X_i \subseteq \mathrm{RES}^*$, $Y_i \subseteq \Gamma^*$.

Proposition 10.2. *The natural map* $K_+(\mathrm{RES}) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*]) \to K_+(\mathrm{RV})$ *is an isomorphism.*

Proof. Surjectivity is Corollary 3.25. We will prove injectivity. In this proof, $X \otimes Y$ will always denote an element of $K_+(\mathrm{RES}) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*])$.

Claim 1. Any element of $K_+(\Gamma[*])$ can be expressed as $\sum_{j=1}^{l} [Y_j] \times \{p_j\}$, for some $Y_j \subseteq \Gamma^{m_j}$, $\dim(Y_j) = m_j$, and $p_j \in \Gamma^{l_j}$.

Proof. Let $Y \subseteq \Gamma^m$ be definable. If $\dim(Y) < m$, then Y can be partitioned into finitely many sets Y_j, each of which lies in some definable affine hypersurface $\sum_{i=1}^{m} \alpha_i x_i = c$, with $\alpha_i \in \mathbb{Q}$, not all 0. In other words $x \mapsto \alpha \cdot x$ is constant on Y_j, where $\alpha = (\alpha_1, \ldots, \alpha_m)$. We may assume that each $\alpha_i \in \mathbb{Z}$ and that they are relatively prime. Then (α) is the first row of a matrix $M \in \mathrm{GL}_m(\mathbb{Z})$. The map $x \mapsto Mx$ takes Y_j to a set of the form $Y'_j \times \{c\}$, $Y'_j \subseteq \Gamma^{m-1}$. Since $[MY_j] = [Y_j]$ in $K_+(\Gamma[*])$, the claim follows by induction. □

Claim 2. Any element of $K_+(\mathrm{RES}) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*])$ can be represented as

$$\sum_{i=1}^{k} X_i \otimes \mathrm{val_{rv}}^{-1} Y_i,$$

where $X_i \subseteq \mathrm{RES}^{n_i}$ and $Y_i \subseteq \Gamma^{m_i}$ are definable sets, and $m_i = \dim Y_i$.

Proof. By the definition of $K_+(\mathrm{RES}) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*])$ and by Claim 1, any element is a sum of tensors $X \otimes \mathrm{val_{rv}}^{-1}(Y \times \{p\})$; using the $\otimes_{K_+(\Gamma^{\mathrm{fin}})}$-relation, $X \otimes \mathrm{val_{rv}}^{-1}(Y \times \{p\}) = (X \times \mathrm{val_{rv}}^{-1}(p)) \otimes Y$. □

Now let X_i, $X'_i \subseteq \mathrm{RES}^*$, Y_i, $Y'_i \subseteq \Gamma^*$ be definable sets, and let

$$F : \dot{\cup}(X_i \times \mathrm{val_{rv}}^{-1}(Y_i)) \to \dot{\cup}(X'_{i'} \times \mathrm{val_{rv}}^{-1}(Y'_{i'}))$$

be a definable isomorphism. Let m be the maximal dimension m of any Y_i or $Y'_{i'}$. Assume the following (by Claim 2):

$$\text{For each } i',\ Y'_{i'} \subseteq \Gamma^{\dim(Y'_{i'})} \text{ and similarly for the } Y_i. \qquad (*)$$

Claim 3. Let P be a complete type of Y_i of dimension m, and Q a complete type of X_i. Then $F(Q \times \mathrm{val_{rv}}^{-1} P) = Q' \times \mathrm{val_{rv}}^{-1} P'$, where Q' is a complete type of some $X'_{i'}$, and P' a complete type type of $Y'_{i'}$.

Moreover, there exist definable sets $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{P}'$, $\widetilde{Q}'$ containing P, Q, P', Q', respectively, such that

(1) F restricts to a bijection $\widetilde{Q} \times \mathrm{val_{rv}}^{-1} \widetilde{P} \to \widetilde{Q}' \times \mathrm{val_{rv}}^{-1} \widetilde{P}'$;
(2) there exist definable bijections $f : \widetilde{P} \to \widetilde{P}'$ and $g : \widetilde{Q} \to \widetilde{Q}'$;
(3) For any $x \in \widetilde{Q}$, $y \in \widetilde{P}$, F restricts to a bijection $\{x\} \times \mathrm{val_{rv}}^{-1}(y) \to \{f(x)\} \times \mathrm{val_{rv}}^{-1}(g(y))$.

Proof. By Lemma 3.17, $\mathrm{val_{rv}}^{-1}(P)$ is a complete type; by the same lemma, $Q \times \mathrm{val_{rv}}^{-1}(P)$ is complete; hence so is $F(Q \times \mathrm{val_{rv}}^{-1}(P))$. We have $F(Q \times \mathrm{val_{rv}}^{-1}(P)) \subseteq (X'_{i'} \times \mathrm{val_{rv}}^{-1}(Y'_{i'}))$ for some i'. Let $Q' = \mathrm{pr}_1(F(Q \times \mathrm{val_{rv}}^{-1}(P)))$, $V' = \mathrm{pr}_2(F(Q \times \mathrm{val_{rv}}^{-1}(P)))$, $P' = \mathrm{val_{rv}}(V') \subseteq Y'_{i'}$. where $\mathrm{pr}_1 : X'_{i'} \times \mathrm{val_{rv}}^{-1}(Y'_{i'}) \to X_i \subseteq \mathrm{RES}$, $\mathrm{pr}_2 : X'_{i'} \times \mathrm{val_{rv}}^{-1}(Y'_{i'}) \to \mathrm{val_{rv}}^{-1}(Y'_{i'})$ are the projections. Then Q', V', P' are complete types. We have $m = \dim(P') \geq \dim(Y'_{i'})$, so by maximality of m, equality holds. We thus have $P' \subseteq \Gamma^{\dim(P')}$. By Lemma 3.17, $Q' \times \mathrm{val_{rv}}^{-1}(P')$ is also complete type. Thus $F(Q \times P) = Q' \times \mathrm{val_{rv}}^{-1} P'$.

By one more use of Lemma 3.17, the function $f_y : x \mapsto \mathrm{pr}_1 F(x, y)$, whose graph is a subset of the stable set $Q \times Q'$, cannot depend on $y \in P$. Thus $f_y = f$, i.e., $F(x, y) = (f(x), \mathrm{pr}_2 F(x, y))$.

Since $Q \times \mathrm{val_{rv}}^{-1}(y)$ is stable, $\mathrm{val_{rv}}\, \mathrm{pr}_2 F$ must be constant on it; so $\mathrm{val_{rv}}\, \mathrm{pr}_2 F(x, y) = g(y)$ on $P \times Q$. This shows that (3) of the "moreover" holds on $P \times Q$. By compactness, it holds on some definable $\widetilde{Q} \times \widetilde{P}$ (and we may take f injective on $\widetilde{Q}$, and g on $\widetilde{P}$). Let $\widetilde{Q}' = f(\widetilde{Q})$, $\widetilde{P}' = g(\widetilde{P})$. Then (1)–(2) hold also. □

Claim 4. Assume $(*)$ holds. Then there exist finitely many definable Y_i^j ($j = 0, \ldots, N_i$) and X_i^j such that $\dim(Y_i^0) < m$, and the conclusion of Claim 3 holds on each $X_i^j \times \mathrm{val_{rv}}^{-1} Y_i^j$ for $j \geq 1$. Moreover, we may take the Y_i^j, X_i^j pairwise disjoint.

Proof. This follows from Claim 3 by compactness; the disjointness can be achieved by noting that if Claim 3(3) holds for $\widetilde{P}$, $\widetilde{Q}$, then it holds for their definable subsets, too. □

We now show that if $\dot{\cup}(X_i \times \mathrm{val_{rv}}^{-1}(Y_i))$ and $\dot{\cup}(X'_{i'} \times \mathrm{val_{rv}}^{-1}(Y'_{i'}))$ are definably isomorphic then $\sum_{i'}[X_{i'}] \otimes [Y_{i'}] = \sum_i [X_i \otimes Y_i]$. We use induction on the maximal dimension m of any Y_i or $Y'_{i'}$, and also on the number of indices i such that $\dim(Y_i) = m$. Say $\dim(Y_1) = m$.

By Claim 2, without changing $\sum_{i'} X'_{i'} \otimes \mathrm{val_{rv}}^{-1}(Y'_{i'}))$ as an element of

$$K_+(\mathrm{RES}) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*]),$$

we can arrange that $\dim(Y_{i'}) = m_{i'}$, i.e., $(*)$ holds. Thus Claims 3 and 4 apply.

The Y_1^j for $j \geq 1$ may be removed from Y_1, if their images are correspondingly excised from the appropriate Y'_j, since $[\widetilde{Q}] \otimes_{K_+(\Gamma^{\mathrm{fin}})} [\widetilde{P}] = [f(\widetilde{Q})] \otimes_{K_+(\Gamma^{\mathrm{fin}})} [g(\widetilde{P})]$. What is left in Y_1 has Γ dimension $< m$, and so by induction the classes are equal.

The injectivity and the proposition follow. □

For applications to VF, we need a version of Proposition 10.2 keeping track of dimensions. Below, the tensor product is in the category of graded semirings.

Corollary 10.3. *The natural map* $K_+(\mathrm{RES}[*]) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*]) \to K_+(\mathrm{RV}[*])$ *is an isomorphism.*

Proof. For each n we have a surjective homomorphism

$$\oplus_{k=1}^n K_+(\mathrm{RES}[k]) \otimes K_+(\Gamma[n-k]) \to K_+(\mathrm{RV}[n]).$$

$K_+\,\mathrm{RV}[n]$ can be identified with a subset of the semiring $K_+\,\mathrm{RV}$, namely, $\{[X] : \dim(X) \leq n\}$. The proof of Proposition 10.2 shows that the kernel is generated by relations of the form

$$(X \times \mathrm{val_{rv}}^{-1}(Y)) \otimes Z = X \otimes (Y \otimes Z)$$

when $Y \in K_+(\Gamma^{\mathrm{fin}})$ and $\dim(X) + \dim(\mathrm{val_{rv}}^{-1}(Y)) + \dim(\mathrm{val_{rv}}^{-1}(Z)) = n$. These relations are taken into account in the ring $K_+(\mathrm{RES}[*]) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*])$, so that the natural map $K_+(\mathrm{RES}[*]) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K_+(\Gamma[*]) \to K_+(\mathrm{RV}[*])$ is injective and hence an isomorphism. □

Recall the classes $e_a = [\{a\}]_1$ in $K(\Gamma[1])$, defined for $a \in \Gamma(\langle\emptyset\rangle)$. They are in $K_+(\Gamma^{\mathrm{fin}})$, hence identified with classes in $K(\mathrm{RES}[1])$, namely, $e_a = [\mathrm{val_{rv}}^{-1}(a)]$. When denoting classes of varieties V over the residue field, we will write $[V]$ for $[V(\mathbf{k})]$, when no confusion can arise.

Definition 10.4. Let $I!$ be the ideal of $K(\mathrm{RES}[*])$ generated by all differences $e_a - e_0$, where $a \in \Gamma(\langle\emptyset\rangle)$. Let $!\mathrm{K}(\mathrm{RES}[*]) = K(\mathrm{RES}[*])/I!$.

By Lemma 9.7(3), the natural homomorphism $K(\mathrm{RES}[*])$ into the localization of $K(\mathrm{RV}[*])$ by all classes e_a factors through $!\mathrm{K}(\mathrm{RES}[*])$.

Since $I!$ is a homogeneous ideal, $!\mathrm{K}(\mathrm{RES}[*])$ is a graded ring.

The theorem that follows, when combined with the canonical isomorphisms $K(\mathrm{VF}[n]) \to K(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}$ and $K(\mathrm{VF}) \to K(\mathrm{RV}[*])/\mathrm{I_{sp}}$,

$$\int\!\!\!\!\!\!\mathbf{R} : K(\mathrm{VF}) \to !\mathrm{K}(\mathrm{RES})[[\mathbb{A}_1(\mathbf{k})]^{-1}],$$

$$\int\!\!\!\!\!\!\mathbf{R}' : K(\mathrm{VF}) \to !\mathrm{K}(\mathrm{RES}).$$

Theorem 10.5.

(1) *There exists a group homomorphism*

$$\mathcal{E}_n : K(\mathrm{RV}[\leq n])/\mathrm{I}_{\mathrm{sp}} \to !\mathrm{K}(\mathrm{RES}[n])$$

with

$$[\mathrm{RV}^{>0}]_1 \mapsto -[\mathbb{A}^{n-1} \times G_m]_n$$

and

$$[X]_k \mapsto [X \times \mathbb{A}^{n-k}]_n$$

for $X \in \mathrm{RES}[k]$.

(2) *There exists a ring homomorphism* $\mathcal{E} : K(\mathrm{RV}[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathrm{K}(\mathrm{RES})[[\mathbb{A}_1]^{-1}]$ *with* $\mathcal{E}([X]_k) = [X]_k/\mathbb{A}^k$ *for* $X \in \mathrm{RES}[k]$.

(3) *There exists a group homomorphism*

$$\mathcal{E}'_n : K(\mathrm{RV}[\leq n])/\mathrm{I}_{\mathrm{sp}} \to !\mathrm{K}(\mathrm{RES}[n])$$

with $[\mathrm{RV}^{>0}]_1 \mapsto 0$ *and* $[X]_k \mapsto [X]_n$ *for* $X \in \mathrm{RES}[k]$.

(4) *There exists a ring homomorphism* $\mathcal{E}' : K(\mathrm{RV}[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathrm{K}(\mathrm{RES})$ *with* $\mathcal{E}([X]_k) = [X]_k$ *for* $X \in \mathrm{RES}[k]$.

Proof.

(1) We first define a homomorphism $\chi[m] : K(\mathrm{RV}[m]) \to !\mathrm{K}(\mathrm{RES}[m])$. By Corollary 10.3,

$$K(\mathrm{RV}[m]) = \oplus_{l=1}^{m} K(\mathrm{RES}[m-l]) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K(\Gamma[l]).$$

Let $\chi_0 = \mathrm{Id}_{K\,\mathrm{RES}[m]}$. For $l \geq 1$, recall the homomorphism $\chi : K(\Gamma[l]) \to \mathbb{Z}$ of Lemma 9.5. It induces $\chi_l : K(\mathrm{RES}[k]) \otimes_{K_+(\Gamma^{\mathrm{fin}})} K(\Gamma[l]) \to !\mathrm{K}(\mathrm{RES}[k])$ by $a \otimes b \mapsto \chi(b) \cdot [G_m]^l \cdot a$.

Define a group homomorphism

$$\chi[m] : K(\mathrm{RV}[m]) \to K(\mathrm{RES}[m]), \quad \chi[m] = \oplus_l \chi_l.$$

We have

$$\chi[m_1 + m_2](ab) = \chi[m_1](a)\chi[m_2](b)$$

when $a \in K(\mathrm{RV}[m_1])$, $b \in K(\mathrm{RV}[m_2])$. This can be checked on homogeneous elements with respect to the grading $\oplus_l K_+(\mathrm{RES}[m-l]) \otimes K_+(\Gamma[l])$.

We compute $\chi[1]([\mathrm{RV}^{>0}]_1) = \chi_1(1 \otimes [\Gamma^{>0}]_1) = -[G_m] \in K(\mathrm{RES}[1])$.

Next, define a group homomorphism $\beta_m : !\mathrm{K}(\mathrm{RES}[m]) \to !\mathrm{K}(\mathrm{RES}[n])$ by $\beta_m([X]) = [X \times \mathbb{A}^{n-m}]$. Define $\gamma : \oplus_{m \leq n} K(\mathrm{RV}[m]) \to !\mathrm{K}(\mathrm{RES}[n])$ by $\gamma = \sum_m \beta_m \circ \chi[m]$. Then γ is a group homomorphism, and $\gamma(a)\gamma(b) = \gamma(ab) \times [\mathbb{A}^n]$ for $a \in K(\mathrm{RV}[m_1])$, $b \in K(\mathrm{RV}[m_2])$, $m_1 + m_2 \leq n$. Again this is easy to verify on homogeneous elements.

Finally, we compute γ on the standard generator $J = [\mathrm{RV}^{>0}]_1 + [1]_0 - [1]_1$ of I_{sp}. Since $\chi[1]([\mathrm{RV}^{>0}]_1) = -[G_m]$, we have

$$\gamma([\mathrm{RV}^{>0}]_1) = \beta_1(-[G_m]) = -[G_m \times \mathbb{A}^{n-1}]_1$$

On the other hand,

$$\gamma([1]_0) = \beta_0([1]_0) = [\mathbb{A}^n]_n,$$
$$\gamma([1]_1) = \beta_1([1]_1) = [\mathbb{A}^{n-1}]_n.$$

Thus $\gamma(J) = [\mathbb{A}^{n-1}]_{n-1} \times (-[G_m]_1 + [\mathbb{A}^1]_1 - [1]_1) = 0$. A homomorphism $K(\mathrm{RV}[\leq n])/\mathrm{I}_{\mathrm{sp}} \to K(\mathrm{RES}[n])$ is thus induced.

(2) For $a \in K(\mathrm{RV}[m])$, let $\mathcal{E}(a) = \beta_m(a)/[\mathbb{A}^m]$. For any large enough n, we have $\mathcal{E}(a) = \mathcal{E}_n(a)/[\mathbb{A}^n]$. The formulas in (1) prove that $\mathcal{E}$ is a ring homomorphism.

(3)–(4) The proof is similar, using χ' from Lemma 9.6 in place of χ of Lemma 9.5, and the identity in place of β_m. □

Corollary 10.6. *The natural morphism* $K(\mathrm{RES}[n]) \to K(\mathrm{RV}[\leq n])/\mathrm{I}_{\mathrm{sp}}$ *has the kernel contained in* $!I$. □

Lemma 10.7. *Let* $\mathbf{T} = \mathrm{ACVF}_{F((t))}$ *or* $\mathbf{T} = \mathrm{ACVF}^{\mathrm{R}}{}_{F((t))}$, *$F$ a field of characteristic* 0, *with* $\mathrm{val}(F) = (0)$, $\mathrm{val}(F((t))) = \mathbb{Z}$, *and* $\mathrm{val}(t) = 1 \in \mathbb{Z}$. *Then there exists a retraction* $\rho_t : K_+(\mathrm{RES}) \to K_+(\mathrm{Var}_F)$. *It induces a retraction* $!\mathrm{K}(\mathrm{RES}) \to K(\mathrm{Var}_F)$.

Proof. Let $t_n \in F((t))^{\mathrm{alg}}$ be such that $t_1 = t$ and $t_{nm}^n = t_m$. For $\alpha = m/n \in \mathbb{Q}$, with $m \in \mathbb{Z}$, $n \in \mathbb{N}$, let $t_\alpha = t_n^m$. Thus $\alpha \to t_a$ is a homomorphism $\mathbb{Q} \to G_m(F((t))^{\mathrm{alg}})$.

Let $V(\alpha) = {\mathrm{val}_{\mathrm{rv}}}^{-1}(\alpha)$. Let $\mathbf{t}_\alpha = \mathrm{rv}(t_\alpha)$. Then $\mathbf{t}_a \in V(\alpha)$.

Let $X \in \mathrm{RES}[n]$. Then for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$, we have $X \subseteq \Pi_{i=1}^n V(\alpha_i)$, where $V(\alpha_i) = {\mathrm{val}_{\mathrm{rv}}}^{-1}(\alpha_i)$. Define $f(x_1, \ldots, x_n) = (x_1/\mathbf{t}_{\alpha_1}, \ldots, x_n/\mathbf{t}_{\alpha_n})$. Then f is $F((t^{1/m}))$-definable for some m, but not in general. Nevertheless, $F(X) =: Y$ is definable. This is because the Galois group $G = \mathrm{Aut}(F^a((t^{1/m}))/F^a((t)))$ extends to a group of valued field automorphisms $\mathrm{Aut}(\mathbf{k}((t^{1/m}))/\mathbf{k}((t)))$ fixing the entire residue field $\mathbf{k}$; while $Y \subseteq \mathbf{k}$; thus G fixes Y pointwise and hence setwise.

The map $X \mapsto Y$ of definable sets described above clearly respects disjoint unions. It also respects definable bijections: if $h : X \to X'$ is a definable bijection, $Y = f(X)$, $Y' = F(Y')$, then fhf^{-1} is an $F((t^{1/\infty}))$-definable bijection $Y \to Y'$; by the Galois argument above, it is, in fact, definable.

The definable subsets of $\mathbf{k}$ are just the F-constructible sets. Thus we have an induced homomorphism $\rho_t : K_+(\mathrm{RES}) \to K_+(\mathrm{Var}_F)$; it is clearly the identity on $K_+(\mathrm{RES})$. It induces a homomorphism $K(\mathrm{RES}) \to K(\mathrm{Var}_F)$.

Finally $\rho_t({\mathrm{val}_{\mathrm{rv}}}^{-1}(\alpha)) = [G_m]$ for any $\alpha \in \mathbb{Q}$; so a homomorphism on $!\mathrm{K}(\mathrm{RES})$ is induced. □

This example can be generalized as follows. Let L be a valued field with residue field F of characteristic 0, $\mathbf{T} = \mathrm{ACVF}_L$ or $\mathrm{ACVF}^{\mathrm{R}}{}_L$. Let $A = \mathrm{res}(L)$, $\mathbf{A} = \mathbb{Q} \otimes A$, and let $t : \mathbf{A} \to G_m(L^a)$ be a monomorphism, with $t(A) \subseteq G_m(L)$. Then there exists a retraction $\rho_t : K_+(\mathrm{RES}) \to K_+(\mathrm{Var}_F)$.

From Theorem 10.5 and Lemma 10.7, we obtain the example discussed in the introduction.

Proposition 10.8. *Let* $\mathbf{T} = \mathrm{ACVF}^{\mathrm{R}}{}_{F((t))}$, *$F$ a field of characteristic* 0, *with* $\mathrm{val}(F) = (0)$ *and* $\mathrm{val}(t) = 1 \in \mathbb{Z}$. *Then there exists a ring homomorphism* $\mathcal{E}_t : K(\mathrm{VF}) \to K(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}]$, *with* $[\mathcal{M}] \mapsto -[G_m]/[G_a]$, $\mathbb{L}([X]_k) \mapsto [X]_k/[\mathbb{A}^k]$ *for* $X \in \mathrm{Var}_F[k]$. *There is also a ring homomorphism* $\mathcal{E}'_t : K(\mathrm{VF}) \to K(\mathrm{Var}_F)$ *with* $\mathbb{L}([X]_k) \mapsto [X]_k$.

10.2 Decomposition of μRV

An analogous decomposition is valid for the measured Grothendieck semiring $\mu_\Gamma\mathrm{RV}$ (Definition 8.13).

Lemma 10.9. *There exists a homomorphism* $K_+ \mu\Gamma[n] \to K_+ \mu_\Gamma\mathrm{RV}[n]$ *with* $[(X, \omega)] \mapsto [(\mathrm{val_{rv}}^{-1}(X), \mathrm{Id}, \omega \circ \mathrm{val_{rv}})]$.

Proof. We have to show that a $\mu\Gamma[n]$-isomorphism $X \to Y$ lifts to a $\mu_\Gamma\mathrm{RV}[n]$-isomorphism. This follows immediately from the definitions. □

Recall $\mu_\Gamma\mathrm{RES}$ from Definition 8.13. Along the lines of Lemma 9.12, we can also describe $K_+ \mu_\Gamma\mathrm{RES}[n]$ as the semigroup of functions with finite support $\Gamma \to K_+(\mathrm{RES}[n])$. We also have the inclusion $K_+ \mu_\Gamma\mathrm{RES}[n] \to K_+ \mu_\Gamma\mathrm{RV}[n]$, $[(X, f)] \mapsto [(X, f, 1)]$.

Let $\mu\Gamma^{\mathrm{fin}}[n]$ be full subcategory of $\mu\Gamma[n]$ whose objects are finite. We have a homomorphism $K_+(\mu\Gamma^{\mathrm{fin}})[n] \to \mu_\Gamma\mathrm{RES}[n]$, $(X, \omega) \mapsto (\mathrm{val_{rv}}^{-1}(X), \mathrm{Id}, \omega \circ \mathrm{val_{rv}})$. As before, we obtain a homomorphism $K_+ \mu_\Gamma\mathrm{RES}[*] \otimes_{K_+(\mu\Gamma^{\mathrm{fin}})} K_+(\mu\Gamma[*]) \to K_+(\mathrm{RV}[*])$.

Let $\mathrm{RES}_{\Gamma\text{-vol}'}$ be the full subcategory of $\mathrm{RV}_{\Gamma\text{-vol}'}$ whose objects are in RES; this is the same as RV except that morphisms must respect $\sum \mathrm{val_{rv}}$. Let $\mathrm{vol}\,\Gamma^{\mathrm{fin}}$ be the subcategory of finite objects of $\mathrm{vol}\,\Gamma$.

Proposition 10.10.

(1) *The natural map* $K_+(\mu_\Gamma\mathrm{RES}[*]) \otimes_{K_+(\mu\Gamma^{\mathrm{fin}})} K_+(\mu\Gamma[*]) \to K_+(\mu_\Gamma\mathrm{RV}[*])$ *is an isomorphism.*
(2) *So is* $K_+(\mathrm{RES}_{\Gamma\text{-vol}'}[*]) \otimes_{K_+(\mathrm{vol}\,\Gamma^{\mathrm{fin}}[*])} K_+(\mathrm{vol}\,\Gamma[*]) \to K_+(\mathrm{RV}_{\Gamma\text{-vol}'}[*])$.
(3) *The decompositions of this section preserve the subsemirings of bounded sets.*

Proof. We first prove surjectivity in (1). By the surjectivity in Corollary 10.3, it suffices to consider a class $c = [(X \times \mathrm{val_{rv}}^{-1}(Y), f, \omega)]$ with $X \in \mathrm{RES}[k]$, $Y \subseteq \Gamma^l$, $f(x, y) = (f_0(x), y)$, and $\omega : X \times (\mathrm{val}\,r^{-1}(Y)) \to \mathrm{RV}$. In fact, as in Proposition 10.2 we may take $\dim(Y) = l$, and inductively we may assume that any class

$[(X' \times Y', f', \omega')]$ with $\dim(Y') < l$ is in the image. Since we may remove a subset of Y of smaller dimension, applying Lemma 3.17 to $\omega : X \times \mathrm{val_{rv}}^{-1}(Y) \to \Gamma$, we may assume $\omega(x, y) = \omega'(\gamma)$ when $\mathrm{val_{rv}}(y) = \gamma$. Now $c = [(X, f_0, 1)] \otimes [(Y, \omega')]$.

The proof of surjectivity in (2) is similar.

The proof of injectivity in (1)–(2) is the same as of Proposition 10.2 and Corollary 10.3. (3) is clear by inspection of the homomorphisms. □

We now deduce Theorem 1.3. For a finite extension L of $\mathbb{Q}_p$, write $\mathrm{vol}_L(U)$ for $\mathrm{vol}_L(U(L))$. Let r be the ramification degree, i.e., $\mathrm{val}(L^*) = (1/r)\mathbb{Z}$. Let $Q = q^r$. The normalization is such that $\mathcal{M}$ has volume 1; so an open ball of valuative radius γ has volume $q^{r\gamma} = Q^\gamma$. Thus the volume of $\mathrm{val_{rv}}^{-1}(\gamma)$ is $(q-1)Q^\gamma$. Also the norm satisfies $|y| = Q^{\mathrm{val}(y)}$.

Proof of Theorem 1.3. For $a \in \Gamma^k$, let $Z(a) = \{x \in \mathcal{O}_L^n : \mathrm{val}(f_1(x)) = a_1 \ldots \mathrm{val}(f_k(x)) = a_k\}$. Then

$$\int_{\mathcal{O}_L^n} |f|^s = \sum_{a \in (\Gamma^{\geq 0})^k} Q^{s \cdot a} \mathrm{vol}_L(Z(a)).$$

According to Propositions 4.5 and 10.10, we can write

$$Z(a) \sim \dot{\bigcup_{i=1}^{\nu}} \mathbb{L}\mathbf{X}_i \times \mathbb{L}\Delta_i(a),$$

where Δ_i is a definable subset of $\Gamma^{k+n_2(i)}$, $h^i : \Delta_i \to \Gamma^k$ the projection to the first k coordinates, $\Delta_i(a) = \{d \in \Gamma^{n_2(i)} : h^i(d) = a\}$, $\mathbf{X}_i = (X_i, f_i) \in \mathrm{RES}[n_1(i)]$, and $\sim$ denotes equivalence up to an admissible transformation. Thus

$$\mathrm{vol}_L(Z(a)) = \mathrm{vol}_L\left(\dot{\bigcup_{i=1}^{\nu}} \mathbb{L}\mathbf{X}_i \times \mathbb{L}\Delta_i(a)\right) = \sum_{i=1}^{\nu} \mathrm{vol}_L(\mathbb{L}\mathbf{X}_i)\mathrm{vol}_L(\mathbb{L}\Delta_i(a)).$$

If $b = (b_1, \ldots, b_{k+n_2(i)}) \in \Delta_i$, let $h_0^i(b)$ be the sum of the last $n_2(i)$ coordinates.

Since $\mathrm{val_{rv}}$ takes only finitely many values on a definable subset of RES, we may assume $\sum \mathrm{val_{rv}}(f(x)) = \gamma(i)$ is constant on $x \in X_i$. Then $\mathrm{vol}_L(\mathbb{L}X_i(L)) = Q^{\gamma(i)}|X_i(L)|$. Thus

$$\int_{\mathcal{O}_L^n} |f|^s = \sum_i |X_i(L)| Q^{\gamma(i)} \sum_{a \in (\Gamma^{\geq 0})^k} Q^{s \cdot a} \mathrm{vol}_L(\mathbb{L}\Delta_i(a)). \tag{10.1}$$

Now $\mathrm{vol}_L(\mathbb{L}\Delta_i(a)) = \sum_{b \in \Delta_i, h(b) = a} (q-1)^{n_2(i)} Q^{h_0(b)}$. Thus

$$\begin{aligned}\sum_{a \in (\Gamma^{\geq 0})^k} Q^{s \cdot a} \mathrm{vol}_L(\mathbb{L}\Delta_i(a)) &= \sum_{b \in \Delta_i} Q^{s_1 h_1^i(b) + \cdots + s_k h_k^i(b)} (q-1)^{n_2(i)} Q^{h_0(b)} \\ &= (q-1)^{n_2(i)} \mathrm{ev}_{h^i, s, Q}(\Delta_i).\end{aligned} \tag{10.2}$$

The theorem follows from equations (10.1)–(10.2). □

Let A be the set of definable points of Γ. Recall that for $X \subseteq \mathrm{RV}$, $[X]_1$ denotes the class $[(X, \mathrm{Id}_X)] \in \mathrm{RV}[1]$ of X with the identity map to RV, and the constant form 1. For $a \in A$, let $\tilde{e}_a = [(\mathrm{val_{rv}}^{-1}(0), \mathrm{Id}, a)] \in \mathrm{RES}[1]$, $f_a = [\{1\}_{\mathbf{k}}, \mathrm{Id}, a] \in \mathrm{RES}[1]$ where a in the third coordinate is the constant function with value a. If a lifts to a definable point d of RV, multiplication by d shows that $\tilde{e}_a = [\mathrm{val_{rv}}^{-1}(a), \mathrm{Id}, 0]$, $f_a = [\{d\}, \mathrm{Id}, 0]$. Note $\tilde{e}_a\tilde{e}_b = \tilde{e}_{a+b}\tilde{e}_0$; and $\tilde{e}_0 = [G_m]$. Let $\tau_a \in \mathrm{RES}[1]$ be the class of $(\mathrm{val_{rv}}^{-1}((a, \infty)), \mathrm{Id}, 0)$. The generating relation of $\mu_\Gamma \mathrm{I_{sp}}$ is thus (τ_0, f_0) (Lemma 8.20(6)). Let $\mathfrak{h}$ be the class of $[(\mathrm{RV}^{>0}, \mathrm{Id}, x^{-1})]$.

Let $!I^0_\mu$ be the ideal of $K(\mu_\Gamma \mathrm{RES}[*])$ generated by the relations $\tilde{e}_{a+b} = [(\mathrm{val_{rv}}^{-1}(a), \mathrm{Id}, b)]$, where $a, b \in A$, b denoting the constant function b. Let $!I_\mu$ be the ideal generated by $!I^0_\mu$ as well as the element $[\mathbb{A}_1]_1$.

Theorem 10.11. *There exist two graded ring homomorphisms*

$$\oint, \oint' : K^{\mathrm{eff}}(\mu_\Gamma \mathrm{VF}[*]) = K(\mu_\Gamma \mathrm{RV}[*])/\mathrm{I}^\mu_{\mathrm{sp}} \to K(\mu_\Gamma \mathrm{RES}[*])/!I_\mu$$

such that the composition

$$K(\mu_\Gamma \mathrm{RES}[*]) \to K(\mu_\Gamma \mathrm{RV}[*])/\mathrm{I}^\mu_{\mathrm{sp}} \to K(\mu_\Gamma \mathrm{RES}[*])/!I_\mu$$

equals the natural projection

$$\pi : K(\mu_\Gamma \mathrm{RES}[*]) \to K(\mu_\Gamma \mathrm{RES}[*])/!I_\mu,$$

with

$$\oint \mathfrak{h} = -[\{0_{\mathbf{k}}\}]_1, \qquad \oint' \mathfrak{h} = 0.$$

Proof. The identification $K^{\mathrm{eff}}(\mu_\Gamma \mathrm{VF}[*]) = K(\mu_\Gamma \mathrm{RV}[*])/\mathrm{I}^\mu_{\mathrm{sp}}$ is given by Theorem 8.28, and we work with $K(\mu_\Gamma \mathrm{RV}[*])/\mathrm{I}^\mu_{\mathrm{sp}}$.

According to Proposition 10.10, we can identify

$$K(\mu_\Gamma \mathrm{RV}[*]) = K(\mu_\Gamma \mathrm{RES}[*]) \otimes_{K_+(\mu\Gamma^{\mathrm{fin}})} K(\mu\Gamma[*]).$$

We first construct two homomorphisms of graded rings $R, R' : K(\mu_\Gamma \mathrm{RV}[*]) \to K(\mu_\Gamma \mathrm{RES}[*])/!I_\mu$. This amounts to finding graded ring homomorphisms $K(\mu\Gamma[*]) \to K(\mu_\Gamma \mathrm{RES}[*])/!I_\mu$, agreeing with π on the graded ring $K_+(\mu\Gamma^{\mathrm{fin}})$. It will be simpler to work with R, R' together, i.e., construct

$$R'' = (R, R') : K(\mu\Gamma[n]) \to (K(\mu_\Gamma \mathrm{RES}[n])/!I_\mu)^2.$$

Recall from Lemma 9.12 the isomorphism

$$\phi : K(\mu\Gamma[n]) \to \mathrm{Fn}(\Gamma, K(\Gamma))[n].$$

Let $\chi'' : K(\Gamma[n]) \to \mathbb{Z}^2$ be the Euler characteristic of Proposition 9.4; so that $\chi'' = (\chi, \chi')$; cf. Lemmas 9.5 and 9.6. We obtain by composition a map $E''_n =$

$(E_n, E'_n) : \mathrm{Fn}(\Gamma, K(\Gamma[n])) \to \mathrm{Fn}(\Gamma, \mathbb{Z})^2$. Here $\mathrm{Fn}(\Gamma, \mathbb{Z})$ is the group of functions $g : \Gamma \to \mathbb{Z}$ such that $g(\Gamma)$ is finite and $g^{-1}(z)$ is a definable subset of Γ (a finite union of definable intervals and points). Thus $\mathrm{Fn}(\Gamma, \mathbb{Z})$ is freely generated as an Abelian group by $\{p_a, q_a, r\}$, where r is the constant function 1, and for $a \in A$, p_a,q_a are the characteristic functions of $\{a\}$, $\{(a, \infty)\}$, respectively. Define $\psi_n : \mathrm{Fn}(\Gamma, \mathbb{Z}) \to K(\mu_\Gamma \mathrm{RES}[*])$:

$$\psi_m(p_a) = [G_m]^{n-1}\tilde{e}_a = [G_m]^n f_a, \qquad \psi_n(q_a) = -[G_m]^n f_a, \qquad \psi_n(r) = 0.$$

For $u \in K(\mu\Gamma[n])$, let $R''(u) = \psi_n(E''_n(\phi(u)))$.

Claim. $R'' : K(\mu\Gamma[*]) \to K(\mu_\Gamma \mathrm{RES}[m])^2$ is a graded ring homomorphism.

Proof. We have already seen that ϕ is a ring homorphism, so it remains to show this for $\psi_* \circ E''_*$. Now by Proposition 9.4, $\chi''(Y) = \chi''(Y')$ iff $[Y] = [Y']$ in the Grothendieck group of DOAG. Hence given families Y_t, $Y_{t'}$ of pairwise disjoint sets with $\chi''(Y_t) = \chi''(Y'_t)$, by Lemma 2.3 we have $\chi''(\cup_t Y_t) = \chi''(\cup_t Y'_t)$. From this and the definition of multiplication in $\mathrm{Fn}(\Gamma, K(\Gamma))[*]$, and the multiplicativity of E''_n, it follows that if $E''_n(f) = E''_n(f')$ and $E''_m(g) = E''_m(g')$ then $E''_{n+m}(fg) = E''_{n+m}(fg)$. In other words, E''_* is a graded homomorphism from into $(\mathrm{Fn}(\Gamma, \mathbb{Z})^2, \star)$ for some uniquely determined multplication $\star$ on $\mathrm{Fn}(\Gamma, \mathbb{Z})^2$. Clearly, $(a, b) \star (c, d) = (a *_1 c, b *_2 d)$ for two operations $*_1$, $*_2$ on $\mathrm{Fn}(\Gamma, \mathbb{Z})$.

Now we can compute these operations explicitly on the generators:

$$p_a * p_b = p_{a+b}, \qquad p_a * q_b = q_{a+b}, \qquad q_a * q_b = -q_{a+b}$$

for both $*_1$ and $*_2$, and

$$r *_1 \tilde{e}_a = r, \qquad r *_1 q_a = -r, \qquad r *_1 r = r,$$
$$r *_2 \tilde{e}_a = -r, \qquad r *_2 q_a = 0, \qquad r *_2 r = -r.$$

Composing with ψ, we see that R'' is, indeed, a graded ring homomorphism. □

Let R, R' be the components of R''.

Claim. R, R', π agree on $K_+(\mu\Gamma^{\mathrm{fin}})$. $R(\tau_0) = R'(\tau_0) = -\tilde{e}_0$.

This is a direct computation. It follows that R, R' induce homomorphisms $K(\mu_\Gamma \mathrm{RV}[*]) \to K(\mu_\Gamma \mathrm{RES}[*])$. Since $\tilde{e}_0 + f_0 = [(\mathbb{A}_1, \mathrm{Id}, 0)]$, modulo $!I_\mu$ both R, R' equalize $\mu_\Gamma \mathrm{I}_{\mathrm{sp}}$, and hence induce homomorphisms on $K(\mu_\Gamma \mathrm{RV}[*])/\mu_\Gamma \mathrm{I}_{\mathrm{sp}} \to K(\mu_\Gamma \mathrm{RES}[*])/!I_\mu$. □

Remark. The construction is heavily, perhaps completely constrained. The value of $\psi_m(p_a)$ is determined by the tensor relation over $K_+(\mu\Gamma^{\mathrm{fin}})$. The value of $\psi_m(q_a)$ is determined by the relation I_{sp}. The choice $\psi(r) = 0$ is not forced, but the multiplicative relation shows that either r or $-r$ is idempotent, so one has a product of two rings, with $\psi(r) = 0$ and with $\psi(r) = \pm 1$. In the latter case we obtain the isomorphisms of Theorem 10.5. Thus the only choice involved is to factor the fibers of an element of $\mathrm{Fn}(\Gamma, K(\Gamma))[n]$ through χ'', i.e., through $K(\mathrm{DOAG})$. It is possible that $K(\Gamma[n]) = K(\mathrm{DOAG}[n])$ (cf. Question 9.9). In this case, $\oint$, $\oint'$, $\not{\mathbb{f}}$, $\not{\mathbb{f}}'$ are injective as a quadruple, and determine $K(\mu\mathrm{VF}[*])$ completely, at least when localized by the volume of a unit ball.

11 Integration with an additive character

Let $\Omega = \mathrm{VF}/\mathcal{M}$. Let $\psi : \mathrm{VF} \to \Omega$ be the canonical map.

Motivation. For any p, $\Omega(\mathbb{Q}_p)$ can be identified with the pth power roots of unity via an additive character on $\mathbb{Q}_p$. For other local fields, the universal ψ we use is tantamount to integration with respect to all additive characters of conductor $\mathcal{M}$ at once. Thus Ω is our motivic analogue of the roots of unity, and the natural map $\mathrm{VF} \to \mathrm{VF}/\mathcal{M}$, an analogue of a generic additive character.

Throughout this paper, we have been able to avoid subtractions and work with semigroups, but here it appears to be essential to work with a group or at least a cancellation semigroup. The reason is that we will introduce, as the essential feature of integration with an additive character, an identification of the integral of a function f with $f + g$ if g is $\mathcal{O}$-invariant. This corresponds to the rule that the sum over a subgroup of a nontrivial character vanishes. Now for any $h : \Omega \to K_+(\mu\mathrm{VF})$, it is easy to construct $h' : \Omega \to K_+(\mu\mathrm{VF})$ such that $h + h'$ is $\mathcal{O}$-invariant. Thus if $f + h = f' + h$ for some h, then $f = f + h + h' = f' + h + h' = f'$. Thus cancellation appears to come by itself.

If we allow all definable sets and volume forms, a great deal of collapsing is caused by the cancellation rule. We thus use the classical remedy and work with bounded sets and volume forms. The setting is flexible and can be compatible with stricter notions of boundedness. This is only a partial remedy in the case of higher-dimensional local fields; cf. Example 12.12.

The theory can be carried out for any of the settings we considered. Let $\mathcal{R}$ be one of these groups or rings, with $\mathcal{D}$ the corresponding data. For instance, $\mathcal{D}$ is the set of pairs (X, ϕ) with X a bounded definable subset of $\mathrm{VF}^n \times \mathrm{RV}^*$, and $\phi : X \to \mathrm{RV}$ is a bounded definable function; $\mathcal{R}$ is the corresponding Grothendieck ring. Similarly, we can take Γ-volumes, or pure isomorphism invariants without volume forms. In this last case there is no point restricting to bounded sets. As we saw, two Euler characteristics into the Grothendieck group of varieties over RES do survive.

In each case, we think of $\mathcal{R}$ as a Grothendieck ring of associated RV-data, modulo a canonical ideal.

Everything can be graded by dimension, but for the moment we have no need to keep track of it, so in the volume case we can take the direct sum over all n or fix one n and omit it from the notation.

The corresponding group for the theory $\mathbf{T}_A$ or $\mathbf{T}_{\langle a \rangle}$ will be denoted $\mathcal{R}_A, \mathcal{R}_a$, etc. When V is a definable set, we let $\mathcal{D}_V$, $\mathcal{R}_V$ denote the corresponding objects over V. For instance, in the case of bounded RV-volumes, $\mathcal{D}_V$ is the set of pairs $(X \subseteq V \times W, \phi : X \to \mathrm{RV}^*)$ such that for any $a \in V$, $(X_a, \phi|X_a)$ with X_a bounded.

If $\mathcal{R}$ is our definable analogue of the real numbers (as recipients of values of p-adic integration), the group ring $\mathcal{C} = \mathcal{R}[\Omega]$ will take the role of the complex numbers. We have a canonical group homomorphism $(\mathrm{VF}, +) \to \Omega \subseteq G_m(\mathcal{C})$, corresponding to a generic additive character.

Integration with an additive character can be presented in two ways: in terms of definable functions $f : X \to \Omega$ (Riemann style), where we wish to evaluate expressions such as $\int_X f(x)\phi(x)$; classically f usually has the form $\psi(h(x))$, where

h is a regular function and ψ is the additive character. Or we can treat definable functions $F : \Omega \to \mathcal{R}$ (Lebesgue style), and evaluation $\int_{\omega \in \Omega} F(\omega)$. We will work with the latter. Given this, to reconstruct a Riemann style integral, given $f : X \to \Omega$, and an $\mathcal{R}$-valued volume form ϕ on X, let

$$F(\omega) = \int_{f^{-1}(\omega)} \phi(x).$$

Then we can define

$$\int_X f(x)\phi(x) = \int_{\omega \in \Omega} \omega F(\omega).$$

It thus suffices to define the integral of a definable function on Ω. Such a function can be interpreted as an $\mathcal{M}$-invariant function on VF. We impose one rule (cancellation): the integral of a function that is constant on each $\mathcal{O}$-class equals zero. The integral is a homomorphism on the group of $\mathcal{M}$-invariant functions $\mathrm{VF} \to \mathcal{R}$, vanishing on the $\mathcal{O}$-invariant ones. We give a full description of the quotient group, showing that the universal homomorphism of this type factors through a similar group on the residue field.

Recall the group $\mathrm{Fn}(V, \mathcal{R})$ of Section 2.2. We will not need to refer to the dimension grading explicitly.

If V is a definable group, V acts on on $\mathrm{Fn}(V, \mathcal{R})$ by translation.

Definition 11.1. For a definable subgroup W of V, let $\mathrm{Fn}(V, \mathcal{R})^W$ be the set of W-invariant elements of $\mathrm{Fn}(V, \mathcal{R})$: they are represented by a definable X, such that if $t \in W$ and $a \in V$ then $X[a]$, $X[a + t]$ represent the same class in $K(\mu\mathrm{VF}_{a,t})[n]$.

Lemma 11.2. *An element of* $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}$ *can be represented by an* $\mathcal{M}$*-invariant* $X \subseteq (\mathrm{VF} \times *)$.

Proof. Let $Y \in \mathcal{D}^{RV}_{\mathrm{VF}}$ represent an element of $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}$. Thus each fiber $Y_a \in \mathcal{D}^{RV}$. By Lemma 3.52, for $\mathbf{a} \in \mathrm{VF}/\mathcal{M}$ one can find $Y'_{\mathbf{a}} \in \mathcal{D}^{RV}$ such that for some $a \in \mathrm{VF}$ with $a + \mathcal{M} = \mathbf{a}$, $Y_a = Y'_{\mathbf{a}}$. As in Lemma 2.3 there exists $Y' \in \mathcal{R}_{\mathrm{VF}/\mathcal{M}}$ such that $Y'_{\mathbf{a}}$ is the fiber of Y' over $\mathbf{a}$. Pulling back to VF gives the required $\mathcal{M}$-invariant representative. □

Since the equivalence is defined in terms of effective isomorphism, Definition 8.2, it is clear that two elements of $\mathcal{D}_\Omega$ are equivalent iff the corresponding pullbacks to $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}$ are equivalent.

The groups $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}$ and $\mathrm{Fn}(\mathrm{VF}/\mathcal{M}, \mathcal{R})$ can thus be identified.

Note that the effective isomorphism agrees with pointwise isomorphism for $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}$, but not for $\mathrm{Fn}(\mathrm{VF}/\mathcal{M}, \mathcal{R})$.

The group we seek to describe is $\mathcal{A} = \mathcal{A}_{\mathbf{T}} = \mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}} / \mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{O}}$. The quotient corresponds to the cancellation rule discussed earlier.

Let $\mathrm{Fn}(\mathbf{k}, \mathcal{R})$ be the Grothendieck group of functions $\mathbf{k} \to \mathcal{R}$, with addition induced from $\mathcal{R}$.

Let $\mathcal{C} = \mathcal{R}[\Omega]$ be the ring of definable functions $\Omega \to \mathcal{R}$ with finite support, convolution product.

Remark. $\mathcal{C}$ embeds into the Galois-invariant elements of the abstract group ring $\mathcal{R}_{\widetilde{T}}[\Omega_{\widetilde{T}}]$, where $\widetilde{T} = \mathbf{T}_{\mathrm{acl}(\emptyset)}$.

The additive group $\mathbf{k} = \mathcal{O}/\mathcal{M}$ is a subgroup of $\Omega = \mathrm{VF}/\mathcal{M}$, and so acts on Ω by translation. It also acts naturally on $\mathrm{Fn}(\mathbf{k}, \mathcal{R})$. This gives two actions on $\mathrm{Fn}(\mathbf{k}, \mathcal{C}) = \mathrm{Fn}(\mathbf{k}, \mathcal{R})[\Omega]$. Let $\mathrm{Fn}(\mathbf{k}, \mathcal{C})_{\mathbf{k}}$ denote the coinvariants with respect to the anti-diagonal action, i.e., the largest quotient on which the two actions coincide.

In general, the upper index denotes invariants, the lower index coinvariants.

$\mathrm{Fn}(\mathrm{VF}, \mathcal{R})$ is the ring of definable functions from VF to $\mathcal{R}$. $\mathrm{Fn}(\mathbf{k}, \mathcal{R})$ is the ring of definable functions from $\mathbf{k}$ to $\mathcal{R}$. $\mathrm{Fn}(\mathbf{k}, \mathcal{C})$ is the ring of definable functions from $\mathbf{k}$ to $\mathcal{C}$; equivalently, it is the set of Galois-invariant elements of the group ring $\mathrm{Fn}(\mathbf{k}, \mathcal{R})[\Omega]$.

The action of $\mathbf{k}$ on $\mathrm{Fn}(\mathbf{k}, \mathcal{C})$ is by translation on $\mathbf{k}$, and negative translation on Ω and hence on $\mathcal{C}$. The term (Const) refers to the image of the constant functions of $\mathrm{Fn}(\mathbf{k}, \mathcal{C})$ in $\mathrm{Fn}(\mathbf{k}, \mathcal{C})_{\mathbf{k}}$. (It is isomorphic to $(\mathcal{C}/\mathbf{k})$.)

Theorem 11.3. *There exists a canonical isomorphism* $\mathrm{Fn}(\mathbf{k}, \mathcal{C})_{\mathbf{k}}/(\mathrm{Const}) \overset{\rightarrow}{\cong} \mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}/\,\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{O}}$.

Proof. Let $\mathcal{A}_{\mathrm{fin}}$ be the subring of $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{M}}$ consisting of functions represented by elements of $\mathrm{Fn}(\mathrm{VF}, \mathcal{D})^{\mathcal{M}}$ whose support projects to a finite subset of $\mathrm{VF}/\mathcal{O}$.

A definable function on $\mathbf{k}$ can be viewed as an $\mathcal{M}$-invariant function on $\mathcal{O}$; this gives

$$\mathrm{Fn}(\mathbf{k}, \mathcal{R}) \overset{\rightarrow}{\cong} \mathrm{Fn}(\mathcal{O}, \mathcal{R})^{\mathcal{M}}. \tag{11.1}$$

On the other hand, we can define a homomorphism

$$\mathrm{Fn}(\mathcal{O}, \mathcal{R})^{\mathcal{M}}[\Omega] \to \mathcal{A}_{\mathrm{fin}} : \sum_{\omega \in W} a(\omega)\omega \mapsto \sum_{\omega \in W} a(\omega)_{\omega}, \tag{11.2}$$

where W is a finite A-definable subset of Ω, $a : W \to \mathrm{Fn}(\mathcal{O}, \mathcal{R})^{\mathcal{M}}$ is an A-definable function, (so that $\sum_{a \in W} a(\omega)\omega$ is a typical element of the group ring $\mathrm{Fn}(\mathcal{O}, \mathcal{R})^{\mathcal{M}}[\Omega]$), and b_{ω} is the translation of b by ω, i.e., $b_{\omega}(x) = b(x - \omega)$.

(11.2) is surjective: Let $f \in \mathcal{A}_{\mathrm{fin}}$ be represented by F, with support Z, a finite union of translates of $\mathcal{O}$. By Lemma 3.39 there exists a finite definable set W, meeting each ball of Z in a unique point. Define $a : W \to \mathrm{Fn}(\mathcal{O}, \mathcal{R})^{\mathcal{M}}$ by

$$a(\omega) = (f|\omega + \mathcal{O})_{-\omega}.$$

Then (11.2) maps $\sum a(\omega)\omega$ to f.

The kernel of (11.2) is the equalizer of the two actions of $\mathbf{k}$. Composing with (11.1), we obtain an isomorphism $(\mathrm{Fn}(\mathbf{k}, \mathcal{R})[\Omega])_{\mathbf{k}} \overset{\rightarrow}{\cong} \mathcal{A}_{\mathrm{fin}}$ or, equivalently,

$$\mathrm{Fn}(\mathbf{k}, \mathcal{C})_{\mathbf{k}} \overset{\rightarrow}{\cong} \mathcal{A}_{\mathrm{fin}}. \tag{11.3}$$

The last ingredient is the homomorphism

$$\mathcal{A}_{\mathrm{fin}} \to \mathcal{A}. \tag{11.4}$$

We need to show that it is surjective, and to describe the kernel.

Using the representation $\mathcal{D}$ of elements of $\mathcal{R}$ by RV-data, an element of $\mathcal{A}$ is represented by an $\mathcal{M}$-invariant definable $W \subset \mathrm{VF} \times \mathrm{RV}^*$.

By Lemma 3.37, for each coset C of $\mathcal{O}$ in VF apart from a finite number, $W \cap (C \times \mathrm{RV}^{n+l})$ is invariant under translation of the first coordinate by elements of $\mathcal{O}$. Thus W is the disjoint sum of an $\mathcal{O}$-invariant set W' and a set $W'' \subset \mathrm{VF} \times \mathrm{RV}^*$ projecting to a finite union Z of cosets of $\mathcal{O}$ in VF, i.e., representing a function in $\mathcal{A}_{\mathrm{fin}}$.

Clearly, $W' \times_{\mathrm{RV}^n} \mathrm{VF}^n$ lies in $\mathrm{Fn}(\mathrm{VF}, \mathcal{R})^{\mathcal{O}}$.

Thus (11.4) is surjective; the kernel is $\mathcal{A}_{\mathrm{fin}}^{\mathcal{O}}$. Composing (11.3),(11.4) we obtain an isomorphism

$$\mathcal{A} \overset{\rightarrow}{\cong} (\mathrm{Fn}(\mathbf{k}, \mathcal{R})[\Omega])_{\mathbf{k}} / (\mathrm{Const}).$$

Using the identification $\mathrm{Fn}(\mathbf{k}, \mathcal{R})[\Omega] \simeq \mathrm{Fn}(\mathbf{k}, \mathcal{C})$, the theorem follows. □

Note that $\mathrm{Fn}(\mathbf{k}, \mathcal{C})^{\mathbf{k}} \simeq \mathcal{C}$, via $\mathrm{Fn}(\mathbf{k}, \mathcal{C}) \simeq \mathrm{Fn}(\mathbf{k} \times \Omega, \mathcal{R})_{\mathrm{fin}}$.

11.1 Definable distributions

$\mathcal{R}$ is graded by dimension (VF-presentation) or ambient dimension (RV-presentation). Write $\mathcal{R} = \oplus_{n \geq 0} \mathcal{R}[n]$.

Let $\mathcal{R}_{df}$ be the dimension-free version: first form the localization $\mathcal{R}[[0]_1^{-1}]$, where $[0]_1$ is the class of the point $1 \in \mathrm{RV}$, as an element of $\mathrm{RV}[1]$. Equivalently, $[0]_1^n$ is the volume of the open n-dimensional polydisc $\mathcal{O}^n$. Let $\mathcal{R}_{df}$ be the zero-dimensional component of this localization. Similarly, define $\mathcal{C}_{df}$ so that $\mathcal{C}_{df} = \mathcal{R}_{df}[\Omega]$. We can also define $K_+(\mathcal{D})_{df}$, and check that the groupification is $\mathcal{R}_{df}$.

Given $a = (a_1, \ldots, a_n) \in \mathrm{VF}^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, let $B(a, \gamma) = \Pi_{i=1}^n B(a_i, \gamma_i)$, where $B(a_i, \gamma_i) = \{c \in \mathrm{VF} : \mathrm{val}(c - a_i) > \gamma_i\}$. Call $B(a, \gamma)$ an open polydisc of dimension γ. If $\gamma \in \Gamma$, let $B(a, \gamma) = B(a, (\gamma, \ldots, \gamma))$ (the open cube of side γ).

Note that $[B(0, \gamma)]$ is invertible in $\mathcal{R}_{df}$, in each dimension. In particular, in dimension 1, $[B(0, \gamma)][B(0, -\gamma)] = [0]_1^2$. Note also that $[B(a, \gamma)] \underset{a}{=} [B(0, \gamma)]$.

We proceed to define integrals of definable functions.

Let U be a bounded definable subset of VF^n. A definable function $f : U \to K_+(\mathcal{D})_{df}$ has the form $[0]_1^{-m} F$, where $F : U \to K_+ \mathcal{D}[m]$ is a definable function, represented by some $\bar{F} \in \mathcal{D}[m + n]_U$. In case $\bar{F}$ can be taken bounded, define

$$\int_U f = [0]_1^{-m+n} [F]_{n+m}.$$

We say that f is *boundedly represented* in this case.

In particular, $\mathrm{vol}(U) = \int_U 1 = [0]_1^{-m} [U]_m$ is treated as a pure number now, without dimension units. (Check the independence of the choices.)

This extends by linearity to $\int_U f$ for $f : U \to \mathcal{R}_{df}$, provided f can be expressed as the difference of two boundedly represented functions $U \to K_+(\mathcal{D})_{df}$.

We now note that averaging twice, with appropriate weighting, is the same as doing it once. The function γ' in the lemmas below corresponds to a partition of U into cubes; $\gamma'(u)$ is the side of the cube around $u \in U$.

Lemma 11.4. *Let U be a bounded open subset of* VF^n*, f a boundedly represented function on U. Let $\gamma' : U \to \Gamma$ be a definable function such that if $u \in U$ and $u' \in B(u, \gamma'(u))$ then $u' \in U$ and $\gamma'(u') = \gamma'(u)$. Then*

$$\int_U f = \int_U \left[\mathrm{vol}(B(u, \gamma'(u)))^{-1} \int_{B(u,\gamma'(u))} f \right].$$

Proof. Let $f = [0]_1^{-m} F$, where $F : U \to K_+ \mathcal{D}[m]$ is bounded. We have $\mathrm{vol}(B(u, \gamma')) = [0]_1^{-n} [\gamma'(u)]^n$ so

$$\mathrm{vol}(B(u, \gamma'))^{-1} = [0]_1^n [\gamma'(u)]^{-n} = [0]_1^{-n} [-\gamma'(u)]^n.$$

Thus, multiplying by $[0]_1^{3n+m}$, we have to show

$$[0]_1^{2n} [F] = [-\gamma'(u)]^n [\{(u, u', z) : u \in U, u' \in B(u, \gamma'(u)), (u', z) \in F\}].$$

Now $u' \in B(u, \gamma'(u))$ iff $u \in B(u', \gamma'(u'))$. Applying the measure-preserving bijection $(u, u', z) \mapsto (u - u', u', z')$ we see that the $[\{(u, u', z) : u \in U, u' \in B(u, \gamma'(u)), (u', z) \in F\}] = [\gamma]_1^n [\{(u', z) : (u', z) \in F]$, so the equality is clear. □

We now define the integral of definable functions into $\mathcal{C}_{df}$. By definition, such a function is a finite sum of products fg, with $f \in \mathrm{Fn}(U, \mathcal{R}_{df})$ and $g \in \mathrm{Fn}(U, \Omega)$. Define

$$\int_U fg = \int_{\omega \in \Omega} \omega \int_{g^{-1}(\omega)} f$$

and extend by linearity.

Note that this is defined as soon as g is boundedly represented. (Again, check the independence of the choices.)

Definition 11.5. A definable distribution on an open $U \subseteq \mathrm{VF}^n$ is a definable function $\mathfrak{d} : U \times \Gamma \to \mathcal{C}_{df}$, such that $\mathfrak{d}(a, \gamma) = \mathfrak{d}(a', \gamma)$ if $B(a, \gamma) = B(a', \gamma)$, and whenever $\gamma' > \gamma$ in each coordinate,

$$\mathfrak{d}(b, \gamma) = \int_{u \in B(b,\gamma)} \mathrm{vol}(B(0, \gamma'))^{-1} \mathfrak{d}(u, \gamma').$$

As in Lemma 11.2, the invariance condition means that $\mathfrak{d}$ can be viewed as a function on open polydiscs, and we will view it this way below.

If $\mathfrak{d}$ takes values in $\mathcal{R}_{df}$, we say it is $\mathcal{R}_{df}$-valued. By definition, $\mathfrak{d}$ can be written as a finite sum $\sum \omega_i \mathfrak{d}_i$, where $\mathfrak{d}_i$ is an $\mathcal{R}_{df}$-valued function; in fact, $\mathfrak{d}_i$ is an $\mathcal{R}_{df}$-valued distribution.

We wish to strengthen the definition of a distribution so as to apply to subpolydiscs of variable size. For this we need a preliminary lemma.

Lemma 11.6. *Let $U = B(a, \gamma)$ be a polydisc. Let $\gamma' : B(a, \gamma) \to \Gamma$ be a definable function such that $\gamma'(u') = \gamma'(u)$ for $u' \in B(u, \gamma'(u))$. Then γ' is bounded on U.*

Proof. Suppose for contradiction that γ' is not bounded on $B(a, \gamma)$; i.e.,

$$(\forall \delta \in \Gamma)(\exists u \in B(a, \gamma))(\gamma'(u) > \delta).$$

This will not change if we add a generic element of Γ to the base, so we may assume $\Gamma(\mathrm{dcl}(\emptyset)) \neq (0)$. By Lemma 3.51, there exists a resolved structure with the same RV-part as $\langle \emptyset \rangle$; hence we may assume **T** is resolved. By Section 6 any VF-generated structure is resolved. By Lemma 3.49, for any $M \models \mathbf{T}$ and $c \in \mathrm{VF}(M)$, $\mathrm{acl}(c)$ is an elementary submodel of M. Consider c with $\mathrm{val}(c) \models p_0$, where p_0 is the generic type at ∞ of elements of Γ, i.e., $p_0|A = \{x > \delta : \delta \in \Gamma(A)\}$. Since

$$\mathrm{acl}(c) \models (\forall \delta \in \Gamma)(\exists u \in B(a, \gamma))(\gamma'(u) > \delta)$$

there exists $e \in \mathrm{acl}(c)$ with $e \in B(a, \gamma)$ and $\gamma'(e) > \mathrm{val}(c)$. By Lemma 5.12, there exists $e_0 \in \mathrm{acl}(\emptyset)$ such that $(c, e) \to (0, e_0)$. In particular, $e_0 \in B(a, \gamma)$. But then since $e \to e_0$ and $\gamma'(e_0) \in \Gamma(\mathrm{acl}(\emptyset))$, we have $e \in B(e_0, \gamma'(e_0))$. Thus $\gamma'(e) = \gamma'(e_0)$. But then $\gamma'(e_0) > \mathrm{val}(c)$, contradicting the choice of c. □

Lemma 11.7.

(1) *Let $\partial : U \times \Gamma \to \mathcal{C}_{df}$ be a definable distribution. Let $\gamma' : U \to \Gamma$ be a definable function with $\gamma'(u) > \gamma$, such that $\gamma'(u') = \gamma'(u)$ for $u' \in B(u, \gamma'(u))$. Then*

$$\partial(b, \gamma) = \int_{u \in B(b,\gamma)} \mathrm{vol}(B(0, \gamma'(u))^{-1} \partial(u, \gamma'(u)). \tag{11.5}$$

(2) *Let ∂_1, dd_2 be definable distributions on U such that for any $x \in U$, for all large enough $\gamma \in \Gamma$, for any $y \in B(x, \gamma)$ and any $\gamma' > \gamma$, $\partial_1(B(y, \gamma')) = \partial_2(B(y, \gamma'))$. Then $\partial_1 = \partial_2$.*

Proof.

(1) To prove (11.5), fix b, γ. We may assume $U = B(b, \gamma)$. Using Lemma 11.6, pick a constant γ'' with $\gamma'' > \gamma'(u)$ for all $u \in B(b, \gamma)$. Use the definition of a distribution with respect to γ'' to compute both $\partial(B(b, \gamma))$ and for each u $\partial(u, \gamma'(u))$, and compare the integrals using Lemma 11.4.

(1) Define $\gamma'(u)$ to be the smallest γ' such that for all $\gamma'' > \gamma'$ and all $y \in B(u, \gamma)$, $\partial_1(B(y, \gamma'')) = \partial_2(B(y, \gamma''))$. It is clear that $\gamma'(u') = \gamma'(u)$ for $u' \in B(u, \gamma'(u))$. (11.5) gives the same integral formula for $\partial_1(b, \gamma)$ and $\partial_2(b, \gamma)$. □

Let ∂ be a definable distribution, and U an arbitrary bounded open set. We can define $\partial(U)$ as follows. For any $x \in U$, let $\rho(x, U)$ be the smallest $\rho \in \Gamma$ such that $B(x, \rho) \subseteq U$. Let $B(x, U) = B(x, \rho(x, U))$; this is the largest open cube around x contained in U. Note that two such cubes $B(x, U)$, $B(x', U)$ are disjoint or equal. Define

$$\mathfrak{d}(U) = \int_{x \in U} \mathrm{vol}(B(x, U))^{-1} \mathfrak{d}(x, \rho(x, U)).$$

More generally, if h is a locally constant function on VF^n into $\mathcal{R}_{df}$ with bounded support, we can define

$$\mathfrak{d}(h) = \int_{x \in \mathrm{VF}^n} h(x)[B(x, h)]^{-1} \mathfrak{d}(x, \rho(x, U)), \tag{11.6}$$

where now $B(x, h) = B(x, \rho(x, U))$ is the largest open cube around x on which h is constant.

Proposition 11.8. *Let $\mathfrak{d}$ be a definable distribution. Then there exists a definable open set $G \subseteq \mathrm{VF}^n$ whose complement Z has dimension $< n$, and a definable function $g : G \to \mathcal{C}_{df}$ such that for any polydisc $U \subseteq G$*

$$\mathfrak{d}(U) = \int_U g.$$

Proof. Since $\mathfrak{d}$ is a finite sum of $\mathcal{R}_{df}$-valued distributions, we may assume it is $\mathcal{R}_{df}$-valued. Given $a \in \mathrm{VF}^n$, we have a function $\alpha_a : \Gamma \to \mathcal{R}_{df}$ defined by $\alpha_a(\rho) = \mathfrak{d}(B(a, \rho))$. Using the RV-description of $\mathcal{R}$, and the stable embeddedness of $\mathrm{RV} \cup \Gamma$, we see that α_a has a canonical code $c(a) \in (\mathrm{RV} \cup \Gamma)^*$.

Let G be the union of all polydiscs W such that c is constant on W. Let $Z = \mathrm{VF}^n \setminus G$. By Lemma 5.13, $\dim(Z) < n$.

Claim. Let W be a polydisc such that c is constant on W. Then for some $r \in \mathcal{R}_{df}$, for any polydisc $U = B(a, \rho) \subseteq W$, $\mathfrak{d}(a, \rho) = r\mathrm{vol}(U)$.

Proof. Since c is constant on W, for some function δ, all ρ and all $b \in W$ with $B(w, \rho) \subseteq W$, we have $\mathfrak{d}(B(w, \rho)) = \delta(\rho)$. By the definition of a distribution we have, for any $a \in W$,

$$\delta(\rho)\mathrm{vol}(B(a, \rho')) \underset{a}{=} \mathrm{vol}B(a, \rho)\delta(\rho').$$

Now $\mathrm{vol}(B(a, \rho)) \underset{a}{=} \mathrm{vol}B(0, \rho)$. Thus $\delta(\rho)\mathrm{vol}(B(0, \rho')) \underset{a}{=} \mathrm{vol}B(0, \rho)\delta(\rho')$. Since this holds for any $a \in W$, by Proposition 3.51 we have

$$\delta(\rho)\mathrm{vol}(B(0, \rho'))) = \mathrm{vol}B(0, \rho)\delta(\rho').$$

Thus $\delta(\rho)/\mathrm{vol}B(0, \rho) = r$ is constant. The claim follows. □

The proposition also follows using Lemma 11.7. □

11.2 Fourier transform

Let ψ be the tautological projection $K \to K/\mathcal{M} = \Omega$.

Let $g : \mathrm{VF}^n \to \mathcal{C}_{df}$ be a definable function, bounded on bounded subsets of VF^n. Define a function $\mathcal{F}(g)$ by

$$\mathcal{F}(g)(U) = \int_{y \in \mathrm{VF}} g(y)(\int_{x \in U} \psi(x \cdot y)).$$

This makes sense since for a given U, $(\int_{x \in U} \psi(x \cdot y))$ vanishes for y outside a certain polydisc (with sides inverse to U). Moreover, we have the following.

Lemma 11.9. $\mathcal{F}(g)$ *is a definable distribution.*

Proof. This follows from Fubini, Lemma 11.4, and chasing the definitions. □

Corollary 11.10. *Fix integers n, d. For all local fields L of sufficiently large residue characteristic, for any polynomial $G \in L[X_1, \dots, X_n]$ of degree $\leq d$, there exists a proper variety V_G of L^n such that $\mathcal{F}(|G|)$ agrees with a locally constant function outside V_G.*

Proof. The proof follows from Lemmas 11.9 and 11.8. □

See [4] for the real case.

12 Expansions and rational points over Henselian fields

We have worked everywhere with the geometry of algebraically closed valued fields, or more generally of **T**, but at a geometric level; all objects and morphisms can be lifted to the algebraic closure, and the quantifiers are interpreted there.

For many purposes, we believe this is the right framework. It includes, for instance, Igusa integrals $\int_{x \in X(F)} |f(x)|^s$, and we will show in a future work how to interpret in it some constructions of representation theory. See also [21].

In other situations, however, one wishes to integrate definable sets over Henselian fields rather than only constructible sets; and to have a change of variable formula for definable maps, as obtained by Denef–Loeser and Cluckers-Loeser (cf. [7]). It turns out that our formalism lends itself immediately to this generalization; we explain in this section how to recover it. The point is that an arbitrary definable set is an RV-union of constructible ones, and the integration theory commutes with RV-unions.

We will consider F that admits quantifier elimination in a language $\mathbb{L}^+$ obtained from the language of **T** *by adding relations to* RV *only*. For example, if $F = \mathrm{Th}(\mathbb{C}((X)))$, F has quantifier elimination in a language expanded with names D_n for subgroups of Γ (with $D_n(F) = n\Gamma(F)$).

There are two steps in moving from F^{alg} to F. We will try to clarify the situation by taking them one at a time. The two steps are to restrict the points to a smaller set (the F-rational points), and they enlarge the language to a larger one (with enough relation symbols for F-quantifier elimination). We will take these steps in the reverse order. In Section 12.1 we show how to extend the results of this paper to expansions of the language in the RV sorts, and in Section 12.3 how to pass to sets of rational points over a Hensel field.

The reader who wishes to restrict attention to constructible integrals (still taking rational points) may skip Section 12.1, taking $\mathbf{T}^+ = \mathbf{T}$ in Section 12.3. In this case one still has a change of variable formula for a constructible change of variable, but not for a definable change of variable. An advantage is that the target ring correspondingly involves the Grothendieck group of constructible sets and maps rather than definable ones, which sometimes has more faithful information; cf. Example 12.12.

12.1 Expansions of the RV sort

Let $\mathbf{T}$ be V-minimal.

Let $\mathbf{T}^+$ be an expansion of $\mathbf{T}$ obtained by adding relations to RV. We assume that every $M \models \mathbf{T}$ embeds into the restriction to the language of $\mathbf{T}$ of some $N \models \mathbf{T}^+$. (As $\mathbf{T}$ is complete, this is actually automatic.) By adding some more basic relations, without changing the class of definable relations, we may assume $\mathbf{T}^+$ eliminates RV-quantifiers. As $\mathbf{T}$ eliminates field quantifiers, and $\mathbf{T}^+$ has no new atomic formulas with VF variables, $\mathbf{T}^+$ eliminates VF-quantifiers, too, and hence all quantifiers.

For instance, $\mathbf{T}+$ may include a name for a subfield of the residue field (say, pseudofinite) or the angular coefficients the Denef–Pas language (where RV is split). Write $+$-definable for $\mathbf{T}^+$-definable; similarly, tp_+ will denote the type in $\mathbf{T}^+$, etc. The unqualified words formula, type, and definable closure will refer to quantifier-free formulas of $\mathbf{T}$.

Lemma 12.1. *Let $M \models \mathbf{T}^+$. Let A be a substructure of M, $c \in M$, $B = A(c) \cap \mathrm{RV}$.*

(1) $\mathrm{tp}(c/A \cup B) \cup {\mathbf{T}^+}_{A\cup B}$ *implies* $\mathrm{tp}_+(c/A \cup B)$.
(2) *Assume c is ${\mathbf{T}^+}_A$-definable. Then $c \in \mathrm{dcl}(A, b)$ for some $b \in A(c) \cap \mathrm{RV}$.*

Proof.

(1) This follows immediately from the quantifier elimination for $\mathbf{T}^+$. Indeed, let $\phi(x) \in \mathrm{tp}_+(c/A \cup B)$. Then ϕ is a Boolean combination of atomic formulas, and it is sufficient to consider the case of ϕ atomic, or the negation of an atomic formula. Now since any basic function $\mathrm{VF}^n \to \mathrm{VF}$ is already in the language of $\mathbf{T}$, every basic function of the language of $\mathbf{T}^+$ denoting a function $VF^n \to \mathrm{RV}$ factors through a $\mathbf{T}$-definable function into RV. Hence the same is true for all terms (compositions of basic functions). And any basic relation is either the equality relation on VF, or else a relation between variables of RV. If ϕ is an equality or inequality between $f(x), g(x)$, it is already in $\mathrm{tp}(c/A)$. Now suppose ϕ is a relation $R(f_1(x), \ldots, f_n(x))$ between elements of RV. Since $B(c) \cap \mathrm{RV} \subseteq B$, the formula $f_i(x) = b_i$ lies in $\mathrm{tp}(c/A \cup B)$ for some $b_i \in B$. On the other hand, $R(b_1, \ldots, b_n)$ is part of ${\mathbf{T}^+}_B$. These formulas together imply $R(f_1(x), \ldots, f_n(x))$.

(2) We must show that $c \in \mathrm{dcl}(A \cup B)$. Let $p = \mathrm{tp}(c/A \cup B)$. By (1), p generates a complete type of $\mathbf{T}^+{}_{A\cup B}$. Since this is the type of c and c is $\mathbf{T}^+{}_A$-definable, and since any model of $\mathbf{T}$ embeds into a model of $\mathbf{T}^+$, p has a unique solution solution in any model of $\mathbf{T}$. Thus $c \in \mathrm{dcl}(A \cup B)$. □

We will now see that any $\mathbf{T}^+$-definable bijection decomposes into $\mathbf{T}$-bijections, and bijections of the form $x \mapsto (x, j(g(x)))$ where g is a $\mathbf{T}$-definable map into RV^m and j is a $\mathbf{T}^+$-definable map on RV.

Corollary 12.2.

(1) *Let P be a $\mathbf{T}^+$-definable set. There exist $\mathbf{T}$-definable $f : \widetilde{P} \to \mathrm{RV}^*$ and a $\mathbf{T}^+$-definable $Q \subseteq \mathrm{RV}^*$ such that $P = f^{-1}Q$.*

(2) *Let P_1, P_2 be $\mathbf{T}^+$-definable sets, and let $F : P_1 \to P_2$ be a $\mathbf{T}^+$-definable bijection. Then there exist $g_i : \widetilde{P_i} \to R_i \subseteq \mathrm{RV}^m$, $R \subseteq \mathrm{RV}^m$, $h_i : R \to R_i$, and a bijection $H : \widetilde{P_1} \times_{g_1,h_1} R \to \widetilde{P_2} \times_{g_2,h_2} R$ over R, all $\mathbf{T}$-definable, and $\mathbf{T}^+$-definable $Q_i \subseteq R_i$, $Q \subseteq R$, and $j_i : Q_i \to Q$ such that $P_i = {g_i}^{-1}Q_i$, $h_i j_i = \mathrm{Id}_{Q_i}$, and for $x \in P_1$,*

$$j_1 g_1(x) = j_2 g_2(F(x)) =: j(x) \quad \text{and} \quad H(x, j(x)) = (F(x), j(x)). \qquad (\diamond)$$

Moreover, if $P_i \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ projects finite-to-one to VF^n, then $R \to R_i$ is finite-to-one.

Proof.

(1) Let $\mathcal{F}$ be the family of all $\mathbf{T}$-definable functions $f : W \to \mathrm{RV}^m$, where W is a definable set.

Claim. If $\mathrm{tp}(c) = \mathrm{tp}(d)$ and $f(c) = f(d)$ for all $f \in \mathcal{F}$ with $c, d \in \mathrm{dom}(f)$, then $c \in P \iff d \in P$.

Proof. We have $\mathrm{tp}(c, f(c)) = \mathrm{tp}(d, f(d)) = \mathrm{tp}(d, f(c))$, so $\mathrm{tp}(c/f(c)) = \mathrm{tp}(d/f(c))$ for all $f \in \mathcal{F}$ with $c \in \mathrm{dom}(f)$, and thus $\mathrm{tp}(c/B) = \mathrm{tp}(d/B)$, where $B = A(c) \cap \mathrm{RV}$. It follows that $\mathrm{tp}_+(c) = \mathrm{tp}_+(d)$ and, in particular, $c \in P \iff d \in P$. □

By compactness, there are $(f_i, W_i)_{i=1}^m \in \mathcal{F}$ such that if $c \in W_i \iff d \in W_i$ and $f_i(c) = f_i(d)$ whenever $c, d \in W_i$, then $c \in P \iff d \in P$. Let $\widetilde{P} = \cup_i W_i$, and extend f_i to $\widetilde{P}$ by $f_i(x) = \infty$ if $x \notin W_i$. Let $f(x) = (f_1(x), \ldots, f_m(x))$. Letting $\widetilde{P} = \cup_i W_i$ and $Q = f(P)$, (1) follows.

(2) Consider first a $\mathbf{T}^+$-type $p = \mathrm{tp}_+(c_1)$, $c_1 \in P_1$. Let $c_2 = F(c_1)$. Using Lemma 3.48, there exists $g_i^p \in \mathcal{F}$ such that $e_i = g_i^p(c_i)$ generates $\mathrm{dcl}(c_i) \cap \mathrm{RV}$. It follows as in Lemma 12.1(1) that e_i generates $\mathrm{dcl}_+(c_i) \cap \mathrm{RV}$. Let e generate $\mathrm{dcl}(c_1, c_2) \cap \mathrm{RV}$; we have $e_i = h_i^p(e)$ for appropriate $\mathbf{T}$-definable h_i^p. Note $\mathrm{dcl}_+(c_1) = \mathrm{dcl}_+(c_2)$, and so $e \in \mathrm{dcl}_+(c_i)$. Now quantifier elimination for $\mathbf{T}^+$ implies the stable embeddedness of RV, in the same way as for ACVF (cf. Section 2.1). By Lemma 2.9 $\mathrm{tp}_+(c_i/e_i)$ implies $\mathrm{tp}_+(c_i/\mathrm{RV})$; in particular, since $e \in \mathrm{dcl}_+(c_i)$ $e = j_i^p(e_i)$ for some $\mathbf{T}^+$-definable j_i^p. By Lemma 12.1(2) over $\mathrm{dcl}(c_1)$, $c_2 \in \mathrm{dcl}(c_1, e)$; similarly, $c_1 \in \mathrm{dcl}(c_2, e)$. Thus there exists a $\mathbf{T}$-definable invertible

H^p with $H^p(c_1, e) = (c_2, e)$. Equations ($\diamond$) have been shown to hold on p. Now g_i extends to a **T**-definable function $g_i : \widetilde{P_i} \to R_i$. By compactness ($\diamond$) holds on some definable neighborhood of p; and by (1) this neighborhood can be taken to have the form ${g_1}^{-1}Q_1$ for some Q_1. Finitely many such neighborhoods cover P_1, and the data can be sewed together as in (1). We thus find $\widetilde{P_1}$, R, R_1, R_2, g_1, g_2, h_1, h_2, H, Q_1, j_1, j_2 such that $h_i j_i(x) = x$ and ($\diamond$) holds on ${g_1}^{-1}Q_1 = P_1$. Let $Q_2 = h_2 j_1 Q_1$; it follows that $P_2 = F(P_1) = {g_2}^{-1}Q_2$.

To prove the last point, since $c_2 \in \mathrm{dcl}(c_1, e)$ we have (Lemma 3.41) $c_2 \in \mathrm{acl}(c_1)$. But $e \in \mathrm{dcl}(c_1, c_2)$ so $e \in \mathrm{acl}(\mathrm{dcl}(c_1))$; and as $e \in \mathrm{RV}^m$ for some m, $e \in \mathrm{acl}(\mathrm{dcl}(e_1))$.

Let VF^+ be the category of $+$-definable subsets of varieties over $\mathrm{VF} \cap \mathrm{dcl}(\emptyset)$, and $+$-definable maps. Define effective isomorphism as in Definition 8.2; let ${K_+}^{\mathrm{eff}}$ denote the Grothendieck group of effective isomorphism classes, and let $[X]$ be the class of X.

Let $\mathrm{RV}^+[*]$ be the category of pairs (Y, f), where Y is a $+$-definable subset of X for some $(X, f) \in \mathrm{Ob}\,\mathrm{RV}[*]$ (Definition 3.66). A morphism $(Y, f) \to (Y', f')$ is a definable bijection $h : Y \to Y'$ such that $f'(h(y)) \in \mathrm{acl}(f(y))$ for $y \in Y$.

Let $K_+(\mathrm{RV}^+[*])$ be the Grothendieck semigroup of isomrphism classes of RV^+; let $\mathrm{I_{sp}}$ be the congruence generated by $(J, 1_1)$, where $J = \{1\}_0 + [\mathrm{RV}^{>0}]_1$.

Proposition 12.3. *There exists a canonical surjective homomorphism of Grothendieck semigroups*

$$\oint : K_+(\mathrm{VF}^+[*]) \to K_+(\mathrm{RV}^+[*])/\mathrm{I_{sp}}$$

determined by

$$\oint [X] = [W]/\mathrm{I_{sp}} \iff [X] = [\mathbb{L}W].$$

Proof. We have to show the following:

(i) Any element of $K_+(\mathrm{VF}^+)$ is effectively isomorphic to one of the form $[\mathbb{L}W]$.
(ii) If $[\mathbb{L}W_1] = [\mathbb{L}W_2]$ then $([W_1], [W_2]) \in \mathrm{I_{sp}}$.

(i) By Corollary 12.2(1), a typical element of $K_+(\mathrm{VF}^+)$ is represented by $P = f^{-1}Q$, where $Q \subseteq \mathrm{RV}^*$is $\mathbf{T}^+$-definable, $f : \widetilde{P} \to \mathrm{RV}^*$ is **T**-definable. For any $a \in \mathrm{RV}^*$, $f^{-1}(a)$ is $\mathbf{T}_a$-definable, and $[f^{-1}(a)] = [\mathbb{L}C_a]$ where $[C_a] = [\int f^{-1}(a)]$. Since $\mathbb{L}$ commutes with RV-disjoint unions, it follows that $[P] = [\mathbb{L}W]$ where $W = \dot\bigcup_{a \in Q} C_a$.

(ii) Assume $[\mathbb{L}W_1] = [\mathbb{L}W_2]$. By Proposition 3.51, the base can be enlarged so as to be made effective, without change to RV; thus to show that $([W_1], [W_2]) \in \mathrm{I_{sp}}$ we may assume $\mathbb{L}W_1, \mathbb{L}W_2$ are isomorphic. Let $f : \mathbb{L}W_1 \to \mathbb{L}W_2$ be an isomorphism. Let $P_i = \mathbb{L}W_i$ and let $\widetilde{P_i}$, R_i, g_i, h_i, R, H, Q, Q_i, j_i be as in Corollary 12.2(2).

Since $P_i = {g_i}^{-1}Q_i = \mathbb{L}W_i$, the maximal $\underset{\mathrm{rv}}{\sim}$-invariant subset of $\widetilde{P_i}$ contains P_i, so we may assume $\widetilde{P_i}$ is $\underset{\mathrm{rv}}{\sim}$-invariant; in other words, $\widetilde{P_i} = \mathbb{L}\widetilde{W_i}$ for some **T**-definable $\widetilde{W_i} \in \mathrm{RV}[*, \cdot]$ containing W_i.

By Lemma 7.8, there exists a special bijection $\sigma : \mathbb{L}\widetilde{W}_i^* \to \mathbb{L}\widetilde{W}_i$ such that $g_i \circ \sigma$ factors through ρ, i.e., for some $e_i : \widetilde{W}_i^* \to R_i$ we have $g_i \circ \sigma = e_i \circ \rho$ on $\mathbb{L}\widetilde{W}_i$. Let $W_i{}^*$ be the pullback of W_i to $\widetilde{W}_i^*$, so that $\sigma(\mathbb{L}W_i^*) = \mathbb{L}W_i = P_i$. Then $([W_i], [W_i{}^*]) \in \mathrm{I_{sp}}$, so it suffices to show that $(W_1^*, W_2^*) \in \mathrm{I_{sp}}$. Since $P_i = g_i{}^{-1}Q_i$, we have $W_i^* = e_i{}^{-1}Q_i$.

For $c \in R$, let $\widetilde{P}_i(c) = \sigma^{-1}g_i{}^{-1}(h_i(c))$, $\widetilde{W}_i(c) = e_i{}^{-1}(h_i(c))$. Then $\widetilde{P}_i(c) = \mathbb{L}\widetilde{W}_i(c)$. Now H induces a bijection $\widetilde{P}_1(c) \to \widetilde{P}_2(c)$. Thus by Proposition 7.25, $(\widetilde{W}_1(c), \widetilde{W}_2(c)) \in \mathrm{I_{sp}}$. In particular, this is true for $c \in Q$; now $h_i : Q \to Q_i$ is a bijection, and $W_i^* = \dot{\bigcup}_{c \in Q} \widetilde{W}_i(c)$. Thus $([W_1^*], [W_2^*]) \in \mathrm{I_{sp}}$. □

Remark. Since the structure on RV in $\mathbf{T}^+$ is arbitrary, we cannot expect the homomorphism of Corollary 12.3 to be injective. We could make it so tautologically by modifying the category RV^+, taking only *liftable* morphisms, i.e., those that lift to VF; we then obtain an isomorphism. In specific cases it may be possible to check that all morphisms are liftable.

12.2 Transitivity

Motivation. Consider a tower of valued fields, such as $\mathbb{C} \le \mathcal{C}((s)) \le \mathcal{C}((s))((t))$. Given a definable set over $\mathcal{C}((s))((t))$, we can integrate with respect to the t-valuation, obtaining data over $\mathcal{C}((s))$ and the value group. The $\mathcal{C}((s))$ can then be integrated with respect to the s-valuation. On the other hand, we can consider directly the $\mathbb{Z}^2$-valued valuation of $\mathcal{C}((s))((t))$, and integrate so as to obtain an answer involving the Grothendieck group of varieties over $\mathbb{C}$. Below we develop the language for comparing these answers, and show that they coincide.

For simplicity we accept here a Denef–Pas splitting, i.e., we expand RV so as to split the sequence $\mathbf{k}^* \to \mathrm{RV}^* \to \Gamma$. Then rv splits into two maps, $\mathrm{ac} : \mathrm{VF}^* \to \mathbf{k}^*$ and $\mathrm{val} : \mathrm{VF}^* \to \Gamma$. This expansion of ACVF(0, 0) is denoted $\mathrm{ACVF^{DP}}$. Note that this falls under the framework of Section 12.1, as will the further expansions below.

Consider two expansions of $\mathrm{ACVF^{DP}}$: (1) Expand the residue field to have the structure of a valued field (itself a model of $\mathrm{ACVF^{DP}}$). (2) Expand the value group to be a lexicographically ordered product of two ordered Abelian groups. Then (1)–(2) yield bi-interpretable theories. In more detail, we have the following:

First expansion. Rename the VF sort as VF_{21}, the residue field as VF_1, and the value group Γ_1. VF_1 carries a field structure; expand it to a model of $\mathrm{ACVF^{DP}}$, with residue field F_0 and value group Γ_0. Let ac_{21}, val_{21} have their natural meanings.

Second expansion. Rename the VF-sort as VF_{20}, the residue field as F_0 and the value group as Γ_{20}. Add a predicate Γ_0 for a proper convex subgroup of Γ_{20}, and a predicate Γ_1 for a complementary subgroup, so that Γ_{20} is identified with the lexicographically ordered $\Gamma_0 \times \Gamma_1$.

Lemma 12.4. *The two theories described above are bi-interpretable. A model of* (1) *can canonically be made into a model of* (2) *with the same class of definable relations, and vice versa.*

Proof. Given (1), let $\mathrm{VF}_{20} = \mathrm{VF}_{21}$ as fields. Define

$$\mathrm{ac}_{20} = \mathrm{ac}_{10} \circ \mathrm{ac}_{21} . \tag{12.1}$$

Let $\Gamma_{20} = \Gamma_1 \times \Gamma_0$, and define $\mathrm{val}_{20} : \mathrm{VF}_{21}^* \to \Gamma_{20}$ by

$$\mathrm{val}_{20}(x) = (\mathrm{val}_{21}(x), \mathrm{val}_{10}(\mathrm{ac}_{21}(x))). \tag{12.2}$$

Conversely, given (2), let $\mathrm{VF}_{21} = \mathrm{VF}_{20}$ as fields;

$$\begin{aligned}
\mathcal{O}_{21} &= \{x \in \mathrm{VF}_{21} : (\exists t \in \Gamma_0)(\mathrm{val}_{20}(x) \geq t)\},\\
\mathcal{M}_{21} &= \{x \in \mathrm{VF}_{21} : (\forall t \in \Gamma_0)(\mathrm{val}_{20}(x) > t)\},\\
\mathrm{VF}_1 &= \mathcal{O}_{21}/\mathcal{M}_{21}.
\end{aligned}$$

Let VF_{21} have the valued field structure with residue field VF_1; note that the value group $\mathrm{VF}_{21}^*/\mathcal{O}_{21}^*$ can be identified with Γ_1. Note that $\ker \mathrm{ac}_{20} \supset 1 + \mathcal{M}_{21}$, so that factors through VF_1^*, and define ac_{10}, ac_{21} so as to make (12.1) hold. Then define val_{21}, val_{10} so that (12.2) holds. □

Let $\mathrm{VF}^+[*]$ denote the category of definable subsets of VF_{21}, equivalently, VF_{20}, in the expansions (1) or (2). According to Proposition 12.3 and Lemma 2.11, we have canonical maps $K_+(\mathrm{VF}^+[*]) \to K_+(\mathrm{RV}_1^+[*])/\mathrm{I_{sp}}$ and $K_+(\mathrm{VF}^+[*]) \to K_+(\mathrm{RV}_2^+[*])/\mathrm{I_{sp}}$, where $\mathrm{RV}_i^+[*]$ denotes the expansion of RV according to (1)–(2), respectively.

By Proposition 8.4 we have canonical maps

$$\begin{aligned}
K_+(\mathrm{VF}^+[*]) &\to K_+(\mathrm{VF}_1[*]) \otimes K_+(\Gamma_{21}[*])/\mathrm{I_{sp}}\\
&\to (K_+(F_0) \otimes K_+(\Gamma_{10})) \otimes K_+(\Gamma_{21})/\mathrm{I_{sp_1}}
\end{aligned} \tag{12.3}$$

for a certain congruence $\mathrm{I_{sp_1}}$. And, on the other hand,

$$\begin{aligned}
K_+(\mathrm{VF}^+[*]) &\to K_+(F_0[*]) \otimes K_+(\Gamma_{20}[*])/\mathrm{I_{sp}}\\
&= K_+(F_0[*]) \otimes (K_+(\Gamma_{10}[*]) \otimes K_+(\Gamma_{21}[*]))/\mathrm{I_{sp_2}}.
\end{aligned} \tag{12.4}$$

For an appropriate $\mathrm{I_{sp_2}}$. The tensor products here are over $\mathbb{Z}$, in each dimension separately.

Using transitivity of the tensor product we identify $(K_+(F_0) \otimes K_+(\Gamma_{10})) \otimes K_+(\Gamma_{21})$ with $K_+(F_0[*]) \otimes (K_+(\Gamma_{10}[*]) \otimes K_+(\Gamma_{21}[*]))$. Then

Theorem 12.5. $\mathrm{I_{sp_1}}, \mathrm{I_{sp_2}}$ *are equal and the maps of* (12.3),(12.4) *coincide.*

Proof. It suffices to show in the opposite direction that the compositions of maps induced by $\mathbb{L}$

$$\begin{aligned}
(K_+(F_0[*]) \otimes K_+(\Gamma_{10}[*]) \otimes K_+(\Gamma_{21}[*]) &\to K_+(\mathrm{VF}_1[*]) \otimes K_+(\Gamma_{21}[*])\\
&\to K_+(\mathrm{VF}^+[*]),
\end{aligned} \tag{12.5}$$

$$\begin{aligned}
K_+(F_0[*]) \otimes K_+(\Gamma_{10}[*]) \otimes K_+(\Gamma_{21}[*]) &\to K_+(F_0[*]) \otimes K_+(\Gamma_{20}[*])\\
&\to K_+(\mathrm{VF}^+[*])
\end{aligned} \tag{12.6}$$

coincide. But this reduces by RV-additivity to the case of points, and by multplicativity to the individual factors F_0, Γ_{21}, Γ_{10}, yielding to an obvious computation in each case. □

12.3 Rational points over a Henselian subfield: Constructible sets and morphisms

Let $\mathbf{T}$ be V-minimal, and $\mathbf{T}^+$ an expansion of $\mathbf{T}$ in the RV sorts.

Let F be an effective substructure of a model of $\mathbf{T}$. Thus $F = (F_{\mathrm{VF}}, F_{\mathrm{RV}})$, with F_{VF} a field, and $\mathrm{rv}(F_{\mathrm{VF}}) = F_{\mathrm{RV}}$; and F is closed under definable functions of $\mathbf{T}$. For example, if $\mathbf{T} = \mathbf{T}^+ = \mathrm{ACVF}(0, 0)$, this is the case iff F_{VF} is a Henselian field and $F_{\mathrm{RV}} = F/\mathcal{M}(F)$; any Hensel field of residue characteristic 0 can be viewed in this way. See Example 12.8.

By a $+$-*constructible* subset of F^n, we mean a set of the form $X(F) = X \cap F^n$, with X a quantifier-free formula of $\mathbf{T}^+$. Let $\mathrm{VF}^+(F)$ be the category of such sets, and $+$-constructible functions between them. The Grothendieck semiring $K_+ \mathrm{VF}^+(F)$ is thus the quotient of $K_+ \mathrm{VF}$ by the semiring congruence

$$I_F = \{([X], [Y]) : X, Y \in \mathrm{Ob}\,\mathrm{VF}^+, X(F) = Y(F)\}.$$

(One can verify this is an ideal; in fact, if $X(F) = Y(F)$ and $X \simeq X'$, then there exists $Y' \simeq Y$ with $X'(F) = Y'(F)$.)

Similarly, we can define I_F^{RV} and form $K_+ \mathrm{RV}(F) \simeq K_+(\mathrm{RV})/I_F^{RV}$. As usual, let $\mathrm{I_{sp}}$ denote the congruence generated by $([1]_0 + [\mathrm{RV}^{>0}]_1, [1]_1)$, and $I_F^{RV} + \mathrm{I_{sp}}$ their sum.

Claim. If $([X], [X']) \in I_F$ then $(\not\int[X], \not\int[X']) \in I_F^{RV} + \mathrm{I_{sp}}$.

Proof. We may assume, changing X within the VF-isomorphism class $[X]$, that $X(F) = X'(F)$. Then $X(F) = (X \cup X')(F) = X'(F)$, and it suffices to show that $(\not\int[X], \not\int[X \cup X']), (\not\int[X'], \not\int[X \cup X']) \in I_F^{RV}$. Thus we may assume $X \subseteq X'$. Let $Z = X' \setminus X$. Then $Z(F) = \emptyset$, and it suffices to show that $(\not\int(Z), \emptyset) \in I_F^{RV}$. Now $\not\int(Z) = [Y]$ for some Y with Z definably isomorphic to $\mathbb{L}Y$. Thus $\mathbb{L}Y(F) = \emptyset$; hence $Y(F) = \emptyset$. Thus $([Y], \emptyset) \in I_F^{RV}$, as required. □

As an immediate consequence, we have the following.

Proposition 12.6. *Assume $F \leq M \models \mathbf{T}$, with F closed under definable functions of $\mathbf{T}$. The homomorphism $\not\int$ of Theorem* 8.8 *induces a homomorphism*

$$\int_F : K_+ \mathrm{VF}^+(F) \to K_+ \mathrm{RV}^+(F)/\mathrm{I_{sp}}. \qquad \square$$

12.4 Quantifier elimination for Hensel fields

Let $\mathbf{T}$ be a V-minimal theory in a language $L_{\mathbf{T}}$, with sorts (VF, RV) (cf. Section 2.1). Assume $\mathbf{T}$ admits quantifier elimination and, moreover, that any definable function is given by a basic function symbol. This can be achieved by an expansion-by-definition of the language.

Let $\mathbf{T}_h = (\mathbf{T})_\forall \cup \{(\forall y \in \mathrm{RV})(\exists x \in \mathrm{VF})(\mathrm{rv}(x) = y)\}$.

A model of $\mathbf{T}_h$ is thus the same as a substructure A of a model of $\mathbf{T}$, such that $\mathrm{RV}(A) = \mathrm{rv}(\mathrm{VF}(A))$.

Lemma 12.7. *Any formula of $L_{\mathbf{T}}$ is $\mathbf{T}$-equivalent to a Boolean combination of formulas in* VF*-variables alone, and formula $\psi(t(x), u)$ where t is a sequence of terms for functions* $\mathrm{VF}^n \to \mathrm{RV}$*, u is a sequence of* RV*-variables, and ψ is a formula of* RV *variables only.*

Proof. This follows from stable embeddedness of RV, Corollary 3.24, Lemma 2.8 and the fact (Lemma 7.10) that definable functions into Γ factor through definable functions into RV. □

Example 12.8. If $\mathbf{T} = \mathrm{ACVF}(0, 0)$, then $\mathbf{T}_h$ is an expansion-by-definition of the theory of Hensel fields of residue characteristic zero.

Proof. We must show that a Henselian valued field is definably closed in its algebraic closure, in the two sorts VF, RV.

Let $K \models T_{Hensel}$, $K \leq M \models \mathrm{ACVF}$. Let $X \subseteq \mathrm{VF}^k \times \mathrm{RV}^l$, $Y \subseteq \mathrm{VF}^{k'} \times \mathrm{RV}^{l'}$ be ACVF_K-definable sets, and $F : X \to Y$ an ACVF_K-definable bijection. We have to show that $F(X \cap K^k \times \mathrm{RV}(K)^l) = Y \cap K^{k'} \times \mathrm{RV}(K)^{l'}$.

K^{alg} is an elementary submodel of M; we may assume $K^{\mathrm{alg}} = M$. By one of the characterizations of Henselianity, the valuation on K extends uniquely to K^{alg}. Hence every field automorphism of M over K is a valued field automorphism. Thus K is the fixed field of $\mathrm{Aut}(M/K)$ (in the sense of valued fields), and hence $K = \mathrm{dcl}(K)$. Since ACVF_K is effective, any definable point of RV lifts to a definable point of VF; so $\mathrm{dcl}(K) \cap \mathrm{RV} = \mathrm{RV}_K$. Thus K is definably closed in M in both sorts. □

Let $L \supset L_{\mathbf{T}}$; assume $L \setminus L_{\mathbf{T}}$ consists of relations and functions on RV only. If $A \leq M \models \mathbf{T}$, let $L_{\mathbf{T}}(A)$ be the languages enriched with constants for each element of A; let $\mathbf{T}_h(A) = \mathbf{T}_A \cup \mathbf{T}_h$, where $\mathbf{T}_A$ is the set of quantifier-free valued field formulas true of A.

Proposition 12.9. $\mathbf{T}_h$ *admits elimination of field quantifiers.*

Proof. Let A be as above. Let Φ_A be the set of $L(A)$-formulas with no VF-quantifiers.

Claim. Let $\phi(x, y) \in \Phi_A$ with x a free VF-variable. Then $(\exists x)\phi(x, y)$ is $\mathbf{T}_h(A)$-equivalent to a formula in Φ_A.

Proof. By the usual methods of compactness and absorbing the y-variables into A, it suffices to prove this when x is the only variable. Assume first that $\phi(x)$ is an $L_{\mathbf{T}}(A)$-formula. By Lemma 4.2, there exists an ACVF-definable bijection between the definable set defined by $\phi(x)$, and a definable set of the form $\mathbb{L}\phi'(x', u)$, where ϕ' is an $L_{\mathbf{T}}(A)$-formula in RV-variables only (including a distinguished variable x' on which $\mathbb{L}$ acts.) By the definition of $\mathbf{T}_h$, in any model of $\mathbf{T}_h$, ϕ has a solution iff $\mathbb{L}\phi'(x', u)$ has a solution. But clearly $\mathbb{L}\phi'(x', u)$ has a solution iff $\phi'(x', u)$ does. Thus $\mathbf{T}_h(A) \models (\exists x)\phi(x) \iff (\exists x', u)\phi'(x', u)$.

Now let $\phi(x)$ be an arbitrary Φ_A formula. Let Ψ be the set of formulas of $L(A)$ involving RV-variables only. Let Θ be the set of conjunctions of formulas of $L_{\mathbf{T}}(A)$ in VF-variables only, and of formulas of the form $\psi(t(x))$, where $\psi \in \Psi$ and t

is a term of $L_{\mathbf{T}}(A)$. The set of disjunctions of formulas in Θ is then closed under Boolean combinations, and under existential RV-quantification. By Lemma 12.7 it includes all $L_{\mathbf{T}}$-formulas, up to equivalence; and also all formulas in RV-variables only. Thus $\phi(x)$ is a disjunction of formulas in Θ, and we may assume $\phi(x) \in \Theta$. Say $\phi = \phi_0(x) \wedge \psi(t(x))$, with $\phi_0 \in L_{\mathbf{T}}(A)$ and $\psi \in \Psi$. By the claim, for some formula $\rho(y)$ of Φ_A, we have $T_h(A) \models \rho(y) \iff (\exists x)(t(x) = y \& \phi_0(x))$. Hence $(\exists x)\phi(x) \iff (\exists y)(\psi(y) \& \rho(y))$. □

Quantifier elimination now follows by induction. □

Remark. Since only field quantifiers are mentioned, this immediately extends to expansions in the field sort.

In particular, one can split the sequence $0 \to \mathbf{k}^* \to \mathrm{RV} \to \Gamma \to 0$ if one wishes. This yields the quantifier elimination [30] in the Denef–Pas language.

The results of Ax-Kochen and Ershov, and the large literature that developed around them, appeared to require methods of "quasi-convergent sequences." It is thus curious that they can also be obtained directly from Robinson's earlier and purely "algebraic" quantifier elimination for ACVF. Note that in the case of ACVF, there is no need to expand the language to obtain QE; and then Lemma 12.7 requires no proof beyond inspection of the language.

12.5 Rational points: Definable sets and morphisms

In this subsection we will work with completions T of $\mathbf{T}_h \cup \{(\exists x \in \Gamma)(x > 0)\}$. These are theories of valued fields of residue characteristic 0, possibly expanded, not necessarily algebraically closed. The language of T is thus the language of $\mathbf{T}^+$. The words formula, type, definable closure will refer to quantifier-free formulas of $\mathbf{T}^+$. Definable closure, types with respect to T are referred to explicitly as dcl^T, $\mathbf{T}\,\mathrm{tp}$, etc.

Let $F \models T$. Since $F \models \mathbf{T}_\forall$, F embeds into a model M' of $\mathbf{T}^+$. Since $\Gamma(F) \neq (0)$, by Proposition 3.51 and Lemma 3.49, there exists $F' \subseteq M'$ containing F, with $\Gamma(F') = \Gamma(F)$, and $M = \mathrm{acl}(F')$ an elementary submodel of M'. Hence F embeds into a model M of $\mathbf{T}^+$ with $\Gamma(F)$ cofinal in $\Gamma(M)$.

Lemma 12.10. *Let $F \models T$, $F \leq M \models \mathbf{T}^+$, $\Gamma(F)$ cofinal in $\Gamma(M)$. Let A be a substructure of M, $c \in F$, $B = A(c) \cap \mathrm{RV} \cap F$,*

(1) $\mathrm{tp}(c/B) \cup T_B$ *implies* $\mathbf{T}\,\mathrm{tp}(c/B)$.
(2) *Assume c is T_A-definable. Then $c \in \mathrm{dcl}(A, b)$ for some $b \in B$.*

Proof.

(1) This follows immediately from the quantifier elimination for T and from Lemma 12.1(1).
(2) We have $B \subseteq \mathrm{dcl}^T(A) \cap \mathrm{RV}$. We must show that $c \in \mathrm{dcl}(A \cup B)$. Let $p = \mathrm{tp}(c/A \cup B)$. By (1), p generates a complete type of $T_{A \cup B}$. Since this is the type of c and c is T_A-definable, some formula P in the language of $T_{A \cup B}$ with $P \in p$

has a unique solution in F. Now the values of F are cofinal in the value group of F^a; so P cannot contain any ball around c. (Any such ball would have an additional point of F, obtained by adding to c some element of large valuation.) Let P' be the set of isolated elements of P; then P' is finite (as is the case for every definable P), $\mathbf{T}_A$-definable, and $c \in P'$. By Lemma 3.9, there exists an $\mathbf{T}_A$-definable bijection $f : P' \to Q$ with $Q \subseteq \mathrm{RV}^n$. Then $f(c) \in \mathrm{dcl}_T(A) = B$, and $c = f^{-1}(f(c)) \in \mathrm{dcl}(A \cup B)$. □

Corollary 12.11. *Two definably isomorphic definable subsets of F have the same class in $K_+ \mathrm{VF}^+(F)$.*

Proof. T-definable bijections are restrictions of $\mathbf{T}^+$-definable bijections. Hence Corollary 12.2 is true with T replacing $\mathbf{T}^+$. □

Thus Proposition 12.6 includes a change-of-variable formalism for definable bijections.

12.6 Some specializations

Tim Mellor's Euler characteristic

Consider the theory RCVF of real closed valued fields. Let $\mathrm{RV}_{\mathrm{RCVF}}$, $\mathrm{RES}_{\mathrm{RCVF}}$, $\mathrm{VAL}_{\mathrm{RCVF}}$ denote the categories of definable sets and maps that lift to bijections of RCVF (on RV and on the residue field, value group, respectively; we do not need to use the sorts of RES other than the residue field here, say, all structures A of interest have Γ_A divisible). From Proposition 12.6 and Corollary 12.11, we obtain an isomorphism: $K(\mathrm{RCVF}) \to K(\mathrm{RV}_{\mathrm{RCVF}})/([0]_1 - [\mathrm{RV}^{>0}]_1 - [0]_0)$.

The residue field is a model of the theory RCF of real closed fields; $K(\mathrm{RCF}) = \mathbb{Z}$ via the Euler characteristic (cf. [37]). Since the ambient dimension grading is respected here, $K(\mathrm{RES}_{\mathrm{RCVF}}) = \mathbb{Z}[t]$.

The value group is a model of DOAG, and moreover, any definable bijection on $\Gamma[n]$ for fixed n lifts to RV and, indeed, to RCVF. This is because the multiplicative group of positive elements is uniquely divisible, and so $SL_n(\mathbb{Q})$ acts on the nth power of this group. By Proposition 9.4, $K(\mathrm{DOAG})[n] = \mathbb{Z}^2$ for each $n \geq 1$, and $K(\mathrm{VAL}_{\mathrm{RCVF}}) = \mathbb{Z}[s]^{(2)} := \{(f, g) \in \mathbb{Z}[s] : f(0) = g(0)\}$.

Thus $K(\mathrm{RV}_{\mathrm{RCVF}}) = \mathbb{Z}[t] \otimes \mathbb{Z}[s]^{(2)} \leq \mathbb{Z}[t, s]^2$; and J is identified with the class $(1, 1) - (0, -s) - (t, t)$. Thus we obtain two homomorphisms $K(\mathrm{RV}_{\mathrm{RCVF}})/J \to \mathbb{Z}[s]$ (one mapping $t \mapsto 1$, the other with $t \mapsto 1 - s$; and as a pair they are injective).

Equivalently, we have found two ring homomorphisms $\chi, \chi' : K(\mathrm{RCVF}) \to \mathbb{Z}[t]$. One of these was found in [27].

Cluckers–Haskell

Take the theory of the p-adics. By Proposition 12.6 and Corollary 12.11 we obtain an isomorphism: $K(\mathrm{pCF}) \to K(\mathrm{RV}_{\mathrm{pCF}})/\mathrm{I}_{\mathrm{sp}}$. However, $\mathrm{RV}_{\mathrm{pCF}}$ is a finite extension of $\mathbb{Z}$, and evidently $K(\mathbb{Z}) = 0$, since $[[0, \infty)] = [[1, \infty)]$. Thus $K(\mathrm{pCF}) = 0$.

12.7 Higher-dimensional local fields

We have seen that the Grothendieck group of definable sets with volume forms loses a great deal of information compared to the semigroup. Over fields with discrete value groups, restricting to bounded sets is helpful; in this way the Grothendieck group retains information about volumes. In the case of higher-dimensional local fields, with value group $A = \mathbb{Z}^n$, simple boundedness is insufficient to save it from collapse. We show that using a simple-minded notion of boundedness is only partly helpful, and loses much of the volume information (all but one $\mathbb{Z}$ factor).

Example 12.12. Let ${K_\mu}^{\mathrm{bdd}}(\mathrm{Th}(\mathbb{C}((s_1))((s_2)))[n])$ be the Grothendieck ring of definable bounded sets and measure-preserving maps in $\mathbb{C}((s_1))((s_2))$ (with $\mathrm{val}(s_1) \ll \mathrm{val}(s_2)$). Let Q^t denote the class of the thin annulus of radius t. In particular, Q^0 is the volume of the units of the valuation ring. Then in ${K_\mu}^{\mathrm{bdd}}(\mathrm{Th}(\mathbb{C}((s_1))((s_2)))[2])$, we have, for example, $(Q^0)^2 = 0$. To see this directly, let

$$Y = \{(x, y) : \mathrm{val}(x) = 0, \mathrm{val}(y) = 0\},$$
$$X = \{(x, y) : 0 < 2\,\mathrm{val}(x) < \mathrm{val}(s_2), \mathrm{val}(x) + \mathrm{val}(y) = 0\}.$$

Then X is bounded. Let $f(x, y) = (x/s_1, s_1 y)$. Then f is a measure-preserving bijection $X \to X' = \{(x, y) : 0 < 2(\mathrm{val}(x) + \mathrm{val}(s_1)) < \mathrm{val}(s_2), \mathrm{val}(x) + \mathrm{val}(y) = 0\}$. But in $\mathbb{C}((s_1))((s_2))$, $2\,\mathrm{val}(x) < \mathrm{val}(s_2)$ iff $2(\mathrm{val}(x) + \mathrm{val}(s_1)) < \mathrm{val}(s_2)$, so $X'(\mathbb{C}((s_1))((s_2))) = X(\mathbb{C}((s_1))((s_2)) \cup Y(\mathbb{C}((s_1))((s_2)))$.

Remark 12.13. $(2[[0, y/2]]-[[0, y]])(2[[0, y/2)]-[[0, y)])$, is a class of the Grothendieck group of Γ that vanishes identically in the $\mathbb{Z}$-evaluation, but not in the $\mathbb{Z}^2$-evaluation.

13 The Grothendieck group of algebraic varieties

Let X, Y be smooth nonsingular curves in $\mathbb{P}^3$, or in some other smooth projective variety Z, and assume $Z \setminus X$, $Z \setminus Y$ are biregularly isomorphic. Say X, Y, Z are defined over $\mathbb{Q}$. Then for almost all p, $|X(\mathbb{F}_p)| = |Y(\mathbb{F}_p)|$, as one may see by counting points of Z, $Z \setminus X$ and subtracting. It follows from Weil's Riemann hypothesis for curves that X, Y have the same genus, from Faltings that X, Y are isomorphic if the genus is 2 or more, and from Tate that X, Y are isogenous if the genus is one. It was this observation that led Kontsevich and Gromov to ask if X, Y must actually be isomorphic. We show that this is the case below.[2]

Theorem 13.1. *Let X, Y be two smooth d-dimensional subvarieties of a smooth projective n-dimensional variety V, and assume $V \setminus X$, $V \setminus Y$ are biregularly isomorphic. Then X, Y are stably birational, i.e., $X \times \mathbb{A}^{n-d}$, $Y \times \mathbb{A}^{n-d}$ are birationally equivalent. If X, Y contain no rational curves, then X, Y are birationally equivalent.*

[2] This already follows from [22], who use different methods.

While we do not obtain a complete characterization in dimensions > 1, the results and method of proof do show that the answer lies in synthetic geometry and is not cohomological in nature.

Let Var_K be the category of algebraic varieties over a field K of characteristic 0.

Let $[X]$ denote the class of a variety X in the Grothendieck semigroup $K_+(\mathrm{Var}_K)$. We allow varieties to be disconnected. As all varieties will be over the same field K, we will write Var for Var_K. Let $K_+ \mathrm{Var}_n$ be the Grothendieck semigroup of varieties of dimension $\leq n$.

For the proof, we view K as a trivially valued subfield of a model of ACVF(0, 0). We work with the theory ACVF_K, so that "definable" means K-definable with quantifier-free ACVF-formulas.

Note that $\mathrm{RES} = \mathbf{k}^*$ in ACVF_K; the only definable point of Γ is 0, so the only definable coset of $\mathbf{k}^*$ is $\mathbf{k}^*$ itself.

The residue map is an isomorphism on K onto a subfield K_{RES} of the residue field $\mathbf{k}$. In particular, any smooth variety V over K lifts canonically to a smooth scheme $V_{\mathcal{O}} = V \otimes_K \mathcal{O}$ over $\mathcal{O}$, with generic fiber $V_{\mathrm{VF}} = V_{\mathcal{O}} \otimes_{\mathcal{O}} \mathrm{VF}$ and special fiber $V_{\mathcal{O}} \otimes_{\mathcal{O}} \mathbf{k} = V \otimes_K \mathbf{k}$. We have a reduction homomorphism $\rho_V : V(\mathcal{O}) \to V(\mathbf{k})$. We will write $V(\mathcal{O})$, $V(\mathrm{VF})$ for $V_{\mathcal{O}}(\mathcal{O})$, $V_{\mathrm{VF}}(\mathrm{VF})$.

Given $k \leq n$ and a definable subset X of RV^* of dimension $\leq k$, let $[X]_k$ be the class of X in $K_+ \mathrm{RV}[k] \subseteq K_+ \mathrm{RV}[\leq n]$. Thus if $\dim(X) = d$ we have $n - d + 1$ classes $[X]_k$, $d \leq k \leq n$, in different direct factors of $K_+ \mathrm{RV}[\leq n]$. We also use $[X]_k$ to denote the image of this class in $K_+ \mathrm{RV}[\leq n]/\mathrm{I_{sp}}$. This abuse of notation is not excessive since for $n \leq N$, $K_+ \mathrm{RV}[\leq n]/\mathrm{I_{sp}}$ embeds in $K_+ \mathrm{RV}[\leq N]/\mathrm{I_{sp}}$ (Lemma 8.7).

Let SD_d be the image of $K_+ \mathrm{RV}[\leq d]$ in $K_+ \mathrm{RV}[\leq N]/\mathrm{I_{sp}}$. Let WD^n_d be the subsemigroup of $\mathrm{RV}[n]$ generated by $\{[X] : \dim(X) \leq d\}$, and use the same letter to denote the image in $\mathrm{RV}[\leq N]/\mathrm{I_{sp}}$. Let $FD^n = SD_{n-1} + WD^n_{n-1}$. We write $a \sim b(FD^n_d)$ for $(\exists u, v \in FD^n_d)(a + u = b + v)$. More generally, for any subsemigroup S' of a semigroup S, write $a \sim b(S')$ for $(\exists u, v \in S)(a + u = b + v)$.

We write $K(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}$ for the groupification of $K_+(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}$.

Lemma 13.2. *Let V be a smooth projective* $\mathbf{k}$*-variety of dimension n, X a definable subset of $V(\mathbf{k})$. Then*

$$\oint [\rho_V{}^{-1}(X)] = [X]_n.$$

Proof. Let $\mathbf{X} = (X, f)$ where $f : X \to \mathrm{RV}^n$ is a finite-to-one map. We have to show that $[\mathbb{L}\mathbf{X}] = [\rho_V{}^{-1}(X)]$ in $K_+(\mathrm{VF}[n])$, i.e., that $\mathbb{L}\mathbf{X}$, $\rho_V{}^{-1}(X)$ are definably isomorphic. By Lemma 2.3 this reduces to the case that X is a point p. Find an open affine neighborhood U of V such that $\rho_V{}^{-1}(p) \subseteq U(\mathcal{O})$, and U admits an étale map $g : V \to \mathbb{A}^n$ over $\mathbf{k}$. Now $U(\mathcal{O}) \simeq \mathcal{O}^n \times_{\mathrm{res}, g} U(\mathbf{k})$. This reduces the lemma to the case of affine space, where it follows from the definition of $\mathbb{L}$. □

Lemma 13.3. *Let X be a K-variety of dimension $\leq d$.*

(1) $\oint(X(\mathrm{VF})) \in SD_d = K_+(\mathrm{RV}[\leq d])/\mathrm{I_{sp}}$.
(2) *If X is a smooth complete variety of dimension d, then $\oint X(\mathrm{VF}) = [X]_d$.*

(3) *If X is a variety of dimension d, then* $\oint X(\mathrm{VF}) \sim [X]_d(FD^d)$.

Proof.

(1) This is obvious, since $\dim(X(\mathrm{VF})) \leq d$.

(2) By Grothendieck's valuative criterion for properness, $X(\mathrm{VF}) = X(\mathcal{O})$. We thus have a map $\rho_V : X(\mathrm{VF}) = X(\mathcal{O}) \to X(\mathbf{k})$. For $\alpha \in X(\mathbf{k})$, let $X_\alpha(\mathrm{VF}) = {\rho_V}^{-1}(\alpha)$. Since X is smooth of dimension d it is covered by Zariski open neighborhoods U admitting an étale map $f_U : U \to \mathbb{A}^d$, defined over K; let $\mathcal{S}$ be a finite family of such pairs (U, f_U), with $\cup_{(U,f_U)\in\mathcal{S}} U = X$. We may choose a definable finite-to-one $f : X \to \mathbb{A}^d$, defined over K, such that for any $x \in X$, for some pair $(U, f_U) \in \mathcal{S}$, $f(x) = f_U(x)$. We have $\mathbb{L}([X]_d) = \mathbb{L}(X, f) = \mathrm{VF}^d \times_{\mathrm{rv},f} X(\mathbf{k})$. We have to show that $\mathbb{L}(X, f)$ is definably isomorphic to $X(\mathrm{VF})$. By Lemma 2.3 it suffices to show that for each $\alpha \in X(\mathbf{k})$, $\mathrm{VF}^d \times_{\mathrm{rv},f} \{a\}$ is α-definably isomorphic to $X_\alpha(\mathrm{VF})$. Now $\mathrm{VF}^d \times_{\mathrm{rv},f} \{a\} \underset{\alpha}{=} {\mathrm{rv}}^{-1}(f(\alpha))$. We have $f(\alpha) = f_U(\alpha)$ for some $(U, f) \in \mathcal{S}$ with $\alpha \in U$. Since f_U is étale, it induces a bijective map $U_\alpha(\mathrm{VF}) \to {\mathrm{rv}}^{-1}(f(\alpha))$. But $X_\alpha(\mathrm{VF}) = U_\alpha(\mathrm{VF})$, so the required isomorphism is proved.

(3) If X, Y are birationally equivalent, then $[X]_d \sim [Y]_d(WD^d_{<d})$, while $X(\mathrm{VF})$, $Y(\mathrm{VF})$ differ by VF-definable sets of dimension $< d$, so

$$\oint(X(\mathrm{VF})) \sim \oint(Y(\mathrm{VF}))(SD_d).$$

Using the resolution of singularities in the following form: every variety is birationally equivalent to a smooth nonsingular one; we are done by (2). With a more complicated induction we should be able to dispense with this use of Hironaka's theorem. □

Lemma 13.4. *Let V be a smooth projective K-variety, X, Y closed subvarieties, Let $F : V \setminus X \to V \setminus Y$ a biregular isomorphism. Let $V_{\mathcal{O}}$, V_{VF}, $V_{\mathbf{k}}$, F_{VF}, etc., be the objects obtained by base change. Then F_{VF} induces a bijection $V(\mathrm{VF}) \setminus X(\mathrm{VF}) \to V(\mathrm{VF}) \setminus Y(\mathrm{VF})$, and*

$$F_{\mathrm{VF}}({\rho_V}^{-1}(X) \setminus X(\mathrm{VF})) = {\rho_V}^{-1}(Y) \setminus Y(\mathrm{VF}).$$

Proof. The first statement follows from the Lefschetz principle since VF is algebraically closed.

Since V is projective, $V(\mathrm{VF}) = V(\mathcal{O})$, and one can define for $v \in V$ the valuative distance $d(v, X)$, namely, the greatest $\alpha \in \Gamma$ such that the image of x in $V(\mathcal{O}/\alpha)$ lies in $X(\mathcal{O}/\alpha)$.

Let $\mathbf{F}$ be the Zariski closure in V^2 of the graph of F. Then $\mathbf{F} \cap (V \setminus X) \times (V \setminus Y)$ is the graph of F. In fact, in any algebraically closed field L, we have

$$\text{if } a \in V(L) \setminus X(L) \quad \text{and} \quad (a, b) \in \mathbf{F}(L), \quad \text{then } b \in V(L) \setminus Y(L), \tag{13.1}$$

and conversely.

Suppose for the sake of contradiction that in some $M \models \mathrm{ACVF}_K$ there exist $a \in {\rho_V}^{-1}(X)$, $b \notin {\rho_V}^{-1}(Y)$, $(a, b) \in \mathbf{F}$. Thus $d(a, X) = \alpha > 0$, $d(b, Y) = 0$. Let

$$C = \{\gamma \in \Gamma : (\forall n \in \mathbb{N}) n\gamma < \alpha\}.$$

We may assume by compactness that $C(M) \neq \emptyset$. Let

$$I = \{y \in \mathcal{O}(M) : \mathrm{val}(y) \notin C\}$$

so that I is a prime ideal of $\mathcal{O}(M)$. Let L be the field of fractions of $\mathcal{O}(M)/I$. Let $\bar{a}, \bar{b}$ be the images of a, b in L. Then $(\bar{a}, \bar{b}) \in \bar{F}$, and $\bar{a} \in X$, $\bar{b} \notin Y$; contradicting (13.1). □

Proof of Theorem 13.1. By Lemma 13.4, there exists a definable bijection ${\rho_V}^{-1}(X) \setminus X \to {\rho_V}^{-1}(Y) \setminus Y$. Applying $\not\int : K(\mathrm{VF}[n]) \to K(\mathrm{RV}[\leq n])/\mathrm{I_{sp}}$, and using Lemmas 13.2 and 13.3, we have $[X]_n - [X]_d = [Y]_n - [Y]_d$. Applying the first retraction $K(\mathrm{RV}[\leq n])/\mathrm{I_{sp}} \to K(\mathrm{RES}[n])$ of Theorem 10.5, we obtain

$$[X_n] - [X \times \mathbb{A}^{n-d}]_n = [Y]_n - [Y \times \mathbb{A}^{n-d}]_n$$

in $!\,K(\mathrm{RES}[n]) = K(\mathrm{Var}_n)$. Thus

$$[X \times \mathbb{A}^{n-d} \dot{\cup} Y]_n + [Z] = [Y \times \mathbb{A}^{n-d} \dot{\cup} X]_n + [Z]$$

for some Z with $\dim(Z) \leq n$, where now the equality is of classes in $K_+ \mathrm{Var}_n$. Counting birational equivalence classes of varieties of dimension n, we see that $X \times \mathbb{A}^{n-d}$, $Y \times \mathbb{A}^{n-d}$ must be birationally equivalent. The last sentence follows from the lemma below. □

Lemma 13.5. *Let X, Y be varieties containing no rational curve. Let U be a variety such that there exists a surjective morphism $\mathbb{A}^m \to U$. If $X \times U$, $Y \times U$ are birationally equivalent, then so are X, Y.*

Proof. For any variety W, let $\mathcal{F}(W)$ be the set of all rational maps $g : \mathbb{A}^1 \to W$. Write $\mathrm{dom}(g)$ for the maximal subset of $\mathbb{A}^1$ where g is regular; so $\mathrm{dom}(g)$ is cofinite in $\mathbb{A}^1$. Let $R_W = \{(g(t), g(t')) \in W^2 : g \in \mathcal{F}(W), t, t' \in \mathrm{dom}(g)\}$. Let E_W be the equivalence relation generated by R_W, on points in the algebraic closure. R_W, E_W may not be constructible in general, but in the case we are concerned with, they are as follows.

Claim. Let $W \subseteq X \times U$ be a Zariski dense open set. Let $\pi : W \to X$ be the projection. Then $\pi(w) = \pi(w')$ iff $(w, w') \in E_W$ iff $(w, w') \in R_W$.

Proof. If $g \in \mathcal{F}(U)$, then $\pi \circ g : \mathrm{dom}(g) \to X$ is a regular map; hence by assumption on X it is constant. It follows that if $(w, w') \in R_U$ then $\pi(w) = \pi(w')$, and hence if $(w, w') \in E_U$ then $\pi(w) = \pi(w')$. Conversely, assume $w', w'' \in W$ and $\pi(w') = \pi(w'')$; then $w' = (x, u')$, $w' = (x, u'')$ for some $x \in X$, $u', u'' \in U$. Let $U_x = \{u \in U : (x, u) \in W\}$. Since W is open, U_x is open in U. Let $h : \mathbb{A}^m \to U$ be a surjective morphism; let $h(v') = u', h(v'') = u''$. The line through v', v'' intersects $h^{-1}(U_x)$ in a nonempty open set. This gives a regular map f from the affine lines, minus finitely many points, into U, passing through u', u''. Thus $t \mapsto (x, f(t))$ gives a rational map from $\mathbb{A}^1$ to W, passing through (w', w''); and so $(w', w'') \in R_U$ and certainly in E_U. □

Using the claim, we prove the lemma. Let $W_X \subseteq X \times U$, $W_Y \subseteq Y \times U$ be Zariski dense open, and $F : W_X \to W_Y$ a biregular isomorphism. Then F takes E_{W_X} to E_{W_Y}. Moving now to the category of constructible sets and maps, quotients by constructible equivalence relations exist, and W_X/E_{W_X} is isomorphic as a constructible set to W_Y/E_{W_Y}. Let $\pi_X : W_X \to X$, $\pi_Y : W_Y \to Y$ be the projections. By the claim, $W_X/E_{W_X} = \pi_X(W_X) =: X'$. Similarly, $W_Y/E_{W_Y} = \pi_Y(W_Y) =: Y'$. Now since W_X, W_Y are Zariski dense, so are X', Y'. Thus X, Y contain isomorphic Zariski dense constructible sets, so they are birationally equivalent. □

Remark. The condition on X, Y may be weakened to the statement that they contain no rational curve through a generic point; i.e., that there exist proper subvarieties $(X_i : i \in I)$ defined over K, such that for any field $L \supset K$, any rational curve on $X \times_K L$ is contained in some $X_i \times_K L$.

Acknowledgments. Thanks to Aviv Tatarsky and Moshe Kaminsky, and to Lou Van den Dries, Clifton Ealy, and Jana Maříková for useful comments and corrections.

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